# Optimal Design of Loyalty Reward Program in a Competitive Duopoly

December 13, 2016

## 1 Introduction

Loyalty programs constitute a huge market and are a major source of revenue for many low margin businesses. Over 48 billion dollars of perceived rewards are issued in the United States alone every year, with every household having over 19 loyalty memberships. These include credit cards, hotel and airline reward programs, and more recently even restaurants, grocery and retail stores. Though forming a big component of the market, there is very less scientific understanding about the design of loyalty reward programs. We aim to address this gap with this research.

One popular form of loyalty reward programs are *frequency reward programs*, where customers earn *points* as currency over spendings with merchants and are able to redeem these points into dollar based rewards after achieving certain threshold point collections. There is extant literature on characterizing customer behavior toward frequency reward programs. Most of the literature is empirical in nature, and relies on psychological behavioral patterns among customers, as opposed to rational economic decision making. In this paper, we consider a competitive duopoly of two merchants where one merchant offers a loyalty reward program and the other offers traditional pricing with discounts and characterize a novel model of customer choice where customers measure their utilities in rational economic terms. In addition, we characterize the optimal reward design choice for the merchant offering the frequency reward program, based on different customer populations: specifically, how should the merchant decide the optimal thresholds and dollar value of rewards to optimize for its revenue share from the participating customer population. One important constraint we impose is that the merchant has to choose a *one design fits all* reward program for the entire participating customer population and is not allowed to personalize the program for different customer segments, while maximizing its overall long term revenue objective.

This is how the remaining of the paper is structured. First we will describe some past work. Then we will go over our contributions and explain how our work builds on top of past literature. In Section 2 we will describe our model followed by the main results in Section 3. We will follow up with a short discussion and future work in Section 4.

## 1.1 Past Literature

## 1.2 Our Contributions

We make both modeling and analytical contributions along the following major directions:

### 1.2.1 Competitive Duopoly Setup

We model a competitive duopoly of two merchants, one of them offering a frequency reward program and the other offering traditional pricing. Both merchants sell an identical good at fixed precommitted prices. The reward program merchant sells the good at a higher price. Customers are drawn from a known population distribution and they measure their utilities in rational economic terms, i.e., they make their purchase decisions to maximize long term discounted rewards. The discount factor is the time value of money, and is constant for all customers. Every customer makes a purchase everyday from either of the two merchants. With each purchase from the reward program merchant, customer gains some fixed number of points, and on achieving the reward redemption threshold, (s)he immediately gains the reward value as a dollar cashback.

### 1.2.2 Customer Parameters

Each customer has two important parameters drawn from the known population distribution: first a visit probability bias with which (s)he purchases the good from the reward program merchant for reasons exogenous to utility maximization, and second a look-ahead factor that controls how far into the future the customer can perceive the rewards. The visit probability bias toward the reward program merchant can be attributed to *excess* *loyalty* which has been argued as a important parameter for the success of any reward program, or it can be attributed to price insensitivity of the customer: whenever the customer is price insensitive, (s)he strictly prefers to purchase from the reward program merchant as (s)he gains points redeemable for rewards in the future. There are many possible reasons for customers' price insensitivity: the reward program merchant could be offering some other monopoly products, or the customer might be getting reimbursed for some purchases as part of corporate perks (eg: corporate travel). As an effect, this visit probability bias controlls how frequently the customers' points increase even when (s)he does not actively choose to make purchases from the reward program merchant. The look-ahead parameter affects the customer behavior dynamics as follows: if the reward is farther than the customer's look-ahead parameter, (s)he is unable to register the future value of that reward and take it into consideration while maximizing long term utility. Both these parameters can be attributed to bounded rationality of customers and have been argued to be important factors toward customer choice dynamics (cite, etc).

### 1.2.3 Customer Choice Dynamics

We formulate the customer choice dynamics as a dynamic program with the state being the number of points collected from the reward program merchant. When the customer does not make biased visits to the reward program merchant, (s)he chooses the merchant maximizing her long term utility: comparing the immediate utility by purchasing the good at cheaper price and the long term utility of waiting and receiving the time discounted reward. The solution to the customer's dynamic program gives conditions for the existence and achievability of a phase transition: a points threshold before which the customer visits the merchant offering rewards only due to the visit probability bias, and after which (s)he always visits the merchant offering rewards till receiving the reward. We show that this phase transition point has the following dependencies:

1. It decreases with increase in the look-ahead parameter, i.e., if the customer can perceive rewards longer into the future, the phase transition for the customer occurs sooner.

2. It decreases with increase in the reward value, and decreases with increase in points threshold required to redeem the reward.

3. It increases with the discount value that the traditional pricing merchant provides, i.e., the cheaper the product from the second merchant, the farther is the phase transition point.

4. (effect w.r.t. to discount factor $\beta$. leaving it for now).

(Discussion relating to past literature on psychological constructs: tipping point literature. purchase acceleration literature)

#### 1.2.4 Reward Program Design

After characterizing the customer behavior dynamics in our model, we optimize over the long run revenues that the reward program merchant achieves. We first show some conditions over the customer population that easily gives better revenues to the reward program merchant over the traditional pricing merchant (can we explain the graph of $(b, p)$ pairs in one or two lines here for this particular result. We provide a framework for optimizing the parameters for the reward program merchant to maximize its revenue objective. We then look at a specific simplified case of proportional budgeting: the reward offered by the reward program merchant if proportional to the product of the distance to the reward and the discount provided by the traditional pricing merchant. We show that Cashback result simplified to two lines here. We characterize conditions for when it's better for the reward program merchant to offer a reward vs not offering any reward (graph of relevant (b,p) pairs). Finally we compare the strategy to offer reward vs offering traditional pricing with discounts itself for the reward program merchant (here the expiration can come into play) and show....

Then, if we have anything, we can talk about multi-tiering strategies.

# 2 Model

We index the two competing merchants selling identical goods as $A$ and $B$. Without loss of generality we assume that $A$ sells the good for a price of 1 dollars while $B$ sells it for $1 - v$ dollars, i.e., $B$ offers a discount of $v$ dollars. $A$ on the other hand offers a reward of value $R$ dollars to a customer after (s)he makes $k$ purchases at $A$.

Every customer purchases an item from either $A$ or $B$ everyday (Can we justify/relax this somehow). We assume customers have a linear homogenous utility in price: at price

$p$ the utility is $\nu(p) = 1 - p$. This reduces to customers getting an immediate utility of 0 from $A$ and $v$ from $B$. Customers have the same time value of money as a discount factor of $\beta$ lying between 0 and 1.

We denote a customer's visit probability bias and look-ahead parameter with $\lambda$ and $t$ respectively. That is with probability $\lambda$ (s)he purchases from $A$ due to externalities and perceives a future reward only if it is within $t$ purchases away. We assume $\lambda$ for a customer to be drawn from a uniform distribution between $[0, b]$, where $b$ is between 0 and 1. And we focus on a simple threshold distribution for the look-ahead parameter $t$:

$$t = \begin{cases} t_1 \geq \Delta, & \text{wp } p, \\ 0, & \text{wp } 1 - p. \end{cases}$$

The above distribution intuitively means that the customers are either myopic and focus only on immediate rewards, or are far-sighted enough. We model the customer's decision problem as a dynamic problem. We index the number of visits the customer makes at store $A$ by $i$, for $0 \leq i \leq k - 1$, and we refer a customer to be in state $i$ after having made $i$ visits to $A$. At state $i$, the customer has two possibilities:

1. With probability $\lambda$, the customer must visit $A$, and she is now in state $i + 1$.
2. With probability $1 - \lambda$, the customer may purchase from $B$ for an immediate utility $v$ and remain in state $i$ or purchase from $A$ for no utility but move to state $i + 1$.

Let $V(i)$ denote the long term expected reward at state $i$. Then we model the decision problem as the following dynamic program.

$$V(i) = \lambda\beta V(i + 1) + (1 - \lambda) \max\{v + \beta V(i), \beta V(i + 1)\} \text{ for } 0 \leq i \leq k - 1$$
$$V(k) = R$$

We show that the decision process exhibits a phase transition; that is prior to some state, the customer will only visit $A$ if (s)he must do so exogenously but after that state, (s)he always decides to go to $A$. This phase transition point is independent of $\lambda$, and depends only on $t$, among the variable customer parameters. Hence we represent this phase transition point as $i_0(t)$.

## 2.1 Merchant Objective

Given the above model of customer dynamics, we define the revenue objectives of $A$ and $B$, where $A$ chooses its parameters whereas $B$ is non strategic. We define the rate of revenue for a merchant from a customer as the expected time averaged revenue that the merchant receives within the customer's lifetime. For simplification we assume merchants do not discount future revenues. As described above, a customer's dynamics are cyclic after each reward cycle. Thus the lifetime dynamics of customer behavior is a regenerative process with independent and identically distributed reward cycle lengths. Let $RoR_A(c)$ and $RoR_B(c)$ denote the expected rate of revenues for merchants $A$ and $B$ respectively from a customer $c$'s lifetime. Let $\tau(t,\lambda)$ denote the total number of visits the customer makes before reaching the phase transition point $i_0(t)$. Then the length of the reward cycle (or total number of visits the customer makes before receiving the reward) is $\tau(t,\lambda)+k-i_0(t)$, as after the phase transition (s)he makes all visits to merchant $A$ only until hitting the reward. In this cycle the number of visits that the customer makes to $A$ are $k$, and to $B$ are $\tau(t,\lambda)-i_0(t)$. The revenue that $A$ earns in one such cycle is $k-R$ and the revenue that $B$ earns is $(1-v)(\tau(t,\lambda)-i_0(t))$. Thus the rate of revenues for $A$ and $B$ from the customer $c$ are as follows:

$$RoR_A(c) = \underset{\tau}{E}\left[\frac{k-R}{\tau(t,\lambda)+k-i_0(t)}\right]$$

$$RoR_B(c) = \underset{\tau}{E}\left[\frac{(\tau(t,\lambda)-i_0(t))(1-v)}{\tau(t,\lambda)+k-i_0(t)}\right]$$

Since the process for a single customer is regenerative, using the reward renewal theorem (CITE), we can take the expectation over the cycle length inside the numerator and denominator respectively. Note that $\underset{\tau}{E}[\tau(t,\lambda)] = \frac{i_0(t)}{\lambda}$ as before reaching the phase transition point, with probability $\lambda$, the customer's visits to $A$ increases by 1 and with probability $1-\lambda$ it stays constant. And then taking the expectation over the customer population the overall rate of revenues for both $A$ and $B$ are as follows:

$$RoR_A = \underset{\lambda,t}{E}\left[\frac{k-R}{i_0(t)/\lambda+k-i_0(t)}\right] \tag{1}$$

$$RoR_B = \underset{\lambda,t}{E}\left[\frac{(i_0(t)\lambda-i_0(t))(1-v)}{i_0(t)/\lambda+k-i_0(t)}\right] \tag{2}$$

# 3 Results

## 3.1 Customer Choice Dynamics

We first show that every customer exhibits the following behavior: until (s)he reaches the phase transition point $i_0(t)$, she visits $A$ only due to the exogeneity paramaeter, and after that (s)he always visits merchant $A$ till she receives the reward. This behavior is cyclic, and repeats after every reward redemption.

**Lemma 3.1.** *$V(i)$ is an increasing function in $i$ if the following condition holds:*

$$R > \frac{(1-\lambda)v}{1-\beta} \tag{3}$$

*And further, $V(i)$ can be evaluated as:*

$$V(i) = \max\left\{\frac{\lambda\beta V(i+1) + (1-\lambda)v}{1-(1-\lambda)\beta}, \beta V(i+1)\right\} \tag{4}$$

*Proof.* First we show that $V(i)$ is an increasing function in $i$ by induction. We first show that if the condition above is satisfied, $V(k-1) < V(k) = R$. Suppose not, so $V(i) \geq R$. Then we have:

$$\begin{aligned}
V(k-1) &= \lambda\beta V(k) + (1-\lambda)(v + \beta V(k-1)) \\
&= \frac{\lambda\beta R + (1-\lambda)v}{1-(1-\lambda)\beta} \\
&< \frac{\lambda\beta R + (1-\beta)R}{1-(1-\lambda)\beta} \\
&= \frac{R(1-(1-\lambda)\beta)}{1-(1-\lambda)\beta} = R
\end{aligned}$$

But this is a contradiction, so $V(k-1) < V(k)$. Now assume $V(i+1) < V(i+2)$ for some $i < k-2$, we will show that this implies $V(i) < V(i+1)$. Suppose not, so $V(i) \geq V(i+1)$. As we did before we may upper bound $V(i)$.

$$\begin{aligned}
V(i) &= \lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)) \\
&\leq (1-\lambda)v + \beta V(i) \\
\iff V(i) &\leq \frac{(1-\lambda)v}{1-\beta}
\end{aligned}$$

But because $V(i+1) < V(i+2)$, we may lower bound $V(i+1)$.

$$\begin{aligned} V(i+1) &\geq \lambda\beta V(i+2) + (1-\lambda)(v + \beta V(i+1)) \\ &= (1-\lambda)v + (1-\lambda)\beta V(i+1) + \lambda\beta V(i+2) \\ &> (1-\lambda) + \beta V(i+1) \\ \iff \quad V(i+1) &> \frac{(1-\lambda)v}{1-\beta} \end{aligned}$$

Again, we have a contradiction, so $V(i) < V(i+1)$, and $V(i)$ is an increasing function in $i$. Now we prove the second claim. We have the following:

$$\begin{aligned} V(i) &= \lambda\beta V(i+1) + (1-\lambda)\max\{v + \beta V(i), \beta V(i+1)\} \\ &= \max\{\lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)), \beta V(i+1)\} \end{aligned}$$

Assuming $V(i)$ is the left term in the above maximum, we may solve the equation for that term.

$$\begin{aligned} V(i) &= \lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)) \\ (1-(1-\lambda)\beta)V(i) &= \lambda\beta V(i+1) + (1-\lambda)v \\ V(i) &= \frac{\lambda\beta V(i+1) + (1-\lambda)v}{1-(1-\lambda)\beta} \end{aligned}$$

And we get our claim. □

Let's combine this theorem with the look-ahead parameter directly

**Theorem 3.1.** *Assuming $V(i)$ is an increasing function in $i$, a phase transition occurs after the consumer makes $i_0$ visits to firm A, which evaluates to:*

$$\begin{aligned} i_0 &= k - \left\lfloor \log_\beta \left( \frac{v}{R(1-\beta)} \right) \right\rfloor \\ &\equiv k - \Delta \end{aligned}$$

*Proof.* First we solve for the condition on $V(i+1)$ for us to choose firm $A$ over $B$ willingly.

$$\begin{aligned}
\beta V(i+1) &> \frac{\lambda\beta V(i+1)+(1-\lambda)v}{1-(1-\lambda)\beta} \\
\iff \beta V(i+1)\left(1-\frac{\lambda}{1-(1-\lambda)\beta}\right) &> \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right)v \\
\iff \beta V(i+1)\left(\frac{1-(1-\lambda)\beta-\lambda}{1-(1-\lambda)\beta}\right) &> \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right)v \\
\iff \beta V(i+1)\left(\frac{(1-\lambda)(1-\beta)}{1-(1-\lambda)\beta}\right) &> \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right)v \\
\iff \beta V(i+1) &> \frac{v}{1-\beta} \\
\iff V(i+1) &> \frac{v}{\beta(1-\beta)}
\end{aligned}$$

Let $i_0$ be the minimum state $i$ such that the above holds, so in particular $V(i_0) \leq \frac{v}{\beta(1-\beta)}$ but $V(i_0+1) > \frac{v}{\beta(1-\beta)}$. We know because $V$ is increasing in $i$ (still need to prove), this point is indeed a phase transition: $V(i) > \frac{v}{\beta(1-\beta)}$ for all $i > i_0$, so after this point, the customer always chooses firm $A$. We may compute $V(i_0)$ easily using this fact.

$$V(i_0) = \beta V(i_0+1) = \cdots = \beta^{k-i_0}V(k) = \beta^{k-i_0}R$$

Thus, we have the following:

$$\begin{aligned}
\beta^{k-i_0} \leq \frac{v}{R\beta(1-\beta)} &< \beta^{k-(i_0+1)} \\
\iff k-i_0 \geq \log_\beta\left(\frac{v}{R\beta(1-\beta)}\right) &> k-(i_0+1) \\
\iff i_0 \leq k-\log_\beta\left(\frac{v}{R(1-\beta)}\right)+1 &< i_0+1 \\
\iff i_0 = k-\left\lfloor \log_\beta\left(\frac{v}{R(1-\beta)}\right)\right\rfloor &\equiv k-\Delta
\end{aligned}$$

□

With the inclusion of the look-ahead factor, the phase transition point depends on it, and we will refer to it as $i_0(t)$. Specifically, the dependence is as follows:

$$i_0(t) = \begin{cases} i_0, & \text{if } t \geq \Delta. \\ k-t, & \text{otherwise.} \end{cases}$$

The above dependence reduces to the following after incorporating the look-ahead distribution:

$$i_0(t) = \begin{cases} i_0, & \text{wp } p, \\ k, & \text{wp } 1-p. \end{cases}$$

Discussion around what do the $i_0$ dependencies mean.

Discussion around setting $i_0 = 0$

## 3.2 Merchant Objective Dynamics

We substitute the value of the phase transition point obtained above in the rate of revenue equations to reevaluate them. And since we assume that $\lambda$ and $t$ are drawn indepent of each other, we can separate the expectation terms and evaluate them sequentially, first over $t$, then over $\lambda$. This reduces the rate of revenues as follows:

$$\begin{aligned}
RoR_A =& \underset{\lambda,t}{E}\left[\frac{k-R}{i_0(t)/\lambda + k - i_0(t)}\right] \\
=& \underset{\lambda}{E}\left[p\cdot\frac{k-R}{i_0/\lambda + k - i_0} + (1-p)\frac{\lambda(k-R)}{k}\right] \\
=& \underset{\lambda}{E}\left[p\cdot\frac{\lambda(k-R)}{k\lambda + i_0(1-\lambda)} + (1-p)\frac{\lambda(k-R)}{k}\right] \\
=& p\cdot\frac{k-R}{(k-i_0)^2}\cdot\left(b(k-i_0) - i_0\log\left(1+\frac{b(k-i_0)}{i_0}\right)\right) + (1-p)\frac{b^2(k-R)}{2k}
\end{aligned}$$

$$\begin{aligned}
RoR_B =& \underset{\lambda,t}{E}\left[\frac{(i_0(t)\lambda - i_0(t))(1-v)}{i_0(t)/\lambda + k - i_0(t)}\right] \\
=& \underset{\lambda}{E}\left[p\cdot\frac{(i_0/\lambda - i_0)(1-v)}{i_0/\lambda + k - i_0} + (1-p)\frac{(k/\lambda - k)(1-v)}{k/\lambda}\right] \\
=& \underset{\lambda}{E}\left[p\cdot\frac{i_0(1-\lambda)(1-v)}{k\lambda + i_0(1-\lambda)} + (1-p)(1-\lambda)(1-v)\right] \\
=& p\cdot\frac{i_0(1-v)}{(k-i_0)^2}\left(k\log\left(1+\frac{b(k-i_0)}{i_0}\right) - b(k-i_0)\right) + (1-p)(b-\frac{b^2}{2})(1-v)
\end{aligned}$$

**Theorem 3.2.** *When is $RoR_A > RoR_B$ at $i_0 = 0$?*

At small values of $b$ the above evaluate to:

$$\begin{aligned}
RoR_A =& p\cdot(k-R)\cdot\frac{b^2}{2i_0} + (1-p)\frac{b^2(k-R)}{2k} \\
=& \frac{b^2(k-R)}{2}\left(\frac{p}{i_0} + \frac{1-p}{k}\right)
\end{aligned}$$

$$
\begin{aligned}
RoR_B =& p \cdot \frac{i_0(1-v)}{(k-i_0)^2}\left(bk\frac{(k-i_0)^2}{i_0} - k\frac{b^2(k-i_0)^2}{2i_0^2}\right) + (1-p)(b-\frac{b^2}{2})(1-v) \\
=& p \cdot (1-v)\left(bk - k\frac{b^2}{2i_0}\right) + (1-p)(b-\frac{b^2}{2})(1-v) \\
=& b(1-v) \cdot \left(pk(1-\frac{b}{2i_0}) + (1-p)(1-\frac{b}{2})\right)
\end{aligned}
$$

Talk about the framework in depth. That this can be used to find optimal $R$ and $k$. Also different distributions can be tested once the framework is ready

### 3.2.1 Proportional Promotion Budgeting

First we look into the case when $A$ sets its reward value $R$ proportional to the product of the distance to the reward $k$ and the discount value $v$ provided by merchant $B$: i.e. $R = \alpha k v$. We refer to this case as proportional promotion budgeting. Note $\alpha$ is a constant here.

**Theorem 3.3.** *Under proportional promotion budgeting, the optimal reward distance that A should set is* $k = \frac{e}{\alpha(1-\beta)}$ *at all values of* $b$ *as long as* $\beta$ *is close to 1.*

I've specified that $\beta$ be close to 1. We do need this for our nice expression. We can also give something when $\beta$ is not close to 1, but I don't think that should be necessary?

*Proof.* Recall the previous expression for $RoR_A$. Maximizing this function is equivalent to maximizing the following:

$$
\begin{aligned}
\max_k\{RoR_A\} \Leftrightarrow & \max_k\left\{\underset{\lambda}{E}\left[\frac{\lambda k}{k-\Delta(1-\lambda)}\right]\right\} \\
\Leftrightarrow & \max_k\left\{\frac{1}{b}\int_0^b \frac{\lambda k}{k-\Delta(1-\lambda)}d\lambda\right\} \\
\Leftrightarrow & \max_k\left\{\frac{k}{\Delta^2 b}\left(\Delta b - (k-\Delta)\log\left(\frac{k-\Delta(1-b)}{k-\Delta}\right)\right)\right\} \\
\Leftrightarrow & \max_k\left\{\frac{k}{\Delta}\left(1 - \frac{k-\Delta}{b\Delta}\log\left(\frac{k-\Delta(1-b)}{k-\Delta}\right)\right)\right\}
\end{aligned}
$$

Now let $\theta = \frac{k}{\Delta}$. Then maximizing the above function is equivalent to maximizing the following function w.r.t. $\theta$. Note that $\theta \geq 1$ because $k \geq \Delta$.

$$
\max_k\{RoR_A\} \Leftrightarrow \max_\theta\{f(\theta)\} \Leftrightarrow \max_\theta\left\{\theta\left(1-\frac{\theta-1}{b}\log\left(1+\frac{b}{\theta-1}\right)\right)\right\} \tag{5}
$$

We will show that $f'(\theta) \leq 0$ for all $\theta$ so maximizing $f$ is equivalent to minimizing $\theta$.

$$
\begin{aligned}
f'(\theta) = \frac{2\theta - 1 + b}{\theta - 1 + b} - \frac{2\theta - 1}{b} \log\left(1 + \frac{b}{\theta - 1}\right) &\leq 0 \\
\iff \left(\frac{2\theta - 1 + b}{\theta - 1 + b}\right)\left(\frac{b}{2\theta - 1}\right) &\leq \log\left(1 + \frac{b}{\theta - 1}\right) \\
\iff \frac{b}{\theta - 1 + b} + \frac{b^2}{(2\theta - 1)(\theta - 1 + b)} &\leq \log\left(1 + \frac{b}{\theta - 1}\right)
\end{aligned}
$$

In the limit of $b = 0$, it is easy to see that $f'(\theta) = 0$ for all $\theta$. We now show that for all $\theta$ and all $0 < b \leq 1$, $\frac{\partial f'(\theta)}{\partial b} \leq 0$.

$$
\begin{aligned}
\frac{\partial f'(\theta)}{\partial b} = \frac{\theta - 1}{(\theta - 1 + b)^2} + \frac{1}{2\theta - 1} \cdot \frac{2b(\theta - 1 + b) - b^2}{(\theta - 1 + b)^2} - \frac{1}{\theta - 1 + b} &\leq 0 \\
\iff \theta - 1 + \frac{2b(\theta - 1 + b) - b^2}{2\theta - 1} &\leq \theta - 1 + b \\
\iff \frac{b(2\theta - 2 + b)}{2\theta - 1} &\leq b \\
\iff 2\theta - 2 + b &\leq 2\theta - 1 \\
\iff b &\leq 1
\end{aligned}
$$

Thus we have shown that for all $\theta$, $f'(\theta) = 0$ as $b0$ and that for all $\theta$ and $0 < b \leq 1$, $f'(\theta) \leq 0$. These together mean that for all $\theta$ and $b \in [0, 1]$, $f'(\theta) \leq 0$. So to maximize $f$, we need to minimize $\theta$.

Let's look at the quantity $\frac{k}{\Delta}$.

$$
\frac{k}{\Delta} = \frac{k}{\log_\beta\left(\frac{1}{\alpha k(1-\beta)}\right)} \sim \frac{k(1 - \beta)}{\log(\alpha k(1 - \beta))}
$$

The above approximation relies on $\beta$ close to 1. Now this value is minimized at $k = \frac{e}{\alpha(1-\beta)}$. Therefore, for all $b$, the optimal value for $k$ is given by $\frac{e}{\alpha(1-\beta)}$, the value for which $\frac{k}{\Delta}$ is minimized.

$\square$

Discussion about why this theorem makes sense - restriction on $\beta$ is realistic anyway and minimizing $\Delta$ makes sense as well. Also, should we put this proof in the appendix and just give outline in paper?

Note that under equal-budgeting, we need $k > \frac{1-\lambda}{1-\beta}$ for $V$ to be increasing. We meet this condition when $k = \frac{e}{\alpha(1-\beta)} \geq \frac{1}{1-\beta} \geq \frac{1-\lambda}{1-\beta}$ (used $\alpha < e$). Figure 1 shows the percentage of visits needed for a "forward-looking" consumer to adopt the reward program as a function of $\beta$. (Here we should add our comments on cash back computations).

Don't think we need this figure anymore - in fact doesn't really make much sense.

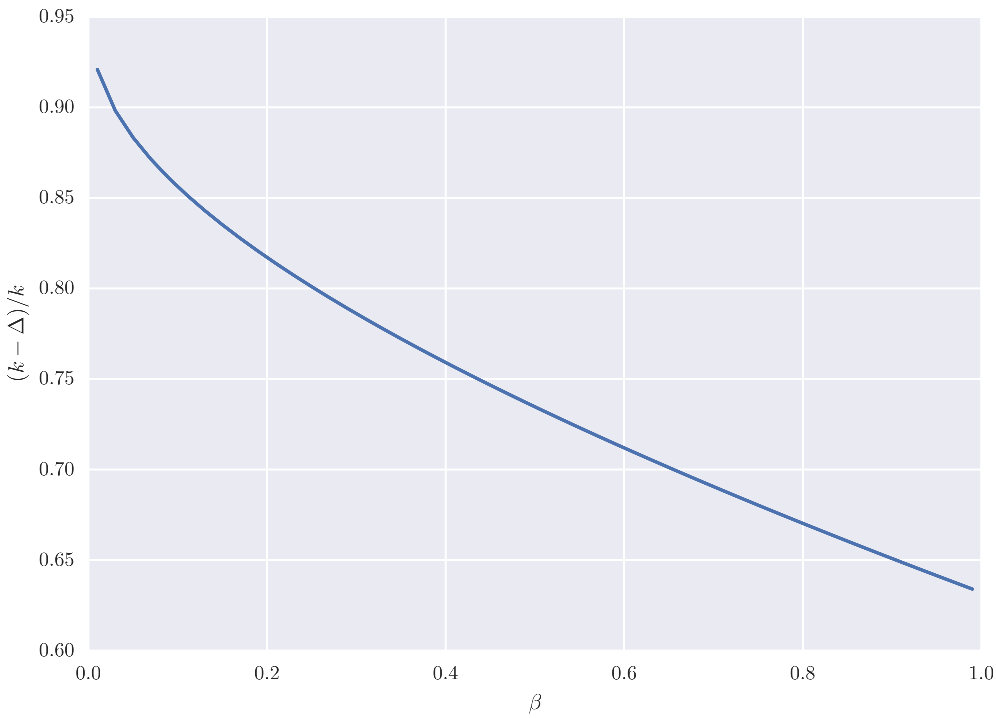


Figure 1: We assume $k = e/(1-\beta)$ and equal-budgeting. The plot shows the percentage of extraneous visits needed for a "forward-looking" consumer needed to adopt the reward program as a function of $\beta$.

Now we consider the expected revenues of each firm under these conditions.

$$\underset{\lambda,t}{E}[RoR_A] = pk(1-v)\frac{1}{b}\int_0^b \frac{\lambda}{k-(1-\lambda)\Delta}\, d\lambda + (1-p)(1-v)\frac{b}{2} \tag{6}$$

$$\underset{\lambda,t}{E}[RoR_B] = pk(1-v)\frac{1}{b}\int_0^b \frac{1-\lambda}{k-(1-\lambda)\Delta}\, d\lambda + (1-p)(1-v)\left(1-\frac{b}{2}\right) \tag{7}$$

First notice that the ratio of expected revenues is independent of $v$, the price difference of the two firms. We observe this behavior in our simulations as well. Figure 2 shows the revenue rates of $A$ and $B$ as a function of $b$ for different values of $v$. We see that the relative rates do not vary with $v$; changing $v$ only changes the absolute revenue rates of each firm, putting more money into the rewards given out. We see that $b$ must be pretty large for firm $A$ to make more money than firm $B$.

Not sure what best figures are now, we can discuss this.

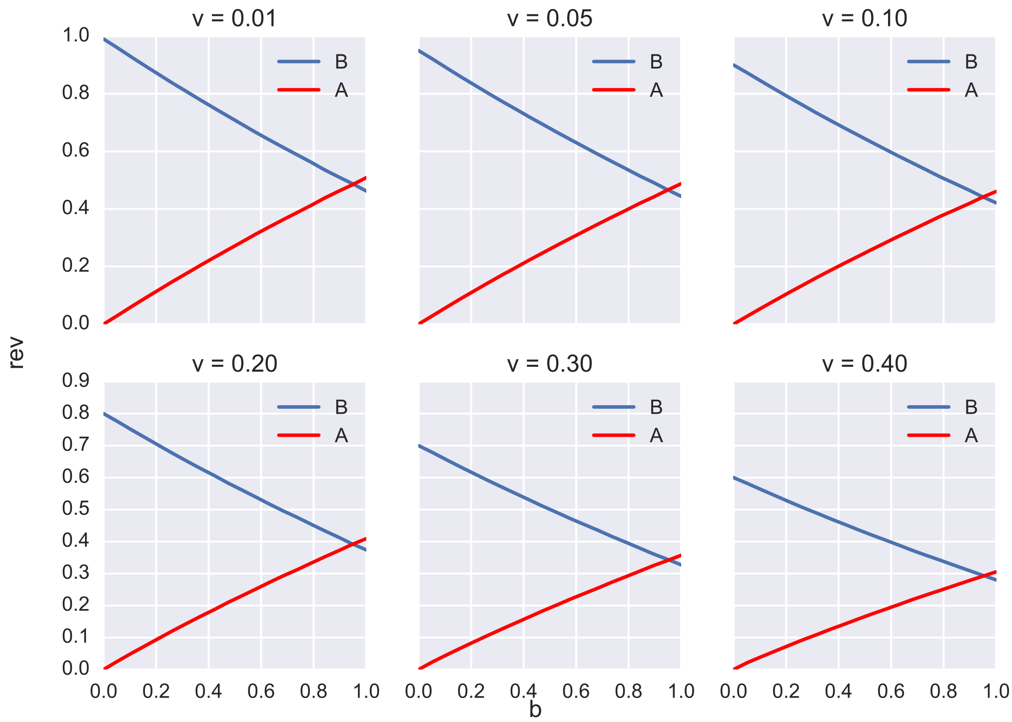


Figure 2: Rates of revenue for $A$ and $B$ with equal-budgeting as a function of $b$ for various $v$. Fixed $p = 0.5$, $\beta = 0.9$ and $k = e/(1-\beta)$.

[Not sure if we should include] We may solve for the $b$ such that the expected revenues are equal, which occurs if when the following holds (excluding work for now).

$$b - \frac{2k(k-\Delta)p}{(1-p)b\Delta^2} \log\left(\frac{k-(1-b)\Delta}{k-\Delta}\right) = 1 - \frac{p}{1-p}\frac{2k-\Delta}{\Delta}$$

Note from the above that $p$, the probability of a consumer being "forward-looking" does affect the ratio of expected rates of revenue. Figure 3 shows simulation results for the rates of revenue of $A$ and $B$ for various values of $p$ with everything else fixed. We see that as $p$ increases, the $b$ required for the reward program to become more profitable than not using one decreases.

#### 3.2.2 Non Strategic Merchant $B$, and Unequal Promotion Budgeting

Now we consider scenarios in which firms $A$ and $B$ have different budgets. First we consider the case where the budgets are still proportional, i.e. $R = \alpha \cdot kv$ for some fixed $\alpha$. Now the expected rate of revenue of $A$ is given by the following, while that of firm $B$

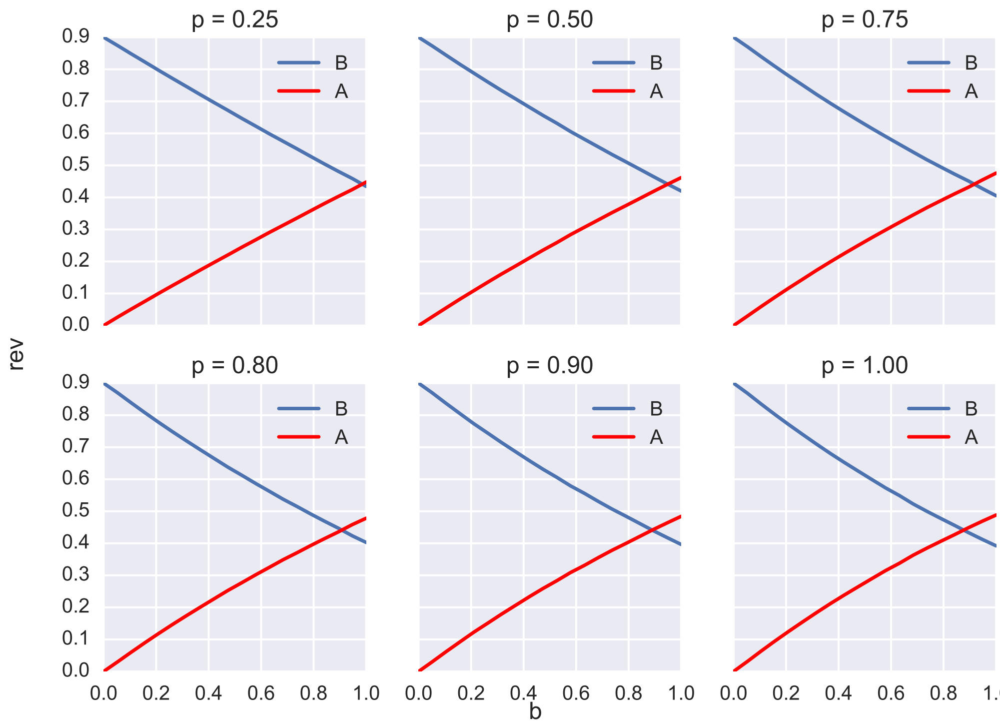


Figure 3: Rates of revenue for $A$ and $B$ with equal-budgeting as a function of $b$ for various $p$. Fixed $v = 0.1$, $\beta = 0.9$ and $k = e/(1-\beta)$.

is unchanged. Therefore the ratio of expected revenue rates may now depend on $v$.

$$\underset{\lambda,t}{E}[RoR_A] = pk(1-\alpha v)\frac{1}{b}\int_0^b \frac{\lambda}{k-(1-\lambda)\Delta}\, d\lambda + (1-p)(1-\alpha v)\frac{b}{2} \tag{8}$$

Following the same logic of Theorem 3.2, the expected revenue rate of $A$ is maximized at $k = \frac{e}{\alpha(1-\beta)}$ at small values of $b$ (need to do again for larger values). For this value of $k$, $\Delta$ is fixed for all $\alpha$ ($\Delta = \lfloor \log_\beta\left(\frac{1}{e}\right)\rfloor$), so we must have $\alpha \leq \frac{e}{\Delta(1-\beta)}$ for $k \geq \Delta$ to hold. When $\beta = 0.9$, this upperbound on $\alpha$ is about 3.

Should proof of this theorem be in the appendix?

**Theorem 3.4.** *Suppose firm A fixes its price at 1, and firm B chooses a price of* $1-v$. *Given a consumer distribution defined by* $p$ *- with probability* $p$*, a consumer is fully forward looking and probability* $1-p$ *the customer does not look ahead at all -* $b$ *- each consumer's monopoly factor to firm A is drawn as* $\lambda \sim Unif(0,b)$ *and* $\beta$ *- the customer's discout factor. Then firm A may choose to give a reward of* $\alpha v < 1$ *to customers after* $k$ *visits. It should run a reward program if the following condition holds.*

$$\frac{1}{b}\left(1-\frac{e-1}{b}\log\left(1+\frac{b}{e-1}\right)\right) \geq \frac{1-(1-p)(1-\alpha v)}{2pe(1-\alpha v)} \tag{9}$$

*Define the function on the left-hand side above as $g(b)$.*

*Proof.* Firm $A$ always sells the good for price 1. If it chooses to run a reward program its expected rate of revenue is given by:

$$\underset{\lambda,t}{E}[RoR_A] = pk(1-\alpha v)\frac{1}{b}\int_0^b \frac{\lambda}{k-(1-\lambda)\Delta}\, d\lambda + (1-p)(1-\alpha v)\frac{b}{2}$$

If it does not run a reward program, then the only visits it will receive are exogenous visits. In this case, its expected rate of revenue is simply $\frac{b}{2}$. We consider a reward program to be profitable if its expected rate of revenue is at least that of the non-reward program expected revenue rate.

$$\begin{aligned} pk(1-\alpha v)\frac{1}{b}\int_0^b \frac{\lambda}{k-(1-\lambda)\Delta}\, d\lambda + (1-p)(1-\alpha v)\frac{b}{2} &\geq \frac{b}{2} \\ \iff \frac{pk(1-\alpha v)}{\Delta}\left(1-\frac{k-\Delta}{b\Delta}\log\left(\frac{k-(1-b)\Delta}{k-\Delta}\right)\right) &\geq \frac{b}{2}(1-(1-p)(1-\alpha v)) \\ \iff pe(1-\alpha v)\left(1-\frac{e-1}{b}\log\left(1+\frac{b}{e-1}\right)\right) &\geq \frac{b}{2}(1-(1-p)(1-\alpha v)) \\ \iff \frac{1}{b}\left(1-\frac{e-1}{b}\log\left(1+\frac{b}{e-1}\right)\right) &\geq \frac{1-(1-p)(1-\alpha v)}{2pe(1-\alpha v)} \end{aligned}$$

Where we have used the work from Theorem 3.2 as well as the fact that the optimal $k$ is given by $\frac{e}{\alpha(1-\beta)}$, making $\Delta \approx \frac{1}{1-\beta}$. $\square$

Note that the above condition on $b$ is rather complicated, so we have plotted it as a function of $b$ below. First we notice that $g(b)$ is decreasing in $b$. So for a fixed evaluation of $x \equiv \frac{1-(1-p)(1-\alpha v)}{2pe(1-\alpha v)}$, we are in one of the following cases:

1. $x \geq g(0)$. So no value of $b$ makes the reward program profitable.

2. $x \leq g(1)$. So any value of $b$ makes the reward program profitable.

3. $x = g(b_0)$ for some $b_0 \in (0,1)$. So the reward program is profitable for all $b \leq b_0$ and not otherwise.

Now we can take a look at the right hand side of the profitability condition. Let $h(p,\alpha,v) = \frac{1-(1-p)(1-\alpha v)}{2pe(1-\alpha v)}$. It is easy to see that for all values of $p$, $\alpha$ and $v$, $\frac{\partial h}{\partial p} < 0$, $\frac{\partial h}{\partial v} > 0$ and $\frac{\partial h}{\partial \alpha} > 0$. These partial derivative signs mean that as $p$ increases (fixing $v$ and $\alpha$), the interval of profitable $b$'s can only increase. This result make sense intuitively - as the $p$ increases, the number of consumers looking ahead does as well, so more people adopt the

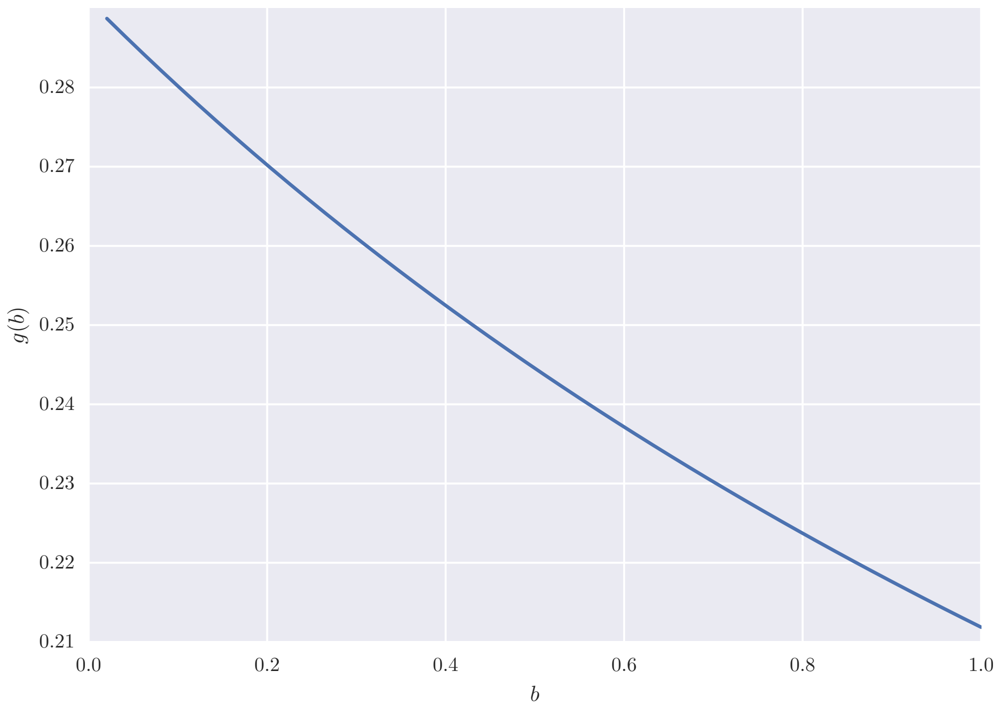


Figure 4: Function governing profitability of reward program for firm $A$ as a function of $b$.

reward program. However, increasing either $\alpha$ or $v$ (keeping others fixed), the interval of profitable $b$'s can only decrease. Thus, increasing the reward while keeping $p$ fixed means that in order for the reward program to remain profitable, the profits earned without the reward program must simultaneously decrease, which occurs with decreasing $b$.

# 4 Conclusions