Concurrent valuation algebras

PhD thesis review seminar

Naso Evangelou-Oost
Advisors: Ian J. Hayes, Larissa Meinicke

14 July 2023

The University of Queensland
Australia

Introduction

- Concurrent & distributed systems are everywhere, yet really hard to reason about.
- Compositional methods mitigate the combinatorial explosion of possibilities when many interacting systems combine.
- Concurrent valuation algebras (CVAs) are. . .
  - compositional, modular, local,
  - algebraic, lattice-based (refinement algebra),
  - able to detect subtle information inconsistency (e.g. sequential inconsistency),
  - algorithmically tractable.

Overview

Background

- Refinement algebras, trace semantics, topological spaces, category theory, simplicial sets. . .

Contextuality in distributed systems

- RAMiCS23 submission (published) (Evangelou-Oost, Bannister and Hayes 2023).
- Specifications of distributed systems via valuation algebras.
- Lattices of distributed specifications.
- Models program behaviours as ***relative*** execution traces (independent of a global clock).
- Manifests sequential consistency as contextuality.

Trace models of concurrent valuation algebras

- ICFEM23 submission (under review) (Evangelou-Oost, Bannister, Meinicke et al. 2023).
- Reformulates the classical definition of ordered valuation algebras (OVAs); introduces Concurrent Valuation Algebras (CVAs).
- Presents 3 trace models representing different paradigms of concurrent/distributed computing, and their interrelations via morphisms.
- Highlights potential for applying local computation framework to model checking.

Computer formalisation

- Computer formalisation of the CVA paper in the proof assistant Isabelle/HOL.

Refinement algebras

- A ***refinement algebra*** is a partially-ordered set $(\Phi, \preceq)$ of specifications/commands with a commutative monoidal (i.e. unital, associative, order-preserving) binary operation

$$\parallel : \Phi \times \Phi \rightarrow \Phi$$

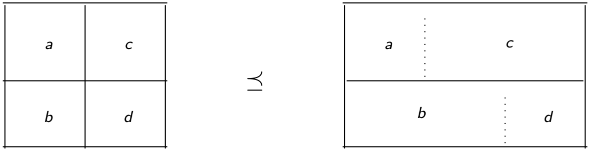
  and a monoidal binary operation

$$\fatsemi : \Phi \times \Phi \rightarrow \Phi,$$

  satisfying a ***weak exchange law.*** for all $a, b, c, d \in \Phi$,

$$(a \parallel b) \fatsemi (c \parallel d) \preceq (a \fatsemi c) \parallel (b \fatsemi d)$$

- $a \preceq b$ means a ***refines*** b; we think of b as the ***abstract specification*** and a as the ***concrete implementation***.
- The weak exchange law abstractly characterises parallel and sequential composition w.r.t. refinement:



- Examples of refinement algebras: Concurrent Kleene Algebra (CKA) (Hoare et al. 2009), Concurrent Refinement Algebra (CRA) (Hayes 2016).

Trace models.

- Specifications $a \in \Phi$ are subsets of ***traces*** $=$ lists of 'observations' $t = [t_1, \ldots, t_n]$.
- Can be coupled to a clock or not. What does the i in t_i mean?
- Decoupling from a global clock $=$ stuttering invariance (traces of states);

$$[1, 1, 2, 3, 3, 3] \sim [1, 2, 2, 3]$$

Two ways to encode stuttering-invariance:

- Stuttering closure (subsets): if the first trace belongs to a specification, so does the second.
- Stuttering reduction (quotients): both traces are identified in a quotient.

Trace models: parallel and sequential products

Parallel & sequential products in trace models depend on what the elements t_i in a trace $t = [t_1, \ldots, t_n]$ mean:

1. ***States:*** t_i ***is a state of the system.***
   - Parallel = sync = $\cap$,
   $$\{[a, b, c]\} \cap \{[a, b, c]\} = \{[a, b, c]\}$$
   - Sequential = gluing concatenation = $\smile$.
   $$\{[a, b, c]\} \smile \{[c, d]\} = \{[a, b, c, d]\}$$
2. ***Actions.*** t_i ***is an action taken by the system.***
   - Parallel = shuffle = $\sqcup\!\sqcup$,
   $$\{[a, b, c]\} \sqcup\!\sqcup \{[d, e]\} = \{[a, b, d, e, c], [d, a, b, c, e], [a, b, d, c, e], \ldots\}$$
   - Sequential = concatenation = $\frown$,
   $$\{[a, b, c]\} \frown \{[d, e]\} = \{[a, b, c, d, e]\}$$

Project goal: localising refinement algebras

Problem area:

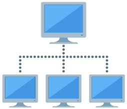
- Using a refinement algebra Φ to model a distributed system means commands $a \in \Phi$ must specify the behaviour over the ***entire state space of the system***.
- If the system is very large, this may be impractical. Moreover, in a modular design process, the overall state space may be unknown.
- For both verification and model checking, decomposing the system into smaller components is imperative for scalability.

Our project's overall goal:

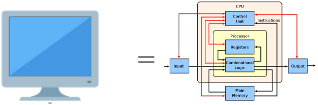
- To identify a useful, and tractable concept of a ***localised refinement algebra*** where specifications need only refer to the relevant subset of the state space, that is suitable for modular, compositional concurrent/distributed systems specifications, and is also computationally tractable.

Topologies encode the connective structure of distributed and concurrent systems:

- Distributed systems: networks.



- Concurrent systems: shared memory and resources.



A ***prealgebra*** is a poset-valued contravariant functor on a topology $\mathscr{T}$,

$$\Phi : \mathscr{T}^{op} \to \mathscr{P}os$$

This consists of the following data:

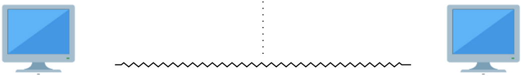
- ***Object map.*** A function sending each domain $A \in \mathscr{T}$ to a poset Φ_A (e.g. of traces, or specifications).
- ***Arrow map.*** A function sending each inclusion $B \subseteq A$ to a monotone ***restriction map***

$$a \mapsto a^{\downarrow B} : \Phi_A \to \Phi_B$$

- ***Transitivity.*** For $C \subseteq B \subseteq A$ and $a \in \Phi_A$, $(a^{\downarrow B})^{\downarrow C} = a^{\downarrow C}$.
- ***Identity law.*** For each $A \in \mathscr{T}$ and $a \in \Phi_A$, $a^{\downarrow A} = a$.

Prealgebra on a distributed system

$$\begin{tikzcd}
& \Phi_{A\cup B} \arrow[dl, "c\mapsto c^{\downarrow A}"'] \arrow[dr, "c\mapsto c^{\downarrow B}"] & \\
\Phi_A \arrow[dr, "a\mapsto a^{\downarrow A\cap B}"'] & & \Phi_B \arrow[dl, "b\mapsto b^{\downarrow A\cap B}"] \\
& \Phi_{A\cap B} &
\end{tikzcd}$$



Summary of RAMiCS23 paper (Evangelou-Oost, Bannister and Hayes 2023).

- Shows how an established concept: ***ordered valuation algebras (OVAs)*** can be used to model distributed systems.
- An OVA is a prealgebra $\Phi : \mathscr{T}^{op} \to \mathscr{P}os$ with an associative binary operation
$$\otimes : \coprod_{A \in \mathscr{T}} \Phi_A \times \coprod_{A \in \mathscr{T}} \Phi_A \to \coprod_{A \in \mathscr{T}} \Phi_A$$
obeying certain axioms.
- Interprets $\otimes$ as synchronisation of valuations defined as subsets of ***relative traces***.
- Defines ***(distributed) specifications*** as subsets of valuations defined over a ***cover*** (i.e. maximal antichain) of $\mathscr{T}$, and shows these form a complete lattice.
- ***Contextuality*** is an abstract form of information inconsistency where a collection of information ***locally agrees but globally disagrees***, that is realisable in a valuation algebra. We show that in this context, contextuality corresponds to ***sequential inconsistency***.
- Gives an example of contextuality in the classic ***dining philosophers*** scenario.

Trace models of concurrent valuation algebras

Summary of ICFEM23 submission (Evangelou-Oost, Bannister, Meinicke et al. 2023).

- Reformulates an OVA in terms of categorical notions, relaxing the commutativity condition, imposing a stability/naturality condition for neutral elements, and imposing a global (Grothendieck) ordering.
- Introduces a ***concurrent valuation algebra (CVA)*** comprising two OVA structures, a weak exchange law, and relations between neutral elements, as a candidate for a ***localised refinement algebra***.
- Presents three trace models of CVAs:
  1. Event trace model with shuffle as parallel and concatenation as parallel.
  2. State trace model with synchronisation as parallel and gluing concatenation as sequential.
  3. Relative state trace model where traces are ***stuttering-reduced***.
- Defines morphisms of CVAs, and shows a surjective morphism from the state model to the relative state model, effectively exhibiting the relative trace model as a quotient of the state model.
- Suggests how to import an efficient ***local computation*** framework for CVAs.

(covariant) Grothendieck construction

Given $\Phi : \mathcal{T}^{op} \to \mathscr{P}os$, combine all the posets Φ_A into one big poset $\int \Phi$.

$$\int \Phi := \{(A, a) \mid A \in \mathcal{T}, a \in \Phi_A\}$$

Partial order $\preceq$:

$$(A, a) \preceq (B, b) \iff B \subseteq A \text{ and } a^{\downarrow B} \leq_{\Phi_B} b$$

Shorthand: write a instead of (A, a) leaving domain implicit; use ***domain*** map d to recover domain

$$\mathrm{d} : \int \Phi \to \mathcal{T}^{op}$$

$$\mathrm{d}a = \text{domain of } a$$

Example.

If $A = \{x, y\}$, $B = \{x\}$, and

$$a = \{[(x_0, y_0), (x_1, y_1)], [(x_0, y_2), (x_1, y_3)]\}, \qquad b = \{[x_0, x_1], [x_2, x_3]\}$$

then,

$$B \subseteq A \text{ and } a^{\downarrow B} = \{[x_0, x_1]\} \subseteq b$$

so, $(A, a) \preceq (B, b)$.

Ordered valuation algebras (OVAs)

Definition. An ***ordered valuation algebra (OVA)*** consists of...

- a prealgebra
$$\Phi : \mathcal{T}^{op} \to \mathcal{P}os$$
- a binary operator
$$\otimes : \textstyle\int \Phi \times \int \Phi \to \int \Phi$$
- a natural transformation
$$\epsilon : 1 \Rightarrow \Phi$$
(i.e. an element $\epsilon_A \in \Phi_A$ for each A such that $\epsilon_A^{\downarrow B} = \epsilon_B$ whenever $B \subseteq A$.)

Obeying the following axioms...

- ***Ordered semigroup.***
$$\otimes \text{ is associative and monotone}$$
- ***Labelling.***
$$\mathrm{d}(a \otimes b) = \mathrm{d}a \cup \mathrm{d}b$$
- ***Neutrality.***
$$\epsilon_{\mathrm{d}a} \otimes a = a = a \otimes \epsilon_{\mathrm{d}a}$$
- ***Combination.***
$$(a \otimes b)^{\downarrow \mathrm{d}a} = a \otimes b^{\downarrow \mathrm{d}a \cap \mathrm{d}b} \text{ and } (a \otimes b)^{\downarrow \mathrm{d}b} = a^{\downarrow \mathrm{d}a \cap \mathrm{d}b} \otimes b$$

Extension

Theorem. Let $(\Phi, \otimes, \epsilon)$ be an OVA. For each inclusion $B \subseteq A$ in $\mathcal{T}$, there is an adjunction/Galois connection

$$a^{\downarrow B} \leq_{\Phi_B} b \iff a \leq_{\Phi_A} \epsilon_a \otimes b$$

We call $b \mapsto \epsilon_a \otimes b$ ***extension of*** b ***to*** A and write $b^{\uparrow A} := \epsilon_a \otimes b$.

Extension takes a specification defined on a domain B ***and lifts it to a domain*** $A \supseteq B$ ***by allowing any behaviour outside*** B***.***

Some facts:

- Extension is a functor: if $C \subseteq B \subseteq A$ and $c \in \Phi_C$, then

$$(c^{\uparrow B})^{\uparrow A} = c^{\uparrow A} \text{ and } c^{\uparrow C} = c$$

- If Φ has the ***strongly neutral property:*** $\forall A, B \in \mathcal{T}$ ***we have*** $\epsilon_A \otimes \epsilon_B = \epsilon_{A \cup B}$, then for all $B \subseteq A$,

$$\epsilon_B^{\uparrow A} = \epsilon_A$$

- Restriction after extension is the identity map, i.e., for $B \subseteq A$ and $b \in \Phi_B$,

$$(b^{\uparrow A})^{\downarrow B} = b$$

- Extension after restriction is a ***closure operator***, i.e., for $B \subseteq A$ and $a \in \Phi_A$,

$$a \leq_{\Phi_A} (a^{\downarrow B})^{\uparrow A} \qquad \text{and} \qquad (((a^{\downarrow B})^{\uparrow A})^{\downarrow B})^{\uparrow A} = (a^{\downarrow B})^{\uparrow A}$$

- If for each $A \in \mathcal{A}$, Φ_A is a complete lattice, then so is $\int \Phi$.

Concurrent valuation algebras (CVAs)

Definition. A ***concurrent valuation algebra (CVA)*** consists of...

- A commutative OVA
$$(\Phi, \|, \delta)$$
- A (generally noncommutative) OVA
$$(\Phi, \fatsemi, \epsilon)$$

Obeying the following axioms...

- ***Weak exchange.***
$$(a \parallel b) \fatsemi (c \parallel d) \preceq (a \fatsemi c) \parallel (b \fatsemi d)$$
- ***Neutral laws.***
$$\epsilon_A \preceq \epsilon_A \parallel \epsilon_A \text{ and } \delta_A \fatsemi \delta_A \preceq \delta_A \quad {}^1$$

[1] These are actually equalities.

- ***Refinement.***

$$a \preceq b \iff \mathrm{d}b \subseteq \mathrm{d}a \text{ and } a^{\downarrow \mathrm{d}b} \leq_{\Phi_{\mathrm{d}b}} b$$

- ***Hoare triple (Hoare logic).***

$$p\ \{a\}\ q := p \mathbin{;} a \preceq q$$

- ***Jones quintuple (rely-guarantee logic).***

$$p\ r\ \{a\}\ g\ q := p\ \{a \parallel r\}\ q \text{ and } a \preceq g$$

Example (concurrency law).
Let $p, p', a, a', q, q' \in \int \Phi$. Then

$$p\ \{a\}\ q \text{ and } p'\ \left\{a'\right\}\ q' \implies (p \parallel p')\ \left\{a \parallel a'\right\}\ (q \parallel q')$$

Proof. Assume $p \mathbin{;} a \preceq q$ and $p' \mathbin{;} a' \preceq q'$. By weak exchange and monotonicity,

$$(p \parallel p') \mathbin{;} (a \parallel a') \preceq (p \mathbin{;} a) \parallel (p' \mathbin{;} a') \preceq q \parallel q'$$

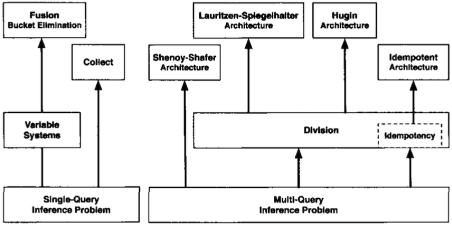
Thus, $(p \parallel p')\ \{a \parallel a'\}\ (q \parallel q')$.

Local computation

Efficient computation is enabled by the ***combination*** law:

$$(a \otimes b)^{\downarrow \mathrm{d}a} = a \otimes b^{\downarrow \mathrm{d}a \cap \mathrm{d}b}$$
$$(a \otimes b)^{\downarrow \mathrm{d}b} = a^{\downarrow \mathrm{d}a \cap \mathrm{d}b} \otimes b$$

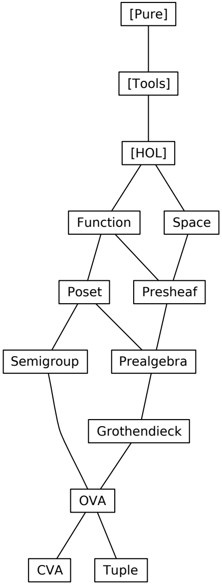
The LHS is usually ***expensive*** to compute, but the RHS can be ***much cheaper*** if $\mathrm{d}a \cap \mathrm{d}b$ is small.



- Local computation framework specialises to standard optimal algorithms in databases, probabilistic graphical models, constraint satisfaction problems, dynamic programming, . . .
- We can already use these algorithms in a CVA on just parallel combinations.
- Need to generalise to noncommutative case & with 2 combine operators.
- Theoretically feasible from the generalised combination rule, but not yet implemented.

Summary.

- Section 2 of (Evangelou-Oost, Bannister, Meinicke et al. 2023), that comprises the abstract theory of CVAs, has been fully formalised in Isabelle/HOL.
- Formalisation of the remaining sections is underway.
- Available online at github.com/nasosev/cva. Collaborators welcome :)
- Acknowledgements: Thank You to the local Isabelle experts for your support: Brae Webb, Callum Bannister, Kait Lam, Scott Heiner ♡



Formalisation strategy

Light embedding of mathematical structures in Isabelle/HOL's simple type system (i.e. without dependent types) using records and validity predicates.

```
record ('x, 'y) Function =
  cod :: "'y set"
  func :: "('x × 'y) set"

definition dom :: "('x, 'y) Function ⇒ 'x set" where
 "dom f ≡ {x. ∃y. (x, y) ∈ func f}"

definition valid_map :: "('x, 'y) Function ⇒ bool" where
"valid_map f ≡
  let
      welldefined = ∀x y. (x, y) ∈ func f ⟶ y ∈ cod f;
      deterministic = ∀x y y'. (x, y) ∈ func f ∧ (x, y') ∈ func f ⟶ y = y'
  in welldefined ∧ deterministic"
```

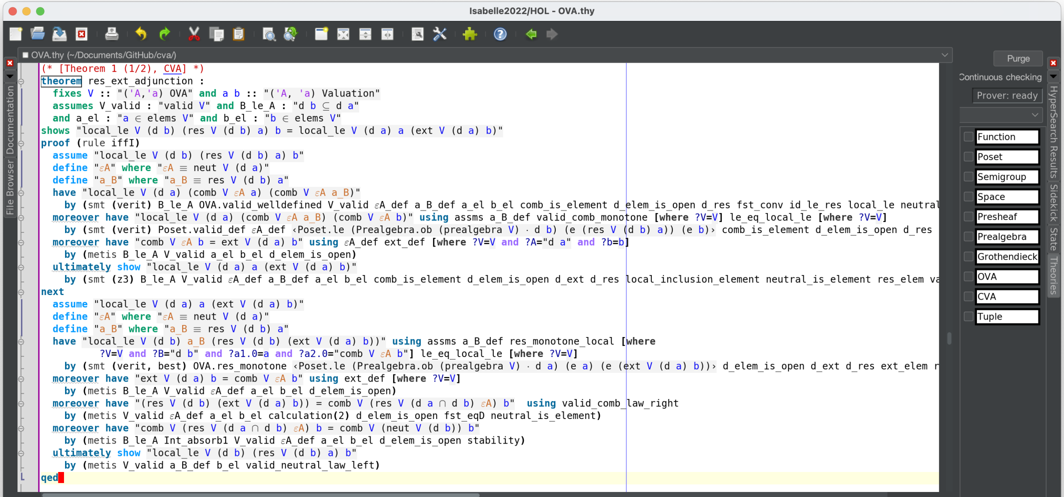
Some other type definitions (corresponding validity predicates not shown) . . .

```
record 'a Semigroup =
    mult :: "('a \<times> 'a,'a) PosetMap"

record ('A, 'a) Prealgebra =
    space :: "'A Space"
    ob :: "('A Open, 'a Poset) Function "
    ar :: "('A Inclusion, ('a, 'a) PosetMap) Function"

record ('A, 'a, 'b) PrealgebraMap =
    nat :: "('A Open, ('a, 'b) PosetMap) Function"
    dom :: "('A, 'a) Prealgebra"
    cod :: "('A, 'b) Prealgebra"

record ('A, 'a) OVA =
    prealgebra :: "('A, 'a) Prealgebra"
    neutral    :: "('A, unit, 'a) PrealgebraMap"
    semigroup  :: "(('A, 'a) Valuation) Semigroup"

record ('A, 'a) CVA =
    par_algebra :: "('A, 'a) OVA"
    seq_algebra :: "('A, 'a) OVA"
```

Theorem 1 formalisation

Theorem (restriction-extension adjunction). Let $(\Phi, \otimes, \epsilon)$ be an OVA. Then for all $a, b \in \int \Phi$ with $\mathrm{d}b \subseteq \mathrm{d}a$,

$$a^{\downarrow \mathrm{d}b} \leq_{\Phi_{\mathrm{d}b}} b \iff a \leq_{\Phi_{\mathrm{d}a}} b^{\uparrow \mathrm{d}a}$$



1. Background
   - Refinement algebras
   - Trace models of refinement algebras
   - Project goal: localising refinement algebras
   - Topologies and presheaves
2. Contextuality in distributed systems
3. Trace models of concurrent valuation algebras
   - (covariant) Grothendieck construction
   - Ordered valuation algebras (OVAs)
   - Concurrent valuation algebras (CVAs), refinement & reasoning
   - Local computation
4. Computer formalisation
5. **Summary & plan for the rest of the project**

Summary & plan for the rest of the project

Summary.

- We reformulated the notion of an OVA, and showed how these can model concurrent/distributed systems, and manifest sequential inconsistency as contextuality.
- We introduced CVAs as candidates for localised refinement algebras, modular/compositional structures for reasoning about concurrent/distributed systems.
- CVAs support classical refinement reasoning methodologies such as Hoare and rely-guarantee logics.
- We described 3 example trace models of CVAs and related them by morphisms.
- We indicated a path to efficient algorithms for inference, static analysis, model checking by extending the classical local computation framework of valuation algebras to CVAs.
- We have a computer formalisation underway in Isabelle/HOL.

Plan.

- Complete the formalisation of CVAs in Isabelle/HOL.
- More models: Aczel trace models with alternative extension operators, graph models, simplicial model, sheaf based models that better capture local properties. . .
- Better understand the implications of nontrivial topologies.
- Generalise the trace model constructions with multioperations.
- Categorical structure of CVAs; connection to monoidal Grothendieck fibrations.
- Push-forward/pullback of OVAs/CVAs on different spaces.

References

- Evangelou-Oost, Nasos, Callum Bannister and Ian J. Hayes (2023). 'Contextuality in Distributed Systems'. In: ***Relational and Algebraic Methods in Computer Science***. Springer International Publishing, pp. 52–68.
- Evangelou-Oost, Nasos, Callum Bannister, Larissa Meinicke et al. (2023). ***Trace models of concurrent valuation algebras***. arXiv: 2305.18017 [cs.LO].
- Hayes, Ian J. (2016). 'Generalised rely-guarantee concurrency: an algebraic foundation'. In: ***Formal Aspects Comput.*** 28.6, pp. 1057–1078. DOI: 10.1007/s00165-016-0384-0. URL: https://doi.org/10.1007/s00165-016-0384-0.
- Hoare, C. A. R. et al. (2009). 'Concurrent Kleene Algebra'. In: ***CONCUR 2009 - Concurrency Theory, 20th International Conference, CONCUR 2009, Bologna, Italy, September 1-4, 2009. Proceedings***. Ed. by Mario Bravetti and Gianluigi Zavattaro. Vol. 5710. Lecture Notes in Computer Science. Springer, pp. 399–414. DOI: 10.1007/978-3-642-04081-8_27. URL: https://doi.org/10.1007/978-3-642-04081-8%5C_27.
- Pouly, Marc and Jürg Kohlas (2012). ***Generic inference: a unifying theory for automated reasoning***. John Wiley & Sons.

Contact: nasoev@proton.me, nasosev.github.io

THANK YOU!