

Incompleteness theorem for physics

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Abstract

We show how Gödel's incompleteness theorems have an analog in quantum theory. Gödel's theorems imply endless opportunities for appending axioms to arithmetic, implicitly showing a role for an entity that writes axioms as logically undetermined strings of symbols. There is an analog of these theorems in physics, to do with the set of explanations of given evidence. We prove that the set of explanations of given evidence is uncountably infinite, thereby showing how contact between theory and experiment depends on activity beyond computation and measurement—a physical activity of logically undetermined symbol handling.

Keywords: evidence vs. explanation, strings of symbols, uncountable explanations, logical incompleteness, guess, symbol-handling agent

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1. Introduction

People writing and reading symbols act as symbol handlers, for example in making use of numerals and symbols of an alphabet. In mathematical logic, symbol handling has attracted serious attention, notably with Gödel's incompleteness theorems. By analyzing the system of symbols involved in the most elementary branch of mathematics, the arithmetic of natural numbers $0, 1, 2, \dots$, Gödel proved that there are always questions in mathematical logic, the answers to which are logically undetermined, calling for additional axioms which in turn generate more such questions [1]. Then came Turing's formulation of the concept of computability in terms of symbol handling, leading to his demonstration of the prevalence of functions of natural numbers that are *uncomputable*. Gödel and Turing's attention to the manipulation of symbols in mathematics opened a field of inquiry into logically undetermined aspects of mathematics, with implications for other fields of endeavor that make use of mathematics, including physics.

In physics, strings of symbols express both evidence and the formulas by which that evidence is explained, formulas that lead to predictions. Recently came the recognition that the expression in symbols has interesting implications for physics [2, 3]. In earlier work, we proved that quantum-theoretic explanations of given evidence are logically undetermined by the evidence, leading to the necessity of guesswork in arriving at an explanation. In the next section we prove a stronger proposition: the set of explanations that fit given evidence is uncountably infinite. Implications of the proposition follow. As discussed in Sec. 3, a corollary to the proof implies an endless open cycle in which symbol-handling agents

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guess explanations and test their implications. Sec. 4 discusses the notion of a symbol handler as exercising capabilities to compute and to guess. To distinguish between the ultimate reach of computation and other capabilities, we introduce a theoretical symbol handler that has the full computational capacity of a Turing machine. In the context of a Turing machine, as discussed in Sec. 5, the uncountable set of explanations has strong implications in relation to what can and cannot be computed. Sec. 6 views agency as pervading the material world. In Sec. 7 we make a few remarks toward future directions for investigation.

2. Uncountable set of explanations of given evidence

As represented in quantum theory, an experiment involves many trials, with an outcome occurring for each trial. Eschewing saying what outcome will occur at a particular trial, quantum theory speaks only of probabilities of outcomes. In explaining probabilities of outcomes, one views a trial as consisting of the preparation of a state, e.g. expressed by a density operator ρ , and a measurement, expressed by a Positive-Operator-Valued Measure (POVM), here denoted M . For some outcome space Ω , with $\tilde{\Omega}$ a σ -algebra of subsets of Ω , one then has for each outcome $\omega \in \tilde{\Omega}$ a non-negative operator $M(\omega)$, the measurement operator for outcome ω . Both the density operator and the measurement operator are operators on some (finite- or infinite-dimensional) Hilbert space \mathcal{H} . By the Born trace rule, the explanation (ρ, M) implies a probability for the outcome ω :

$$\Pr(\omega) = \text{tr}[\rho M(\omega)]. \quad (1)$$

The whole outcome space Ω is the union of all outcomes. By definition, Ω is an element of the σ -algebra $\tilde{\Omega}$, and $M(\Omega) = \mathbf{1}$, the identity operator on \mathcal{H} .

Here is the issue. Books on quantum mechanics teach us to calculate probabilities from given states and measurement operators, using the Born trace rule; however, experiments with unexpected results present the “inverse situation” (as in quantum decision theory). One is given probabilities abstracted from evidence on the workbench, and one seeks “blackboard” explanations in terms of states and measurement operators. In practice, an investigator’s education and habits of thought may limit the choice of explanations, but logic alone leaves open a vast region of explanations that exactly fit the given probabilities.

Theorem: The set of inequivalent explanations that exactly fit given probabilities is uncountably infinite.

Proof: The idea behind the proof is to exploit tensor products of Hilbert spaces. For example, an experiment with two detectors can be explained using measurement operators that are tensor products, one factor of the tensor product for each detector. Ignoring outcomes of one detector coarsens the experiment; one explains the remaining evidence by setting the measurement operator for the ignored factor to the identity operator. In this way a explanation of evidence is condensed into a simpler explanation of a condensation of the evidence. This process can be reversed: any explanation of evidence from an experiment can be seen as a condensation of any of a multitude of possible extensions of the experiment. Such an extension entails augmenting the Hilbert space $\mathcal{H}^{(0)}$ by another Hilbert-space factor $\mathcal{H}^{(1)}$ to get $\mathcal{H}^{(0)} \otimes \mathcal{H}^{(1)}$.

The proof proceeds in two steps. In the first step, we prove that for any explanation there are always two more inequivalent explanations, as follows. Suppose that $\tilde{\Omega}^{(0)}$ is a σ -algebra of outcomes for an explanation of an experiment, and that probabilities $\Pr(\omega^{(0)})$ are abstracted from results for each outcome $\omega^{(0)} \in \tilde{\Omega}^{(0)}$. Suppose further that one is given an explanation $(\rho^{(0)}, M^{(0)})$ with the operators on a Hilbert space $\mathcal{H}^{(0)}$ that exactly imply the given probabilities. Then there are always two more inequivalent explanations that give the same probabilities. Both of these explanations involve a tensor product Hilbert space of $\mathcal{H}^{(0)}$ with a second factor $\mathcal{H}^{(1)}$. The explanations for outcome $\omega^{(0)}$ are $(\rho^{(0)} \otimes \rho_1^{(1)}, M^{(0)}(\omega^{(0)}) \otimes M^{(1)}(\Omega^{(1)}))$ and $(\rho^{(0)} \otimes \rho_2^{(1)}, M^{(0)}(\omega^{(0)}) \otimes M^{(1)}(\Omega^{(1)}))$, where $\rho_1^{(1)}$ and $\rho_2^{(1)}$ are distinct density operators on $\mathcal{H}^{(1)}$. Because of the equality $M^{(1)}(\Omega^{(1)}) = \mathbf{1}^{(1)}$, these explanations are blind to the second factor of the density operator, which is why they give the same probabilities as does the given explanation. But the two explanations are inequivalent, because they extend to an expanded experiment which provides for attending to two or more distinct outcomes in $\tilde{\Omega}^{(1)}$, for which the two extended explanations imply different probabilities.

Now for the second step. As a mathematical construct, the augmentation by one more tensor-product factor can be repeated without end, resulting in a set of mutually inequivalent explanations, one for each of the infinite sequences j_1, j_2, \dots :

$$\left(\rho^{(0)} \otimes \bigotimes_{n=1}^{\infty} \rho_{j_n}^{(n)}, M^{(0)}(\omega^{(0)}) \otimes \bigotimes_{n=1}^{\infty} M^{(n)}(\Omega^{(n)}) \right) \quad (2)$$

The explanations with the infinite sequence of factors differ independently for each factor. Thus the set of explanations has the cardinality of the set of infinite binary fractions, a set that by the Cantor diagonal argument is uncountable, mean-

ing it cannot be mapped 1-to-1 to the set of natural numbers, or, less formally, that it cannot be listed even on an infinitely extendable Turing tape. Expression (2) displays an uncountable subset of the set of inequivalent explanations that match a given evidence. It follows that the whole set of inequivalent explanations must also be uncountable. Q.E.D.

Corollary: There is no logical ground to exclude any of the uncountable set of potential explanations of given evidence prior to additional evidence not yet on hand.

Remarks:

1. The multiplicity of explanations has nothing to do with imperfections in the fit between evidence and its explanation; as proved above, it holds even in the ideal case in which one demands an exact fit. Requiring only an approximate fit, as is common practice, makes room for even more explanations of any given evidence.
2. Quantum-state tomography claims to determine a quantum state from evidence. To this claim we respond that quantum state tomography assumes that the measurement operators are known, and, by the Theorem, this knowledge cannot be obtained from evidence alone. In addition, quantum state tomography assumes some finite dimension of the Hilbert space, a dimension underivable from evidence.
3. It can also be noted that evidence is expressed by probabilities that are functions of parameters; we think of parameter values as settings of knobs, such as a knob by which to vary a magnetic field strength or a knob to vary the angle of a polarizer. Then not only the evidence but also the density operator and the POVM depend on knob settings. The proof of the Theorem goes through the same way for each knob setting[2, 4].
4. For the physicist struggling to come up with *some* explanation of unexpected evidence, worrying about the possibility of other explanations may seem superfluous; yet being aware that something outside of logic is required to make an explanation liberates one from futile efforts to derive an explanation by logic alone.

After introducing Turing machines, we will discuss implications of the uncountability of the set of explanations of given evidence in more detail, but here is a first hint. An impediment to assigning a suitable probability measure to the set arises, because the set of explanations is both uncountable and devoid of a natural

metric. The real numbers are uncountable but come with a metric, and the metric allows for probability measures that in effect assign a probability not to a particular real number but to an interval of real numbers. (Intervals expressing values of measurable variables are appropriate because two real numbers that are sufficiently close are experimentally indistinguishable.) A countably infinite set requires no metric in order for non-zero probabilities to be assigned to its elements, because its elements can be ordered by natural numbers. For any $0 < r < 1$, an symbol-handling agent can assign a probability $(1 - r)r^n$ to the n -th element of the countable set. (As probabilities must, these sum to 1.)

In contrast to the metric structure of real numbers and the countability of natural numbers, explanations of given evidence are discrete and uncountable. They are like infinite sequences of distinct digits. If a digit in one sequence differs from a corresponding digit in another sequence, the sequences are distinct, regardless of how far along the sequence the difference appears. Because of its discrete topology on an uncountable set, we have the following.

Proposition 1: Any probability measure on the set of explanations of given evidence must assign zero probability to all but a countable subset of its elements.

By Prop. 1 and the Corollary to the Theorem, we have

Proposition 2: Neither an explanation nor a probability measure on explanations can be logically determined by the evidence explained.

3. Open cycle of guessing and testing

Out of a potential for uncountable inequivalent explanations, physicists write down particular explanations. By the Corollary, the writing of symbols that introduce an explanation of given evidence takes something beyond logic and evidence. This ‘something’ can reasonably be called a **guess**. The guess, neither derived mathematically on the blackboard nor generated from an experiment on the work bench, comes from “somewhere else.” Guessed explanations, expressed in mathematical symbols, feed into the development of experimental devices and into the design of future experiments. Different guessed explanations lead to different experimental designs, leading to different bodies of evidence that call for more explanations and hence more guesses. Any particular explanation is essentially certain to require revision when tested over enough of its extensions. We thus arrive at

Proposition 3: The regularities that physicists find in the material world defy any final expression and any final explanation.

With Prop. 3, we see that quantum theory implies an endless evolution of physics as an activity of symbol-handling agents “guessing and testing” in an open cycle with no possibility of completion.

Attending to symbol handling clarifies a distinction between an occurrence of an event of low probability and what we term a *surprise*. Via the Born trace rule, an explanation assigns probabilities to a (possibly infinite) list of possible outcomes. One can think of this list of possibilities as a string of symbols recorded in some agent’s computer file. In this way we link the concept of *possible* to the contents of an agent’s memory. In contrast, our dictionary defines *possible* as: able to happen although not certain to ... [5], without reference to an agent or to any memory. By referencing possibilities to an agent’s list we define a *surprise* as a reaction of an agent to an event not on the list of possibilities—the occurrence of “an unknown unknown.” Historically, (as in the disaster at the Three-Mile Island nuclear plant), people experience outcomes not on their lists of possibilities, and they react to such outcomes, thereby experiencing surprise. (Experiencing surprise is distinct from facing measurement uncertainty. *Uncertainty*, denotes a spread in a probability distribution ensuing from some measurement model [6], involving a list of possible outcomes, and not the introduction of a possibility not in the list. Symbol-handling agents operating in the open cycle of guessing and testing encounter surprises that call for revising their guesses and the explanations that ensue from them; their assumptions evolve. In 2005 we wrote of “the tree of assumptions”[2]:

Some guesses get tested (one speaks of *hypotheses*), but testing a guess requires other guesses not tested. By way of example, to guide the choice of a density operator by which to model the light emitted by a laser, one sets up the laser, filters, and a detector on a bench to produce experimental outcomes. But to arrive at any but the coarsest properties of a density operator one needs, in addition to these outcomes, a model of the detector, and concerning this model, there must always be room for doubt; we can try to characterize the detector better, but for that we have to assume a model for one or more sources of light. When we link bench and blackboard, we work in the high branches of a tree of assumptions, holding on by metaphors, where we can let go of one assumption only by taking hold of others.

4. Agency in physics

With the recognition of an open cycle of guessing and testing, the blackboard of theory and the workbench of experiment can no longer encompass all of physics: there is a gap between them. In this gap, something else comes into play, in order for symbol-handling agents (people or perhaps other organisms) to link evidence to particular explanations. How, then, are we to think about a physical world now recognized as inseparable from symbol-handling agents in interaction with other parts of this world?

Based on the recognition that we ourselves as physicists act as symbol-handling agents, and that we are organisms alive in the physical world, we wonder what other aspects of behavior, especially biological behavior, might fruitfully be explained by invoking symbol-handling agents as elements of description. *Symbols* and *symbol-handling agents* are terms of description available at widely varying levels of detail. I may see myself as handling symbols, and I may inquire into evidence of symbol handling on the part of mitochondria within my cells. We propose that symbol-handling agents enter physical explanations.

Mainstream physics, emphasizing particles and fields, has no explicit vocabulary for symbol handlers or their symbols. Recently, however, a crack opened in physics for discussions of agents and symbols, for example in the work of Fuchs and Schack [7, 8] and Briegel [9]. In these and other current examples, *agency* and *agent* name a variety of notions with a complex history, partly in opposition to obsolete notions of objectivity that traces back at least to Descartes [10]. Some biologically-oriented notions of agents are introduced in biosemiotics [11] and in code biology [12, 13, 14].

What capabilities are to be ascribed to a ‘symbol handling agent’? We think of such an agent as taking steps, one after another, and as equipped with a memory. Each “next step” of an agent is influenced both by the contents of its memory and by an inflow of symbols from an environment that includes other agents, and also by a logically undetermined “oracle” external to the agent [15]. Guesses come from an agent interacting with an oracle, the workings of which we refrain from trying to penetrate. How to elaborate this proposal remains open; presumably investigators will conceive of a variety of expressions of ‘symbol-handlers for applications varying from descriptions of viruses to descriptions of human mentality.

In order to distinguish between what can be computed and what must come from beyond computation (as guesses from interaction with an oracle), we imagine the extreme case of a symbol handler that, while open to guessing, possesses maximal computational capacity. Thus we are unconcerned with practical lim-

its on computing imposed by limits on memory or by limits on the rate at which a symbol handler computes, leading us to assume that the symbol handler has the ultimate computational capability of a Turing machine. The Turing machine, however, requires modification to offer a place for guesses from interaction with an oracle and for communication with other such machines.

Fortunately for our purposes, in a side remark to his 1936 paper, Turing briefly introduced an alternative machine called a *choice machine*, contrasted with the usual Turing machine that Turing called an a-machine:

If at each stage the motion of a machine . . . is completely determined by the [memory] configuration, we shall call the machine an “automatic machine” (or a-machine). For some purposes we might use machines (choice machines or c-machines) whose motion is only partially determined by the configuration When such a machine reaches one of these ambiguous configurations, it cannot go on until some arbitrary choice has been made by an external operator. This would be the case if we were using machines to deal with axiomatic systems. [16].

We picture a symbol handler equipped with a c-machine modified to take part in a communications network by transmitting symbols to other such machines. We call the modified c-machine a Choice Machine. We posit that on occasion an “oracle” writes a symbol onto the scanned square of the Turing tape of the symbol-handler’s Choice Machine *privately*, in the sense that the symbol remains unknown to other symbol handlers unless and until the symbol-handling agent that receives the chosen symbol reports it to others [17]. There is no limit to the number of symbols that the oracle can write, and we make no rule of separating data from program, so that what comes from the oracle can influence both data and programs.

5. Appreciating uncountability

While we previously linked agency to multiple explanations [3], now we have the proof that the set of explanations of given evidence is *uncountably* infinite. The indefinitely extendable symbol-holding tape of the Choice Machine gives an interesting way to appreciate this uncountable set, as follows. For a finite set, one can write down numerical names for the elements, one after another: element 1, element 2, etc. For countably infinite sets one cannot quite do this, but one imagines a correspondence of names of the elements of the set with the natural

numbers. One thinks “as if” names of the elements could all be written one after another on a Turing tape, without ever running out of symbol-holding squares of the tape. A Turing machine, programmed to get to a the name of a particular element, will always reach it.

By definition, an *uncountable set* cannot have the names of its elements written on even an unlimited tape, so that even in the theoretical limit of unlimited memory and unlimited computational speed, an agent’s Choice Machine cannot be relied on to eventually arrive at every explanation that exhibits some property of interest. To arrive an explanation or even a subset of explanations requires a reach beyond what can be computed by any variety of Turing machine. Because the set of potential explanations is uncountable, when you guess an explanation, you do more than pick from a list: something is created.

Remarks:

1. Gödel’s first incompleteness theorem implies that arithmetic generates an infinite list of questions, each answerable by an agent putting forth an answer that inserts one of two axioms, extending arithmetic in an unending structure of binary choices. Thus by the diagonal argument, the set of extended arithmetics is uncountable. Gödel used a fixed countable system of symbols to make his proof from which it follows that the set of extended arithmetical systems is uncountable. Similarly, we use a fixed countable system to prove that the set of explanations of given evidence is uncountable.
2. By definition, an axiom is logically undetermined and is thus a species of a guess made by some agent. Recalling the metaphor of the ‘tree of assumptions’, the axioms that settle Gödel’s logically undecidable questions represent uncountable branches (perhaps better said, branchlets) of that tree.

6. Is agency pervasive?

A major major fork in the tree of assumptions can now be seen. Physicists exhibit agency, and if physicists are part of the material world then some physical material is under the influence of agency. We now ask: is all physical matter influenced by agency? Are the stars made out of only dead matter, which is to say, are they uninfluenced by agency, or not? If the stars are shaped partly by agency, that is, by acts of agents, agents that are by definition capable of uncomputable behavior, a new area for investigation opens up. Habit in physics weighs against this view, but that habit depends on the notion that although “now we see through

a glass darkly,” there is some ultimate end picture toward which we approach. In contrast, the implication of the proof of uncountable explanations of any given evidence, along with the recognition of symbol-handling as part of the material world, decides for the other major branch: a physical world pervaded by agency.

In a world pervaded by agency, the big-bang theory still serves to explain a body of cosmological evidence; we do not criticize that explanation, but we recognize that there is potentially an uncountable set of other explanations that also explain the evidence. One no longer can hope to relate the world of physical experience to a single system of logic, comparable to what Gödel used to prove the incompleteness theorems, but rather the world of physical experience is to be seen as something for which explanations, like extended arithmetics, form an uncountable set.

7. Discussion

Proven above is the Theorem that all explanations are necessarily open to testing and to needs for revision, so that any path of inquiry remains unclosely open; there can be no final answers compatible with quantum mechanics. How is one to respond? One is left with a question of faith. Is the conflict of quantum theory with final answers something to respond to with despair or with joy? That choice no science of which we know can resolve: it is up to the person.

For a next step along our path of inquiry, we plan to follow up on the expression of evidence and its explanations by flows of strings of symbols in cycles of guessing and testing. It might be supposed symbols flow within some fixed spacetime. On the contrary, however, we are interested in the establishment, maintenance, and dissolution of flows symbols among agents as enabling an uncountable set of other systems of spatio-temporal management, differing from the spacetimes of special or general relativity, systems of spatio-temporal management that accommodate to the circumstances in which the agents operate. For example, animal behavior, as in E. Coli [18], frog vision [19], and depth vision of praying mantis [20] come to mind as places to start to look for interesting systems of spatio-temporal management.

Readers of an earlier draft offered several comments and criticisms deserving of the following responses.

7.1. Non-triviality of tensor products

Is a density operator that is a tensor product trivial? One can always write a density operator for each of two unrelated experiments as a tensor product with

one factor for each experiment, and it is fair to call such a tensor product ‘trivial’; however, products of density operators enter explanations non-trivially, for example when two parameters of a single experiment enter distinct factors of a tensor product of density operators. In that case a choice of a parameter can have an effect that is unexpressed in a simpler explanation, but has a dramatic effect when a tensor product is invoked in a more encompassing explanation.

For example, this situation occurs in the BB84 protocol for Quantum Key Distribution (QKD), which relies on quantum uncertainty to claim security against undetected eavesdropping. In BB84[21], Alice transmits a sequence of laser-generated, single-photon light pulses to Bob, from which Bob and Alice extract a key. For each pulse, Alice “sets a knob” to choose randomly among four types of light pulses, each characterized by a polarization vector and a frequency spectrum. If the spectra are identical, the experiment can be explained by a density operator on the two-dimensional vector space of polarizations, as assumed in much of the QKD literature. In a two dimensional space, there must be substantial overlap among the four polarizations, forcing substantial uncertainty on any eavesdropper’s measurements. The claim of QKD security depends on this uncertainty. However, a physically more informed explanation accounts for light frequency, for which it introduces another factor, of infinite dimension, so that the density operator becomes a tensor product of a density operator expressing polarization with a density operator expressing the frequency spectrum of the light pulse. By “setting a knob”, Alice determines both a polarization angle *and a frequency*.

The simpler explanation, employing the density operator on the 2-dimensional polarization space, gives an adequate description of eavesdropping under the condition that the eavesdropper makes no use of frequency discrimination. But the tensor-product explanation is critical to showing what a better equipped eavesdropper can learn. In some popular implementations, each type of light pulse comes from a different laser. It is readily shown that if the lasers are imperfectly aligned in frequency, the overlaps among the four light states approach zero, so that the QKD system is seen have no security at all[22].

7.2. Entanglement

Here is an example of two explanations of given evidence, the first of which involves no entanglement, while the second involves an entangled state. Consider a simple case of evidence with two outcomes: $\omega_1^{(0)}$ and $\omega_2^{(0)}$, and with probabilities given, for some $0 \leq a \leq 1$, by

$$\Pr(\omega_1^{(0)}) = a, \text{ and } \Pr(\omega_2^{(0)}) = 1 - a. \quad (3)$$

These probabilities accord with an explanation α involving a Hilbert space of just two real dimensions, $\mathcal{H}^{(0)} = \mathbb{R}^2$, with a basis $\{|x^{(0)}\rangle, |y^{(0)}\rangle\}$, so that

$$|x^{(0)}\rangle\langle x^{(0)}| + |y^{(0)}\rangle\langle y^{(0)}| = \mathbf{1}^{(0)}, \quad (4)$$

with

$$\rho_\alpha = (\sqrt{a}|x^{(0)}\rangle + \sqrt{1-a}|y^{(0)}\rangle)(\sqrt{a}\langle x^{(0)}| + \sqrt{1-a}\langle y^{(0)}|); \quad (5)$$

$$M_\alpha^{(0)}(\omega_1^{(0)}) = |x^{(0)}\rangle\langle x^{(0)}| \text{ and} \quad (6)$$

$$M_\alpha^{(0)}(\omega_2^{(0)}) = |y^{(0)}\rangle\langle y^{(0)}| \quad (7)$$

An alternate explanation β of the probabilities given in (3) asserts an entangled state. The explanation β involves an additional tensor-product space $\mathcal{H}^{(1)} = \mathbb{R}^2$, again a real vector space of dimension 2, with basis vectors $\{|x^{(1)}\rangle, |y^{(1)}\rangle\}$:

$$\begin{aligned} \rho_\beta &= (\sqrt{a}|x^{(0)}\rangle|x^{(1)}\rangle + \sqrt{1-a}|y^{(0)}\rangle|y^{(1)}\rangle) \\ &\quad (\sqrt{a}\langle x^{(0)}|\langle x^{(1)}| + \sqrt{1-a}\langle y^{(0)}|\langle y^{(1)}|) \end{aligned} \quad (8)$$

$$M_\beta = M_\alpha^{(0)} \otimes \mathbf{1}^{(1)} \quad (9)$$

7.3. Concept of an explanation involving an infinite string of symbols

Can it make sense to admit explanations expressed as (countably) infinite strings of symbols? One can view (2) as a mapping from infinite binary sequences to explanations. Looked at that way, an explanation is then a particular infinite bit sequence together with the mapping. (The mapping is expressed by a finite number of symbols (it fits on the page!), but the bit sequence is (countably) infinite.) What leads us to propose such a form as an explanation is the conceptual separation between theory on the blackboard and experimental activity on the workbench: we no longer see practical constraints on the workbench as constraining the forms admissible for theory. A precedent for the lack of constraint is the Turing machine with its infinite tape.

The separation between explanations on the blackboard and evidence on the workbench established by the Theorem is already provided by a weaker theorem in [2] which implies a countable (not an uncountable) set of explanations of given evidence. This multiplicity shows that wave functions and linear operators cannot be found on the workbench of evidence, but instead are imaginative entities to be seen on the blackboard of mathematics. What to make of this separation of explanations from anything that can be seen on the workbench? From a standpoint that recognizes agency, explanations on the blackboard appear as strings of symbols

written by agents, and are acceptable entities toward which to devote theoretical attention. So, if one wants to build a mathematical theory of explanations, what should be the building blocks?

To choose the “building blocks” for such a theory, it may help to notice the use of number systems in physics. In that use one tolerates a separation between the mathematical properties by which number systems are defined and experience with physical acts of computation. The limited memory of any actual computer puts a limit on the range of integers that the computer can deal with one-to-one, so the computer cannot represent the whole set of integers. Instead it represents a subset of integers, and appends to this subset some way of dealing with underflow and overflow. Besides the well-known issue of round-off errors in floating-point representations of the integers, one gets a second effect, more interesting for present purposes. For arithmetic operations on some integers x , $(x + x) - x$ overflows, while $x + (x - x)$ does not overflow, but gives x ; the associative law of addition suffers exceptions. (Try it in Matlab for $x = 10^{308}$). When dealing with integers near the limits of the capacity of the computer, a programmer has to deal with this and other exceptions to the “laws of arithmetic.” Should physicists use those laws or should they instead use the much more complicated behavior of expressions of number as actually implementable in computer hardware on the workbench? For physics, the answer is driven by the drive for elegance and simplicity: “An important task of the theoretical physicist lies in distinguishing between trivial and nontrivial discrepancies between theory and experiment.”[23, p3] The mathematical idea of integers—that every integer has a successor and a predecessor—so charms the mind that the gap between the mathematically defined integers and their limited realization in computers is no fatal stroke against their use in the constructing of theory.

Now back to building blocks for a theory of explanations. We start with a theoretical abstraction, parallel to the that of the integers: in explaining given evidence, one can always take one more factor into account. Do we accept this abstraction or not? Were we to hew too closely to what can be realized, we expect to eventually bog down if we keep trying to add factors realizable on the workbench; however, to account for this limitation of the workbench, we must deal with the vexing question of “what is the maximum number of factors?”, which depends on the technology on hand and, and so evolves with that technology. For theoretical purposes we ignore such a limit, which, once done, opens the way to theoretical portrayals of explanations as in (2).

7.4. Testing extended explanations

A question was raised about testing extensions of explanations of the form (2). Before they are extended, those explanations (2) are no harder to test than is the explanation (1), for the simple reason that all these explanations, with the extra factors of the POVM set to unit operators ($M^{(n)}(\Omega^{(n)}) = \mathbf{1}^{(n)}$), assert exactly the same probabilities. The extended explanations obtained by attending to finer outcomes $\omega^{(n)} \subset \Omega^{(n)}$ for some values of n are testable in principle, to the extent of selecting any particular factor, say the m -th. To test the extended explanation with respect to the m -th factor, all the factors $n \neq m$ are ignored, as expressed by unit POVM factors, so that (2) reduces to

$$\left(\rho^{(0)} \otimes \rho_{j_m}^{(m)}, M^{(0)}(\omega^{(0)}) \otimes M^{(m)}(\omega^{(m)}) \right), \quad (10)$$

The trace of (10) is the marginal probability obtained by ignoring outcomes of all sectors except sector 0 and sector m , and the distinctions in probability do not become small as m increases.

7.5. Restriction to finite sets

Some readers may still be uncomfortable with explanations involving infinite tensor products, so we address what remains of our argument if only finite sets are admitted. Suppose that we allow only some finite number N of tensor-product factors, and, correspondingly, only N extensions of arithmetic. Also, we truncate the Turing tape to make a computer of finite memory capacity. Then there are 2^N extended arithmetics. Similarly, the number of alternative explanations is $\geq 2^N$. Thus, recording the names of all the alternative explanations or of all the extended arithmetics places a demand on computer memory that increases exponentially with N . The parallel between Gödel incompleteness and explanations is still present in the exponential behaviors, but, with the restriction to finite sets, the parallel is expressed by *complexity*, rather than by the (to us) mathematically more appealing formulation in terms of computability.

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