Lesson–4

Complex Numbers

AHL 1.13

Content	Guidance, clarification and syllabus links
Modulus–argument (polar) form: $z = r(\cos\theta + i\sin\theta) = r\text{cis}\theta$ Euler form: $z = re^{i\theta}$ Sums, products and quotients in Cartesian, polar or Euler forms and their geometric interpretation.	The ability to convert between Cartesian, modulus-argument (polar) and Euler form is expected.

AHL 1.14

Content	Guidance, clarification and syllabus links
Complex conjugate roots of quadratic and polynomial equations with real coefficients.	Complex roots occur in conjugate pairs.
De Moivre's theorem and its extension to rational exponents.	Includes proof by induction for the case where $n \in \mathbb{Z}^+$; awareness that it is true for $n \in \mathbb{R}$.
Powers and roots of complex numbers.	**Link to:** sum and product of roots of polynomial equations (AHL 2.12), compound angle identities (AHL 3.10).

- **A** Complex numbers
- **B** The sum of two squares factorisation
- **C** Operations with complex numbers
- **D** Equality of complex numbers
- **E** Properties of complex conjugates

Any number of the form $a+bi$ where $a, b \in \mathbb{R}$ and $i=\sqrt{-1}$, is called a **complex number**.

REAL AND IMAGINARY PARTS

If $z=a+bi$ where $a, b \in \mathbb{R}$, then:

- a is the **real part** of z, written $\mathcal{Re}(z)$
- b is the **imaginary part** of z, written $\mathcal{Im}(z)$.

For example:

- If $z=2+3i$, then $\mathcal{Re}(z)=2$ and $\mathcal{Im}(z)=3$.

COMPLEX CONJUGATES

Suppose $z=a+bi$ where $a, b \in \mathbb{R}$.

The **complex conjugate** of z is $z^*=a-bi$.

$\mathcal{Re}(z^*)=\mathcal{Re}(z)$ and $\mathcal{Im}(z^*)=-\mathcal{Im}(z)$

$$(a+bi)+(c+di)=(a+c)+(b+d)i \qquad \textbf{addition}$$
$$(a+bi)-(c+di)=(a-c)+(b-d)i \qquad \textbf{subtraction}$$
$$(a+bi)(c+di)=ac+adi+bci+bdi^2 \qquad \textbf{multiplication}$$

Two complex numbers are **equal** when their **real parts** are equal and their **imaginary parts** are equal.

$$a+bi=c+di \;\Leftrightarrow\; a=c \;\text{ and }\; b=d.$$

MODULUS

The **modulus** of the complex number $z = a + bi$ is the length of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$, which is the real number $|z| = \sqrt{a^2 + b^2}$.

PROPERTIES OF MODULUS

- $|z^*| = |z|$
- $|z|^2 = zz^*$
- $|z_1 z_2| = |z_1||z_2|$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$
- $|z_1 z_2 z_3 \ldots . z_n| = |z_1||z_2||z_3| \ldots . |z_n|$ and $|z^n| = |z|^n$ for $n \in \mathbb{Z}^+$.

$$(2+3i) \pm (3+4i) = (2 \pm 3) + (3 \pm 4)i$$

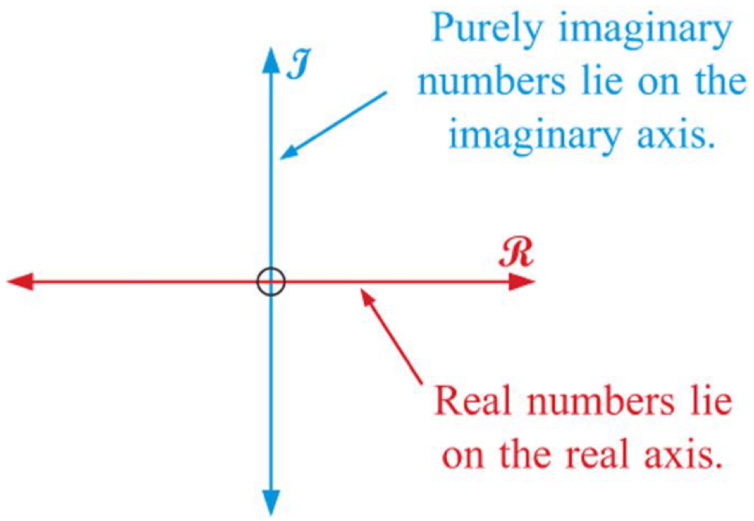
$$= \underline{\underline{5+7i}}$$

$$(2+3i)-(3+4i) = \underline{\underline{-1-i}}$$

Complex Numbers–2

- **A** The complex plane
- **B** Modulus and argument
- **C** Geometry in the complex plane
- **D** Polar form
- **E** Euler's form
- **F** De Moivre's theorem
- **G** Roots of complex numbers

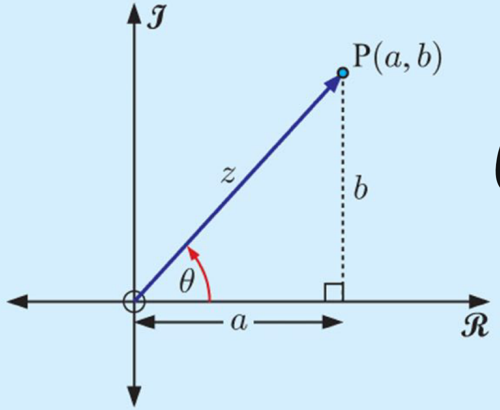
Complex plane or Argand plane:

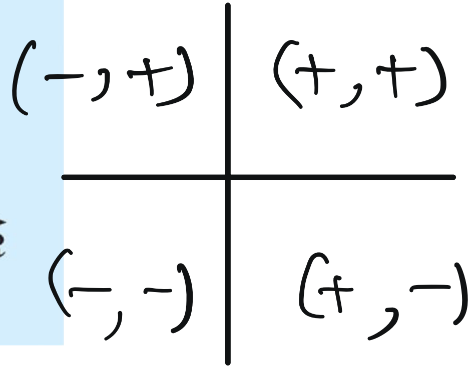


ARGUMENT

Suppose the complex number $z = a + bi$ is represented by the vector $\overrightarrow{OP}$.

The **argument** of z, or simply $\arg z$, is the angle θ, where $-\pi < \theta \leqslant \pi$ is measured anticlockwise between the positive real axis and $\overrightarrow{OP}$.





Real numbers (other than zero) have argument 0 or π.

Purely imaginary numbers have argument $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

If $z = a + bi$ then

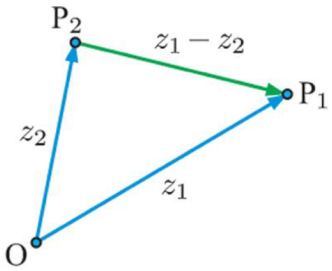
$$\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$$

GEOMETRY IN THE COMPLEX PLANE

DISTANCES

Suppose P_1 and P_2 are two points in the complex plane which correspond to the complex numbers z_1 and z_2.

$$\begin{aligned}
\overrightarrow{P_2P_1} &= \overrightarrow{P_2O} + \overrightarrow{OP_1} \\
&= -z_2 + z_1 \\
&= z_1 - z_2 \\
\therefore \quad |z_1 - z_2| &= |\overrightarrow{P_2P_1}| \\
&= \text{distance between } P_1 \text{ and } P_2.
\end{aligned}$$



If $z_1 \equiv \overrightarrow{OP_1}$ and $z_2 \equiv \overrightarrow{OP_2}$ then $|z_1 - z_2|$ is the distance between the points P_1 and P_2.

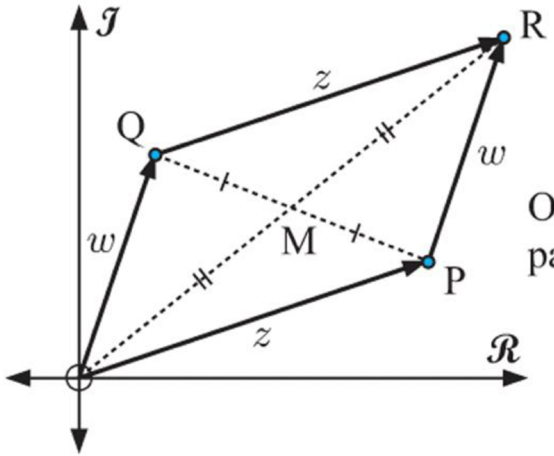
MIDPOINTS

The parallelogram alongside is formed using complex numbers w and z.

The diagonals of the parallelogram are $\overrightarrow{OR}$ and $\overrightarrow{PQ}$, and these diagonals bisect each other.

Now $\quad \overrightarrow{OR} \equiv w + z$

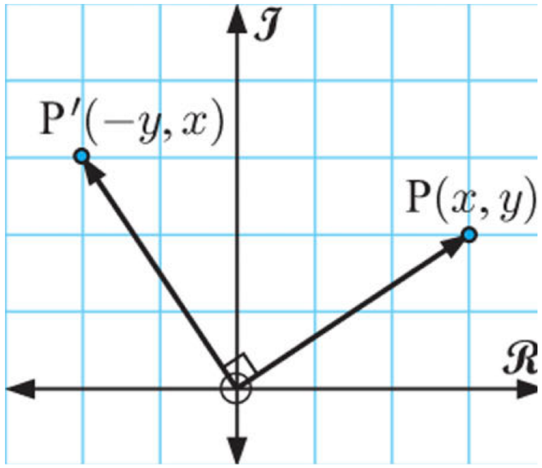
$$\therefore \quad \overrightarrow{OM} \equiv \frac{w+z}{2}.$$



OPRQ is a parallelogram

Transformation of z to iz is shown in this figure.
If z = x + iy, then

$$iz = i\left(x + iy\right)$$
$$= ix + i^2 y$$
$$= -y + ix$$



$$Now\ |z| = \sqrt{x^2 + y^2}$$
$$and\ |iz| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} \quad \Rightarrow \quad OP' = OP$$

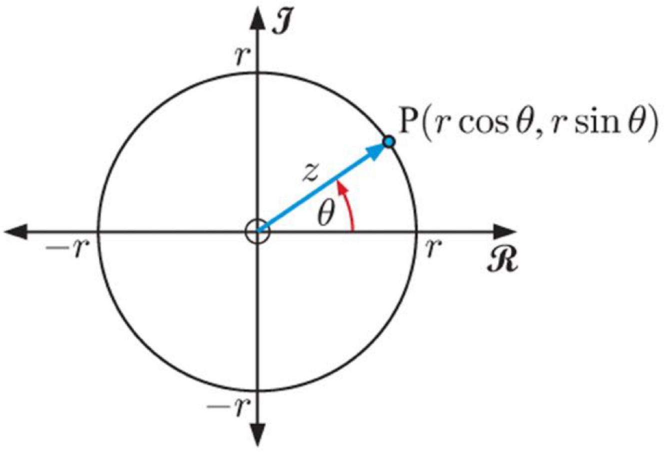
POLAR FORM

We have seen that the **Cartesian form** of a complex number is $z = a + bi$.

Polar form of z is based on the Modulus and argument of z.

Any point P which lies on a circle with centre O(0, 0) and radius r, has Cartesian coordinates $(r\cos\theta, r\sin\theta)$.

$\therefore$ on the Argand plane, the complex number represented by $\overrightarrow{\text{OP}}$ is $z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and $\theta = \arg z$.



We define $\boldsymbol{\operatorname{cis}\theta = \cos\theta + i\sin\theta}$

so $z = |z|\operatorname{cis}\theta$.

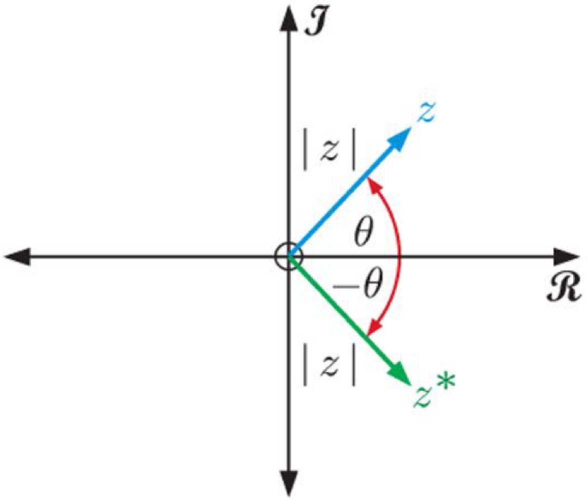
A complex number z has **polar form** $z = |z|\operatorname{cis}\theta$ where $|z|$ is the **modulus** of z, θ is the **argument** of z, and $\operatorname{cis}\theta = \cos\theta + i\sin\theta$.

Polar form is also called **modulus-argument form**.

Polar form is very helpful in dealing with multiplication and division of complex numbers.

The conjugate of z has the same length as z, and its argument is $(-\theta)$.

If $z = |z| \operatorname{cis}\theta$, then $z^* = |z| \operatorname{cis}(-\theta)$.



PROPERTIES OF cis θ

Three useful properties of cis θ are:

- $\text{cis}\,\theta \times \text{cis}\,\phi = \text{cis}(\theta + \phi)$
- $\dfrac{\text{cis}\,\theta}{\text{cis}\,\phi} = \text{cis}(\theta - \phi)$
- $\text{cis}(\theta + k2\pi) = \text{cis}\,\theta$ for all $k \in \mathbb{Z}$.

$\cos\theta = \cos(\theta + k2\pi)$

$\sin\theta = \sin(\theta + k2\pi)$

MULTIPLICATION OF COMPLEX NUMBERS

Suppose $z = |z| \operatorname{cis}\theta$ and $w = |w| \operatorname{cis}\phi$

Then $zw = |z| \operatorname{cis}\theta \times |w| \operatorname{cis}\phi$

$$= \underbrace{|z||w|}_{\text{non-negative}} \operatorname{cis}(\theta + \phi) \qquad \{\text{property of cis}\}$$

$\therefore \quad |zw| = |z||w|$ and $\arg zw = \theta + \phi = \arg z + \arg w$.

If a complex number z is multiplied by $r \operatorname{cis}\theta$ then its modulus is *multiplied* by r and its argument is *increased* by θ.

Q(1) (No Calculator)

[Maximum mark: 7]

The complex numbers w and z satisfy the equations

$$\frac{z}{w} = \mathrm{i},$$

$$w^* + 2z = 4 + 5\mathrm{i}.$$

Find w and z in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{Z}$.

$$z = iw$$

$$\therefore \quad w^* + 2z = 4 + 5i \text{ ——— (i)}$$

Let $w = x + yi \rightarrow w^* = x - yi$

$$\therefore \quad z = i(x + yi) = xi + yi^2 = \underline{\underline{-y + xi}}$$

$\therefore$ eqn. (i) will be

$$(w^* + 2z) = 4+5i$$

$$(x - yi) + 2(-y + xi) = 4+5i$$

$$(x-2y) + (2x-y)i = 4+5i$$

Comparing real & imaginary parts, we have

$$x - 2y = 4 \quad \text{—— (ii)}$$

$$2x - y = 5 \quad \text{—— (iii)}$$

Solving (ii) & (iii), we have

$$x = 2, \; y = -1 \Rightarrow \therefore \boxed{w = 2 - i}$$

$$\& \boxed{z = 1+2i}$$

EULER'S FORM

For any $\theta \in \mathbb{R}$, $\boldsymbol{e^{i\theta} = \cos\theta + i\sin\theta}$.

This identity allows us to write any complex number $z = |z| \operatorname{cis}\theta$ in the **Euler form** $z = |z| e^{i\theta}$.

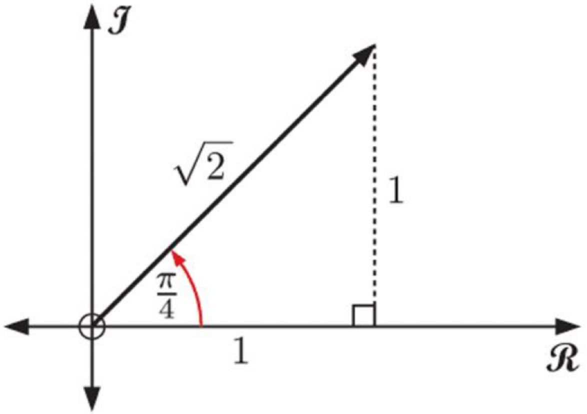
For example, consider $z = 1 + i$.

$|z| = \sqrt{2}$ and $\theta = \frac{\pi}{4}$,

$\therefore \quad 1 + i = \sqrt{2} \operatorname{cis} \frac{\pi}{4} = \sqrt{2} e^{i\frac{\pi}{4}}$

So, $\sqrt{2} \operatorname{cis} \frac{\pi}{4}$ is the **polar form** of $1 + i$

and $\sqrt{2} e^{i\frac{\pi}{4}}$ is the **Euler form** of $1 + i$.



14E.1

2 Write:

a $-1+i$ in polar form and Euler form

b $3\operatorname{cis}\left(-\frac{\pi}{6}\right)$ in Cartesian form and Euler form

c $2e^{i\frac{2\pi}{3}}$ in Cartesian form and polar form.

(a) Let $z_1 = -1+i$

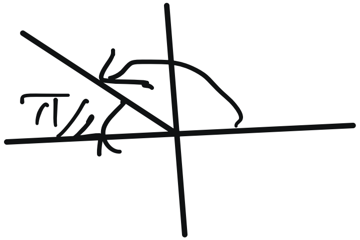
$\therefore |z_1| = \sqrt{2}$ & $\arg(z_1) = \frac{3\pi}{4}$

In polar form:

$z_1 = \sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

$\therefore$ In Euler form

$z_1 = \sqrt{2}\, e^{i\left(\frac{3\pi}{4}\right)}$



b $3\operatorname{cis}\left(-\frac{\pi}{6}\right)$ in Cartesian form and Euler form

Let $Z_2 = 3\operatorname{cis}\left(-\frac{\pi}{6}\right)$

$$Z_2 = r(\cos\theta + i\sin\theta)$$

$$= 3\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$= 3\left(\frac{\sqrt{3}}{2} - i\left(\frac{1}{2}\right)\right)$$

$$= \frac{3\sqrt{2}}{2} - \frac{3}{2}i \quad \text{(Cartesian form)}$$

$$\& \; Z_2 = 3e^{i\left(-\frac{\pi}{6}\right)} \quad \text{(Euler Form)}$$

c $2e^{i\frac{2\pi}{3}}$ in Cartesian form and polar form. (HW)

DE MOIVRE'S THEOREM

Polar form enables us to easily calculate powers of complex numbers.

For example, if $z = |z| \operatorname{cis}\theta$

then
$$\begin{aligned} z^2 &= |z|\operatorname{cis}\theta \times |z|\operatorname{cis}\theta \\ &= |z|^2 \operatorname{cis}(\theta+\theta) \\ &= |z|^2 \operatorname{cis} 2\theta \end{aligned}$$

and
$$\begin{aligned} z^3 &= z^2 z \\ &= |z|^2 \operatorname{cis} 2\theta \times |z|\operatorname{cis}\theta \\ &= |z|^3 \operatorname{cis}(2\theta+\theta) \\ &= |z|^3 \operatorname{cis} 3\theta \end{aligned}$$

The generalisation of this process is **De Moivre's Theorem**:

$$(|\boldsymbol{z}|\,\mathbf{cis}\,\boldsymbol{\theta})^n = |\boldsymbol{z}|^n\,\mathbf{cis}\,\boldsymbol{n\theta} \quad \text{for all} \quad n \in \mathbb{Q}.$$

Q(2) (No Calculator)

[Maximum mark: 6]

Consider the complex number $z = \frac{w_1}{w_2}$ where $w_1 = \sqrt{2} + \sqrt{6}\mathrm{i}$ and $w_2 = 3 + \sqrt{3}\mathrm{i}$.

(a) Express w_1 and w_2 in modulus-argument form and write down

(i) the modulus of z;

(ii) the argument of z. [4]

(b) Find the smallest positive integer value of n such that z^n is a real number. [2]

(a) $w_1 = \sqrt{2} + \sqrt{6}\,i$

$\therefore |w_1| = \sqrt{2+6} = 2\sqrt{2}$ & $\arg(w_1) = \frac{\pi}{3}$

$w_2 = 3 + \sqrt{3}\,i$

$|w_2| = \sqrt{12} = 2\sqrt{3}$, $\arg(w_2) = \frac{\pi}{6}$

(i) $z = \dfrac{w_1}{w_2}$

$\therefore |z| = \left|\dfrac{w_1}{w_2}\right| = \dfrac{|w_1|}{|w_2|} = \dfrac{2\sqrt{2}}{2\sqrt{3}} = \dfrac{\sqrt{2}}{\sqrt{3}} = \boxed{\dfrac{\sqrt{6}}{3}}$

(ii) $\arg(z) = \arg\left(\dfrac{w_1}{w_2}\right) = \arg(w_1) - \arg(w_2)$

$= \dfrac{\pi}{3} - \dfrac{\pi}{6}$

$\therefore \arg(z) = \boxed{\dfrac{\pi}{6}}$

(b) n^{th} power of z is a real number,

i.e. z^n is a real number.

If $\arg(z^n) = k\pi$, $k \in \mathbb{Z}$

$$n \arg(z) = k\pi$$

$$n \frac{\pi}{6} = k\pi$$

$$\Rightarrow \boxed{n = 6}$$

is smallest positive integer

ROOTS OF COMPLEX NUMBERS

SOLVING $z^n = c$

Consider an equation of the form $z^n = c$ where n is a positive integer and c is a complex number.

The **nth roots of the complex number c** are the n solutions of $z^n = c$.

For example, the 4th roots of $2i$ are the four solutions of $z^4 = 2i$.

- There are **exactly** n nth roots of c.
- If $c \in \mathbb{R}$, the complex roots must occur in conjugate pairs.
- If $c \notin \mathbb{R}$, the complex roots do not all occur in conjugate pairs.
- The roots of z^n will all have the same modulus which is $|c|^{\frac{1}{n}}$.
- On an Argand diagram, the roots all lie on a circle with radius $|c|^{\frac{1}{n}}$.
- The roots on the circle $r = |c|^{\frac{1}{n}}$ will be equally spaced around the circle.
  If you join the points corresponding to the roots, you will obtain a regular polygon.

1 Find the three cube roots of 1 using:

a factorisation **b** the "nth roots method"

(a) $z^3 = 1$

$\therefore z^3 - 1^3 = 0$ $\qquad (a^3 - b^3 = (a-b)(a^2+ab+b^2))$

$(z-1)(z^2+z+1) = 0$

$z = 1$, $\quad z^2+z+1 = 0 \Rightarrow z = \dfrac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)} = \dfrac{-1\pm\sqrt{-3}}{2}$

$\therefore z = \dfrac{-1 \pm \sqrt{3}i}{2}$

$\therefore z = 1$, $\quad z = \dfrac{-1+\sqrt{3}i}{2}$, $\quad z = \dfrac{-1-\sqrt{3}i}{2}$

are three cube roots of unity.

b the "*n*th roots method"

$$z^3 = 1 = 1\,\mathrm{cis}(0 + k2\pi), \quad k \in \mathbb{Z} \text{ (polar form)}$$

$$\therefore z = \left[1\,\mathrm{cis}(0 + k2\pi)\right]^{1/3}$$

$$z = 1^{1/3}\,\mathrm{cis}\left(\frac{0 + k2\pi}{3}\right)$$

$$z = 1\,\mathrm{cis}\left(\frac{k2\pi}{3}\right)$$

For

$k = 0 \rightarrow$ $z_1 = 1\,\mathrm{cis}(0) = 1$ ✓

$k = 1 \rightarrow$ $z_2 = 1\,\mathrm{cis}\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ ✓

$k = 2 \rightarrow$ $z_3 = 1\,\mathrm{cis}\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ ✓

are three cube roots of unity.

6 Solve the following equations, and display the solutions for each on an Argand diagram:

5.1 **a** $z^3 = 2 + 2i$ **b** $z^3 = -2 + 2i$ **c** $z^2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$$(a)\quad z^3 = 2 + 2i = 2\sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4} + k2\pi\right),\ k \in \mathbb{Z}$$

$$\therefore z = \left[2\sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4} + k2\pi\right)\right]^{1/3}$$

$$z = (2\sqrt{2})^{1/3}\,\mathrm{cis}\left(\frac{\pi}{12} + \frac{k2\pi}{3}\right)$$

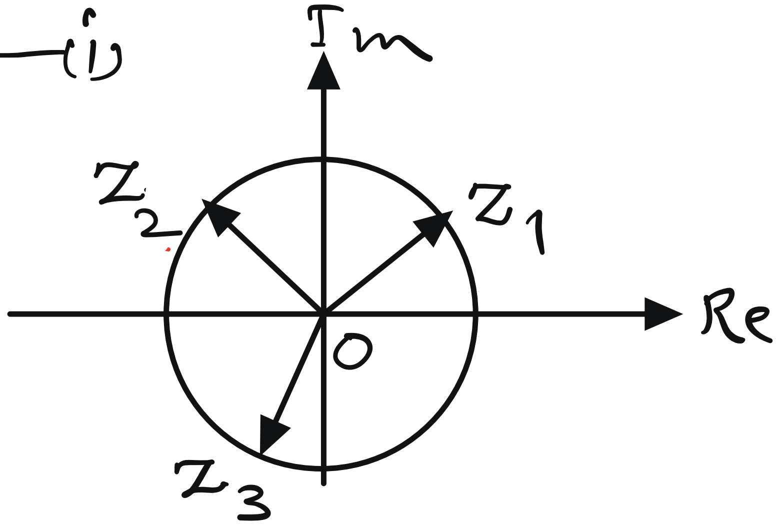
$$z = \sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{12} + \frac{k8\pi}{12}\right) \text{ — (i)}$$

For

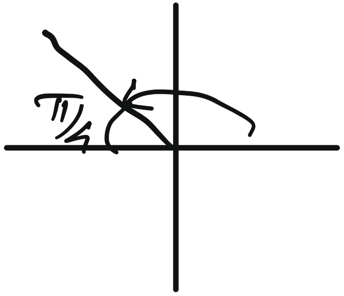
$k = -1 \rightarrow z_3 = \sqrt{2}\,\mathrm{cis}\left(-\frac{7\pi}{12}\right)$

$k = 0 \rightarrow z_1 = \sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{12}\right)$

$k = 1 \rightarrow z_2 = \sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4}\right)$



b $z^3 = -2 + 2i$



$$z^3 = -2 + 2i$$

$$|z^3| = \sqrt{8} = 2\sqrt{2}$$

$$\arg(z^3) = \frac{3\pi}{4}$$

$$\therefore z^3 = 2\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4} + k2\pi\right), \quad k \in Z$$

$$z = \left[2\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4} + k2\pi\right)\right]^{1/3}$$

$$z = (2\sqrt{2})^{1/3}\,\text{cis}\left(\frac{3\pi}{12} + \frac{k2\pi}{3}\right) \quad \text{(De Moivre)}$$

$$z = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{12} + k\frac{8\pi}{12}\right)$$

b $z^3 = -2 + 2i$

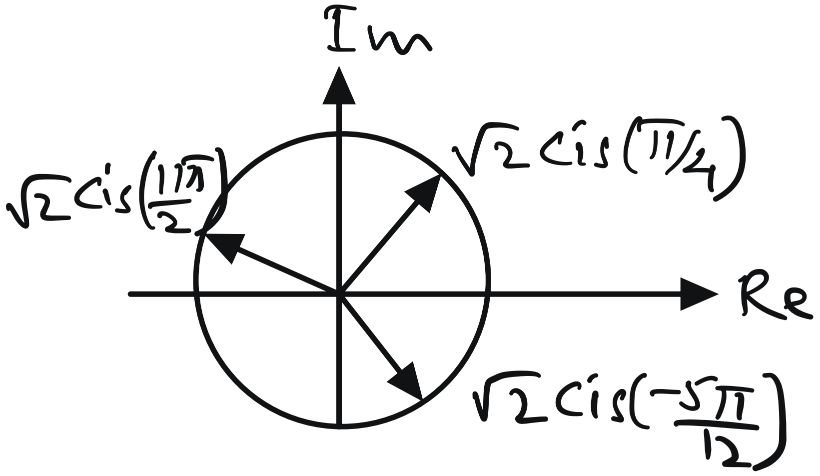
$$\therefore z = \sqrt{2}\ \text{cis}\left(\frac{3\pi}{12} + k\,\frac{8\pi}{12}\right)$$

For

$k = -1 \longrightarrow z_1 = \sqrt{2}\,\text{cis}\left(\frac{3\pi - 8\pi}{12}\right) = \sqrt{2}\,\text{cis}\left(\frac{-5\pi}{12}\right)$

$k = 0 \longrightarrow z_2 = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

$k = 1 \longrightarrow z_3 = \sqrt{2}\,\text{cis}\left(\frac{11\pi}{12}\right)$



c $z^2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ (HW)

Vieta's Formula for Generalized Higher Degree Polynomials

Consider a polynomial of degree n, $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$, with complex coefficients and having complex roots, $r_1, r_2, \dots, r_{n-1}, r_n$. The $(n-k)^{th}$ coefficient a_{n-k} is related to a signed sum of all possible sub products of roots, taken k at-a-time as follows.

$$\sum_{i_1<i_2<\dots<i_k\leq n} r_{i_1} r_{i_2} \dots r_{i_k} = (-1)^k \frac{a_{n-k}}{a_k} \text{ for } k = 1, 2, 3, \dots, n.$$

Proof: Consider a polynomial, $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$, of degree n that has complex roots $r_1, r_2, \dots, r_{n-1}, r_n$. The expressions on the left side of Vieta's formula are the elementary symmetric functions of $r_1, r_2, \dots, r_{n-1}, r_n$. Comparing the coefficients in the equation, $a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = a_n(x-r_1)(x-r_2)\dots(x-r_n)$.

Hence, Vieta's formula gives the equation to find the sum of the roots.

$$\sum_{i=r}^{n} r_i = -\frac{a_{n-1}}{a_n}$$

$$\boxed{\text{Sum of roots} = -\frac{a_{n-1}}{a_n}}$$

Also, the following equation can find the products of the roots.

$$r_1, r_2, \dots, r_{n-1}, r_n = (-1)^n \frac{a^o}{a^n}$$

$$\boxed{\text{Product of roots} = (-1)^n \frac{a_0}{a_n}}$$

Q(5) (No Calculator) May 2023 P1

Do **not** write solutions on this page.

11. [Maximum mark: 22]

Consider the complex number $u = -1 + \sqrt{3}\mathrm{i}$.

(a) By finding the modulus and argument of u, show that $u = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$. [3]

(b) (i) Find the smallest positive integer n such that u^n is a real number.

(ii) Find the value of u^n when n takes the value found in part (b)(i). [5]

(c) Consider the equation $z^3 + 5z^2 + 10z + 12 = 0$, where $z \in \mathbb{C}$.

(i) Given that u is a root of $z^3 + 5z^2 + 10z + 12 = 0$, find the other roots.

(ii) By using a suitable transformation from z to w, or otherwise, find the roots of the equation $1 + 5w + 10w^2 + 12w^3 = 0$, where $w \in \mathbb{C}$. [9]

(d) Consider the equation $z^2 = 2z^*$, where $z \in \mathbb{C}$, $z \neq 0$.

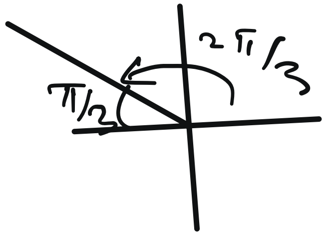
By expressing z in the form $a + b\mathrm{i}$, find the roots of the equation. [5]

Consider the complex number $u = -1 + \sqrt{3}\,\mathrm{i}$.

(a) By finding the modulus and argument of u, show that $u = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$. [3]

$$|u| = \sqrt{4} = 2$$

$$\arg(u) = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$$



$$\therefore \arg(u) = \frac{2\pi}{3}$$

$$\therefore u = |u|\, e^{i(\arg u)}$$

$$\therefore \boxed{u = 2\, e^{i\left(\frac{2\pi}{3}\right)}}$$

(b) (i) Find the smallest positive integer n such that u^n is a real number.

(ii) Find the value of u^n when n takes the value found in part (b)(i). [5]

(i) u^n is a real number

$\therefore \arg(u^n) = k\pi, \quad k \in \mathbb{Z}$

$\left[u^n = 2^n e^{i\left(\frac{2\pi}{3}n\right)}\right]$

$\therefore$ From part (a)

$$\frac{2\pi}{3}n = \underline{\underline{k}}\pi$$

$$\Rightarrow \boxed{n = 3}$$

(ii) $u^3 = 2^3 e^{i\left(\frac{2\pi}{\cancel{3}}(\cancel{3})\right)} = 2^3 e^{i(2\pi)} = 8(\underset{(1)}{\cos 2\pi} + i\underset{0}{\sin 2\pi})$

$$\therefore \boxed{u^3 = 8}$$

(c) Consider the equation $z^3 + 5z^2 + 10z + 12 = 0$, where $z \in \mathbb{C}$.

a_n a_{n-1}

(i) Given that u is a root of $z^3 + 5z^2 + 10z + 12 = 0$, find the other roots.

(ii) By using a suitable transformation from z to w, or otherwise, find the roots of the equation $1 + 5w + 10w^2 + 12w^3 = 0$, where $w \in \mathbb{C}$. [9]

(i) Since $u = -1+\sqrt{3}i$ is a roots

$\therefore$ other root is $-1-\sqrt{3}i$

Let z_3 be the Third z_3

$\therefore$ Sum of roots $= -\frac{a_{n-1}}{a_n} = -\frac{5}{1} = -5$

$\therefore (-1+\cancel{\sqrt{3}i}) + (-1-\cancel{\sqrt{3}i}) + z_3 = -5$

$-2 + z_3 = -5$

$\therefore z_3 = \boxed{-3}$

$\therefore$ The roots are $-1+\sqrt{3}i, -1-\sqrt{3}i, -3$ are three roots.

(ii) By using a suitable transformation from z to w, or otherwise, find the roots of the equation $1 + 5w + 10w^2 + 12w^3 = 0$, where $w \in \mathbb{C}$. [9]

Since the roots of the following equ.

$$z^3 + 5z^2 + 10z + 12 = 0$$

are $u, u^*, -3$

$\therefore (z-u)(z-u^*)(z+3) = 0$

Since coeff. of $1+5w+10w^2+12w^3=0$ has same coeff.

$\therefore (1-uw)(1-u^*w)(1+3w)=0$

$\therefore 1-uw=0, \quad 1-u^*w=0$

$$w_1 = \frac{1}{u}, \quad w_2 = \frac{1}{u^*}, \quad w_3 = -\frac{1}{3}$$

$w_1 = \frac{1}{u}$, $w_2 = \frac{1}{u^*}$, $w_3 = -\frac{1}{3}$

$u = 2e^{i\frac{2\pi}{3}}$.

$$\therefore w_1 = \frac{1}{2e^{i\left(\frac{2\pi}{3}\right)}} = \boxed{\frac{1}{2}e^{-i\left(\frac{2\pi}{3}\right)}}$$

$$w_2 = \frac{1}{u^*} = \frac{1}{2e^{-i\left(\frac{2\pi}{3}\right)}} = \boxed{\frac{1}{2}e^{i\left(\frac{2\pi}{3}\right)}}$$

$$w_3 = \boxed{-\frac{1}{3}}$$

are the roots of

$$1 + 5w + 10w^2 + 12w^3 = 0$$

(d) Consider the equation $z^2 = 2z^*$, where $z \in \mathbb{C}$, $z \neq 0$.

By expressing z in the form $a + bi$, find the roots of the equation. [5]

$$z^2 = 2z^*$$

$$z = a + bi$$

$$\therefore z^* = a - bi$$

$$\& \; z^2 = (a+bi)^2 = (a^2-b^2) + (2ab)i$$

$$\therefore (a^2-b^2) + (2ab)i = 2(a-bi) = 2a - 2bi$$

$\therefore$ comparing real & imaginary parts, we have

$$\therefore a^2 - b^2 = 2a \quad \& \quad 2ab = -2b \Rightarrow \boxed{a = -1}$$

$$(-1)^2 - b^2 = 2(-1) \Rightarrow 1 - b^2 = -2 \Rightarrow b^2 = 3 \Rightarrow b = \pm\sqrt{3}$$

$\therefore$ roots of eqn. are

$$\boxed{-1 \pm \sqrt{3}\, i}$$

Next Lesson–5

Mathematical Induction:

AHL 1.15

Content	Guidance, clarification and syllabus links
Proof by mathematical induction.	Proof should be incorporated throughout the course where appropriate. Mathematical induction links specifically to a wide variety of topics, for example complex numbers, differentiation, sums of sequences and divisibility.
Proof by contradiction.	**Examples:** Irrationality of $\sqrt{3}$; irrationality of the cube root of 5; Euclid's proof of an infinite number of prime numbers; if a is a rational number and b is an irrational number, then $a+b$ is an irrational number.
Use of a counterexample to show that a statement is not always true.	**Example:** Consider the set P of numbers of the form $n^2+41n+41,\ \ n \in \mathbb{N}$, show that not all elements of P are prime. **Example:** Show that the following statement is not always true: there are no positive integer solutions to the equation $x^2+y^2=10$. It is not sufficient to state the counterexample alone. Students must explain why their example is a counterexample.

End of Lesson–4

That's it. I hope you enjoyed the lesson.

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