Čech cohomology of a cover

MATH6006: Algebraic topology

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The Mayer-Vietoris sequence

Let $\mathcal{U} := \{U, V\}$ be an open cover of a smooth manifold M. The sequence of inclusions

$$M \xleftarrow{j_U \amalg j_V} U \coprod V \underset{i_V}{\overset{i_U}{\leftleftarrows}} U \cap V$$

induces an exact sequence of chain complexes, the ***Mayer-Vietoris sequence***

$$0 \longrightarrow \Omega^\bullet(M) \xrightarrow{r=(j_U^*, j_V^*)} \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{\delta = i_V^* - i_U^*} \Omega^\bullet(U \cap V) \longrightarrow 0$$

and a long exact sequence in cohomology

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(M) \arrow[r, "r^*"] & H^q(U) \oplus H^q(V) \arrow[r, "\delta^*"] & H^q(U \cap V) \arrow[dll, "d^*"] \\
 & H^{q+1}(M) \arrow[r, "r^*"] & H^{q+1}(U) \oplus H^{q+1}(V) \arrow[r, "\delta^*"] & H^{q+1}(U \cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Reformulating the Mayer-Vietoris sequence

Let

$$\begin{aligned} C^0(\mathcal{U},\Omega^q) &:= \Omega^q(U)\oplus\Omega^q(V), \\ C^1(\mathcal{U},\Omega^q) &:= \Omega^q(U\cap V), \\ C^p(\mathcal{U},\Omega^q) &:= 0,\ p\geqslant 2. \end{aligned}$$

With the anti-commuting differential operators

$$\begin{aligned} (-1)^\bullet d : C^\bullet(\mathcal{U},\Omega^\bullet) &\to C^\bullet(\mathcal{U},\Omega^{\bullet+1}), && \text{(exterior derivative)}, \\ \delta : C^\bullet(\mathcal{U},\Omega^\bullet) &\to C^{\bullet+1}(\mathcal{U},\Omega^\bullet), && \text{(difference)}. \end{aligned}$$

$(C^\bullet(\mathcal{U},\Omega^\bullet), d, \delta)$ is a ***double complex***.

Reformulating the Mayer-Vietoris sequence (ii)

$$
\begin{aligned}
C^0(\mathcal{U}, \Omega^q) &:= \Omega^q(U) \oplus \Omega^q(V),\\
C^1(\mathcal{U}, \Omega^q) &:= \Omega^q(U \cap V),\\
C^p(\mathcal{U}, \Omega^q) &:= 0,\, p \geqslant 2.
\end{aligned}
$$

The complex $(C^\bullet(\mathcal{U}, \Omega^\bullet), d, \delta)$:

$$
\begin{tikzcd}
\vdots & \vdots & \vdots \\
\Omega^2(U) \oplus \Omega^2(V) \arrow[u, "d"] \arrow[r, "\delta"] & \Omega^2(U \cap V) \arrow[u, "-d"] \arrow[r] & 0 \arrow[u] \\
\Omega^1(U) \oplus \Omega^1(V) \arrow[u, "d"] \arrow[r, "\delta"] & \Omega^1(U \cap V) \arrow[u, "-d"] \arrow[r] & 0 \arrow[u] \\
\Omega^0(U) \oplus \Omega^0(V) \arrow[u, "d"] \arrow[r, "\delta"] & \Omega^0(U \cap V) \arrow[u, "-d"] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

Reformulating the Mayer-Vietoris sequence (iii)

Let

$$T^n(C) := \bigoplus_{p+q=n} C^p(\mathcal{U}, \Omega^q),$$
$$D := (-1)^p d + \delta.$$

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots \\
T^2(C) & \Omega^2(U) \oplus \Omega^2(V) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^2(U \cap V) \arrow[r] \arrow[u, "-d"] & 0 \arrow[u] \\
T^1(C) & \Omega^1(U) \oplus \Omega^1(V) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^1(U \cap V) \arrow[r] \arrow[u, "-d"] & 0 \arrow[u] \\
T^0(C) & \Omega^0(U) \oplus \Omega^0(V) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^0(U \cap V) \arrow[r] \arrow[u, "-d"] & 0 \arrow[u]
\end{tikzcd}$$

Proposition

$(T^\bullet(C), D)$ is a complex called the ***total complex*** of $(C^\bullet(\mathcal{U}, \Omega^\bullet), d, \delta)$:

Proof.

$D^2 = d^2 + \delta d - d\delta + \delta^2 = 0.$ $\square$

Reformulating the Mayer-Vietoris sequence (iv)

Theorem

$T^\bullet(C)$ computes the de Rham cohomology of M:

$$H^\bullet(T(C)) \simeq H^\bullet_{\mathrm{dR}}(M).$$

Proof sketch.

Step 1. Note the natural map

$$r : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(U) \oplus \Omega^\bullet(V) \subset T^\bullet(C)$$

is a ***chain map***, i.e. the following diagram commutes:

$$\begin{tikzcd} \Omega^{\bullet+1}(M) \arrow[r, "r"] & T^{\bullet+1} \\ \Omega^\bullet(M) \arrow[u, "d"] \arrow[r, "r"] & T^\bullet \arrow[u, "D"'] \end{tikzcd}$$

(Check: $Dr = (\delta + (-1)^p d)r = dr = rd$.) Thus r induces a map in cohomology

$$r^* : H^\bullet_{\mathrm{dR}}(M) \longrightarrow H^\bullet(T(C)).$$

Step 2. A quick diagram chase then shows r^* is bijective. $\square$

Let

- $\mathcal{I}$ be a countable ordered set,
- $\mathcal{U} := \{U_i\}_{i\in\mathcal{I}}$ an open cover of M,
- $U_{a_0\dots a_p} := \bigcap_{i=0}^{p} U_{a_i}$.

The sequence of inclusions $\partial_i : U_{a_0\dots a_p} \hookrightarrow U_{a_0\dots\hat{a}_i\dots a_p}$ (omits index a_i),

$$M \longleftarrow \coprod_{a_0} U_{a_0} \;\underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}}\; \coprod_{a_0<a_1} U_{a_0 a_1} \;\overset{\partial_0}{\underset{\partial_2}{\overset{\partial_1}{\Lleftarrow}}}\; \coprod_{a_0<a_1<a_2} U_{a_0 a_1 a_2} \;\Lleftarrow\; \cdots$$

by functoriality of $\Omega^\bullet$, induces a sequence of restrictions of forms

$$\Omega^\bullet(M) \longrightarrow \prod_{a_0} \Omega^\bullet(U_{a_0}) \;\underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}}\; \prod_{a_0<a_1} \Omega^\bullet(U_{a_0 a_1}) \;\overset{\delta_0}{\underset{\delta_2}{\overset{\delta_1}{\Rrightarrow}}}\; \prod_{a_0<a_1<a_2} \Omega^\bullet(U_{a_0 a_1 a_2}) \;\Rrightarrow\; \cdots$$

Generalising to a countable cover (ii)

Define

$$\delta : \prod \Omega^\bullet(U_{a_0 \dots a_{p-1}}) \longrightarrow \prod \Omega^\bullet(U_{a_0 \dots a_p})$$
$$\delta := \delta_0 - \delta_1 + \delta_2 - \cdots \delta_p$$

Proposition

The ***generalised Mayer-Vietoris sequence***

$$0 \longrightarrow \Omega^\bullet(M) \xrightarrow{r} \prod \Omega^\bullet(U_{a_0}) \xrightarrow{\delta} \prod \Omega^\bullet(U_{a_0 a_1}) \xrightarrow{\delta} \cdots$$

is a complex, i.e. $\delta^2 = 0$.

Generalising to a countable cover (iii)

Proof.

Denote

$$\omega_{a_0\dots a_p} := \omega|_{U_{a_0\dots a_p}}$$

and let

$$\omega \in \prod_{a_0<\dots<a_p} \Omega^\bullet(U_{a_0\dots a_p}).$$

Then

$$(\delta\omega)_{a_0\dots a_p} := \sum_{i=0}^{p} (-1)^i \omega_{a_0\dots\hat{a}_i\dots a_p}|_{U_{a_0\dots a_p}},$$

and

$$\begin{aligned}
(\delta^2\omega)_{a_0\dots a_{p+2}} &= \sum (-1)^j (\delta\omega)_{a_0\dots\hat{a}_j\dots a_{p+2}} \\
&= \sum_{j<i} (-1)^i (-1)^j \omega_{a_0\dots\hat{a}_j\dots\hat{a}_i\dots a_{p+2}} + \sum_{i<j} (-1)^i (-1)^{j-1} \omega_{a_0\dots\hat{a}_i\dots\hat{a}_j\dots a_{p+2}} \\
&= 0.
\end{aligned}$$

□

The generalised Mayer-Vietoris sequence

Proposition

The generalised Mayer-Vietoris sequence is exact:

$$0 \longrightarrow \Omega^\bullet(M) \xrightarrow{r} \prod \Omega^\bullet(U_{a_0}) \xrightarrow{\delta} \prod \Omega^\bullet(U_{a_0 a_1}) \xrightarrow{\delta} \cdots$$

Proof.

Step 1. Exactness at the start of the chain:
Clearly $\Omega^\bullet(M) = \text{global forms} = \ker \delta^0$ since a cochain in $\prod \Omega^\bullet(U_{a_0})$ is a global form $\iff$ it agrees on the overlaps.

Step 2. Exactness for the rest of the chain:
We show there exists a ***homotopy operator*** h on the chain so

$$\delta h + h\delta = 1$$

holds, which clearly implies the vanishing of cohomology.

The generalised Mayer-Vietoris sequence (ii)

Proof.

Let

- $\{\varrho_a\}$ be a partition of unity subordinate to $\mathcal{U} := \{U_a\}$,
- $\omega \in \prod \Omega^\bullet(U_{a_0 \dots a_p})$,
- $$h : \prod \Omega^\bullet(U_{a_0 \dots a_p}) \longrightarrow \prod \Omega^\bullet(U_{a_0 \dots a_{p-1}}),$$
  $$(h\omega)_{a_0 \dots a_{p-1}} := \sum_a \varrho_a \omega_{a a_0 \dots a_{p-1}},$$

Then

$$\begin{aligned}
(h\delta\omega)_{a_0 \dots a_p} &= \sum_a \varrho_a (\delta\omega)_{a a_0 \dots a_p} = \sum_{i,a} (-1)^i \varrho_a (\delta\omega)_{a a_0 \dots \hat{a}_i \dots a_p} \\
(\delta h\omega)_{a_0 \dots a_p} &= \sum_i (-1)^i (h\omega)_{a_0 \dots \hat{a}_i \dots a_p} = \sum_a \varrho_a \omega_{a_0 \dots a_p} + \sum_{a,i} (-1)^{i+1} \varrho_a (\delta\omega)_{a a_0 \dots \hat{a}_i \dots a_p} \\
&= \omega_{a_0 \dots a_p} - (h\delta\omega)_{a_0 \dots a_p},
\end{aligned}$$

thus

$$\delta h + h\delta = 1.$$

□

The generalised Mayer-Vietoris principle

Denote

$$C^p(\mathcal{U}, \Omega^q) := \prod \Omega^q(U_{a_0\dots a_p})$$

(The p-cochains of the cover $\mathcal{U}$ with values in the q-forms.)
With differentials d and δ, the double complex formed called the ***Čech-de Rham complex***

$$(C^\bullet(\mathcal{U}, \Omega^\bullet), d, \delta).$$

Attaching $\ker \delta^0 = \Omega^\bullet(M)$ as a column on the left via the chain map r gives an example of an ***augmented double complex***

$$\Omega^\bullet(M) \xrightarrow{r} C^\bullet(\mathcal{U}, \Omega^\bullet).$$

$$\begin{tikzcd}
& \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Omega^2(M) \arrow[u, "d"] \arrow[r, "r"] & C^0(\mathcal{U}, \Omega^2) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathcal{U}, \Omega^2) \arrow[u, "-d"] \arrow[r, "\delta"] & C^2(\mathcal{U}, \Omega^2) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & \Omega^1(M) \arrow[u, "d"] \arrow[r, "r"] & C^0(\mathcal{U}, \Omega^1) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathcal{U}, \Omega^1) \arrow[u, "-d"] \arrow[r, "\delta"] & C^2(\mathcal{U}, \Omega^1) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & \Omega^0(M) \arrow[u, "d"] \arrow[r, "r"] & C^0(\mathcal{U}, \Omega^0) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathcal{U}, \Omega^0) \arrow[u, "-d"] \arrow[r, "\delta"] & C^2(\mathcal{U}, \Omega^0) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

The exactness of the generalised Mayer-Vietoris sequence means this augmented double complex has exact rows.

The generalised Mayer-Vietoris principle (ii)

As before, let

$$T^n(C) := \bigoplus_{p+q=n} C^p(\mathcal{U}, \Omega^q),$$
$$D := (-1)^p d + \delta.$$

and form the total complex $(T^\bullet(C), D)$.

Lemma

If all the rows of an augmented double complex

$$0 \longrightarrow (A^\bullet, d) \xrightarrow{r} (B^\bullet, d, \delta)$$

are exact, then the cohomology of the first column is isomorphic to the cohomology of the total complex:

$$H^\bullet(A) \simeq H^\bullet(T(B)).$$

Proof.

As r is a chain map it induces a map r^* in cohomology.
Step 1. Show r^* is surjective.
Step 2. Show r^* is injective. $\square$

The generalised Mayer-Vietoris principle (iii)

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & A^2 \arrow[r, "r"] \arrow[u, "d"] & B^{0,2} \arrow[r, "\delta"] \arrow[u, "d"] & B^{1,2} \arrow[r, "\delta"] \arrow[u, "-d"] & B^{2,2} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & A^1 \arrow[r, "r"] \arrow[u, "d"] & B^{0,1} \arrow[r, "\delta"] \arrow[u, "d"] & B^{1,1} \arrow[r, "\delta"] \arrow[u, "-d"] & B^{2,1} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & A^0 \arrow[r, "r"] \arrow[u, "d"] & B^{0,0} \arrow[r, "\delta"] \arrow[u, "d"] & B^{1,0} \arrow[r, "\delta"] \arrow[u, "-d"] & B^{2,0} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots
\end{tikzcd}
$$

$$T^n(B) = \bigoplus_{n=p+q} B^{p,q}$$

$$D = (-1)^p d + \delta$$

The generalised Mayer-Vietoris principle (iv)

Corollary

The restriction map

$$r : \Omega^\bullet(M) \longrightarrow C^0(\mathcal{U}, \Omega^\bullet)$$

induces an isomorphism in cohomology

$$r^* : H^\bullet_{\mathrm{dR}}(M) \xrightarrow{\simeq} H^\bullet(T(C)).$$

The Čech cohomology of a cover

Recall the augmented Čech-de Rham complex:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Omega^2(M) \arrow[u, "d"] \arrow[r, "r"] & C^0(\mathcal{U},\Omega^2) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathcal{U},\Omega^2) \arrow[u, "-d"] \arrow[r, "\delta"] & C^2(\mathcal{U},\Omega^2) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & \Omega^1(M) \arrow[u, "d"] \arrow[r, "r"] & C^0(\mathcal{U},\Omega^1) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathcal{U},\Omega^1) \arrow[u, "-d"] \arrow[r, "\delta"] & C^2(\mathcal{U},\Omega^1) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & \Omega^0(M) \arrow[u, "d"] \arrow[r, "r"] & C^0(\mathcal{U},\Omega^0) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathcal{U},\Omega^0) \arrow[u, "-d"] \arrow[r, "\delta"] & C^2(\mathcal{U},\Omega^0) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

The Čech cohomology of a cover (ii)

We can also augment each column by the kernel of the bottom d via the inclusion

$$i : C^\bullet(\mathcal{U}, \mathbb{R}) := \ker d^0 \longrightarrow C^0(\mathcal{U}, \Omega^0)$$

Each $C^p(\mathcal{U}, \mathbb{R})$ is the space of ***locally constant functions*** on the p-fold intersections.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Omega^2(M) \arrow[u, "d"] \arrow[r, "r"] & C^0(\mathcal{U}, \Omega^2) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathcal{U}, \Omega^2) \arrow[u, "-d"] \arrow[r, "\delta"] & C^2(\mathcal{U}, \Omega^2) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & \Omega^1(M) \arrow[u, "d"] \arrow[r, "r"] & C^0(\mathcal{U}, \Omega^1) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathcal{U}, \Omega^1) \arrow[u, "-d"] \arrow[r, "\delta"] & C^2(\mathcal{U}, \Omega^1) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & \Omega^0(M) \arrow[u, "d"] \arrow[r, "r"] & C^0(\mathcal{U}, \Omega^0) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\mathcal{U}, \Omega^0) \arrow[u, "-d"] \arrow[r, "\delta"] & C^2(\mathcal{U}, \Omega^0) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
 & & C^0(\mathcal{U}, \mathbb{R}) \arrow[u, "i"] \arrow[r, "\delta"] & C^1(\mathcal{U}, \mathbb{R}) \arrow[u, "i"] \arrow[r, "\delta"] & C^2(\mathcal{U}, \mathbb{R}) \arrow[u, "i"] \arrow[r, "\delta"] & \cdots \\
 & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The bottom row is called the ***Čech complex of the cover*** $\mathcal{U}$, and its cohomology $H^\bullet(\mathcal{U}, \mathbb{R})$ is the ***Čech cohomology of*** $\mathcal{U}$.

De Rham's theorem

- Note: taking the total complex is invariant under the transpose:

$$C^p(\mathcal{U}, \Omega^q) \mapsto C^q(\mathcal{U}, \Omega^p),$$

- So the columns of the augmented complex of the last slide are exact, we could apply the lemma to show an isomorphism between the de Rham cohomology and the Čech cohomology through the total cohomology of the Čech-de Rham complex:

$$H^\bullet(\mathcal{U}, \mathbb{R}) \overset{?}{\simeq} H^\bullet(T(C)) \simeq H^\bullet_{\mathrm{dR}}(M).$$

- The failure of the pth column

$$\begin{array}{ccccccc} 0 \longrightarrow & C^p(\mathcal{U}, \mathbb{R}) & \longrightarrow & C^p(\mathcal{U}, \Omega^1) & \longrightarrow & C^p(\mathcal{U}, \Omega^2) & \longrightarrow \cdots \\ & = & & = & & = & \\ & \text{locally const. functions on } U_{a_0 \dots a_p} & & \prod \Omega^0(U_{a_0 \dots a_p}) & & \prod \Omega^1(U_{a_0 \dots a_p}) & \end{array}$$

to be exact, is measured by the

$$\prod_{q \geqslant 1} H^q_{\mathrm{dR}}(U_{a_0 \dots a_p}).$$

- By Poincaré's lemma, if all intersections are contractible then all columns of the augmented complex are exact. This proves:

De Rham's theorem (ii)

Theorem (de Rham)

If $\mathcal{U}$ is a good cover of M, then the de Rham cohomology is isomorphic to the Čech cohomology of the good cover:

$$H^\bullet_{\mathrm{dR}}(M) \simeq H^\bullet(\mathcal{U}, \mathbb{R}).$$

Corollary

The Čech cohomology $H^\bullet(\mathcal{U}, \mathbb{R})$ is the same for all good covers $\mathcal{U}$ of M.

Corollary

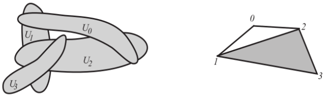
As every manifold has a good cover, the de Rham cohomology is a topological invariant, i.e. it does not depend on the differential structure.

Remark

The de Rham theorem shows an isomorphism of ***groups***, but we can also give the Čech complex a product structure and show an isomorphism of ***graded algebras***.

Čech cohomology. . .

- . . . computes cohomology by combinatorics of a cover (instead of geometry/analysis of the space).
- . . . generalises to compute or approximate diverse cohomology theories (eg. sheaf cohomology: replace $\Omega^\bullet$ by a sheaf).
- . . . is the cohomology of a simplicial approximation to a space. (The Čech cohomology of a cover is the simplicial cohomology of the ***geometric realisation*** of its ***nerve***.)



An open cover and the geometric realization of its nerve.

Thank you.

References:

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