The Move Group of the Three–Ring Puzzle is $\mathbf{S_{12}}$

Abstract

We analyze the well-known three–ring ("Machinarium") puzzle with twelve fixed positions arranged on three intersecting circles. Each legal move is a 60° rotation of one circle, hence a 6-cycle on the six positions on that circle. We prove that the group G generated by the three rotations is the full symmetric group S_{12}. We give a short proof via Jordan's theorem (in a very weak form) and, independently, a constructive proof that avoids Jordan altogether: we show the action is 2-transitive, exhibit an explicit 3-cycle, then generate A_{12} from conjugates and conclude S_{12} by parity.

1 The model and notation

There are three circles, labelled A, B, C, each intersecting the other two in exactly two positions. Altogether there are 12 fixed *positions*.

Definition 1.1 (Position set and labels)**.** Let the position set be

$$\Omega = \{a_1, a_2, b_1, b_2, c_1, c_2, ab_1, ab_2, bc_1, bc_2, ca_1, ca_2\}.$$

Here a_1, a_2 (resp. b_1, b_2, c_1, c_2) are the two positions unique to circle A (resp. B, C). The two intersections $A \cap B$ contribute ab_1, ab_2, the two intersections $B \cap C$ give bc_1, bc_2, and the two intersections $C \cap A$ give ca_1, ca_2.

Conventions. We compose permutations *right-to-left*: $(\alpha\beta)(x) = \alpha(\beta(x))$. We write permutations in disjoint cycle notation. We adopt the commutator convention $[x, y] := xyx^{-1}y^{-1}$. We write $\mathrm{S}(\Omega)$ for the full symmetric group on Ω and $\mathrm{A}(\Omega)$ for its alternating subgroup; when $|\Omega| = 12$ we identify these with S_{12} and A_{12}.

Definition 1.2 (Generators)**.** Let $g_A, g_B, g_C \in \mathrm{S}(\Omega)$ be the permutations induced by a geometrically positive 60° rotation of circles A, B, C respectively. Around each circle the six positions occur in cyclic order, hence

$$\begin{aligned} g_A &= (a_1\ ca_1\ ab_1\ a_2\ ab_2\ ca_2)\,, \\ g_B &= (b_1\ ab_1\ bc_1\ b_2\ bc_2\ ab_2)\,, \\ g_C &= (c_1\ bc_1\ ca_1\ c_2\ ca_2\ bc_2)\,. \end{aligned}$$

We write $G = \langle g_A, g_B, g_C \rangle \leq \mathrm{S}(\Omega) \cong \mathrm{S}_{12}$ for the *move group*.

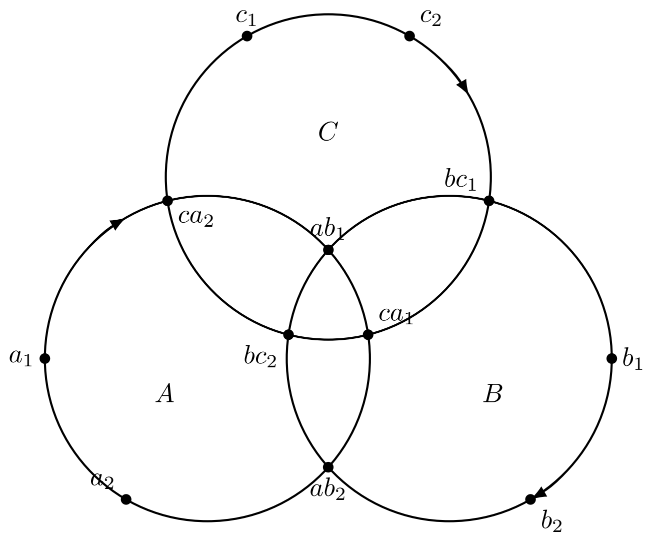

Figure 1: Schematic of the three rings and the 12 labelled positions. Points ab_i lie on $A \cap B$, bc_i on $B \cap C$, and ca_i on $C \cap A$. The arrow snippets indicate the common positive direction used in Definition 1.2. (Figure not to scale.)

2 First properties of G

Lemma 2.1 (Transitivity)**.** *The action of G on Ω is transitive.*

Proof. Starting from a_1, powers of g_A reach every position on circle A: $a_1, ca_1, ab_1, a_2, ab_2, ca_2$. From ab_1 and ab_2 the moves $g_B^{\pm 1}$ reach b_1, b_2, bc_1, bc_2; from ca_1, ca_2, the moves $g_C^{\pm 1}$ reach c_1, c_2, bc_1, bc_2. Thus the G-orbit of a_1 is all of Ω. □

Lemma 2.2 (Explicit 3-cycle factors)**.** *The group G has elements that decompose as products of two disjoint 3-cycles; explicitly,*

$$[g_A, g_B] = (a_2\ bc_1\ ab_1)\ (b_1\ ab_2\ ca_2)\,, \quad [g_B, g_C] = (b_2\ ca_1\ bc_1)\ (c_1\ bc_2\ ab_2)\,,$$

and

$$[g_C, g_A] = (a_1\ ca_2\ bc_2)\ (c_2\ ab_1\ ca_1)\,.$$

Proof. A direct calculation suffices; we record one in detail. Track each symbol through $g_A g_B g_A^{-1} g_B^{-1}$. For example:

$$ab_1 \xrightarrow{g_B^{-1}} b_1 \xrightarrow{g_A^{-1}} b_1 \xrightarrow{g_B} ab_2 \xrightarrow{g_A} ca_2,$$

so $ab_1 \mapsto ca_2$. Continuing with $ca_2 \mapsto b_1$ and $b_1 \mapsto ab_1$ shows $(b_1\ ab_2\ ca_2)$ is one factor. Repeating for the remaining moved points yields the displayed decompositions. (The other two commutators are analogous.) □

3 A short proof via primitivity and Jordan

Definition 3.1 (Blocks and primitivity)**.** A nonempty subset $\Delta \subseteq \Omega$ is a *block of imprimitivity* for G if for each $g \in G$ either $g(\Delta) = \Delta$ or $g(\Delta) \cap \Delta = \varnothing$. The action is *primitive* if the only blocks are singletons and Ω.

Lemma 3.2 (Primitivity)**.** *The action of G on Ω is primitive.*

Proof. Assume $\varnothing \neq \Delta \subsetneq \Omega$ is a block. Since G is transitive (Lemma 2.1), we may assume $a_1 \in \Delta$. Since a_1 lies only on circle A, both $g_B^{\pm 1}$ and $g_C^{\pm 1}$ fix a_1; hence they lie in $G_{a_1} \subseteq G_\Delta$. Therefore $g_B(\Delta) = \Delta = g_C(\Delta)$, i.e. the block Δ is invariant under the point stabilizer G_{a_1}. By Lemma 4.2 (proved in Section 4), G_{a_1} acts transitively on $\Omega \setminus \{a_1\}$. It follows that the only G-blocks containing a_1 are $\{a_1\}$ and Ω. Since Δ is assumed nontrivial, necessarily $\Delta = \Omega$, a contradiction. Thus $\Delta = \Omega$, contradicting nontriviality. Hence no nontrivial block exists. $\square$

We now invoke a classical result in a form tailored to our situation.

Proposition 3.3 (Jordan, special case)**.** *Let $G \leq \mathrm{S}_n$ act primitively with $n \geq 6$. If G contains a 3-cycle, then $\mathrm{A}_n \leq G$.*

(Idea of proof). Jordan's original theorem states: if a primitive $G \leq \mathrm{S}_n$ contains a p-cycle for some prime $p \leq n-3$, then $G \supseteq \mathrm{A}_n$. The case $p = 3$ used here follows by the same argument; see standard texts (e.g. Cameron; Dixon–Mortimer). One can also give a short direct proof: the normal closure of any 3-cycle in a primitive group of degree $n \geq 6$ is transitive and contains a point stabilizer in its support, forcing it to contain all 3-cycles and hence A_n. (A completely constructive alternative avoiding Jordan appears in Section 4.) $\square$

Theorem 3.4 (Main theorem)**.** *With $G = \langle g_A, g_B, g_C \rangle$ as in Definition 1.2, we have $G = \mathrm{S}_{12}$.*

Proof. By Lemma 3.2, G is primitive. By Lemma 4.4 (proved in Section 4), G contains a 3-cycle. Proposition 3.3 then yields $\mathrm{A}_{12} \leq G$. Since each g_* is a 6-cycle and hence an *odd* permutation, $G \nsubseteq \mathrm{A}_{12}$. Therefore $G = \mathrm{S}_{12}$. $\square$

4 A constructive proof avoiding Jordan

We now give a completely explicit route to S_{12} that does not appeal to Jordan's theorem.

Step 1: 2-transitivity (point stabilizer transitivity)

Definition 4.1. The action of G is 2-*transitive* if for any ordered pairs (x, y), (x', y') of distinct points there exists $g \in G$ with $g(x) = x'$ and $g(y) = y'$. Equivalently, for a fixed point x, the stabilizer G_x acts transitively on $\Omega \setminus \{x\}$.

Lemma 4.2. *The stabilizer G_{a_1} acts transitively on $\Omega \setminus \{a_1\}$. Hence the action of G is 2-transitive.*

Proof. Since a_1 lies only on circle A, both $g_B^{\pm 1}$ and $g_C^{\pm 1}$ fix a_1; thus they lie in G_{a_1}. From the cycles in Definition 1.2, starting from bc_2 we obtain

$$bc_2 \xrightarrow{g_B^{-1}} b_2, \quad bc_2 \xrightarrow{g_B} ab_2, \quad bc_2 \xrightarrow{g_B^2} b_1, \quad bc_2 \xrightarrow{g_B^3} ab_1,$$

and

$$bc_2 \xrightarrow{g_C} c_1, \xrightarrow{g_C^2} bc_1, \xrightarrow{g_C^3} ca_1, \xrightarrow{g_C^{-1}} ca_2, \xrightarrow{g_C^{-2}} c_2.$$

To reach a_2 while fixing a_1, consider

$$h := g_A^{-1}\, g_B\, g_A.$$

Since $g_A^6 = 1$, we have $g_A^{-1} = g_A^5$. Also $bc_2 \notin \operatorname{supp}(g_A)$, so $g_A(bc_2) = bc_2$; and $g_A(a_1) = ca_1 \notin \operatorname{supp}(g_B)$, hence $h \in G_{a_1}$. Moreover

$$h(bc_2) = g_A^{-1}\big(g_B(g_A(bc_2))\big) = g_A^{-1}\big(g_B(bc_2)\big) = g_A^{-1}(ab_2) = a_2.$$

Therefore the G_{a_1}-orbit of bc_2 contains every point of $\Omega \setminus \{a_1\}$, proving the claim. □

Step 2: Many 3-cycles and generation of A_{12}

We first pin down an explicit 3-cycle.

Lemma 4.3. *The commutator $[g_C, g_A]$ has the 3-cycle $\tau = (a_1\ ca_2\ bc_2)$ as a factor in its disjoint cycle decomposition.*

Proof. This is the third identity in Lemma 2.2. □

Lemma 4.4 (Isolating a single 3-cycle)**.** *Set $\tau = (a_1\, ca_2\, bc_2)$ and $\sigma = (c_2\, ab_1\, ca_1)$, so that $[g_C, g_A] = \tau\,\sigma$ by Lemma 2.2. There exists $h \in G_{a_1}$ such that h fixes a_1, ca_2, bc_2, c_2 and swaps ab_1 with ca_1; for example,*

$$h \;=\; g_B\,(g_A g_B g_A^{-1})\,(g_A^{-1} g_B^{-1} g_A)^2\,(g_A g_B g_A^{-1})\,g_B^{-1}.$$

Then $h\sigma h^{-1} = \sigma^{-1}$ and $h\tau h^{-1} = \tau$, hence

$$(\tau\sigma)\,h\,(\tau\sigma)\,h^{-1} = \tau^2 \in G,$$

so in particular $\tau \in G$.

Lemma 4.5 (Fixed-point conjugates of τ)**.** *Let $G = \langle g_A, g_B, g_C \rangle \le S_\Omega$ be as above and set $\tau = (a_1\, ca_2\, bc_2)$. Then G_{a_1} acts 2-transitively on $\Omega \setminus \{a_1\}$. Consequently, for every ordered pair (x, y) of distinct elements of $\Omega \setminus \{a_1\}$ there exists $h \in G_{a_1}$ with*

$$h\,\tau\,h^{-1} = (a_1\, x\, y).$$

Proof. All of g_B, $g_A g_B g_A^{-1}$, $g_A^5 g_B g_A^{-5}$, and $g_A^2 g_C g_A^{-2}$ fix a_1, hence

$$H \;:=\; \langle\, g_B,\ g_A g_B g_A^{-1},\ g_A^5 g_B g_A^{-5},\ g_A^2 g_C g_A^{-2}\,\rangle \;\le\; G_{a_1}.$$

The supports of these four 6-cycles are

$$\begin{aligned}
\operatorname{supp}(g_A^2 g_C g_A^{-2}) &= \{c_1, bc_1, ca_1, c_2, bc_2, a_2\},\\
\operatorname{supp}(g_B) &= \{b_1, ab_1, bc_1, b_2, bc_2, ab_2\},\\
\operatorname{supp}(g_A g_B g_A^{-1}) &= \{b_1, bc_1, b_2, bc_2, a_2, ca_2\},\\
\operatorname{supp}(g_A^5 g_B g_A^{-5}) &= \{b_1, bc_1, b_2, bc_2, a_2, ca_1\},
\end{aligned}$$

whose union is $\Omega \setminus \{a_1\}$ and which intersect pairwise; thus H is transitive on $\Omega \setminus \{a_1\}$.

To prove 2-transitivity, it suffices to show that the stabilizer H_{ca_2} is transitive on $\Omega \setminus \{a_1, ca_2\}$. Note that g_B and

$$h \ := \ g_A^5 g_B g_A^{-5} = (bc_2 \ a_2 \ b_1 \ ca_1 \ bc_1 \ b_2)$$

both fix ca_2. Hence $\langle g_B, h\rangle$ moves bc_2 to ab_2 and to a_2, and permutes the 8 points

$$\{b_1, ab_1, bc_1, b_2, bc_2, ab_2, a_2, ca_1\}.$$

To bring in the remaining points, consider the commutator

$$\kappa \ := \ [g_C, h] = g_C \, h \, g_C^{-1} \, h^{-1} \ \in H_{ca_2}.$$

A direct check shows

$$\kappa(ca_2) = ca_2, \qquad \kappa(c_1) = a_2, \qquad \kappa(c_2) = ca_1,$$

so κ fixes ca_2 while sending c_1 and c_2 into the above 8-point orbit of $\langle g_B, h\rangle$. Therefore H_{ca_2} is transitive on $\Omega \setminus \{a_1, ca_2\}$.

Since H is transitive on $\Omega \setminus \{a_1\}$, all point stabilizers H_x $(x \neq a_1)$ are conjugate in H, hence each H_x is transitive on $\Omega \setminus \{a_1, x\}$. Thus H (and therefore G_{a_1}) is 2-transitive on $\Omega \setminus \{a_1\}$.

Given distinct $x, y \in \Omega \setminus \{a_1\}$, choose $u \in H$ with $u(ca_2) = x$ and then $v \in H_x$ with $v(u(bc_2)) = y$. Setting $h := vu \in H \leq G_{a_1}$ we obtain

$$h \, \tau \, h^{-1} = (a_1 \, x \, y),$$

as required. □

Lemma 4.6 (Generating A_n from 3-cycles sharing a point)**.** *For $n \geq 5$ the subgroup of A_n generated by the set $\{(1 \, i \, j) : 2 \leq i < j \leq n\}$ is A_n.*

Proof. Let H be the subgroup generated by all $(1 \, i \, j)$ with $2 \leq i < j \leq n$. Using right-to-left composition one checks

$$(1 \, i \, j) \, (1 \, j \, k) = (i \, j \, k).$$

Hence, from cycles of the form $(1 \, i \, j)$ we obtain *all* 3-cycles $(i \, j \, k)$ with $2 \leq i, j, k \leq n$. But A_n is generated by 3-cycles, so $H = \mathrm{A}_n$. □

Proposition 4.7. *The subgroup G contains A_{12}.*

Proof. By Lemma 4.5, G contains every 3-cycle of the form $(a_1 \ x \ y)$ with x, y distinct in $\Omega \setminus \{a_1\}$. Relabelling a_1 as 1 and the other points as $2, \ldots, 12$, Lemma 4.6 implies that these 3-cycles generate A_{12}. Thus $\mathrm{A}_{12} \leq G$. □

Theorem 4.8 (Main theorem, constructive proof)**.** *We have $G = \mathrm{S}_{12}$.*

Proof. By Proposition 4.7, $\mathrm{A}_{12} \leq G$. Each generator g_A, g_B, g_C is a 6-cycle and hence an *odd* permutation; therefore G is not contained in A_{12}. It follows that $G = \mathrm{S}_{12}$. □

5 Remarks and small variations

Remark 5.1 (Computational check)**.** The commutator identities in Lemma 2.2 can be verified by hand as above, or quickly by a computer algebra system modeling the three 6-cycles from Definition 1.2.

Remark 5.2 (Why Jordan is overkill here)**.** Jordan's theorem streamlines the argument once primitivity and the presence of a 3-cycle are known. In this special puzzle one can avoid it entirely by the explicit 2-transitivity of G (Lemma 4.2) and the elementary generation Lemma 4.6. This proves, in effect, the case "$p = 3$" of Jordan's theorem for this concrete configuration.

Remark 5.3 (Generalizations)**.** The same strategy applies to other ring-intersection puzzles whenever one can (i) exhibit enough elements in a point stabilizer to make it transitive on the complement, and (ii) produce a single 3-cycle. The parity argument then upgrades A to S.

Acknowledgements

The labelling convention in Definition 1.1 follows the natural geometry of three rings A, B, C with pairwise two-point intersections. The figure is schematic and not to scale; only incidences and labels matter for the proofs.