TTK4125 - Exercise 2

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Problems

Preliminary

Our Static Parameter Estimator (SPM) is given by: $y = \theta^*\phi(t)$, where θ is an estimate of θ^* which is an unknown constant. We define the parameter estimation error as:

$\tilde{\theta}(t) = \theta(t) - \theta^*$

We then have the estimation error as:

$\varepsilon(t) = y(t) - \theta\phi = -\tilde{\theta}\phi(t)$

The given Lyapunov function is:

$V(t) = \frac{1}{2\gamma}\tilde{\theta}^2$

Which is positive definite, and its derivative is:

$\dot{V}(t) = \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} = -\frac{1}{\gamma}\tilde{\theta}\dot{\theta} = \frac{1}{\gamma}\tilde{\theta}(-\gamma\phi^2\tilde{\theta}) = -\phi^2\tilde{\theta}^2 = -\varepsilon^2 \leq 0$

Which is negative semi-definite. The update law is given as:

$\dot{\theta} = \gamma\phi\varepsilon = -\gamma\phi^2\tilde{\theta}$

1. $\theta \in \mathcal{L}_\infty$

   This claim is the same as the condition: $sup_{t\geq 0}|f(t)| < \infty \iff f \in \mathcal{L}_\infty$ We have a positive definite function $V(t)$ with a semi-negative definite derivative $\dot{V}(t)$, given the update law. This means that since V is non-increasing and so is $\tilde{\theta}$, and since θ^* is constant, then $\theta(t)$ must be non-increasing aswell, implying that it's bounded. implying that $\theta \in \mathcal{L}_\infty$.

2. $\varepsilon \in \mathcal{L}_2$

   Following the argumentation presented in lecture 2, we have:
   $\int_0^\infty |\varepsilon|^2 dt = \int_0^\infty -\dot{V} dt = V(0) - V(t)_\infty < \infty$
   Implying that $\varepsilon \in \mathcal{L}_2$.

3. $\varepsilon \in \mathcal{L}_\infty$ given that $\phi \in \mathcal{L}_\infty$

   Meaning we need $\sup_{t\geq 0|\varepsilon|} < \infty$ which would imply $sup_{t\geq 0}|\tilde{\theta}\phi| < \infty$. We have already shown that $\tilde{\theta} \in \mathcal{L}_\infty$, and since $\phi \in \mathcal{L}_\infty$ is given, then the product of the two must also be bounded, which implies that $\varepsilon \in \mathcal{L}_\infty$.

4. $\varepsilon \leftrightarrow 0$ given $\phi, \dot{\phi} \in \mathcal{L}_\infty$

   By Lemma A.4.7, if $\varepsilon, \dot{\varepsilon} \in \mathcal{L}$, and $\varepsilon \in \mathcal{L}_p$ for some $p \in [1, \infty)$, then

$\varepsilon \to 0$ as $t \to \infty$.
We have already shown that $\varepsilon \in \mathcal{L}_2$, so we need to show that $\varepsilon, \dot{\varepsilon} \in \mathcal{L}_\infty$.
Given that $\phi \in \mathcal{L}_\infty$ we have that $\varepsilon \in \mathcal{L}_\infty$ already.
Now, $sup_{t\geq 0}|\dot{\varepsilon}| \Rightarrow sup_{t\geq 0}|-\dot{\theta}\phi - \tilde{\theta}\dot{\phi}|$
Already, $\theta \in \mathcal{L}_\infty$
And with $\dot{\theta} = \gamma\varepsilon\phi$ and now if $\phi \in \mathcal{L}_\infty$ is given, then $\varepsilon \in \mathcal{L}_\infty$ giving $\dot{\theta} \in \mathcal{L}_\infty$.

Resulting in $\dot{\theta}, \theta \in \mathcal{L}_\infty$ using $\phi, \dot{\phi} \in \mathcal{L}_\infty$ and $\varepsilon, \dot{\varepsilon} \in \mathcal{L}_\infty$ and $\varepsilon \in \mathcal{L}_2$. Now by using "Barbalat's Lemma", we can say that $\varepsilon \to 0$ as $t \to \infty$.

From this we cannot conclude that $\theta \to \theta^*$.
From $\varepsilon = y - \theta\phi$ we see that this only tends to zero if either $\phi \to 0$ or $\tilde{\theta} \to 0$.
This shows that we can reach zero estimation error without zero parameter estimation error. In this case if $\varepsilon = 0, \dot{\theta} = 0$, our θ is not updating, even though it might be wrong. We can see this relation from
$\dot{\tilde{\theta}} = -\gamma\tilde{\theta}\phi^2$
So we need $\phi \neq 0$ for $\dot{\tilde{\theta}}$ to be driving $\tilde{\theta}$ to change.
This is known as persisten excitation.