Optimal Design of Loyalty Reward Program in a Competitive Duopoly

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1   Introduction

Loyalty programs constitute a huge market and are a major source of revenue for many low margin businesses. Over 48 billion dollars of perceived rewards are issued in the United States alone every year, with every household having over 19 loyalty memberships. These include credit cards, hotel and airline reward programs, and more recently even restaurants, grocery and retail stores. Though forming a big component of the market, there is very less scientific understanding about the design of loyalty reward programs. We aim to address this gap with this research.

One popular form of loyalty reward programs are *frequency reward programs*, where customers earn *points* as currency over spendings with merchants and are able to redeem these points into dollar based rewards after achieving certain threshold point collections. There is extant literature on characterizing customer behavior toward frequency reward programs. Most of the literature is empirical in nature, and relies on psychological behavioral patterns among customers, as opposed to rational economic decision making. In this paper, we consider a competitive duopoly of two merchants where one merchant offers a loyalty reward program and the other offers traditional pricing with discounts and characterize a novel model of customer choice where customers measure their utilities in rational economic terms. In addition, we characterize the optimal reward design choice for the merchant offering the frequency reward program, based on different customer populations: specifically, how should the merchant decide the optimal thresholds and dollar value of rewards to optimize for its revenue share from the participating customer

population. One important constraint we impose is that the merchant has to choose a *one design fits all* reward program for the entire participating customer population and is not allowed to personalize the program for different customer segments, while maximizing its overall long term revenue objective.

This is how the remaining of the paper is structured. First we will describe some past work. Then we will go over our contributions and explain how our work builds on top of past literature. In Section 2 we will describe our model followed by the main results in Section 3. We will follow up with a short discussion and future work in Section 4.

1.1 Past Literature

1.2 Our Contributions

We make both modeling and analytical contributions along the following major directions:

1.2.1 Competitive Duopoly Setup

We model a competitive duopoly of two merchants, one of them offering a frequency reward program and the other offering traditional pricing. Both merchants sell an identical good at fixed precommitted prices. The reward program merchant sells the good at a higher price. Customers are drawn from a known population distribution and they measure their utilities in rational economic terms, i.e., they make their purchase decisions to maximize long term discounted rewards. The discount factor is the time value of money, and is constant for all customers. Every customer makes a purchase everyday from either of the two merchants. With each purchase from the reward program merchant, customer gains some fixed number of points, and on achieving the reward redemption threshold, (s)he immediately gains the reward value as a dollar cashback.

1.2.2 Customer Parameters

Each customer has two important parameters drawn from the known population distribution: first a visit probability bias with which (s)he purchases the good from the reward program merchant for reasons exogenous to utility maximization, and second a look-ahead factor that controls how far into the future the customer can perceive the rewards. The visit probability bias toward the reward program merchant can be attributed to *excess*

loyalty which has been argued as a important parameter for the success of any reward program, or it can be attributed to price insensitivity of the customer: whenever the customer is price insensitive, (s)he strictly prefers to purchase from the reward program merchant as (s)he gains points redeemable for rewards in the future. There are many possible reasons for customers' price insensitivity: the reward program merchant could be offering some other monopoly products, or the customer might be getting reimbursed for some purchases as part of corporate perks (eg: corporate travel). As an effect, this visit probability bias controlls how frequently the customers' points increase even when (s)he does not actively choose to make purchases from the reward program merchant. The look-ahead parameter affects the customer behavior dynamics as follows: if the reward is farther than the customer's look-ahead parameter, (s)he is unable to register the future value of that reward and take it into consideration while maximizing long term utility. Both these parameters can be attributed to bounded rationality of customers and have been argued to be important factors toward customer choice dynamics (cite, etc).

1.2.3 Customer Choice Dynamics

We formulate the customer choice dynamics as a dynamic program with the state being the number of points collected from the reward program merchant. When the customer does not make biased visits to the reward program merchant, (s)he chooses the merchant maximizing her long term utility: comparing the immediate utility by purchasing the good at cheaper price and the long term utility of waiting and receiving the time discounted reward. The solution to the customer's dynamic program gives conditions for the existence and achievability of a phase transition: a points threshold before which the customer visits the merchant offering rewards only due to the visit probability bias, and after which (s)he always visits the merchant offering rewards till receiving the reward. We show that this phase transition point has the following dependencies:

1. It decreases with increase in the look-ahead parameter, i.e., if the customer can perceive rewards longer into the future, the phase transition for the customer occurs sooner.

2. It decreases with increase in the reward value, and decreases with increase in points threshold required to redeem the reward.

3. It increases with the discount value that the traditional pricing merchant provides, i.e., the cheaper the product from the second merchant, the farther is the phase transition point.

4. (effect w.r.t. to discount factor β. leaving it for now).

(Discussion relating to past literature on psychological constructs: tipping point literature. purchase acceleration literature)

1.2.4 Reward Program Design

After characterizing the customer behavior dynamics in our model, we optimize over the long run revenues that the reward program merchant achieves. We first show some conditions over the customer population that easily gives better revenues to the reward program merchant over the traditional pricing merchant (can we explain the graph of (b, p) pairs in one or two lines here for this particular result. We provide a framework for optimizing the parameters for the reward program merchant to maximize its revenue objective. We then look at a specific simplified case of proportional budgeting: the reward offered by the reward program merchant if proportional to the product of the distance to the reward and the discount provided by the traditional pricing merchant. We show that Cashback result simplified to two lines here. We characterize conditions for when it's better for the reward program merchant to offer a reward vs not offering any reward (graph of relevant (b,p) pairs). Finally we compare the strategy to offer reward vs offering traditional pricing with discounts itself for the reward program merchant (here the expiration can come into play) and show....

Then, if we have anything, we can talk about multi-tiering strategies.

2   Model

We index the two competing merchants selling identical goods as A and B. Without loss of generality we assume that A sells the good for a price of 1 dollars while B sells it for $1 - v$ dollars, i.e., B offers a discount of v dollars. A on the other hand offers a reward of value R dollars to a customer after (s)he makes k purchases at A.

Every customer purchases an item from either A or B everyday (Can we justify/relax this somehow) (Does it matter that it's everyday? We just care about tracking every

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purchase of the item the customer makes, doesn't have to be regular I think). We assume customers have a linear homogenous utility in price: at price p the utility is $\nu(p) = 1 - p$. This reduces to customers getting an immediate utility of 0 from A and v from B. Customers have the same time value of money as a discount factor of β lying between 0 and 1.

We denote a customer's visit probability bias and look-ahead parameter with λ and t respectively. That is with probability λ (s)he purchases from A due to externalities and perceives a future reward only if it is within t purchases away. We assume λ for a customer to be drawn from a uniform distribution between $[0, b]$, where b is between 0 and 1. And we focus on a simple threshold distribution for the look-ahead parameter t:

$$
t = \begin{cases} t_1, & \text{wp } p, \\ 0, & \text{wp } 1 - p. \end{cases}
$$

The above distribution intuitively means that the customers are either myopic and focus only on immediate rewards, or are far-sighted enough (we assume t_1 is large). We model the customer's decision problem as a dynamic problem. We index the number of visits the customer makes at store A by i, for $0 \le i \le k - 1$, and we refer a customer to be in state i after having made i visits to A. At state i, the customer has two possibilities:

1. With probability λ, the customer must visit A, and she is now in state $i + 1$.

2. With probability $1 - \lambda$, the customer may purchase from B for an immediate utility v and remain in state i or purchase from A for no utility but move to state $i + 1$.

Let $V(i)$ denote the long term expected reward at state i. Then we model the decision problem as the following dynamic program.

$$
V(i) = \lambda \beta V(i + 1) + (1 - \lambda) \max\{v + \beta V(i), \beta V(i + 1)\} \text{ for } 0 \le i \le k - 1
$$
$$
V(k) = R
$$

We show that the decision process exhibits a phase transition; that is prior to some state, the customer will only visit A if (s)he must do so exogenously but after that state, (s)he always decides to go to A. This phase transition point is independent of λ, and depends only on t, among the variable customer parameters. Hence we represent this phase transition point as $i_0(t)$.

2.1 Merchant Objective

Given the above model of customer dynamics, we define the revenue objectives of A and B, where A chooses its parameters whereas B is non strategic. We define the rate of revenue for a merchant from a customer as the expected time averaged revenue that the merchant receives within the customer's lifetime. For simplification we assume merchants do not discount future revenues. As described above, a customer's dynamics are cyclic after each reward cycle. Thus the lifetime dynamics of customer behavior is a regenerative process with independent and identically distributed reward cycle lengths. Let $RoR_A(c)$ and $RoR_B(c)$ denote the expected rate of revenues for merchants A and B respectively from a customer c's lifetime. Let $\tau(t, \lambda)$ denote the total number of visits the customer makes before reaching the phase transition point $i_0(t)$. Then the length of the reward cycle (or total number of visits the customer makes before receiving the reward) is $\tau(t, \lambda) + k - i_0(t)$, as after the phase transition (s)he makes all visits to merchant A only until hitting the reward. In this cycle the number of visits that the customer makes to A are k, and to B are $\tau(t, \lambda) - i_0(t)$. The revenue that A earns in one such cycle is $k - R$ and the revenue that B earns is $(1 - v)(\tau(t, \lambda) - i_0(t))$. Thus the rate of revenues for A and B from the customer c are as follows:

$$RoR_A(c) = \mathop{E}_{\tau} \left[\frac{k - R}{\tau(t, \lambda) + k - i_0(t)} \right]$$

$$RoR_B(c) = \mathop{E}_{\tau} \left[\frac{(\tau(t, \lambda) - i_0(t))(1 - v)}{\tau(t, \lambda) + k - i_0(t)} \right]$$

Since the process for a single customer is regenerative, using the reward renewal theorem (CITE), we can take the expectation over the cycle length inside the numerator and denominator respectively. Note that $\mathop{E}_{\tau}[\tau(t, \lambda)] = \frac{i_0(t)}{\lambda}$ as before reaching the phase transition point, with probability λ, the customer's visits to A increases by 1 and with probability $1 - \lambda$ it stays constant. And then taking the expectation over the customer population the overall rate of revenues for both A and B are as follows:

$$RoR_A = \mathop{E}_{\lambda,t} \left[\frac{k - R}{i_0(t)/\lambda + k - i_0(t)} \right] \tag{1}$$

$$RoR_B = \mathop{E}_{\lambda,t} \left[\frac{(i_0(t)\lambda - i_0(t))(1 - v)}{i_0(t)/\lambda + k - i_0(t)} \right] \tag{2}$$

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3 Results

3.1 Customer Choice Dynamics

We first show that every customer exhibits the following behavior: until (s)he reaches the phase transition point $i_0(t)$, she visits A only due to the exogeneity paramaeter, and after that (s)he always visits merchant A till she receives the reward. This behavior is cyclic, and repeats after every reward redemption.

Lemma 3.1. *$V(i)$ is an increasing function in i if the following condition holds:*

$$R > \frac{(1-\lambda)v}{1-\beta} \tag{3}$$

And further, $V(i)$ can be evaluated as:

$$V(i) = \max\left\{\frac{\lambda\beta V(i+1) + (1-\lambda)v}{1-(1-\lambda)\beta}, \beta V(i+1)\right\} \tag{4}$$

Proof. First we show that $V(i)$ is an increasing function in i by induction. We first show that if the condition above is satisfied, $V(k-1) < V(k) = R$. Suppose not, so $V(i) \geq R$. Then we have:

$$
\begin{aligned}
V(k-1) &= \lambda\beta V(k) + (1-\lambda)(v + \beta V(k-1)) \\
&= \frac{\lambda\beta R + (1-\lambda)v}{1-(1-\lambda)\beta} \\
&< \frac{\lambda\beta R + (1-\beta)R}{1-(1-\lambda)\beta} \\
&= \frac{R(1-(1-\lambda)\beta)}{1-(1-\lambda)\beta} = R
\end{aligned}
$$

But this is a contradiction, so $V(k-1) < V(k)$. Now assume $V(i+1) < V(i+2)$ for some $i < k-2$, we will show that this implies $V(i) < V(i+1)$. Suppose not, so $V(i) \geq V(i+1)$. As we did before we may upper bound $V(i)$.

$$
\begin{aligned}
V(i) &= \lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)) \\
&\leq (1-\lambda)v + \beta V(i) \\
\iff V(i) &\leq \frac{(1-\lambda)v}{1-\beta}
\end{aligned}
$$

But because $V(i+1) < V(i+2)$, we may lower bound $V(i+1)$.

$$V(i+1) \geq \lambda\beta V(i+2) + (1-\lambda)(v + \beta V(i+1))$$
$$= (1-\lambda)v + (1-\lambda)\beta V(i+1) + \lambda\beta V(i+2)$$
$$> (1-\lambda) + \beta V(i+1)$$
$$\Longleftrightarrow V(i+1) > \frac{(1-\lambda)v}{1-\beta}$$

Again, we have a contradiction, so $V(i) < V(i+1)$, and $V(i)$ is an increasing function in i. Now we prove the second claim. We have the following:

$$V(i) = \lambda\beta V(i+1) + (1-\lambda)\max\{v + \beta V(i), \beta V(i+1)\}$$
$$= \max\{\lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i)), \beta V(i+1)\}$$

Assuming $V(i)$ is the left term in the above maximum, we may solve the equation for that term.

$$V(i) = \lambda\beta V(i+1) + (1-\lambda)(v + \beta V(i))$$
$$(1 - (1-\lambda)\beta)V(i) = \lambda\beta V(i+1) + (1-\lambda)v$$
$$V(i) = \frac{\lambda\beta V(i+1) + (1-\lambda)v}{1 - (1-\lambda)\beta}$$

And we get our claim. $\square$

Probably want some exposition between the lemma and the theorem.

Theorem 3.1. *Suppose $V(i)$ is an increasing function in i and consider a customer with look-ahead parameter t. A phase transition occurs after (s)he makes $i_0(t)$ visits to firm A, where $i_0(t)$ is given by:*

$$i_0(t) = \begin{cases} k - \Delta \equiv i_0, & \text{if } t \geq \Delta. \\ k - t, & \text{otherwise.} \end{cases} \tag{5}$$

with

$$\Delta = \left\lfloor \log_\beta\left(\frac{v}{R(1-\beta)}\right) \right\rfloor \tag{6}$$

Proof. Still need to fix this proof a little bit I think. I will try to work on that.

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First we solve for the condition on $V(i+1)$ for us to choose firm A over B willingly.

$$\beta V(i+1) > \frac{\lambda \beta V(i+1) + (1-\lambda)v}{1-(1-\lambda)\beta}$$

$$\iff \beta V(i+1)\left(1 - \frac{\lambda}{1-(1-\lambda)\beta}\right) > \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right)v$$

$$\iff \beta V(i+1)\left(\frac{1-(1-\lambda)\beta-\lambda}{1-(1-\lambda)\beta}\right) > \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right)v$$

$$\iff \beta V(i+1)\left(\frac{(1-\lambda)(1-\beta)}{1-(1-\lambda)\beta}\right) > \left(\frac{1-\lambda}{1-(1-\lambda)\beta}\right)v$$

$$\iff \beta V(i+1) > \frac{v}{1-\beta}$$

$$\iff V(i+1) > \frac{v}{\beta(1-\beta)}$$

Let i_0 be the minimum state i such that the above holds, so in particular $V(i_0) \leq \frac{v}{\beta(1-\beta)}$ but $V(i_0+1) > \frac{v}{\beta(1-\beta)}$. We know because V is increasing in i, this point is indeed a phase transition: $V(i) > \frac{v}{\beta(1-\beta)}$ for all $i > i_0$, so after this point, the customer always chooses firm A. We may compute $V(i_0)$ easily using this fact.

$$V(i_0) = \beta V(i_0+1) = \cdots = \beta^{k-i_0}V(k) = \beta^{k-i_0}R$$

Thus, we have the following:

$$\beta^{k-i_0} \leq \frac{v}{R\beta(1-\beta)} < \beta^{k-(i_0+1)}$$

$$\iff k - i_0 \geq \log_\beta\left(\frac{v}{R\beta(1-\beta)}\right) > k - (i_0+1)$$

$$\iff i_0 \leq k - \log_\beta\left(\frac{v}{R(1-\beta)}\right) + 1 < i_0 + 1$$

$$\iff i_0 = k - \left\lfloor \log_\beta\left(\frac{v}{R(1-\beta)}\right)\right\rfloor \equiv k - \Delta$$

The above dependence reduces to the following after incorporating the look-ahead distribution:

$$i_0(t) = \begin{cases} i_0, & \text{wp } p, \\ k, & \text{wp } 1-p. \end{cases}$$

$\square$

Discussion around what do the i_0 dependencies mean.

Discussion around setting $i_0 = 0$

3.2 Merchant Objective Dynamics

We substitute the value of the phase transition point obtained above in the rate of revenue equations to reevaluate them. And since we assume that λ and t are drawn independent of each other, we can separate the expectation terms and evaluate them sequentially, first over t, then over λ. This reduces the rate of revenues as follows:

$$
\begin{aligned}
RoR_A =& \mathop{E}_{\lambda,t} \left[\frac{k-R}{i_0(t)/\lambda + k - i_0(t)} \right] \\
=& \mathop{E}_{\lambda} \left[p \cdot \frac{k-R}{i_0/\lambda + k - i_0} + (1-p)\frac{\lambda(k-R)}{k} \right] \\
=& \mathop{E}_{\lambda} \left[p \cdot \frac{\lambda(k-R)}{k\lambda + i_0(1-\lambda)} + (1-p)\frac{\lambda(k-R)}{k} \right] \\
=& p \cdot \frac{k-R}{b(k-i_0)^2} \cdot \left(b(k-i_0) - i_0 \log\left(1 + \frac{b(k-i_0)}{i_0}\right) \right) + (1-p)\frac{b(k-R)}{2k} \\
=& p \cdot \frac{k-R}{b\Delta^2} \cdot \left(b\Delta - (k-\Delta) \log\left(1 + \frac{b\Delta}{k-\Delta}\right) \right) + (1-p)\frac{b(k-R)}{2k} \\
=& p \cdot \frac{k-R}{\Delta} \cdot \left(1 - \frac{k-\Delta}{b\Delta} \log\left(1 + \frac{b\Delta}{k-\Delta}\right) \right) + (1-p)\frac{b(k-R)}{2k}
\end{aligned}
$$

$$
\begin{aligned}
RoR_B =& \mathop{E}_{\lambda,t} \left[\frac{(i_0(t)\lambda - i_0(t))(1-v)}{i_0(t)/\lambda + k - i_0(t)} \right] \\
=& \mathop{E}_{\lambda} \left[p \cdot \frac{(i_0/\lambda - i_0)(1-v)}{i_0/\lambda + k - i_0} + (1-p)\frac{(k/\lambda - k)(1-v)}{k/\lambda} \right] \\
=& \mathop{E}_{\lambda} \left[p \cdot \frac{i_0(1-\lambda)(1-v)}{k\lambda + i_0(1-\lambda)} + (1-p)(1-\lambda)(1-v) \right] \\
=& p \cdot \frac{i_0(1-v)}{b(k-i_0)^2} \left(k \log\left(1 + \frac{b(k-i_0)}{i_0}\right) - b(k-i_0) \right) + (1-p)(1-\frac{b}{2})(1-v) \\
=& p \cdot \frac{(k-\Delta)(1-v)}{b\Delta^2} \left(k \log\left(1 + \frac{b\Delta}{k-\Delta}\right) - b\Delta \right) + (1-p)(1-\frac{b}{2})(1-v) \\
=& p \cdot \frac{(k-\Delta)(1-v)}{\Delta} \left(\frac{k}{b\Delta} \log\left(1 + \frac{b\Delta}{k-\Delta}\right) - 1 \right) + (1-p)(1-\frac{b}{2})(1-v)
\end{aligned}
$$

3.2.1 Proportional Promotion Budgeting

First we look into the case when A sets its reward value R proportional to the product of the distance to the reward k and the discount value v provided by merchant B: i.e. $R = \alpha k v$. We refer to this case as proportional promotion budgeting. Note α is a constant here.

Theorem 3.2. *Under proportional promotion budgeting, the optimal reward distance that A should set is $k = \frac{e}{\alpha(1-\beta)}$ at all values of b as long as β is close to 1.*

Proof. Recall the previous expression for RoR_A. Maximizing this function is equivalent to maximizing the following:

$$\max_k\{RoR_A\} \Leftrightarrow \max_k \left\{ \frac{k}{\Delta} \left(1 - \frac{k-\Delta}{b\Delta} \log \left(\frac{k - \Delta(1-b)}{k - \Delta} \right) \right) \right\}$$

Now let $\theta = \frac{k}{\Delta}$. Then maximizing the above function is equivalent to maximizing the following function w.r.t. θ. Note that $\theta \geq 1$ because $k \geq \Delta$.

$$\max_k\{RoR_A\} \Leftrightarrow \max_\theta\{f(\theta)\} \Leftrightarrow \max_\theta \left\{ \theta \left(1 - \frac{\theta-1}{b} \log \left(1 + \frac{b}{\theta-1} \right) \right) \right\}$$

We will show that $f'(\theta) \leq 0$ for all θ so maximizing f is equivalent to minimizing θ.

$$
\begin{aligned}
f'(\theta) &= \frac{2\theta - 1 + b}{\theta - 1 + b} - \frac{2\theta - 1}{b} \log \left(1 + \frac{b}{\theta-1} \right) \\
&= \frac{2\theta-1}{b} \cdot \left(\left(\frac{2\theta-1+b}{\theta-1+b} \right) \left(\frac{b}{2\theta-1} \right) - \log \left(1 + \frac{b}{\theta-1} \right) \right) \\
&= \frac{2\theta-1}{b} \cdot \left(\frac{b}{\theta-1+b} + \frac{b^2}{(2\theta-1)(\theta-1+b)} - \log \left(1 + \frac{b}{\theta-1} \right) \right)
\end{aligned}
$$

Let $g(b, \theta) = \frac{b}{\theta-1+b} + \frac{b^2}{(2\theta-1)(\theta-1+b)} - \log \left(1 + \frac{b}{\theta-1} \right)$. In the limit of $b \to 0$, it is easy to see that $f'(\theta) = g(b, \theta) = 0$ for all θ in the domain. We now show that for all θ in the domain and all $0 < b \leq 1$, $\frac{\partial g(b,\theta)}{\partial b} \leq 0$.

$$
\begin{aligned}
\frac{\partial g(b,\theta)}{\partial b} &= \frac{\theta-1}{(\theta-1+b)^2} + \frac{1}{2\theta-1} \cdot \frac{2b(\theta-1+b) - b^2}{(\theta-1+b)^2} - \frac{1}{\theta-1+b} \leq 0 \\
&\iff \theta - 1 + \frac{2b(\theta-1+b) - b^2}{2\theta-1} \leq \theta - 1 + b \\
&\iff \frac{b(2\theta-2+b)}{2\theta-1} \leq b \\
&\iff 2\theta - 2 + b \leq 2\theta - 1 \\
&\iff b \leq 1
\end{aligned}
$$

Thus we have shown that for all θ, $g(b, \theta) = 0$ as $b \to 0$ and that for all θ and $0 < b \leq 1$, $g(b, \theta)$ is decreasing. These together mean that for all θ and $b \in [0, 1]$, $g(b, \theta) \leq 0$. Which implies that $f'(\theta) \leq 0$ for all θ in the domain. So to maximize f, we need to minimize θ.

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Now let's look at the quantity $\theta = \frac{k}{\Delta}$.

$$\frac{k}{\Delta} = \frac{k}{\log_\beta \left(\frac{1}{\alpha k (1-\beta)} \right)} \sim \frac{k(1-\beta)}{\log(\alpha k(1-\beta))}$$

The above approximation relies on β close to 1. Now this value is minimized at $k = \frac{e}{\alpha(1-\beta)}$. Therefore, for all b, the optimal value for k is given by $\frac{e}{\alpha(1-\beta)}$, the value for which $\frac{k}{\Delta}$ is minimized and takes the value $\frac{e}{\alpha}$.

$\square$

An interesting point to observe in the above proof is that maximizing the revenue objective is equivalent to minimizing $\frac{k}{\Delta}$ or maximizing $\frac{\Delta}{k}$. This quantity $\frac{\Delta}{k}$ is effectively the *zone of influence*, i.e., the ratio of number of visits required for a "forward-looking" customer to adopt a reward program and the total distance to the reward. In short, if the merchant minimizes the *zone of influence*, it maximizes its revenue objective. Merchants often offer reward currency for free (give some examples of airlines issuing free credit or accelerating rate at which points are issued). We are characterizing the time until which it is most beneficial to offer such accelerations. The condition that β be close to 1 is not very restrictive, as we expect it to be near 1 in most cases (maybe get a citation for this). Note that because $k \geq \Delta$, the above also shows $\alpha \leq e$. Finally, observe that, we need $R > \frac{(1-\lambda)v}{1-\beta}$ for V to be increasing. We meet this condition with proportional budgeting when $k = \frac{e}{\alpha(1-\beta)}$ as $R = \alpha k v = \frac{ev}{1-\beta} \geq \frac{v}{1-\beta} \geq \frac{(1-\lambda)v}{1-\beta}$. Also note that if α is 1, then the value of k corresponds to a cashback percentage between 2 and 4 as β ranges between 0.95 and 0.9. This value is realistic to what is observed in practice too.

Now we evaluate the expected rate of revenues of each merchant under these conditions of proportional budgeting and optimal k.

$$
\begin{aligned}
RoR_A =& pk \cdot \frac{1 - \alpha v}{\Delta} \cdot \left(1 - \frac{k - \Delta}{b\Delta} \log \left(1 + \frac{b\Delta}{k - \Delta} \right) \right) + (1 - p) \frac{bk(1 - \alpha v)}{2k} \\
=& (1 - \alpha v) \left(p \frac{e}{\alpha} \left(1 - \frac{e - \alpha}{b\alpha} \log \left(1 + \frac{b\alpha}{e - \alpha} \right) \right) + (1 - p) \frac{b}{2} \right)
\end{aligned}
$$

$$RoR_B = p \cdot \frac{(k-\Delta)(1-v)}{\Delta}\left(\frac{k}{b\Delta}\log\left(1+\frac{b\Delta}{k-\Delta}\right)-1\right)+(1-p)(1-\frac{b}{2})(1-v)$$
$$= (1-v)\left(p \cdot \frac{e-\alpha}{\alpha}\left(\frac{e}{b\alpha}\log\left(1+\frac{b\alpha}{e-\alpha}\right)-1\right)\right)$$
$$= (1-v)\left(p\frac{e}{\alpha}\left(\frac{e-\alpha}{b\alpha}\log\left(1+\frac{b\alpha}{e-\alpha}\right)-\frac{e-\alpha}{e}\right)+(1-p)(1-\frac{b}{2})\right)$$

Observe that both the above equations have a left term and a right term. The left term is the rate of revenue obtained from "forward-looking" customers whereas the right term is from the myopic customers. As α ranges between 0 and e, the value on the left term increases from 0 for RoR_A and decreases to 0 for RoR_B. That is by controlling the reward budget ratio, merchant A is able to gain the entire "forward-looking" customer base. The next figure illustrates the gap in rate of revenue $(RoR_A - RoR_B)$ as follows: for different values of α we plot the region of b and p pairs where the gap is positive. We observe that there is a clear threshold of b and p beyond which the revenue gap is always positive. This is easily explained by the increasing nature of the revenue gap with b and p. But more interestingly, the threshold value of b and p decreases as α is increased toward e.

As $\alpha \to 0$, $RoR_A \to b/2$, i.e., the revenue earned is only due to visit probability bias. Observe how RoR_A varies with α. The marginal revenue term $(1-\alpha v)$ decreases with α as the merchant is giving higher rewards to customers, whereas the market share term increases as the merchant gains more "forward-looking" customers with increase in reward budget. Also note that $\alpha = 0$ is equivalent to the merchant not running any reward program. We next investigate the question of when it is better for the merchant to offer a reward program as opposed to doing traditional pricing. That is for what values of α, b, p, v is it appropriate for merchant A to offer a reward program. In other words, when is the rate of revenue of merchant A greater than its rate of revenue at $\alpha = 0$.

Is this the best way to frame the next results? It seems like we are thinking of α fixed and then the firm must decide to either offer the reward program or not; but here, we are changing the choice from reward program vs not to our previously fixed α vs $\alpha = 0$. This is just very subtle semantics, just something to think about.
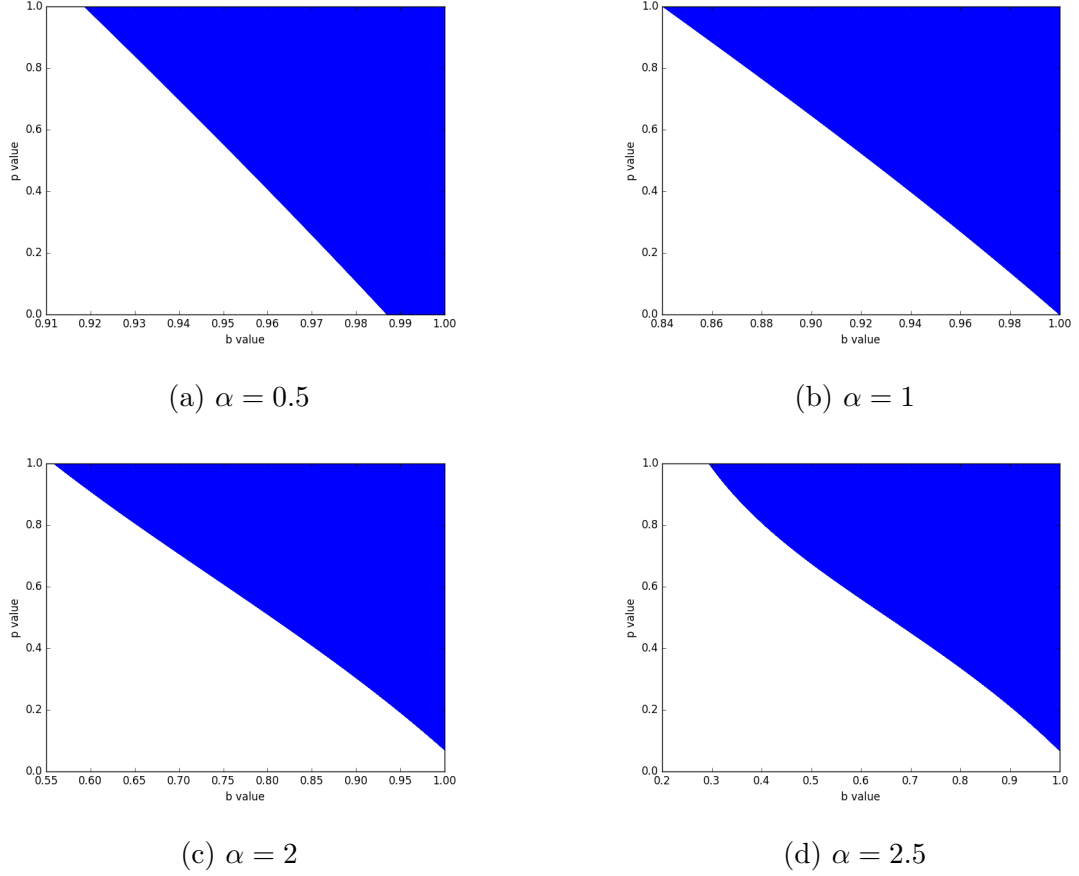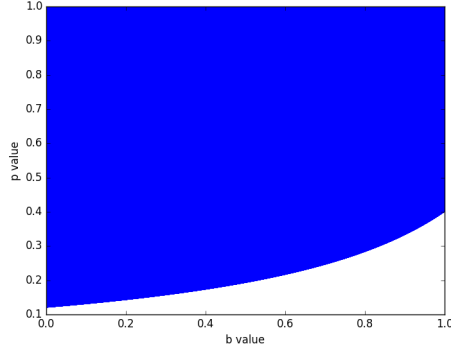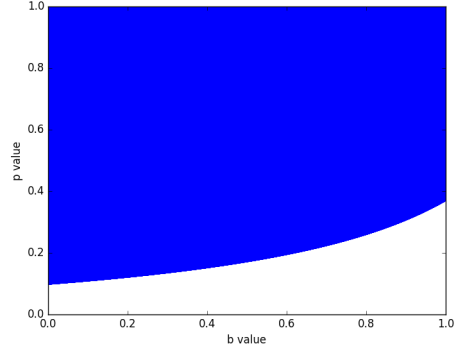
(a) $\alpha = 0.5$

(b) $\alpha = 1$

(c) $\alpha = 2$

(d) $\alpha = 2.5$

Figure 1: Achievable region of (b, p) pairs where rate of revenue is higher for merchant A for different values of budget proportions α. This achievable region is maximized at α close to e.

$$(1 - \alpha v) \left(p\frac{e}{\alpha} \left(1 - \frac{e - \alpha}{b\alpha} \log \left(1 + \frac{b\alpha}{e - \alpha} \right) \right) + (1 - p)\frac{b}{2} \right) > \qquad (1 - p)b/2$$

$$\implies (1 - \alpha v)p\frac{e}{\alpha} \left(1 - \frac{e - \alpha}{b\alpha} \log \left(1 + \frac{b\alpha}{e - \alpha} \right) \right) > \qquad \alpha v(1 - p)b/2$$

$$\implies (1 - \alpha v)p\frac{e}{\alpha} \left(\frac{b\alpha}{2(e - \alpha)} - \frac{b^2\alpha^2}{3(e - \alpha)^2} \right) > \qquad \alpha v(1 - p)b/2$$

$$\implies (1 - \alpha v)\frac{pe}{e - \alpha} \left(\frac{1}{2} - \frac{b\alpha}{3(e - \alpha)} \right) > \qquad \alpha v(1 - p)/2$$

$$\implies \left(\frac{1}{\alpha v} - 1 \right) \frac{e}{e - \alpha} \left(\frac{1}{2} - \frac{b\alpha}{3(e - \alpha)} \right) > \qquad \frac{1 - p}{2p}$$
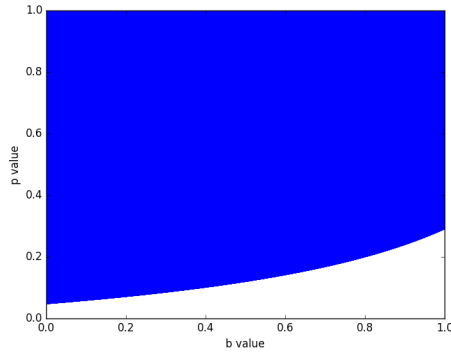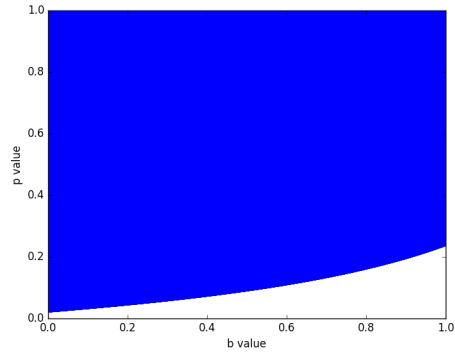
From above we get an interesting sufficient cutoff for b: $b < 3(e - \alpha)/2$. LHS is decreasing in v and decreasing in b. RHS is decreasing in p.

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(a) $\alpha = 0.5$

(b) $\alpha = 1$

(c) $\alpha = 2$

(d) $\alpha = 2.5$

Figure 2: Achievable region of (b, p) pairs where rate of revenue is higher with a reward program than it is without. In these plots $v = 0.05$ and $\beta = 0.95$.

Don't think the above is necessary anymore? But will keep for now.

Talk about the framework in depth. That this can be used to find optimal R and k. Also different distributions can be tested once the framework is ready

Just as we can look at values of b and p such that $RoR_A > RoR_B$, we can characterize the (b, p) pairs for which $RoR_A > \frac{b}{2}$. These pairs correspond to customer distributions for which a reward program is more profitable than not. Figure 4 shows the pairs for different values of α, keeping $v = 0.05$ and $\beta = 0.95$ fixed. Observe that under these conditions, the firm should choose to run a reward program most of the time. Also notice that reward programs begin to lose profitability as b increases; larger b values mean that people make more exogenous visits, so a reward program is no longer needed to entice visits.

Probably need to expand on above, will think about how to do that. Also, need to figure out when to add expiration comment.

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For any fixed α, the exact conditions on p, b and v for $RoR_A > \frac{b}{2}$ are rather complex. We will focus on one particular simple case: $\alpha \to e$.

Theorem 3.3. *As $\alpha \to e$, $RoR_A > \frac{b}{2}$ if the following condition on b holds:*

$$b < \frac{2p}{p + \frac{e \cdot v}{1 - e \cdot v}} \tag{7}$$

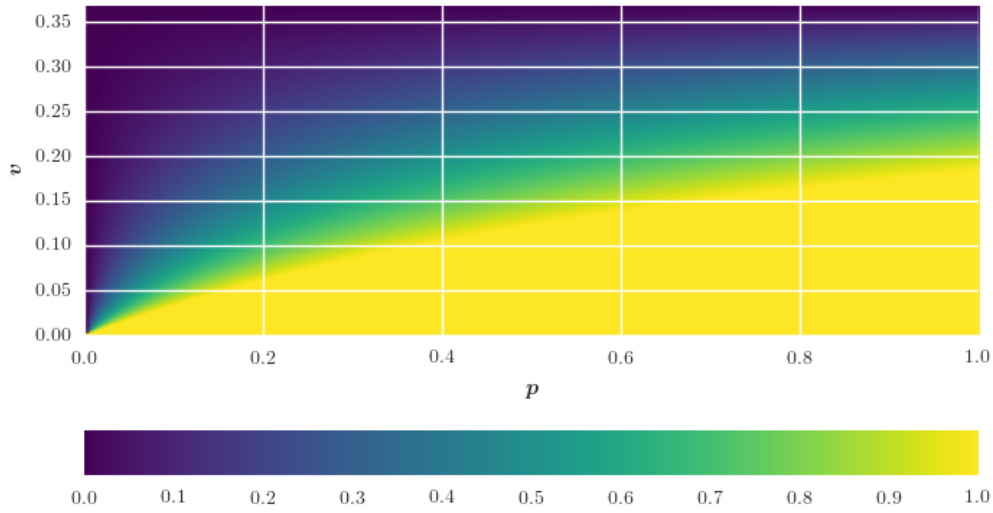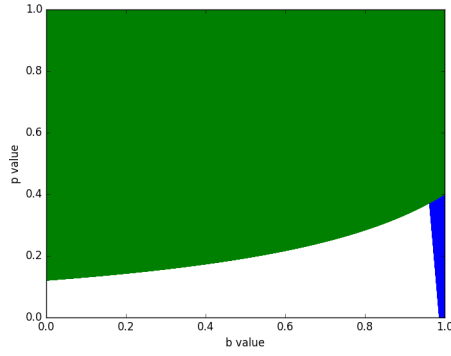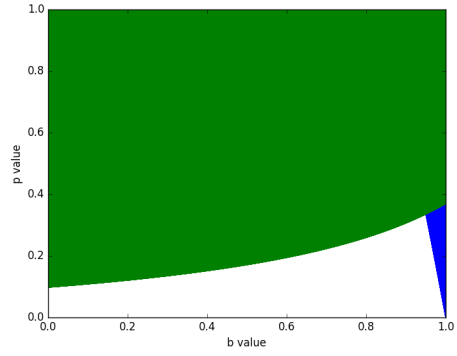*Proof.* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Figure 3: Upper bound on b for reward program to be more profitable than no reward program for different values of p and v with $\alpha \to e$.
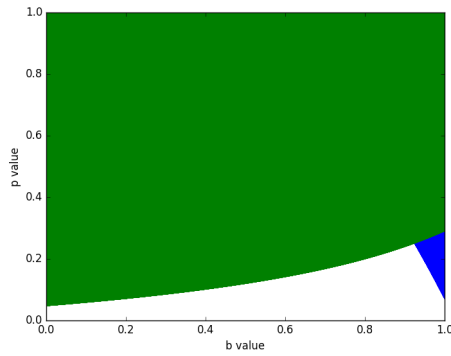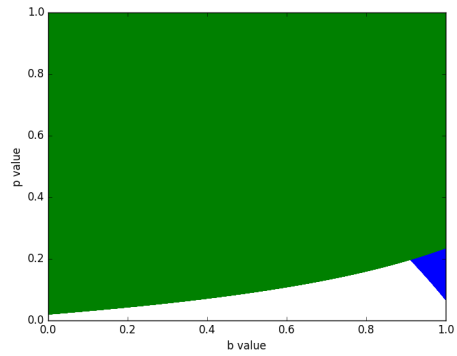
4 Conclusions

(a) $\alpha = 0.5$

(b) $\alpha = 1$

(c) $\alpha = 2$

(d) $\alpha = 2.5$

Figure 4: Just adding these in for reference. I think if we do want to do both together, we should not have them be so dense, or we need to do some other method of plotting. Also, not sure these are the best representatives, maybe we keep the other plots and then plot both together for different v's and α's?