Trace models of concurrent valuation algebras

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Introduction

- **Challenges in Reasoning:** Concurrent and distributed systems are prevalent but complex to analyse.
- **Compositional Methods:** These methods address the exponential growth of possibilities in interacting systems.
- **Concurrent Valuation Algebras (CVAs):**
  - *Attributes:* Compositional, modular, local, algebraic, lattice-based.
  - *Generalisation:* Generalises models like CSP (Communicating Sequential Processes), CKA (Concurrent Kleene Algebra), CRA (Concurrent Refinement Algebra).
  - *Advantages:* Detects subtle information inconsistencies (RAMiCS23 paper), supports Hoare and rely-guarantee logics.

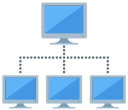
Project Goal:

- ***Developing a Localised Refinement Algebra*** for modular and compositional specifications in concurrent/distributed systems, generalising existing models, and supporting standard program logics.

Topologies and Prealgebras

Encoding Connective Structures:

- **Distributed Systems:** Represented as networks.



- **Concurrent Systems:** Characterised by shared memory and resources.

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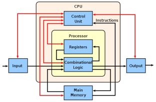


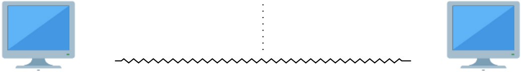
Prealgebra: A poset-(partially ordered set) valued contravariant functor on a topology $\mathscr{T}$ that captures the connective structure of a concurrent/distributed system.

$$\Phi : \mathscr{T}^{op} \to \mathscr{P}os$$

- **Object Map:** Maps each domain $A \in \mathscr{T}$ to a poset Φ_A. ***(Φ_A is the poset of specifications on A.)***
- **Arrow Map:** Maps each inclusion $B \subseteq A$ to a monotone restriction map $a \mapsto a^{\downarrow B} : \Phi_A \to \Phi_B$. ***(Focuses a specification $a \in \Phi_A$ to a subdomain $B \subseteq A$, yielding a specification $a^{\downarrow B}$.)***
- **Transitivity & Identity Law:** For $C \subseteq B \subseteq A$, $(a^{\downarrow B})^{\downarrow C} = a^{\downarrow C}$; for each A and $a \in \Phi_A$, $a^{\downarrow A} = a$.

Prealgebra on a distributed system

$$
\begin{tikzcd}
 & \Phi_{A\cup B} \arrow[dl, "c \mapsto c^{\downarrow A}"'] \arrow[dr, "c \mapsto c^{\downarrow B}"] & \\
\Phi_A \arrow[dr, "a \mapsto a^{\downarrow A\cap B}"'] & & \Phi_B \arrow[dl, "b \mapsto b^{\downarrow A\cap B}"] \\
 & \Phi_{A\cap B} &
\end{tikzcd}
$$



(Covariant) Grothendieck Construction

Combining Posets: Given a prealgebra $\Phi : \mathcal{T}^{op} \to \mathcal{P}os$, the Grothendieck construction combines all posets Φ_A into a single poset $\int \Phi$.

Definition of $\int \Phi$:

$$\int \Phi := \{(A, a) \mid A \in \mathcal{T}, a \in \Phi_A\}$$

Partial 'Grothendieck' Order $\preceq$:

$$(A, a) \preceq (B, b) \iff B \subseteq A \text{ and } a^{\downarrow B} \leq_{\Phi_B} b$$

Shorthand Notation:

- Write a instead of (A, a), with the domain being implicit.
- Use the ***domain*** map d to recover the domain of an element:

$$\mathrm{d} : \int \Phi \to \mathcal{T}^{op}$$
$$\mathrm{d}a = \mathrm{d}(A, a) = A$$

Definition: An ***ordered valuation algebra (OVA)*** comprises:

- A ***prealgebra***:

$$\Phi : \mathcal{T}^{op} \to \mathcal{P}os$$

- A ***binary operator***:

$$\otimes : \int \Phi \times \int \Phi \to \int \Phi$$

- A ***neutral element*** ϵ, comprising an element $\epsilon_A \in \Phi_A$ for each $A \in \mathcal{T}$, such that $\epsilon_A^{\downarrow B} = \epsilon_B$ for all $B \subseteq A$.

OVA Axioms:

- ***Ordered Semigroup:*** $\otimes$ is associative and monotone.
- ***Labelling:***

$$\mathrm{d}(a \otimes b) = \mathrm{d}a \cup \mathrm{d}b$$

- ***Neutrality:***

$$\epsilon_{\mathrm{d}a} \otimes a = a = a \otimes \epsilon_{\mathrm{d}a}$$

- ***Combination:***

$$(a \otimes b)^{\downarrow \mathrm{d}a} = a \otimes b^{\downarrow \mathrm{d}a \cap \mathrm{d}b} \text{ and } (a \otimes b)^{\downarrow \mathrm{d}b} = a^{\downarrow \mathrm{d}a \cap \mathrm{d}b} \otimes b$$

Extension in Ordered Valuation Algebras

Theorem: For an OVA $(\Phi, \otimes, \epsilon)$, for each inclusion $B \subseteq A$ in $\mathcal{T}$, there is a Galois connection: $\forall\ a \in \Phi_A,\ b \in \Phi_B$,

$$a^{\downarrow B} \leq_{\Phi_B} b \iff a \leq_{\Phi_A} \epsilon_{\mathrm{d}a} \otimes b$$

Define the ***extension of*** b ***to*** A as

$$b^{\uparrow A} := \epsilon_{\mathrm{d}a} \otimes b$$

Interpretation: Extension allows lifting a specification from a domain B to a larger domain A by permitting any behaviour outside B.

Key Facts:

- **Functorial Nature of Extension:** If $C \subseteq B \subseteq A$, $(c^{\uparrow B})^{\uparrow A} = c^{\uparrow A}$ and $c^{\uparrow C} = c$.
- **Restriction-Extension Identity:** $(b^{\uparrow A})^{\downarrow B} = b$.
- **Closure Properties:** Extension after restriction is a closure operator, ensuring $a \leq_{\Phi_A} (a^{\downarrow B})^{\uparrow A}$.
- **Lattice Completeness:** If each Φ_A is a complete lattice, $\int \Phi$ is also complete.

Concurrent Valuation Algebras (CVAs)

Definition: A ***concurrent valuation algebra (CVA)*** comprises:

- A ***commutative OVA***:

$$(\Phi, \|, \mathtt{run})$$

- A (usually noncommutative) ***OVA***:

$$(\Phi, \fatsemi, \mathtt{skip})$$

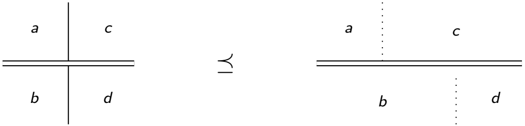
CVA Axioms:

- ***Weak Exchange:***

$$(a \parallel b) \fatsemi (c \parallel d) \preceq (a \fatsemi c) \parallel (b \fatsemi d)$$

- ***Neutral Laws:***

$$\mathtt{skip}_A \preceq \mathtt{skip}_A \parallel \mathtt{skip}_A \text{ and } \mathtt{run}_A \fatsemi \mathtt{run}_A \preceq \mathtt{run}_A$$



Refinement & Reasoning in CVAs

- ***Refinement:***

$$a \preceq b \iff \mathrm{d}b \subseteq \mathrm{d}a \text{ and } a^{\downarrow \mathrm{d}b} \leq_{\Phi_{\mathrm{d}b}} b$$

- ***Hoare Triple (Hoare Logic):***

$$p\ \{a\}\ q := p \fatsemi a \preceq q$$

- ***Jones Quintuple (Rely-Guarantee Logic):***

$$p\ r\ \{a\}\ g\ q := p\ \{a \parallel r\}\ q \text{ and } a \preceq g$$

Example (Concurrency Law): Let $p, p', a, a', q, q' \in \int \Phi$. Then:

$$p\ \{a\}\ q \text{ and } p'\ \left\{a'\right\}\ q' \implies (p \parallel p')\ \left\{a \parallel a'\right\}\ (q \parallel q')$$

Proof: Assume $p \fatsemi a \preceq q$ and $p' \fatsemi a' \preceq q'$. By weak exchange and monotonicity:

$$(p \parallel p') \fatsemi (a \parallel a') \preceq (p \fatsemi a) \parallel (p' \fatsemi a') \preceq q \parallel q'$$

Thus, $(p \parallel p')\ \{a \parallel a'\}\ (q \parallel q')$.

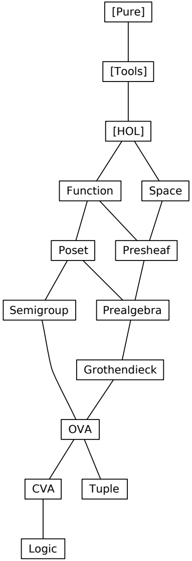
Models of Concurrent Valuation Algebras (CVAs)

- ***Action Trace Model:*** Traces as lists of atomic actions $t = [t_1, \ldots, t_n]$. Sequential composition as concatenation, parallel composition as interleaving.
- ***State Trace Model:*** Traces as nonempty lists of states $t = [t_1, \ldots, t_n]$. Sequential composition as gluing concatenation, parallel composition as synchronisation.
- ***Relative Trace Model:*** 'Stuttering-reduced' nonempty lists of states $t = [t_1, \ldots, t_n]$. Uses gluing concatenation and synchronisation for sequential and parallel compositions, respectively.
- ***CSP (Communicating Sequential Processes) Trace Model*:*** The domains of the topology are CSP alphabets.
- ***CKA (Concurrent Kleene Algebra) Algebraic Model*:*** The underlying topology is a one-point space.
- ***CRA (Concurrent Refinement Algebra) Algebraic Model*:*** Also based on a one-point space topology.

Computer Formalisation of CVAs

Summary:

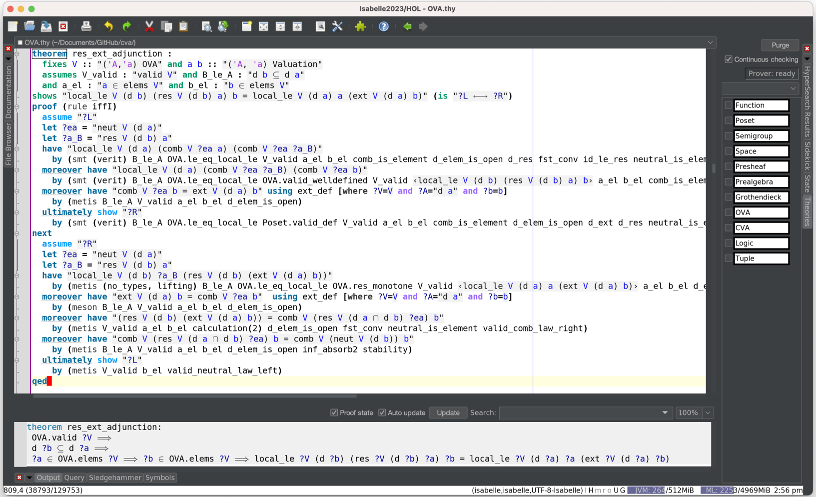
- **Formalisation in Isabelle/HOL:** Sections 2 and 3 of (ICFEM23 paper), covering abstract CVA theory, fully formalised.
- **Extension to Logics:** Includes formalisation of Hoare and Rely-Guarantee logics in CVAs, beyond the scope of the paper.
- **Open Source Collaboration:** The work is available at github.com/nasosev/cva. Collaborators are welcome :)



Theorem 1 Formalisation

Theorem (restriction-extension adjunction). Let $(\Phi, \otimes, \epsilon)$ be an OVA. Then for all $a, b \in \int \Phi$ with $\mathrm{d}b \subseteq \mathrm{d}a$,

$$a^{\downarrow \mathrm{d}b} \leq_{\Phi_{\mathrm{d}b}} b \iff a \leq_{\Phi_{\mathrm{d}a}} b^{\uparrow \mathrm{d}a}$$



Summary & Future Work

Summary:

- Introduced CVAs as a compositional algebra for concurrent/distributed systems.
- Highlighted CVAs' duoidal structure, refinement capabilities, and support for Hoare and rely-guarantee logics.
- Demonstrated multiple concrete models of CVAs relevant to concurrency paradigms.
- Indicated potential applications of local computation theory, with implications for static analysis and model checking.

Future Work: Exploration of further applications and theoretical extensions of CVAs.

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THANK YOU!