HODGE THEORY

MATH6007: Riemannian geometry

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Outline

Exterior algebra

Definition (Exterior algebra)

The ***exterior algebra*** ΛV of an $\mathbb{R}$-vector space V is the algebra generated by

- $+$ addition and $\cdot$ scalar multiplication;
- $\wedge : V \times V \to V$ ***exterior product***,

such that

- $\wedge$ is antisymmetric:

$$a \wedge a = 0 \qquad \forall a \in V;$$

- ΛV is an associative algebra—i.e. with compatible structures of a ring and an $\mathbb{R}$-vector space:

$$x \cdot (a \wedge b) = (x \cdot a) \wedge b = a \wedge (x \cdot b) \qquad \forall x \in \mathbb{R},\ \forall a, b \in V.$$

Structure and properties of the exterior algebra

- ΛV is ***graded-commutative***:

$$\Lambda V = \bigoplus_{k \in \mathbb{N}} \Lambda^k V, \qquad \Lambda^k V := \operatorname{Span}\left\{a_1 \wedge \cdots \wedge a_k : a_i \in \Lambda^1 V\right\}$$

$$a \wedge b = (-1)^{\deg a \deg b}(b \wedge a) \in \Lambda^{\deg a + \deg b} V$$

- If $(\varepsilon_i)_{i=1}^n$ is a basis of V, then a basis of ΛV is

$$\{\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k} : 1 \leqslant i_1 < \cdots < i_k \leqslant n\}$$

Exercise

$$\Lambda^0 V \simeq \Lambda^n A \simeq \mathbb{R}, \qquad \Lambda^1 V \simeq \Lambda^{n-1} V \simeq V, \qquad \Lambda^{k>n} V \simeq 0,$$

$$\operatorname{Dim} \Lambda^k V = \binom{n}{k}, \qquad \operatorname{Dim} \Lambda V = 2^n.$$

Exterior algebra of an inner product space

Let V be an n-dimensional $\mathbb{R}$-inner product space. Extend its inner product to ΛV by setting

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_l \rangle := \begin{cases} \operatorname{Det} \langle v_i, w_j \rangle_{ij} & : k = l \\ 0 & : k \neq l. \end{cases}$$

Exercise

If $(\varepsilon_i)_{i=1}^n$ is an orthonormal basis of V, the basis $\{\varepsilon_I\}$ is orthonormal for ΛV.

Hodge star

Definition (Orientation)

A choice of component in $\Lambda^n V \setminus \{0\}$ is an ***orientation*** of ΛV.

Definition (Hodge star)

On an ***oriented inner product space*** V there is a linear isometry (ex.), the ***Hodge star***

$$\star : \Lambda^k V \to \Lambda^{n-k} V,$$

acting on ***any*** orthonormal basis by

$$\star(e_1 \wedge \cdots \wedge e_k) = \pm e_{k+1} \wedge \cdots \wedge e_n$$

$$\star(e_1 \wedge \cdots \wedge e_n) = \pm 1, \qquad \star 1 = \pm e_1 \wedge \cdots \wedge e_n$$

where $\pm = +$ if $e_1 \wedge \cdots \wedge e_n$ is in the orientation component and $-$ otherwise.

Properties of the Hodge star

Exercise (Property A)

On $\Lambda^k V$,

$$\star\star = (-1)^{k(n-k)}.$$

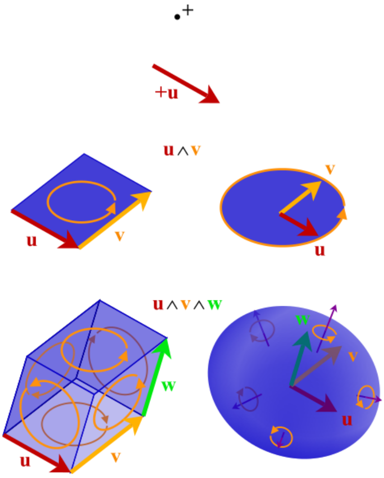
Exercise (Property B)

For $v, w \in \Lambda^k V$,

$$\langle v, w\rangle = \star(v \wedge \star w) = \star(w \wedge \star v).$$

Geometry of an exterior algebra

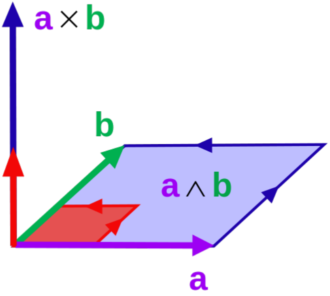
- a scalar $v \in \Lambda^0 V \simeq \mathbb{R}$ is an oriented magnitude;
- a vector $v \in \Lambda^1 V \simeq V$ is an oriented length;
- a ***bivector*** $v \wedge w \in \Lambda^2 V$ is an oriented area;
- a ***trivector*** $v \wedge w \wedge u \in \Lambda^3 V$ is an oriented volume;
- ...



Cross product in $\mathbb{R}^3$

The cross product in $\mathbb{R}^3$ is really just

$$v \times w := \star(v \wedge w) : \Lambda^2\mathbb{R}^3 \to \Lambda^1\mathbb{R}^3$$



From now on, (M, g) is a compact oriented riemannian n-manifold.

de Rham complex

Definition (de Rham complex)

The ***de Rham complex*** of a smooth manifold M is a *differential graded algebra*

$$C^\infty(M) \simeq \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \cdots$$

i.e., an exterior algebra $\Omega^\bullet(M) = \bigwedge_{C^\infty(M)} \Omega^k(M)$ with $\mathbb{R}$-linear *differential*

$$d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$$

$$\mathrm{dd} = 0, \qquad \mathrm{d}(a \wedge b) = \mathrm{d}a \wedge b + (-1)^{\deg a} a \wedge \mathrm{d}b$$

Definition (de Rham cohomology)

The ***de Rham cohomology*** of M is $H^\bullet(X) := \frac{\operatorname{Ker} \mathrm{d}}{\operatorname{Im} \mathrm{d}}$.

Hodge star on $\Omega^\bullet(M)$

Each cotangent space $\mathrm{T}_m^* M$ is an oriented inner product space. Then

$$\star : \Lambda^k(\mathrm{T}_m^* M) \to \Lambda^{n-k}(\mathrm{T}_m^* M).$$

extends fibrewise to smooth forms, and maps smooth forms to smooth forms (ex.):

$$\star : \Omega^k(M) \to \Omega^{n-k}(M).$$

Codifferential

Definition (Codifferential)

$$\delta := (-1)^{n(k+1)+1} \star \mathrm{d} \star \qquad : \Omega^k(M) \to \Omega^{k-1}(M)$$

Proposition

$$\delta\delta = 0.$$

Proof.

Ignoring signs,

$$\delta\delta = (\star \mathrm{d} \star)(\star \mathrm{d} \star) = \star \mathrm{d}(\star\star)\mathrm{d}\star = \star(\mathrm{dd})\star = \star(0)\star = 0. \qquad \square$$

Laplacian

Definition (Laplacian)

$$\Delta := \delta \mathrm{d} + \mathrm{d}\delta \qquad : \Omega^k(M) \to \Omega^k(M)$$

$$\begin{tikzcd}
\Omega^0(M) \arrow[loop above, "\Delta"] \arrow[r, shift left, "\mathrm{d}"] \arrow[rrrr, bend right, leftrightarrow, "\star"'] & \Omega^1(M) \arrow[loop above, "\Delta"] \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\mathrm{d}"] \arrow[rr, bend right, leftrightarrow, "\star"'] & \cdots \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\mathrm{d}"] & \Omega^{n-1}(M) \arrow[loop above, "\Delta"] \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\mathrm{d}"] & \Omega^n(M) \arrow[loop above, "\Delta"] \arrow[l, shift left, "\delta"]
\end{tikzcd}$$

The Laplacian in $\mathbb{R}^2$

Let x, y the standard coordinate maps on $\mathbb{R}^2$, and $\omega = \mathrm{d}x \wedge \mathrm{d}y$.

$$\star \mathrm{d}x = \mathrm{d}y, \qquad \star \mathrm{d}y = -\mathrm{d}x$$

Let $f \in \Omega^0(\mathbb{R}^2)$. Then $\Delta f = \delta \mathrm{d}f + \mathrm{d}\delta f = \delta \mathrm{d}f$, and

$$\begin{aligned}
\mathrm{d}f &= \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y, \qquad \star \mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}y - \frac{\partial f}{\partial y}\mathrm{d}x, \\
\mathrm{d}\star \mathrm{d}f &= \mathrm{d}\left(\frac{\partial f}{\partial x}\mathrm{d}y\right) - \mathrm{d}\left(\frac{\partial f}{\partial y}\mathrm{d}x\right) \\
&= \left(\frac{\partial^2 f}{\partial x^2}(\mathrm{d}x + \mathrm{d}y) \wedge \mathrm{d}y + \frac{\partial f}{\partial x}\mathrm{d}\mathrm{d}y\right) - \left(\frac{\partial^2 f}{\partial y^2}(\mathrm{d}x + \mathrm{d}y) \wedge \mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}\mathrm{d}x\right) \\
&= \frac{\partial^2 f}{\partial x^2}\mathrm{d}x \wedge \mathrm{d}y - \frac{\partial^2 f}{\partial y^2}\mathrm{d}y \wedge \mathrm{d}x, = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}x \wedge \mathrm{d}y, \\
\star \mathrm{d}\star \mathrm{d}f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}, \\
\Delta f = \delta \mathrm{d}f &= \left((-1)^{2(1+1)+1} \star \mathrm{d}\star\right)\mathrm{d}f = -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = \text{the usual Laplacian!.}
\end{aligned}$$

Hodge inner product

Proposition

$$\langle a, b\rangle := \int_M a \wedge \star b \qquad : \Omega^k(M) \times \Omega^k(M) \to \mathbb{R}.$$

is an inner product.

Proof.

Apply Property B. $\square$

Property C

Proposition 1 (Property C)

$$\star\,\Delta = \Delta\,\star.$$

Proof.

Ignoring signs!!!,

$$\begin{aligned}
\star\,\Delta &= \star\mathrm{d}\delta + \star\delta\mathrm{d} \\
&= \star\mathrm{d}(\star\mathrm{d}\star) + \star(\star\mathrm{d}\star)\mathrm{d} \\
&= (\star\mathrm{d}\star)\mathrm{d}\,\star + (\star\star)\mathrm{d}\star\mathrm{d} \\
&= \delta\mathrm{d}\,\star + \mathrm{d}\star\mathrm{d}(\star\star) \qquad \text{(Pro. A x2)} \\
&= \delta\mathrm{d}\,\star + \mathrm{d}(\star\mathrm{d}\star)\star \\
&= \delta\mathrm{d}\,\star + \mathrm{d}\delta\star = \Delta\,\star.
\end{aligned}$$

□

Property D

Proposition (d is adjoint to δ) (Property D)

$$\langle \mathrm{d}a, b\rangle = \langle a, \delta b\rangle, \qquad a \in \Omega^{k-1}(M), \quad b \in \Omega^k(M).$$

Proof.

Check note that (ex.) $\mathrm{d}\star b = (-1)^{-k} \star \delta b$. Then

$$\mathrm{d}(a \wedge \star b) = \mathrm{d}a \wedge \star b + (-1)^{k-1} a \wedge \mathrm{d}\star b = \mathrm{d}a \wedge \star b - a \wedge \star\delta b.$$

By definition of d. Integrate the LHS by Stoke's ($\int_M \mathrm{d}\omega = \int_{\partial M} \omega$),

$$\int_M \mathrm{d}(a \wedge \star b) = \int_{\partial M} = a \wedge \star b = 0. \qquad (\partial M = \varnothing)$$

Integrate the RHS, use definition of $\langle \cdot, \cdot \rangle$:

$$\int_M (\mathrm{d}a \wedge \star b - a \wedge \star\delta\star b) = \langle \mathrm{d}a, b\rangle - \langle a, \delta b\rangle. \qquad \square$$

Δ is self-adjoint

Corollary

$$\langle \Delta\, a, b \rangle = \langle a, \Delta\, b \rangle, \qquad a, b \in \Omega^k(M).$$

$\square$

Property E

Proposition (Property E)

$$\Delta\, a = 0 \iff \mathrm{d}a = 0 = \delta a.$$

Proof.

$(\Longleftarrow)$ is by definition. Now by Pro. D,

$$\langle \Delta\, a, a \rangle = \langle (\delta \mathrm{d} + \mathrm{d}\delta) a, a \rangle = \langle \delta a, \delta a \rangle + \langle \mathrm{d}a, \mathrm{d}a \rangle,$$

so $(\Longrightarrow)$ by positive-definiteness. □

Corollary

Every harmonic map ($\Delta\, f = 0$) on a compact connected oriented riemannian manifold is constant. □

Hodge Theorems

Theorem (cf. [War83, The. 6.8])

Every $a \in \Omega^k(M)$ can be written uniquely as a sum

$$a = b + \mathrm{d}c + \delta e, \qquad \Delta b = 0.$$

□

Theorem

$$\operatorname{Ker} \Delta_k \simeq H^k(M).$$

Proof.

1. $a \in H^k(M) \implies 0 = \mathrm{d}a = \mathrm{d}b + \mathrm{d}\mathrm{d}c + \mathrm{d}\delta e \implies \mathrm{d}\delta e = 0$ (Pro. E)
2. $0 = \mathrm{d}\delta e \implies 0 = \langle \mathrm{d}\delta e, e\rangle = \langle \delta e, \delta e\rangle \implies \delta e = 0$ (Pro. D, pos.-def.)
3. $\implies a = b + \mathrm{d}c.$
4. $\implies [a] = [b]$ and if $\Delta\, a = 0$ then $\mathrm{d}c = 0$ by uniqueness. □

Poincaré duality

Theorem (Poincaré duality for de Rham cohomology)

The bilinear form

$$\begin{aligned} H^k(M) \times H^{n-k}(M) &\to \mathbb{R} \\ ([a],[b]) &\mapsto \int_M a \wedge b \end{aligned}$$

is a perfect (i.e. nonsingular) pairing, and thus induces an isomorphism

$$H^k(M) \simeq H^{n-k}(M).$$

Poincaré duality: proof

Goal: given nonzero $[a] \in H^k(M)$, find nonzero $[b] \in H^{n-k}(M)$ such that $([a],[b]) \neq 0$.

Proof.

1. Let $[a] \in H^k(M) \setminus \{0\}$.
2. Assume a is harmonic representative of $[a]$ (Hodge The.)
3. $[a] \neq 0 \implies a \neq 0$.
4. $\star\Delta = \Delta\star \implies \star a$ also harmonic. (Pro. C)
5. $\star a$ harmonic $\implies d(\star a) = 0$. (Pro. E)
6. $\implies \exists[\star a] \in H^{n-k}(M)$.
7. $([a],[\star a]) \mapsto \int_M a \wedge \star a = \langle a, a\rangle \neq 0$. (def., pos-def. of $\langle\cdot,\cdot\rangle$)
8. $\therefore$ pairing is nonsingular $\implies$ isomorphism. $\square$

Application to curvature

Theorem ([Gol98, The. 3.2.1, p. 87])

If (M, g) is a compact oriented riemannian n-manifold with positive-definite Ricci curvature, then

$$\operatorname{Dim} H^1(M) = 0.$$

□

Corollary

If (M, g) is as above, then

$$\operatorname{Ker} \Delta_1 \simeq \operatorname{Ker} \Delta_{n-1} \simeq 0.$$

Proof.

Follows directly by Poincaré duality and Hodge Theorem B. □

THANK YOU!

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