We have reduced the problem of revenue maximization (for firm A) under proportional budgeting to the following maximization problem.

$$\max_{\alpha} \left\{ (1-\alpha v) E_{\lambda} \left[p \frac{\lambda}{1-\frac{\alpha}{e}(1-\lambda)} + (1-p)\lambda \right] \right\} \tag{1}$$

Furthermore, because any reward scheme may be written as a proportial budgeting scheme for appropriate k and α values, the above optimization also gives results beyond just proportional budgeting. The optimization problem may be written as:

$$\max_{\alpha} \left\{ (1-\alpha v) \int_0^1 \left(p \frac{\lambda}{1-\frac{\alpha}{e}(1-\lambda)} + (1-p)\lambda \right) f(\lambda) d\lambda \right\} \tag{2}$$

where $f(\lambda)$ is the pdf of λ, supported on $[0, 1]$.

One method to evalute the above was to use the Taylor series expansion of moments of random variables. Let $g(\lambda)$ be the function in the integrand above (exluding the pdf), so in our case, the method reduces to:

$$\max_{\alpha} \{ (1-\alpha v) E_{\lambda}(g(\lambda) \} \approx \max_{\alpha} \left\{ (1-\alpha v) \left(g(\mu_{\lambda}) + \frac{g''(\mu_{\lambda})}{2} \sigma_{\lambda}^2 \right) \right\}$$

However, I'm not sure what conditions on the distribution we need to ignore the higher order terms of the series expansion.

First I want to examine the problem with simple delta distributions, i.e. λ constant. There are three cases we care about.

1. ($\lambda = 0$) Here we are maximizing $(1-\alpha v)\left(p\frac{0}{1-\frac{\alpha}{e}}\right)$. This function is 0 for all values of α (even as $\alpha \to 0$). Thus, this result shows (matching our intuition) that some excess loyalty is needed for anyone to adopt the reward program in the model - just another way to think about it.

2. ($\lambda = 1$) Here we are maximizing $(1-\alpha v)\lambda$, which occurs at $\alpha = 0$. Here again we just get an obvious result: if all customers have excess loyalty 1, the firm will get all business and should not offer any reward scheme.

3. ($0 < \lambda < 1$) I have been messing with this for a bit. It is quite a messy expression. Through plotting, I've seen some that are maximized at $\alpha \to 0$ and some at $\alpha \to e$ (generally when λ is large and small, respectively). I have yet to see conditions that give α something in between.

I figure this may be the easiest problem we can think about solving, so this is what I've been trying to do first.