Application of Neural Networks to Conformal Field Theory

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What is a CFT?

The conformal group is the set of transformations of spacetime that preserve angles (but not necessarily distances). A conformal field theory is a quantum(classical) field theory that is invariant under the conformal group.

$$x'^{\mu} = x^{\mu} + a^{\mu} \qquad \text{(Translation)}$$

$$x'^{\mu} = \alpha x^{\mu} \qquad \text{(Dilation)}$$

$$x'^{\mu} = M^{\mu}_{\nu} x^{\nu} \qquad \text{(Rotation)}$$

$$x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2} \qquad \text{(SCT)}$$

Why CFTs are important?

- Some of the most important equations in physics are conformally invariant. E.g, Maxwell's equation of free EM fields in (3+1)D.
- Useful for study of RG flows, phase transitions and critical phenomena of statistical mechanical systems such as Ising Model.
- High energy phenomenological motivations.
- AdS/CFT correspondence (gauge-gravity duality).
- String theory- If string theory is correct, then in some sense conformal invariance is one of the most fundamental features of reality.

Outline of this Work

- **Warm Up: Discrete Symmetries**
  - Odd-Even Classification
  - Point Cloud Symmetry Classification
- **CFT Classification Problems**
  - Scale vs Conformal Invariance
  - Scale vs Conformal Invariance using only Correlators
  - Scale vs Conformal Invariance in Embedding Space
  - Scale vs Conformal Invariance using only Correlators in Embedding Space
  - Scale vs Conformal Invariance for spinning 3-pt function
  - Scale vs Conformal Invariance for spinning 3-pt function using only correlators
- **CFT Regression Problem**: Predicting the OPE Coefficient

We give a *proof of concept*.

Warm Up: Odd-Even & Point Cloud Symmetry

- **Odd-Even Symmetry:**
  Given $f(x)$. Let $x \longrightarrow -x$. Define

$$f(-x) = -f(x) \quad \textit{odd}$$

$$f(-x) = f(x) \quad \textit{even}$$

- **Point Cloud Symmetry:**
  Define

$$(x, y, z) \longrightarrow (x, y, -z) \quad \textit{reflection}$$

$$(x, y, z) \longrightarrow (-x, -y, -z) \quad \textit{antipodal}$$

$$(x_1, y_1, z_1) \longrightarrow (x_2, y_2, z_2) \quad \textit{null}$$

Contd.: Odd-Even & Point Cloud Symmetry

Generating training data

1. Start with random numbers
2. Do mathematical operations to give them an appropriate form
3. Add the labels
4. Arrange them in handy order
5. Save the data as a file from which can be totally recovered

Kids park

Contd.: Odd-Even & Point Cloud Symmetry

Odd Even Classifier

Point Cloud	X	Y
Even Function	$(\mathbf{x}, \mathbf{f}(\mathbf{x}),\ -\mathbf{x}, \mathbf{f}(\mathbf{x}))$	$(1, 0)$
Odd Function	$(\mathbf{x}, \mathbf{f}(\mathbf{x}),\ -\mathbf{x}, -\mathbf{f}(\mathbf{x}))$	$(0, 1)$

```
_________________________________________________________________
Layer (type)                 Output Shape              Param #
=================================================================
dense_1 (Dense)              (None, 20)                4020
_________________________________________________________________
dense_2 (Dense)              (None, 10)                210
_________________________________________________________________
dropout_1 (Dropout)          (None, 10)                0
_________________________________________________________________
dense_3 (Dense)              (None, 2)                 22
=================================================================
Total params: 4,252
Trainable params: 4,252
Non-trainable params: 0
_________________________________________________________________
```

Relu

Relu

Softmax

SGD

Binary
Cross entropy

Contd.: Odd-Even & Point Cloud Symmetry

3D symmetries

Point Cloud Symmetry	X	Y
Plane Reflection	$(\mathbf{x},\mathbf{y},\mathbf{z}) \cup (\mathbf{x},\mathbf{y},-\mathbf{z})$	$(1,0,0)$
Antipodal Reflection	$(\mathbf{x},\mathbf{y},\mathbf{z}) \cup (-\mathbf{x},-\mathbf{y},-\mathbf{z})$	$(0,1,0)$
Null	$(\mathbf{x}_1,\mathbf{y}_1,\mathbf{z}_1) \cup (\mathbf{x}_2,\mathbf{y}_2,\mathbf{z}_2)$	$(0,0,1)$

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_________________________________________________________________
Layer (type)                 Output Shape              Param #
=================================================================
dense_1 (Dense)              (None, 50)                7550
_________________________________________________________________
dense_2 (Dense)              (None, 20)                1020
_________________________________________________________________
dropout_1 (Dropout)          (None, 20)                0
_________________________________________________________________
dense_3 (Dense)              (None, 3)                 63
=================================================================
Total params: 8,633
Trainable params: 8,633
Non-trainable params: 0
_________________________________________________________________
```

Relu

Relu

Softmax

SGD

Categorical cross entropy

Contd.: Learning curves

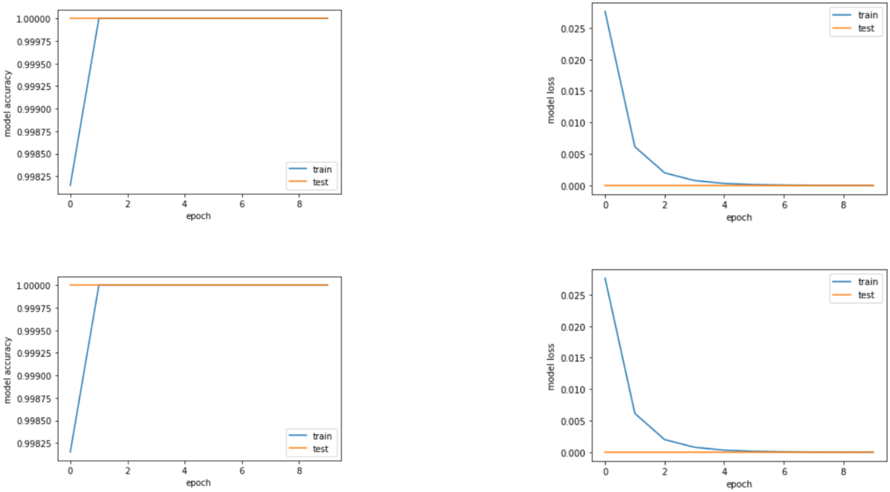

Figure: Learning curves for Point cloud symmetries(Top) and Odd Even(Bottom)

Scale vs Conformal Invariance: 2-point functions

- A 2-point function is in general a function of two space-time points

$$f(x_1, x_2)$$

- In a D-dimensional quantum field theory with Poincare symmetry containing **scalar** operators $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ with scaling dimensions $\Delta_1, \Delta_2, \Delta_3$ respectively,
  Scale invariant 2-point function is constrained to be:

$$< \mathcal{O}_{\Delta_i}(x_1)\mathcal{O}_{\Delta_j}(x_2) >= \frac{c_{ij}}{|x_1 - x_2|^{\Delta_i + \Delta_j}}$$

- Conformal invariant 2-point function is further constrained to

$$c_{ij} = \delta_{ij}$$

Scale vs Conformal Invariance: 3-point functions

- **3-pt functions put more prominent restrictions.**
- Only scale invariant 3-pt function has the form:

$$< \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3) >= \sum_{\{a,b,c\}} \frac{c_{abc}}{|x_1 - x_2|^a |x_2 - x_3|^b |x_3 - x_1|^c}$$

with the constraint

$$a + b + c = \Delta_1 + \Delta_2 + \Delta_3$$

- Conformal invariant 3-pt function is further constrained to:

$$< \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3) >= \frac{\lambda_{123}}{|x_1 - x_2|^{\alpha_{123}} |x_2 - x_3|^{\alpha_{231}} |x_3 - x_1|^{\alpha_{312}}}$$

with

$$\alpha_{ijk} = \Delta_i + \Delta_j - \Delta_k$$

SI vs CI: Setting up the Machine Learning Problem

Scale Vs Conformal Invariance

Symmetry	X	Y
Scale Invariance	$\left(\mathrm{x}_{12}, \mathrm{x}_{23}, \mathrm{x}_{31}, \Delta_1, \Delta_2, \Delta_3, f^{(3)}_{scale}\right)_{(100)}$	$(1, 0)$
Conformal Invariance	$\left(\mathrm{x}_{12}, \mathrm{x}_{23}, \mathrm{x}_{31}, \Delta_1, \Delta_2, \Delta_3, f^{(3)}_{cft}\right)_{(100)}$	$(0, 1)$

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_________________________________________________________________
Layer (type)                 Output Shape              Param #
=================================================================
dense_4 (Dense)              (None, 50)                350050
_________________________________________________________________
dense_5 (Dense)              (None, 20)                1020
_________________________________________________________________
dropout_2 (Dropout)          (None, 20)                0
_________________________________________________________________
dense_6 (Dense)              (None, 2)                 42
=================================================================
Total params: 351,112
Trainable params: 351,112
Non-trainable params: 0
_________________________________________________________________
```

Relu

Sigmoid

Softmax

SGD

Binary
Cross entropy

SI vs CI: Setting up the Machine Learning Problem

Scale Vs Conformal Invariance with Correlator

Symmetry	X	Y
Scale Invariance	$\left(\mathbf{x}_{12}, \mathbf{x}_{23}, \mathbf{x}_{31}, f^{(2)}_{\Delta_1}, f^{(2)}_{\Delta_2}, f^{(2)}_{\Delta_3}, f^{(3)}_{\mathrm{scale}}\right)_{(100)}$	$(1,0)$
Conformal Invariance	$\left(\mathbf{x}_{12}, \mathbf{x}_{23}, \mathbf{x}_{31}, f^{(2)}_{\Delta_1}, f^{(2)}_{\Delta_2}, f^{(2)}_{\Delta_3}, f^{(3)}_{\mathrm{conformal}}\right)_{(100)}$	$(0,1)$

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_________________________________________________________________
Layer (type)                 Output Shape              Param #
=================================================================
dense_1 (Dense)              (None, 40)                280040
_________________________________________________________________
dense_2 (Dense)              (None, 10)                410
_________________________________________________________________
dropout_1 (Dropout)          (None, 10)                0
_________________________________________________________________
dense_3 (Dense)              (None, 2)                 22
=================================================================
Total params: 280,472
Trainable params: 280,472
Non-trainable params: 0
_________________________________________________________________
```

Sigmoid

Sigmoid

Sigmoid

SGD

Binary
Cross entropy

SI vs CI: Results

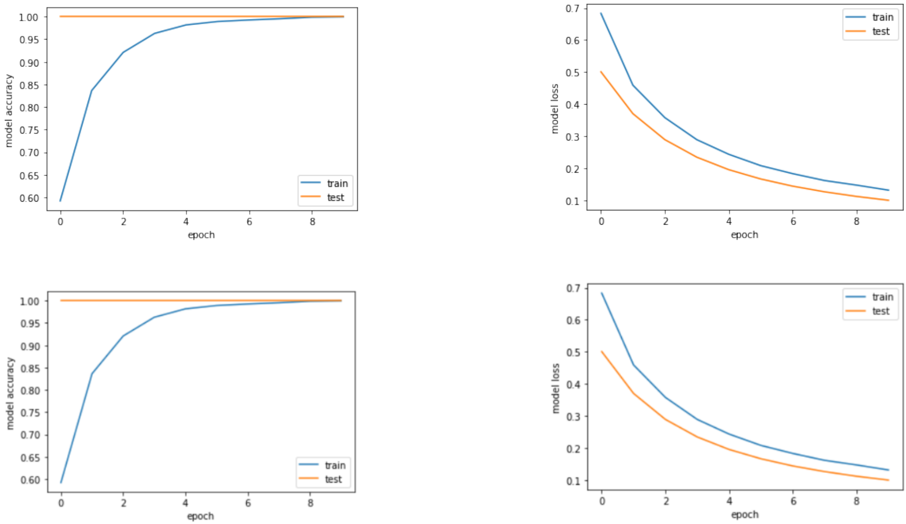

Figure: Learning curves for Scalar vs Conformal invariance(Top) and Scalar vs Conformal invariance with correlator(Bottom)

Embedding Space

- The conformal group in D dimensions is isomorphic to SO(D+1,1).

  Conformal invariants in D dim $\leftrightarrow$ Lorentz invariants in (D+1,1) dim
- Coordinates in (D+1,1) space are related to D space coordinates

$$X^M \equiv (X^+, X^-, X^\mu) = (1, x^2, x^\mu)$$

- **2-pt function** in (D+1,1) space:

$$< \Phi(X)\Phi(Y) > \propto \frac{1}{(X.Y)^\Delta}$$

- **3-pt function**:

$$< \Phi_1(X_1)\Phi_2(X_2)\Phi_3(X_3) > \propto \frac{1}{(X_1.X_2)^{\alpha_{123}}(X_2.X_3)^{\alpha_{231}}(X_3.X_1)^{\alpha_{312}}}$$

Scale vs Conformal Invariance in Embedded Coordinates

Symmetry	X							Y
Scale Inv.	$X_1.X_2$	$X_2.X_3$	$X_3.X_1$	Δ_1	Δ_2	Δ_3	$f^{(3)}_{scale}$	(1,0)
Conformal Inv.	$X_1.X_2$	$X_2.X_3$	$X_3.X_1$	Δ_1	Δ_2	Δ_3	$f^{(3)}_{cft}$	(0,1)

```
_________________________________________________________________
Layer (type)                 Output Shape              Param #
=================================================================
dense_1 (Dense)              (None, 50)                350050
_________________________________________________________________
dense_2 (Dense)              (None, 20)                1020
_________________________________________________________________
dropout_1 (Dropout)          (None, 20)                0
_________________________________________________________________
dense_3 (Dense)              (None, 2)                 42
=================================================================
Total params: 351,112
Trainable params: 351,112
Non-trainable params: 0
_________________________________________________________________
```

Sigmoid

Sigmoid

Sigmoid

Adam

Binary
Cross entropy

SI vs CI: Framing the problem in Embedding Space

Scale vs Conformal Invariance with correlator in Embedded Coordinates

Symmetry	X							Y
Scale Inv.	$X_1.X_2$	$X_2.X_3$	$X_3.X_1$	$f^{(2)}_{\Delta_1}$	$f^{(2)}_{\Delta_2}$	$f^{(2)}_{\Delta_3}$	$f^{(3)}_{scale}$	(1,0)
Conformal Inv.	$X_1.X_2$	$X_2.X_3$	$X_3.X_1$	$f^{(2)}_{\Delta_1}$	$f^{(2)}_{\Delta_2}$	$f^{(2)}_{\Delta_3}$	$f^{(3)}_{cft}$	(0,1)

```
_________________________________________________________________
Layer (type)                 Output Shape              Param #
=================================================================
dense_1 (Dense)              (None, 50)                350050
_________________________________________________________________
dense_2 (Dense)              (None, 20)                1020
_________________________________________________________________
dropout_1 (Dropout)          (None, 20)                0
_________________________________________________________________
dense_3 (Dense)              (None, 2)                 42
=================================================================
Total params: 351,112
Trainable params: 351,112
Non-trainable params: 0
_________________________________________________________________
```

Sigmoid

Sigmoid

Sigmoid

Adam

Binary
Cross entropy

New Results! SI vs CI: Classification in Embedding Space

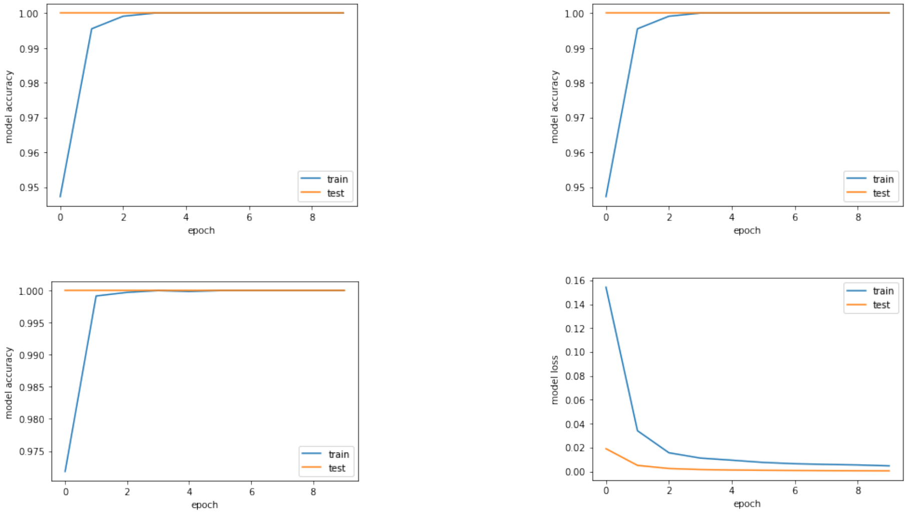

Figure: Learning curves for Scalar vs Conformal invariance(Top) and Scalar vs Conformal invariance with correlator(Bottom) in embedded space.

Open problem: What about fields with spin?

- All our discussion so far was limited to scalar operators.
- **Spinning 2-point Function**:
  Scale Invariant Spin(l=1)-Spin(l=1)
  $$< \phi_{\Delta_1 \mu}(x)\phi_{\Delta_2 \nu}(y) >= \frac{I_{\mu\nu}(x-y)}{(x-y)^{\Delta_1+\Delta_2}}$$
  where
  $$I_{\mu\nu}(x-y) = \delta_{\mu\nu} + \beta\frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2}$$
  Demanding conformal invariance, we have $\beta = -2$ & $\Delta_1 = \Delta_2$.
- **Spinning 3-point Function**:
  Conformal invariantscalar-scalar-spin(l=1) 3-pt function
  $$< \phi_{\Delta_1}(x_1)\phi_{\Delta_2}(x_2)\Phi_{\Delta_3 \mu}(x_3) >\propto \frac{R_\mu(x,y|z)}{|x_1-x_2|^{\alpha_{123}}|x_2-x_3|^{\alpha_{231}}|x_3-x_1|^{\alpha_{312}}}$$
  where
  $$R_\mu(x,y|z) = \frac{|y-z||x-z|}{|x-y|}\left(\frac{(x-z)_\mu}{|x-z|^2} - \frac{(y-z)_\mu}{|y-z|^2}\right)$$

Scale Vs Conformal Invariance in system with spin 1

Symmetry	X							Y
Scale Inv.	x_{12}	x_{23}	x_{31}	Δ_1	Δ_2	Δ_3	$f^{(3)}_{scale\,\mu=1,\ldots,N}$	(1,0)
Conformal Inv.	x_{12}	x_{23}	x_{31}	Δ_1	Δ_2	Δ_3	$f^{(3)}_{cft\;\mu=1,\ldots,N}$	(0,1)

```
_________________________________________________________________
Layer (type)                 Output Shape              Param #
=================================================================
Input (Dense)                (None, 80)                8080
_________________________________________________________________
Relu (Dense)                 (None, 40)                3240
_________________________________________________________________
dense_1 (Dense)              (None, 4)                 164
=================================================================
Total params: 11,484
Trainable params: 11,484
Non-trainable params: 0
_________________________________________________________________
```

relu

Sigmoid

Softmax

Adam

Binary Cross entropy

SI vs CI: Setting up the problem for spinning correlators

Scale Vs Conformal Invariance with correlator in systems with spin 1

Sym	X							Y
SI	x_{12}	x_{23}	x_{31}	$f^{(2)}_{\Delta_1}$	$f^{(2)}_{\Delta_2}$	$f^{(2)}_{\Delta_3\ \mu,\nu=1,\dots,N}$	$f^{(3)}_{scale\ \mu=1,\dots,N}$	$(1,0)$
CI	x_{12}	x_{23}	x_{31}	$f^{(2)}_{\Delta_1}$	$f^{(2)}_{\Delta_2}$	$f^{(2)}_{\Delta_3\ \mu,\nu=1,\dots,N}$	$f^{(3)}_{cft\ \mu=1,\dots,N}$	$(0,1)$

```
_________________________________________________________________
Layer (type)                 Output Shape              Param #
=================================================================
dense_1 (Dense)              (None, 60)                960060
_________________________________________________________________
dense_2 (Dense)              (None, 30)                1830
_________________________________________________________________
dropout_1 (Dropout)          (None, 30)                0
_________________________________________________________________
dense_3 (Dense)              (None, 2)                 62
=================================================================
Total params: 961,952
Trainable params: 961,952
Non-trainable params: 0
```

Relu

Sigmoid

Softmax

Adam

Binary
Cross entropy

New Results! SI vs CI: Scalar-Scalar-Spin 1

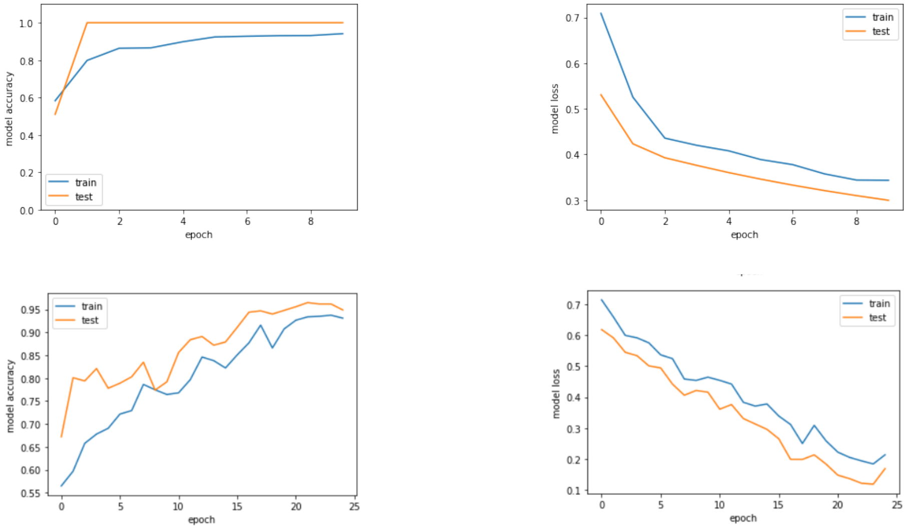

Figure: Learning curves for Scalar vs Conformal invariance(Top) and Scalar vs Conformal invariance with correlator(Bottom) in systems with spin 1.

Conformal Blocks

- 4-pt function of 4 equal dimension scalars can be written as

$$< \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) >= \frac{g(u,v)}{|x_1 - x_2|^{2\Delta_\phi}|x_3 - x_4|^{2\Delta_\phi}}$$

  where

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}; \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- $g(u,v)$ can be written as

$$g(u,v) = 1 + \sum_{\mathcal{O}(l,\Delta)} \lambda_{\mathcal{O}}^2 G_{\mathcal{O}}(u,v)$$

  where $G_{\mathcal{O}} \equiv G_{\Delta}^{(l)}$ It is called the conformal block expansion.

1D Conformal Block as Toy Model

- Consider a 1D scalar conformal block expansion

$$f(z) = \sum_{\{h\}} c_h z^h 2F1(h, h; 2h; z)$$

- For our analysis
  - Assume: Scalar operators with $h \in \mathrm{Z}$.[Not so physical].
  - Truncate at n=5.

$$f(z) = \sum_{n=0}^{5} c_n z^n 2F1(n, n; 2n; z)$$

OPE Coefficient: Setting up the Regression problem

OPE Coefficiencient (Regression problem)

OPE Coefficients	X	Y
$c_n \in (0,1) \forall n \leq 5$	$\{f\,(x_i)\}$	$\{c_2\, z^2\, {}_2F_1\,(2,2,4;z) \quad \mid \quad z \in \{-0.8, -0.4, 0.4, 0.8\}\ ,\}$

$$\{f \quad (x_i) \quad \mid \quad x_i = -0.8 + i\,\frac{1.6}{N}\,,\, i \in [0, N-1]\}$$ N=100

```
Layer (type)                 Output Shape              Param #
=================================================================
Input (Dense)                (None, 80)                8080
_________________________________________________________________
Relu (Dense)                 (None, 40)                3240
_________________________________________________________________
dense_1 (Dense)              (None, 4)                 164
=================================================================
Total params: 11,484
Trainable params: 11,484
Non-trainable params: 0
_________________________________________________________________
```

Relu

Sigmoid

Softmax

RMSprop
(1e-6)

MSE

OPE Coefficient: Regression Results

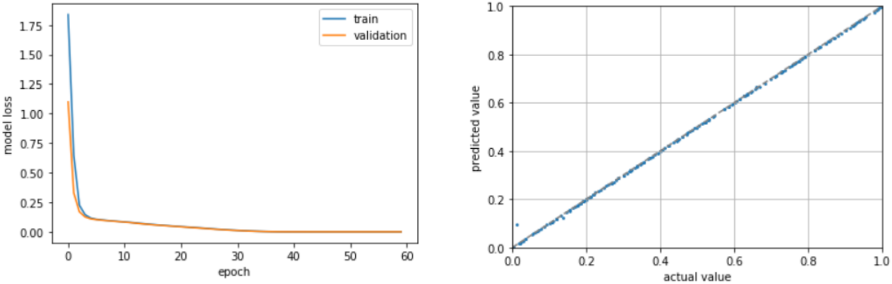

Figure: Loss plot(left) and actual vs predicted result(right) for OPE coefficient calculation.

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Future Work: Exciting Journey Ahead!

- Extend the scale-conformal invariance classification to higher spin operators and test the accuracy with increasing spin.
- Do more analysis with conformal blocks in higher space-time dimension CFTs.
- Extend the OPE coefficient prediction to spinning conformal blocks.
- Classification and regression problems in Ising model and other unitary minimal model CFTs.
- **Machine learn the CFT data**.

References

1. H. Chen, Y. He, S. Lal, M.Z. Zaz, "Machine Learning Etudes in Conformal Field Theory", arXiv:2006.16114 (2020).
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4. P. Mehta, M. Bukov, C.-H. Wang, A. G. Day, C. Richardson, C. K. Fisher et al., A high-bias, low-variance introduction to machine learning for physicists, Physics Reports 810 (2019).
5. H. Osborn F.A. Dolan. Conformal four point functions and the operator product expansion.Nucl.Phys., B599:459–496 (2001)