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Homological aspects of Morse-Bott theory

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Abstract

Morse theory is a set of techniques that relates analysis to topology. The main insight is that the critical set of a generic real-valued map on a smooth manifold depends on topological properties of the manifold. At the most basic level, this relationship manifests through inequalities between the number of nondegenerate critical points of a given index and the corresponding Betti number. Classical Morse theory applies to maps whose critical points are all nondegenerate—a condition that constrains the critical points to be isolated. Raoul Bott developed a generalisation of Morse theory that applies to maps whose critical sets are submanifolds. This thesis explores the use of Morse theory and Morse-Bott theory in homological computations.

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Chapter 0

Introduction

Every mathematician has a secret weapon.
Mine is Morse theory.

—Raoul Bott [Nan12]

The aim of this thesis is to give a self-contained exposition of the homological aspects of *classical Morse theory*[1], and of an extension to it due to Raoul Bott[2], known as *Morse-Bott theory*. This chapter will briefly describe Morse theory and the context for its extension through Morse-Bott theory. It will then summarise the various applications of this theoretical framework through a brief review of the literature. Next, a chapter outline delineates the main topics of the discussion. Finally, the conditions for the development of this thesis are introduced in a statement of our approach to the content.

0.1 Morse theory and Morse-Bott theory

Broadly speaking, Morse theory concerns the interaction between topology and analysis. More specifically, Morse theory is a body of techniques and insights that relates analytical data of generic smooth real-valued maps

$$\varphi : M \to \mathbb{R}$$

on a manifold M to basic algebro-topological invariants. Bott described Morse theory as 'a beautiful and natural extension of the minimum principle for a continuous function on a compact space' ([Bot82])—that is, that such a map attains a minimum and a maximum value. Morse theory refines and extends this principle to a relationship between the *Betti numbers* $\beta_i(M)$ of a compact manifold M—numerical invariants that, roughly speaking, count its i-dimensional 'holes'—to the *Morse numbers* $\mu_i(\varphi)$, which count the number of critical points of the map φ of each *index* i. In its most primitive form, this relationship

[1]Named after Harold Calvin Marston Morse (March 24, 1892 – June 22, 1977).
[2]Raoul Bott (September 24, 1923 – December 20, 2005).

manifests in a set of inequalities

$$\beta_i(M) \leqslant \mu_i(\varphi), \qquad i \in \mathbb{N}.$$

The channel between analysis and topology that Morse theory exposes is reciprocal: maps whose critical structure is readily understood may be used to probe the topology of M, or if the topology is already known, topological data may then be used to deduce the behaviour of maps on M that may be otherwise difficult to calculate or understand. Both vantage points have led to monumental breakthroughs throughout the latter half of the 20th century. Notable examples are John Milnor's construction of exotic spheres [MS74], Bott's discovery of the periodicity in the homotopy groups of Lie groups [Bot59; Bot60], and Stephen Smale's proofs of the h-cobordism theorem and the Generalised Poincaré conjecture in dimensions greater than four [Sma61; Sma62].

The simplest context for Morse theory is when the manifold M is compact, finite-dimensional, and without boundary, and when the map φ has only isolated and *nondegenerate* critical points—these are points at which the qualitative behaviour of φ is determined by only the linear and quadratic terms in its Taylor expansion. Such a map is called *Morse*, and it is to the study of Morse maps that the term *classical Morse theory* refers. It often happens that naturally arising maps have a built-in symmetry which puts them outside of the bounds of the classical theory, as their critical sets will then generally form positive-dimensional submanifolds. This situation was addressed by Bott's extension to Morse theory (now referred to as *Morse-Bott theory*) in his seminal work [Bot54], which widens the class of susceptible maps to those satisfying a more relaxed nondegeneracy condition. Bott famously utilised this generalised theory in his periodicity theorems, and in various other ground-breaking results throughout his career; for example, his determination with Michael Atiyah of the cohomological structure of the moduli space of algebraic bundles on a Riemann surface [AB83].

Morse theory remains an area of active research, and far-reaching extensions beyond Morse-Bott theory have been developed. In the setting of smooth manifolds, *Morse homology* is the homology of a chain complex generated by the critical points of a Morse map with a boundary operator defined in terms of gradient flow lines, and *Morse-Bott homology* is the corresponding extension to the Morse-Bott setting [BH10]—both theories coinciding with the ordinary homology of the manifold. *Picard–Lefschetz theory* adapts Morse theory to holomorphic maps on complex manifolds [Nic11, Sec. 5, pp. 251–295], and *Floer homology* to symplectic and infinite-dimensional manifolds [Don02]. *Discrete Morse theory* is an adaptation to finite and simplicial structures that has powerful implications in computer science and applied topology, especially in the nascent field of topological data analysis [For02] where it is extensively used in popular computational topology packages [Nan19; Har19]. *String topology* is an innovative area of research that investigates the topology of free loop spaces [CS09], born out of Bott's original work on the loop spaces of Lie groups [Bot60]. In fact, large swathes of classical algebraic topology can be reformulated from the Morse theoretic lens of critical points and gradient flows between them [Ger17].

0.2 Chapter outline

In Chapter 1 the foundational definitions and facts needed for the development of Morse theory in Chapter 2 are reviewed. Topological spaces are the central focus of this chapter, which begins with their most general and abstract formulation and leads to a description of the special subclasses of spaces that are most relevant to Morse theory: CW complexes and smooth manifolds. Some key constructions on these such as attaching spaces and colimit spaces are described. Along the way, the requisite algebraic topology, i.e. the homotopy and homology groups of a space, and various tools to compute these, is introduced.

Chapter 2 concerns classical Morse theory. This begins with an introduction to the analytical details of critical points and follows with a key technical fact about these, known as the *Morse lemma*. Then the cornerstone results of the theory, the *Morse theorems A* and *B*, and the structure theorem are discussed. Theorems A and B relate the local topology of M to the local critical structure of a real-valued smooth map on M, while the structure theorem is a global statement that relates the topology of the entire manifold to a Morse map on it. A complete proof is only given for the latter, as this is the most crucial for applications. The reader is referred to Milnor's iconic exposition [Mil63] for proofs of the theorems in the cases where only outlines are given. After proving the structure theorem, a simple but powerful result known as the *Morse inequalities* is derived, and various properties and consequences of this are discussed.

Chapter 3 concerns Bott's extension to Morse theory, which relaxes the nondegeneracy criterion on critical points to address maps whose critical sets form positive-dimensional submanifolds but still satisfy a generalised form of nondegeneracy. This generalised criterion amounts to measuring degeneracy in only the orthogonal directions to the critical submanifolds. The reformulation of the main theorems of Chapter 2 to this setting involves the language of *fibre bundles*, so a beginning section is devoted to an introduction of this concept. The extensions to the main theorems are then presented, before moving on to special properties and applications.

0.3 Statement of approach

Since its publication in 1963, Milnor's *Morse theory* [Mil63] has remained the standard reference for classical Morse theory, and the major theorems of Chapter 2 are treated in Milnor's text. The approach taken in Chapter 2 was first to expand on the proof of the central result, Theorem 2.20, especially where some details regarding the important case of the critical set having infinite cardinality were suppressed in Milnor's treatment (the significant result that every smooth manifold has the homotopy type of a CW complex depends on this case). This required a prolonged survey of the properties of colimits and homotopy groups in Chapter 1. A second aim of Chapter 2 was to consolidate simple applications of the Morse inequalities (Section 2.5), including several that were not found in the literature. In Chapter 3, a principle intent was to clarify the role of orientability assumptions in the formulation of the Morse-Bott inequalities (Section 3.3), which have frequently been misstated [Rot16], and to emphasize their application to the computation of Euler characteristics (Section 3.4).

Morse theory is an eclectic subject that draws from the theory and techniques of many disciplines. To present a self-contained exposition with modest prerequisites, we chose to solely address the homological aspects of the theory, and consequently, the discriminatory power of the presented results is limited at the level of homological equivalence. Notable facets of the classical theory that are outside of the scope of our treatment are the dynamical systems approach, known as *Morse-Smale dynamics*, and the *handle presentation theorem*, which concerns handlebody decompositions (the smooth analogue of CW decompositions) and other diffeomorphism invariants; excellent references for these are [Mat02; Nic11].

It is hoped that the reader will find this thesis to be a valuable and approachable introduction to the basic elements of Morse theory, and that after reading, he or she will be better equipped to pursue this fascinating topic further. The reader is assumed to be familiar with analysis, manifolds, and general and algebraic topology at the undergraduate level. Standard references to these topics are respectively [Spi65; Lee13; Mun75; Hat02]. The primary references for Chapters 2 and 3 are [Mil63; BH04b; Nic11; Bot82; Bot54; AB83].

Chapter 1

Preliminaries

This preliminary chapter reviews the background that will be assumed in Chapters 2 and 3. A section is devoted to each of the important varieties of objects relevant to Morse theory: *CW complexes* (Section 1.3), and *smooth manifolds* (Section 1.5); as well as to the algebraic tools that will be used to understand them: *homotopy groups* (Section 1.2) and *homology groups* (Section 1.4). CW complexes and smooth manifolds are both instances of *topological spaces*, whose basic properties are the subject of Section 1.1.

1.1 Topological spaces

The role of a *topological space* is to establish the most general setting in which the concept of continuity is realisable. Specifically, a topological space is a set equipped with an additional structure, its *topology*, that prescribes how the points of the space cohere. The classical study of topological spaces is known as *general* or *point-set topology*. It is assumed that most of the basic elements of this theory are familiar: the product, subspace and quotient topologies, bases, compactness, countability, separation axioms, boundedness, connectedness, closure, etc. A standard reference for these concepts is [Mun75]. Recalled here are only the definitions and facts that will be most frequently referred to.

Definition 1.1 (Topological space)**.** A ***topological space*** is a pair $(X, \mathcal{T}_X)$, where X is a set and $\mathcal{T}_X \subseteq 2^X$ is a set of subsets of X, the ***open sets***, that are closed under:

1. finite intersections;
2. arbitrary unions. ▪

Remark 1.2. As the space X itself is the intersection of zero subsets, it is open, and as $\varnothing$ is the union of zero subsets, it is also open. Hence, the minimum number of open sets of a topology is two, i.e. when $\mathcal{T}_X = \{\varnothing, X\}$—this is the ***codiscrete*** topology. At the opposite extreme, the ***discrete*** topology is the power set of X, $\mathcal{T}_X = 2^X$, in which every subset is open. ▪

Definition 1.3 (Coarser, finer)**.** If $\mathcal{T}$ and $\mathcal{S}$ are topologies on X and $\mathcal{T} \subseteq \mathcal{S}$, then $\mathcal{T}$ is ***coarser*** than $\mathcal{S}$ and $\mathcal{S}$ is ***finer*** than $\mathcal{T}$. ▪

Topological spaces comprise an eclectic variety of objects, including graphs, manifolds, metric spaces, fractal structures, solution sets of polynomial or differential equations, and a vast expanse of pathological structures that defy classification. The structure-preserving maps between topological spaces are *continuous maps*.

Definition 1.4 (Continuous map)**.** A map $f : X \to Y$ of topological spaces is ***continuous*** if the preimage of each open set in Y is an open set in X, i.e. if the preimage $f^{-1} : 2^Y \to 2^X$ restricts to a map

$$f^{-1} : \mathcal{T}_Y \to \mathcal{T}_X.$$

▪

Remark 1.5. Some comments on terminology: the complement of an open set is a ***closed*** set, and all the definitions have equivalent counterparts phrased in terms of closed sets. When there is no risk of confusion, *space* refers to a topological space, and *map* refers to a continuous map. In the sequel, the word *map* should always be assumed to be the natural one in the given context; e.g. a *map* between vector spaces is a linear transformation, etc. ▪

Proposition 1.6. *The composition of continuous maps is continuous.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps between topological spaces. Let $U \in \mathcal{T}_Z$ be an open set of Z. First note that

$$x \in (g \circ f)^{-1}(U) \iff g(f(x)) \in U \iff f(x) \in g^{-1}(U) \iff x \in f^{-1}\Big(g^{-1}(U)\Big),$$

hence,

$$(g \circ f)^{-1}(U) = f^{-1}\Big(g^{-1}(U)\Big).$$

By continuity of g, $g^{-1}(U)$ is open in Y. By continuity of f,

$$(f \circ g)^{-1}(U) = f^{-1}\Big(g^{-1}(U)\Big)$$

is open in X. It follows that $g \circ f : X \to Z$ is continuous. ■

The following property is perhaps the most basic invariant of a topological space.

Definition 1.7 (Connected space)**.** A topological space X is ***connected*** if it cannot be written as a union $X = U \cup V$ for open sets $U, V \in \mathcal{T}_X$. Otherwise, it is ***disconnected***. ▪

A stronger and more applicable form of connectedness is *path-connectedness*. A ***path*** in a space X is a continuous map $\mathbb{I} \to X$ from the closed unit interval $\mathbb{I} := [0, 1]$ to X. Two points $x, y \in X$ are said to be connected by a path if there exists a path $f : \mathbb{I} \to X$ with $f(0) = x$ and $f(1) = y$. The space X is then ***path-connected*** if every pair of points in X can be connected by a path. For the spaces of relevance to Morse theory—CW complexes and smooth manifolds—connectedness is equivalent to path-connectedness (see respectively [Hat02, App., p. 522], [Lee13, Pro. 1.11, p. 8]), so in these contexts the terms may be used synonymously.

As with vector spaces, it is convenient to represent the topology of a space in terms of a generating subcollection, also called a *basis*.

Definition 1.8 (Basis)**.** A subcollection $\mathcal{B} \subseteq \mathcal{T}_X$ of a topological space X is a ***basis*** for the topology $\mathcal{T}_X$, if for all $U \in \mathcal{T}_X$ and $x \in U$, there is a basis member $B \in \mathcal{B}$ for which $x \in B \subseteq U$. ▪

Example 1.9 (Open ball basis of euclidean space)**.** An ***open ball*** with centre $x \in \mathbb{R}^n$ and radius $\varrho > 0$ is the set

$$B(x, \varrho) := \{y \in \mathbb{R}^n : \|x - y\| < \varrho\},$$

where

$$\|\cdot\| : \mathbb{R}^n \to \mathbb{R}, \qquad x =: (x_i)_{i=1}^n \mapsto \sqrt{\sum_{i=1}^n x_i^2}$$

is the standard norm on $\mathbb{R}^n$. A basis for the standard topology on $\mathbb{R}^n$ is

$$\mathcal{B} := \{B(x, \varrho) : x \in \mathbb{R}^n, \varrho > 0\}.$$ ▪

It is easily shown ([Tu11, Pro. A.7, p. 321]) that Definition 1.8 is equivalent to requiring that the basis be a generating subcollection of the topology under the operation of union—i.e. a subcollection $\mathcal{B}$ is a basis if and only if every member of $\mathcal{T}_X$ is a union of members of $\mathcal{B}$. The possibly more natural notion of a subcollection $\mathcal{C} \subseteq \mathcal{T}_X$ that generates $\mathcal{T}_X$ under the operations of union *and* finitary intersections is a ***subbasis***. Of course, a basis is a subbasis.

The topological spaces that admit finite bases are practically devoid of interest. The next simplest case is of spaces that admit countable bases, which will be a defining property of smooth manifolds (Definition 1.61).

Definition 1.10 (Second-countable space)**.** A topological space is ***second-countable*** if it admits a countable basis. ▪

Example 1.11 (Countable basis for euclidean space [Tu11, Pro. A.11, p. 323])**.** A countable basis for $\mathbb{R}^n$ is given by the collection of open balls with rational centres and rational radii,

$$\mathcal{B} := \{B(x, \varrho) : x \in \mathbb{Q}^n, \varrho \in \mathbb{Q}_+\}.$$ ▪

The most frequently used constructions on topological spaces are the subspace, product, quotient, and direct limit.

Definition 1.12 (Subspace)**.** Let A be a subset of a topological space X and let

$$\mathcal{T}_A := \{U \cap A : U \in \mathcal{T}_X\}.$$

By the distributive properties of the union and intersection,

$$\bigcup_i (U_i \cap A) = \left(\bigcup_i U_i\right) \cap A, \qquad \bigcap_i (U_i \cap A) = \left(\bigcap_i U_i\right) \cap A,$$

hence, $\mathcal{T}_A$ is closed under unions and finite intersections, and so defines a topology for A, the ***subspace topology***, and A is a ***subspace*** of X. ▪

An obvious but important property of second-countable spaces is the following:

Proposition 1.13. *A subspace of a second-countable space is second-countable.* ■

Definition 1.14 (Product space)**.** If X and Y be topological spaces, then it is easily shown (see e.g. [Tu11, Pro. A.20, p. 326]) that the collection

$$\mathcal{B} := \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

forms a basis, and the topology it generates is the ***product topology*** on the ***product space*** $X \times Y$ [Mun75, Sec. 15, Def, p. 86]. ▪

Recall that an ***equivalence relation*** on a set A is a relation that is reflexive, symmetric, and transitive. The ***equivalence class*** $[a]$ of an element $a \in A$ is the set of all elements that are equivalent to a. An equivalence relation $\sim$ partitions A into disjoint equivalence classes, and there is a natural surjection $p : A \twoheadrightarrow A/\sim$ sending each element $a \in A$ to its equivalence class $[a]$.

Definition 1.15 (Quotient space)**.** Let X be a topological space and $\sim$ an equivalence relation with natural projection $p : X \twoheadrightarrow X/\sim$. Let

$$\mathcal{T}_{X/\sim} := \left\{U \subseteq X/\sim \, : p^{-1}(U) \in \mathcal{T}_X\right\}.$$

By commutativity of the preimage with unions and intersections,

$$p^{-1}\left(\bigcup_i U_i\right) = \bigcup_i p^{-1}(U_i), \qquad p^{-1}\left(\bigcap_i U_i\right) = \bigcap_i p^{-1}(U_i),$$

it follows that $\mathcal{T}_{X/\sim}$ is closed under finite intersections and arbitrary unions. Then $\mathcal{T}_{X/\sim}$ is a topology, the ***quotient topology*** on the ***quotient space*** $X/\sim$. ▪

Note that the quotient map $p : X \twoheadrightarrow X/\sim$ is automatically continuous.

Definition 1.16 (Direct limit space)**.** If

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots$$

is a sequence of inclusions of topological spaces, their ***direct limit*** or ***colimit***, denoted

$\mathrm{Colim}_{n\in\mathbb{N}} X_n$, is the space whose underlying set is

$$X := \bigcup_{n\in\mathbb{N}} X_n$$

and whose topology is the ***final*** topology, i.e. the finest topology for which the inclusion maps $X_n \hookrightarrow X$ are all continuous. Equivalently, if $i_n : X_n \hookrightarrow X$ denotes the inclusion map for some $n \in \mathbb{N}$, then a subset $U \subseteq X$ is open if and only if $i_n^{-1}(U) \subseteq X_n$ is open for all $n \in \mathbb{N}$. ▪

Remark 1.17. Each of the above constructions (i.e. the subspace, product, quotient, and direct limit spaces) are examples of *universal* constructions known as *limits* and *colimits*. The fact that these constructions always exist is due to *bicompleteness* of the category of topological spaces [Rie17, Pro. 3.5.2, p. 100]. ▪

Numerous *separation axioms* are defined for topological spaces. The most frequently employed is the *Hausdorff* property, that will apply to manifolds (Section 1.5).

Definition 1.18 (Hausdorff space)**.** A topological space X is ***Hausdorff*** if for every distinct pair of points $x, y \in X$, there exists disjoint open sets $U, V \in \mathcal{T}_X$ with $x \in U$ and $y \in V$. ▪

Proposition 1.19. *A subspace of a Hausdorff space is Hausdorff.*

Proof. Let X be a Hausdorff space, $A \subseteq X$, and $x, y \in A$. As X is Hausdorff, there are disjoint open neighbourhoods U of x and V of y in X. Then $U \cap A$ and $V \cap A$ are disjoint open neighbourhoods of x and y respectively in A. ∎

Another fundamental property that topological spaces may possess is *compactness*. Compact spaces, while generally infinite, retain many of the qualities of finite spaces. If X is a topological space, then an ***open cover*** of X is a collection $\{U_i\}_i$ of open sets U_i for which $X \subseteq \bigcup_i U_i$. A ***subcover*** of an open cover is a subcollection whose union still contains X.

Definition 1.20 (Compact space)**.** A topological space X is ***compact*** if every open cover of X has a finite subcover. ▪

Some familiar and easily derived examples of compact spaces are the following:

Example 1.21 (Compact spaces)**.** Every closed interval in $\mathbb{R}$ is compact. Every closed disk $\mathbb{D}^n := \{x \in \mathbb{R}^n : \|x\| \leqslant 1\}$ is compact, and so are their boundary spheres $\partial\mathbb{D}^n =: \mathbb{S}^n = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Every finite union of compact subsets is compact. Every finite space is compact. ▪

Several basic and important properties of compact spaces follow.

Proposition 1.22. *Every closed subset of a compact topological space is compact.*

Proof. Let $\{U_i\}_i$ be an open cover of a closed subset A of a topological space X. The collection $\{U_i\}_i \cup \{X \setminus A\}$ is then an open cover of X that admits a finite subcover, also covering A. Hence, A is compact. ■

Proposition 1.23. *The continuous image of a compact set is compact.*

Proof. Let $f : X \to Y$ a continuous map and $A \subseteq X$ a compact subset of X. Suppose $\{U_i\}_i$ is an open cover of $f(A)$. By continuity of f, each preimage $f^{-1}(U_i)$ is open in X. Moreover,

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}\left(\bigcup_i U_i\right) = \bigcup_i f^{-1}(U_i),$$

hence, $\{f^{-1}(U_i)\}_i$ is an open cover of A. By compactness of A, there is a finite subcover $\left\{f^{-1}\left(U_{i_j}\right)\right\}_{i_j}$ for which

$$A \subseteq \bigcup_j f^{-1}\left(U_{i_j}\right) = f^{-1}\left(\bigcup_j U_{i_j}\right),$$

so that $f(A) \subseteq \bigcup_j U_{i_j}$. It follows that the image $f(A)$ is compact. ■

Arbitrary unions of compact spaces clearly have no chance at being compact, so it is somewhat surprising that arbitrary products (with the product topology) are always compact; this nontrivial result is known as the *Tychonoff theorem*.

Theorem 1.24 (Tychonoff theorem [Mun75, The. 37.3, p. 234])**.** *The product of compact topological spaces is compact.* ■

For any category of mathematical object, a basic question is if two objects are the same. In the context of topological spaces, 'sameness' means *homeomorphism*:

Definition 1.25 (Homeomorphism)**.** Two topological spaces X and Y are ***homeomorphic*** if there exists a continuous map $f : X \to Y$ with a continuous inverse $f^{-1} : Y \to X$. ▪

The notation $X \simeq Y$ and $f : X \xrightarrow{\simeq} Y$ will denote a homeomorphism, and the same notation will be used more generally to refer to an isomorphism in any category of objects with such a notion.

Classifying spaces up to homeomorphism is well known to be completely intractable. There are two complementary approaches to mitigate the difficulty. One is to restrict the class of spaces to a better-behaved subclass, such as the CW complexes (Section 1.3) or manifolds (Section 1.5). The other is to develop invariants based on coarser equivalence relations that may at least (potentially) distinguish between nonhomeomorphic spaces. One of the most important of such invariants are the *homotopy groups* of a topological space.

1.2 Homotopy

While homeomorphism is an equivalence relation of topological spaces under continuous deformations, *homotopy* is a corresponding equivalence between continuous maps. Recall that $\mathbb{I}$ denotes the closed interval $[0,1]$ and $\mathbb{I}^n = [0,1]^n$ will denote the n-dimensional unit cube for each $n \in \mathbb{N}$. Of course, $\mathbb{I}^n$ is homeomorphic to the closed n-disk $\mathbb{D}^n$.

Definition 1.26 (Homotopy)**.** Let $f, g : X \to Y$ be continuous maps between topological spaces X and Y. A ***homotopy*** between f and g is a continuous map

$$h : \mathbb{I} \times X \to Y$$

for which

$$h(0, \cdot\,) = f(\,\cdot\,), \qquad h(1, \cdot\,) = g(\,\cdot\,).$$

If there exists such a map h, then f is ***homotopic*** to g, written $f \sim g$. ▪

It is common to write the first argument of a homotopy as a subscript, so that Definition 1.26 would read: $h_0 = f$, $h_1 = g$. This custom will be adopted from now.

The notion of homotopic maps suggests the following equivalence relation of spaces:

Definition 1.27 (Homotopy equivalence)**.** Two topological spaces X and Y are ***homotopy equivalent*** if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ for which

$$g \circ f \sim \mathbb{1}_X, \qquad f \circ g \sim \mathbb{1}_Y\,. \tag{1.1}$$

This relation is denoted $X \sim Y$, and $f : X \xrightarrow{\sim} Y$ to indicate that f induces a homotopy equivalence. ▪

Observe that if the notation $\sim$ is replaced with $=$ in Equation (1.1), then the definition of homeomorphism is recovered. It is in this sense that homotopy equivalence approximates homeomorphism. A particularly simple homotopy equivalence that will be central to the formulation of Morse theory is the following:

Definition 1.28 (Deformation retract)**.** A subspace A of a topological space X is a ***deformation retract*** if there exists a homotopy

$$h : \mathbb{I} \times X \to X$$

for which

$$h(0,x) = x, \qquad h(\tau,a) \in A, \qquad h(1,x) \in A,$$

for all $x \in X$, $a \in A$, and $\tau \in \mathbb{I}$. ▪

If a space deformation retracts to a point, then it is ***contractible***. For instance, every euclidean space $\mathbb{R}^n$ and every disk $\mathbb{D}^n$ is contractible via the homotopy

$$\mathbb{I} \times X \to X, \qquad (\tau, x) \mapsto (1-\tau)x,$$

and hence, $\mathbb{R}^n$ and $\mathbb{D}^n$ are of the same homotopy type. This demonstrates that compactness is not generally preserved under homotopy equivalence.

The *homotopy groups* of a space are a sequence of abstract groups that encode information about the 'holes' in a space. Informally, the homotopy group of nth degree detects where and how an n-dimensional lasso will get caught when thrown from an anchored point. A ***pointed topological space*** (X, x) is a space with a distinguished basepoint $x : * \to X$, and a ***pointed map*** $f : (X, x) \to (Y, y)$ between pointed topological spaces is a continuous map that preserves the basepoints, i.e. for which $f(x) = y$. The homotopy groups are defined only for pointed spaces (the basepoint being the anchor point of the lasso), however, for path-connected spaces the choice of basepoint is irrelevant, and so in this context it is often suppressed from the notation.

Definition 1.29 (Homotopy groups)**.** Let $X = (X, x)$ be a pointed topological space. For each $n \in \mathbb{N}_+$, the nth ***homotopy group*** $\pi_n(X)$ of X is the set of homotopy classes of maps from the n-cube $\mathbb{I}^n$ to X that send the faces $\partial\mathbb{I}^n$ of the n-cube to the basepoint x, denoted

$$\pi_n(X) := [(\mathbb{I}^n, \partial\mathbb{I}^n), (X, x)].$$

If $f, g : \mathbb{I}^n \to X$ are maps represented by respectively $[f], [g] \in \pi_n(X)$, then their product is their 'concatenation' in the first coordinate, traversed at double-speed:

$$(g \cdot f)(\tau_1, \ldots, \tau_n) := \begin{cases} f(2\tau_1, \tau_2, \ldots, \tau_n) & : \tau_1 \in \left[0, \frac{1}{2}\right] \\ g(2\tau_1 - 1, \tau_2, \ldots, \tau_n) & : \tau_1 \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

(see Figure 1.1). The inverse f^{-1} of f has the direction of traversal in its first coordinate reversed,

$$f^{-1}(\tau_1, \ldots, \tau_n) = f(-\tau_1, \tau_2, \ldots, \tau_n).$$

This concatenation of maps passes to equivalence classes by

$$[g \cdot f] := [g] \cdot [f].$$

It is easily verified that this is well-defined, and moreover that in the passage to equivalence classes, these definitions endow $\pi_n(X)$ with the structure of a group, i.e. with an associative invertible product, and a unit element corresponding to the equivalence class of the constant map $\mathbb{I}^n \twoheadrightarrow x$ (see [BT82, Cha. 17, p. 206] for further details). ▪

For $n = 0$, $\pi_0(X)$ is the pointed set of path components of X, but this does not generally have a group structure. It is easily shown that π_n is abelian if $n \geqslant 2$ ([BT82, Pro 17.1, p. 207]), but the first homotopy group π_1, also called the ***fundamental group***, may be nonabelian.

Moreover, if $f : (X, x) \to (Y, y)$ is a pointed map of topological spaces, then it is easily verified that f induces a group homomorphism for each $n \in \mathbb{N}_+$, sending the equivalence

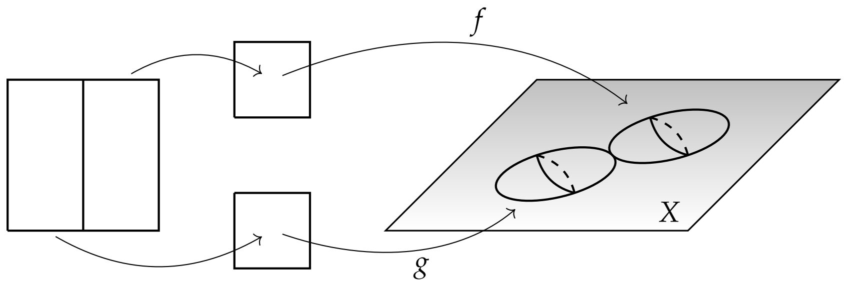


Figure 1.1. Product of elements in $\pi_n(X)$.

class of a continuous map $g : \mathbb{S}^n \to X$ to the equivalence class of $f \circ g : \mathbb{S}^n \to Y$, i.e.

$$\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, y), \qquad [g] \mapsto [f \circ g].$$

The ***loop space*** $\Omega(X)$ of a pointed space X is the mapping space of continuous pointed maps from the circle to X, i.e.

$$\Omega(X) := X^{\mathbb{S}^1},$$

where the topology is the *compact-open*[1] topology. The higher homotopy groups may also be constructed inductively through the loop spaces:

Proposition 1.30 (Homotopy groups [BT82, Pro. 17.2, p. 208])**.** *For a pointed topological space X, there is an isomorphism*

$$\pi_n(X) \simeq \pi_{n-1}(\Omega X)$$

for all $n \in \mathbb{N}_+$. ∎

On products, the behaviour of the homotopy groups is as simple as could be hoped:

Proposition 1.31. *For a product $\prod X_i$ of path-connected pointed topological spaces,*

$$\pi_n\left(\prod_i X_i\right) \simeq \prod_i \pi_n(X_i) \tag{1.2}$$

for all $n \in \mathbb{N}$.

Proof. A map $Y \to \prod_i X_i$ is the same thing as a collection of maps $\{Y \to X_i\}_i$. Taking Y to be $\mathbb{S}^n$ and $\mathbb{I} \times \mathbb{S}^n$ gives the required bijection Equation (1.2). But as $(f_i)_i (g_i)_i = (f_i g_i)_i$, this bijection is moreover a group isomorphism. ∎

Example 1.32 (Homotopy groups of spheres [BT82, Pro. 17.9, Pro. 17.10, pp. 214–215])**.**

[1] The ***compact-open*** topology on a mapping space X^Y is the topology generated by the subbasis of subsets $U^C \subseteq X^Y$ that map a compact subspace $C \subseteq Y$ to an open set $U \in T_X$.

The homotopy groups of the 0-sphere are all trivial, and of the circle are

$$\pi_i\left(\mathbb{S}^1\right) = \begin{cases} \mathbb{Z} & : i = 1 \\ 0 & : i > 1. \end{cases}$$

For higher spheres, $\pi_n(\mathbb{S}^n) = \mathbb{Z}$ holds, and it is also true that $\pi_i(\mathbb{S}^n) = 0$ for all $i < n$ (see Corollary 1.44), but the underlying pattern in the homotopy groups of spheres is mysterious; for example, the values of $\pi_i\left(\mathbb{S}^2\right)$ for $i > 64$ are unknown [Hat02, Sec. 4.1, p. 339]. ▪

In Definition 1.16, the colimit of a sequence of topological spaces was defined, and it was mentioned that this is an example of a more general universal construct in Remark 1.17. Colimits of groups are also well-defined (by cocompleteness of the category of groups; see [Rie17, Cha. 3, p. 100]). If

$$G_0 \hookrightarrow G_1 \hookrightarrow \cdots \tag{1.3}$$

is a sequence of group homomorphisms, let $f_{ij} : G_i \to G_j$ denote the induced map by these inclusions whenever $i < j$, and let $f_{ii} := \mathbb{1}_{G_i}$ be the identity map for each i. The colimit of the sequence (1.3) is the group defined by

$$\operatorname*{Colim}_{i \in \mathbb{N}} G_i := \frac{\bigoplus_{i \in \mathbb{N}} G_i}{H},$$

where H is the subgroup generated by all elements of the form $g - f_{ij}(g)$ for $g \in G_i \leqslant \bigoplus_{i \in \mathbb{N}} G_i$. Heuristically, elements of the direct sum of the G_i are identified in the colimit if and only if they 'ultimately become equal' via the sequence of inclusions. The following result, which will be recalled in the proof of the main theorem of Chapter 2, establishes that homotopy groups commute with colimits.

Lemma 1.33 ([May99, Cha. 4, p. 67])**.** *Let X be the colimit of a sequence of inclusions $X_i \hookrightarrow X_{i+1}$ of pointed spaces. Then the natural[2] map*

$$\operatorname*{Colim}_{i \in \mathbb{N}} \pi_n(X_i) \to \pi_n(X)$$

is an isomorphism for each $n \in \mathbb{N}$.

Proof. As a sphere is compact, its image under every map $\mathbb{S}^n \to X$ is compact by Proposition 1.23, and hence, contained in one of the X_i. The result follows. ■

1.3 CW complexes

Definition 1.1 is impressive in its capacity to distil the concept of space to its essence. However, the economy of this definition comes at the cost of admitting a vast variety of

[2] *Natural* refers to *natural transformation*, discussed at the end of Section 1.4.

pathological examples that may not be intelligibly incorporated into the theory. To make progress, a subclass of topological spaces that are not too wild to reason with but still include enough interesting spaces to study must be isolated. Milnor has argued [Mil59] that the most appropriate such class for the purposes of homotopy theory are the *CW complexes*. These are spaces that are built up inductively by *attaching cells*. The vital property of CW complexes is that a continuous map that induces isomorphisms of homotopy groups in all dimensions (a ***weak equivalence***) is necessarily a homotopy equivalence (Theorem 1.45).

Definition 1.34 (Attaching cells)**.** Let X be a topological space and $a : \partial\mathbb{D}^n \to X$ a continuous map from the boundary of the n-disk $\partial\mathbb{D}^n \simeq \mathbb{S}^{n-1}$ to X (note that $\mathbb{S}^{-1} = \varnothing$). Set $X \cup_a \mathbb{D}^n$ to be the space obtained from the disjoint union of X and the n-disk by identifying the boundary of the disk with its image under a, i.e.

$$X \cup_a \mathbb{D}^n := \frac{X \coprod \mathbb{D}^n}{\partial\mathbb{D}^n \sim a(\partial\mathbb{D}^n)}$$

(see Figure 1.2). Then $X \cup_a \mathbb{D}^n$ is the ***attaching space*** of X and the n-disk, and a is the ***attaching map***. The topology on $X \cup_a \mathbb{D}^n$ is the quotient topology (Definition 1.15). The image of $\mathbb{D}^n$ is a ***closed*** n***-cell*** in X, and the image of its interior $\operatorname{Int}\mathbb{D}^n = \mathbb{D}^n \smallsetminus \partial\mathbb{D}^n$ is an ***open*** n***-cell*** in X, or simply an n***-cell***, often denoted e^n. Each attaching map a extends to a ***characteristic map*** $b : \mathbb{D}^n \to X$ that is a homeomorphism $\operatorname{Int}\mathbb{D}^n \xrightarrow{\cong} e^n$ from the interior of the disk to the corresponding open cell e^n in X. Note that while an open n-cell is open in X_n, it is generally not open in X. If the attaching map is unknown or unimportant, then the attaching space is equivalently written $X \cup_{\partial\mathbb{D}^n} \mathbb{D}^n$, or even $X \cup \mathbb{D}^n$.

If there is a (possibly infinite) set of attaching maps

$$\{a_i : \partial\mathbb{D}^n \to X\}_i,$$

then this may instead be viewed as a single continuous map whose domain is the disjoint union of the domain spheres,

$$a : \coprod_i \partial D^n \to X.$$

Then all the cells may be attached simultaneously by setting, in an analogous way:

$$X \cup_a \left(\coprod_i \mathbb{D}^n\right) := \frac{X \amalg (\coprod_i \mathbb{D}^n)}{\coprod_i \partial\mathbb{D}^n \sim a(\coprod_i \partial D^n)}.$$ ▪

Definition 1.35 (CW complex)**.** A ***CW complex*** is a topological space X with an increasing sequence of closed subspaces

$$X^0 \hookrightarrow X^1 \hookrightarrow \cdots \hookrightarrow X$$

for which:

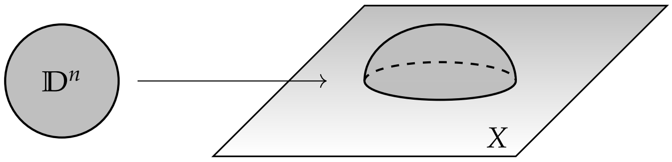


Figure 1.2. Attaching a cell $\mathbb{D}^n$ to a space X.

1. X^0 is a discrete set;
2. X^n is obtained from X^{n-1} by attaching n-cells, for $n \in \mathbb{N}_+$;
3. X is the colimit of the inclusions $X^0 \hookrightarrow X^1 \hookrightarrow \cdots$.

The subspace $X^n \subseteq X$ is the ***n*-skeleton** of X, and a continuous map $f : X \to Y$ of CW complexes is ***cellular*** if it maps n-skeletons to n-skeletons, i.e. if $f(X^n) \subseteq Y^n$ for each $n \in \mathbb{N}$. A ***subcomplex*** A of a CW complex X is a subspace that is also a *CW* complex, for which the image of the composite of each characteristic map $\mathbb{D}^n \to A$ with the inclusion $A \hookrightarrow X$ is a cell of X. A ***finite subcomplex*** is a subcomplex that contains only finitely many cells. ▪

If the process in Definition 1.35 terminates at some finite n, i.e. $X^k = X^n$ for all $k > n$ and $X^n \neq X^{n-1}$, then X is ***finite-dimensional*** of ***dimension*** n, which is the maximum dimension of cells in X (and in this this case, condition 3 is redundant).

Remark 1.36. A closely related construct is that of a ***cell complex***, which is the same as a CW complex except without the requirement that X^n be obtained from X^{n-1} by attaching n-cells: the cells may be attached in any order. This distinction will be relevant in the formulation of the structure theorem of classical Morse theory (Theorem 2.20). ▪

The letter 'W' in *CW complex* refers to *weak topology*, which is archaic terminology for the colimit topology, and the 'C' refers to *closure-finiteness*, which is the property that the closure $\mathrm{Cl}\, e^n$ of each open cell e^n is contained in a finite complex. As CW complexes are Hausdorff [Hat02, Pro. A.3, p. 522] and the closure of a compact Hausdorff space is compact [Mun75, The. 26.2, p. 165], closure-finiteness for CW complexes is a corollary of the following result:

Proposition 1.37 ([Hat02, Pro. A.1, p. 520])**.** *A compact subspace of a CW complex is contained in a finite subcomplex.* ■

Another basic property that will follow from the results in Chapter 2 is the following:

Proposition 1.38 (cf. [Hat02, Cor. A.8, Cor. A.10, p. 527])**.** *The homology groups of a finite CW complex have finite rank.* ■

Below are several examples of CW structures on familiar spaces.

Example 1.39 (Euclidean spaces)**.** The real line $\mathbb{R}$ has a 1-dimensional CW structure with the integers $\mathbb{Z} \hookrightarrow \mathbb{R}$ as 0-cells and the closed intervals $\{[n, n+1] : n \in \mathbb{Z}\}$ as 1-cells. This construction generalises to give $\mathbb{R}^n$ a CW structure in an analogous way, where the n-cells are n-cubes whose vertices are 0-cells in the integral lattice $\mathbb{Z}^n$. ▪

Example 1.40 (Spheres)**.** The n-sphere $\mathbb{S}^n$ for $n \in \mathbb{N}$ has a CW decomposition with one 0-cell and one n-cell, given by $X^0 = *$ and $X^n = * \cup_a \mathbb{D}^n$, where $a : \mathbb{S}^{n-1} \to *$ is the constant map. This is equivalent to the quotient space construction $\mathbb{S}^n = \frac{\mathbb{D}^n}{\partial D^n}$.

Another CW decomposition of the sphere is obtained by taking the attaching map to be the inclusion map $i : \mathbb{S}^{n-1} = \partial\mathbb{D}^{n-1} \hookrightarrow \mathbb{D}^{n-1}$ itself; this corresponds to gluing the two n-disks along their boundaries:

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, hook, "i"] \arrow[d, hook, "i"'] & \mathbb{D}^n \arrow[d, hook] \\ \mathbb{D}^n \arrow[r, hook] & \mathbb{D}^n \cup_i \mathbb{D}^n = \mathbb{S}^n. \end{tikzcd}$$

By induction, a CW structure of $\mathbb{S}^n$ with two cells in each dimension $i \in \{0, \ldots, n\}$ is obtained. ▪

The last example demonstrates that a topological space may be given nonequivalent CW structures.

Example 1.41 (Projective spaces)**.** The real projective space $\mathbb{RP}^n$ is realised as a quotient space of the n-sphere by the $\mathbb{Z}_2$ action identifying antipodal points,

$$\mathbb{RP}^n := \frac{\mathbb{S}^n}{x \sim -x}.$$

Giving $\mathbb{S}^n$ the second CW structure of Example 1.40, it is easily verified that the $\mathbb{Z}_2$-action restricts to a homeomorphism of the two cells in each dimension, thus giving $\mathbb{RP}^n$ a CW structure with a single i-cell for $i \in \{0, \ldots, n\}$.

The complex projective space $\mathbb{CP}^n$ is realised as a quotient of $\mathbb{S}^{2n+1} \subset \mathbb{C}^n$ by the circle action $z \mapsto e^{i\vartheta}z$, for $e^{i\vartheta} \in \mathbb{S}^1$. In Example 2.58 a CW structure on $\mathbb{CP}^n$ with one cell in each even dimension $i \in \{0, 2, \ldots, 2n\}$ is deduced with Morse theory. ▪

Definition 1.35 allows for CW complexes with cells in infinitely many dimensions. Each of the last examples of finite CW complexes have infinite-dimensional analogues:

Example 1.42 (Infinite-dimensional CW complexes)**.** In each of Examples 1.39 to 1.41, a CW complex was produced via an inductive process in a finite number of steps. Allowing the respective processes to continue indefinitely, the infinitary analogues of those spaces are denoted $\mathbb{R}^\infty$, $\mathbb{S}^\infty$, $\mathbb{RP}^\infty$, and $\mathbb{CP}^\infty$. ▪

A key property of CW complexes that will play a crucial role in the proof of the main theorem of Chapter 2 (Theorem 2.20) is that every continuous map between CW complexes is a cellular map 'up to homotopy':

Theorem 1.43 (Cellular approximation theorem [Hat02, The. 4.8, p. 349])**.** *Every continuous map $f : X \to Y$ of CW complexes is homotopic to a cellular map.* ■

Corollary 1.44. *$\pi_m(\mathbb{S}^n) = 0$ if $m < n$ for $n, m \in \mathbb{N}$.*

Proof. Let each sphere have the first CW structure given in Example 1.40, with the 0-cell as basepoint. Then every pointed map $\mathbb{S}^m \to \mathbb{S}^n$ is homotopic to a cellular map by Theorem 1.43, and hence, constant if $m < n$. ■

It is important to note that the homotopy groups do not even distinguish spaces up to homotopy; an example is given by the pair $\mathbb{RP}^2$ and $\mathbb{S}^3 \times \mathbb{RP}^\infty$, which have isomorphic homotopy groups in all dimensions but are not homotopy equivalent [Hat02, The. 4.5, p. 348]. It is in this respect that CW complexes are distinguished:

Theorem 1.45 (Whitehead's theorem [Hat02, The. 4.5, p. 346])**.** *If a continuous map $f : X \to Y$ between connected CW complexes induces isomorphisms*

$$f_* : \pi_n(X) \to \pi_n(Y)$$

for all $n \in \mathbb{N}_+$, then f is a homotopy equivalence. Moreover, if f is an inclusion of a subcomplex $X \hookrightarrow Y$, then X is a deformation retract of Y.

Remark 1.46. A topological space X is ***dominated*** by a topological space Y if there are continuous maps

$$X \xhookrightarrow{i} Y \xrightarrow{r} X$$

for which

$$r \circ i \sim \mathbb{1}_X .$$

Theorem 1.45 applies more generally to topological spaces dominated by CW complexes [Whi49a, The. 1, p. 215]. ▪

Remark 1.47. In Theorem 1.45, it is essential that one has the map f; it is not enough that the homotopy groups of two CW complexes coincide, as the remarks preceding the theorem illustrate in the case of $\mathbb{RP}^2$ and $\mathbb{S}^3 \times \mathbb{RP}^\infty$. Nevertheless, with this caveat understood, the homotopy groups play a role of complete invariants in the class of CW complexes. ▪

Recorded below is a simple fact that will be recalled in the proof of the main theorem in Chapter 2 (Theorem 2.20).

Lemma 1.48 ([May99, Cha. 10, Sec. 2, p. 74])**.** *The colimit of a sequence of inclusions of subcomplexes $X_n \hookrightarrow X_{n+1}$ in CW complexes is a CW complex that contains each of the X_n as a subcomplex.* ■

1.4 Homology

The homotopy groups of a space are notoriously difficult to compute. One of the outstanding problems of mathematics is to understand the homotopy groups of spheres,

$\pi_n(\mathbb{S}^m)$ for $n, m \in \mathbb{N}_+$. In fact, there does not exist a noncontractible ***simply-connected***—i.e. with $\pi_1 = 0$—compact manifold whose homotopy groups are all known [May99, p. 67]. Another fundamental invariant of a space are its *homology groups*, which are substantially easier to compute. On one hand, the homology groups serve as a sort of linear approximation to the homotopy groups. On the other hand, homology groups may detect features that homotopy groups do not—a principal example is *orientability* (see Section 1.5). The homology groups relate to the homotopy groups via a map known as the *Hurewicz homomorphism* (see e.g. [Hat02, Pro. 4.36, p. 369]). Various formulations of homology exist, but for 'reasonable' spaces they coincide. The quintessential example is *singular homology*.

The map
$$(x_1, \dots, x_n) \mapsto \{x_1, \dots, x_n, 0\}$$
induces an inclusion of euclidean spaces $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ for each $n \in \mathbb{N}$. Viewing each $\mathbb{R}^n$ as a vector subspace of $\mathbb{R}^{n+1}$, the infinite euclidean space is then the infinite-dimensional vector space
$$\mathbb{R}^\infty = \bigcup_{n \in \mathbb{N}} \mathbb{R}^n$$
(cf. Example 1.42). Let e_i denote the ith standard basis vector of $\mathbb{R}^n$, i.e. the vector with 1 in the ith coordinate and 0 elsewhere, and let e_0 denote the origin.

Definition 1.49 (Singular homology)**.** The ***standard n-simplex*** is the subset
$$\Delta^n := \left\{ \sum_{i=0}^{n} \tau_i e_i : \sum_{i=0}^{n} \tau_i = 1, \tau_i \geqslant 0 \right\}.$$
If X is a topological space, then a ***singular n-simplex*** in X is a continuous map
$$\Delta^n \to X,$$
and a ***singular n-chain*** in X is a finite $\mathbb{Z}$-linear combination of singular n-simplices in X. Together, the singular chains form an abelian group $C_n(X)$, freely generated by the singular n-simplices. There are ***face maps***
$$\partial_n^k : \Delta^{n-1} \to \Delta^n, \qquad \sum_{i=0}^{n-1} \tau_i e_i \mapsto \sum_{i=0}^{k-1} \tau_i e_i + \sum_{i=k+1}^{n} \tau_{i-1} e_i.$$
(see Figure 1.3). The $\mathbb{N}$-graded abelian group of singular chains,
$$C_\bullet(X) := \bigoplus_{n \in \mathbb{N}} C_n(X)$$

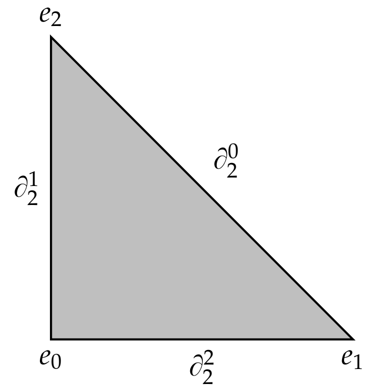


Figure 1.3. Images of face maps on the standard 1-simplex.

is equipped with ***boundary maps*** ∂_n that act on the generators by

$$s \mapsto \sum_{i=0}^{n} (-1)^i s \circ \partial_n^i,$$

and linearly extends to a group homomorphism

$$\partial_n : C_n(X) \to C_{n-1}(X).$$

Moreover,

$$(\partial_n \circ \partial_{n+1})(s) = \sum_{j=0}^{n} \sum_{i=0}^{n+1} (-1)^{i+j} s \circ \partial_{n+1}^i \circ \partial_n^j = 0,$$

as if $i < j$, then the (i, j) and $(j-1, i)$ summands add to zero. It follows that $\operatorname{Im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n$. The pair

$$(C_\bullet(X), \partial) := (C_n(X), \partial_n)_{n \in \mathbb{N}}$$

is a ***chain complex***. If $\mathbb{A}$ is a commutative unital ring, then tensoring with $\mathbb{A}$ gives the corresponding ***chain complex with coefficients*** in $\mathbb{A}$,

$$C_\bullet(X; \mathbb{A}) := C_\bullet(X) \otimes \mathbb{A}.$$

The nth ***singular homology*** group of X with coefficients in $\mathbb{A}$ is then defined:

$$H_n(X; \mathbb{A}) := \frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}}, \qquad \partial : C_n(X; \mathbb{A}) \to C_{n-1}(X; \mathbb{A}).$$

The resulting $\mathbb{N}$-graded $\mathbb{A}$-module is denoted

$$H_\bullet(X; \mathbb{A}) := \bigoplus_{n \in \mathbb{N}} H_n(X; \mathbb{A}). \qquad \blacksquare$$

Customarily, the notation $H_\bullet(X)$ stands for the homology of X with $\mathbb{Z}$-coefficients, however in Chapters 2 and 3, attention will be restricted to homology with field coefficients. For legibility, unless otherwise stated, $H_\bullet(X)$ will refer to $H_\bullet(X;\mathbb{F})$ for an arbitrary field $\mathbb{F}$.

Proposition 1.50. *Let X be a nonempty path-connected topological space. Then $H_0(X;\mathbb{A}) = \mathbb{A}$.*

Proof. Define the ***degree*** map on 0-chains by

$$\varepsilon : C_0(X;\mathbb{A}) \to \mathbb{A}, \qquad \sum_i n_i s_i \mapsto \sum_i n_i.$$

This map is clearly surjective as X is nonempty. If $s : \Delta^1 \to X$ is a singular 1-simplex, then

$$(\varepsilon \circ \partial_1)s = \varepsilon(e_1 - e_0) = 1 - 1 = 0,$$

and it follows that $\operatorname{Im}\partial_1 \subseteq \operatorname{Ker}\varepsilon$. On the other hand, suppose

$$\varepsilon\left(\sum_i n_i s_i\right) = \sum_i n_i = 0, \tag{1.4}$$

and let $x \in X$ be a point and $s_0 : \Delta^0 \to x$ the singular 0-simplex with image x. By path-connectedness of X, there are paths

$$f_i : \mathbb{I} \to X, \qquad f(0) = s_0(e_0), \qquad f(1) = s_i(e_1).$$

The paths f_i are then singular 1-simplices with boundaries $\partial f_i = s_i - s_0$. Hence, by Equation (1.4),

$$\partial\left(\sum_i n_i f_i\right) = \sum_i n_i s_i - \sum_i n_i s_0 = \sum_i n_i s_i,$$

so $\sum_i n_i s_i$ is a boundary, and it follows $\operatorname{Ker}\varepsilon \subseteq \operatorname{Im}\partial_1$. It follows that $\operatorname{Ker}\varepsilon = \operatorname{Im}\partial_1$, and so there is a short exact sequence

$$0 \hookrightarrow \partial C_1(X;\mathbb{A}) \hookrightarrow C_0(X;\mathbb{A}) \xrightarrow{\varepsilon} \mathbb{A} \twoheadrightarrow 0,$$

and hence,

$$H_0(X;\mathbb{A}) = \frac{\operatorname{Ker}\partial_0}{\operatorname{Im}\partial_1} = \frac{C_0(X;\mathbb{A})}{\operatorname{Ker}\varepsilon} = \frac{\partial C_1(X;\mathbb{A}) \oplus \mathbb{A}}{\operatorname{Ker}\varepsilon} = \frac{\operatorname{Ker}\varepsilon \oplus \mathbb{A}}{\operatorname{Ker}\varepsilon} = \mathbb{A}. \qquad \blacksquare$$

For a continuous map $f : X \to Y$ between topological spaces, precomposition with a singular simplex $s : \Delta^n \to X$ on X gives a singular simplex $f \circ s : \Delta^n \to Y$ on Y. Linear extension then induces a homomorphism $C_n(X) \to C_n(Y)$, and in fact a homomorphism

$$H_n(f) : H_n(X) \to H_n(Y),$$

that commutes with compositions, in the sense that if $g : Y \to Z$ is another continuous map, then

$$H_n(g \circ f) = H_n(g) \circ H_n(f),$$

and that preserves identities,

$$H_n(\mathbb{1}_X) = \mathbb{1}_{H_n(X)}$$

([Hat02, Pro. 2.19, p. 111]). This behaviour is ***functoriality***, and $H_n(\,\cdot\,;\mathbb{A})$ is an example of a ***functor*** from topological spaces to abelian groups. The homotopy groups $\pi_n(\,\cdot\,,\,\cdot\,)$ are also functors, from pointed topological spaces to groups. Crucially, the singular homology of a space is a homotopy invariant, so in particular, singular homology may enable the differentiation between spaces of differing homotopy type.

Theorem 1.51 ([Hat02, The. 2.10, p. 112]). *Let $f, g : X \to Y$ be homotopic maps between topological spaces X and Y. Then f and g induce the same map in homology:*

$$H_\bullet(f) = H_\bullet(g) : H_\bullet(X) \to H_\bullet(Y).$$

In particular, a homotopy equivalence induces an isomorphism in homology. ■

The term 'singular' refers to the singularities that the singular simplices generally have, being merely continuous images of standard simplices; while the chain complex itself is a free module with a (generically) uncountable basis of these singular simplices. This makes singular homology an unwieldy tool to carry out computations. The merit of singular homology is rather in its generality and theoretical simplicity. For CW complexes, a much more practical homology theory exists, *cellular homology*, which is isomorphic to the singular homology. [Hat02, Sec 2.2, p. 137]. For the purposes of the ensuing discussion, only the formal properties of homology are relevant; these are summarised in the *Eilenberg–Steenrod axioms* that are listed at the end of this section (Theorem 1.59).

A generalisation of homology applies to ***topological pairs*** of spaces (X, A), where X is a space and $A \subseteq X$ is a subspace.

Definition 1.52 (Relative homology). Let (X, A) be a topological pair where $i : A \hookrightarrow X$ is the inclusion, and let $\mathbb{A}$ be a commutative unital ring. The inclusion i induces an $\mathbb{N}$-graded $\mathbb{A}$-module homomorphism by

$$i_* : C_\bullet(A;\mathbb{A}) \to C_\bullet(X;\mathbb{A}), \qquad c \mapsto c \circ i.$$

Denote

$$C_n(X, A;\mathbb{A}) := \frac{C_n(X;\mathbb{A})}{i_*(C_n(A;\mathbb{A}))}.$$

It is easily checked that ∂ fixes $C_\bullet(A;\mathbb{A})$ and hence, induces a map $\partial' : C_\bullet(X, A;\mathbb{A}) \to C_{\bullet-1}(X, A;\mathbb{A})$ that satisfies $\partial' \circ \partial' = 0$; thus, the ***relative homology*** of the pair (X, A) is well-defined, and is denoted

$$H_\bullet(X, A;\mathbb{A}) := \frac{\operatorname{Ker}\partial'}{\operatorname{Im}\partial'}.$$

Note that $C_\bullet(X,\varnothing;\mathbb{A}) = C_\bullet(X;\mathbb{A})$ and $H_\bullet(X,\varnothing;\mathbb{A}) = H_\bullet(X;\mathbb{A})$, so the relative definitions properly generalise Definition 1.49. ▪

A singular $(n+1)$-chain $c \in C_n(X,A;\mathbb{A})$ is then a relative A-cycle if $c \in C_n(X;\mathbb{A})$ is a chain whose boundary ∂c, while not vanishing, is in $C_{n-1}(X,A;\mathbb{A})$. That is, it vanishes only up to contributions from A.

In many cases, the relative homology of a pair (X,A) coincides with the homology of the quotient space $\frac{X}{A}$. A sufficient condition for this to occur is the following:

Definition 1.53 (Good pair)**.** A topological pair (X,A) is a ***good pair*** if:

1. A is closed inside X;
2. A is a deformation retract of a neighbourhood in X. ▪

Proposition 1.54 ([Hat02, Pro. 2.22, p. 124])**.** *Let (X,A) be a good pair. Then the quotient map*

$$(X,A) \twoheadrightarrow \left(\frac{X}{A},*\right)$$

induces isomorphisms in homology

$$H_n(X,A;\mathbb{A}) \xrightarrow{\simeq} H_n\left(\frac{X}{A},*;\mathbb{A}\right) =: \widetilde{H}_n\left(\frac{X}{A};\mathbb{A}\right)$$

for all $n \in \mathbb{N}$. ■

The groups $\widetilde{H}_\bullet(X;\mathbb{A}) := H_\bullet(X,*;\mathbb{A})$ are the ***reduced homology***, and are defined for a nonempty topological space X as the homology of the augmented chain complex

$$\cdots \longrightarrow C_1(X;\mathbb{A}) \xrightarrow{\partial} C_0(X;\mathbb{A}) \xrightarrow{\varepsilon} \mathbb{A} \longrightarrow 0,$$

where $\varepsilon : \sum_i n_i s_i \mapsto \sum_i n_i$ is the degree map defined in Proposition 1.50. The added group $\mathbb{A}$ at the end of this augmented chain complex is generated by the unique map $\varnothing \to X$ from the empty simplex to X. These reduced groups differ only in degree zero,

$$H_n(X;\mathbb{A}) = \begin{cases} \widetilde{H}_n(X;\mathbb{A}) \oplus \mathbb{A}, & n = 0 \\ \widetilde{H}_n(X;\mathbb{A}), & n \neq 0. \end{cases}$$

Moreover, if (X,A) is a topological pair with $A \neq \varnothing$, then there is no difference between reduced and unreduced homology, i.e. $\widetilde{H}_\bullet(X,A;\mathbb{A}) = H_\bullet(X,A;\mathbb{A})$ [Hat02, Sec. 2.1, p. 118]. It often happens that an equality in homology is true for all dimensions except zero, where an exceptional case is needed; the role of the reduced groups is to alleviate this awkwardness.

The following example describes the homology groups of spheres, which clarify the comparative simplicity of this invariant compared with the homotopy groups (cf. Example 1.32).

Example 1.55 (Homology of a sphere (cf. [Hat02, Cor. 2.14, p.114])). The reduced homology of the sphere is

$$\widetilde{H}_i(\mathbb{S}^n;\mathbb{A}) = \begin{cases} \mathbb{A} & : i = n \\ 0 & : i \neq n. \end{cases}$$ ▪

The major limitation in the computability of homotopy groups is the lack of a general tool to express them as a 'sum of parts'.[3] For homology, such a tool is available:

Theorem 1.56 (Mayer-Vietoris sequence [Hat02, Sec. 2.2, p. 149]). *Let $\{A, B\}$ be an open cover of a topological space X with nonempty intersection. Then there is a long exact sequence in homology (coefficients suppressed):*

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A \cap B) \longrightarrow \cdots$$ ■

For CW complexes, another such tool is the following:

Theorem 1.57 (Künneth formula [Hat02, Cor. 3B.7, p. 276]). *Let X and Y be CW complexes spaces and $\mathbb{F}$ a field. Then there is a natural isomorphism*

$$H_\bullet(X \times Y; \mathbb{F}) \xrightarrow{\cong} H_\bullet(X; \mathbb{F}) \otimes_{\mathbb{F}} H_\bullet(Y; \mathbb{F})$$ ■

An important consequence of the Mayer-Vietoris sequence applied to attaching cells, which will be used repeatedly (and sometimes implicitly) in the sequel, is the following:

Proposition 1.58 (cf. [BT82, Pro. 17.12, p. 219]). *Attaching an n-cell to a space X does not alter its homology except in degrees $n-1$ or n. Moreover, there is an exact sequence (coefficients suppressed):*

$$0 \rightarrowtail H_n(X) \rightarrowtail H_n(X \cup_a \mathbb{D}^n) \longrightarrow \mathbb{A} \xrightarrow{a_*} H_{n-1}(X) \twoheadrightarrow H_{n-1}(X \cup_a \mathbb{D}^n) \twoheadrightarrow 0,$$

where $a_* : H_{n-1}(\mathbb{S}^{n-1}) \to H_{n-1}(X)$ is the induced map. Hence, the inclusion $X \hookrightarrow X \cup_a \mathbb{D}^n$ induces a surjection in degree $n-1$ and an injection in degree n.

Proof. Let

$$U := X \cup_a \mathbb{D}^n \setminus *, \qquad V := \left\{ x \in \mathbb{D}^n : \|x\| < \frac{1}{2} \right\},$$

where $*$ is the origin of the cell $\mathbb{D}^n$. Then U is homotopy equivalent to X, V is contractible and $\{U, V\}$ is an open cover of $X \cup_a \mathbb{D}^n$. By the Mayer-Vietoris sequence (Theorem 1.56), the following sequence is exact:

$$\cdots \longrightarrow H_i(\mathbb{S}^{n-1}) \longrightarrow H_i(X) \oplus H_i(V) \longrightarrow H_i(X \cup_a \mathbb{D}^n) \longrightarrow H_{i-1}(\mathbb{S}^{n-1}) \longrightarrow \cdots.$$

[3] Actually, a homotopy analogue of Theorem 1.56 exists, but only for π_1; it is known as *van Kampen's theorem* [Hat02, The. 1.20, p. 43].

Then if $i \neq n-1$ or $i \neq n$, there is an isomorphism $H_i(X \cup_a \mathbb{D}^n) = H_i(X)$ (recall Example 1.55). For $i = n$,

$$0 \rightarrowtail H_n(X) \rightarrowtail H_n(X \cup_a \mathbb{D}^n) \longrightarrow H_{n-1}\big(\mathbb{S}^{n-1}\big) \xrightarrow{a_*} H_{n-1}(X) \twoheadrightarrow H_{n-1}(X \cup_a \mathbb{D}^n) \twoheadrightarrow 0,$$

and the result follows. ∎

For relative homology, there is a ***connecting map*** $\delta : H_\bullet(\,\cdot\,,\,\cdot\,) \to H_{\bullet-1}(\,\cdot\,)$ which is an example of a ***natural transformation***: this is a map between functors that is canonical, in the sense that the following diagram is commutative for every continuous map $f : (X, A) \to (Y, B)$ of pairs of topological spaces (i.e. a continuous map $f : X \to Y$ for which $f(A) \subseteq B$) and every $n \in \mathbb{N}$:

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{H_n(f)} & H_n(Y,B) \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle\delta} \\ H_{n-1}(A) & \xrightarrow[H_{n-1}(f)]{} & H_{n-1}(B). \end{array}$$

There is a set of axioms, the *Eilenberg-Steenrod axioms*, that define the archetype of homology, in that any functor satisfying them is called a *homology functor*. The singular homology functor satisfies these axioms, and moreover, all the facts and properties described in this section follow formally from the axioms, and hence, apply to any homology functor [Hat02, Sec. 2.3, p. 160].

Theorem 1.59 (Eilenberg–Steenrod axioms (cf. [May99, Sec. 13.1, p. 95]))**.** *Let G be an abelian group and $n \in \mathbb{N}$. The nth singular homology is a functor $H_n(\,\cdot\,,\,\cdot\,; G)$ from pairs of topological spaces to abelian groups, and there are natural transformations*

$$\delta : H_n(X, A; G) \to H_{n-1}(A; G).$$

for each topological pair (X, A). Moreover, these functors and natural transformations satisfy the following properties:

1. **DIMENSION** *If $X \simeq *$, then* $H_n(X; G) := \begin{cases} G & : n = 0 \\ 0 & : n \neq 0. \end{cases}$

2. **EXACTNESS** *The sequence*

$$\cdots \longrightarrow H_n(A; G) \longrightarrow H_n(X; G) \longrightarrow H_n(X, A; G) \xrightarrow{\delta} H_{n-1}(A; G) \longrightarrow \cdots$$

is exact, where the unlabelled arrows are induced from the inclusions $A \hookrightarrow X$ and $(X, \varnothing) \hookrightarrow (X, A)$.

3. **EXCISION** *If* $B \subseteq A \subseteq X$ *and* $\operatorname{Cl} B \subseteq \operatorname{Int} A$*, then the inclusion*

$$(X \setminus B, A \setminus B) \hookrightarrow (X, A)$$

induces an isomorphism

$$H_\bullet(X \setminus B, A \setminus B; G) \xrightarrow{\simeq} H_\bullet(X, A; G).$$

4. **ADDITIVITY** *If* $(X, A) := \coprod_i (X_i, A_i)$ *is a disjoint union of pairs, then the inclusions* $(X_i, A_i) \hookrightarrow (X, A)$ *induce an isomorphism*

$$\bigoplus_i H_\bullet(X_i, A_i; G) \xrightarrow{\simeq} H_\bullet(X, A; G).$$

5. **HOMOTOPY** *If* $f, g : (X, A) \to (Y, B)$ *are homotopic maps, then*

$$H_\bullet(f; G) = H_\bullet(g; G) : H_\bullet(X, A; G) \to H_\bullet(Y, B; G). \qquad \blacksquare$$

Remark 1.60. The natural transformation δ of Theorem 1.59 is closely related to the boundary map ∂. By a basic result of homological algebra [Hat02, The. 2.12, p.113] (sometimes known as the *snake lemma*), an exact sequence of chain complexes (coefficients suppressed):

$$0 \longrightarrow C_\bullet(A) \xrightarrow{\ i\ } C_\bullet(X) \xrightarrow{\ q\ } C_\bullet(X, A) \longrightarrow 0,$$

induces the long exact sequence of property 2 of Theorem 1.59,

$$\cdots \longrightarrow H_n(A) \xrightarrow{\ i_*\ } H_n(X) \xrightarrow{\ q_*\ } H_n(X, A) \xrightarrow{\ \delta\ } H_{n-1}(A) \longrightarrow \cdots.$$

The induced maps act in the following way:

$$\begin{aligned} i_*(c + \partial C_{n+1}(A)) &= c + \partial C_{n+1}(X) \\ q_*(c + \partial C_{n+1}(X)) &= c + \partial C_{n+1}(X) + C_n(A) \\ \delta(c + \partial C_{n+1}(X) + C_n(A)) &= \partial c + \partial C_n(A). \end{aligned}$$

In other words, the connecting map δ is exactly the boundary map ∂ applied to the cosets in $H_n(X, A)$. For this reason, from now on both maps will be called ∂. ▪

1.5 Smooth manifolds

Roughly speaking, a *manifold* is a topological space that is locally homeomorphic to a euclidean space, and a *smooth manifold* is a manifold on which one can do calculus. A manifold is then an exceptionally regular topological space. In particular, the *dimension* of a manifold is a well-defined invariant. Many of the classical examples of spaces from geometry may be given the structure of a smooth manifold; some that have already been mentioned are the euclidean spaces, the spheres, and projective spaces. There are notable exclusions: for example, a disk of positive dimension is not a manifold as it has a boundary, and many other important spaces would be manifolds if it were not for isolated singularities. Nevertheless, the theory of manifolds serves as a guidepost and point of departure in the broader study of topology and geometry.

Definition 1.61 (Smooth manifold)**.** A ***smooth manifold of dimension*** n, or an n***-manifold*** consists of the following data:

1. a second-countable Hausdorff space M;
2. an open cover $\{U_i\}_i$ of M;
3. a collection of ***charts***, which are homeomorphisms indexed by $\{U_i\}_i$,

$$\left\{x_i : U_i \xrightarrow{\simeq} \mathbb{R}^n\right\}_i,$$

   such that each image $x_i(U_i)$ is open in $\mathbb{R}^n$, and if $U_i \cap U_j \neq \varnothing$, then the ***transition map***

$$x_j \circ x_i^{-1} : x_i\big(U_i \cap U_j\big) \to x_j\big(U_i \cap U_j\big)$$

   is smooth (i.e. C^∞).

The collection of charts is a ***smooth atlas*** of M. ▪

If in Definition 1.61 the requirement of smooth transition maps is dropped, then the structure defined is a ***topological manifold***. There are other simple and natural variations: manifolds with boundary (the closed disk $\mathbb{D}^n$ is an example) and infinite-dimensional manifolds to name two—and Morse theory has extensions to both these settings (cf. resp. [BNR16; Cha93]). However, these are outside of the scope of this discussion, so in the sequel, the term *manifold* will always refer to a smooth manifold (finite-dimensional and without boundary).

Example 1.62 (Smooth manifolds)**.** Each of Examples 1.39 to 1.41: euclidean space $\mathbb{R}^n$, the sphere $\mathbb{S}^n$, and the real and complex projective spaces $\mathbb{RP}^n$ and $\mathbb{CP}^n$, for each $n \in \mathbb{N}$, may be given the structure of a smooth manifold. ▪

Definition 1.63 (Smooth map)**.** Let M be a smooth m-manifold and N a smooth n-manifold. A continuous map $f : M \to N$ is ***smooth*** if for all charts $x : U \to \mathbb{R}^m$ on M and $y : V \to R^n$ on N, the composition

$$y \circ f \circ x^{-1} : \mathbb{R}^m \to \mathbb{R}^n$$

is smooth. ▪

A smooth map whose inverse is also smooth is a ***diffeomorphism***—this is the appropriate notion of 'sameness' for smooth manifolds.

It was remarked earlier that compact topological spaces form a particularly nice and well-behaved subclass of topological spaces, and a similar remark was made for manifolds. It is then unsurprising that *compact manifolds* are especially nice and well-behaved, and these will play a privileged role in the ensuing discussion of Morse theory. Of the examples just mentioned in Example 1.62, all but euclidean space are compact manifolds.

Example 1.64 (Product manifold)**.** Let M and N be smooth manifolds with smooth atlases

$$\left\{x_i : U_i \xrightarrow{\simeq} \mathbb{R}^m\right\}_i, \qquad \left\{y_j : V_j \xrightarrow{\simeq} \mathbb{R}^n\right\}_j$$

respectively for $m, n \in \mathbb{N}$. Then their product $M \times N$ is a smooth mn-manifold with smooth atlas

$$\left\{x_i \times y_j : U_i \times V_j \xrightarrow{\simeq} \mathbb{R}^m \times \mathbb{R}^n\right\}_{ij},$$

and whose natural projections

$$\mathrm{Proj}_1 : M \times N \twoheadrightarrow M, \qquad \mathrm{Proj}_2 : M \times N \twoheadrightarrow N$$

are smooth maps [Lee13, Exa. 1.34, p. 21]. ▪

The key property that differentiates smooth manifolds from their topological counterparts is the existence of *tangent spaces*. At each point of a manifold, the tangent space is the $\mathbb{R}$-algebra of *linear derivations* at that point. Every smooth map between smooth manifolds induces an $\mathbb{R}$-linear map at each tangent space, its *differential*.

In a precise sense, the tangent space at a point of a manifold is the best linear approximation of the manifold at that point, and correspondingly, the differential of a smooth map at a point is the best linear approximating map. It is through this linearisation that topological problems can be translated to linear ones, and the tools of calculus can be invoked.

Definition 1.65 (Germ)**.** Let M be a smooth manifold. The set of smooth real-valued maps $M \to \mathbb{R}$ is denoted $C^\infty(M)$. The ***germ*** of a smooth map in $C^\infty(M)$, defined in a neighbourhood of a point $m \in M$, is the equivalence class of maps which coincide on some (possibly smaller) neighbourhood of m, i.e. via the (easily verified) equivalence relation

$$(\varphi, U) \sim (\psi, V) \iff \exists W \subseteq U \cap V : \varphi\big|_W = \psi\big|_W,$$

where U and V are neighbourhoods of m and $\varphi \in C^\infty(U)$ and $\psi \in C^\infty(V)$. The set of germs of smooth real-valued maps at m is denoted $C^\infty_m(M)$, and with the operations of addition and multiplication of maps and scalar multiplication, has the structure of an $\mathbb{R}$-algebra. ▪

Definition 1.66 (Tangent space)**.** Let M be a smooth manifold and $m \in M$. A ***derivation*** or ***tangent vector*** at m is an $\mathbb{R}$-linear map

$$v : C^\infty_m(M) \to \mathbb{R}$$

for which

$$v(\varphi\psi) = (v\varphi)\psi(m) + \varphi(m)v(\psi).$$

for all $\varphi, \psi \in C^\infty_m(M)$. The set of all derivations at m is the ***tangent space*** at m, denoted $\mathrm{T}_m\, M$, and has the structure of an $\mathbb{R}$-vector space. ▪

Definition 1.67 (Differential)**.** Let $f : M \to N$ be a smooth map of manifolds. At each point $m \in M$, f induces an $\mathbb{R}$-linear map, the ***differential*** of f at m,

$$\mathrm{d}f_m : \mathrm{T}_m\, M \to \mathrm{T}_{f(m)}\, N,$$

that acts on a derivation $v \in \mathrm{T}_m\, M$ by

$$(\mathrm{d}f_m v)\varphi := v(\varphi \circ f) \in \mathbb{R}, \qquad \varphi \in C^\infty_{f(m)}(M).$$ ▪

Remark 1.68*.* Although tangent vectors are formally defined as maps on germs, in practice there is no need to distinguish between a germ and a representative map for the germ, and so a tangent vector may equally be viewed as a map $C^\infty(M) \to \mathbb{R}$. ▪

Definition 1.69 (Immersion, submersion)**.** Let $f : M \to N$ be a smooth map of manifolds. If at each $m \in M$ the differential $\mathrm{d}f_m : \mathrm{T}_m\, M \to \mathrm{T}_{f(m)}\, N$ is:

1. injective, then f is an ***immersion***;
2. surjective, then f is a ***submersion***. ▪

Definition 1.70 (Smooth embedding)**.** A smooth map $f : M \to N$ is a ***smooth embedding*** if:

1. it is an immersion;
2. the image $f(M)$ with the subspace topology is homeomorphic to M under f. ▪

In the sequel, the term ***smooth submanifold***, or simply ***submanifold***, will refer to a subset of a manifold whose inclusion map is a smooth embedding.[4]

The next proposition, known as the *chain rule*, says that the tangent space construction is functorial (cf. Section 1.4).

[4] Beware that there is little consensus in the literature on this terminology: some authors use *submanifold* or possibly *immersed submanifold* to refer to the image of an injective immersion (which in the subspace topology, has no reason to itself be a manifold). See [Tu11, Rem., p. 122].

Proposition 1.71 (Chain rule). *The tangent space construction* T *is a functor from pointed smooth manifolds to* $\mathbb{R}$*-vector spaces, i.e.*

$$\mathrm{d}(\mathbb{1}_M)_m = \mathbb{1}_{\mathrm{T}_m M}$$

and if

$$f : (M, m) \to (N, n), \qquad g : (N, n) \to (O, o)$$

are pointed smooth maps, then

$$\mathrm{d}(g \circ f)_m = \mathrm{d}g_{f(m)} \circ \mathrm{d}f_m.$$

Proof. Let (M, n) be a pointed smooth manifold. First note that the identity map $\mathbb{1}_M$ induces the identity on $\mathrm{T}_m M$, as by definition, for all $v \in \mathrm{T}_p M$, $\varphi \in C^\infty_p(M)$

$$(\mathrm{d}(\mathbb{1}_M)_m v)\varphi = v(\varphi \circ \mathbb{1}_M) = v(\varphi).$$

If

$$f : (M, m) \to (N, n), \qquad g : (N, n) \to (O, o)$$

are pointed smooth maps, then let $v \in \mathrm{T}_m M$ and $\varphi \in C^\infty_{g(f(p))}(O)$. On one hand,

$$\big(\mathrm{d}(g \circ f)_m v\big)\varphi = v(\varphi \circ g \circ f),$$

and on the other,

$$\big((\mathrm{d}g_{f(m)} \circ \mathrm{d}f_m)v\big)\varphi = \big(\mathrm{d}g_{f(m)}(\mathrm{d}f_m v)\big)\varphi = (\mathrm{d}f_m v)(\varphi \circ g) = v(\varphi \circ g \circ f),$$

hence,

$$\mathrm{d}(g \circ f)_m = \mathrm{d}g_{f(m)} \circ \mathrm{d}f_m.$$ ∎

Definition 1.72 (Vector field). A ***vector field*** is a smooth map

$$V : C^\infty(M) \to C^\infty(M)$$

that is a derivation on the $\mathbb{R}$-algebra $C^\infty(M)$ of smooth real-valued maps on M, i.e. an $\mathbb{R}$-linear map satisfying the Leibniz rule

$$V(\varphi\psi) = \psi V(\varphi) + \varphi V(\psi), \tag{1.5}$$

for all $\varphi, \psi \in C^\infty(M)$. ▪

The set of all vector fields, denoted $\mathfrak{X}(M)$, has the structure of a $C^\infty(M)$-module. Moreover, there is an antisymmetric $\mathbb{R}$-bilinear mapping, the ***Lie bracket***,

$$[\,\cdot\,,\,\cdot\,] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

which acts by

$$[V, W]\varphi := V(W\varphi) - W(V\varphi)$$

for all $\varphi \in C^\infty(M)$. With the Lie bracket, $\mathfrak{X}(M)$ has the complementary structure of an $\mathbb{R}$-algebra. (Although the composition of vector fields $V \circ W$ is a mapping $C^\infty(M) \to C^\infty(M)$, it does not generally satisfy Equation (1.5) and hence, does not induce an operation on $\mathfrak{X}(M)$.)

In Chapter 3, the concept of *orientability* of a manifold will be integral to the development of Morse-Bott theory. An *orientation* of a manifold is a generalisation the concept of clockwise/anticlockwise in $\mathbb{R}^2$ or left-handedness/right-handedness in $\mathbb{R}^3$, and common sense suggests that orientation must then be a property that is invariant under rotations and reversed by reflections. An *orientable manifold* will be one for which an orientation is possible. A familiar counterexample is the real projective plane $\mathbb{RP}^2$.

Let $\mathbb{A}$ be a commutative unital ring and M an n-manifold for $n \in \mathbb{N}$. For each $m \in M$, let $U \subseteq \mathbb{R}^n$ be a coordinate chart around m. By the excision, exactness, and homotopy properties of Theorem 1.59, there are isomorphisms

$$H_i(M, M \smallsetminus \{m\}; \mathbb{A}) \xrightarrow{\simeq} H_i(U, U \smallsetminus \{m\}; \mathbb{A}) \xrightarrow{\simeq} \widetilde{H}_{i-1}(\mathbb{S}^{n-1}; \mathbb{A}).$$

for each $i \in \mathbb{N}$, where $\mathbb{S}^{n-1}$ is a sphere centred on m. As a rotation of $\mathbb{R}^n$ is homotopic to the identity, the induced map on $\widetilde{H}_{i-1}(\mathbb{S}^{n-1}; \mathbb{A})$ is the identity. It is also easily checked that a reflection induces $-\mathbb{1}$ on $\widetilde{H}_{i-1}(\mathbb{S}^{n-1}; \mathbb{A})$. This observation inspires the following definition.

Definition 1.73 (Orientability)**.** For an n-manifold M and $\mathbb{A}$ a unital ring, an ***A-fundamental class at a subspace*** $A \subseteq M$ is an element $\omega \in H_n(M, M \smallsetminus A; \mathbb{A})$ such that, for each $a \in A$, the image of ω under the map induced by the inclusion $(M, M \smallsetminus A) \hookrightarrow (M, M \smallsetminus \{a\})$,

$$H_n(M, M \smallsetminus A; \mathbb{A}) \xrightarrow{\simeq} H_n(M, M \smallsetminus \{a\}; \mathbb{A}) \simeq \mathbb{A}$$

generates $\mathbb{A}$ (i.e. is a unit). If $A = M$, then ω is simply an $\mathbb{A}$***-fundamental class of*** M.

An $\mathbb{A}$***-orientation of*** M is a set of pairs $\{U_i, \omega_i\}_i$ such that $\{U_i\}_i$ is an open cover of M and each ω_i is an $\mathbb{A}$-fundamental class at U_i, and moreover, the following consistency condition is met: if $U_i \cap U_i \neq \varnothing$, then the images of ω_i and ω_j under the maps induced by the inclusions

$$M \smallsetminus U_i \hookrightarrow M \smallsetminus U_i \cap U_i, \qquad M \smallsetminus U_j \hookrightarrow M \smallsetminus U_i \cap U_i$$

coincide in $H_n(M \smallsetminus U_i \cap U_j; \mathbb{A})$. ∎

If a manifold is $\mathbb{Z}$-orientable, then as there is a unique homomorphism from $\mathbb{Z}$ to every other ring $\mathbb{A}$[5], composing with this map gives an $\mathbb{A}$-orientation of M, hence,

[5]This is because $\mathbb{Z}$ is *cofinal* in the category of unital rings.

$\mathbb{Z}$*-orientability* and $\mathbb{Z}$*-orientations* are referred to as simply *orientability* and *orientations* (see e.g. [May99, Cha. 20.3, Cor., p. 157]). If a manifold is not $\mathbb{Z}$-orientable, then it is ***nonorientable***. On the other hand, it is trivial that every manifold is $\mathbb{Z}_2$-orientable.

A crown jewel in the theory of manifolds is the following theorem, that expresses a symmetry in the homology groups of an orientable compact manifold.

Theorem 1.74 (Homological Poincaré duality [Hat02, Sec. 3.3, p. 230])**.** *Let* $\mathbb{F}$ *be a field and* M *a compact oriented* n*-manifold. Then there is an isomorphism*

$$H_k(M; \mathbb{F}) \simeq H_{n-k}(M; \mathbb{F})$$

for all $k \in \mathbb{N}$*.*[6] ∎

Note that the orientability assumption may be dropped by taking $\mathbb{F} := \mathbb{Z}_2$. A trivial corollary of Theorem 1.74 is then that the homology of every n-manifold vanishes in degree above n.

[6]Poincaré duality is more naturally formulated as an isomorphism between the homology group of degree k and the *cohomology* ring of degree $n - k$, and this relation moreover holds with coefficients in an arbitrary commutative unital ring.

Chapter 2

Classical Morse theory

The study of Morse theory begins in this chapter. Critical points and their *nondegeneracy* are introduced in Section 2.1, and *Morse maps* are then defined as smooth $\mathbb{R}$-valued maps, all whose critical points are nondegenerate. Following is a statement of the *Morse lemma*, a key technical result that asserts the existence of a convenient chart in a neighbourhood of a nondegenerate critical point. Section 2.2 presents the cornerstone results of Morse theory, referred to as the *Morse theorems A* and *B*. From these is deduced the *structure theorem* in Section 2.3, which establishes that when a Morse map $\varphi : M \to \mathbb{R}$ obeys a certain regularity condition, a CW complex of the same homotopy type of M can be constructed, whose cells are in bijective correspondence with the critical points of φ. Section 2.4 cites several existence results, and outlines how these imply that every smooth manifold has the homotopy type of a CW complex. Section 2.5 then deduces the *Morse inequalities*—a system of inequalities of formal polynomials that encode, on one side, the critical structure of a Morse map, and the homological structure of the manifold on the other. The chapter concludes in Section 2.6, which concerns $\mathbb{F}$-*perfection* and *F-completability*—special properties of Morse maps that strengthen their connection to the homology of the underlying manifold.

2.1 Morse maps

Morse theory applies to a generic subset of the smooth $\mathbb{R}$-valued maps $C^\infty(M)$ on a smooth manifold M known as *Morse maps*. These are maps whose critical points are not 'too flat'—an idea that is formalised in the property of *nondegeneracy* of a critical point. In the definitions below, $\varphi : M \to \mathbb{R}$ is taken to be a smooth map on a smooth manifold M.

The differential of a smooth map between manifolds was defined in Definition 1.67. The differential of φ at a point $m \in M$ is then an $\mathbb{R}$-linear map

$$\mathrm{d}\varphi_m : \mathrm{T}_m\, M \to \mathrm{T}_{\varphi(m)}\, \mathbb{R} \simeq \mathbb{R}.$$

For computations, it is necessary to understand how the differential acts in local coordinates. Let

$$x = (x_i)_{i=1}^n : U \to \mathbb{R}^n$$

be a chart centred on m, i.e. for which $x(m) = x_i(m) = 0$ for all $i \in \{1, \dots, n\}$ where $x_i := \mathrm{Proj}_i\, x$ are the coordinate projections. The basis associated to this chart is the list of tangent vectors $\left(\frac{\partial}{\partial x_i}\right)_{i=1}^n$ defined by $\frac{\partial}{\partial x_i} x_j = \delta_{ij}$ (see e.g. [Tu11, Pro. 8.9, p. 90]). The differential then acts on this basis by

$$\mathrm{d}\varphi_m\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial \varphi}{\partial x_i}(m) = \frac{\partial\left(\varphi \circ x^{-1}\right)}{\partial r_i}(0),$$

where $\left(\frac{\partial}{\partial r_i}\right)_{i=1}^n$ is the ***standard basis*** on $\mathbb{R}^n$, associated to the ***standard coordinates*** $(r_i)_{i=1}^n$, which are characterised by $x_i = r_i \circ x$ and $\frac{\partial}{\partial r_i} r_j = \delta_{ij}$ (see [Tu11, Sec. 6.6, p. 67] and [Tu11, Pro. 8.11, p. 91] for further details).

Remark 2.1. It is a common notational shortcut to both suppress the chart x and to conflate the standard euclidean coordinates $(r_i)_i$ with the coordinates $(x_i)_i$ of the chart. This custom will be adopted in computations. ▪

Definition 2.2 (Critical/regular point/value)**.** A point $c \in M$ is a ***critical point*** of $\varphi : M \to \mathbb{R}$ if the differential at c vanishes, i.e. if

$$\mathrm{d}\varphi_c = 0 \in \mathrm{T}_c^* M.$$

The set of critical points of φ is denoted $\mathrm{Cr}\,\varphi$, and its complement $M \setminus \mathrm{Cr}\,\varphi$ are the ***regular points*** of φ. The set $\varphi(\mathrm{Cr}\,\varphi)$ are the ***critical values*** of φ, and its complement $\mathbb{R} \setminus \varphi(\mathrm{Cr}\,\varphi)$ are the ***regular values*** of φ. ▪

Definition 2.3 (Hessian)**.** The ***Hessian*** of the map φ at a critical point $c \in \mathrm{Cr}\,\varphi$ is the $\mathbb{R}$-bilinear map

$$\mathrm{d}^2\varphi_c : \mathrm{T}_c M \times \mathrm{T}_c M \to \mathbb{R}, \qquad (v, w) \mapsto (V(W\varphi))(c), \tag{2.1}$$

where $V, W \in \mathfrak{X}(M)$ are arbitrary vector fields satisfying $V(c) = v$ and $W(c) = w$. To show this is well-defined, note that

$$(V(W\varphi))(c) = v(W\varphi),$$

which is independent of the vector field V extending v. But as c is a critical point,

$$(V(W\varphi))(c) - (W(V\varphi))(c) =: ([V, W]\varphi)(c) = \mathrm{d}\varphi_c([v, w]) = 0$$

(recall Definition 1.72). Hence, the definition (2.1) is also independent of the vector field W extending w. Moreover, this shows that the Hessian is a symmetric $\mathbb{R}$-bilinear form.

A local description is as follows: in a chart $x = (x_i)_{i=1}^n : U \to \mathbb{R}^n$ centred on m, the

Hessian acts on the basis $\left(\frac{\partial}{\partial x_i}\right)_{i=1}^{n}$ of $\mathrm{T}_c M$ by

$$\mathrm{d}^2\varphi_c\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(c) = \frac{\partial^2(\varphi \circ x^{-1})}{\partial r_i \partial r_j}(0).$$ ▪

Definition 2.4 (Nondegenerate critical point)**.** A critical point $c \in \operatorname{Cr}\varphi$ is ***nondegenerate*** if the Hessian $\mathrm{d}\varphi_c$ at c is nondegenerate as a bilinear form, i.e.

$$\mathrm{d}^2\varphi_c(v, w) = 0 \ \forall w \in \mathrm{T}_c M \iff v = 0.$$

This condition is equivalent to requiring that the matrix

$$\left[\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(c)\right]_{ij}$$

corresponding to the Hessian in some chart $x : U \to \mathbb{R}^n$ be nonsingular. (The latter formulation is independent of the given chart, as coordinate transformations correspond to conjugation by a nonsingular Jacobian matrix [Spi65, The. 5.2, pp. 111–113].) ▪

Definition 2.5 (Morse map)**.** If $\varphi : M \to \mathbb{R}$ is a smooth map for which $\operatorname{Cr}\varphi$ consists only of nondegenerate critical points, then φ is a ***Morse map***. ▪

Definition 2.6 (Index of a critical point)**.** The ***index*** $\lambda_c(\varphi)$ (or just λ_c) of a nondegenerate critical point $c \in \operatorname{Cr}\varphi$ is the maximal dimension of a subspace on which $\mathrm{d}^2\varphi_c$ is negative-definite.

Equivalently, this is the number of negative eigenvalues (counting multiplicity) in a matrix corresponding to $\mathrm{d}^2\varphi_c$ in some chart. This is well-defined, as the matrix corresponding to $\mathrm{d}^2\varphi_c$ is symmetric, so it has real eigenvalues, and by Sylvester's Law of Inertia [Car17, The. 9.13, p. 313] the number of negative eigenvalues of such a matrix is invariant under conjugation (i.e. coordinate transformations). ▪

The interpretation of the index of a nondegenerate critical point c is the number of independent directions along which the map φ simultaneously decreases from c.

Example 2.7 (Minima and maxima)**.** Every nondegenerate maximum of a smooth map $\varphi : M \to \mathbb{R}$ on a smooth n-manifold M has index n, as the eigenvalues of the Hessian at a maximum are all negative. Similarly, every nondegenerate minimum has index 0. ▪

Remark 2.8. It should be checked that all the definitions above are valid in the smooth category, i.e. that they are invariant under diffeomorphism. It is a straightforward if tedious exercise to do so. ▪

The significance of the nondegeneracy condition stems from the following result, known as the *Morse lemma*, which shows that the map $\varphi : M \to \mathbb{R}$ takes a special canonical form in a neighbourhood of a nondegenerate critical point. An elementary proof of this result, using the same method as that for diagonalising quadratic forms, is found in [Mil63, Lem. 2.2, p. 6].

Lemma 2.9 (Morse lemma [Mil63, Lem. 2.2, p. 6])**.** *Let $\varphi : M \to \mathbb{R}$ be a smooth map on an n-manifold M for $n \in \mathbb{N}_+$. If $c \in \operatorname{Cr}\varphi$ is a nondegenerate critical point, then there is a chart*

$$x = (x_i)_{i=1}^n : U \to \mathbb{R}^n$$

centred on c for which

$$\varphi = \varphi(c) - \sum_{i=1}^{\lambda_c} x_i^2 + \sum_{i=\lambda_c+1}^{n} x_i^2. \tag{2.2}$$

∎

Shortly, it will be shown that it is reasonable to imagine φ as a height map, i.e. as the projection onto a coordinate axis of an embedding of M into euclidean space (see Remark 2.27). Then the Morse lemma indicates that, under such a projection, every critical point has a neighbourhood that looks like a saddle (generically) or a bowl (at minima and maxima). The first important consequence of Lemma 2.9 is that critical points of a Morse map are isolated:

Corollary 2.10. *Let $\varphi : M \to \mathbb{R}$ be a Morse map. Then $\operatorname{Cr}\varphi$ is a discrete set.*

Proof. Let $c \in \operatorname{Cr}\varphi$ be a critical point. By Lemma 2.9, there is a chart $(x_i)_{i=1}^n : U \to \mathbb{R}^n$ centred at c for which Equation (2.2) holds. Then $\mathrm{d}\varphi_c = 0$ implies

$$\frac{\partial\varphi}{\partial x_i} = \pm 2x_i = 0, \qquad i \in \{0, \dots, n\},$$

but this is the point c. ∎

With the additional assumption of compactness, it then follows that the critical set is finite:

Corollary 2.11. *Let $\varphi : M \to \mathbb{R}$ be a Morse map on a compact manifold M. Then $\operatorname{Cr}\varphi$ is finite.*

Proof. An infinite subset of a compact space must have a limit point in that space [Mun75, Thm. 28.1, p. 179]. By Corollary 2.10, $\operatorname{Cr}\varphi$ has no limit point in M, and thus, $\operatorname{Cr}\varphi$ is finite. ∎

Below is an example and a nonexample of a Morse map.

Example 2.12 (Height map on a sphere)**.** Let

$$\mathbb{S}^n := \left\{x \in \mathbb{R}^{n+1} \ : \|x\| = 1\right\}$$

be the unit n-sphere in $\mathbb{R}^{n+1}$ for some $n \in \mathbb{N}$, and consider the projection onto the 'vertical' axis,

$$\varphi := \operatorname{Proj}_{n+1} : \mathbb{S}^n \twoheadrightarrow [-1, 1], \qquad (x_i)_{i=1}^{n+1} \mapsto x_{n+1}.$$

(see Figure 2.1). The south and north poles,

$$c_1 = (0, \dots, -1), \qquad c_2 = (0, \dots, 1)$$

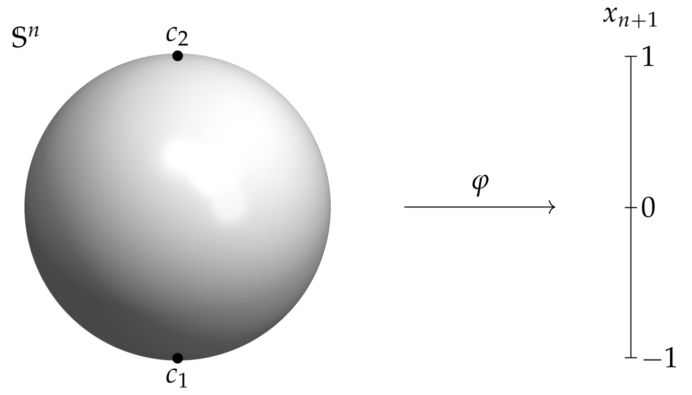


Figure 2.1. Height map on a sphere.

are clearly critical points. Away from these points, there are charts such as the one centred at $(1, 0, \ldots, 0)$ where φ takes the form

$$\left(\sqrt{1 - \sum_{i=2}^{n+1} x_i^2}, x_2, \ldots, x_{n+1}\right) \mapsto x_{n+1}$$

(where for brevity, the custom of omitting chart maps from the notation and letting the x_i stand alternately for coordinate maps or points of euclidean space has been adopted; see also Remark 2.1). It is then clear that the differential is nonvanishing away from the poles, and thus, all other points are regular. On a convenient chart centred on the north pole c_2, φ takes the form

$$\varphi = \sqrt{1 - \sum_{i=1}^{n} x_i^2}.$$

The corresponding Hessian matrix is then easily seen to be the negative of the identity matrix,

$$\mathrm{d}^2\varphi_c = -I_{n\times n},$$

hence, c_2 is nondegenerate of index n. A similar chart at the south pole c_1 reveals that it is nondegenerate and of index 0. It follows that φ is a Morse map. ∎

Example 2.13 (Squared height map on a sphere)**.** Now let

$$\psi := \varphi^2 : \mathbb{S}^2 \twoheadrightarrow \mathbb{I}$$

be the squared height map of Example 2.12. Each point of the equator

$$C := \left\{(x_i)_{i=1}^{n+1} : x_{n+1} = 0\right\} \subset \mathbb{S}^n$$

is then a minimum point of ψ, so by Corollary 2.11, ψ is not a Morse map (see Figure 2.2). ∎

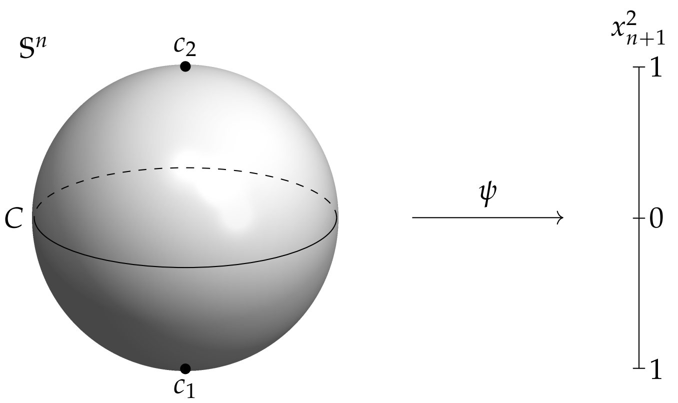


Figure 2.2. Squared height map on a sphere.

The last example shows that there is no reason to expect the product (or sum) of Morse maps to again be Morse. (For the sum, note that φ is Morse if and only if $-\varphi$ is.)

2.2 Theorems A and B

This section discusses the cornerstone theorems of classical Morse theory, referred to as the *Morse theorems A* and *B* by Bott [Bot82], which relate the local topology of a manifold to neighbourhoods of the nondegenerate critical points. Given a Morse map $\varphi : M \to \mathbb{R}$, the ***sublevel*** and ***interlevel*** sets will be denoted respectively

$$M^\alpha := \varphi^{-1}((-\infty, \alpha]), \qquad M_\alpha^\beta := \varphi^{-1}([\alpha, \beta])$$

for some $[\alpha, \beta] \subset \mathbb{R}$. The Morse theorems describe how the homotopy type of the sublevel set M^α varies as $\alpha \in \mathbb{R}$ increases. The map $\varphi : M \to \mathbb{R}$ may be visualised as a measuring of the height of the manifold fixed in place relative to a rising body of water, where $\alpha \in \mathbb{R}$ is the water level and M^α is the portion of the manifold submerged (cf. Remark 2.27). When α is below the minimum critical value, the manifold is still suspended above the water. As α increases past the minimum critical value, bowl-shaped caps of the manifold protrude under water, by Lemma 2.9. These correspond to neighbourhoods of the minimum points of index zero.

As the water level α increases further through regular values, the manifold sinks deeper under water, but its topology beneath the surface does not change. This is the statement of Theorem A. When the water level reaches the next critical value, the topology beneath the water surface must change, and the change is determined by the indices of the corresponding critical points. This is the statement of Theorem B.

The formulation of the main results of Morse theory require that the map φ satisfies an additional criterion—that of being *exhaustive*. The assumption of this property turns out to be mild (see Proposition 2.26 and Remark 2.27).

Definition 2.14 (Exhaustive map)**.** If $\varphi : M \to \mathbb{R}$ is a smooth map for which M^α is compact for all $\alpha \in \mathbb{R}$, then φ is ***exhaustive***. ▪

Note that if M is compact, then every Morse map $\varphi : M \to \mathbb{R}$ is automatically exhaustive, as the preimage of a closed subset is closed by continuity (Definition 1.4 and Remark 1.5), and a closed subset of a compact space is compact (Proposition 1.22).

Theorem 2.15 (Morse theorem A [Mil63, The. 3.1, p. 12])**.** *Let $\varphi : M \to \mathbb{R}$ be an exhaustive smooth map, and $[\alpha, \beta] \subset \mathbb{R}$ a closed interval containing only regular values of φ. Then there is a homotopy equivalence*

$$M^{\beta} \sim M^{\alpha}.$$

To describe the proof of Theorem 2.15, the concept of a *riemannian manifold* is required.

Definition 2.16 (Riemannian manifold)**.** A ***riemannian manifold*** is a pair $(M, \langle \cdot , \cdot \rangle)$ where M is a smooth manifold and for each $m \in M$,

$$\langle \cdot , \cdot \rangle_m : \mathrm{T}_m M \times \mathrm{T}_m M \to \mathbb{R}$$

is an inner product that is smoothly varying, i.e. for all smooth vector fields $X, Y \in \mathfrak{X}(M)$, the map

$$m \mapsto \langle X, Y \rangle_m$$

is smooth. ▪

It is a basic fact that every smooth manifold can be equipped with a riemannian metric (see e.g. [Tu17, Theorem 1.12, p. 6]).

Proof sketch of Theorem 2.15. Let $\langle \cdot , \cdot \rangle$ be a riemannian metric on M. The ***gradient*** $\nabla \psi \in \mathfrak{X}(M)$ of a smooth map $\psi \in C^{\infty}(M)$ is uniquely characterised by the equation

$$\langle \nabla \psi, V \rangle = V(\psi)$$

for all vector fields $V \in \mathfrak{X}(M)$. The absence of critical points in the preimage $\varphi^{-1}([\alpha, \beta])$ implies that

$$V := -\frac{\nabla \varphi}{\|\nabla \varphi\|}$$

is a well-defined vector field on $\varphi^{-1}([\alpha, \beta])$. One may then extend V (arbitrarily) to a vector field on all of M, whose integral curves give a deformation retract (Definition 1.28) of M^{β} onto M^{α} (see Figure 2.3). ■

For a complete proof of Theorem 2.15 (which is only slightly longer than the sketch above), see [Mil63, The. 3.1, p. 12].

An interesting application of Theorem 2.15, known as *Reeb's sphere theorem*, was the catalyst for Milnor's discovery of *exotic* 7-spheres, i.e. smooth manifolds that are homeomorphic but not diffeomorphic to the standard 7-sphere [Mil56].

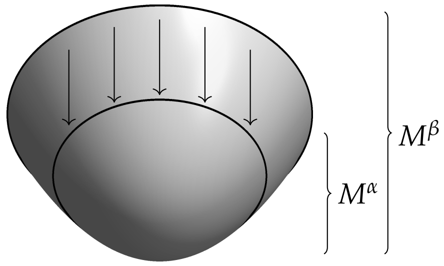


Figure 2.3. Deformation retract from a gradient flow.

Proposition 2.17 (Morse map with two critical points (Reeb [Ree46])). *Let $\varphi : M \to \mathbb{R}$ be a Morse map on a compact n-manifold M. If φ has exactly two critical points, then M is homeomorphic to the n-sphere $\mathbb{S}^n$.*

Proof. By the extreme value principle for a compact space [Mun75, The. 27.4, p. 174], the two critical points are a minimum and a maximum with indices 0 and n, respectively (cf. Example 2.7). Then by the Morse lemma (Lemma 2.9), for ε such that $0 < \varepsilon \ll 1$, the inter-level sets M_0^ε and $M_{1-\varepsilon}^1$ are closed n-cells. By Theorem 2.15, M_0^ε is homeomorphic to $M_0^{1-\varepsilon}$. Then M is homeomorphic to two n-cells glued along their boundary (cf. Example 1.40), i.e. an n-sphere

$$M \simeq M_0^{1-\varepsilon} \cup_a M_{1-\varepsilon}^1 \simeq \mathbb{S}^n,$$

for some attaching map a. ∎

Remark 2.18. Proposition 2.17 holds even when φ is degenerate, but the proof is harder; for details, see [Mil64, The. 1, pp. 165–183]. ▪

Next, the Morse theorem B is presented, which says that if $\alpha \in \mathbb{R}$ varies through a critical value, then the topology of the sublevel set M^α must change. A stronger version of this theorem known as the *handle presentation theorem* holds, which keeps track of the smooth structure and describes the diffeomorphism type of M via a *handlebody* decomposition (the smooth analogue of a CW decomposition); this is the approach taken in, for example, [Mat02; Nic11]. As the diffeomorphism type is irrelevant to homological calculations, that version is not presented here.

Theorem 2.19 (Morse theorem B [Mil63, The. 3.2, Rem. 3.3, pp. 14–19]). *Let $\varphi : M \to \mathbb{R}$ a smooth map and $\gamma \in \varphi(\operatorname{Cr}\varphi)$ a critical value for which $\varphi^{-1}(\gamma) = \{c_1, \ldots, c_k\}$ consists of k nondegenerate critical points for some $k \in \mathbb{N}_+$. If $\varepsilon > 0$ is such that $\varphi^{-1}([\gamma - \varepsilon, \gamma + \varepsilon])$ is compact and whose only critical points are fibres of γ, then there is a homotopy equivalence*

$$M^{\gamma+\varepsilon} \sim M^{\gamma-\varepsilon} \cup \mathbb{D}^{\lambda_{c_1}} \cup \cdots \cup \mathbb{D}^{\lambda_{c_k}},$$

where the unions denote cell attachments by unspecified attaching maps.

Proof sketch. [1] It will be assumed that exactly one critical point $c \in \operatorname{Cr}\varphi$ corresponds

[1] This synopsis of Milnor's proof was informed by the discussion in [BT82, pp. 222–223].

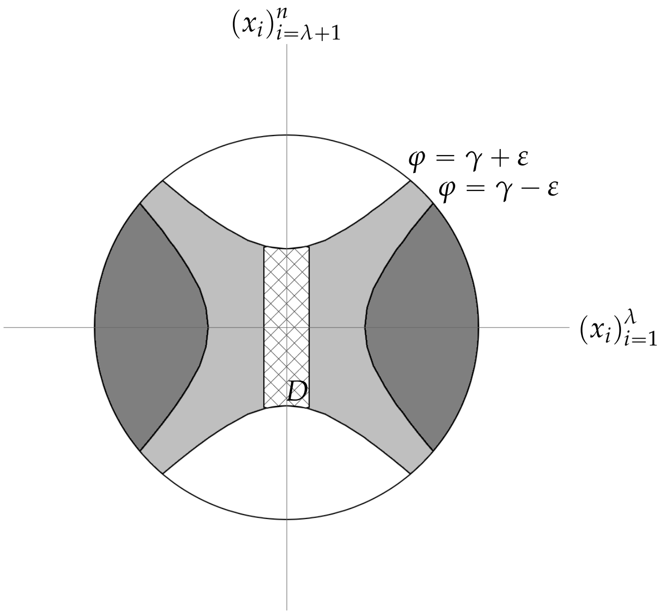


Figure 2.4. The shaded region is $M^{\gamma+\varepsilon}$.

to the critical value $\gamma := \varphi(c)$; the general case is proved by showing that it is always possible to make a perturbation of φ to a Morse map whose critical points have the same multiplicity and indices, but whose critical values are in bijective correspondence with the critical points.

By Lemma 2.9, there is a chart $(x_i)_{i=1}^n : U \to \mathbb{R}^n$ in which the level sets of $M^{\gamma-\varepsilon}$ and $M^{\gamma+\varepsilon}$ take the forms (see Figure 2.4)

$$M^{\gamma-\varepsilon} \cap U = \left\{ -\sum_{i=1}^{\lambda_c} x_i^2 + \sum_{i=\lambda_c+1}^{\operatorname{Dim} M} x_i^2 \leqslant -\varepsilon \right\}, \qquad M^{\gamma+\varepsilon} \cap U = \left\{ -\sum_{i=1}^{\lambda_c} x_i^2 + \sum_{i=\lambda_c+1}^{\operatorname{Dim} M} x_i^2 \leqslant \varepsilon \right\}.$$

Let

$$D := \left\{ m \in M : \varphi(m) \leqslant \varepsilon + \gamma, \sum_{i=1}^{\lambda_c} x_i^2 \leqslant \delta \right\} \subset U$$

for some δ for which $0 < \delta \ll \varepsilon$. Then D has the homotopy type of a cell $\mathbb{D}^{\lambda_c}$. Setting

$$C := \operatorname{Cl}\left(M^{\gamma+\varepsilon} \setminus D\right),$$

it will then follow from Theorem 2.15 that C deformation retracts onto $M^{\gamma-\varepsilon}$ (see Figure 2.5). As $M^{\gamma+\varepsilon}$ is obtained from D by attaching C, there is a homotopy equivalence

$$M^{\gamma+\varepsilon} \sim M^{\gamma-\varepsilon} \cup_{\mathbb{S}^{\lambda_c-1}} \mathbb{D}^{\lambda_c},$$

as required. ■

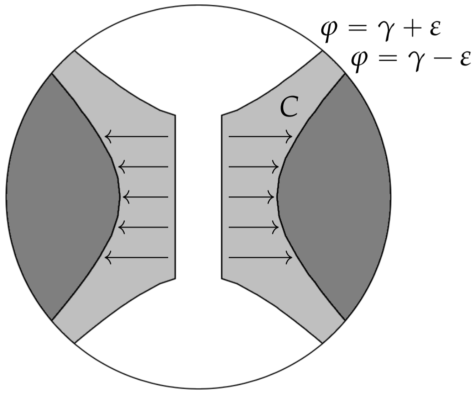


Figure 2.5. The shaded region C deformation retracts onto $M^{\gamma-\varepsilon}$.

2.3 Structure theorem

Theorems A and B gave a description of the local topology of a manifold (up to homotopy) in terms of the local analytic properties of a smooth map $\varphi : M \to \mathbb{R}$. The structure theorem that follows systematically applies these theorems over the entire manifold and aggregates the data obtained to give a description of the global topology of M. The outline of the approach is simple: inductively apply Theorems 2.15 and 2.19 over the sequence of critical values to obtain a CW complex homotopy equivalent to M. There is a technical subtlety: indices of the critical points as they appear in M^α may not be a nondecreasing function of α, so that the resulting complex may not be a CW complex, but a *cell* complex (see Remark 1.36). There are several ways of dealing with this. One is to perturb the Morse map $\varphi : M \to \mathbb{R}$ to a Morse map $\psi : M \to \mathbb{R}$ that satisfies

$$\psi(c) = \lambda_c(\psi),$$

for each $c \in \operatorname{Cr}\psi$. Such a map is ***self-indexing***, and a theorem of Smale shows that a Morse map may always be perturbed to a self-indexing Morse map [Nic11, The. 2.33, p. 73]. A second approach is to use cellular approximation (Theorem 1.43) to exhibit a homotopy equivalence from the cell complex to the CW complex, and it is this approach that is taken here, following [Mil63, The. 3.5, pp. 20–24].

Theorem 2.20 (Structure theorem). *Let $\varphi : M \to \mathbb{R}$ be an exhaustive Morse map. Then M has the homotopy type of a CW complex with exactly one λ_c-cell $\mathbb{D}^{\lambda_c}$ for each critical point $c \in \operatorname{Cr}\varphi$.*

The proof depends on several technical lemmas about attaching cells in a topological space.

Lemma 2.21 (Whitehead [Whi49b, Lem. 5, pp. 48–92]). *Let X be a topological space and*

$$a_0 : \mathbb{S}^{k-1} \to X, \qquad a_1 : \mathbb{S}^{k-1} \to X$$

homotopic maps for some $k \in \mathbb{N}_+$. Then the identity map $\mathbb{1}_X : X \to X$ extends to a homotopy *equivalence*

$$X \cup_{a_0} \mathbb{D}^k \xrightarrow{\sim} X \cup_{a_1} \mathbb{D}^k.$$

Proof. [2] Denote the corresponding characteristic maps also by

$$a_0 : \mathbb{D}^k \to X \cup_{a_0} \mathbb{D}^k, \qquad a_1 : \mathbb{D}^k \to X \cup_{a_1} \mathbb{D}^k,$$

and let $a : \mathbb{I} \times \mathbb{S}^{k-1} \to X$ be a homotopy from a_0 to a_1. Define

$$f : X \cup_{a_0} \mathbb{D}^k \to X \cup_{a_1} \mathbb{D}^k$$

by

$$f\Big|_X := \mathbb{1}_X, \qquad f\Big|_{\mathbb{D}^k} := a_0(\tau u) \mapsto \begin{cases} a_1(2\tau u) & : \tau \in \left[0, \frac{1}{2}\right], u \in \partial\mathbb{D}^k \\ a_{2-2\tau}(u) & : \tau \in \left(\frac{1}{2}, 1\right], u \in \partial\mathbb{D}^k, \end{cases}$$

where $u \in \partial\mathbb{D}^k$ is viewed as a unit vector, and define similarly

$$g : X \cup_{a_1} \mathbb{D}^k \to X \cup_{a_0} \mathbb{D}^k$$

by

$$g\Big|_X := \mathbb{1}_X, \qquad g\Big|_{\mathbb{D}^k} := a_1(\tau u) \mapsto \begin{cases} a_0(2\tau u) & : \tau \in \left[0, \frac{1}{2}\right], u \in \partial\mathbb{D}^k \\ a_{2\tau-1}(u) & : \tau \in \left(\frac{1}{2}, 1\right], u \in \partial\mathbb{D}^k. \end{cases}$$

It is easily seen that both f and g are well-defined and continuous. Now for all $u \in \partial\mathbb{D}^k$,

$$\begin{aligned} (g \circ f)(a_0(\tau u)) &= \begin{cases} (g \circ a_1)(2\tau u) & : \tau \in \left[0, \frac{1}{2}\right] \\ (g \circ a_{2-2\tau})(u) & : \tau \in \left(\frac{1}{2}, 1\right] \end{cases} \\ &= \begin{cases} a_0(4\tau u) & : \tau \in \left[0, \frac{1}{4}\right] \\ a_{4\tau-1}(u) & : \tau \in \left(\frac{1}{4}, \frac{1}{2}\right] \\ a_{2-2\tau}(u) & : \tau \in \left(\frac{1}{2}, 1\right]. \end{cases} \end{aligned}$$

For all $u \in \partial\mathbb{D}^k$, define a map

$$h : \mathbb{I} \times \left(X \cup_{a_0} \mathbb{D}^k\right) \to X \cup_{a_0} \mathbb{D}^k$$

by

$$h_\varsigma\Big|_X := \mathbb{1}_X, \qquad h_\varsigma\Big|_{\mathbb{D}^k} := a_0(\tau u) \mapsto \begin{cases} a_0((4-3\varsigma)\tau u) & : \tau \in \left[0, \frac{1}{4-3\varsigma}\right] \\ a_{(4-3\varsigma)\tau-1}(u) & : \tau \in \left(\frac{1}{4-3\varsigma}, \frac{2}{4-3\varsigma}\right] \\ a_{\frac{1}{2}(4-3\varsigma)(1-\tau)}(u) & : \tau \in \left(\frac{2-\varsigma}{4-3\varsigma}, 1\right]. \end{cases}$$

It is easily checked that h is well-defined and continuous, and hence, that h is a homotopy

[2] This proof was informed by [BH04b, Lem. 3.29, p. 70].

with

$$h_0 = g \circ f, \qquad h_1 = \mathbb{1}_{X \cup_{a_0} \mathbb{D}^k}.$$

Similarly, define

$$\widetilde{h} : \mathbb{I} \times \left(X \cup_{a_1} \mathbb{D}^k \right) \to X \cup_{a_1} \mathbb{D}^k$$

by

$$\widetilde{h}_\varsigma \Big|_X := \mathbb{1}_X, \qquad \widetilde{h}_\varsigma \Big|_{\mathbb{D}^K} := a_1(\tau u) \mapsto \begin{cases} a_1((4-3\varsigma)\tau u) & : \tau \in \left[0, \frac{1}{4-3\varsigma}\right] \\ a_{2-(4-3\varsigma)\tau}(u) & : \tau \in \left(\frac{1}{4-3\varsigma}, \frac{2}{4-3\varsigma}\right] \\ a_{1-\frac{1}{2}(4-3\varsigma)(1-\tau)}(u) & : \tau \in \left(\frac{2-\varsigma}{4-3\varsigma}, 1\right]. \end{cases}$$

Then it is again easily checked that $\widetilde{h}$ is a homotopy with

$$\widetilde{h}_0 = f \circ g, \qquad \widetilde{h}_1 = \mathbb{1}_{X \cup_{a_1} \mathbb{D}^k}.$$

It follows that f is a homotopy equivalence

$$f : X \cup_{a_0} \mathbb{D}^k \xrightarrow{\sim} X \cup_{a_1} \mathbb{D}^k. \qquad \blacksquare$$

Lemma 2.22. *Let $f : X \to Y$ be a continuous map between topological spaces X and Y. If f has a left homotopy inverse $g : Y \to X$ and a right homotopy inverse $h : Y \to X$, then f is a homotopy equivalence, and g is a two-sided homotopy inverse of f.*

Proof. The relations

$$g \circ f \sim \mathbb{1}_X, \qquad f \circ h \sim \mathbb{1}_Y,$$

together imply

$$g \sim g \circ (f \circ h) = (g \circ f) \circ h \sim h.$$

Hence,

$$f \circ h \sim f \circ g \sim \mathbb{1}_Y,$$

so that g is a two-sided homotopy inverse of f. $\blacksquare$

Lemma 2.23 (P. Hilton [Mil63, Lem. 3.7, p. 21])**.** *Let X be a topological space and*

$$a : \mathbb{S}^{k-1} \to X$$

an attaching map for some $k \in \mathbb{N}_+$. Then every homotopy equivalence $f : X \xrightarrow{\sim} Y$ extends to a homotopy equivalence

$$X \cup_a \mathbb{D}^k \xrightarrow{\sim} Y \cup_{f \circ a} \mathbb{D}^k.$$

Proof. Define a map

$$\widetilde{f} : X \cup_a \mathbb{D}^k \to Y \cup_{f \circ a} \mathbb{D}^k, \qquad \widetilde{f}\Big|_X := f, \qquad \widetilde{f}\Big|_{\mathbb{D}^k} := \mathbb{1}_{\mathbb{D}^k}.$$

Let g be the homotopy inverse of f, and define similarly

$$\widetilde{g} : Y \cup_{f \circ a} \mathbb{D}^k \to X \cup_{g \circ f \circ a} \mathbb{D}^k, \qquad \widetilde{g}\Big|_Y := g, \qquad \widetilde{g}\Big|_{\mathbb{D}^k} := \mathbb{1}_{\mathbb{D}^k}\,.$$

As there is a homotopy $g \circ f \circ a \sim a$, Lemma 2.21 implies there is a homotopy equivalence

$$h : X \cup_{g \circ f \circ a} \mathbb{D}^k \xrightarrow{\sim} X \cup_a \mathbb{D}^k.$$

It will first be shown that the composition

$$h \circ \widetilde{g} \circ \widetilde{f} : X \cup_a \mathbb{D}^k \to X \cup_a \mathbb{D}^k$$

is homotopic to the identity $\mathbb{1}_{X \cup_a \mathbb{D}^k}$. First, let r denote a homotopy between $g \circ f$ and the identity, i.e.

$$r : \mathbb{I} \times X \to X, \qquad r_0 := g \circ f, \qquad r_1 := \mathbb{1}_X\,.$$

For $u \in \partial \mathbb{D}^k$ a unit vector, it then follows

$$h \circ \widetilde{g} \circ \widetilde{f}\Big|_X = g \circ f, \qquad h \circ \widetilde{g} \circ \widetilde{f}\Big|_{\mathbb{D}^k} := \tau u \mapsto \begin{cases} 2\tau u & : \tau \in \left[0, \frac{1}{2}\right] \\ (r_{2-2\tau} \circ a)(\tau u) & : \tau \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

The required homotopy,

$$s : \mathbb{I} \times \left(X \cup_a \mathbb{D}^k\right) \to X \cup_a \mathbb{D}^k, \qquad s_0 = h \circ \widetilde{g} \circ \widetilde{f}, \qquad s_1 = \mathbb{1}_{X \cup_a \mathbb{D}^k},$$

will then be given by

$$s_\varsigma\Big|_X := r_\varsigma, \qquad s_\varsigma\Big|_{\mathbb{D}^k} := \tau u \mapsto \begin{cases} \frac{2}{1+\varsigma}\tau u & : \tau \in \left[0, \frac{1+\varsigma}{2}\right], \\ \left(r_{2-2\tau+\varsigma} \circ a\right)(u) & : \tau \in \left(\frac{1+\varsigma}{2}, 1\right]. \end{cases}$$

It is easily checked that s_ς is well-defined and continuous and so it follows $\widetilde{f}$ has as left homotopy inverse $h \circ \widetilde{g}$, and a similar proof shows that $\widetilde{g}$ has a left homotopy inverse. As h is a homotopy equivalence, it has a left inverse, so that

$$h \circ \left(\widetilde{g} \circ \widetilde{f}\right) \sim \mathbb{1}_{X \cup_a \mathbb{D}^k} \implies \left(\widetilde{g} \circ \widetilde{f}\right) \circ h \sim \mathbb{1}_{X \cup_{gfa} \mathbb{D}}\,.$$

As $\widetilde{g}$ has a left inverse,

$$\widetilde{g} \circ \left(\widetilde{f} \circ h\right) \sim \mathbb{1}_{X \cup_{gfa} \mathbb{D}} \implies \left(\widetilde{f} \circ h\right) \circ \widetilde{g} \sim \mathbb{1}_{X \cup_{fa} \mathbb{D}}\,.$$

Then $\widetilde{f}$ has a right inverse $h \circ \widetilde{g}$, so by Lemma 2.22, $\widetilde{f}$ has a two-sided inverse and is thus a homotopy equivalence. ■

Proof of Theorem 2.20. As φ is exhaustive, it follows by Corollary 2.10 that its critical values can be arranged in an increasing sequence

$$\gamma_0 < \gamma_1 < \gamma_2 < \cdots$$

that has no limit point. If $\alpha < \gamma_0$, then $M^\alpha = \varnothing$, and if $\alpha \in (\gamma_0, \gamma_1)$, then M^α is a disjoint union of points, and so in either case is a CW complex. Let $\alpha \in \mathbb{R} \setminus \varphi(\operatorname{Cr}\varphi)$ and let γ be the smallest $\gamma_i > \alpha$. Suppose for induction that there is a homotopy equivalence to a CW complex

$$f : M^\alpha \xrightarrow{\sim} X.$$

By Theorem 2.19, for some ε for which $0 < \varepsilon \ll 1$, there is a homotopy equivalence

$$M^{\gamma+\varepsilon} \sim M^{\gamma-\varepsilon} \cup_{a_1} \mathbb{D}^{\lambda_{c_1}} \cdots \cup_{a_m} \mathbb{D}^{\lambda_{c_m}},$$

where $\{c_1, \ldots, c_k\} := \varphi^{-1}(\gamma) \cap \operatorname{Cr}\varphi$, and $(a_1, \ldots, a_m)$ are unspecified attaching maps. Then by Theorem 2.15, there is a homotopy equivalence

$$g : M^{\gamma-\varepsilon} \xrightarrow{\sim} M^\alpha.$$

By cellular approximation (Theorem 1.43), each composition $f \circ g \circ a_i$ is homotopic to a cellular map

$$b_i : \mathbb{S}^{\lambda_{c_i}-1} \to X^{\lambda_{c_i}-1}$$

into the $(\lambda_{c_i} - 1)$-skeleton of X. By Lemmas 2.22 and 2.23, there is then a homotopy equivalence

$$M^{\gamma+\varepsilon} \sim X \cup_{b_1} \mathbb{D}^{\lambda_{c_1}} \cdots \cup_{b_m} \mathbb{D}^{\lambda_{c_m}} =: X_i,$$

where the right-hand side is a CW complex. Moreover, it is easily seen that there is a deformation retract $M^{\gamma+\varepsilon} \xrightarrow{\sim} M^\alpha$ (see e.g. [Mil63, Rem. 3.4, p. 20]). By induction over the critical values, it follows that each M^{α_i} for $\alpha_i \in (\gamma_{i-1}, \gamma_i)$ has the homotopy type of a CW complex X_i. If M is compact, then there are only finitely many critical values, and the proof is complete. If M is not compact but $\operatorname{Cr}\varphi \subset M^\alpha$ for some $\alpha \in \mathbb{R}$, then a similar proof to Theorem 2.15 shows that M^α is a deformation retract of M, so that proof is again complete.

If there are infinitely many critical points, then the construction above gives an infinite sequence of homotopy equivalences $(f_i)_i$, each extending the previous:

$$\begin{tikzcd}
M^{\alpha_0} \arrow[r, hook] \arrow[d, "f_0"', "\wr"] & M^{\alpha_1} \arrow[r, hook] \arrow[d, "f_1"', "\wr"] & \cdots \\
X_0 \arrow[r, hook] & X_1 \arrow[r, hook] & \cdots .
\end{tikzcd}$$

Let

$$M := \operatorname*{Colim}_{i \in \mathbb{N}} M^{\alpha_i}, \qquad X := \operatorname*{Colim}_{i \in \mathbb{N}} X_i, \qquad f := \operatorname*{Colim}_{i \in \mathbb{N}} f_i,$$

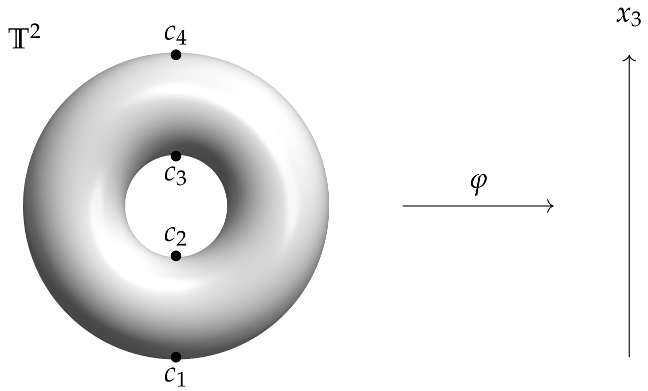


Figure 2.6. Height map on an upright torus.

where the colimit of the maps f_i is the map whose graph $\{(m, f(m)) : m \in M\}$ is the union of the graphs of the f_i. As $X_i \sim M^{\alpha_i}$ for all $i \in \mathbb{N}$, then $\pi_n(X_i) \simeq \pi_n(M^{\alpha_i})$ for all $n \in \mathbb{N}_+$. Hence, by Lemma 1.33, the map f induces isomorphisms

$$\pi_n(M) \xrightarrow{\simeq} \operatorname*{Colim}_{i \in \mathbb{N}} \pi_n(M^{\alpha_i}) \xrightarrow{\simeq} \operatorname*{Colim}_{i \in \mathbb{N}} \pi_n(X_i) \xrightarrow{\simeq} \pi_n(X)$$

for each $n \in \mathbb{N}_+$. By Lemma 1.48, X is itself a CW complex, and so trivially is dominated by a CW complex (recall Remark 1.46). It can be shown that M is also dominated by a CW complex[3]; see e.g. [Mil63, p. 24], [Mil59, The. 1, p. 272], [Han51, The. 3.3, p. 392]. Now both X and M are dominated by CW complexes and f induces isomorphisms of their homotopy groups in all dimensions, so by the generalised version of Whitehead's theorem (Theorem 1.45 and Remark 1.46)

$$f : M \xrightarrow{\sim} X$$

is a homotopy equivalence. This completes the proof. ■

Example 2.24 (Height map on an upright torus)**.** A height map

$$\varphi : \mathbb{T}^2 \to \mathbb{R}, \qquad (x_1, x_2, x_3) \mapsto x_3$$

on the standard 2-torus $\mathbb{T}^2 \subset \mathbb{R}^3$ standing vertically and tangent to the plane, as depicted in Figure 2.6, has four critical points: a minimum c_1, two saddles c_2 and c_3, and a maximum c_4, that are all easily verified to be nondegenerate. Via Theorem 2.20, the corresponding CW decomposition is

$$\mathbb{T}^2 = * \cup_{\mathbb{S}^0} \mathbb{D}^1 \cup_{\mathbb{S}^0} \mathbb{D}^1 \cup_{\mathbb{S}^1} \mathbb{D}^2.$$

The first two attachments result in a figure-8 (a *wedge sum* of two circles) that forms the

[3] In fact, this fact alone easily implies M has the homotopy type of a CW complex; see [Hat02, Pro. A.11, p. 528].

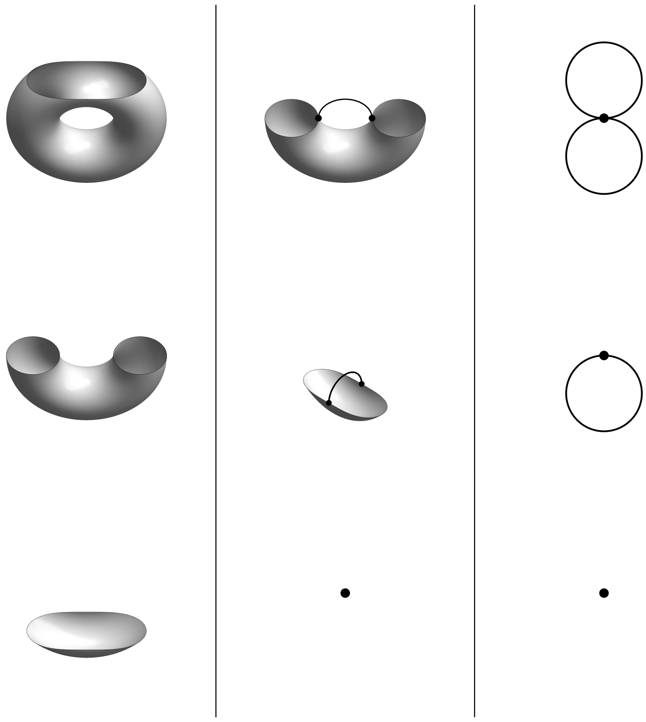


Figure 2.7. CW decomposition of a standing torus via the height map. Rows are homotopy equivalent. Left column: sublevel set M^α for $\alpha \in \mathbb{R}$ ascending. Middle column: sublevel set with cell attached. Right column: CW complex.

1-skeleton of the complex. The last map attaches the surface of the torus to its 1-skeleton (see Figure 2.7).

The map φ in this example is self-indexing, so there was no need to apply cellular approximation on the attaching maps, as in the proof of Theorem 2.20. ▪

Remark 2.25. Theorems 2.19 and 2.20 are nonconstructive, in that they assert the existence of attaching maps but do not specify them. This may seem like a significant shortcoming, as knowing only the number of cells in each dimension of a CW complex generally gives little information about its homotopy type. In Section 2.5 it will be demonstrated that despite this limitation, in many cases much interesting data may be extracted with the help of these theorems. ▪

2.4 Existence of Morse maps

The power of Morse theory is contingent on the availability of Morse maps for arbitrary manifolds. This brief section overviews several key results that ensure that not only are Morse maps always available, but that every smooth $\mathbb{R}$-valued map is arbitrarily close to a Morse map. Another reassuring fact is that every Morse map may be viewed as a 'height' map in a precise sense, and this is what motivates the repeated use of these in examples. This section concludes by applying Theorem 2.20 to obtain a fundamental topological result on the structure of smooth manifolds: that *every* smooth manifold has the homotopy type of a CW complex.

Proposition 2.26 (cf. [Nic11, Cor. 1.25, p. 21]). *Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold of $\mathbb{R}^n$ not containing the origin for some $n \in \mathbb{N}$. Then for almost all $v \in \mathbb{R}^n$ (in the Lebesgue measure), the height map $\eta_v : M \to \mathbb{R}$ and the distance map $\delta_v : M \to \mathbb{R}$, given by respectively*

$$\eta_v := \langle v, \cdot \rangle, \qquad \delta_v := \langle v - \cdot, v - \cdot \rangle,$$

are Morse (where $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the standard inner product on $\mathbb{R}^n$). Moreover, if M is a closed subset of $\mathbb{R}^n$, then δ_v is exhaustive. ■

Remark 2.27. In view of the Whitney embedding theorem [Whi57, Theorem 1A, p. 113], which ensures that every smooth n-manifold can be smoothly embedded as a closed subset of $\mathbb{R}^{2n+1}$, Proposition 2.26 then implies exhaustive Morse maps exist on all smooth manifolds.

Moreover, every Morse map $\varphi : M \to \mathbb{R}$ can be defined as a height map relative to some smooth embedding. To see this, let $i : M \hookrightarrow \mathbb{R}^n$ be a smooth embedding for some $n \in \mathbb{N}$. A second smooth embedding is then given by

$$\widetilde{i} : M \hookrightarrow \mathbb{R} \times \mathbb{R}^n, \qquad m \mapsto (\varphi(m), i(m)).$$

If $(e_i)_{i=0}^n$ denotes the canonical basis of $\mathbb{R} \times \mathbb{R}^m$, then φ is the projection

$$m \mapsto \eta_{e_0}(m) = \left\langle e_0, \widetilde{i}(m) \right\rangle,$$

which is the height relative to the e_0 axis. ▪

The next result ensures that not only are Morse maps available, they are abundant.

Theorem 2.28 ([BH04b, The. 5.27, p. 139]). *Let $\psi : M \to \mathbb{R}$ be a smooth map. Then for all $\varepsilon > 0$ there exists a Morse map φ such that,*

$$\sup \{ |\varphi(m) - \psi(m)| : m \in M \} < \varepsilon.$$ ■

Remark 2.29. Theorem 2.28 says that the set of Morse maps on a smooth manifold is dense in the set of all smooth maps in the *uniform topology*. If M is compact, then the uniform topology agrees with the compact-open topology (cf. Footnote 1). For details on the topology of mapping spaces, see [Mun75, Cha. 7, p. 263]. ▪

The following fundamental result follows directly from Theorem 2.20 and Proposition 2.26, and the Whitney embedding theorem [Whi57, Theorem 1A, p. 113].

Theorem 2.30. *Every smooth manifold has the homotopy type of a CW complex. Every compact smooth manifold has the homotopy type of a finite CW complex.* ■

Theorem 2.30 is a powerful regularity result on the topology of smooth manifolds. It is worth noting that corresponding theorem for topological manifolds is considerably harder to prove [May99, p. 151].

2.5 Morse inequalities

In this section, the *Morse inequalities* are deduced. These are a system of inequalities of formal polynomials relating numerical invariants of the topology of a smooth manifold to the critical structure of a Morse map on it. These inequalities comprise a readily applicable tool to either deduce the homological structure of a manifold from a Morse map, or to characterise the possible Morse maps that a manifold may admit. The proofs of the Morse inequalities presented in this section are based on those in [Nic11, Sec. 2.3, pp. 57–62].

It will be convenient to first establish some terminology and notation. Throughout, $\mathbb{F}$ will denote an arbitrary field. The ring of ***formal Laurent series*** is the set

$$\mathbb{Z}((t)) := \left\{ \sum_{i\in\mathbb{Z}} n_i t^i : n_i \in \mathbb{Z}\ \forall i, n_i = 0\ \forall i \ll 0 \right\},$$

i.e. formal power series with integral coefficients that have only finitely many nonzero negative powers. The ring operations are ordinary addition and multiplication of power series. It is easily checked that the following relation on $\mathbb{Z}((t))$ defines a ***partial ordering***, i.e, a reflexive, antisymmetric, and transitive relation:

$$A(t) \preceq B(t) :\equiv\ \exists N(t) \in \mathbb{N}((t)) : B(t) - A(t) = (1+t)N(t), \tag{2.3}$$

where $\mathbb{N}((t))$ is the subset (not subring) of $\mathbb{Z}((t))$ of formal Laurent series with nonnegative coefficients. The following properties of this order relation are immediate:

Proposition 2.31. *Let $A(t)$, $B(t)$, and $C(t) \in \mathbb{Z}((t))$ and $D(t) \in \mathbb{N}((t))$. Then:*

1. *$A(t) \preceq B(t)$ if and only if $A(t) + C(t) \preceq B(t) + C(t)$;*
2. *if $A(t) \preceq B(t)$, then $A(t)D(t) \preceq B(t)D(t)$.* ■

By considering the inequality Equation (2.3) at each power of t, the following 'weak' inequalities are deduced.

Proposition 2.32 (Abstract weak Morse inequalities)**.** *Let*

$$A(t) := \sum_{i\in\mathbb{Z}} a_i t^i, \qquad B(t) := \sum_{i\in\mathbb{Z}} b_i t^i$$

be formal Laurent series for which $A(t) \preceq B(t)$. Then

$$a_i \leqslant b_i$$

for all $i \in \mathbb{Z}$. ■

By associating a formal power series to both the homology groups of a manifold and the critical set of a Morse map, a more refined set of inequalities will be obtained.

A $\mathbb{Z}$-graded $\mathbb{F}$-vector space

$$A_\bullet := \bigoplus_{i\in\mathbb{Z}} A_i$$

for which $\operatorname{Dim} A_i < \infty$ for all $i \in \mathbb{Z}$, and $A_i = 0$ for all $i \ll 0$ is ***admissible***.

Definition 2.33 (Poincarè series). If $A_\bullet$ is an admissible $\mathbb{Z}$-graded $\mathbb{F}$-vector space, then its ***Poincaré series*** is the formal Laurent series

$$\mathscr{P}_{A_\bullet}(t) := \sum_{i\in\mathbb{Z}} (\operatorname{Dim} A_i) t^i \in \mathbb{Z}((t)).$$ ▪

Lemma 2.34 (Subadditivity). *Let $A_\bullet$, $B_\bullet$, and $C_\bullet$ be admissible $\mathbb{Z}$-graded $\mathbb{F}$-vector spaces that admit a long exact sequence*

$$\cdots \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow A_{i-1} \longrightarrow \cdots .$$

Then

$$\mathscr{P}_{B_\bullet}(t) \le \mathscr{P}_{A_\bullet}(t) + \mathscr{P}_{C_\bullet}(t).$$

Proof. By exactness and splitting,

$$A_i \simeq (\operatorname{Ker} A_i \to B_i) \oplus (\operatorname{Im} A_i \to B_i) \simeq (\operatorname{Ker} A_i \to B_i) \oplus (\operatorname{Ker} B_i \to C_i), \tag{2.4}$$

and similarly,

$$B_i \simeq (\operatorname{Ker} B_i \to C_i) \oplus (\operatorname{Ker} C_i \to A_{i-1}) \tag{2.5}$$

$$C_i \simeq (\operatorname{Ker} C_i \to A_{i-1}) \oplus (\operatorname{Ker} A_{i-1} \to B_{i-1}) \tag{2.6}$$

for each $i \in \mathbb{Z}$. Denote $n_i := \operatorname{Dim}(\operatorname{Ker} A_i \to B_i)$. Taking dimensions and subtracting Equation (2.5) from Equations (2.4) and (2.6) then gives

$$\operatorname{Dim} A_i - \operatorname{Dim} B_i + \operatorname{Dim} C_i = n_i + n_{i-1}.$$

By summing over all $i \in \mathbb{Z}$, it then follows that

$$(\mathscr{P}_{A_\bullet}(t) + \mathscr{P}_{C_\bullet}(t)) - \mathscr{P}_{B_\bullet}(t) = (1+t)\sum_{i\in\mathbb{Z}} n_i t^i.$$ ■

Definition 2.35 (Betti numbers, Poincaré polynomial). For X a compact topological space with finite-dimensional[4] homology in all degrees, the ***Betti numbers*** of X are the dimensions of its homology groups,

$$\beta_i(X;\mathbb{F}) := \operatorname{Dim} H_i(X;\mathbb{F}), \qquad i \in \mathbb{N},$$

[4]This condition is sometimes absent in the literature (e.g. [Nic11, p. 48]), but it is necessary: a striking testament is an example of Barratt and Milnor [BM62] of a compact finite-dimensional space with homology groups of uncountable dimension in infinitely many degrees.

and the ***Poincaré series*** of X is the *generating function*

$$\mathscr{P}_{X;\mathbb{F}}(t) := \sum_{i\in\mathbb{N}} \beta_i(X;\mathbb{F})t^i.$$

For a topological pair (X, A), the ***relative Poincaré series*** is defined analogously:

$$\mathscr{P}_{X,A;\mathbb{F}}(t) := \sum_{i\in\mathbb{N}} (\operatorname{Dim} H_i(X, A; \mathbb{F}))t^i.$$

If these series happen to be polynomials (e.g. whenever X is a finite-dimensional compact manifold), then they are referred to as ***Poincarè polynomials***. ▪

Definition 2.36 (Morse numbers, Morse series)**.** If $\varphi : M \to \mathbb{R}$ is a Morse map on a compact manifold M, then the critical set is finite by Corollary 2.11, and so the series

$$\mathscr{M}_\varphi(t) := \sum_{c\in\operatorname{Cr}\varphi} t^{\lambda_c(\varphi)} =: \sum_{\lambda\in\mathbb{N}} \mu_\lambda(\varphi)t^\lambda,$$

is a polynomial, the ***Morse polynomial*** of φ, where the coefficients $\mu_\lambda(\varphi)$ (or just μ_λ) for $\lambda \in \mathbb{N}$ are the ***Morse numbers*** of φ. ▪

It is now possible to state and prove the Morse inequalities:

Theorem 2.37 (Morse inequalities)**.** *Let $\varphi : M \to \mathbb{R}$ be a Morse map on a compact manifold M. Then*

$$\mathscr{P}_{M;\mathbb{F}}(t) \preceq \mathscr{M}_\varphi(t).$$

Proof. Let $\gamma_1 < \gamma_2 < \cdots < \gamma_k$ be the critical values $\{\gamma_1, \ldots, \gamma_k\} := \varphi(\operatorname{Cr}\varphi)$, and let $\{\alpha_0, \ldots, \alpha_k\} \subset \mathbb{R} \setminus \varphi(\operatorname{Cr}\varphi)$ be regular values for which

$$\alpha_0 < \gamma_1, \qquad \alpha_k > \gamma_k, \qquad \alpha_i \in [\gamma_i, \gamma_{i+1}], \qquad i \in \{0, \ldots, k\}.$$

From the long exact sequences of the pairs $(M^{\alpha_i}, M^{\alpha_{i-1}})$ (Theorem 1.59),

$$\cdots \longrightarrow H_n(M^{\alpha_{i-1}}) \longrightarrow H_n(M^{\alpha_i}) \longrightarrow H_n(M^{\alpha_i}, M^{\alpha_{i-1}}) \xrightarrow{\partial} H_{n-1}(M^{\alpha_{i-1}}) \longrightarrow \cdots$$

it follows that the homology $H_\bullet(M^{\alpha_i}, M^{\alpha_{i-1}})$ is finite-dimensional in all degrees, and moreover by Lemma 2.34,

$$\mathscr{P}_{M^{\alpha_i}} \preceq \mathscr{P}_{M^{\alpha_i},M^{\alpha_{i-1}}} + \mathscr{P}_{M^{\alpha_{i-1}}}.$$

By Proposition 2.31, summing over $i \in \{1, \ldots, k\}$ gives

$$\begin{aligned} \sum_{i=1}^{k} \mathscr{P}_{M^{\alpha_i}} &\preceq \sum_{i=1}^{k} \mathscr{P}_{M^{\alpha_i}, M^{\alpha_{i-1}}} + \sum_{i=1}^{k} \mathscr{P}_{M^{\alpha_{i-1}}} \\ \implies \mathscr{P}_{M^{\alpha_k}} - \mathscr{P}_{M^{\alpha_0}} &\preceq \sum_{i=1}^{k} \mathscr{P}_{M^{\alpha_i}, M^{\alpha_{i-1}}}. \end{aligned}$$

As $M^{\alpha_0} = \varnothing$ and $M^{\alpha_k} \simeq M$, this becomes

$$\mathscr{P}_M \preceq \sum_{i=1}^{k} \mathscr{P}_{M^{\alpha_i}, M^{\alpha_{i-1}}}.$$

Denote the critical points in the fibre of γ_i by

$$\mathrm{Cr}_i := \varphi^{-1}(\gamma_i) \cap \mathrm{Cr}\,\varphi.$$

By the structure theorem (Theorem 2.20),

$$M^{\alpha_i} \sim M^{\alpha_1 - 1} \cup \bigcup_{c \in \mathrm{Cr}_i} \mathbb{D}^{\lambda_c}.$$

The excision and additivity properties of homology (Theorem 1.59) then give an isomorphism of graded homology (coefficients suppressed):

$$H_\bullet\left(M^{\alpha_i}, M^{\alpha_i - 1}\right) \simeq H_\bullet\left(\coprod_{c \in \mathrm{Cr}_i} \left(\mathbb{D}^{\lambda_c}, \mathbb{S}^{\lambda_c}\right)\right) \simeq \bigoplus_{c \in \mathrm{Cr}_i} H_\bullet\left(\mathbb{D}^{\lambda_c}, \mathbb{S}^{\lambda_c}\right) \simeq \bigoplus_{c \in \mathrm{Cr}_i} \widetilde{H}_\bullet\left(\mathbb{S}^{\lambda_c}\right).$$

Recalling (Example 1.55) that the reduced homology of the sphere is

$$\widetilde{H}_j\left(\mathbb{S}^{\lambda}\right) = \begin{cases} \mathbb{F} & : i = \lambda \\ 0 & : i \neq \lambda, \end{cases}$$

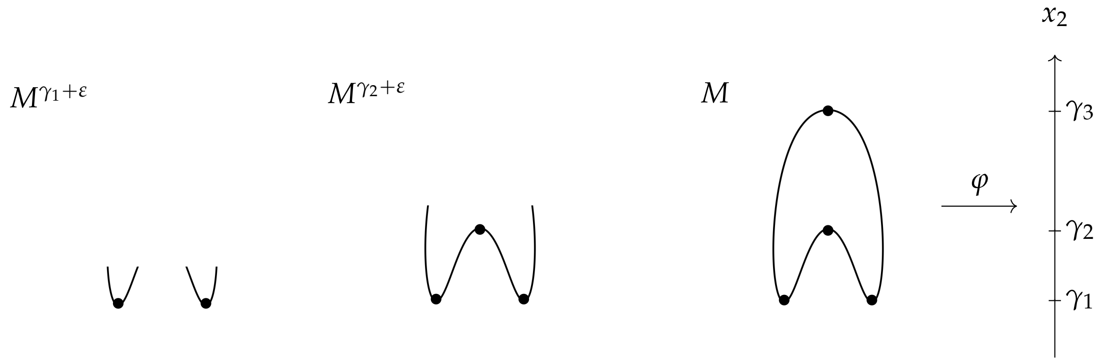
it then follows that

$$\mathscr{P}_{M^{\alpha_i}, M^{\alpha_{i-1}}}(t) = \sum_{c \in \mathrm{Cr}_i} t^{\lambda_c}.$$

Summing over $i \in \{1, \ldots, k+1\}$ then gives the result:

$$\mathscr{P}_M(t) \preceq \sum_{i=1}^{k+1} \mathscr{P}_{M^{\alpha_i}, M^{\alpha_{i-1}}}(t) = \sum_{c \in \mathrm{Cr}\,\varphi} t^{\lambda_c} =: \mathscr{M}_\varphi(t). \qquad \blacksquare$$

Remark 2.38. Bott provides the following intuition for the proof of Theorem 2.37 in [Bot82, pp. 340–341]: consider the attaching sphere $\mathbb{S}^{\lambda-1}$ in the attaching space $M^{\gamma-\varepsilon} \cup_{\mathbb{S}^{\lambda-1}} \mathbb{D}^{\lambda}$ of Theorem 2.19. The image of this sphere corresponds to a $(\lambda - 1)$-cycle in $M^{\gamma-\varepsilon}$. By



$\mathscr{M} = 2, \quad \mathscr{P} = 2.$ $\quad$ $\mathscr{M} = 2 + t, \quad \mathscr{P} = 1, \quad \Delta = 1 + t.$ $\quad$ $\mathscr{M} = 2 + 2t, \quad \mathscr{P} = 1 + t, \quad \Delta = 0.$

Figure 2.8. Height map on a circle with four critical points. First image: the sublevel set $M^{\gamma_1+\varepsilon}$ has the homotopy type of two points. Second image: attaching a 1-cell kills a homology class in H_0. Third image: attaching a 1-cell creates a new class in H_1.

Proposition 1.58, only the homology in degrees $\lambda - 1$ or λ may be affected, and in fact there are only two possibilities:

1. ***The sphere bounds a chain in*** $M^{\gamma-\varepsilon}$**.** Then the attached cell creates a new nontrivial λ-cycle in $H_\lambda(M^{\gamma+\varepsilon}; \mathbb{F})$ (and the corresponding critical point is then $\mathbb{F}$-*completable*—see Definition 2.53).

2. ***The sphere does not bound a chain.*** Then the cycle corresponding to the sphere represents a nontrivial class in $H_{\lambda-1}(M^{\gamma-\varepsilon}; \mathbb{F})$, and this cycle is killed by the attaching cell; i.e. the corresponding class is zero in $H_{\lambda-1}(M^{\gamma+\varepsilon}; \mathbb{F})$

The effect on the difference of the Poincaré polynomial $\Delta\mathscr{P} := \mathscr{P}_{M^\beta;\mathbb{F}} - \mathscr{P}_{M^\gamma;\mathbb{F}}$ is then $\Delta\mathscr{P} = t^\lambda$ in the first case, and $\Delta\mathscr{P} = -t^{\lambda-1}$ in the second. On the other hand, the difference in the Morse polynomial $\Delta\mathscr{M} := \mathscr{M}_\varphi^{\gamma+\varepsilon} - \mathscr{M}_\varphi^{\gamma-\varepsilon}$ is t^λ in either case. The corresponding difference $\Delta = \Delta\mathscr{M} - \Delta\mathscr{P}$ is then respectively 0 or $t^{\lambda-1}(1+t)$, and the Morse inequalities follow by induction over the critical values (see Figure 2.8). ▪

Before moving on to some applications, it will be useful to recall the statement of the abstract weak Morse inequalities (Proposition 2.32) into the present context:

Proposition 2.39 (Weak Morse inequalities)**.** *Let $\varphi : M \to \mathbb{R}$ be a Morse map on a compact manifold. Then the Morse numbers are bounded below by the Betti numbers, i.e.*

$$\beta_i(M; \mathbb{F}) \leqslant \mu_i(\varphi)$$

for every field $\mathbb{F}$ and for all $i \in \mathbb{N}$.[5][6] ■

---

[5]The Ljustemik-Schnirelmann category $\operatorname{Cat} M$ of a compact smooth manifold M is a homotopy invariant that gives a sharper lower bound to the number of critical points of *every* smooth map $\varphi : M \to \mathbb{R}$ (i.e. not necessarily nondegenerate), but it is significantly harder to compute [CP86].

[6]The inequality $\sum_{i\in\mathbb{N}} \beta_i(M; \mathbb{F}) \leqslant \#\operatorname{Cr}\varphi$ is what motivated the Arnold conjecture of symplectic geometry,

In light of Poincaré duality (Theorem 1.74), the weak inequalities may be mildly refined:

Corollary 2.40. *Let $\varphi : M \to \mathbb{R}$ be a Morse map on a compact n-manifold for $n \in \mathbb{N}$. Then*

$$\beta_i(M;\mathbb{Z}_2) \leqslant \min\{\mu_i(\varphi), \mu_{n-i}(\varphi)\}, \qquad i \in \mathbb{N}. \qquad \blacksquare$$

If M is compact, evaluating the Morse polynomial at special values may produce interesting analytical or topological data. For instance, at $t := 1$,

$$\mathscr{M}_\varphi(1) = \#\operatorname{Cr}\varphi,$$

an analytical quantity fully dependent on φ. On the other hand, evaluating the Morse polynomial at $t = -1$ computes a basic topological invariant, the ***Euler characteristic*** of M,

$$\chi_M := \sum_{i\in\mathbb{N}} (-1)^i \beta_i(X;\mathbb{F}),$$

that is, of course, independent of φ.

Corollary 2.41. *Let $\varphi : M \to \mathbb{R}$ be a Morse map on a compact manifold M. Then the Morse polynomial computes the Euler characteristic[7]., i.e.*

$$\mathscr{M}_\varphi(-1) = \chi_M.$$

Proof. By Theorem 2.37, $\mathscr{P}_{M;\mathbb{F}}(t) \preceq \mathscr{M}_\varphi(t)$ means there exists a polynomial $N(t) \in N((t))$ with positive integral coefficients for which

$$\mathscr{M}_\varphi(-t) - \mathscr{P}_{M;\mathbb{F}}(t) = (1+t)N(t).$$

Evaluating at $t = -1$ then gives

$$\mathscr{M}_\varphi(-1) - \mathscr{P}_{M;\mathbb{F}}(-1) = (1-1)N(-1) = 0,$$

and the result follows. ■

Corollary 2.41 has an immediate and fundamental topological implication:

Corollary 2.42. *The Euler characteristic is independent of the homology coefficients.*[8] ■

Remark 2.43. Corollary 2.42 is not at all apparent from the definition of the Euler characteristic as the alternating sum of Betti numbers, which themselves *do* depend on the

that bounds below the number of fixed points of a Hamiltonian symplectomorphism by the sum of the Betti numbers [Arn65].

[7]This is a special case of the Hopf Index theorem for vector fields of the form $\nabla\varphi$ [BT82, The. 11.25, p. 129].

[8]This actually follows for coefficients in every commutative unital ring.

homology coefficients.[9] An example of this dependence is given by $\mathbb{RP}^2$, whose Poincaré polynomials over the fields $\mathbb{Q}$ and $\mathbb{Z}_2$ differ (see Remark 2.59):

$$\mathscr{P}_{\mathbb{RP}^2;\mathbb{Q}}(t) = 1, \qquad \mathscr{P}_{\mathbb{RP}^2;\mathbb{Z}_2}(t) = 1 + t + t^2. \qquad ▪$$

The remainder of this section explores simple consequences and applications of the Morse inequalities.

Corollary 2.44. *A compact smooth manifold of odd dimension has vanishing Euler characteristic.*

Proof. Let $\varphi : M \to \mathbb{R}$ be a Morse map on a compact n-manifold. Note that $-\varphi$ is also a Morse map and $\mu_\lambda(\varphi) = \mu_{n-\lambda}(-\varphi)$ for each $\lambda \in \mathbb{N}$, which follows directly from the relevant definitions. Now Corollary 2.41 implies

$$\chi_M = \sum_{\lambda=0}^{n} \mu_\lambda(\varphi)(-1)^\lambda = \sum_{\lambda=0}^{n} \mu_\lambda(\varphi)(-1)^{n-\lambda}.$$

The two sums differ by a factor of $(-1)^n$, thus $\chi_M = 0$ if n is odd. ■

Corollary 2.45. *A Morse map on a compact smooth manifold of odd dimension has an even number of critical points.*

Proof. By Corollaries 2.41 and 2.44,

$$\mathscr{M}_\varphi(-1) = \chi_M = 0.$$

But clearly

$$\#\operatorname{Cr}\varphi = \mathscr{M}_\varphi(1) = \mathscr{M}_\varphi(-1) \pmod 2,$$

so, the result follows. ■

There are many results in the other direction: for example, if either M is a compact oriented manifold with $\operatorname{Dim} M = 2 \pmod 4$, or if $M = \partial W$ for a compact manifold W, then χ_M is even (see respectively [May99, Pro., p. 166], [May99, Cor., p. 168]), and the reasoning of the last proof implies that in these cases the number of critical points $\#\operatorname{Cr}\varphi$ of every Morse map $\varphi : M \to \mathbb{R}$ is also even.

Corollary 2.46. *Let M and N be compact smooth manifolds. Then the Euler characteristic is multiplicative, i.e,*

$$\chi_{M\times N} = \chi_M \chi_N.$$

Proof. Let $\varphi : M \to \mathbb{R}$ and $\psi : N \to \mathbb{R}$ be Morse maps, and define

$$\vartheta : M \times N \to \mathbb{R}, \qquad (m, n) \mapsto \varphi(m) + \psi(n).$$

[9] The dependence on the coefficient field is only by its characteristic, and if the homology groups are *torsion-free*, then there is no dependence. The precise effect of the coefficients is described by the *universal coefficients theorem*; see e.g. [Hat02, Sec. 3.A, pp. 261–268].

Clearly

$$\mathrm{d}\vartheta_{m,n} = 0 \iff \mathrm{d}\varphi_m = 0 \in \mathrm{T}^*_m M \text{ and } \mathrm{d}\psi_m = 0 \in \mathrm{T}^*_n N,$$

hence, $\operatorname{Cr}\vartheta = \operatorname{Cr}\varphi \times \operatorname{Cr}\psi$. Moreover, for a critical point $(c_M, c_N) \in \operatorname{Cr}\vartheta$, the Hessian matrix will take the block form

$$\mathrm{d}^2\vartheta_{c_M,c_N} = \begin{bmatrix} \mathrm{d}^2\varphi_{c_M} & 0 \\ 0 & \mathrm{d}^2\psi_{c_N} \end{bmatrix}.$$

It then follows

$$\lambda_{(c_M,c_N)}(\vartheta) = \lambda_{c_M}(\varphi) + \lambda_{c_N}(\psi) \implies \mu_\lambda(\vartheta) = \sum_{i+j=\lambda} \mu_i(\varphi)\mu_j(\psi),$$

and hence,

$$\mathscr{M}_\vartheta(t) = \mathscr{M}_\varphi(t)\mathscr{M}_\psi(t).$$

Setting $t = -1$ then gives the result. ■

The ***connected sum*** $M \sharp N$ of connected smooth n-manifolds M and N is the smooth manifold obtained by deleting an open n-disk $\mathbb{D}^n$ from both M and N, and smoothly identifying the boundary spheres $\partial\mathbb{D}^n$ by some choice of diffeomorphism [Lee13, Exa. 9.31, p. 225]. This operation is not uniquely defined—even up to homotopy—without specifying the diffeomorphism. A counterexample is given by $\mathbb{CP}^2 \sharp \mathbb{CP}^2$ and $\mathbb{CP}^2 \sharp -\mathbb{CP}^2$ (where the minus sign denotes reversal of orientation), which are not even homotopy equivalent [KPT18, p. 1]. (If the manifolds are oriented, then there are at most two possibilities up to homeomorphism, depending on whether the smooth attaching map preserves or reverses orientation [Lee11, pp. 164–165].) Below we outline a derivation of the formula for the Euler characteristic of a connected sum of compact manifolds via the Morse inequalities. In particular, our argument suggests that at least the Euler characteristic is independent of the choice of attaching map.

Example 2.47 (Euler characteristic of a connected sum)**.** Let M and N be connected compact smooth n-manifolds for some $n \in \mathbb{N}$, embedded into a sufficiently high-dimensional euclidean space $\mathbb{R}^m$ (by Whitney's embedding theorem [Whi57, Theorem 1A, p. 113]). By Proposition 2.26, there exists a vector $v \in \mathbb{R}^m$ for which

$$\vartheta := \operatorname{Proj}_v : M \amalg N \to \mathbb{R}$$

is Morse, and the restrictions

$$\varphi := \vartheta\Big|_M, \qquad \psi := \vartheta\Big|_N$$

are then also Morse. By letting M remain fixed and performing a connect sum $M \sharp N$ of a small neighbourhood of a maximum point of M and a minimum point on N, it is plausible that for some v, the induced map

$$\eta := \mathrm{Proj}_v : M \sharp N \to \mathbb{R}$$

is Morse. If that is true, then η will have Morse numbers equal to the sum of the Morse numbers of φ and ψ, except for one fewer of degree 0, and one fewer of degree n, corresponding to the critical points contained in the deleted disks:

$$\mu_\lambda(\eta) = \begin{cases} \mu_\lambda(\varphi) + \mu_\lambda(\psi) & : \lambda \neq 0, n \\ \mu_\lambda(\varphi) + \mu_\lambda(\psi) - 1 & : \lambda = 0 \\ \mu_\lambda(\varphi) + \mu_\lambda(\psi) - 1 & : \lambda = n. \end{cases}$$

Then

$$\mathscr{M}_\eta(t) = \mathscr{M}_\varphi(t) + \mathscr{M}_\psi(t) - 1 - t^n,$$

and hence,

$$\chi_{M \sharp N} = \chi_M + \chi_N - \big(1 + (-1)^n\big) = \chi_M + \chi_N - \chi_{\mathbb{S}^n},$$

(where the Euler characteristic of the n-sphere was deduced in Example 2.56.) This argument could perhaps be made rigorous with a similar approach to that used in the proof of Theorem 2.19. ▪

Example 2.48 (Extrema on the circle)**.** Let $\mathbb{S}^1$ be smoothly embedded in $\mathbb{R}^2$ and $\varphi : M \to \mathbb{R}$ a Morse map. As the Euler characteristic of the circle is zero, the number of local minima of φ equals the number of local maxima, as

$$0 = \chi_{\mathbb{S}^1} = \mathscr{M}_\varphi(-1) = \mu_0(\varphi) - \mu_1(\varphi),$$

by Corollary 2.41. ▪

The ***genus*** g of a connected compact oriented surface (i.e. 2-manifold) is the number of ‘holes’ in the surface. The connected compact orientable surface of genus g is the g-holed torus, and is customarily denoted M_g.

Example 2.49 (Height map on M_g)**.** Let M_g be embedded vertically in $\mathbb{R}^3$, similarly to Figure 2.6, and let $\varphi : M_g \to \mathbb{R}$ be the height map. It is easy to see that M_g has $2 + 2g$ critical points and that all of them are nondegenerate. Their indices correspond to the minimum critical point of index 0, the maximum of index 2, and $2g$ saddle points of index 1. The Morse polynomial is then

$$\mathscr{M}_\varphi(t) = 1 + 2gt + t^2,$$

and the relationship between genus and Euler characteristic is thus,

$$\chi = 2 - 2g.$$ ▪

An example in a similar spirit to Reeb's sphere theorem (Proposition 2.17) is the following:

Proposition 2.50 (Morse map with three critical points [EK62, Lem., p. 21])**.** *Let $\varphi : M \to \mathbb{R}$ be a Morse map on a compact n-manifold M for $n \in \mathbb{N}$ with three critical points. Then n is even, and the indices of the critical points are* $0, \frac{n}{2}, n$.[10]

Proof. The case $n = 0$ is ruled out (as 0-manifolds do not admit Morse maps), and $n = 1$ is also by Example 2.48. Also note that M must be connected, as a continuous map on a disjoint union of two compact manifolds has at least four critical points by the extreme value theorem [Mun75, The. 27.4, p. 174]. Let $\mathbb{F} := \mathbb{Z}_2$ be the field of two elements. As M is compact and connected,

$$\beta_0(M; \mathbb{Z}_2) = \beta_n(M; \mathbb{Z}_2) = 1.$$

Taken with

$$\# \operatorname{Cr} \varphi = \mathscr{M}_\varphi(1) = 3,$$

the weak Morse inequalities (Proposition 2.39) then imply

$$\mu_\lambda(\varphi) = \beta_\lambda(M; \mathbb{Z}_2), \qquad \lambda \in \{0, \ldots, n\}.$$

As remarked in the proof of Corollary 2.44, $\mu_\lambda(\varphi) = \mu_{n-\lambda}(-\varphi)$, and so the same argument applied to $-\varphi$ gives

$$\mu_\lambda(-\varphi) = \beta_{n-\lambda}(M; \mathbb{Z}_2), \qquad \lambda \in \{0, \ldots, n\},$$

and the result follows. ∎

2.6 $\mathbb{F}$-perfect Morse maps

Depending on the point of view, the Morse polynomial may be seen as estimating the Poincaré polynomial, or vice versa. Later examples will demonstrate that the estimates of the Morse inequalities are in fact sharp, i.e. equality may be attained. The following terminology describes this situation.

Definition 2.51 ($\mathbb{F}$-perfect Morse map)**.** A Morse map $\varphi : M \to \mathbb{R}$ is $\mathbb{F}$***-perfect*** if the Morse and Poincaré polynomials coincide, i.e. if

$$\mathscr{M}_\varphi = \mathscr{P}_{M;\mathbb{F}}.$$ ▪

Remark 2.52. If a manifold M has ***torsion***, i.e. if its homology has elements of finite order (e.g. because of the presence of nonorientable cycles (cf. Definition 1.73)), then its Betti

[10] It also follows that M has the homotopy type of an $\frac{n}{2}$-sphere with an n-sphere attached, and the cohomology ring (over every coefficient ring) of a projective plane over the real, complex, quaternionic or octonionic numbers [EK62, pp. 21–24].

numbers, and hence $\mathscr{P}_{M;\mathbb{F}}$, depends on the coefficient field $\mathbb{F}$ (recall Remark 2.43). It follows that if a manifold admits a perfect Morse map, then it is automatically torsion-free. ▪

Several criteria for and consequences of this property follow.

Definition 2.53 ($\mathbb{F}$-completable critical point/Morse map)**.** Let $c \in \operatorname{Cr}\varphi$ be a critical point of a Morse map $\varphi : M \to \mathbb{R}$, and let ε be such that $0 < \varepsilon \ll 1$. Denote by

$$[c] \in H_{\lambda_c}\left(M^{\varphi(c)+\varepsilon}, M^{\varphi(c)-\varepsilon}; \mathbb{F}\right) \xrightarrow{\partial} H_{\lambda_c - 1}\left(M^{\varphi(c)-\varepsilon}; \mathbb{F}\right)$$

the nontrivial homology class corresponding to c in the long exact sequence of the pair $\left(M^{\varphi(q)+\varepsilon}, M^{\varphi(q)-\varepsilon}\right)$ (cf. Proposition 1.58). Then c is $\mathbb{F}$***-completable*** if its image under ∂ vanishes. The Morse map φ is $\mathbb{F}$***-completable*** if every one of its critical points is $\mathbb{F}$-completable. ▪

If a critical point $c \in \operatorname{Cr}\varphi$ is $\mathbb{F}$-completable, then the Morse map φ is perfectly 'efficient' in its estimation at c, as recalling the proof of Theorem 2.37 and Remark 2.38, in this case no extraneous terms are contributed to the Morse polynomial from c. It follows that if every critical point of the map φ is $\mathbb{F}$-completable, then φ is $\mathbb{F}$-perfect.

Proposition 2.54. *Let $\varphi : M \to \mathbb{R}$ be an $\mathbb{F}$-completable Morse map on a compact manifold M. Then φ is $\mathbb{F}$-perfect.*

Proof. As in the proof of Theorem 2.37, let $\gamma_1 < \gamma_2 < \cdots < \gamma_k$ be the critical values $\{\gamma_1, \ldots, \gamma_k\} := \varphi(\operatorname{Cr}\varphi)$, and $\{\alpha_0, \ldots, \alpha_k\} \subset \mathbb{R} \setminus \varphi(\operatorname{Cr}\varphi)$ regular values for which

$$\alpha_0 < \gamma_1, \qquad \alpha_k > \gamma_k, \qquad \alpha_i \in [\gamma_i, \gamma_{i+1}], \qquad i \in \{0, \ldots, k\}.$$

By the definition of an $\mathbb{F}$-completable map, the connecting maps ∂ in the long exact sequences of the pairs $(M^{\alpha_i}, M^{\alpha_{i-1}})$ vanish, giving short exact sequences

$$0 \rightarrowtail H_n(M^{\alpha_{i-1}}; \mathbb{F}) \rightarrowtail H_n(M^{\alpha_i}; \mathbb{F}) \twoheadrightarrow H_n(M^{\alpha_i}, M^{\alpha_{i-1}}; \mathbb{F}) \xrightarrow{\partial} 0,$$

and hence, by splitting, isomorphisms

$$H_n(M^{\alpha_i}; \mathbb{F}) \simeq H_n(M^{\alpha_{i-1}}; \mathbb{F}) \oplus H_n(M^{\alpha_i}, M^{\alpha_{i-1}}; \mathbb{F}), \qquad n \in \mathbb{N}.$$

Taking Poincaré polynomials and summing over $i \in \{1, \ldots, k\}$ then gives

$$\mathscr{P}_{M;\mathbb{F}} = \mathscr{P}_{M^{\alpha_k};\mathbb{F}} - \mathscr{P}_{M^{\alpha_0};\mathbb{F}} = \sum_{i=1}^{k} \mathscr{P}_{M^{\alpha_i}, M^{\alpha_{i-1}};\mathbb{F}} = \mathscr{M}_\varphi.$$

(where the last equality was an intermediate result in the proof of Theorem 2.37). ■

Proposition 2.55 (Morse's lacunary principle)**.** *Let $\varphi : M \to \mathbb{R}$ be a Morse map on a compact manifold M. If there are no consecutive critical indices, i.e.*

$$|\lambda_c - \lambda_d| \neq 1 \qquad \forall c, d \in \operatorname{Cr}\varphi,$$

then φ is $\mathbb{F}$-perfect for every field $\mathbb{F}$.

Proof. Suppose φ is not $\mathbb{F}$-perfect. Then there is a nonzero $N(t) \in \mathbb{N}((t))$ for which

$$\mathscr{M}_\varphi(t) - \mathscr{P}_{M;\mathbb{F}}(t) = (1+t)N(t). \tag{2.7}$$

Consecutive powers of t occur in the right side of Equation (2.7), and thus also on the left. But $\mathscr{M}_\varphi$ dominates $\mathscr{P}_{M;\mathbb{F}}$ in each coefficient by Proposition 2.39, so consecutive powers of t occur in $\mathscr{M}_\varphi$. ∎

Example 2.56 (Homology of certain sphere products)**.** The height map on the unit n-sphere

$$\varphi := \operatorname{Proj}_{x_{n+1}} : \mathbb{S}^n \to \mathbb{R}$$

of Example 2.12 satisfies the lacunary condition of Proposition 2.55, so is $\mathbb{F}$-perfect and computes the Betti numbers by $\mathscr{M}_\varphi(t) = \mathscr{P}_{M;\mathbb{F}}(t) = 1 + t^n$. Let $m \in \mathbb{N}$ be such that $|n - m| \geqslant 2$, and define

$$\vartheta : \mathbb{S}^n \times \mathbb{S}^m \to \mathbb{R}, \qquad (m_1, m_2) \mapsto \varphi(m_1) + \psi(m_2),$$

where $\psi : \mathbb{S}^m \to \mathbb{R}$ is the height map on $\mathbb{S}^m$. By the proof of Corollary 2.46, ϑ is a Morse map with

$$\mathscr{M}_\vartheta = \mathscr{M}_\varphi \mathscr{M}_\psi = (1+t^n)(1+t^m) = 1 + t^n + t^m + t^{nm},$$

which also satisfies the lacunary condition, and so coincides with the Poincaré polynomial by Proposition 2.55:

$$\mathscr{P}_{\mathbb{S}^n \times \mathbb{S}^m;\mathbb{F}}(t) = 1 + t^n + t^m + t^{nm}.$$ ▪

Remark 2.57. The conclusions of Example 2.56 remain valid without any assumption of the lacunary condition; indeed, Poincaré polynomials of CW complexes are multiplicative, as a direct consequence of the Künneth formula (Theorem 1.57). ▪

Example 2.58 (Perfect Morse map on complex projective space)**.** The complex projective space $\mathbb{CP}^n$ for $n \in \mathbb{N}$ is the space of directions through the origin of $\mathbb{C}^{n+1}$ up to phase (cf. Example 1.41). By projecting each direction onto the unit sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ and dividing by the phase shifting $\mathbb{S}^1$-action,

$$e^{i\vartheta} \cdot \left(z_j\right)_{j=1}^{n+1} = \left(e^{i\vartheta} z_j\right)_{j=1}^{n+1},$$

there is an identification

$$\mathbb{CP}^n \simeq \frac{\mathbb{S}^{2n+1}}{\mathbb{S}^1}.$$

Now define a map

$$\widetilde{\varphi} : \mathbb{S}^{2n+1} \to \mathbb{R}, \qquad (z_j)_{j=1}^{n+1} \mapsto \sum_{j=1}^{n+1} j\|z_j\|^2.$$

This map is invariant under the $\mathbb{S}^1$-action, and so the quotient projection induces a smooth map $\varphi : \mathbb{CP}^n \to \mathbb{R}$ [Lee13, Theorem 21.10, pp. 544–545]. By the method of Lagrange multipliers ([Bot82, p. 338]), or by analysing φ within a suitable atlas ([BH04b, Exa. 3.7, p. 49]), the critical points are found to correspond to the coordinate axes, where the eigenvalues of the Hessian $d^2\varphi$ at the jth critical point are

$$\{j, 1-j, \ldots, n-j\}$$

with multiplicity 2 (over the reals). Hence, the index of the jth critical point is $2j$. The Morse polynomial of φ is then

$$\mathscr{M}_\varphi(t) = \sum_{j=0}^{n} t^{2j},$$

so, the lacunary condition of Proposition 2.55 is met, and φ is $\mathbb{F}$-perfect, and hence,

$$\mathscr{P}_{M;\mathbb{F}}(t) = \mathscr{M}_\varphi(t).$$

In particular, the Euler characteristic is

$$\chi_{\mathbb{CP}^n} = \sum_{j=0}^{n} (-1)^{2j} = n+1. \qquad \blacksquare$$

Remark 2.59. The homology groups of real projective space are ([Hat02, Exa. 2.50, p. 154])

$$H_i(\mathbb{RP}^n;\mathbb{F}) = \begin{cases} \mathbb{F} & : i = 0 \\ \frac{\mathbb{F}}{2\mathbb{F}} & : i \text{ odd }, 1 \leqslant i \leqslant n-1 \\ \{\alpha \in \mathbb{F} : 2\alpha = 0\} & : i \text{ even }, 1 \leqslant i \leqslant n \\ 0 & : \text{else.} \end{cases}$$

The corresponding map of Example 2.58 on $\mathbb{RP}^n$ gives the Morse polynomial

$$\mathscr{M}_\varphi(t) = \sum_{i=0}^{n} t^i,$$

which is then $\mathbb{Z}_2$-perfect but not $\mathbb{Q}$-perfect, so perfection certainly depends on the coefficients. $\blacksquare$

Chapter 3

Morse-Bott theory

The Morse maps on a manifold M form a dense subset of $C^\infty(M)$ in the uniform topology (Theorem 2.28). However, the nondegeneracy condition on critical points that constrains them to be isolated excludes important classes of maps. If for example, the map $\varphi : M \to \mathbb{R}$ possesses a symmetry

$$\varphi(g \cdot m) = \varphi(m), \qquad \forall g \in G, \ \forall m \in M$$

where G is a *Lie group* (i.e. an abstract group with the compatible structure of a smooth manifold; see Section 3.4), the *orbit* $\{g \cdot c : g \in G\}$ of a critical point $c \in \operatorname{Cr} \varphi$ will consist entirely of critical points. Unless G is discrete or c is a fixed point of the G-action, the orbit of c will not be an isolated set and so φ will not be Morse. By the results of Section 2.4, φ may always be perturbed to a Morse map, but this is unlikely to preserve the symmetry invariance. Morse-Bott theory is an extension of the classical techniques of Chapter 2 to maps whose critical points may not be isolated, but still conform to a more relaxed form of nondegeneracy. Historically, this generalisation to Morse theory has been profitably exploited, most notably by Bott, who it is named after, in his application of Morse theory to the topology of Lie groups [Bot59; Bot60].

The aim of this final chapter is to present a motivational overview of Morse-Bott theory. Section 3.1 introduces the basic language of fibre bundles that is used throughout the discussion. The concept of orientability of a vector bundle is emphasised, and this leads to a statement of the *Thom isomorphism*, an essential ingredient in the development of the Morse-Bott inequalities that are the subject of Section 3.3. Section 3.2 generalises the main definitions (*nondegenerate critical manifolds, Morse-Bott maps*) and theorems (the *Morse-Bott lemma* and the *Morse-Bott theorem B*) of Chapter 2. Section 3.4 explores several applications of the Morse-Bott inequalities to the computation of Euler characteristics. The chapter concludes in Section 3.5, which extends the concept of $\mathbb{F}$-perfection and $\mathbb{F}$-completability to Morse-Bott maps.

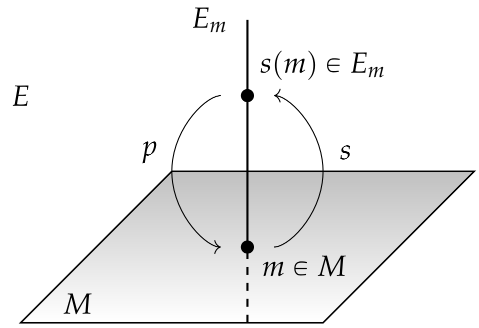


Figure 3.1. Section of a fibre bundle.

3.1 Fibre bundles

Morse-Bott theory will generalise Morse theory to maps for which the Hessian at a critical point need only be nondegenerate on a subspace of each tangent space that is orthogonal (with respect to some riemannian metric) to the critical submanifold to which it belongs. The collection of all such subspaces is the *normal bundle* of the submanifold, and is an example of a *fibre bundle*. Just as a manifold generalises euclidean space, in that locally it is homeomorphic to a euclidean space, a fibre bundle generalises a product space, in that locally it is homeomorphic to a product.

Definition 3.1 (Fibre bundle)**.** A ***fibre bundle with fibre*** F is a continuous surjection $p : E \twoheadrightarrow M$ of topological manifolds for which every $m \in M$ has a neighbourhood U of m for which there is a *fibre-preserving* homeomorphism

$$f_U : p^{-1}(U) \xrightarrow{\simeq} U \times F,$$

i.e. a homeomorphism for which the triangle below is commutative:

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\simeq", "f_U"'] \arrow[dr, two heads, "p"'] & U \times F \arrow[d, two heads, "\mathrm{Proj}_1"] \\ & U. \end{tikzcd}$$

In particular,

$$\left(f_U \circ p^{-1}\right)(m) \subseteq \mathrm{Proj}_1^{-1}(m)$$

for each $m \in U$. The collection $\{f_U\}_U$ is a ***local trivialisation*** of the fibre bundle.

This assemblage is abbreviated by either $E \twoheadrightarrow M$ or $F \hookrightarrow E \twoheadrightarrow M$ (even though there is not a single map $F \hookrightarrow E$ but one for each $m \in M$). The space M is the ***base space***, E is the ***total space***, and each $p^{-1}(m) =: E_m \simeq F$ for $m \in M$ is a ***fibre***. ■

Definition 3.2 (Bundle map)**.** Let $E_1 \twoheadrightarrow M_1$ and $E_2 \twoheadrightarrow M_2$ be fibre bundles (not necessarily with the same fibre). A ***bundle map*** from $E_1 \twoheadrightarrow M_1$ to $E_2 \twoheadrightarrow M_2$ is a pair of continuous maps $E_1 \to E_2$ and $M_1 \to M_2$ for which the square

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, two heads] & E_2 \arrow[d, two heads] \\ M_1 \arrow[r] & M_2 \end{tikzcd}$$

commutes. If $M :\simeq M_1 \simeq M_2$ and $M_1 \xrightarrow{\simeq} M_2 = \mathbb{1}_M$ is the identity map, then the bundle map is ***over*** M. Hence, a bundle map from $E_1 \twoheadrightarrow M$ to $E_2 \twoheadrightarrow M$ over M is represented by a commutative triangle

$$\begin{tikzcd} E_1 \arrow[r] \arrow[dr, two heads] & E_2 \arrow[d, two heads] \\ & M. \end{tikzcd}$$ ∎

Definition 3.3 (Bundle isomorphism)**.** Two fibre bundles $E_1 \twoheadrightarrow M_1$ and $E_2 \twoheadrightarrow M_2$ are ***isomorphic*** if there exist continuous maps $E_1 \xrightarrow{\simeq} E_1$ and $M_1 \xrightarrow{\simeq} M_2$ making the square below commute

$$\begin{tikzcd} E_1 \arrow[r, shift left, "\simeq"] \arrow[d, two heads] & E_2 \arrow[l, shift left, "\simeq"] \arrow[d, two heads] \\ M_1 \arrow[r, shift left, "\simeq"] & M_2. \arrow[l, shift left, "\simeq"] \end{tikzcd}$$ ∎

Definition 3.4 (Section)**.** A ***section*** of a fibre bundle $p : E \twoheadrightarrow M$ is a continuous map $s : M \rightarrowtail E$ for which the composition $p \circ s = \mathbb{1}_M$ is the identity on M (see Figure 3.1). ∎

Definition 3.5 (Subbundle)**.** Let $p : E \twoheadrightarrow M$ be a fibre bundle. A fibre bundle $p' : E' \to M'$ for which $E' \hookrightarrow E$ and $M' \hookrightarrow M$ are submanifolds and $p' = p\big|_{E'}$ is a ***subbundle*** of $p : E \twoheadrightarrow M$. ∎

Definition 3.6 (Pullback bundle)**.** Given a fibre bundle $p : E \twoheadrightarrow M$ and a continuous map $f : M' \twoheadrightarrow M$, the ***pullback bundle*** along f is the fibre bundle $p' : f^*E \twoheadrightarrow M$ with fibre F and total space

$$f^*E := \big\{ \big(m', e\big) \in M' \times E : f\big(m'\big) = p(e) \big\},$$

and where $p' := \mathrm{Proj}_1 : f^*E \to M'$, so that the following square commutes:

$$\begin{tikzcd} f^*E \arrow[r, two heads, "\mathrm{Proj}_2"] \arrow[d, two heads, "\mathrm{Proj}_1"'] & E \arrow[d, two heads, "p"] \\ M' \arrow[r, "f"'] & M. \end{tikzcd}$$ ∎

Listed below are several familiar examples of fibre bundles.

Example 3.7 (Trivial/trivialisable bundle)**.** A fibre bundle $F \hookrightarrow F \times M \twoheadrightarrow M$ is ***trivial*** or a ***product bundle***, and is the same thing as the product manifold $F \times M$ (Example 1.64). A fibre bundle that is isomorphic to a trivial bundle is ***trivialisable***. ∎

A torus is then an example of a trivial fibre bundle $\mathbb{S}^1 \hookrightarrow \mathbb{S}^1 \times \mathbb{S}^1 \twoheadrightarrow \mathbb{S}^1$. The simplest nontrivial fibre bundle is the Möbius band:

Example 3.8 (Möbius band)**.** The Möbius band is a fibre bundle $\mathbb{D}^1 \hookrightarrow E \twoheadrightarrow \mathbb{S}^1$ for which $E \neq \mathbb{S}^1 \times \mathbb{D}^1$ (as this is a cylinder), hence, the Möbius band is a nontrivial fibre bundle. ▪

Example 3.9 (Covering space)**.** A ***covering space*** is a fibre bundle $p : E \twoheadrightarrow M$ for which each fibre E_m for $m \in M$ is a discrete set. ▪

Definition 3.10 (Vector bundle)**.** Let $\mathbb{F}$ be a fixed field. An $\mathbb{F}$***-vector bundle*** is a fibre bundle $V \hookrightarrow E \twoheadrightarrow M$ with fibre V an $\mathbb{F}$-vector space, such that each homeomorphism $f_U : p^{-1}(U) \xrightarrow{\simeq} U \times F$ in a local trivialisation $\{f_U\}_U$ restricts to an $\mathbb{F}$-linear isomorphism on the fibres,

$$f_U\Big|_{E_m} : E_m \xrightarrow{\simeq} \{m\} \times V$$

for all $m \in M$. The ***rank*** of the vector bundle is the dimension of V, denoted

$$\operatorname{Rank}(V \hookrightarrow E \twoheadrightarrow M) := \operatorname{Dim} V. \qquad ▪$$

A ***real vector bundle*** is one whose typical fibre V is an $\mathbb{R}$-vector space, and a ***complex vector bundle*** is one whose typical fibre is a $\mathbb{C}$-vector space.

Definition 3.11 (Direct sum bundle)**.** Let $p_1 : E_1 \twoheadrightarrow M$ and $p_2 : E_2 \twoheadrightarrow M$ be two vector bundles over M. Then their ***direct sum*** is the vector bundle $E_1 \oplus E_2 \twoheadrightarrow M$, whose total space is the topological subspace

$$E_1 \oplus E_2 := \{(v_1, v_2) \in E_1 \times E_2 : p_1(v_1) = p_2(v_2)\} \subseteq E_1 \times E_2,$$

and whose projection map is

$$p : E_1 \oplus E_2 \twoheadrightarrow M, \qquad (v_1, v_2) \mapsto p_1(v_1) = p_2(v_2),$$

and whose fibres are the direct sum of the fibres of E_1 and E_2, i.e.

$$(E_1 \oplus E_2)_m = (E_1)_m \oplus (E_2)_m, \qquad \forall m \in M. \qquad ▪$$

Recall that a submersion is a smooth map $f : M \to N$ between smooth manifolds whose differentials $\mathrm{d}f_m : \mathrm{T}_m M \to \mathrm{T}_{f(m)} N$ are surjective at each point $m \in M$ (Definition 1.69).

Definition 3.12 (Smooth fibre bundle)**.** Let $F \hookrightarrow E \twoheadrightarrow M$ be a fibre bundle. If

1. F, E, and M are smooth manifolds;
2. the fibre inclusions $F \hookrightarrow E$ are smooth maps;
3. the projection $E \twoheadrightarrow M$ is a smooth submersion;

4. the local trivialisations $f_U : p^{-1}(U) \xrightarrow{\simeq} U \times F$ are diffeomorphisms,

then $F \hookrightarrow E \twoheadrightarrow M$ is a ***smooth fibre bundle***. ▪

Definitions 3.10 and 3.12 are compatible, and their combination defines a ***smooth vector bundle***.

Example 3.13 (Tangent bundle)**.** Let M be a smooth n-manifold. Then the ***tangent bundle*** of M

$$\mathrm{T}\, M := \bigcup_{m \in M} \mathrm{T}_m\, M \twoheadrightarrow M, \qquad \mathrm{T}_m\, M \mapsto m$$

is a smooth vector bundle of rank n, whose fibres are the tangent spaces $\mathrm{T}_m\, M$ for each $m \in M$ (see e.g. [Tu11, Cha. 12, p. 129]). ▪

Example 3.14 (Normal bundle)**.** A sequence

$$0 \rightarrowtail E \rightarrowtail F \twoheadrightarrow G \twoheadrightarrow 0$$

of fibre bundles over a manifold M is ***exact*** if it is fibrewise exact, i.e. if the sequence

$$0 \rightarrowtail E_m \rightarrowtail F_m \twoheadrightarrow G_m \twoheadrightarrow 0$$

is exact at each $m \in M$. Let M be a smooth manifold and $i : N \to M$ an immersion (recall Definition 1.69). The normal bundle $\mathrm{T}_{M/N}$ of N is defined via the short exact sequence of vector bundles

$$0 \rightarrowtail \mathrm{T}\, N \rightarrowtail i^*\mathrm{T}\, M \twoheadrightarrow \mathrm{T}_{M/N} \twoheadrightarrow 0.$$

By exhibiting an arbitrary riemannian metric on M, it is a simple exercise to show that every exact sequence of smooth vector bundles splits to an (orthogonal) direct sum (see e.g. [Lee18, Cha. 2, p. 16]), hence,

$$i^*\mathrm{T}\, M \simeq \mathrm{T}\, N \oplus \mathrm{T}_{M/N}.$$

The normal bundle $\mathrm{T}_{M/N}$ of a submanifold $N \hookrightarrow M$ is then characterised by the fibrewise quotient

$$\mathrm{T}_{M/N} = \frac{i^*\mathrm{T}\, M}{\mathrm{T}\, N} = \coprod_{m \in M} \frac{\mathrm{T}_m\, M}{\mathrm{T}_m\, N}.$$ ▪

For a real vector bundle $p : E \twoheadrightarrow M$ and an arbitrary metric $\|\cdot\|$ on the manifold E, the ***unit disk bundle*** $\mathbb{D}(E)$ is the subbundle whose fibres restrict to a unit disk,

$$\mathbb{D}(E) := \bigcup_{m \in M} \left\{ e \in p^{-1}(m) : \|e\| \leqslant 1 \right\}$$

and

$$\mathbb{S}(E) := \partial\mathbb{D}(E) = \bigcup_{m \in M} \left\{ e \in p^{-1}(m) : \|e\| = 1 \right\}$$

is the ***unit sphere bundle*** that bounds $\mathbb{D}(E)$. The following (nonstandard[1]) definition generalises orientability for a manifold (Definition 1.73) to real vector bundles.

Definition 3.15 (Orientable real vector bundle)**.** Fix an arbitrary field $\mathbb{F}$. A real vector bundle $E \twoheadrightarrow M$ is $\mathbb{F}$***-orientable*** if there exists a generator

$$\tau \in H_n(\mathbb{D}(E); \mathbb{F})$$

that is in the image of every of the maps

$$H_n(\mathbb{D}(E)_m; \mathbb{F}) \to H_n(\mathbb{D}(E); \mathbb{F})$$

induced by the inclusions $\mathbb{D}(E)_m \hookrightarrow \mathbb{D}(E)$ for all $m \in M$. Such an element τ will be referred to as a ***Thom class***. ▪

In the sequel, concerns of orientability will complicate the application of Morse-Bott theory. If we are willing to work with $\mathbb{Z}_2$-coefficients, then the following proposition permits us to dismiss these concerns.

Proposition 3.16 (cf. [Hat02, Sec. 4D.10, p. 442])**.** *Every disk bundle has a Thom class with $\mathbb{Z}_2$-coefficients.* ■

Example 3.17 (Orientable vector bundles)**.** Every complex vector bundle is $\mathbb{Q}$-orientable. Every real vector bundle over a simply connected space is $\mathbb{Q}$-orientable (cf. [BT82, Pro. 11.4, 11.5, p. 116]). ▪

The following famous and nontrivial theorem, which will be integral to the formulation of the Morse-Bott inequalities in Section 3.3, describes the homology of a smooth manifold in terms of the homology of a disk bundle over it.

Theorem 3.18 (Thom isomorphism (cf. [Hat02, Cor. 4D.9, p. 441]))**.** *Let $\mathbb{F}$ be a field, M a smooth manifold, and $p : E \twoheadrightarrow X$ an $\mathbb{F}$-orientable real vector bundle of rank r. Then there is an isomorphism*

$$H_{\bullet - r}(M; \mathbb{F}) \simeq H_\bullet(\mathbb{D}(E), \mathbb{S}(E); \mathbb{F}).$$ ■

3.2 Nondegenerate critical manifolds

The generalisation of nondegenerate critical points to submanifolds was pioneered by Bott in [Bot54]. The aim is to extend the theory developed thus far to maps whose critical sets are disjoint unions of submanifolds. On critical submanifolds, the differential vanishing will force the map $\varphi : M \to \mathbb{R}$ to be constant, and nondegeneracy is

[1]Orientability of vector bundles is more naturally defined in terms of cohomology, and the Thom class refers to an element of the corresponding cohomology ring. Because cohomology is not discussed in this thesis, we have used Lefschetz duality to obtain an equivalent homological definition; see e.g. [Hat02, The. 3.43, p. 254].

inescapable. The insight of Bott was that, by ignoring the directions tangential to the critical submanifolds, i.e. by restricting the Hessian to their normal bundles, the notion of nondegeneracy remains valid and consistent. Indeed, the entire apparatus of the classical theory developed in Chapter 2 may be recovered into this more relaxed setting.

The content of the proceeding sections follows [Nic11, Sec. 2.6, pp. 83–86] and [AB83, pp. 528–531].

Definition 3.19 (Nondegenerate critical submanifold)**.** Let $\varphi : M \to \mathbb{R}$ be a smooth map. If C is a connected submanifold of M for which the restriction of the differential to C vanishes, i.e,

$$\mathrm{d}\varphi\Big|_C = 0,$$

and moreover, the kernel of the Hessian coincides with the tangent space at each point of C, i.e.

$$\operatorname{Ker} \mathrm{d}^2\varphi_c = \mathrm{T}_c\, C, \qquad \forall c \in C,$$

then C is a ***nondegenerate critical submanifold*** with respect to φ. ▪

Definition 3.20 (Morse-Bott map)**.** Let $\varphi : M \to \mathbb{R}$ be a smooth map. If the critical set $\operatorname{Cr}\varphi$ is a disjoint union of connected nondegenerate submanifolds, then φ is a ***Morse-Bott*** map. ▪

The first step in generalising Morse theory to this setting is to obtain an appropriate analogue of the Morse Lemma (Lemma 2.9). Suppose that $C \in \pi_0(\operatorname{Cr}\varphi)$ is a nondegenerate critical submanifold of a smooth map $\varphi : M \to \mathbb{R}$. At each point $c \in C$, the kernel of the Hessian coincides with the tangent space at c. Hence, the Hessian restricts to nondegenerate symmetric $\mathbb{R}$-bilinear form on each normal space:

$$\mathrm{d}^2\varphi_c : \frac{T_c M}{T_c C} \times \frac{T_c M}{T_c C} \to \mathbb{R}.$$

By extending fibrewise, a nondegenerate symmetric $\mathbb{R}$-bilinear form $\mathrm{d}^2\varphi_C$ on the normal bundle $\mathrm{T}_{M/C}$ is obtained. This observation suggests the following generalisation of Lemma 2.9, whose proof may be found in [Hör07, App. C.6, pp. 502–503] or [BH04a].

Lemma 3.21 (Morse-Bott lemma)**.** *Let $C \in \pi_0(\operatorname{Cr}\varphi)$ be a critical submanifold of a Morse-Bott map $\varphi : M \to \mathbb{R}$. Then there exists an open neighbourhood U of $C \subseteq \mathrm{T}_{M/C}$ and a smooth open embedding $i : U \hookrightarrow M$ such that*

$$i\Big|_C = \mathbb{1}_C, \qquad i^*\varphi = \varphi(C) + \mathrm{d}^2\varphi_C.$$

Moreover, given an arbitrary riemannian metric on M, there is an orthogonal splitting

$$\mathrm{T}_{M/C} =: \mathrm{T}^-_{M/C} \oplus \mathrm{T}^+_{M/C}$$

where $\mathrm{d}^2\varphi_C$ is negative-definite on $\mathrm{T}^-_{M/C}$ and positive-definite on $\mathrm{T}^+_{M/C}$. Denoting the restrictions

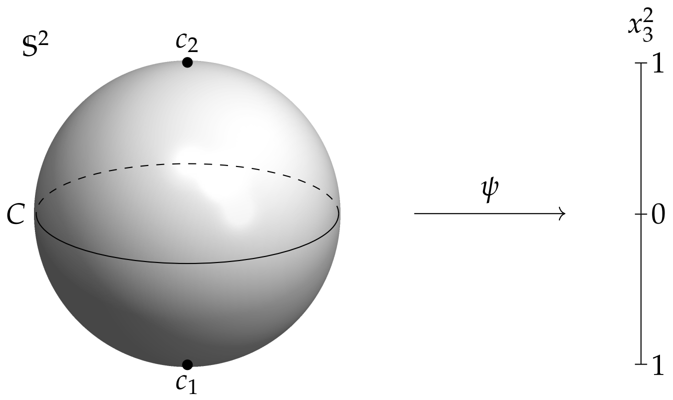


Figure 3.2. Squared height map on the 2-sphere.

of the Hessian to these summands

$$x_- := \mathrm{d}^2\varphi_C\Big|_{\mathrm{T}^-_{M/C}}, \qquad x_+ := \mathrm{d}^2\varphi_C\Big|_{\mathrm{T}^+_{M/C}},$$

then,

$$i^*\varphi = \varphi(C) + x_- + x_+.$$

The topological type of the **negative normal bundle** $\mathrm{T}^-_{M/C}$ *is independent of the various choices, and so is an invariant of* φ. ■

The preceding result permits the definition of the *index* of a critical submanifold, analogously to the classical case.

Definition 3.22 (Index)**.** Let $C \in \pi_0(\mathrm{Cr}\,\varphi)$ be a critical submanifold of a Morse-Bott map $\varphi : M \to \mathbb{R}$. The ***index*** $\lambda_C(\varphi)$ (or just λ_C) of C is the rank of its negative normal bundle,

$$\lambda_C(\varphi) := \mathrm{Rank}\,\mathrm{T}^-_{M/C}.$$ ▪

This definition is consistent with Definition 2.6, as if $\{c\} \in \pi_0(\mathrm{Cr}\,f)$ is a 0-dimensional critical submanifold, then its normal bundle coincides with the tangent space at c, i.e. $\mathrm{T}_{M/\{c\}} = \mathrm{T}_c\,M$, and it then follows from Lemma 3.21 that its negative normal bundle $\mathrm{T}^-_{M/c}$ coincides with the span of the negative eigenspaces of $\mathrm{d}^2\varphi_c$, agreeing with the classical index.

Example 3.23 (Constant map)**.** Every constant map $\varphi : m \mapsto \alpha \in \mathbb{R}$ is Morse-Bott, as then M itself is the unique critical submanifold. The index of M is then 0, as $\mathrm{T}_{M/M} = 0$. ▪

Example 3.24 (Morse map)**.** Every Morse map $\varphi : M \to \mathbb{R}$ is also a Morse-Bott map, as $\mathrm{Cr}\,\varphi$ may be identified with a disjoint union of 0-dimensional nondegenerate critical submanifolds in an obvious way. ▪

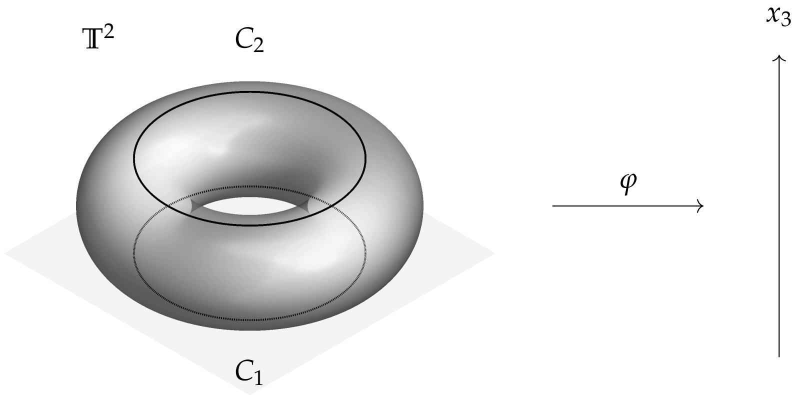


Figure 3.3. Height map on a standing torus.

Example 3.25 (Squared height map on a sphere)**.** Consider again the squared height map on the sphere of Example 2.13, and for simplicity take $n := 2$:

$$\varphi : \mathbb{S}^2 \to \mathbb{R}, \qquad (x_1, x_2, x_3) \mapsto x_3^2.$$

It is easily checked that the critical submanifolds are the south pole $(0, 0, -1)$ and north pole $(0, 0, 1)$ (of dimensions 0), and the equator $C := \{(x_1, x_2, x_3) \in \mathbb{S}^2 : x_3 = 0\}$ (of dimension 1). The argument to show the poles are nondegenerate of index 2 is no different from the classical case. To see that the equator is also nondegenerate, consider for example the chart around the point $(1, 0, 0) \in C$ given by

$$\left(\sqrt{1 - x_2^2 - x_3^2}, x_2, x_3\right) \mapsto (x_2, x_3).$$

The map φ in this chart takes the form

$$\left(\sqrt{1 - x_2^2 - x_3^2}, x_2, x_3\right) \mapsto x_3^2.$$

whose degeneracy in the classical sense is manifest in the absence of the term $\pm x_2^2$. However, x_2 corresponds in this chart to the tangential direction of the equator. Hence, in the x_3 direction, i.e. on the normal bundle of C, φ is nondegenerate. It follows that φ is a Morse-Bott map (see Figure 3.2). ▪

The last example also illustrates that critical submanifolds may be of different dimensions.

Example 3.26 (Height map on a horizontal torus)**.** The critical set of the height map $\varphi : \mathbb{T}^2 \to \mathbb{R}$ on a horizontal torus is two copies of $\mathbb{S}^1$, and it is easily checked that restriction $\mathrm{d}^2\varphi\big|_C$ to the normal directions of these critical circles $C \in \pi_0(\operatorname{Cr}\varphi)$ is nonsingular, showing that φ is Morse-Bott (see Figure 3.3). ▪

A powerful property of Morse-Bott maps that does not hold for Morse maps, which will be utilised in Section 3.4, is the following:

Proposition 3.27 (cf. Atiyah-Bott [AB83, Pro. 1.3, p. 531]). *Let $p : E \twoheadrightarrow M$ be a smooth fibre bundle. Then $\varphi : M \to \mathbb{R}$ is Morse-Bott if and only if the pullback bundle $p^*\varphi : E \to \mathbb{R}$ is Morse-Bott. Moreover, there is a bijection*

$$\pi_0(\mathrm{Cr}\,\varphi) \simeq \pi_0(\mathrm{Cr}\,p^*\varphi)$$

such that for each $C \in \pi_0(\mathrm{Cr}\,\varphi)$,

$$\lambda_C(\varphi) = \lambda_{p^{-1}(C)}(p^*\varphi). \qquad \blacksquare$$

The following generalisation of Theorem 2.19 follows by an analogous proof.

Theorem 3.28 (Morse-Bott Theorem B [Nic11, The. 2.44, p. 85]). *Let $\varphi : M \to \mathbb{R}$ be an exhaustive Morse-Bott map and $\gamma \in \varphi(\mathrm{Cr}\,\varphi)$ a critical value for which $\varphi^{-1}(\gamma) \cap \mathrm{Cr}\,\varphi = \{C_1, \ldots, C_k\}$ is a finite set of critical submanifolds. For each $i \in \{1, \ldots, k\}$, denote by $\mathbb{D}^-(C_i)$ the unit disk bundle of T^-_{M/C_i} (with respect to an arbitrary metric on T^-_{M/C_i}). Then for ε such that $0 < \varepsilon \ll 1$, there is a homotopy equivalence*

$$M^{\gamma+\varepsilon} \sim M^{\gamma-\varepsilon} \cup \mathbb{D}^-(C_1) \cup \cdots \cup \mathbb{D}^-(C_k),$$

where the union denotes the space obtained from $M^{\gamma+\varepsilon}$ by attaching the disk bundles $\mathbb{D}^-(C_i)$ to $M^{\gamma-\varepsilon}$ along their boundaries $\mathbb{S}^-(C_i) := \partial\mathbb{D}^-(C_i)$. Moreover, for every field $\mathbb{F}$, there is an isomorphism of graded homology:

$$H_\bullet\left(M^{\gamma+\varepsilon}, M^{\gamma-\varepsilon}; \mathbb{F}\right) \simeq \bigoplus_{i=1}^{k} H_\bullet(\mathbb{D}(C), \mathbb{S}(C); \mathbb{F}). \tag{3.1}$$

$\blacksquare$

3.3 Morse-Bott inequalities

This section derives the analogue of the Morse inequalities (Theorem 2.37) in the Morse-Bott setting. A subtlety that arises that is not present in the classical case, is that care must be paid to issues of orientability. Indeed, the Morse-Bott inequalities do not hold for arbitrary Morse-Bott maps, but only for $\mathbb{F}$-*orientable* Morse-Bott maps, to be defined below.

In fact, there has been a history of mistaken applications of Morse-Bott theory in the literature due to missing or inadequate formulations of the criterion for orientability: variously, authors have attempted to derive the Morse-Bott inequalities with either no assumptions of orientability, an assumption of orientability of the critical submanifolds, or an assumption of orientability of the ambient manifold. In [Rot16], it is shown that none of these assumptions are sufficient, by exhibiting a Morse-Bott map on an orientable manifold with orientable critical submanifolds that does not satisfy the Morse-Bott inequalities (Theorem 3.32.) (see also [Gue02, Sec. 2.5, p. 18] and the errata [BH00; Hur10]). The significance of the following definition will become evident in the proof of Theorem 3.32, which intimately relies on the Thom isomorphism (Theorem 3.18).

Definition 3.29 ($\mathbb{F}$-orientable Morse-Bott map)**.** Let $\mathbb{F}$ be a field and $\varphi : M \to \mathbb{R}$ a Morse-Bott map. Then φ is $\mathbb{F}$***-orientable***, if for every critical submanifold $C \in \operatorname{Cr}\varphi$, the negative normal bundle $\mathrm{T}^{-}_{M/C}$ is $\mathbb{F}$-orientable. ▪

Definition 3.30 (Morse-Bott polynomial)**.** Let $\mathbb{F}$ be a field and $\varphi : M \to \mathbb{R}$ a Morse-Bott map on a compact manifold M. The ***Morse-Bott*** polynomial of φ with respect to $\mathbb{F}$ is

$$\mathscr{M}_{\varphi;\mathbb{F}}(t) := \sum_{C \in \pi_0(\operatorname{Cr}\varphi)} t^{\lambda_C(\varphi)} \mathscr{P}_{C;\mathbb{F}}(t). \qquad ▪$$

Remark 3.31. In the case that φ is a Morse map, the definition coincides with Definition 2.36, as the Poincaré polynomial of a point is $\mathscr{P}_{*;\mathbb{F}} = 1$. Also note that, unlike the Morse polynomial, the Morse-Bott polynomial itself depends on the homology coefficients. ▪

Theorem 3.32 (Morse-Bott inequalities)**.** *Let $\varphi : M \to \mathbb{R}$ be an $\mathbb{F}$-orientable Morse-Bott map on a compact manifold M. Then*

$$\mathscr{P}_{M;\mathbb{F}} \preceq \mathscr{M}_{\varphi;\mathbb{F}}.$$

Proof. Let $\gamma_1 < \gamma_2 < \cdots < \gamma_k$ be the critical values $\{\gamma_1, \ldots, \gamma_k\} := \varphi(\operatorname{Cr}\varphi)$, and let $\{\alpha_0, \ldots, \alpha_k\} \subset \mathbb{R} \setminus \varphi(\operatorname{Cr}\varphi)$ be regular values for which

$$\alpha_0 < \gamma_1, \qquad \alpha_k > \gamma_k, \qquad \alpha_i \in [\gamma_i, \gamma_{i+1}], \qquad i \in \{0, \ldots, k\}.$$

As each of the negative disk bundles $\mathbb{D}^-(C)$ for $C \in \pi_0(\operatorname{Cr}\varphi)$ is $\mathbb{F}$-orientable, by Equation (3.1) and the Thom isomorphism (Theorem 3.18),

$$H_\bullet\left(M^{\alpha_i}, M^{\alpha_i - 1}; \mathbb{F}\right) \simeq \bigoplus_{C \in \varphi^{-1}(\gamma_i) \cap \pi_0(\operatorname{Cr}\varphi)} H_{\bullet - \lambda_C}(C; \mathbb{F}), \qquad i \in \{0, \ldots, k\}.$$

Hence, summing over i gives

$$\sum_{i=1}^{k} \mathscr{P}_{M^{\alpha_i}, M^{\alpha_i - 1};\mathbb{F}}(t) = \sum_{C \in \pi_0(\operatorname{Cr}\varphi)} t^{\lambda_C} \mathscr{P}_{C;\mathbb{F}}(t) =: \mathscr{M}_{\varphi;\mathbb{F}}(t).$$

But by an analogous argument to Theorem 2.37 (see also [Nic11, Cor 2.46, p.86]),

$$\mathscr{P}_{M;\mathbb{F}} \preceq \sum_{i=1}^{k} \mathscr{P}_{M^{\alpha_i}, M^{\alpha_i - 1};\mathbb{F}},$$

so, the result follows. ■

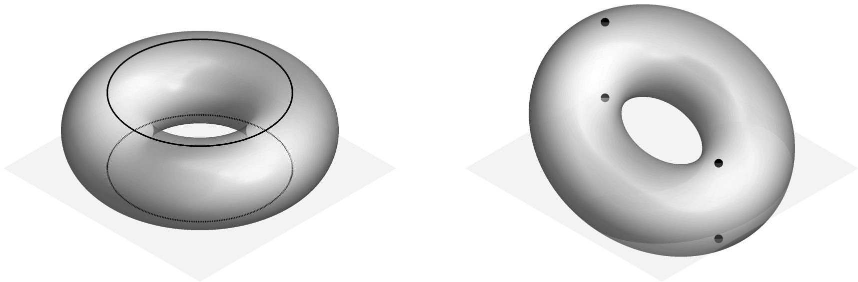


Figure 3.4. Perturbing a Morse-Bott map gives a Morse map with the same Morse polynomial (critical sets are black).

The generalisation of Corollary 2.41, which follows directly from Theorem 3.32, is that the Euler characteristic of a manifold is computed by the Euler characteristics of its critical submanifolds:

Corollary 3.33. *Let $\varphi : M \to \mathbb{R}$ be an $\mathbb{F}$-orientable Morse-Bott map on a compact manifold M. Then*

$$\chi_M = \sum_{C \in \pi_0(\mathrm{Cr}\,\varphi)} (-1)^{\lambda_C(\varphi)} \chi_C. \qquad \blacksquare$$

Remark 3.34. The following intuition for the Morse-Bott inequalities is given in [Gue02, Sec. 2.5, p. 18]. If φ is a Morse-Bott map, then by the discussion in Section 2.4, a small generic perturbation will yield a Morse map ψ. Each critical submanifold $C \in \mathrm{Cr}\,\varphi$ will (generically) split into a finite set of critical points, where for each Betti number $\beta_i(C; \mathbb{F})$ of C, there will be $\beta_i(C; \mathbb{F})$ critical points c of index

$$\lambda_c(\psi) = i + \lambda_C(\varphi).$$

This corresponds to simply reorganising the terms of the Morse-Bott polynomial, and so $\mathscr{M}_\psi = \mathscr{M}_{\varphi;\mathbb{F}}$. This situation, in the case of Example 3.26, is illustrated in Figure 3.4. ▪

3.4 Euler characteristics of fibre bundles

In Chapter 2, the multiplicative property of Euler characteristics was deduced from the Morse inequalities (Corollary 2.46). To illustrate the relative power of the Morse-Bott inequalities, a vastly stronger statement about the Euler characteristic of fibre bundles may be obtained, by a direct application of Theorem 3.32.

Proposition 3.35. *Let $F \hookrightarrow E \twoheadrightarrow M$ be a smooth fibre bundle of compact manifolds M and E. Then $\chi_E = \chi_F \chi_M$.*

Proof. Let $\varphi : M \to \mathbb{R}$ be a Morse map and $p : E \twoheadrightarrow M$ the projection. By Proposition 3.27, the pullback $p^*\varphi : E \to \mathbb{R}$ is Morse-Bott and whose critical submanifolds are given by the fibres of the critical points of φ, i.e. there is an index-preserving bijection $\pi_0(\mathrm{Cr}\,p^*\varphi) \simeq$

$\pi_0(\operatorname{Cr}\varphi)$. As φ is automatically $\mathbb{Z}_2$-orientable (Proposition 3.16), Corollary 3.33 applies to give

$$\chi_E = \sum_{C \in \pi_0(\operatorname{Cr} p^*\varphi)} (-1)^{\lambda_C(p^*\varphi)} \chi_C = \sum_{c \in \pi_0(\operatorname{Cr}\varphi)} (-1)^{\lambda_c(\varphi)} \chi_F = \chi_M \chi_F,$$

as claimed. ■

Of course, product bundles (Example 3.7) and covering maps (Example 3.9) are subsumed in Proposition 3.35.

Example 3.36 (Euler characteristic of a covering space). Let $F \hookrightarrow E \twoheadrightarrow M$ be a covering space of compact manifolds E and M. As the fibre F is a finite set (by compactness), its Euler characteristic is a fixed natural number $n \in \mathbb{N}_+$, and so by Proposition 3.35,

$$\chi_E = n\chi_M.$$ ▪

Proposition 3.35 is applied below to show the vanishing of Euler characteristics of the classical compact ***Lie groups***—abstract groups that have the complementary structure of a smooth manifold and whose group operations are smooth automorphisms. To do this, two key results are employed on ***homogeneous spaces***—smooth manifolds equipped with a ***transitive action*** of a Lie group G, i.e. an action

$$G \times M \to M$$

such that for all $m, n \in M$, there is a $g \in G$ for which $(g, m) \mapsto n$. For each point m, the ***orbit*** of m under G is

$$\operatorname{Orbit} m := \{n \in M : \exists g, (g, m) \mapsto n\},$$

and the ***stabiliser*** of m is

$$\operatorname{Stab} m := \{g \in G : (g, m) \mapsto m\}.$$

For homogeneous spaces there is the following version of the *orbit-stabiliser theorem* from the theory of abstract group actions:

Theorem 3.37 (Smooth orbit-stabaliser theorem (cf. [Lee13, The. 21.18, p. 552])). *Let M be a G-homogeneous space under the smooth transitive action of a Lie group G. Then there are diffeomorphisms*

$$\frac{G}{\operatorname{Stab} m} \simeq \operatorname{Orbit} m \simeq M$$

for each $m \in M$. ■

The following result then ensures the quotient projection of Theorem 3.37 is smooth:

Theorem 3.38 (cf. [War83, The. 3.58, p. 120]). *Let M be a G-homogeneous space, and let H*

denote the stabiliser of a point of M. Then the quotient projection

$$\frac{G}{H} \twoheadrightarrow M$$

is a smooth fibre bundle.[2] ■

Example 3.39 (Euler characteristics of the orthogonal groups)**.** The orthogonal group $\mathrm{O}(n)$ for $n \in \mathbb{N}$ is the Lie group of $\mathbb{R}$-linear isometries (i.e. distance preserving transformations) of euclidean space $\mathbb{R}^n$. Equivalently, this is the group of orthogonal matrices,

$$\mathrm{O}(n) = \left\{ A : A^{-1} = A^\top \right\} \subset \mathrm{GL}(n;\mathbb{R}),$$

where the group operation is matrix multiplication.

It is easily seen that the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is fixed by linear isometries, so a $\mathrm{O}(n)$-action is induced on $\mathbb{S}^{n-1}$. Moreover, this induced action is transitive, as every unit vector $u_1 \in \mathbb{S}^{n-1}$ can be completed to an orthonormal basis $(u_i)_{i=1}^n$ of $\mathbb{R}^n$, and then the corresponding matrix $A := [u_1, \dots, u_n]$ is an element of $\mathrm{O}(n)$ satisfying $Ae_1 = u_1$, where $(e_i)_{i=1}^n$ is the standard basis of $\mathbb{R}^n$. The stabiliser of e_1 is then the subset of matrices whose first column is e_1,

$$\operatorname{Stab} e_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} : B \in \mathrm{O}(n-1) \right\},$$

and this group is manifestly isomorphic to $\mathrm{O}(n-1)$. Theorem 3.37 then applies to give

$$\mathbb{S}^{n-1} = \operatorname{Orbit} e_1 = \frac{\mathrm{O}(n)}{\mathrm{O}(n-1)}.$$

Then from Theorem 3.38 it follows there are smooth fibre bundles

$$\mathrm{O}(n) \hookrightarrow \mathrm{O}(n+1) \twoheadrightarrow \mathbb{S}^n. \tag{3.2}$$

Now, $\mathrm{O}(n)$ is the continuous preimage of the closed set $\{I_{n\times n}\}$ (containing the identity matrix) under the map $A \mapsto A^\top A^{-1}$. Also, $\mathrm{O}(n)$ is clearly a bounded subset of $\mathbb{R}^{n^2}$, as for every orthogonal matrix $A \in \mathrm{O}(n)$, $A^\top A^{-1} = I_{n\times n}$ implies $\|A\| = 1$ (with the norm $\|\cdot\|$ induced by the inclusion $\mathrm{O}(n) \hookrightarrow \mathbb{R}^{n^2}$). Hence, by the Heine-Borel theorem [Spi65, Cor. 1.7, p. 10], $\mathrm{O}(n)$ is compact. Now Proposition 3.35 may be applied to obtain

$$\chi_{\mathrm{O}(n+1)} = \chi_{\mathrm{O}(n)}\chi_{\mathbb{S}^n}.$$

Recalling that $\chi_{\mathbb{S}^n} = 1 + (-1)^n$ from Example 2.56, there holds:

$$\chi_{\mathrm{O}(n)} = \begin{cases} 2 & : n = 1 \\ 0 & : n \geqslant 2. \end{cases}$$

[2]Such a fibre bundle is a *G-principal bundle*.

A similar approach applied to the unitary and symplectic groups gives the fibre bundles

$$\mathrm{U}(n-1) \hookrightarrow \mathrm{U}(n) \twoheadrightarrow \mathbb{S}^{2n-1}$$
$$\mathrm{Sp}(n-1) \hookrightarrow \mathrm{Sp}(n) \twoheadrightarrow \mathbb{S}^{4n-1},$$

showing that their Euler characteristics vanish for all $n \in \mathbb{N}_+$. ▪

In fact, the complete homological structure of the classical Lie groups in Example 3.39 was determined by Theodore Frankel [Fra65] by showing that the trace map $A \mapsto \operatorname{Re}\operatorname{Tr} A$ is a *perfect* Morse-Bott map (these are discussed in the following section) and calculating its critical structure. Liviu Nicolaescu exhibits an analogous perfect Morse-Bott map in [Nic11, Sec. 3.2, p. 114] to determine the homology of complex Grassmannians.

3.5 $\mathbb{F}$-perfect Morse-Bott maps

As in the classical case, the most favourable context for applying the Morse-Bott inequalities is when the Morse-Bott polynomial coincides with the Poincaré polynomial, i.e. when

$$\mathscr{M}_{\varphi;\mathbb{F}} = \mathscr{P}_{M;\mathbb{F}},$$

and in this case $\mathscr{M}_{\varphi;\mathbb{F}}$ is again called $\mathbb{F}$***-perfect***. Many of the successful applications of Morse-Bott theory have relied on a *carefully* chosen $\mathbb{F}$-perfect Morse-Bott map; indeed, $\mathbb{F}$-perfect maps are now not hard to come by, as every constant map is $\mathbb{F}$-perfect! As in the classical case, a sufficient condition for a Morse-Bott map to be $\mathbb{F}$-perfect is if it is $\mathbb{F}$-*completable*.

Definition 3.40 ($\mathbb{F}$-completable Morse-Bott map [AB83, p. 531])**.** [3] Let $\mathbb{F}$ be a field, $\varphi : M \to \mathbb{R}$ a Morse-Bott map on a compact manifold M, and ε such that $0 < \varepsilon \ll 1$. Then φ is $\mathbb{F}$***-completable*** if for each critical value $\gamma \in \varphi(\operatorname{Cr}\varphi)$ and each critical submanifold $C \in \varphi^{-1}(\gamma) \cap \operatorname{Cr}\varphi$, the dashed arrow in the commutative diagram

$$\begin{tikzcd}
H_\bullet(\mathbb{D}^-(C);\mathbb{F}) \arrow[r] & \widetilde{H}_\bullet(\mathbb{D}^-(C),\mathbb{S}^-(C);\mathbb{F}) \arrow[r, "\partial"] & \widetilde{H}_{\bullet-1}(\mathbb{S}^-(C);\mathbb{F}) \arrow[d] \\
 & H_{\bullet-\lambda_C}(C;\mathbb{F}) \arrow[u, "\wr"] \arrow[r, dashed] & \widetilde{H}_{\bullet-1}(M^{\gamma-\varepsilon};\mathbb{F})
\end{tikzcd}$$

is the zero map. (In this diagram, the top row is extracted from the long exact sequence of the pair $(\mathbb{D}^-(C),\mathbb{S}^-(C))$ (Theorem 1.59), the upward arrow is the Thom isomorphism (Theorem 3.18) and the downward arrow is induced from the inclusion $\mathbb{S}^-(C) \hookrightarrow M^{\gamma-\varepsilon}$.) ▪

Remark 3.41. Note that Definition 2.53 is recovered if φ is Morse, as then the Thom isomorphism degenerates to $H_{\bullet-\lambda_c}(*;\mathbb{F}) \xrightarrow{\simeq} \widetilde{H}_\bullet(\mathbb{S}^{\lambda_c};\mathbb{F})$, which is a triviality. ▪

[3] This notion of $\mathbb{F}$-completability was integral to the *equivariant* extension of Morse-Bott theory carried out in the seminal paper [AB83].

Using Equation (3.1) and arguing exactly as in Proposition 2.54, the following result is obtained (see also [Nic11, The. 2.48, p. 86]).

Proposition 3.42. *Let $\varphi : M \to \mathbb{R}$ be an $\mathbb{F}$-completable, $\mathbb{F}$-orientable Morse-Bott map on a compact manifold M. Then φ is $\mathbb{F}$-perfect, i.e.*

$$\mathscr{M}_{\varphi;\mathbb{F}} = \mathscr{P}_{M;\mathbb{F}}. \qquad \blacksquare$$

An analogue of the lacunary principle (Proposition 2.55) for the Morse-Bott setting is possible only when the Betti numbers and the indices of the critical submanifolds are all even:

Corollary 3.43 (Morse-Bott lacunary principle)**.** *Let $\varphi : M \to \mathbb{R}$ be an $\mathbb{F}$-orientable Morse-Bott map on a compact manifold M, such that for every critical submanifold $C \in \pi_0(\operatorname{Cr}\varphi)$ the index $\lambda_C(\varphi)$ of C is even, and each $\mathscr{P}_{C;\mathbb{F}}$ is even, i.e.*

$$\beta_n(C;\mathbb{F}) \neq 0 \implies n \in 2\mathbb{Z}.$$

Then φ is $\mathbb{F}$-perfect.

The following proof is provided in [Nic11, Cor. 2.49, p.87]:

Proof. Using the notation of Theorem 3.32, from Theorem 3.28 and the long exact sequence of the pairs $(M^{\alpha_i}, M^{\alpha_{i-1}})$,

$$\cdots \longrightarrow H_j(M^{\alpha_{i-1}}) \longrightarrow H_j(M^{\alpha_i}) \longrightarrow H_j(M^{\alpha_i}, M^{\alpha_{i-1}}) \xrightarrow{\partial} H_{j-1}(M^{\alpha_{i-1}}) \longrightarrow \cdots,$$

it follows by induction over the sequence $(\alpha_i)_i$ that if j is odd, then $\beta_j(M^{\alpha_i};\mathbb{F}) = 0$, and if j is even, then there are short exact sequences

$$0 \rightarrowtail H_j(M^{\alpha_{i-1}}) \rightarrowtail H_j(M^{\alpha_i}) \twoheadrightarrow H_j(M^{\alpha_i}, M^{\alpha_{i-1}}) \twoheadrightarrow 0.$$

Then φ is $\mathbb{F}$-completable, and the claim follows by Proposition 3.42. $\blacksquare$

We note that a more transparent proof is possible:

Alternative proof. If the hypothesis holds, then each power of t in

$$\mathscr{M}_{\varphi;\mathbb{F}}(t) - \mathscr{P}_{M;\mathbb{F}}(t) := \sum_{C \in \pi_0(\operatorname{Cr}\varphi)} t^{\lambda_C(\varphi)} \mathscr{P}_{C;\mathbb{F}}(t) - \sum_{n \in \mathbb{N}} \beta_n(M;\mathbb{F}) t^n$$

is clearly even, by Theorem 3.32. Hence, if $N(t) \in \mathbb{N}((t))$, then the relation

$$\mathscr{M}_{\varphi;\mathbb{F}}(t) - \mathscr{P}_{M;\mathbb{F}}(t) = (1+t)N(t)$$

implies $N(t) = 0$, and thus, $\mathscr{M}_{\varphi;\mathbb{F}}(t) = \mathscr{P}_{M;\mathbb{F}}(t)$, i.e. φ is $\mathbb{F}$-perfect. $\blacksquare$

Example 3.44 (Height map on a horizontal torus)**.** The critical submanifolds of the height map φ on the horizontal torus of Example 3.26 (see Figure 3.3) were two copies of the circle: one a minimum of index 0, and the other a maximum of index 1. The Morse-Bott polynomial then coincides with the Poincaré polynomial:

$$\mathscr{M}_{\varphi;\mathbb{F}}(t) = t^0(1+t) + t(1+t) = 1 + 2t + t^2 = \mathscr{P}_{\mathrm{T}^2;\mathbb{F}}(t),$$

and so φ is an $\mathbb{F}$-perfect Morse-Bott map. ▪

For an example of a Morse-Bott map that is not perfect, we revisit the squared height map on the sphere:

Example 3.45 (Squared height map on the n-sphere)**.** The squared height map on the n-sphere of Example 2.13 (see Figure 2.2) was

$$\varphi : \mathbb{S}^n \to \mathbb{R}, \qquad (x_i)_{i=1}^{n+1} \mapsto x_{n+1}^2.$$

The critical submanifolds are the south and north poles c_1 and c_2, that are maximums and thus of index n, and the equator C that is a minimum of index 0. The respective contributions to the Morse-Bott polynomial of φ are then

$$t^n \mathscr{P}_{c_1;\mathbb{F}}(t) = t^n, \qquad t^n \mathscr{P}_{c_2;\mathbb{F}}(t) = t^n, \qquad t^0 \mathscr{P}_{C;\mathbb{F}}(t) = 1 + t^{n-1}.$$

The Morse-Bott polynomial is therefore

$$\mathscr{M}_{\varphi;\mathbb{F}}(t) = 1 + t^{n-1} + 2t^n,$$

whereas the Poincaré polynomial is, of course, $\mathscr{P}_{\mathbb{S}^n;\mathbb{F}}(t) = 1 + t^n$ (cf. Example 2.56). ▪

List of figures

List of notations

$\mathbb{1}$	—	identity map
$\mathbb{A}$	—	unital ring
$\mathbb{F}$	—	field
$\mathbb{N}$	—	natural numbers $0, 1, 2, \ldots$
$\mathbb{N}_+$	—	positive natural numbers $1, 2, 3, \ldots$
$\mathbb{Z}_n$	—	integers mod n
$\mathbb{I}^n = [0, 1]^n$	—	closed unit cube in $\mathbb{R}^n$
$\mathbb{D}^n$	—	closed unit disk in $\mathbb{R}^n$
$\mathbb{S}^n = \partial\mathbb{D}^{n+1}$	—	boundary of the closed unit disk in $\mathbb{R}^n$
$* = \mathbb{D}^0$	—	point
$\ll$	—	much less than
$\subset$	—	proper subset
$A \setminus B$	—	set-theoretic difference of A from B
#	—	cardinality of a set
B^A	—	set of all maps from A to B
2^A	—	power set of A
Cl	—	topological closure
Int	—	topological interior
∂	—	algebraic or manifold boundary
A^*	—	dual space of the vector space A
Rank	—	rank of an abelian group, matrix, vector bundle
$A^\top$	—	transpose of the matrix A
$\oplus$	—	direct sum
$\otimes$	—	tensor product
$\hookrightarrow$	—	embedding, inclusion
$\rightarrowtail$	—	injection
$\twoheadrightarrow$	—	surjection
$\xrightarrow{\sim}$	—	homotopy equivalence
$\xrightarrow{\simeq}$	—	isomorphism
$\sim$	—	equivalent
$\simeq$	—	isomorphic

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