Homological aspects of Morse-Bott theory

Nasos Evangelou-Oost

Advisor: Prof Joseph Grotowski

University of Queensland

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Outline

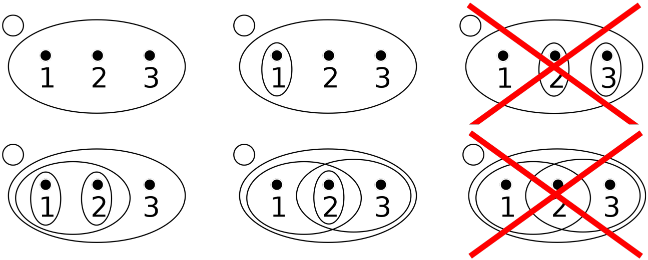
Topological spaces

Definition

A ***topological space*** is a pair $(X, \mathcal{T})$, where:

- X is a set;
- $\mathcal{T}$ is a set of subsets of X, the ***topology*** or ***open sets***, which is closed under:
  - finite intersections;
  - unions.

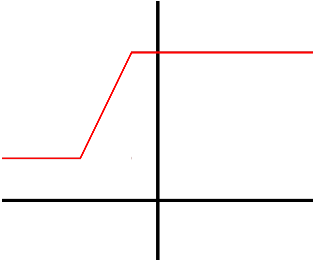
In particular, $\emptyset$ and $X \in \mathcal{T}$.



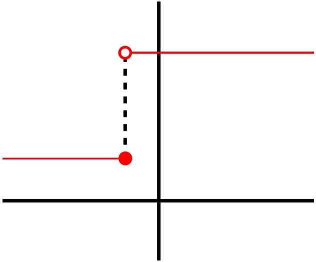
Continuity

Definition

$f : X \longrightarrow Y$ is ***continuous*** if the preimage of every open set in Y is an open set in X.



continuous map $\mathbb{R} \longrightarrow \mathbb{R}$



discontinuous map $\mathbb{R} \longrightarrow \mathbb{R}$

Homotopy

Definition

For continuous maps

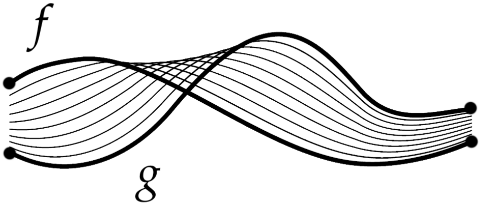
$$f, g : X \longrightarrow Y$$

A ***homotopy*** from f to g is a continuous map

$$h : [0,1] \times X \longrightarrow Y$$

such that

$$h(0,x) = f(x), \qquad h(1,x) = g(x).$$



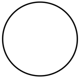
Manifolds

Definition

- ***Manifold***: a space that is *locally homeomorphic* to $\mathbb{R}^n$ for a fixed $n \in \mathbb{N}_0$.
- ***Smooth manifold***: 'a manifold on which you can do calculus'.

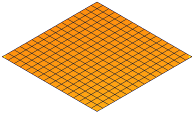
- 1-manifolds

line

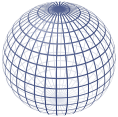


circle

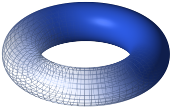
- 2-manifolds (surfaces)



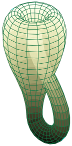
plane



sphere



torus



Klein bottle

...

- 3-manifolds, 4-manifolds, ...

Compactness

Definition

A topological space X is ***compact*** if every cover $\bigcup_i U_i \supseteq X$ of open sets $\{U_i\}_i$ has a finite subcollection that still covers X.

Example

Except for the line and the plane, all manifolds on the last slide are compact.

Topological equivalence

Definition

If there exists continuous maps

$$f : X \longrightarrow Y, \qquad g : Y \longrightarrow X$$

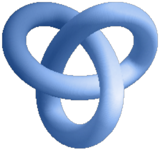
such that

- $$f \circ g \sim \mathbb{1}_X \text{ and } g \circ f \sim \mathbb{1}_Y,$$
  then X is ***homotopy equivalent*** to Y.
- $$f \circ g = \mathbb{1}_X \text{ and } g \circ f = \mathbb{1}_Y,$$
  then X is ***homeomorphic*** to Y.

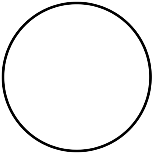
Topological equivalence: examples

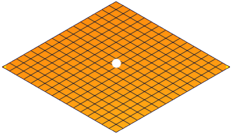
1. homeomorphic: ‘the same up to bending (but no tearing)’;


$\simeq$

$\simeq$


2. homotopy equivalent: ‘the same up to bending, expanding, contracting’.


$\sim$

$\sim$


Homology

Fix a field $\mathbb{F}$. There is a map $H_\bullet(\,\cdot\,)$, ***homology***, that associates...

- ... a topological space X to a sequence of $\mathbb{F}$-vector spaces

$$H_0(X), H_1(X), \ldots$$

  where $H_i(X)$ has a basis of the i-dimensional 'holes' of X.

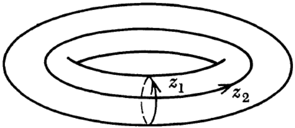
- ... a continuous map $f : X \longrightarrow Y$ to a sequence of $\mathbb{F}$-linear maps

$$\begin{aligned} H_0(f) : H_0(X) &\longrightarrow H_0(Y), \\ H_1(f) : H_1(X) &\longrightarrow H_1(Y), \\ &\vdots \end{aligned}$$

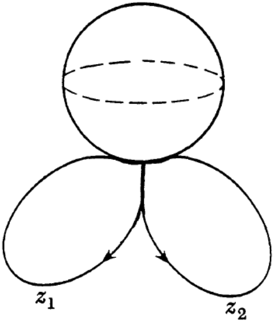
-
  - If $f \sim g$ then $H_\bullet(f) = H_\bullet(g)$;
  - if $X \sim Y$ then $H_\bullet(X) = H_\bullet(Y)$.

Homology, cont.

$H_\bullet$ 'linearises' the topology, so some information is lost:



torus



sphere with two circles attached (fake torus)

These two are not homotopy equivalent, but
$H_\bullet(\textit{torus}) = H_\bullet(\textit{fake torus})$.

Betti numbers

Definition

Let M be a compact manifold M. The ith ***Betti number*** of M is

$$\beta_i := \dim H_i(M; \mathbb{F}).$$

- These count the number of 'holes' in each dimension.
- For an n-manifold, the Betti numbers only go up to n.

Definition

The ***Poincaré polynomial of*** M is

$$\mathscr{P}_{M;\mathbb{F}}(t) := \beta_0 + \beta_1 t + \cdots + \beta_n t^n$$

Example

$\mathscr{P}_{\mathbb{R}^n;\mathbb{R}} = 1, \qquad \mathscr{P}_{S^n;\mathbb{R}} = 1 + t^n, \qquad \mathscr{P}_{T^2;\mathbb{R}} = 1 + 2t + t^2,$
$\mathscr{P}_{K;\mathbb{R}} = 1 + t, \qquad \mathscr{P}_{K;\mathbb{Z}_2} = 1 + 2t + t^2.$

Smooth maps and critical points

Definition

A map $f : M \longrightarrow \mathbb{R}$ on a smooth manifold M is ***smooth*** if locally, the partial derivatives

$$\frac{\partial^i f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}, \qquad i := i_1 + \cdots + i_n$$

exist and are continuous for all $i_1, \ldots, i_n \in \mathbb{N}_0$.

Definition

$c \in M$ is a ***critical point*** if $\mathrm{d}f(c) = 0$.
Locally: if the Jacobian vanishes at c:

$$\left[\frac{\partial f}{\partial x_i}(c)\right]_i = [0]_i.$$

Nondegenerate critical point

$f : M \longrightarrow \mathbb{R}$ is a smooth map on a smooth manifold M.

Definition

$c \in M$ is a ***nondegenerate critical point*** if $\mathrm{d}c = 0$ and locally

$$\det H \neq 0, \qquad H := \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(c) \right]_{ij}.$$

If the Hessian is nonsingular at c then f is 'not too flat' at c.

Definition

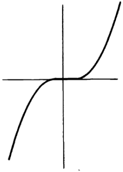
The ***index*** λ_c is the number of negative eigenvalues of H.

λ_c is the number of independent directions along which f will decrease from c.

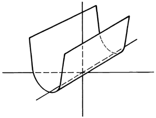
Examples of degenerate and nondegenerate critical points

(c = origin.)

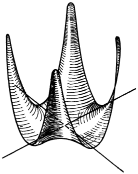
DEGENERATE



$f(x) = x^3$

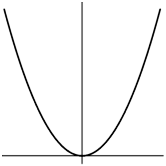


$f(x, y) = x^2$

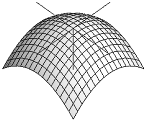


$f(x, y) = x^2y^2$

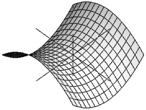
NONDEGENERATE



$f(x) = x^2$
$\lambda_0 = 0$



$f(x, y) = -x^2 - y^2$
$\lambda_0 = 2$



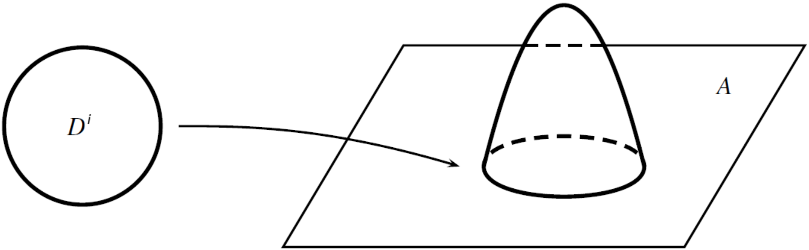
$f(x, y) = x^2 - y^2$
$\lambda_0 = 1$

Idea of Morse theory

Marston Morse

Nondegenerate critical points of real-valued maps are linked to the topology.

Attaching disks



Definition

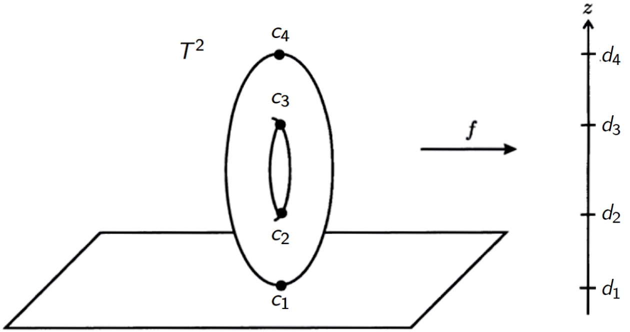
A space X is obtained from a space A by ***attaching an i-disk*** with a map $\alpha : \partial D^i \longrightarrow A$ if

$$X = \frac{A \coprod D^i}{\partial D^i \sim \alpha(\partial D^i)}.$$

Example: the upright torus

The height map $f := \mathrm{Proj}_z$ of a torus embedded in $\mathbb{R}^3$ standing tangent to the xy-plane has 4 nondegenerate critical points with indices $\lambda = 0, 1, 1, 2$.

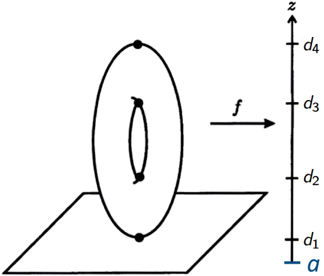
$$d_i := f(c_i)$$



;

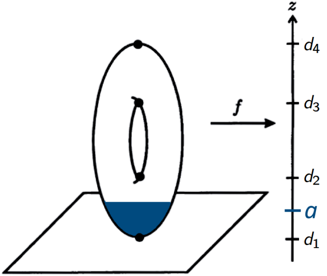
$$M^a := f^{-1}(-\infty, a]$$

$a \in$	$M^a \sim$	attach
$(-\infty, d_1)$	$\varnothing$	



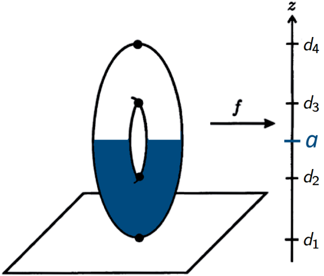
$$M^a := f^{-1}(-\infty, a]$$

$a \in$	$M^a \sim$	attach
$(-\infty, d_1)$	$\varnothing$	
(d_1, d_2)	 $\sim$ •	D^0

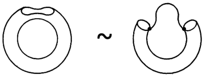


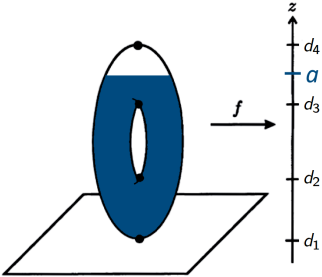
$$M^a := f^{-1}(-\infty, a]$$

$a \in$	$M^a \sim$	attach
$(-\infty, d_1)$	$\varnothing$	
(d_1, d_2)		D^0
(d_2, d_3)		D^1

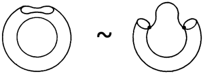
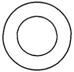


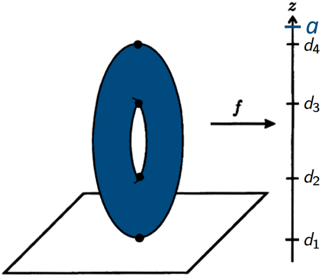
$$M^a := f^{-1}(-\infty, a]$$

$a \in$	$M^a \sim$	attach
$(-\infty, d_1)$	$\varnothing$	
(d_1, d_2)		D^0
(d_2, d_3)		D^1
(d_3, d_4)		D^1



$$M^a := f^{-1}(-\infty, a]$$

$a \in$	$M^a \sim$	attach
$(-\infty, d_1)$	$\varnothing$	
(d_1, d_2)		D^0
(d_2, d_3)		D^1
(d_3, d_4)		D^1
(d_4, ∞)		D^2



The Morse theorems

Suppose $f^{-1}[a, b]$ is compact $\forall [a, b] \subset \mathbb{R}$. Then:

Theorem (Morse [1934])

If $f^{-1}[a, b]$ contains no critical points, then

$$M^b \sim M^a. \qquad \square$$

Theorem (Morse [1934])

If $f^{-1}[a, b]$ contains exactly one nondegenerate critical point of index λ, then

$$M^b \sim M^a \cup D^\lambda. \qquad \square$$

Morse inequalities

Definition

If $f : M \longrightarrow \mathbb{R}$ has only nondegenerate critical points it is a ***Morse map***.

Definition

Let $\mu_\lambda := \#$ critical points of index λ of a Morse map $f : M \longrightarrow \mathbb{R}$ on a compact manifold M. The ***Morse polynomial*** of f is

$$\mathscr{M}_f(t) := \mu_0 + \mu_1 t + \cdots + \mu_n t^n,$$

Theorem (Morse [1934])

$$\mathscr{M}_f(t) - \mathscr{P}_{M;\mathbb{F}}(t) = (1+t)N(t)$$

where $N(t)$ is a polynomial with coefficients in $\mathbb{N}_0$. $\square$

Euler characteristic

Definition

For M a compact manifold, the ***Euler characteristic*** is

$$\chi_M := \sum_{i=0}^{n} \beta_i(M)(-1)^i = \mathscr{P}_{M;\mathbb{F}}(-1).$$

From the Morse inequalities

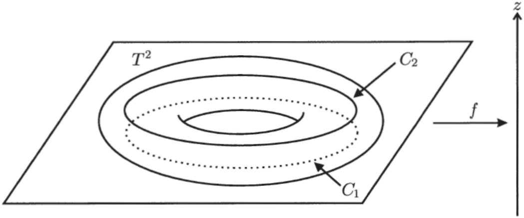
$$\mathscr{M}_f(t) - \mathscr{P}_{M;\mathbb{F}}(t) = (1+t)N(t),$$

we have

$$\mathscr{M}_f(-1) = \chi_M.$$

Morse-Bott theory

Raoul Bott



Many 'natural' maps cannot be Morse because their critical points are not isolated.

Definition

$f : M \longrightarrow R$ is ***Morse-Bott*** if the critical points are a disjoint union of (orientable) submanifolds such that the Hessian is nondegenerate restricted to the normal directions.

Morse-Bott inequalities

$f : M \longrightarrow \mathbb{R}$ is Morse-Bott (critical set are disjoint nondegenerate submanifolds).

Definition

The ***Morse-Bott polynomial of*** f is

$$\mathscr{M}_{f;\mathbb{F}}(t) := \sum_{C \in \pi_0(\operatorname{Cr} f)} t^{\lambda_C} \mathscr{P}_{C;\mathbb{F}}(t).$$

Theorem (Bott [1954])

$$\mathscr{M}_{f;\mathbb{F}}(t) - \mathscr{P}_{M;\mathbb{F}}(t) = (1+t)N(t) \quad \text{for some } N(t) \in \mathbb{N}_0[t]. \qquad \square$$

Other applications

Historical:

- Milnor [1956]: discovery of exotic spheres.
- Bott [1959]: Periodicity Theorem: π_n of infinite classical Lie groups are periodic.
- Smale [1961]: Generalised Poincaré conjecture in Dim $\geqslant 5$: every homotopy sphere is a sphere.
- Witten [1982]: Supersymmetric quantum field theory.

Modern:

- String topology [1999–]: algebraic structures on homology of free loop spaces.
- Discrete Morse theory [2002–] (topological data analysis, persistent homology, combinatorial topology)

THANK YOU!

Figures from:

Augustin Banyaga and David Hurtubise.
Lectures on Morse homology, volume 29 of *Kluwer Texts in the Mathematical Sciences*.
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