Optimal Design of Loyalty Reward Program in a Competitive Duopoly

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1 Introduction

2 Model

Consider a competitive duopoly of two stores, A and B, selling the same item. Without loss of generality we assume that store A sells the item for a price of 1 dollars while store B sells it for $1 - v$ dollars, i.e., B offers a discount of v dollars. Store A on the other hand offers a reward of value R dollars to a customer after (s)he makes k purchases at A. Our goal is to understand the dynamics of competition between A and B with respect to setting of the reward and discount parameters to maximize their respective revenues over a distribution of customers under certain assumptions on their behavior.

2.1 Customer Choice Model

First we assume that every customer purchases an item from either A or B everyday. We assume that there is some exogenous probability, λ, during each purchase that forces the customer to go to store A. This λ is a customer specific parameter and is drawn from a uniform distribution between $[0, b]$, where b is between 0 and 1. Let $0 < \beta \leq 1$ denote the discounting factor of future money. We assume customers to have a linear homogenous utility in price: at price p the utility is $\nu(p) = 1 - p$. This reduces to customers getting an immediate utility of 0 from A and v from B.

We model the customer's decision problem as a dynamic problem. We index the number of visits the customer makes at store A by i, for $0 \leq i \leq k - 1$, and we refer a customer to be in state i after having made i visits to A. At state i, the customer has two possibilities:

1. With probability λ, the customer must visit A, and she is now in state $i+1$.
2. With probability $1-\lambda$, the customer may purchase from B for an immediate utility v and remain in state i or purchase from A for no utility but move to state $i+1$.

Let $V(i)$ denote the long term expected reward at state i. Then we may model the decision problem as the following dynamic program.

$$V(i) = \lambda\beta V(i+1) + (1-\lambda)\max\{v + \beta V(i), \beta V(i+1)\} \text{ for } 0 \le i \le k-1$$
$$V(k) = R$$

We will show that the decision process exhibits a phase transition; that is prior to some state i_0, the customer will only visit A if she must do so exogenously but after i_0, she always decides to go to A.

Finally, we assume the customer has a look-ahead factor t, which models how many purchases ahead the customer looks ahead when making her current decision. This value will affect the phase transition of the decision process. Consider a distribution T describing the look-ahead factor for consumers. We will focus on threshold distributions; for example, with probability p the look-ahead is t_1 and with probability $1-p$, the look-ahead is t_0. With the addition of this look-ahead factor, the phase transition point will now depend on it, and we will refer to it as $i_0(t)$.

2.2 Merchant Objective and Competition

Given the above model of customer dynamics, and that customer exogeneity and look ahead parameters are drawn from a known distribution, the two merchants try competing over the customer base to maximize their long run revenues. We define the rate of revenue for a merchant from a customer as the expected time averaged revenue that the merchant receives within the customer's lifetime. For simplification we assume merchants do not discount future revenues. As described above, a customer's dynamics are cyclic after each reward cycle. Thus the lifetime dynamics of customer behvior is a regenerative process with independent and identically distributed reward cycle lengths. Let $RoR_A(c)$ and $RoR_B(c)$ denote the expected rate of revenues for merchants A and B respectively from a customer c's lifetime. Let $\tau(t,\lambda)$ denote the total number of visits the customer makes before reaching the phase transition point $i_0(t)$. Then the length of the reward cycle (or total number of visits the customer makes before receiving the reward) is $\tau(t,\lambda)+k-i_0(t)$, as after the phase transition (s)he makes all visits to merchant A only until hitting the reward. In this cycle the number of visits that the customer makes to A are k, and to B are $\tau(t,\lambda)-i_0(t)$. Thus the rate of revenues are as follows:

$$RoR_A(c) = \underset{\tau}{E}\left[\frac{k-R}{\tau(t,\lambda)+k-i_0(t)}\right] \tag{1}$$

$$RoR_B(c) = \underset{\tau}{E}\left[\frac{(\tau(t,\lambda)-i_0(t))(1-v)}{\tau(t,\lambda)+k-i_0(t)}\right] \tag{2}$$

The goal of each merchant is to set their reward or pricing parameters so as to maximize the expected value of rate of revenue over the entire customer population. Before that, since the process for a single customer is regenerative, using the reward renewal theorem (CITE), we can take the expectation over the cycle length inside the numerator and denominator respectively. Note that $\underset{\tau}{E}[\tau(t,\lambda)] = \frac{i_0(t)}{\lambda}$ as before reaching the phase transition point, with probability λ, the customer's visits to A increases by 1 and with probability $1-\lambda$ it stays constant. Then the objectives of the two merchants are:

$$\max_{R,k}\{RoR_A\} = \max_{R,k}\left\{\underset{\lambda,t}{E}\left[\frac{k-R}{i_0(t)/\lambda+k-i_0(t)}\right]\right\} \tag{3}$$

$$\max_{v}\{RoR_B\} = \max_{v}\left\{\underset{\lambda,t}{E}\left[\frac{(\tau(t,\lambda)-i_0(t))(1-v)}{i_0(t)/\lambda+k-i_0(t)}\right]\right\} \tag{4}$$

3 Results

3.1 Customer Choice Dynamics

We first show that every customer exhibits the following behavior: until (s)he reaches the phase transition point $i_0(t)$, she visits A only due to the exogeneity paramaeter, and after that (s)he always visits merchant A till she receives the reward. This behavior is cyclic, and repeats after the first reward redemption.

Lemma 3.1. *$V(i)$ is an increasing function in i if the following condition holds:*

$$R > \frac{(1-\lambda)v}{1-\beta} \tag{5}$$

And further, $V(i)$ can be evaluated as:

$$V(i) = \max\left\{\frac{\lambda\beta V(i+1) + (1-\lambda)v}{1-(1-\lambda)\beta}, \beta V(i+1)\right\} \tag{6}$$

Proof. First we show that $V(i)$ is an increasing function in i by induction. We first show that if the condition above is satisfied, $V(k-1) < V(k) = R$. Suppose not, so $V(i) \geq R$. Then we have:

$$\begin{aligned}
V(k-1) &= \lambda\beta V(k) + (1-\lambda)(v + \beta V(k-1)) \\
&= \frac{\lambda\beta R + (1-\lambda)v}{1-(1-\lambda)\beta} \\
&< \frac{\lambda\beta R + (1-\beta)R}{1-(1-\lambda)\beta} \\
&= \frac{R(1-(1-\lambda)\beta)}{1-(1-\lambda)\beta} = R
\end{aligned}$$

But this is a contradiction, so $V(k-1) < V(k)$. Now assume $V(i+1) < V(i+2)$ for some $i < k-2$, we will show that this implies $V(i) < V(i+1)$. Suppose not, so $V(i) \geq V(i+1)$. As we did before we may upper bound $V(i)$.

$$\begin{aligned}
V(i) &= \lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)) \\
&\leq (1-\lambda)v + \beta V(i) \\
\iff V(i) &\leq \frac{(1-\lambda)v}{1-\beta}
\end{aligned}$$

But because $V(i+1) < V(i+2)$, we may lower bound $V(i+1)$.

$$\begin{aligned}
V(i+1) &\geq \lambda\beta V(i+2) + (1-\lambda)(v + \beta V(i+1)) \\
&= (1-\lambda)v + (1-\lambda)\beta V(i+1) + \lambda\beta V(i+2) \\
&> (1-\lambda) + \beta V(i+1) \\
\iff V(i+1) &> \frac{(1-\lambda)v}{1-\beta}
\end{aligned}$$

Again, we have a contradiction, so $V(i) < V(i+1)$, and $V(i)$ is an increasing function in i. Now we prove the second claim. We have the following:

$$\begin{aligned}
V(i) &= \lambda\beta V(i+1) + (1-\lambda)\max\{v + \beta V(i), \beta V(i+1)\} \\
&= \max\{\lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)), \beta V(i+1)\}
\end{aligned}$$

Assuming $V(i)$ is the left term in the above maximum, we may solve the equation for that term.

$$\begin{aligned} V(i) &= \lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)) \\ (1-(1-\lambda)\beta)V(i) &= \lambda\beta V(i+1) + (1-\lambda)v \\ V(i) &= \frac{\lambda\beta V(i+1) + (1-\lambda)v}{1-(1-\lambda)\beta} \end{aligned}$$

And we get our claim. □

Theorem 3.1. *Assuming $V(i)$ is an increasing function in i, a phase transition occurs after the consumer makes i_0 visits to firm A, which evaluates to:*

$$\begin{aligned} i_0 &= k - \left\lfloor \log_\beta \left(\frac{v}{R(1-\beta)} \right) \right\rfloor \\ &\equiv k - \Delta \end{aligned}$$

Proof. First we solve for the condition on $V(i+1)$ for us to choose firm A over B willingly.

$$\begin{aligned} & \beta V(i+1) > \frac{\lambda\beta V(i+1) + (1-\lambda)v}{1-(1-\lambda)\beta} \\ \iff & \beta V(i+1)\left(1 - \frac{\lambda}{1-(1-\lambda)\beta}\right) > \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right) v \\ \iff & \beta V(i+1)\left(\frac{1-(1-\lambda)\beta-\lambda}{1-(1-\lambda)\beta}\right) > \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right) v \\ \iff & \beta V(i+1)\left(\frac{(1-\lambda)(1-\beta)}{1-(1-\lambda)\beta}\right) > \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right) v \\ \iff & \beta V(i+1) > \frac{v}{1-\beta} \\ \iff & V(i+1) > \frac{v}{\beta(1-\beta)} \end{aligned}$$

Let i_0 be the minimum state i such that the above holds, so in particular $V(i_0) \leq \frac{v}{\beta(1-\beta)}$ but $V(i_0+1) > \frac{v}{\beta(1-\beta)}$. We know because V is increasing in i (still need to prove), this point is indeed a phase transition: $V(i) > \frac{v}{\beta(1-\beta)}$ for all $i > i_0$, so after this point, the customer always chooses firm A. We may compute $V(i_0)$ easily using this fact.

$$V(i_0) = \beta V(i_0+1) = \cdots = \beta^{k-i_0} V(k) = \beta^{k-i_0} R$$

Thus, we have the following:

$$\begin{aligned} & \beta^{k-i_0} \leq \frac{v}{R\beta(1-\beta)} < \beta^{k-(i_0+1)} \\ \iff & k - i_0 \geq \log_\beta\left(\frac{v}{R\beta(1-\beta)}\right) > k - (i_0+1) \\ \iff & i_0 \leq k - \log_\beta\left(\frac{v}{R(1-\beta)}\right) + 1 < i_0 + 1 \\ \iff & i_0 = k - \left\lfloor \log_\beta\left(\frac{v}{R(1-\beta)}\right)\right\rfloor \equiv k - \Delta \end{aligned}$$

□

Now we assume the look-ahead factor of a customer is drawn from some distribution $t \sim T$. The phase transition of the customer's DP will now depend on t.

$$i_0(t) = \begin{cases} i_0, & \text{if } t \geq \Delta. \\ k - t, & \text{otherwise.} \end{cases}$$

In this section, we focus on a very simple threshold distribution given by the following.

$$t = \begin{cases} t_1 \geq \Delta, & \text{wp } p, \\ 0, & \text{wp } 1-p. \end{cases}$$

3.2 Merchant Objective Dynamics

3.2.1 Non Strategic Merchant B, and Equal Promotion Budgeting

First we look into the case when merchant B does not strategize over its discount value v, and when merchant A sets its reward parameters so as to have equal budgets for promosionts as B: i.e. $R = kv$.

Theorem 3.2. *Given $\lambda \sim Unif(0, b)$, merchant A sets its reward parameter $k = \frac{e}{1-\beta}$ at small values of b and $k = \frac{e^{(1-\beta)t_1}}{1-\beta}$ for larger values.*

Proof. First merchant A's objective evaluates to the following:

$$\begin{aligned}
\max_k\{RoR_A\} \Leftrightarrow & \max_k\left\{\underset{\lambda}{E}\left[\frac{\lambda k}{k-\Delta(1-\lambda)}\right]\right\} \\
\Leftrightarrow & \max_k\left\{\frac{1}{b}\int_0^b \frac{\lambda k}{k-\Delta(1-\lambda)}d\lambda\right\} \\
\Leftrightarrow & \max_k\left\{\frac{k}{\Delta^2 b}\left(\Delta b-(k-\Delta)\log\left(\frac{k-\Delta(1-b)}{k-\Delta}\right)\right)\right\} \\
\Leftrightarrow & \max_k\left\{\frac{k}{\Delta}\left(1-\frac{k-\Delta}{b\Delta}\log\left(\frac{k-\Delta(1-b)}{k-\Delta}\right)\right)\right\}
\end{aligned}$$

Now let $\theta = \frac{\Delta}{k}$. Then maximizing the above function is equivalent to maximizing the following function w.r.t. θ with keeping in mind the range that θ can follow.

$$\max_k\{RoR_A\} \Leftrightarrow \max_\theta\{f(\theta)\} \Leftrightarrow \max_\theta\left\{\frac{1}{\theta}\left(1-\frac{1-\theta}{b\theta}\log\left(1+\frac{b\theta}{1-\theta}\right)\right)\right\} \tag{7}$$

Let's look at the quantity $\frac{\Delta}{k}$.

$$\frac{\Delta}{k} = \frac{\log_\beta\left(\frac{1}{k(1-\beta)}\right)}{k} \sim \frac{\log(k(1-\beta))}{k(1-\beta)}$$

Now this value is maximized at $k = \frac{e}{1-\beta}$ and minimized at the maximum possible value of k which is bounded above by $\frac{e^{(1-\beta)t_1}}{1-\beta}$ from the assumption of $t_1 \geq \Delta$.

When b is small, the function $f(\theta)$ can be approximated by taking the second degree terms for the term inside the log. This gives:

$$\begin{aligned}
f(\theta) \sim & \frac{1}{\theta}\left(1-\frac{1-\theta}{b\theta}\left(\frac{b\theta}{1-\theta}-\frac{\left(\frac{b\theta}{1-\theta}\right)^2}{2}\right)\right) \\
= & \frac{b}{2(1-\theta)}
\end{aligned}$$

Clearly $f(\theta)$ is maximized when θ is maximized. This happens as shown above at $k = \frac{e}{1-\beta}$

Whereas when b is large, $f'(\theta)$ can be shown to be negative. Hence we get our result. $\square$

Note that under equal-budgeting, we need $k > \frac{1-\lambda}{1-\beta}$ for V to be increasing. We meet this condition when $k = \frac{e}{1-\beta} > \frac{1}{1-\beta} \geq \frac{1-\lambda}{1-\beta}$. Figure 1 shows the percentage of visits needed for a "forward-looking" consumer to adopt the reward program as a function of β. (Here we should add our comments on cash back computations).

Now we consider the expected revenues of each firm under these conditions.

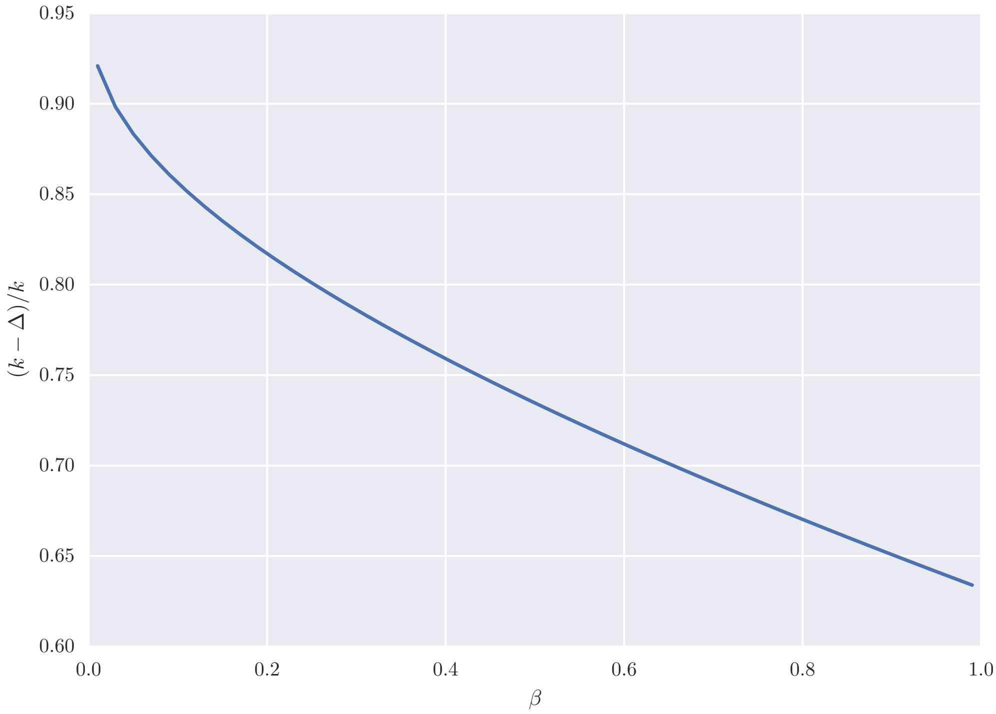

Figure 1: We assume $k = e/(1-\beta)$ and equal-budgeting. The plot shows the percentage of extraneous visits needed for a "forward-looking" consumer needed to adopt the reward program as a function of β.

$$\underset{\lambda,t}{E}[RoR_A] = pk(1-v)\frac{1}{b}\int_0^b \frac{\lambda}{k-(1-\lambda)\Delta}\, d\lambda + (1-p)(1-v)\frac{b}{2} \tag{8}$$

$$\underset{\lambda,t}{E}[RoR_B] = pk(1-v)\frac{1}{b}\int_0^b \frac{1-\lambda}{k-(1-\lambda)\Delta}\, d\lambda + (1-p)(1-v)\left(1-\frac{b}{2}\right) \tag{9}$$

First notice that the ratio of expected revenues is independent of v, the price difference of the two firms. We observe this behavior in our simulations as well. Figure 2 shows the revenue rates of A and B as a function of b for different values of v. We see that the relative rates do not vary with v; changing v only changes the absolute revenue rates of each firm, putting more money into the rewards given out. We see that b must be pretty large for firm A to make more money than firm B.

[Not sure if we should include] We may solve for the b such that the expected revenues are equal, which occurs if when the following holds (excluding work for now).

$$b - \frac{2k(k-\Delta)p}{(1-p)b\Delta^2}\log\left(\frac{k-(1-b)\Delta}{k-\Delta}\right) = 1 - \frac{p}{1-p}\frac{2k-\Delta}{\Delta}$$

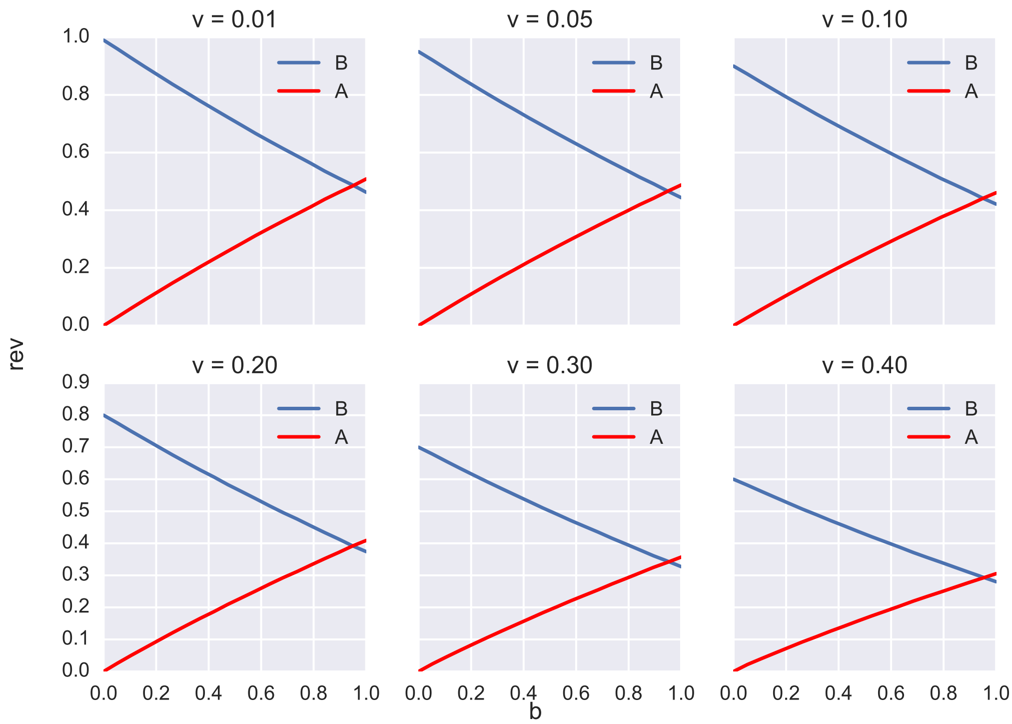

Figure 2: Rates of revenue for A and B with equal-budgeting as a function of b for various v. Fixed $p = 0.5$, $\beta = 0.9$ and $k = e/(1-\beta)$.

Note from the above that p, the probability of a consumer being "forward-looking" does affect the ratio of expected rates of revenue. Figure 3 shows simulation results for the rates of revenue of A and B for various values of p with everything else fixed. We see that as p increases, the b required for the reward program to become more profitable than not using one decreases.

3.2.2 Non Strategic Merchant B, and Unequal Promotion Budgeting

Now we consider scenarios in which firms A and B have different budgets. First we consider the case where the budgets are still proportional, i.e. $R = \alpha \cdot kv$ for some fixed α. Now the expected rate of revenue of A is given by the following, while that of firm B is unchanged. Therefore the ratio of expected revenue rates may now depend on v.

$$\underset{\lambda,t}{E}[RoR_A] = pk(1-\alpha v)\frac{1}{b}\int_0^b \frac{\lambda}{k-(1-\lambda)\Delta}\, d\lambda + (1-p)(1-\alpha v)\frac{b}{2} \tag{10}$$

Following the same logic of Theorem 3.2, the expected revenue rate of A is maximized at $k = \frac{e}{\alpha(1-\beta)}$ at small values of b (need to do again for larger values). For this value of

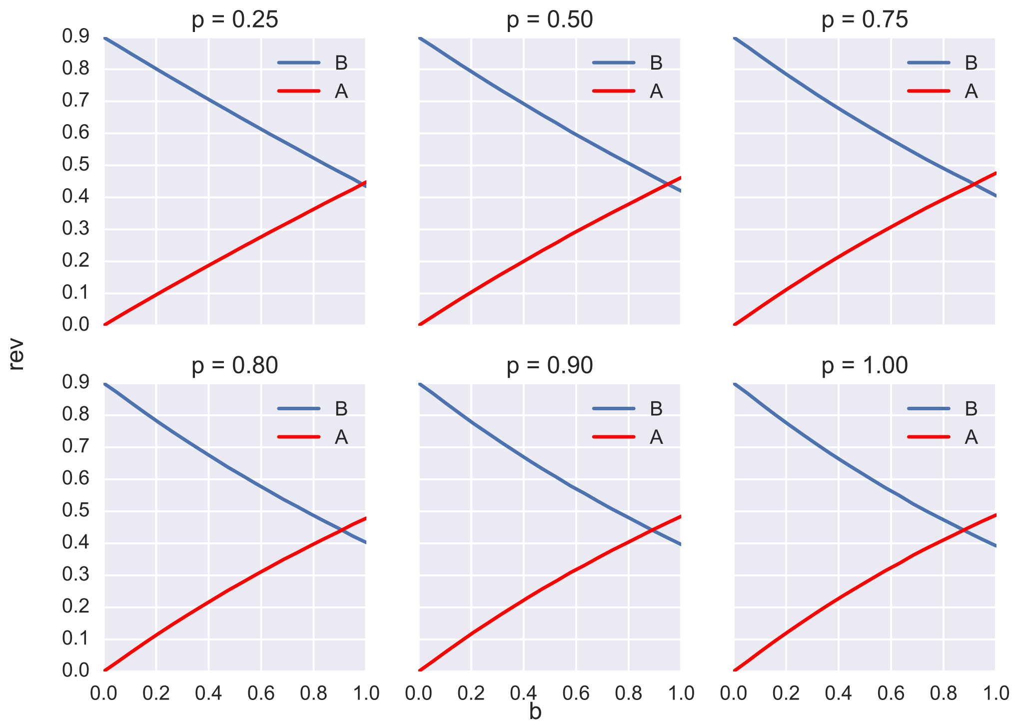

Figure 3: Rates of revenue for A and B with equal-budgeting as a function of b for various p. Fixed $v = 0.1$, $\beta = 0.9$ and $k = e/(1-\beta)$.

k, Δ is fixed for all α ($\Delta = \left\lfloor \log_\beta \left(\frac{1}{e}\right) \right\rfloor$), so we must have $\alpha \leq \frac{e}{\Delta(1-\beta)}$ for $k \geq \Delta$ to hold. When $\beta = 0.9$, this upperbound on α is about 3.

Theorem 3.3. *Suppose firm A fixes its price at 1, and firm B chooses a price of $1-v$. Given a consumer distribution defined by p - with probability p, a consumer is fully forward looking and probability $1-p$ the customer does not look ahead at all - b - each consumer's monopoly factor to firm A is drawn as $\lambda \sim Unif(0,b)$ and β - the customer's discout factor. Then firm A may choose to give a reward of $\alpha v < 1$ to customers after k visits. It should run a reward program if the following condition holds.*

$$\frac{1}{b}\left(1 - \frac{e-1}{b}\log\left(1 + \frac{b}{e-1}\right)\right) \geq \frac{1-(1-p)(1-\alpha v)}{2pe(1-\alpha v)} \tag{11}$$

Define the function on the left-hand side above as $g(b)$.

Proof. Firm A always sells the good for price 1. If it chooses to run a reward program its expected rate of revenue is given by:

$$\mathop{E}_{\lambda,t}[RoR_A] = pk(1-\alpha v)\frac{1}{b}\int_0^b \frac{\lambda}{k-(1-\lambda)\Delta}\, d\lambda + (1-p)(1-\alpha v)\frac{b}{2}$$

If it does not run a reward program, then the only visits it will receive are exogenous visits. In this case, its expected rate of revenue is simply $\frac{b}{2}$. We consider a reward program to be profitable if its expected rate of revenue is at least that of the non-reward program expected revenue rate.

$$\begin{aligned} & pk(1-\alpha v)\frac{1}{b}\int_0^b \frac{\lambda}{k-(1-\lambda)\Delta}\, d\lambda + (1-p)(1-\alpha v)\frac{b}{2} \geq \frac{b}{2} \\ \iff & \frac{pk(1-\alpha v)}{\Delta}\left(1-\frac{k-\Delta}{b\Delta}\log\left(\frac{k-(1-b)\Delta}{k-\Delta}\right)\right) \geq \frac{b}{2}(1-(1-p)(1-\alpha v)) \\ \iff & pe(1-\alpha v)\left(1-\frac{e-1}{b}\log\left(1+\frac{b}{e-1}\right)\right) \geq \frac{b}{2}(1-(1-p)(1-\alpha v)) \\ \iff & \frac{1}{b}\left(1-\frac{e-1}{b}\log\left(1+\frac{b}{e-1}\right)\right) \geq \frac{1-(1-p)(1-\alpha v)}{2pe(1-\alpha v)} \end{aligned}$$

Where we have used the work from Theorem 3.2 as well as the fact that the optimal k is given by $\frac{e}{\alpha(1-\beta)}$, making $\Delta \approx \frac{1}{1-\beta}$. $\square$

Note that the above condition on b is rather complicated, so we have plotted it as a function of b below. First we notice that $g(b)$ is decreasing in b. So for a fixed evaluation of $x \equiv \frac{1-(1-p)(1-\alpha v)}{2pe(1-\alpha v)}$, we are in one of the following cases:

1. $x \geq g(0)$. So no value of b makes the reward program profitable.

2. $x \leq g(1)$. So any value of b makes the reward program profitable.

3. $x = g(b_0)$ for some $b_0 \in (0, 1)$. So the reward program is profitable for all $b \leq b_0$ and not otherwise.

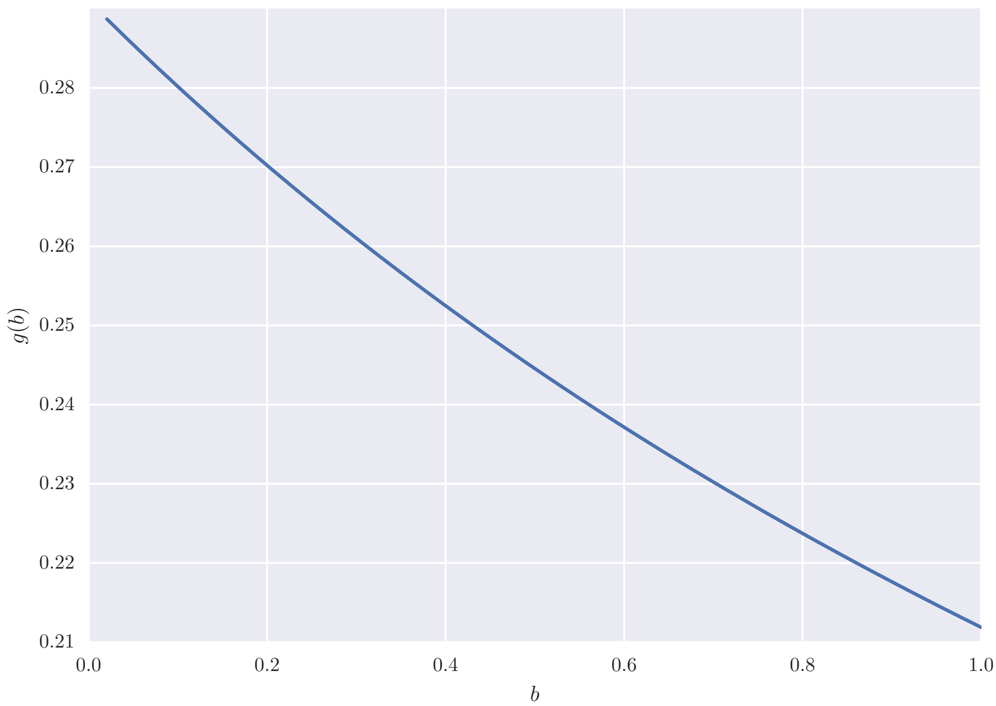

Figure 4: Function governing profitability of reward program for firm A as a function of b.