DeepBayes Summer School Application, 2018

Theoretical Assignment

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1 Problem 1

The random variable ξ has Poisson distribution with the parameter λ. If $\xi = k$ we perform k Bernoulli trials with the probability of success p. Let us define the random variable η as the number of successful outcomes of Bernoulli trials. Prove that η has Poisson distribution with the parameter $p\lambda$.

Solution

As any random variable ξ has Poisson distribution with the parameter λ if $P(\xi = k) = \exp(-\lambda)\frac{\lambda^k}{k!}$. If η defines the number of successful outcomes of Bernoulli trials, we can calculate the following probability.

$$P(\eta = k_k|\xi = k_1) = \begin{cases} C_{k_1}^{k_k} \cdot p^{k_k}(1-p)^{k_1-k_k}, & \text{if } k_k <= k_1 \\ 0, & \text{otherwise} \end{cases}$$

So we have conditional probability for every k. According to the law of total probability, we now have:

$$P(\eta = k_k) = \sum_{i=1}^{N} P(\eta = k|\xi = k_i)P(\xi = k_i) = 0 + \sum_{i=k}^{N} P(\eta = k|\xi = k_i)P(\xi = k_i)$$

Now as we consider the values of $P(\eta = k_k|\xi = k_1)$ and $P(\xi = k_i)$ we can calculate the following.

$$P(\eta = k_k) = \sum_{i=k}^{N} \frac{k_i!}{k_k!(k_i-k_k)!} \cdot p^{k_k}(1-p)^{k_i-k_k} \exp(-\lambda)\frac{\lambda^{k_i}}{k_i!} = \sum_{i=k}^{N} \frac{p^{k_k}(1-p)^{k_i-k_k}}{k_k!(k_i-k_k)!} \cdot \exp(-\lambda)\lambda^{k_i}$$

Now, let's rename the variables – $k_k = k$, $k_i - k = n$. Then:

$$P(\eta = k) = \sum_{n=0}^{N} \frac{p^k(1-p)^n}{k!n!} \cdot \exp(-\lambda)\lambda^{n+k} = \exp(-\lambda)\frac{(p\lambda)^k}{k!}\sum_{n=0}^{N}\frac{(1-p)^n\lambda^n}{n!} = \exp(-\lambda)\frac{(p\lambda)^k}{k!} \cdot \exp((1-p)\lambda) = \exp(-p\lambda)\frac{(p\lambda)^k}{k!}$$

As a result, we can see, that random variable η has Poisson distribution with the parameter $p\lambda$.

2 Problem 2

A strict reviewer needs t_1 minutes to check assigned application to Deep|Bayes summer school, where t_1 has normal distribution with parameters $\mu_1 = 30$, $\sigma_1 = 10$. While a kind reviewer needs t_2 minutes to check an application, where t_2 has normal distribution with parameters $\mu_2 = 20$, $\sigma_2 = 5$. For each application the reviewer is randomly selected with 0.5 probability. Given that the time of review $t = 10$, calculate the conditional probability that the application was checked by a kind reviewer.

Solution

Consider we have $t_1 \sim \mathcal{N}(\mu_1 = 30, \sigma_1^2 = 100)$ and $t_2 \sim \mathcal{N}(\mu_2 = 20, \sigma_2^2 = 25)$.

As the time of review was $t = 10$ we need to calculate conditional probability, that the reviewer was pretty kind. Let's call this event F, from *fortunate* ;)

Using the `Bayes' theorem`:

$$P(F|t=10) = \frac{P(t=10|F)P(F)}{P(t=10)}$$

We can get $P(t = 10|F)$ from the distribution of t_2 as it is exactly $p_{t_2}(10)$ and $p(F)$ is simply 0.5. The remaining unknown variable is $P(t = 10)$, but it can be easily calculated according to the the law of total probability.

$$P(t=10) = P(t=10|F)P(F) + P(t=10|\bar{F})p(\bar{F})$$

And now we know everything! With the help from python and `scipy.stats.norm.pdf(x, μ, σ)` we calculate the probability. And it is equal $0.6666666666666667 \approx 0.7$. Not that bad, hah?