

1 Problem description

Consider a duopoly with two stores, A and B , selling the same item. Store A sells the item for a price p while store B sells it for price $p^+ = p + \epsilon$ for some $\epsilon > 0$. Store B also gives a reward R to a customer after making k purchases at B . Our goal is to find the optimal k for store B to maximize its rate of revenue under certain assumptions on customer behavior.

We assume that there is some exogenous probability, λ , during each purchase that forces the customer to go to store B . Let $0 < \beta \leq 1$ denote the discounting factor of future money. (NOTE/QUESTION: in the previous write-up, β was called the expiry factor - is this still the correct interpretation? If so, it seems like β should be close to zero but in the discounting interpretation β should be close to one) For simplicity, we assume the customer has the following utility as a function of price paid: $u(p) = v > 0$ and $u(p^+) = 0$.

We model the customer's decision problem as a dynamic problem. We index the number of visits the customer makes at store B by i , for $0 \leq i \leq k-1$, and we refer a customer to be in state i after having made i visits to B . At state i , the customer has two possibilities:

1. With probability λ , the customer must visit B , and she is now in state $i+1$.
2. With probability $1-\lambda$, the customer may purchase from A for utility v and remain in state i or purchase from B for no utility but move to state $i+1$.

Let $V(i)$ denote the long term expected reward at state i . Then we may model the decision problem as the following dynamic program.

$$V(i) = \lambda\beta V(i+1) + (1-\lambda) \max\{v + \beta V(i), \beta V(i+1)\} \text{ for } 0 \leq i \leq k-1$$
$$V(k) = R$$

We will show that the decision process exhibits a phase transition; that is prior to some state i_0 , the customer will only visit B if she must do so exogenously but after i_0 , she always decides to go to B .

Finally, we assume the customer has a look-ahead factor t , which models how many purchases ahead the customer looks ahead when making her current decision. This value will affect the phase transition of the decision process. Consider a distribution T describing the look-ahead factor for consumers. We will focus on threshold distributions; for example, with probability p the look-ahead is t_1 and with probability $1-p$, the look-ahead is t_0 .

2 Solving the DP and the phase transition

NEED TO: find conditions under which $V(i)$ is increasing in i and provide proof (these will be different than in the original write-up). The following lemma would be written as a part of this lemma.

Lemma 2.1. *We may write the DP as*

$$V(i) = \max \left\{ \frac{\lambda\beta V(i+1) + (1-\lambda)v}{1 - (1-\lambda)\beta}, \beta V(i+1) \right\}$$

Proof. We have the following:

$$\begin{aligned} V(i) &= \lambda\beta V(i+1) + (1-\lambda) \max\{v + \beta V(i), \beta V(i+1)\} \\ &= \max\{\lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)), \beta V(i+1)\} \end{aligned}$$

Assuming $V(i)$ is the left term in the above maximum, we may solve the equation for that term.

$$\begin{aligned} V(i) &= \lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)) \\ (1 - (1-\lambda)\beta)V(i) &= \lambda\beta V(i+1) + (1-\lambda)v \\ V(i) &= \frac{\lambda\beta V(i+1) + (1-\lambda)v}{1 - (1-\lambda)\beta} \end{aligned}$$

□

Theorem 2.1. *A phase transition occurs after the consumer makes i_0 visits to firm B , which evaluates to:*

$$\begin{aligned} i_0 &= k - \left\lceil \log_{\frac{1}{\beta}} \left(\frac{R\beta(1-\beta)}{v} \right) \right\rceil \\ &\equiv k - \Delta \end{aligned}$$

(Note: we may need conditions as we did before - thing to check)

Proof. First we solve for the condition on $V(i+1)$ for us to choose firm B over A willingly.

$$\begin{aligned} \beta V(i+1) &> \frac{\lambda\beta V(i+1) + (1-\lambda)v}{1 - (1-\lambda)\beta} \\ \iff \beta V(i+1) \left(1 - \frac{\lambda}{1 - (1-\lambda)\beta} \right) &> \left(\frac{1-\lambda}{1 - (1-\lambda)\beta} \right) v \\ \iff \beta V(i+1) \left(\frac{1 - (1-\lambda)\beta - \lambda}{1 - (1-\lambda)\beta} \right) &> \left(\frac{1-\lambda}{1 - (1-\lambda)\beta} \right) v \\ \iff \beta V(i+1) \left(\frac{(1-\lambda)(1-\beta)}{1 - (1-\lambda)\beta} \right) &> \left(\frac{1-\lambda}{1 - (1-\lambda)\beta} \right) v \\ \iff \beta V(i+1) &> \frac{v}{1-\beta} \\ \iff V(i+1) &> \frac{v}{\beta(1-\beta)} \end{aligned}$$

Let i_0 be the minimum state i such that the above holds, so in particular $V(i_0) \leq \frac{v}{\beta(1-\beta)}$ but $V(i_0 + 1) > \frac{v}{\beta(1-\beta)}$. We know because V is increasing in i (still need to prove), this point is indeed a phase transition: $V(i) > \frac{v}{\beta(1-\beta)}$ for all $i > i_0$, so after this point, the customer always chooses firm B . We may compute $V(i_0)$ easily using this fact.

$$V(i_0) = \beta V(i_0 + 1) = \dots = \beta^{k-i_0} V(k) = \beta^{k-i_0} R$$

Thus, we have the following:

$$\begin{aligned}
& \beta^{k-i_0} \leq \frac{v}{R\beta(1-\beta)} < \beta^{k-(i_0+1)} \\
& \iff \left(\frac{1}{\beta}\right)^{k-i_0} \geq \frac{R\beta(1-\beta)}{v} > \left(\frac{1}{\beta}\right)^{k-(i_0+1)} \\
& \iff k - i_0 \geq \log_{\frac{1}{\beta}} \left(\frac{R\beta(1-\beta)}{v} \right) > k - (i_0 + 1) \\
& \iff i_0 + 1 > k - \log_{\frac{1}{\beta}} \left(\frac{R\beta(1-\beta)}{v} \right) \geq i_0 \\
& \iff i_0 = k - \left\lceil \log_{\frac{1}{\beta}} \left(\frac{R\beta(1-\beta)}{v} \right) \right\rceil \equiv k - \Delta
\end{aligned}$$

□

3 Look-ahead, threshold distribution

Now we assume the look-ahead factor of a customer is drawn from some distribution $t \sim T$. The phase transition of the customer's DP will now depend on t .

$$i_0(t) = \begin{cases} i_0, & \text{if } t \geq \Delta. \\ k - t, & \text{otherwise.} \end{cases}$$

Assuming a customer look-ahead distribution and a fixed reward size, R , we want to choose a k to maximize the revenue per reward cycle. That is we want to maximize the quantity given by revenue of B during the k visits over the total number of purchases (at both A and B) to reach k visits at B . For simplicity we assume $p^+ = 1$, so the revenue of B per reward cycle is $k - R$. The expected total number of purchases per reward cycle is $\frac{i_0(t)}{\lambda} + (k - i_0(t))$, where the first term represents the expected number of visits needed to reach the phase transition with exogenous visits to B and the second term is just the remaining visits to B to receive the reward.

Note that we can think of the length of the reward cycle as a random variable: length of cycle = $\tau + k - i_0(t)$ where τ is a random variable representing the number of visits needed to hit the phase transition $E(\tau) = \frac{i_0(t)}{\lambda}$. Ideally, we like to maximize the following obje