Optimal Design of Loyalty Reward Program in a Competitive Duopoly

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1 Introduction

2 Model

Consider a competitive duopoly of two stores, A and B, selling the same item. Without loss of generality we assume that store A sells the item for a price of 1 dollars while store B sells it for $1 - v$ dollars, i.e., B offers a discount of v dollars. Store A on the other hand offers a reward of value R dollars to a customer after (s)he makes k purchases at A. Our goal is to understand the dynamics of competition between A and B with respect to setting of the reward and discount parameters to maximize their respective revenues over a distribution of customers under certain assumptions on their behavior.

2.1 Customer Choice Model

First we assume that every customer purchases an item from either A or B everyday. We assume that there is some exogenous probability, λ, during each purchase that forces the customer to go to store A. This λ is a customer specific parameter and is drawn from a uniform distribution between $[0, b]$, where b is between 0 and 1. Let $0 < \beta \leq 1$ denote the discounting factor of future money. We assume customers to have a linear homogenous utility in price: at price p the utility is $\nu(p) = 1 - p$. This reduces to customers getting an immediate utility of 0 from A and v from B.

We model the customer's decision problem as a dynamic problem. We index the number of visits the customer makes at store A by i, for $0 \leq i \leq k - 1$, and we refer a customer to be in state i after having made i visits to A. At state i, the customer has two possibilities:

1. With probability λ, the customer must visit A, and she is now in state $i+1$.

2. With probability $1-\lambda$, the customer may purchase from B for an immediate utility v and remain in state i or purchase from A for no utility but move to state $i+1$.

Let $V(i)$ denote the long term expected reward at state i. Then we may model the decision problem as the following dynamic program.

$$V(i) = \lambda\beta V(i+1) + (1-\lambda)\max\{v + \beta V(i), \beta V(i+1)\} \text{ for } 0 \le i \le k-1$$
$$V(k) = R$$

We will show that the decision process exhibits a phase transition; that is prior to some state i_0, the customer will only visit A if she must do so exogenously but after i_0, she always decides to go to A.

Finally, we assume the customer has a look-ahead factor t, which models how many purchases ahead the customer looks ahead when making her current decision. This value will affect the phase transition of the decision process. Consider a distribution T describing the look-ahead factor for consumers. We will focus on threshold distributions; for example, with probability p the look-ahead is t_1 and with probability $1-p$, the look-ahead is t_0. With the addition of this look-ahead factor, the phase transition point will now depend on it, and we will refer to it as $i_0(t)$.

2.2 Merchant Objective and Competition

Given the above model of customer dynamics, and that customer exogeneity and look ahead parameters are drawn from a known distribution, the two merchants try competing over the customer base to maximize their long run revenues. We define the rate of revenue for a merchant from a customer as the expected time averaged revenue that the merchant receives within the customer's lifetime. For simplification we assume merchants do not discount future revenues. As described above, a customer's dynamics are cyclic after each reward cycle. Thus the lifetime dynamics of customer behvior is a regenerative process with independent and identically distributed reward cycle lengths. Let $RoR_A(c)$ and $RoR_B(c)$ denote the expected rate of revenues for merchants A and B respectively from a customer c's lifetime. Let $\tau(t, \lambda)$ denote the total number of visits the customer makes before reaching the phase transition point $i_0(t)$. Then the length of the reward cycle (or total number of visits the customer makes before receiving the reward) is $\tau(t, \lambda)+k-i_0(t)$, as after the phase transition (s)he makes all visits to merchant A only till hitting the reward. In this cycle the number of visits that the customer makes to A are k, and to B are $\tau(t, \lambda) - i_0(t)$. Thus the rate of revenues are as follows:

$$RoR_A(c) = \underset{\tau}{E}\left[\frac{k-R}{\tau(t,\lambda)+k-i_0(t)}\right] \tag{1}$$

$$RoR_B(c) = \underset{\tau}{E}\left[\frac{(\tau(t,\lambda)-i_0(t))(1-v)}{\tau(t,\lambda)+k-i_0(t)}\right] \tag{2}$$

The goal of each merchant is to set their reward or pricing parameters so as to maximize the expected value of rate of revenue over the entire customer population. Before that, since the process for a single customer is regenerative, using the reward renewal theorem (CITE), we can take the expectation over the cycle length inside the numerator and denominator respectively. Note that $\underset{\tau}{E}[\tau(t, \lambda)] = \frac{i_0(t)}{\lambda}$ as before reaching the phase transition point, with probability λ, the customer's visits to A increases by 1 and with probability $1 - \lambda$ it stays constant. Then the objectives of the two merchants are:

$$\max_{R,k}\{RoR_A\} = \max_{R,k}\left\{\underset{\lambda,t}{E}\left[\frac{k-R}{i_0(t)/\lambda+k-i_0(t)}\right]\right\} \tag{3}$$

$$\max_{v}\{RoR_B\} = \max_{v}\left\{\underset{\lambda,t}{E}\left[\frac{(\tau(t,\lambda)-i_0(t))(1-v)}{i_0(t)/\lambda+k-i_0(t)}\right]\right\} \tag{4}$$

3 Results

3.1 Customer Choice Dynamics

We first show that every customer exhibits the following behavior: until (s)he reaches the phase transition point $i_0(t)$, she visits A only due to the exogeneity paramaeter, and after that (s)he always visits merchant A till she receives the reward. This behavior is cyclic, and repeats after the first reward redemption.

NEED TO: find conditions under which $V(i)$ is increasing in i and provide proof (these will be different than in the original write-up). The following lemma would be written as a part of this lemma.

Lemma 3.1. *We may write the DP as*

$$V(i) = \max\left\{\frac{\lambda\beta V(i+1) + (1-\lambda)v}{1-(1-\lambda)\beta}, \beta V(i+1)\right\}$$

Proof. We have the following:

$$\begin{aligned} V(i) &= \lambda\beta V(i+1) + (1-\lambda)\max\{v + \beta V(i), \beta V(i+1)\} \\ &= \max\{\lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)), \beta V(i+1)\} \end{aligned}$$

Assuming $V(i)$ is the left term in the above maximum, we may solve the equation for that term.

$$\begin{aligned} V(i) &= \lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)) \\ (1-(1-\lambda)\beta)V(i) &= \lambda\beta V(i+1) + (1-\lambda)v \\ V(i) &= \frac{\lambda\beta V(i+1) + (1-\lambda)v}{1-(1-\lambda)\beta} \end{aligned}$$

□

Theorem 3.1. *A phase transition occurs after the consumer makes i_0 visits to firm B, which evaluates to:*

$$\begin{aligned} i_0 &= k - \left\lfloor \log_\beta\left(\frac{v}{R(1-\beta)}\right)\right\rfloor \\ &\equiv k - \Delta \end{aligned}$$

(Note: we may need conditions as we did before - thing to check)

Proof. First we solve for the condition on $V(i+1)$ for us to choose firm B over A willingly.

$$\begin{aligned} \beta V(i+1) &> \frac{\lambda\beta V(i+1) + (1-\lambda)v}{1-(1-\lambda)\beta} \\ \iff \beta V(i+1)\left(1 - \frac{\lambda}{1-(1-\lambda)\beta}\right) &> \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right)v \\ \iff \beta V(i+1)\left(\frac{1-(1-\lambda)\beta-\lambda}{1-(1-\lambda)\beta}\right) &> \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right)v \\ \iff \beta V(i+1)\left(\frac{(1-\lambda)(1-\beta)}{1-(1-\lambda)\beta}\right) &> \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right)v \\ \iff \beta V(i+1) &> \frac{v}{1-\beta} \\ \iff V(i+1) &> \frac{v}{\beta(1-\beta)} \end{aligned}$$

Let i_0 be the minimum state i such that the above holds, so in particular $V(i_0) \leq \frac{v}{\beta(1-\beta)}$ but $V(i_0+1) > \frac{v}{\beta(1-\beta)}$. We know because V is increasing in i (still need to prove), this point is indeed a phase transition: $V(i) > \frac{v}{\beta(1-\beta)}$ for all $i > i_0$, so after this point, the customer always chooses firm B. We may compute $V(i_0)$ easily using this fact.

$$V(i_0) = \beta V(i_0+1) = \cdots = \beta^{k-i_0} V(k) = \beta^{k-i_0} R$$

Thus, we have the following:

$$\begin{aligned}
&\beta^{k-i_0} \leq \frac{v}{R\beta(1-\beta)} < \beta^{k-(i_0+1)} \\
\iff\ & k - i_0 \geq \log_\beta\left(\frac{v}{R\beta(1-\beta)}\right) > k-(i_0+1) \\
\iff\ & i_0 \leq k - \log_\beta\left(\frac{v}{R(1-\beta)}\right) + 1 < i_0 + 1 \\
\iff\ & i_0 = k - \left\lfloor \log_\beta\left(\frac{v}{R(1-\beta)}\right)\right\rfloor \equiv k - \Delta
\end{aligned}$$

□

Now we assume the look-ahead factor of a customer is drawn from some distribution $t \sim T$. The phase transition of the customer's DP will now depend on t.

$$i_0(t) = \begin{cases} i_0, & \text{if } t \geq \Delta. \\ k-t, & \text{otherwise.} \end{cases}$$

In this section, we focus on a very simple threshold distribution given by the following.

$$t = \begin{cases} t_1 \geq \Delta, & \text{wp } p, \\ 0, & \text{wp } 1-p. \end{cases}$$

3.2 Merchant Objective Dynamics

3.2.1 Non Strategic Merchant B, and Equal Promotion Budgeting

First we look into the case when merchant B does not strategize over its discount value v, and when merchant A sets its reward parameters so as to have equal budgets for promosionts as B: i.e. $R = kv$.

Theorem 3.2. *Given $\lambda \sim Unif(0,b)$, merchant A sets its reward parameter $k = \frac{e}{1-\beta}$ at small values of b and $k = \frac{e^{(1-\beta)t_1}}{1-\beta}$ for larger values.*

Proof. First merchant A's objective evaluates to the following:

$$\begin{aligned}
\max_k\{RoR_A\} \Leftrightarrow & \max_k\left\{\underset{\lambda}{E}\left[\frac{\lambda k}{k-\Delta(1-\lambda)}\right]\right\} \\
\Leftrightarrow & \max_k\left\{\frac{1}{b}\int_0^b \frac{\lambda k}{k-\Delta(1-\lambda)}d\lambda\right\} \\
\Leftrightarrow & \max_k\left\{\frac{k}{\Delta^2 b}\left(\Delta b-(k-\Delta)\log\left(\frac{k-\Delta(1-b)}{k-\Delta}\right)\right)\right\} \\
\Leftrightarrow & \max_k\left\{\frac{k}{\Delta}\left(1-\frac{k-\Delta}{b\Delta}\log\left(\frac{k-\Delta(1-b)}{k-\Delta}\right)\right)\right\}
\end{aligned}$$

Now let $\theta = \frac{\Delta}{k}$. Then maximizing the above function is equivalent to maximizing the following function w.r.t. θ with keeping in mind the range that θ can follow.

$$\max_k\{RoR_A\} \Leftrightarrow \max_\theta\{f(\theta)\} \Leftrightarrow \max_\theta\left\{\frac{1}{\theta}\left(1-\frac{1-\theta}{b\theta}\log\left(1+\frac{b\theta}{1-\theta}\right)\right)\right\} \tag{5}$$

Let's look at the quantity $\frac{\Delta}{k}$.

$$\frac{\Delta}{k} = \frac{\log_\beta\left(\frac{1}{k(1-\beta)}\right)}{k} \sim \frac{\log(k(1-\beta))}{k(1-\beta)}$$

Now this value is maximized at $k = \frac{e}{1-\beta}$ and minimized at the maximum possible value of k which is bounded above by $\frac{e^{(1-\beta)t_1}}{1-\beta}$ from the assumption of $t_1 \geq \Delta$.

When b is small, the function $f(\theta)$ can be approximated by taking the second degree terms for the term inside the log. This gives:

$$\begin{aligned}
f(\theta) \sim & \frac{1}{\theta}\left(1-\frac{1-\theta}{b\theta}\left(\frac{b\theta}{1-\theta}-\frac{\left(\frac{b\theta}{1-\theta}\right)^2}{2}\right)\right) \\
= & \frac{b}{2(1-\theta)}
\end{aligned}$$

Clearly $f(\theta)$ is maximized when θ is maximized. This happens as shown above at $k = \frac{e}{1-\beta}$

Whereas when b is large, $f'(\theta)$ can be shown to be negative. Hence we get out result. □

Lemma 3.2. *The objective function for the above look-ahead distribution is given by:*

$$f(k) = \frac{\lambda(k-R)p}{k-(1-\lambda)\Delta} + \frac{\lambda(k-R)(1-p)}{k-(1-\lambda)t_0}$$

Proof.

$$
\begin{aligned}
E_t\left(\frac{k-R}{\frac{i_0(t)}{\lambda}+k-i_0(t)}\right) &= \frac{(k-R)p}{\frac{i_0}{\lambda}+k-i_0} + \frac{(k-R)(1-p)}{\frac{k-t_0}{\lambda}+k-(k-t-0)} \\
&= \frac{\lambda(k-R)p}{i_0+\lambda(k-i_0)} + \frac{\lambda(k-R)(1-p)}{k-t_0+t_0\lambda} \\
&= \frac{\lambda(k-R)p}{k-\Delta+\lambda(\Delta)} + \frac{\lambda(k-R)(1-p)}{k-t_0+t_0\lambda} \\
&= \frac{\lambda(k-R)p}{k-(1-\lambda)\Delta} + \frac{\lambda(k-R)(1-p)}{k-(1-\lambda)t_0} \equiv f(k)
\end{aligned}
$$

□

We wish to maximize this objective function for $k > \Delta$ (otherwise i_0 would be negative). Next, we will characterize the conditions under which we can maximize the function and what the maxima are.

Lemma 3.3. *If $(1-\lambda)t_0 \leq R \leq (1-\lambda)\Delta$, the above objective function has real-valued critical points.*

Proof. First we differentiate $f(k)$.

$$
\begin{aligned}
\frac{df}{dk} &= \frac{\lambda(k-(1-\lambda)\Delta)-\lambda(k-R)p}{(k-(1-\lambda)\Delta)^2} + \frac{\lambda(1-p)(k-(1-\lambda)t_0)-\lambda(k-R)(1-p)}{(k-(1-\lambda)t_0)^2} \\
&= \frac{\lambda p(R-(1-\lambda)\Delta)}{(k-(1-\lambda)\Delta)^2} + \frac{\lambda(1-p)(R-(1-\lambda)t_0)}{(k-(1-\lambda)t_0)^2}
\end{aligned}
$$

Setting equal to zero and solving for k we get the following. Let $c_1 = R-(1-\lambda)\Delta$ and $c_2 = R-(1-\lambda)t_0$.

$$pc_1(k-(1-\lambda)t_0)^2 = -(1-p)c_2(k-(1-\lambda)R)^2$$

$$\iff (pc_1)k^2 - (2pc_1(1-\lambda))k + (pc_1(1-\lambda)^2t_0^2 = (-(1-p)c_2)k^2 + (2(1-p)(1-\lambda)c_2\Delta)k - ((1-p)c_2$$

$$\iff (pc_1+(1-p)c_2)k^2 - 2(1-\lambda)(pc_1t_0+(1-p)c_2\Delta)k + (1-\lambda)^2(pc_1t_0^2+(1-p)c_2\Delta^2) = 0$$

For the above to have real-valued solutions, we need:

$$4(1-\lambda)^2(pc_1t_0+(1-p)c_2\Delta)^2 - 4(pc_1+(1-p)c_2)(1-\lambda)^2(pc_1t_0^2+(1-p)c_2\Delta^2) \geq 0$$

$$\iff p^2c_1^2t_0^2 + (1-p)^2c_2^2\Delta^2 + 2p(1-p)c_1c_2t_0\Delta - p^2c_1^2t_0^2 + (1-p)^2c_2^2\Delta^2 + p(1-p)c_1c_2t_0^2 + p(1-p)c_1c_2$$

$$\iff p(1-p)c_1c_2(2t_0\Delta - \Delta^2 - t_0^2) \geq 0$$

$$\iff -p(1-p)c_1c_2(t_0-\Delta)^2 \geq 0$$

$$\iff -p(1-p)c_1c_2 \geq 0$$

$$\iff c_1c_2 \leq 0$$

The constraint that $c_1c_2 \leq 0$ means $(R-(1-\lambda)\Delta)(R-(1-\lambda)t_0) \leq 0$. Because $R \geq 0$, $(1-\lambda) \geq 0$ and $t_0 < \Delta$, we also have that $(R-(1-\lambda)\Delta) < (R-(1-\lambda)t_0)$. So we must have:

$$(R-(1-\lambda)\Delta) \leq 0 \leq (R-(1-\lambda)t_0)$$
$$\iff (1-\lambda)t_0 \leq R \leq (1-\lambda)\Delta$$

□

Note that Δ depends on R, so the above inequality is more complicated than as written. TODO - see if we can get a nice inequality for R not in terms of Δ. (Can also add a sample plot here).

Now need to check that this is actually a maximum.

4 Question that we need to answer

Here is something that is concerning me. Look at the objective function again.

$$f(k) = \frac{\lambda p(k-R)}{k-(1-\lambda)\Delta} + \frac{\lambda(1-p)(k-R)}{k-(1-\lambda)t_0}$$

We want to maximize this function over values of k greater than Δ. We know that $t_0 < \Delta$ so the denominator of the first term will always be less than that of the second term. Note even more that when k is really close to Δ the first term dominates and will be really large. Obviously, when $k = (1-\lambda)\Delta < \Delta$, $f(k)$ will be infinite (as long as $R \neq \Delta$). And as k increases from here, f will decrease. I believe this objective function will always be maximized at $k = \Delta$ unless R is also very close to Δ - which as we've seen on restrictions on R, may not be possible. We may need to tweek our objective function.

Some futher investigation on restrictions on R to get a maximum from critical point. First we look at the second derivative of f.

$$f''(k) = -\frac{2\lambda p(R-(1-\lambda)\Delta)}{(k-(1-\lambda)\Delta)^3} - \frac{2\lambda(1-p)(R-(1-\lambda)t_0)}{(k-(1-\lambda)t_0)^3}$$

We know the denominators will always be positive on our domain. We also know that for a critical point to be real we must have $(R-(1-\lambda)\Delta) \leq 0 \leq (R-(1-\lambda)t_0)$. So for a real-valued critical point to be a maximum, we must have:

$$-\frac{2\lambda p(R-(1-\lambda)\Delta)}{(k-(1-\lambda)\Delta)^3}-\frac{2\lambda(1-p)(R-(1-\lambda)t_0)}{(k-(1-\lambda)t_0)^3}<0$$
$$\iff \frac{2\lambda p|R-(1-\lambda)\Delta|}{(k-(1-\lambda)\Delta)^3}<\frac{2\lambda(1-p)(R-(1-\lambda)t_0)}{(k-(1-\lambda)t_0)^3}$$

Again, the denominator of the first term will always be smaller than that of the second term, so will be difficult (more restrictions on R and Δ) for above to be true.

Furthermore, we need the critical point of f to be greater than Δ. The critical points of f are given by:

$$x=\frac{(1-\lambda)(pc_1t_0+(1-p)c_2\Delta)\pm(1-\lambda)|t_0-\Delta|(-p(1-p)c_1c_2)^{\frac{1}{2}}}{pc_1+(1-p)c_2}$$

And we need:

$$\frac{(1-\lambda)(pc_1t_0+(1-p)c_2\Delta)+(1-\lambda)|t_0-\Delta|(-p(1-p)c_1c_2)^{\frac{1}{2}}}{pc_1+(1-p)c_2}\geq\Delta$$

Again, this gives more restrictions on R and Δ. I still need to solve for all these restrictions, but I wanted to write this up so far to let you know a concern.

5 When should a store offer a reward?

Notice the revenue rate function $f(k)$, fixing β, t_0, p, R and λ, approaches λ as $k\to\infty$. This limit makes sense, as k approaching ∞ means that no reward will be given, so all visits to B will be exogenous, with a revenue rate of λ. One question that we may find interesting is when it is impossible to beat the revenue rate of λ (supposing R is fixed) and thus not offer a reward at all. Even if fixing R is not very realistic, understanding the objective function as a function of λ may give us insight into what to set R.

Let's fix β, p, R and let $t_0=0$ for now (for simplicity - can go in later and add more general case next). We wish to find the set of $\lambda\in[0,1]$ such that $f(k)\leq\lambda$ on $k\geq\Delta$: the λ's for which we can not beat revenue rate of λ. I believe that there will always be some λ_0 such that for all $\lambda<\lambda_0$, we can increase the revenue rate by offering a reward and for all $\lambda\geq\lambda_0$, we cannot increase the revenue rate by offering a reward. (Note: I have not finished with the full proof of this yet, but going to write-up some ideas so far).

Lemma 5.1. *Fix $R > 0$, β, p, v and $t_0 = 0$. Then $f(\Delta) \leq \lambda$ if and only if $\lambda \geq \frac{p(\Delta-R)}{\Delta-(\Delta-R)(1-p)}$. Note that when equality holds in the second inequality, it does in the first as well.*

Proof. We have:

$$\begin{aligned} f(\Delta) &= \frac{\lambda p(\Delta - R)}{\Delta - (1-\lambda)\Delta} + \frac{\lambda(1-p)(\Delta - R)}{\Delta} \\ &= \frac{p(\Delta - R)}{\Delta} + \frac{\lambda(1-p)(\Delta - R)}{\Delta} \\ &= \frac{(\Delta - R)(\lambda(1-p)+p)}{\Delta} \end{aligned}$$

Then:

$$\begin{aligned} &\frac{(\Delta - R)(\lambda(1-p)+p)}{\Delta} \leq \lambda \\ \iff\ & \lambda - \frac{(\Delta - R)\lambda(1-p)}{\Delta} \geq \frac{p(\Delta - R)}{\Delta} \\ \iff\ & \lambda\left(\frac{\Delta - (\Delta - R)(1-p)}{\Delta}\right) \geq \frac{p(\Delta - R)}{\Delta} \\ \iff\ & \lambda \geq \frac{p(\Delta - R)}{\Delta - (\Delta - R)(1-p)} \end{aligned}$$

Note for the last step above, we need $\Delta-(\Delta-R)(1-p) > 0$. But because $\Delta, R, (1-p) \geq 0$ then if $(\Delta - R) \leq 0$, the condition will always be satisfied (if $(\Delta - R) = 0$, $\Delta = R > 0$ and value is positive). And if $(\Delta - R) > 0$, $(1-p)(\Delta - R) \leq (\Delta - R) < \Delta$, so $\Delta - (\Delta - R)(1-p) > 0$. □

I believe the above threshold $\frac{p(\Delta-R)}{\Delta-(\Delta-R)(1-p)}$ is our desired λ_0. To show this we need to show that $f(\Delta)$ is a necessary and sufficient condition for $f(k)$ not exceeding λ on $k \geq \Delta$. Clearly it is a necessary condition, so we just need to show it is sufficient. I believe the best way to do this is to show that for these λ, no critical points that are maxima exist in the range $[\Delta, \infty)$. If this is the case, then the max will occur at the the endpoints, which we have already argued will not be greater than λ. First we prove some lemmas about the roots of f', assuming they are real-valued.

Lemma 5.2. *If $p(R-(1-\lambda)\Delta)+(1-p)R > 0$, the objective function $f(k)$ (when $t_0 = 0$) has at least one real-valued critical point with value at least $(1-\lambda)\Delta$.*

Proof. First note that we need $R-(1-\lambda)\Delta \leq 0 \leq R-(1-\lambda)t_0 = R$ for f' to have real roots. The roots of f' are given by:

$$x = \frac{(1-\lambda)((1-p)R\Delta) \pm (1-\lambda)\Delta(-p(1-p)(R-(1-\lambda)\Delta)R)^{\frac{1}{2}}}{p(R-(1-\lambda)\Delta)+(1-p)R} \equiv C \pm D$$

We will show that $C = \frac{(1-\lambda)((1-p)R\Delta)}{p(R-(1-\lambda)\Delta)+(1-p)R} \geq (1-\lambda)\Delta$. Then f' has at most one root less than $(1-\lambda)\Delta$; because if $C \geq (1-\lambda)\Delta$, for every D it is not possible for both $C+D$ and $C-D$ to be less than $(1-\lambda)\Delta$. Note that $(1-\lambda)(1-p)R\Delta \geq 0$ and $p(R-(1-\lambda)\Delta) \leq 0$ and by assumption, the denominator of C is positive as well. Therefore, we have:

$$\begin{aligned} C &= \frac{(1-\lambda)(1-p)R\Delta}{p(R-(1-\lambda)\Delta)+(1-p)R} \\ &\geq \frac{(1-\lambda)(1-p)R\Delta}{(1-p)R} = (1-\lambda)\Delta \end{aligned}$$

because $(1-p)R \geq (1-p)R + p(R-(1-\lambda)\Delta) > 0$. $\square$

Note that the above lemma applies to any choice of parameters as long as that denominator is positive. Hopefully we can use it to understand the objective function in more general settings as well.

Lemma 5.3. *If $\lambda \geq \frac{p(\Delta-R)}{\Delta-(\Delta-R)(1-p)}$, the objective function $f(k)$ (when $t_0=0$) has at most one real-valued critical point greater than Δ.*

Proof. Again note that we need $R-(1-\lambda)\Delta \leq 0 \leq R-(1-\lambda)t_0 = R$ for f' to have real roots. The roots of f' are given by:

$$x = \frac{(1-\lambda)((1-p)R\Delta) \pm (1-\lambda)\Delta(-p(1-p)(R-(1-\lambda)\Delta)R)^{\frac{1}{2}}}{p(R-(1-\lambda)\Delta)+(1-p)R} \equiv C \pm D$$

We will show that $C = \frac{(1-\lambda)((1-p)R\Delta)}{p(R-(1-\lambda)\Delta)+(1-p)R} \leq \Delta$. Then f' has at most one root great than Δ; because if $C \leq \Delta$, for every D it is not possible for both $C+D$ and $C-D$ to be greater than Δ. Note that $(1-\lambda)(1-p)R\Delta \geq 0$ and $p(R-(1-\lambda)\Delta) \leq 0$. If $p(R-(1-\lambda)\Delta)+(1-p)R < 0$, then $C < 0 \leq \Delta$ and we are done. So we focus on the case that $p(R-(1-\lambda)\Delta)+(1-p)R > 0$.

We will think of C as a function of λ.First we show that for $\lambda' = \frac{p(\Delta-R)}{\Delta-(\Delta-R)(1-p)}$, $C(\lambda') = \Delta$. It is a straightforward computation to see that $(1-\lambda') = \frac{R}{\Delta-(\Delta-R)(1-p)}$. And we have:

$$
\begin{aligned}
C(\lambda') &= \frac{(1-\lambda')(1-p)R\Delta}{p(R-(1-\lambda')\Delta)+(1-p)R} \\
&= \frac{(1-p)R^2\Delta}{\Delta-(\Delta-R)(1-p)} \cdot \frac{1}{p\left(R-\frac{R\Delta}{\Delta-(\Delta-R)(1-p)}\right)+(1-p)R} \\
&= \frac{(1-p)R^2\Delta}{\Delta-(\Delta-R)(1-p)} \cdot \frac{1}{\left(\frac{-pR(\Delta-R)(1-p)}{\Delta-(\Delta-R)(1-p)}\right)+(1-p)R} \\
&= \frac{(1-p)R^2\Delta}{-pR(\Delta-R)(1-p)+(1-p)R(\Delta-(\Delta-R)(1-p)))} \\
&= \frac{(1-p)R^2\Delta}{(1-p)R\Delta-(pR(\Delta-R)(1-p)+(1-p)R(\Delta-R)(1-p))} \\
&= \frac{(1-p)R^2\Delta}{(1-p)R\Delta-(1-p)R(\Delta-R)} = \frac{(1-p)R^2\Delta}{(1-p)R^2} = \Delta
\end{aligned}
$$

Next notice that when the denominator is positive, C is a decreasing function in $\lambda < 1$; increasing λ decreases the value of the nominator and increases the value of the denominator (the first term becomes less negative so the whole thing becomes more positive). So by these two facts, $C(\lambda) \leq \Delta$ for $\lambda \geq \lambda'$ as defined above.

Finally, we need that the denominator is non-zero. So we need $R \neq p(1-\lambda)\Delta$. But we already have for real roots that $R \geq (1-\lambda)\Delta$. So for $p < 1$, we have a non-zero denominator. □

Now we must prove that $f(\Delta) \leq \lambda$ is a sufficient condition the maximum of f on $k \geq \Delta$ being no greater than λ.

Lemma 5.4. *Fix $R > 0$, β, p, v and $t_0 = 0$. If $f(\Delta) \leq \lambda$, then $f(k) \leq \lambda$, $\forall k \geq \Delta$.*

Proof. We have that $f(\Delta) \leq \lambda$ and $f(k) \to \lambda$ as $k \to \infty$. Then if we show that no strict maximum occurs on the interval (Δ, ∞), we have shown the claim. We will break the proof down in cases based on the sign on the derivative of f at delta. Note that because $f(\lambda) \leq \Delta$, we have a λ such that our previous lemma applies.

Case 1: ($f'(\Delta) = 0$) For the derivative of f at Δ to be zero, no critical points of f may occur after Δ. To see this, we make use of our previous lemma; if Δ is the larger of the two roots of f' then we are done, and by the previous result, if it is the smaller of the two roots, it must be the case that both roots are actually Δ. Because no critical points occur after Δ, there are no strict maximum in the interval (Δ, ∞).

Case 2: $(f'(\Delta) < 0)$ By our previous lemma, only one critical point may occur after Δ, so by the negative sign of $f'(\Delta)$, that critical point may only be a minimum.

Case 3: $(f'(\Delta) > 0)$ (NOT COMPLETED YET, so here are some notes) Here we want to show that in fact no critical points occur after Δ, so no strict maximum may occur on the interval of interest. I am still working on this proof - I think it won't take too much longer. $\square$