Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

6 May 2024 —TZ1

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

1. [Maximum mark: 27]

This question considers two possible models for the occurrence of random events in a computer game.

In a new computer game, each time a player performs an action, there is a random chance that the action will be *boosted*, meaning that it provides a benefit to the player.

The designer of this computer game is considering two possible models for when to boost an action.

In the first model, the probability that an action will be boosted is constant.

(a) Suppose the probability that an action will be boosted is 0.1.

(i) Find the probability that the first boost occurs on the third action. [2]

(ii) Find the probability that at least one boost occurs in the first six actions. [3]

(b) Suppose the probability that an action will be boosted is p, where $0 < p < 1$.

(i) Explain why the probability that the first boost occurs on the x^{th} action is $p(1-p)^{x-1}$. [1]

Let X be the number of actions until the first boost occurs.

(ii) Hence, write down an expression, using sigma notation, for $\mathrm{E}(X)$ in terms of x and p. [1]

Consider the sum of an infinite geometric sequence, with first term a and common ratio r $(|r| < 1)$,

$$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}.$$

(c) (i) By differentiating both sides of the above equation with respect to r, find an expression for $\sum_{n=1}^{\infty} nar^{n-1}$ in terms of a and r. [4]

(ii) Hence, show that $\mathrm{E}(X) = \dfrac{1}{p}$. [2]

(This question continues on the following page)

In the first model, the probability that an action will be boosted is constant.

(a) Suppose the probability that an action will be boosted is 0.1.

(i) Find the probability that the first boost occurs on the third action.

(ii) Find the probability that at least one boost occurs in the first six actions.

(a) (i) No boost × No boost × boost.

$$= (1-0.1) \times (1-0.1) \times 0.1$$

$$= 0.9 \times 0.9 \times 0.1$$

$$= \boxed{0.081}$$

(ii) $n = 6, \quad p = 0.1 \Rightarrow X \sim B(n, p)$

$X \sim B(6, 0.1)$

$P(X \geq 1) = \boxed{0.469}$

Casio CG-50

Lower = 1

upper = 999

Num. = 6

$p = 0.1$

(b) Suppose the probability that an action will be boosted is p, where $0 < p < 1$.

(i) Explain why the probability that the first boost occurs on the x^{th} action is $p(1-p)^{x-1}$. [1]

No boost × No boost . . . . $(x-1)$ time × boost

$= (1-p)(1-p) \cdots (x-1) \text{ times} \times p$

$= (1-p)^{x-1} p$

$= p(1-p)^{x-1}$

Let X be the number of actions until the first boost occurs.

(ii) Hence, write down an expression, using sigma notation, for $\mathrm{E}(X)$ in terms of x and p. [1]

$E(X) = \sum_{i=1}^{n} x_i p_i$ formula.

$$\boxed{E(X) = \sum_{x=1}^{\infty} x\, p(1-p)^{x-1}}$$

Consider the sum of an infinite geometric sequence, with first term a and common ratio r $(|r| < 1)$,

$$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}.$$

(c) (i) By differentiating both sides of the above equation with respect to r, find an expression for $\sum_{n=1}^{\infty} nar^{n-1}$ in terms of a and r. [4]

$$\frac{d}{dr}\left(a+ar+ar^2+ar^3+\cdots\right) = \frac{d}{dr}\left(\frac{a}{1-r}\right)$$

$$= \frac{d}{dr}\left(a(1-r)^{-1}\right)$$

$$0 + a + 2ar + 3ar^2 + \cdots = a(-1)(1-r)^{-2} \times (-1)$$

$$= + \frac{a}{(1-r)^2}$$

$$\therefore \sum_{n=1}^{\infty} nar^{n-1} = \boxed{\frac{a}{(1-r)^2}}$$

(ii) Hence, show that $\mathrm{E}(X) = \dfrac{1}{p}$.

$$E(x) = \sum x_i p_i$$

Here $a = p$, $r = 1-p$

$$\therefore E(x) = p \times \frac{1}{(1-(1-p))^2} = \frac{p}{(p)^2}$$

$$\therefore \boxed{E(x) = \frac{1}{p}}$$

(This question continues on the following page)

(Question 1 continued)

It can be shown that $\mathrm{Var}(X) = \dfrac{1-p}{p^2}$.

(d) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ when $p = 0.1$. [2]

In the designer's second model, the initial probability that an action is boosted is 0.2, and each time an action occurs that is not boosted, the probability that the next action is boosted increases by 0.2. After an action has been boosted, the probability resets to 0.2 for the next action.

(e) Show that the probability that the first boost occurs on the third action is 0.288. [2]

Let Y be the number of actions until the first boost occurs.

(f) Explain why $Y \leq 5$. [1]

The following table shows the probability distribution of Y.

y	1	2	3	4	5
$\mathrm{P}(Y = y)$	0.2	m	0.288	n	0.0384

(g) (i) Find the value of m and the value of n. [2]

(ii) Show that $\mathrm{E}(Y) = 2.5104$. [2]

(iii) Find $\mathrm{Var}(Y)$. [2]

(h) (i) Use the expression given in (c)(ii) to find the value of p for which $\mathrm{E}(X) = \mathrm{E}(Y)$. [1]

(ii) Find $\mathrm{Var}(X)$ for this value of p. [1]

(iii) Hence determine, with a reason, which model provides a more consistent experience for the player with respect to boosted actions. [1]

(Question 1 continued)

$\left\{ \begin{array}{l} 4:30 - 5:30 \\ \\ \end{array} \right.$

It can be shown that $\mathrm{Var}(X) = \dfrac{1-p}{p^2}$.

(d) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ when $p = 0.1$. [2]

$$E(x) = \frac{1}{0.1} = \boxed{10}$$

$$\& \ Var(X) = \frac{1-0.1}{(0.1)^2} = \boxed{90}$$

In the designer's second model, the initial probability that an action is boosted is 0.2, and each time an action occurs that is not boosted, the probability that the next action is boosted increases by 0.2. After an action has been boosted, the probability resets to 0.2 for the next action.

(e) Show that the probability that the first boost occurs on the third action is 0.288. [2]

Initially $p = 0.2$, $q = 1 - 0.2 = 0.8$

1^{st} not boost & 2^{nd} boost.

$0.8 \times (0.2 + 0.2) = 0.4$

1^{st} not boost $\times$ 2^{nd} not boost $\times$ 3^{rd} boost.

$\underset{p=0.4}{}$ $\underset{p=0.6}{}$

$0.8 \times (1 - 0.4) \times (0.6)$

$P(3^{rd} \text{ boost}) = 0.8 \times 0.6 \times 0.6 = \boxed{0.288}$

Let Y be the number of actions until the first boost occurs.

(f) Explain why $Y \leq 5$. [1]

$p=0.2$ $p=0.4$ $p=0.6$ $p=0.8$ $p=1$

Not boost × Not boost × Not boost × Not boost × boost

$(1-0.2) \times (1-0.4)$ $(1-0.6) \times (1-0.8) \times 1$

1 2 3 4 5

$\therefore$ First boost must occur in 5^{th} action

hence $Y \leq 5$

The following table shows the probability distribution of Y.

y	1	2	3	4	5
$P(Y=y)$	0.2	m	0.288	n	0.0384

0.32 0.1536

(g) (i) Find the value of m and the value of n. [2]

(ii) Show that $E(Y) = 2.5104$. [2]

(iii) Find $Var(Y)$. [2]

(i) $P(Y=2) = \text{no boost} \times \text{boost}$

$= (1-0.2) \times 0.4$

$= 0.8 \times 0.4$

$m = \boxed{0.32}$

$P(Y=4) = \text{no boost} \times \text{no boost} \times \text{no boost} \times \text{boost}$

$= (1-0.2) \times (1-0.4) \times (1-0.6) \times 0.8$

$= 0.8 \times 0.6 \times 0.4 \times 0.8$

$\therefore n = \boxed{0.1536}$

(ii) $E(Y) = 1 \times 0.2 + 2 \times 0.32 + 3 \times 0.288 + 4 \times 0.1536 + 5 \times 0.0384$

$E(Y) = \boxed{2.5104}$

(iii) $Var(Y) = E(Y^2) - (E(Y))^2$

$= (1^2 \times 0.2 + 2^2 \times 0.32 + 3^2 \times 0.288 + 4^2 \times 0.1536 + 5^2 \times 0.0384) - (2.5104)^2$

$= 1.18749\ldots$

$Var(Y) = \boxed{1.19}$

(h) (i) Use the expression given in (c)(ii) to find the value of p for which $\mathrm{E}(X) = \mathrm{E}(Y)$. [1]

(ii) Find $\mathrm{Var}(X)$ for this value of p. [1]

(iii) Hence determine, with a reason, which model provides a more consistent experience for the player with respect to boosted actions. [1]

(i) $E(x) = \frac{1}{p}$

$\therefore E(x) = E(y)$

$\frac{1}{p} = 2.5104$

$\therefore \boxed{p = 0.398}$

(ii) $\mathrm{Var}(X) = \frac{1-p}{p^2}$.

$= \frac{(1 - 0.39834289\ldots)}{(0.39834289\ldots)^2}$

$\boxed{Var(x) = 3.79}$

(iii) $Var(x) = 3.79$

$Var(Y) = 1.19$ ✓

$\therefore Var(x) > Var(Y)$

$\therefore$ 2nd model is more consistent.