Optimal Design of Loyalty Reward Program in a Competitive Duopoly

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1 Introduction

Loyalty programs constitute a huge market and are a major source of revenue for many low margin businesses. Over 48 billion dollars of perceived rewards are issued in the United States alone every year, with every household having over 19 loyalty memberships. These include credit cards, hotel and airline reward programs, and more recently even restaurants, grocery and retail stores. Though forming a big component of the market, there is very less scientific understanding about the design of loyalty reward programs. We aim to address this gap with this research.

One popular form of loyalty reward programs are *frequency reward programs*, where customers earn *points* as currency over spendings with merchants and are able to redeem these points into dollar based rewards after achieving certain threshold point collections. There is extant literature on characterizing customer behavior toward frequency reward programs. Most of the literature is empirical in nature, and relies on psychological behavioral patterns among customers, as opposed to rational economic decision making. In this paper, we consider a competitive duopoly of two merchants where one merchant offers a loyalty reward program and the other offers traditional pricing with discounts and characterize a novel model of customer choice where customers measure their utilities in rational economic terms. In addition, we characterize the optimal reward design choice for the merchant offering the frequency reward program, based on different customer populations: specifically, how should the merchant decide the optimal thresholds and dollar value of rewards to optimize for its revenue share from the participating customer population. One important constraint we impose is that the merchant has to choose a *one design fits all* reward program for the entire participating customer population and is not allowed to personalize the program for different customer segments, while maximizing its overall long term revenue objective.

This is how the remaining of the paper is structured. First we will describe some past work. Then we will go over our contributions and explain how our work builds on top of past literature. In Section 2 we will describe our model followed by the main results in Section 3. We will follow up with a short discussion and future work in Section 4.

1.1 Past Literature

1.2 Our Contributions

We make both modeling and analytical contributions along the following major directions:

1.2.1 Competitive Duopoly Setup

We model a competitive duopoly of two merchants, one of them offering a frequency reward program and the other offering traditional pricing. Both merchants sell an identical good at fixed precommitted prices. The reward program merchant sells the good at a higher price. Customers are drawn from a known population distribution and they measure their utilities in rational economic terms, i.e., they make their purchase decisions to maximize long term discounted rewards. The discount factor is the time value of money, and is constant for all customers. Every customer makes a purchase everyday from either of the two merchants. With each purchase from the reward program merchant, customer gains some fixed number of points, and on achieving the reward redemption threshold, (s)he immediately gains the reward value as a dollar cashback.

1.2.2 Customer Parameters

Each customer has two important parameters drawn from the known population distribution: first a visit probability bias with which (s)he purchases the good from the reward program merchant for reasons exogenous to utility maximization, and second a look-ahead factor that controls how far into the future the customer can perceive the rewards. The visit probability bias toward the reward program merchant can be attributed to *excess* *loyalty* which has been argued as a important parameter for the success of any reward program, or it can be attributed to price insensitivity of the customer: whenever the customer is price insensitive, (s)he strictly prefers to purchase from the reward program merchant as (s)he gains points redeemable for rewards in the future. There are many possible reasons for customers' price insensitivity: the reward program merchant could be offering some other monopoly products, or the customer might be getting reimbursed for some purchases as part of corporate perks (eg: corporate travel). As an effect, this visit probability bias controlls how frequently the customers' points increase even when (s)he does not actively choose to make purchases from the reward program merchant. The look-ahead parameter affects the customer behavior dynamics as follows: if the reward is farther than the customer's look-ahead parameter, (s)he is unable to register the future value of that reward and take it into consideration while maximizing long term utility. Both these parameters can be attributed to bounded rationality of customers and have been argued to be important factors toward customer choice dynamics (cite, etc).

1.2.3 Customer Choice Dynamics

We formulate the customer choice dynamics as a dynamic program with the state being the number of points collected from the reward program merchant. When the customer does not make biased visits to the reward program merchant, (s)he chooses the merchant maximizing her long term utility: comparing the immediate utility by purchasing the good at cheaper price and the long term utility of waiting and receiving the time discounted reward. The solution to the customer's dynamic program gives conditions for the existence and achievability of a phase transition: a points threshold before which the customer visits the merchant offering rewards only due to the visit probability bias, and after which (s)he always visits the merchant offering rewards till receiving the reward. We show that this phase transition point has the following dependencies:

1. It decreases with increase in the look-ahead parameter, i.e., if the customer can perceive rewards longer into the future, the phase transition for the customer occurs sooner.

2. It decreases with increase in the reward value, and decreases with increase in points threshold required to redeem the reward.

3. It increases with the discount value that the traditional pricing merchant provides, i.e., the cheaper the product from the second merchant, the farther is the phase transition point.

4. (effect w.r.t. to discount factor β. leaving it for now).

(Discussion relating to past literature on psychological constructs: tipping point literature. purchase acceleration literature)

1.2.4 Reward Program Design

After characterizing the customer behavior dynamics in our model, we optimize over the long run revenues that the reward program merchant achieves. We first show some conditions over the customer population that easily gives better revenues to the reward program merchant over the traditional pricing merchant (can we explain the graph of (b, p) pairs in one or two lines here for this particular result. We provide a framework for optimizing the parameters for the reward program merchant to maximize its revenue objective. We then look at a specific simplified case of proportional budgeting: the reward offered by the reward program merchant if proportional to the product of the distance to the reward and the discount provided by the traditional pricing merchant. We show that Cashback result simplified to two lines here. We characterize conditions for when it's better for the reward program merchant to offer a reward vs not offering any reward (graph of relevant (b,p) pairs). Finally we compare the strategy to offer reward vs offering traditional pricing with discounts itself for the reward program merchant (here the expiration can come into play) and show....

Then, if we have anything, we can talk about multi-tiering strategies.

2 Model

We index the two competing merchants selling identical goods as A and B. Without loss of generality we assume that A sells the good for a price of 1 dollars while B sells it for $1 - v$ dollars, i.e., B offers a discount of v dollars. A on the other hand offers a reward of value R dollars to a customer after (s)he makes k purchases at A. We investigate only the case we refer to as "proportional promotion budgeting" wherein this reward R is proportional to the product of the distance to the reward k and the discount v provided by B. That is $R = \alpha k v$ where α is assumed to be a constant.

We assume customers purchase the item from either A or B everyday, i.e., we ignore the heterogeneity in frequency of purchases among the customers in our model and leave it for future work. We assume customers have a linear homogenous utility in price: at price p the utility is $\nu(p) = 1 - p$. This reduces to customers getting an immediate utility of 0 from A and v from B. Customers have the same time value of money as a discount factor of β lying between 0 and 1 for all customers.

We denote a customer's visit probability bias and the look-ahead parameter with λ and t respectively. That is with probability λ (s)he purchases from A due to externalities and perceives a future reward only if it is within t purchases away. We assume λ for a customer to be drawn from a uniform distribution between $[0, b]$, where b is between 0 and 1. And we focus on a simple threshold distribution for the look-ahead parameter t:

$$t = \begin{cases} t_1, & \text{wp } p, \\ 0, & \text{wp } 1 - p. \end{cases}$$

The above distribution intuitively means that the customers are either myopic and focus only on immediate rewards, or are far-sighted enough (we assume t_1 is large). We model the customer's decision problem as a dynamic problem. We index the number of purchases the customer makes from A until reward by i, for $0 \leq i \leq k-1$, and we refer a customer to be in state i after having made i purchases from A. At state i, the customer has two possibilities:

1. With probability λ, the customer must visit A, and she is now in state $i+1$.

2. With probability $1-\lambda$, the customer may purchase from B for an immediate utility v and remain in state i or purchase from A for no immediate utility but move to state $i+1$.

Let $V(i)$ denote the long term expected reward at state i. Then we model the decision problem as the following dynamic program.

$$V(i) = \lambda\beta V(i+1) + (1-\lambda)\max\{v + \beta V(i), \beta V(i+1)\} \text{ for } 0 \leq i \leq k-1$$

$$V(k) = R$$

We show that the decision process exhibits a phase transition; that is prior to some state, the customer purchases from A only if (s)he must do so exogenously but after that state, (s)he always decides to purchase from A. This phase transition point is independent of λ, and depends only on t, among the variable customer parameters. Hence we represent this phase transition point as $i_0(t)$.

2.1 Merchant Objective

Given the above model of customer dynamics, we define the revenue objectives of A and B, where A chooses its parameters whereas B is non strategic. We define the rate of revenue for a merchant from a customer as the expected time averaged revenue that the merchant receives within the customer's lifetime. For simplification we assume merchants do not discount future revenues. As described above, a customer's dynamics are cyclic after each reward cycle. Thus the lifetime dynamics of customer behavior is a regenerative process with independent and identically distributed reward cycle lengths. Let $RoR_A(c)$ and $RoR_B(c)$ denote the expected rate of revenues for A and B respectively from a customer c's lifetime. Let $\tau(t,\lambda)$ denote the total number of purchases the customer makes before reaching the phase transition point $i_0(t)$. Then the length of the reward cycle (or total number of purchases the customer makes before receiving the reward) is $\tau(t,\lambda) + k - i_0(t)$, as after the phase transition (s)he makes all purchases from A only until hitting the reward. In this cycle the number of visits that the customer makes to A are k, and to B are $\tau(t,\lambda) - i_0(t)$. The revenue that A earns in one such cycle is $k - R$ and the revenue that B earns is $(1-v)(\tau(t,\lambda) - i_0(t))$. Thus the rate of revenues for A and B from the customer c are as follows:

$$RoR_A(c) = \underset{\tau}{E}\left[\frac{k-R}{\tau(t,\lambda)+k-i_0(t)}\right]$$

$$RoR_B(c) = \underset{\tau}{E}\left[\frac{(\tau(t,\lambda)-i_0(t))(1-v)}{\tau(t,\lambda)+k-i_0(t)}\right]$$

Since the process for a single customer is regenerative, using the reward renewal theorem (CITE), we can take the expectation over the cycle length inside the numerator and denominator respectively. Note that $\underset{\tau}{E}[\tau(t,\lambda)] = \frac{i_0(t)}{\lambda}$ as before reaching the phase transition point, with probability λ, the number of purchases by the customer from A increases by 1 and with probability $1-\lambda$ it stays constant. And then taking the expectation over the customer population the overall rate of revenues for both A and B are as follows:

$$RoR_A = \underset{\lambda,t}{E}\left[\frac{k-R}{i_0(t)/\lambda + k - i_0(t)}\right] \tag{1}$$

$$RoR_B = \underset{\lambda,t}{E}\left[\frac{(i_0(t)\lambda - i_0(t))(1-v)}{i_0(t)/\lambda + k - i_0(t)}\right] \tag{2}$$

3 Results

3.1 Customer Choice Dynamics

We first show that every customer exhibits the following behavior: until (s)he reaches the phase transition point $i_0(t)$, she purchases from A only due to the exogeneity paramater, and after that (s)he always purchases from A till she receives the reward. This behavior is cyclic, and repeats after every reward redemption.

Lemma 3.1. *$V(i)$ is an increasing function in i if the following condition holds:*

$$R > \frac{(1-\lambda)v}{1-\beta} \tag{3}$$

And further, $V(i)$ can be evaluated as:

$$V(i) = \max\left\{\frac{\lambda\beta V(i+1) + (1-\lambda)v}{1-(1-\lambda)\beta}, \beta V(i+1)\right\} \tag{4}$$

Proof. First we show that $V(i)$ is an increasing function in i by induction. We first show that if the condition above is satisfied, $V(k-1) < V(k) = R$. Suppose not, so $V(i) \geq R$. Then we have:

$$\begin{aligned}
V(k-1) &= \lambda\beta V(k) + (1-\lambda)(v + \beta V(k-1)) \\
&= \frac{\lambda\beta R + (1-\lambda)v}{1-(1-\lambda)\beta} \\
&< \frac{\lambda\beta R + (1-\beta)R}{1-(1-\lambda)\beta} \\
&= \frac{R(1-(1-\lambda)\beta)}{1-(1-\lambda)\beta} = R
\end{aligned}$$

But this is a contradiction, so $V(k-1) < V(k)$. Now assume $V(i+1) < V(i+2)$ for some $i < k-2$, we will show that this implies $V(i) < V(i+1)$. Suppose not, so $V(i) \geq V(i+1)$.

As we did before we may upper bound $V(i)$.

$$\begin{aligned} V(i) &= \lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)) \\ &\leq (1-\lambda)v + \beta V(i) \\ \iff V(i) &\leq \frac{(1-\lambda)v}{1-\beta} \end{aligned}$$

But because $V(i+1) < V(i+2)$, we may lower bound $V(i+1)$.

$$\begin{aligned} V(i+1) &\geq \lambda\beta V(i+2) + (1-\lambda)(v + \beta V(i+1)) \\ &= (1-\lambda)v + (1-\lambda)\beta V(i+1) + \lambda\beta V(i+2) \\ &> (1-\lambda) + \beta V(i+1) \\ \iff V(i+1) &> \frac{(1-\lambda)v}{1-\beta} \end{aligned}$$

Again, we have a contradiction, so $V(i) < V(i+1)$, and $V(i)$ is an increasing function in i. Now we prove the second claim. We have the following:

$$\begin{aligned} V(i) &= \lambda\beta V(i+1) + (1-\lambda)\max\{v + \beta V(i), \beta V(i+1)\} \\ &= \max\{\lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)), \beta V(i+1)\} \end{aligned}$$

Assuming $V(i)$ is the left term in the above maximum, we may solve the equation for that term.

$$\begin{aligned} V(i) &= \lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)) \\ (1-(1-\lambda)\beta)V(i) &= \lambda\beta V(i+1) + (1-\lambda)v \\ V(i) &= \frac{\lambda\beta V(i+1) + (1-\lambda)v}{1-(1-\lambda)\beta} \end{aligned}$$

And we get our claim. □

Now if the expected reward of the customer increases with the number of purchases made from A, we expect that at some number of purchases it becomes profitable for the customer to choose to purchase from A as opposed to B. We characterize this phase transition point in the following theorem.

Theorem 3.1. *Suppose $V(i)$ is an increasing function in i and consider a customer with look-ahead parameter t. A phase transition occurs after (s)he makes $i_0(t)$ visits to firm*

A, where $i_0(t)$ is given by:

$$i_0(t) = \begin{cases} k - \Delta \equiv i_0, & \textit{if } t \geq \Delta. \\ k - t, & \textit{otherwise.} \end{cases} \tag{5}$$

with

$$\Delta = \left\lfloor \log_\beta \left(\frac{v}{R(1-\beta)} \right) \right\rfloor \tag{6}$$

Proof. Still need to fix this proof a little bit I think. I will try to work on that.

First we solve for the condition on $V(i+1)$ for us to choose firm A over B willingly.

$$\begin{aligned}
\beta V(i+1) &> \frac{\lambda\beta V(i+1) + (1-\lambda)v}{1-(1-\lambda)\beta} \\
\iff \beta V(i+1)\left(1 - \frac{\lambda}{1-(1-\lambda)\beta}\right) &> \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right)v \\
\iff \beta V(i+1)\left(\frac{1-(1-\lambda)\beta-\lambda}{1-(1-\lambda)\beta}\right) &> \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right)v \\
\iff \beta V(i+1)\left(\frac{(1-\lambda)(1-\beta)}{1-(1-\lambda)\beta}\right) &> \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right)v \\
\iff \beta V(i+1) &> \frac{v}{1-\beta} \\
\iff V(i+1) &> \frac{v}{\beta(1-\beta)}
\end{aligned}$$

Let i_0 be the minimum state i such that the above holds, so in particular $V(i_0) \leq \frac{v}{\beta(1-\beta)}$ but $V(i_0+1) > \frac{v}{\beta(1-\beta)}$. We know because V is increasing in i, this point is indeed a phase transition: $V(i) > \frac{v}{\beta(1-\beta)}$ for all $i > i_0$, so after this point, the customer always chooses firm A. We may compute $V(i_0)$ easily using this fact.

$$V(i_0) = \beta V(i_0+1) = \cdots = \beta^{k-i_0}V(k) = \beta^{k-i_0}R$$

Thus, we have the following:

$$\begin{aligned}
\beta^{k-i_0} \leq \frac{v}{R\beta(1-\beta)} &< \beta^{k-(i_0+1)} \\
\iff k - i_0 \geq \log_\beta\left(\frac{v}{R\beta(1-\beta)}\right) &> k-(i_0+1) \\
\iff i_0 \leq k - \log_\beta\left(\frac{v}{R(1-\beta)}\right) + 1 &< i_0 + 1 \\
\iff i_0 = k - \left\lfloor \log_\beta\left(\frac{v}{R(1-\beta)}\right)\right\rfloor &\equiv k - \Delta
\end{aligned}$$

The above dependence reduces to the following after incorporating the look-ahead distribution:

$$i_0(t) = \begin{cases} i_0, & \text{wp } p, \\ k, & \text{wp } 1-p. \end{cases}$$

$\square$

Note that the phase transition point is independent of λ, the customer's visit probability bias toward the merchant. As we would expect, it increases with the look-ahead parameter, and with the price discount offered by merchant B. And decreases with increase in the reward value R and decrease in the distance to reward k. The variation with the discount factor β is interesting: we can show that for any $\frac{R}{v} \geq 1$ there exists a $\beta \in [0,1]$ that minimizes the phase transition point i_0 for "forward-looking" customers. This means that customers who are more patient have longer time frame to transition and so do customers who are less patient than the optimal value. We refer to the ratio of number of visits required for a "forward-looking" customer to adopt a reward program and the total distance to the reward as the "influence zone". Intuitively this is the fraction of visits that the merchant wants to influence the customer by offering exogenous means of earning additional points like bonus miles in airlines, or accelerated earnings, as discussed in the introduction. Next we find the optimal k for minimizing this influence zone.

Remark 1. *Influence zone is minimized at* $k = \frac{e}{\alpha(1-\beta)}$ *under proportional promotion budgeting.*

Proof. As defined the influence zone is $\frac{i_0}{k} = \frac{k-\Delta}{k} = 1 - \frac{\Delta}{k}$. Thus minimizing the influence zone is equivalent to minimizing $\frac{k}{\Delta}$.

$$\frac{k}{\Delta} = \frac{k}{\log_\beta\left(\frac{1}{\alpha k(1-\beta)}\right)} \sim \frac{k(1-\beta)}{\log(\alpha k(1-\beta))}$$

The above approximation relies on β close to 1. Now this value is minimized at $k = \frac{e}{\alpha(1-\beta)}$. Therefore, for all b, the optimal value for k is given by $\frac{e}{\alpha(1-\beta)}$, the value for which $\frac{k}{\Delta}$ is minimized and takes the value $\frac{e}{\alpha}$. At this value the influence zone takes the value $1 - \frac{\alpha}{e}$. $\square$

Note that if α is 1, then the value of k corresponds to a cashback between 2% and 4% as β ranges between 0.95 and 0.9. This value is realistic to what is observed in practice.

3.2 Merchant Objective Dynamics

We substitute the value of the phase transition point obtained above in the rate of revenue equations to reevaluate them. And since we assume that λ and t are drawn independent of each other, we can separate the expectation terms and evaluate them sequentially, first over t, then over λ. This reduces the rate of revenues as follows:

$$
\begin{aligned}
RoR_A =& \underset{\lambda,t}{E}\left[\frac{k-R}{i_0(t)/\lambda + k - i_0(t)}\right] \\
=& \underset{\lambda}{E}\left[p \cdot \frac{k-R}{i_0/\lambda + k - i_0} + (1-p)\frac{\lambda(k-R)}{k}\right] \\
=& \underset{\lambda}{E}\left[p \cdot \frac{\lambda(k-R)}{k\lambda + i_0(1-\lambda)} + (1-p)\frac{\lambda(k-R)}{k}\right] \\
=& p \cdot \frac{k-R}{b(k-i_0)^2} \cdot \left(b(k-i_0) - i_0 \log\left(1 + \frac{b(k-i_0)}{i_0}\right)\right) + (1-p)\frac{b(k-R)}{2k} \\
=& p \cdot \frac{k-R}{b\Delta^2} \cdot \left(b\Delta - (k-\Delta)\log\left(1 + \frac{b\Delta}{k-\Delta}\right)\right) + (1-p)\frac{b(k-R)}{2k} \\
=& p \cdot \frac{k-R}{\Delta} \cdot \left(1 - \frac{k-\Delta}{b\Delta}\log\left(1 + \frac{b\Delta}{k-\Delta}\right)\right) + (1-p)\frac{b(k-R)}{2k}
\end{aligned}
$$

$$
\begin{aligned}
RoR_B =& \underset{\lambda,t}{E}\left[\frac{(i_0(t)\lambda - i_0(t))(1-v)}{i_0(t)/\lambda + k - i_0(t)}\right] \\
=& \underset{\lambda}{E}\left[p \cdot \frac{(i_0/\lambda - i_0)(1-v)}{i_0/\lambda + k - i_0} + (1-p)\frac{(k/\lambda - k)(1-v)}{k/\lambda}\right] \\
=& \underset{\lambda}{E}\left[p \cdot \frac{i_0(1-\lambda)(1-v)}{k\lambda + i_0(1-\lambda)} + (1-p)(1-\lambda)(1-v)\right] \\
=& p \cdot \frac{i_0(1-v)}{b(k-i_0)^2}\left(k\log\left(1 + \frac{b(k-i_0)}{i_0}\right) - b(k-i_0)\right) + (1-p)(1-\frac{b}{2})(1-v) \\
=& p \cdot \frac{(k-\Delta)(1-v)}{b\Delta^2}\left(k\log\left(1 + \frac{b\Delta}{k-\Delta}\right) - b\Delta\right) + (1-p)(1-\frac{b}{2})(1-v) \\
=& p \cdot \frac{(k-\Delta)(1-v)}{\Delta}\left(\frac{k}{b\Delta}\log\left(1 + \frac{b\Delta}{k-\Delta}\right) - 1\right) + (1-p)(1-\frac{b}{2})(1-v)
\end{aligned}
$$

The above framework can be used for optimizing for the reward parameters to maximize A's rate of revenue, and correspondingly for the discount parameter for B, for varying distributions of the customer population. We leave the competitive study where merchant B could strategize on its discount value v for future work. And in this work we only consider proportional promotion budgeting as mentioned in Section 2, where $R = \alpha k v$ and α is a constant.

Theorem 3.2. *Under proportional promotion budgeting, the optimal reward distance that A should set is $k = \frac{e}{\alpha(1-\beta)}$ at all values of b as long as β is close to 1.*

Proof. Recall the previous expression for RoR_A. Maximizing this function is equivalent to maximizing the following:

$$\max_k\{RoR_A\} \Leftrightarrow \max_k \left\{ \frac{k}{\Delta}\left(1 - \frac{k-\Delta}{b\Delta}\log\left(\frac{k-\Delta(1-b)}{k-\Delta}\right)\right)\right\}$$

Now let $\theta = \frac{k}{\Delta}$. Then maximizing the above function is equivalent to maximizing the following function w.r.t. θ. Note that $\theta \geq 1$ because $k \geq \Delta$.

$$\max_k\{RoR_A\} \Leftrightarrow \max_\theta\{f(\theta)\} \Leftrightarrow \max_\theta \left\{\theta\left(1 - \frac{\theta-1}{b}\log\left(1+\frac{b}{\theta-1}\right)\right)\right\}$$

We will show that $f'(\theta) \leq 0$ for all θ so maximizing f is equivalent to minimizing θ.

$$\begin{aligned} f'(\theta) &= \frac{2\theta-1+b}{\theta-1+b} - \frac{2\theta-1}{b}\log\left(1+\frac{b}{\theta-1}\right) \\ &= \frac{2\theta-1}{b}\cdot\left(\left(\frac{2\theta-1+b}{\theta-1+b}\right)\left(\frac{b}{2\theta-1}\right) - \log\left(1+\frac{b}{\theta-1}\right)\right) \\ &= \frac{2\theta-1}{b}\cdot\left(\frac{b}{\theta-1+b} + \frac{b^2}{(2\theta-1)(\theta-1+b)} - \log\left(1+\frac{b}{\theta-1}\right)\right) \end{aligned}$$

Let $g(b,\theta) = \frac{b}{\theta-1+b} + \frac{b^2}{(2\theta-1)(\theta-1+b)} - \log\left(1+\frac{b}{\theta-1}\right)$. In the limit of $b \to 0$, it is easy to see that $f'(\theta) = g(b,\theta) = 0$ for all θ in the domain. We now show that for all θ in the domain and all $0 < b \leq 1$, $\frac{\partial g(b,\theta)}{\partial b} \leq 0$.

$$\begin{aligned} \frac{\partial g(b,\theta)}{\partial b} &= \frac{\theta-1}{(\theta-1+b)^2} + \frac{1}{2\theta-1}\cdot\frac{2b(\theta-1+b)-b^2}{(\theta-1+b)^2} - \frac{1}{\theta-1+b} \leq 0 \\ &\iff \theta-1+\frac{2b(\theta-1+b)-b^2}{2\theta-1} \leq \theta-1+b \\ &\iff \frac{b(2\theta-2+b)}{2\theta-1} \leq b \\ &\iff 2\theta-2+b \leq 2\theta-1 \\ &\iff b \leq 1 \end{aligned}$$

Thus we have shown that for all θ, $g(b,\theta) = 0$ as $b \to 0$ and that for all θ and $0 < b \le 1$, $g(b,\theta)$ is decreasing. These together mean that for all θ and $b \in [0,1]$, $g(b,\theta) \le 0$. Which implies that $f'(\theta) \le 0$ for all θ in the domain. So to maximize f, we need to minimize θ.

Observe that minimizing $\theta = \frac{k}{\Delta}$ is equivalent to minimizing the influence zone. As shown in Remark 1, this happens at $k = \frac{e}{\alpha(1-\beta)}$. □

An interesting point to observe above is that maximizing the revenue objective is equivalent to minimizing the influence zone, i.e., under our model, maximizing revenue is equivalent to the time until which it is most beneficial to offer reward point accelerations. The condition that β be close to 1 is not very restrictive, as we expect it to be near 1 in most cases (maybe get a citation for this). Note that because $k \ge \Delta$, the above also shows $\alpha \le e$. Finally, observe that, we need $R > \frac{(1-\lambda)v}{1-\beta}$ for V to be increasing. We meet this condition with proportional budgeting when $k = \frac{e}{\alpha(1-\beta)}$ as $R = \alpha k v = \frac{ev}{1-\beta} \ge \frac{v}{1-\beta} \ge \frac{(1-\lambda)v}{1-\beta}$.

Now we evaluate the expected rate of revenues of each merchant under these conditions of proportional budgeting and optimal k.

$$
\begin{aligned}
RoR_A =& pk \cdot \frac{1-\alpha v}{\Delta} \cdot \left(1 - \frac{k-\Delta}{b\Delta} \log\left(1 + \frac{b\Delta}{k-\Delta}\right)\right) + (1-p)\frac{bk(1-\alpha v)}{2k} \\
=& (1-\alpha v)\left(p\frac{e}{\alpha}\left(1 - \frac{e-\alpha}{b\alpha}\log\left(1+\frac{b\alpha}{e-\alpha}\right)\right) + (1-p)\frac{b}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
RoR_B =& p \cdot \frac{(k-\Delta)(1-v)}{\Delta}\left(\frac{k}{b\Delta}\log\left(1+\frac{b\Delta}{k-\Delta}\right) - 1\right) + (1-p)(1-\frac{b}{2})(1-v) \\
=& (1-v)\left(p \cdot \frac{e-\alpha}{\alpha}\left(\frac{e}{b\alpha}\log\left(1+\frac{b\alpha}{e-\alpha}\right) - 1\right)\right) \\
=& (1-v)\left(p\frac{e}{\alpha}\left(\frac{e-\alpha}{b\alpha}\log\left(1+\frac{b\alpha}{e-\alpha}\right) - \frac{e-\alpha}{e}\right) + (1-p)(1-\frac{b}{2})\right)
\end{aligned}
$$

Observe that both the above equations have a left term and a right term. The left term is the rate of revenue obtained from "forward-looking" customers whereas the right term is from the myopic customers. As α ranges between 0 and e, the value on the left term increases from 0 for RoR_A and decreases to 0 for RoR_B. That is by controlling the reward budget ratio, merchant A is able to gain the entire "forward-looking" customer base.

As $\alpha \to 0$, $RoR_A \to b/2$, i.e., the revenue earned is only due to visit probability bias. Observe how RoR_A varies with α. The marginal revenue term $(1 - \alpha v)$ decreases with α as the merchant is giving higher rewards to customers, whereas the market share term increases as the merchant gains more "forward-looking" customers with increase in reward budget. Also note that $\alpha = 0$ is equivalent to the merchant not running any reward program.

Figure 1 illustrates the region in terms of the customer parameters (b, p) where $RoR_A > RoR_B$ (indicated in blue) and $RoR_A > \frac{b}{2}$ (indicated in yellow) for different values of α, keeping $v = 0.05$ and $\beta = 0.95$ fixed.. That is, the region where A earns higher revenue rate as compared to B and the region where it is profitable for A to run a reward program as opposed to traditional pricing. The blue region shows that there is a clear threshold of b and p beyond which $RoR_A > RoR_B$. This is explained by the increasing nature of the revenue gap $(RoR_A - RoR_B)$ with b and p. But more interestingly, the threshold value of b and p decreases as α is increased toward e. Whereas the green region shows that the firm should choose to run a reward program most of the time except for when b is large: larger b values mean that customers make more exogenous visits, so a reward program is no longer needed to entice visits, but only causes decrease in the profitability of the reward program merchant.

Probably need to expand on above, will think about how to do that. Also, need to figure out when to add expiration comment.

Is this the best way to frame the next results? It seems like we are thinking of α fixed and then the firm must decide to either offer the reward program or not; but here, we are changing the choice from reward program vs not to our previously fixed α vs $\alpha = 0$. This is just very subtle semantics, just something to think about.

For any fixed α, the exact conditions on p, b and v for $RoR_A > RoR_B$ and $RoR_A > \frac{b}{2}$ are rather complex. We will focus on one particular simple case: $\alpha \to e$.

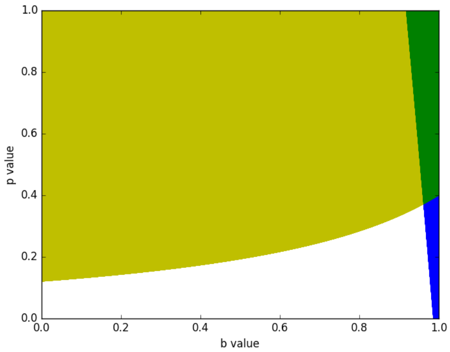
Theorem 3.3. *As $\alpha \to e$, $RoR_A > RoR_B$ if the following condition on b holds:*

$$b > 2 \cdot \frac{(1 - v) - \frac{p}{1-p} \cdot (1 - ev)}{(1 - v) + (1 - ev)} \tag{7}$$

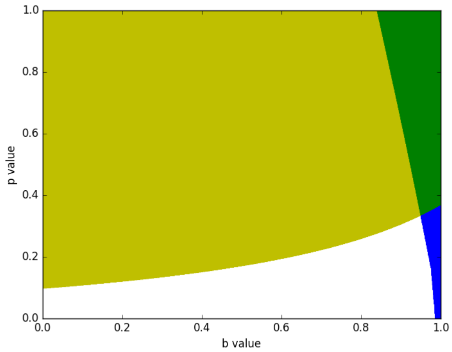
Proof. □

Theorem 3.4. *As $\alpha \to e$, $RoR_A > \frac{b}{2}$ if the following condition on b holds:*

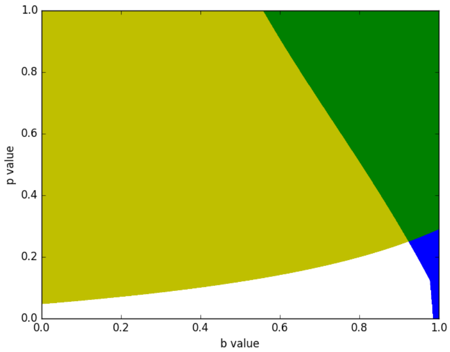
$$b < \frac{2p}{p + \frac{e \cdot v}{1 - e \cdot v}} \tag{8}$$



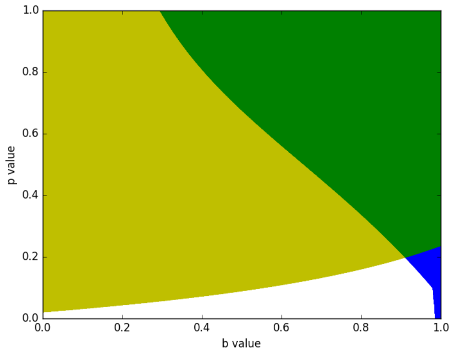
(a) $\alpha = 0.5$



(b) $\alpha = 1$



(c) $\alpha = 2$



(d) $\alpha = 2.5$

Figure 1: Regions where $RoR_A > RoR_B$ (blue), where $RoR_A > \frac{b}{2}$ (yellow) and where both are true (green) for different values of α. In all cases, $\beta = 0.95$ and $v = 0.05$.

Proof. $\square$

The above theorem gives a nice upperbound on b for the optimal reward program being more profitable than no reward program. Notice that when $\frac{e \cdot v}{1 - e \cdot v} \leq p$, this upperbound is 1, and there is no restriction on b. This case happens for large p values and/or small v values. Figure 1 shows the upperbound on b for all values of p and v ($v < e$); for each (p, v) pair, a firm should run a reward program for all values of b below the upperbound shown in the figure. We see as p increases (more people adopt reward programs), the region of allowable b values increases.

The previous $\alpha \to e$ discussion is a special case of a more general result.

Theorem 3.5. *Fix $\alpha \in (0, e)$. For any (p, v) pair, there exists some upper bound $b_0 \in [0, 1]$ such that for all $b \leq b_0$, $RoR_A \geq \frac{b}{2}$.*

We delay the proof of this theorem to first prove a helpful lemma. It is a straightfor-

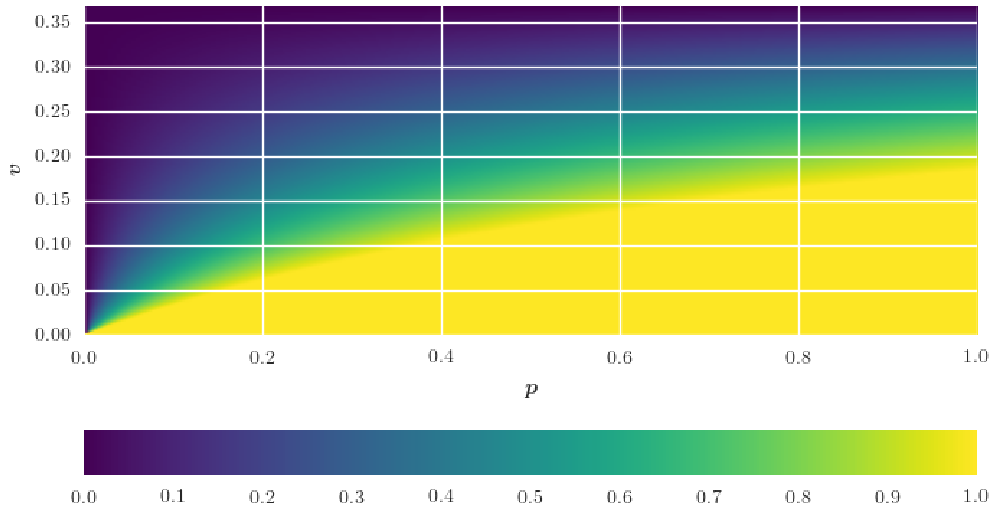


Figure 2: Upper bound on b for reward program to be more profitable than no reward program for different values of p and v with $\alpha \to e$.

ward computation to see that the condition of $RoR_A \geq \frac{b}{2}$ is equivalent to:

$$\frac{1}{b}\left(1 - \frac{e-\alpha}{b\alpha}\log\left(1 + \frac{b\alpha}{e-\alpha}\right)\right) \geq \frac{\alpha(1-(1-p)(1-\alpha v))}{2pe(1-\alpha v)}$$
$$\iff g(b;\alpha) \geq h(p,v;\alpha)$$

where we have defined functions $g(b)$ and $h(p,v)$ for fixed α for the above inequalities.

Lemma 3.2. *For a fixed α, $g(b)$ is decreasing for all $b \in [0,1]$.*

Proof. We take the derivative of g:

$$g'(b) = \frac{2(e-\alpha)}{b^3\alpha}\log\left(1+\frac{b\alpha}{e-\alpha}\right) - \frac{1}{b^2} - \frac{1}{b^2\left(1+\frac{b\alpha}{e-\alpha}\right)} \leq 0$$
$$\iff \frac{2(e-\alpha)}{b\alpha}\log\left(1+\frac{b\alpha}{e-\alpha}\right) \leq 1 + \frac{1}{1+\frac{b\alpha}{e-\alpha}}$$
$$\iff \frac{2\log(1+x)}{x} \leq 1 + \frac{1}{1+x}$$

where $x = \frac{b\alpha}{e-\alpha}$, and as $b \in [0,1]$, $x \in [0, \frac{\alpha}{e-\alpha}$. We can see that as $x \to 0$, the above inequality is an equality. Need to show above, it is true by plotting. I was trying to show like we had done before; we want to show $L(x) \leq R(x)$ for all $x \geq 0$, we know they are equal at $x = 0$, so show that $L'(x) \leq R'(x)$ for all x (both are decreasing functions though). $\square$

Thus, $g(b)$ is decreasing in b, so for any (p, v) pair, we may compute $h(p, v; \alpha)$, which will then fall into one of the following three cases.

1. $h(p, v; \alpha) \geq g(0)$. So no value of b makes the reward program profitable.

2. $h(p, v; \alpha) \leq g(1)$. So any value of b makes the reward program profitable.

3. $h(p, v; \alpha) = g(b_0)$ for some $b_0 \in (0, 1)$. So the reward program is profitable for all $b \leq b_0$ and not otherwise.

Thus, the above lemma and discussion proves our theorem; for fixed α and any (p, v) pair, there is some upperbound on b s.t. $RoR_A \leq \frac{b}{2}$.

I can also look at derivatives of h with respect to p and v - had this before, somewhat interesting. I also need to be careful about notation, I will think about the best way to use g and h

4 Conclusions

References