Combinatorial structures on non-crossing partitions

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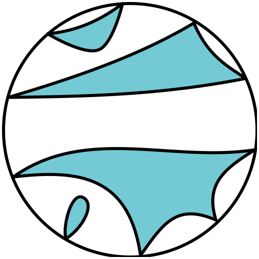
Overview

1. Non-crossing partitions

2. Linear combinatorial species
   - "Non-crossing substitution"

What are non-crossing partitions?

Arrange the points of a finite totally ordered set in increasing order around a circle and connect them with strings that do not cross each other.



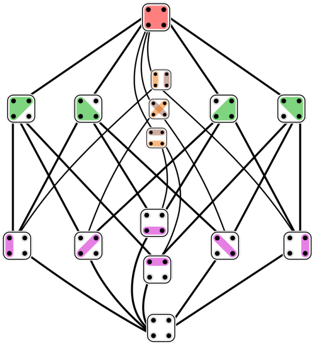
A non-crossing partition is the set of connected components of this picture.

The non-crossing partitions on a set are counted by the Catalan number

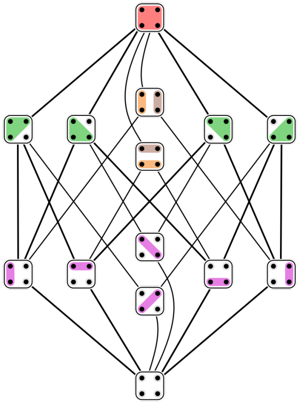
$$C_n = \frac{1}{n+1}\binom{2n}{n} = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots.$$

The lattice of non-crossing partitions

The non-crossing partitions on a set form a lattice when partially ordered by refinement (but not a sublattice of Part). It has more symmetry than Part (e.g. it is self-dual).



Part[4]



NC[4]

What are combinatorial species?

A combinatorial species is a gadget that, given any finite set, produces a set of discrete structures of a certain kind. The structures could be:

- trees,
- graphs,
- functions,
- relations,
- posets,
- permutations,
- partitions,
- finite geometries,
- finite groups,
- . . . .

One of many variations of species are linear species, whose structures depend on a given total order on the underlying set. More precisely, . . .

Definition for linear species

A linear species is a functor

$$\mathsf{F} : \mathsf{Lin} \to \mathsf{FinSet},$$

where

$$\mathsf{Lin} = \text{category of finite totally ordered sets \& increasing bijections},$$
$$\mathsf{FinSet} = \text{category of finite sets \& functions}.$$

- For a finite totally ordered set I, we say $\mathsf{F}[I]$ is the set of F-structures on I.
- For an increasing bijection $\lambda : I \to I'$, we say

  $$\mathsf{F}[\lambda] : \mathsf{F}[I] \to \mathsf{F}[I']$$

  is the transport of F-structures along λ.

Simple examples of linear species

Examples

0	Empty species
1	Species of the empty set
X	Species of singletons
Set	Species of (totally ordered) sets
List	Species of lists
Cyc	Species of oriented cycles
Comp	Species of linear set partitions
Part	Species of set partitions
NC	Species of non-crossing partitions

Associated generating functions

To each linear species F we associate

- an ordinary generating function

$$f(x) = \sum_{n \geq 0} |\mathrm{F}[n]|\, x^n;$$

- an exponential generating function

$$F(x) = \sum_{n \geq 0} |\mathrm{F}[n]|\, \frac{x^n}{n!},$$

where $|\mathrm{F}[n]|$ is the number of elements of $\mathrm{F}[I]$ for any totally ordered set I of n elements.

Aim: "Find a simple expression or identity for $f(x)$ or $F(x)$."

Associated generating functions for examples

Examples (E.g.f.s for examples (& o.g.f. for NC))

$$0(x) = 0;$$

$$1(x) = 1;$$

$$\mathrm{X}(x) = x;$$

$$\mathrm{Set}(x) = e^x;$$

$$\mathrm{List}(x) = \frac{1}{1-x};$$

$$\mathrm{Cyc}(x) = -\log(1-x);$$

$$\mathrm{Comp}(x) = \frac{1+e^{2x}}{2};$$

$$\mathrm{Part}(x) = e^{e^x-1};$$

$$\mathrm{NC}(x) = e^{2x}(I_0(2x) - I_1(2x));$$

$$\mathrm{nc}(x) = \frac{1-\sqrt{1-4x}}{2x}$$

(I_n is the modified Bessel function of the 1st kind.)

Operations on linear species

Combinatorial operations on species	$\longleftrightarrow$	**Calculus on generating functions**
F'		$\frac{d}{dx}F(x)$
$\int \mathsf{F}$		$\int_0^x F(x)\,\mathrm{d}x$
$\mathsf{F} + \mathsf{G}$		$F(x) + G(x)$
$\mathsf{F} \times \mathsf{G}$		$F(x) \times G(x)$
$\mathsf{F} \cdot_{\leq} \mathsf{G}$		$f(x)g(x)$
$\mathsf{F} \cdot \mathsf{G}$		$F(x)G(x)$
$\mathsf{F} * \mathsf{G}$		$\int_0^x F(\xi)G(x-\xi)\,\mathrm{d}\xi$
$\mathsf{F} \circ_{\leq} \mathsf{G}$		$f(g(x))$
$\mathsf{F} \circ \mathsf{G}$		$F(G(x))$
$\vdots$		$\vdots$

Definitions for operations

$$F'[I] \coloneqq F[\{I\} +_{\leq} I]$$

$$\left(\int F\right)[I] \coloneqq F[I \setminus \min I]$$

$$(F + G)[I] \coloneqq F[I] \sqcup G[I]$$

$$(F \times G)[I] \coloneqq F[I] \times G[I]$$

$$(F \cdot_{\leq} G)[I] \coloneqq \bigsqcup_{I_1 +_{\leq} I_2 = I} F[I_1] \times G[I_2]$$

$$(F \cdot G)[I] \coloneqq \bigsqcup_{I_1 + I_2 = I} F[I_1] \times G[I_2]$$

$$(F * G)[I] \coloneqq F \cdot_{\leq} X \cdot_{\leq} G$$

If $G[\emptyset] = \emptyset$,

$$(F \circ_{\leq} G)[I] \coloneqq \bigsqcup_{\pi \in \mathrm{Comp}[I]} F[\pi] \times \prod_{p \in \pi} G[p]$$

$$(F \circ G)[I] \coloneqq \bigsqcup_{\pi \in \mathrm{Part}[I]} F[\pi] \times \prod_{p \in \pi} G[p]$$

Samples of combined species

Examples

$$\mathsf{Set} = 1 + \mathsf{Set}_{+}$$

$$\mathsf{List}' = \mathsf{List}^2$$

$$\mathsf{Part} = \mathsf{Set} \circ \mathsf{Set}_{+}$$

$${\mathsf{Set}_{\mathsf{even}}}^2 = 1 + {\mathsf{Set}_{\mathsf{odd}}}^2$$

$${\mathsf{Alt}_{\mathsf{odd}}}' = 1 + {\mathsf{Alt}_{\mathsf{odd}}}^2$$

$${\mathsf{Alt}_{\mathsf{even}}}' = \mathsf{Alt}_{\mathsf{odd}} \cdot \mathsf{Alt}_{\mathsf{even}}$$

$$\mathsf{Graph}_{\mathsf{even}} = 1 + \int \mathsf{Graph}$$

A new operation: "non-crossing substitution"

If $\mathsf{G}[\emptyset] = \emptyset$,

$$(\mathsf{F} \diamond \mathsf{G})[I] := \bigsqcup_{\pi \in \mathsf{NC}[I]} \mathsf{F}[\pi] \times \prod_{p \in \pi} \mathsf{G}[p]$$

Proposition

$$(F \diamond G)(x) = \mathcal{L}^{-1}\left\{ \frac{F\left(s\,g\left(\frac{x}{s}\right)\right)}{s^2} \right\}(1),$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform.

Worked example of non-crossing substitution

Example (lists of non-crossing sets)

How many ways to partition a set into lists of non-crossing subsets?

$$F(x) = \mathrm{List}(x) = \frac{1}{1-x},\ g(x) = \mathrm{set}_+(x) = \frac{x}{1-x};$$

$$(F \diamond G)(x) = \mathcal{L}^{-1}\left\{s^{-2}F\left(s\,G\left(s^{-1}x\right)\right)\right\}(1) = \mathcal{L}^{-1}\left\{\frac{1}{s^2\left(1-\frac{x}{1-\frac{x}{s}}\right)}\right\}(1)$$

$$= \left(e^{\frac{tx}{1-x}+t-1}\right)\Big|_{t=1} = e^{\frac{x}{1-x}} = (\mathrm{Set}\circ\mathrm{List}_+)(x).$$

We have proved an identity:

$$\mathrm{List}\diamond\mathrm{Set}_+ = \mathrm{Set}\circ\mathrm{List}_+.$$

And the answer to the question is the coefficient of x^n in the e.g.f. $e^{\frac{x}{1-x}}$:

$$1, 1, 3, 13, 73, 501, 4051, 37633, 394353, 4596553, \ldots.$$

Plan for the project

- Find a better formula for "non-crossing substitution"; find interesting examples.
- Consider cyclic group acting on linear structures: Cyclic Sieving Phenomenon, "cyclic species"?

References

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Thank You.