The category of opetopes and the category of opetopic sets

Eugenia Cheng

Department of Pure Mathematics, University of Cambridge
E-mail: e.cheng@dpmms.cam.ac.uk

October 2002

Abstract

We give an explicit construction of the category **opetope** of opetopes. We prove that the category of opetopic sets is equivalent to the category of presheaves over **opetope**

Contents

Introduction

In [3], Baez and Dolan give a definition of weak n-category in which the underlying shapes of cells are 'opetopes' and the underlying data is given by 'opetopic sets'. The idea is that opetopic sets should be presheaves over the category of opetopes. However Baez and Dolan do not explicitly construct the category of opetopes, so opetopic sets are defined directly

1

instead. A relationship between this category of opetopic sets and a category of presheaves is alluded to but not proved.

The main result of this paper is that the category of opetopic sets is equivalent to the category of presheaves over the category of opetopes. However, we do not use the opetopic definitions exactly as given in [3] but continue to use the modifications given in our earlier work ([6], [5]). In these papers we use a generalisation along lines which the original authors began, but chose to abandon for reasons which are unclear. This generalisation enables us, in [6], to exhibit a relationship with the work of Hermida, Makkai and Power ([7]) and, in [5], with the work of Leinster ([13]). Given these useful results, we continue to study the modified theory in this work.

We begin, in Section 1 by giving an explicit construction of the category of opetopes. The idea is as follows. In [6] we constructed, for each $k \geq 0$, a category $\mathbb{C}_k$ of k-opetopes. For the category **Opetope** of opetopes of all dimensions, each category $\mathbb{C}_k$ should be a full subcategory of **Opetope**; furthermore there should be 'face maps' exhibiting the constituent m-opetopes, or 'faces' of a k-opetope, for $m \leq k$. We refer to the m-opetope faces as m-faces. Note that there are no degeneracy maps.

The $(k-1)$-faces of a k-opetope α should be the $(k-1)$-opetopes of its source and target; these should all be distinct. Then each of these faces has its own $(k-2)$-faces, but all these $(k-2)$-opetopes should not necessarily be considered as distinct $(k-2)$-faces in α. For α is a configuration for composing its $(k-1)$-faces at their $(k-2)$-faces, so the $(k-2)$-faces should be identified with one another at places where composition is to occur. That is, the composite face maps from these $(k-2)$-opetopes to α should therefore be equal. Some further details are then required to deal with isomorphic copies of opetopes.

Recall that a 'configuration' for composing $(k-1)$-opetopes is expressed as a tree (see [6]) whose nodes are labelled by the $(k-1)$-opetopes in question, with the edges giving their inputs and outputs. So composition occurs along each edge of the tree, via an object-morphism label, and thus the tree tells us which $(k-1)$-opetopes are identified.

In order to express this more precisely, we first give a more formal description of trees (Section 1.2). In fact, this leads to an abstract description of trees as certain Kelly-Mac Lane graphs; this is the subject of [4]. The results of Section 1.2 thus arise as preliminary results in [4] and we refer the reader to this paper for the full account and proofs.

In Section 2, we examine the theory of opetopic sets. We begin by following through our modifications to the opetopic theory to include the theory of opetopic sets. (Our previous work has only dealt with the theory of opetopes.) We then use results of [12] to prove that the category of opetopic sets is indeed equivalent to the category of presheaves on $\mathcal{O}$, the category of opetopes defined in Section 1. This is the main result of this work.

Finally, a comment is due on the notion of 'multitope' as defined in [7].

In this work, Hermida, Makkai and Power begin a definition of n-category explicitly analogous to that of [3], the analogous concepts being 'multitopes' and 'multitopic sets'. In [6] we prove that 'opetopes and multitopes are the same up to isomorphism', that is, for each $k \geq 0$ the category of k-opetopes is equivalent to the (discrete) category of k-multitopes. In [7], Hermida, Makkai and Power do go on to give an explicit definition of the analogous category **Multitope**, of multitopes. Given the above equivialences, and assuming the underlying idea is the same, this would be equivalent to the category **Opetope**, but we do not attempt to prove it in this work.

Terminology

i) Since we are concerned chiefly with *weak n-categories*, we follow Baez and Dolan ([3]) and omit the word 'weak' unless emphasis is required; we refer to *strict n-categories* as 'strict n-categories'.

ii) We use the term 'weak n-functor' for an n-functor where functoriality holds up to coherent isomorphisms, and 'lax' functor when the constraints are not necessarily invertible.

iii) In [3] Baez and Dolan use the terms 'operad' and 'types' where we use 'multicategory' and 'objects'; the latter terminology is more consistent with Leinster's use of 'operad' to describe a multicategory whose 'objects-object' is 1.

iv) In [7] Hermida, Makkai and Power use the term 'multitope' for the objects constructed in analogy with the 'opetopes' of [3]. This is intended to reflect the fact that opetopes are constructed using operads but multitopes using multicategories, a distinction that we have removed by using the term 'multicategory' in both cases. However, we continue to use the term 'opetope' and furthermore, use it in general to refer to the analogous objects constructed in each of the three theories. Note also that Leinster uses the term 'opetope' to describe objects which are analogous but not *a priori* the same; we refer to these as 'Leinster opetopes' if clarification is needed.

v) We regard sets as sets or discrete categories with no notational distinction.

1 The category of opetopes

In this section we give an explicit construction of the category **Opetope** of opetopes. This construction will enable us, in Section 2, to prove that the category of opetopic sets is in fact a presheaf category.

We begin with a brief account of the trees used to construct higher-dimensional opetopes from lower-dimensional ones; we refer the reader to [4] for the full account, with proofs and examples.

1.1 Informal description of trees

Recall the trees introduced in [6] to describe the morphisms of a slice multicategory. These are 'labelled combed trees' with ordered nodes. In fact, we will first consider the *unlabelled* version of such trees, since the labelled version follows easily. For example the following is a tree:
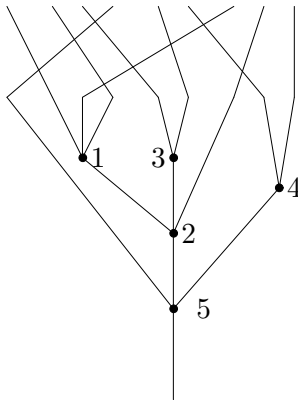
Explicitly, a tree $T = (T, \rho, \tau)$ consists of

i) A planar tree T

ii) A permutation $\rho \in \mathbf{S}_l$ where l = number of leaves of T

iii) A bijection $\tau : \{\text{nodes of } T\} \longrightarrow \{1, 2, \ldots, k\}$ where k = number of nodes of T; equivalently an ordering on the nodes of T.

Note that there is a 'null tree' with no nodes

.

1.2 Formal description of trees

In this section we give a formal description of the above trees, characterising them as connected graphs with no closed loops (in the conventional sense of 'graph'). This will enable us, in Section 1.4, to determine which faces of faces are identified in an opetope.

Note that the material in this section is presented fully in [4]. It enables us to express a tree as a Kelly-Mac Lane graph; it also enables us to show that all *allowable* Kelly-Mac Lane graphs of the correct shape arise in this way.

We consider a tree with k nodes $N_1, \ldots, N_k$ where N_i has m_i inputs and one output. Let N be a node with $(\sum_i m_i) - k + 1$ inputs; N will be used to represent the leaves and root of the tree.

Then a tree is given by a bijection

$$\coprod_i \{\text{inputs of } N_i\} \coprod \{\text{output of } N\} \longrightarrow \coprod_i \{\text{output of } N_i\} \coprod \{\text{inputs of } N\}$$

since each input of a node is either connected to a unique output of another node, or it is a leaf, that is, input of N. Similarly each output of a node is either attached to an input of another node, or it is the root, that is, output of N.

We express this formally as follows.

Lemma 1.1 *Let T be a tree with nodes $N_1, \ldots, N_k$, where N_i has inputs $\{x_{i1}, \ldots, x_{im_i}\}$ and output x_i. Let N be a node with inputs $\{z_1, \ldots, z_l\}$ and output z, with*

$$l = (\sum_{i=1}^{k} m_i) - k + 1.$$

Then T is given by a bijection

$$\alpha \; : \; \coprod_i \{x_{i1}, \ldots, x_{im_i}\} \coprod \{z\} \longrightarrow \coprod_i \{x_i\} \coprod \{z_1, \ldots, z_l\}.$$

For the converse, every such bijection gives a graph, but it is not necessarily a tree. We need to ensure that the resulting graph has no closed loops; the use of the 'formal' node N then ensures connectedness. We express this formally as follows.

Lemma 1.2 *Let $N_1, \ldots, N_k, N$ be nodes where N_i has inputs $\{x_{i1}, \ldots, x_{im_i}\}$ and output x_i, and N has inputs $\{z_1, \ldots, z_l\}$ and output z, with $l = (\sum_{i=1}^{k} m_i) - k + 1$. Let α be a bijection*

$$\coprod_i \{x_{i1}, \ldots, x_{im_i}\} \coprod \{z\} \longrightarrow \coprod_i \{x_i\} \coprod \{z_1, \ldots, z_l\}.$$

Then α defines a graph with nodes $N_1, \ldots, N_k$.

5

Lemma 1.3 *Let α be a graph as above. Then α has a closed loop if and only if there is a non-empty sequence of indices*

$$\{t_1, \ldots, t_n\} \subseteq \{1, \ldots, k\}$$

such that for each $2 \leq j \leq n$

$$\alpha(x_{t_j b_j}) = x_{t_{j-1}}$$

for some $1 \leq b_j \leq m_j$, and

$$\alpha(x_{t_1 b_1}) = x_{t_n}$$

for some $1 \leq b_1 \leq m_1$.

Corollary 1.4 *A tree with nodes $N_1, \ldots, N_k$ is precisely a bijection α as in Lemma 1.2, such that there is no sequence of indices as in Lemma 1.3.*

1.3   Labelled trees

For the construction of opetopes we require the 'labelled' version of the trees presented in Section 1.1. A tree labelled in a category $\mathbb{C}$ is a tree as above, with each edge labelled by a morphism of $\mathbb{C}$ considered to be pointing 'down' towards the root.

Proposition 1.5 *Let $N_1, \ldots, N_k, N$ be nodes where N_i has inputs*

$$\{x_{i1}, \ldots, x_{im_i}\}$$

and output x_i, and N has inputs $\{z_1, \ldots, z_l\}$ and output z, with

$$l = (\sum_{i=1}^{k} m_i) - k + 1.$$

Then a labelled tree with these nodes is given by a bijection

$$\alpha : \coprod_i \{x_{i1}, \ldots, x_{im_i}\} \coprod \{z\} \longrightarrow \coprod_i \{x_i\} \coprod \{z_1, \ldots, z_l\}$$

satisfying the conditions as above, together with, for each

$$y \in \coprod_i \{x_{i1}, \ldots, x_{im_i}\} \coprod \{z\}$$

a morphism $f \in \mathbb{C}$ giving the label of the edge joining y and $\alpha(y)$. Then y is considered to be labelled by the object $\mathrm{cod}(f)$ and $\alpha(y)$ by the object $\mathrm{dom}(f)$.

Proof.   Follows immediately from Corollary 1.4 and the definition. $\square$

1.4   The category of opetopes

In our earlier work ([6]) we constructed for each $k \geq 0$ the category $\mathbb{C}_k$ of k-opetopes. We now construct a category **Opetope** of opetopes of all dimensions whose morphisms are, essentially, face maps. Each category $\mathbb{C}_k$ is to be a full subcategory of **Opetope**, and there are no morphisms from an opetope to one of lower dimension.

We construct the category **Opetope** $= \mathcal{O}$ as follows. Write $\mathcal{O}_k = \mathbb{C}_k$.

For the objects:

$$\text{ob } \mathcal{O} = \coprod_{k \geq 0} \mathcal{O}_k.$$

The morphisms of $\mathcal{O}$ are given by generators and relations as follows.

- Generators

1) For each morphism $f : \alpha \longrightarrow \beta \in \mathcal{O}_k$ there is a morphism

$$f : \alpha \longrightarrow \beta \in \mathcal{O}.$$

2) Let $k \geq 1$ and consider $\alpha \in \mathcal{O}_k = o(I^{k+}) = \text{elt}\,(I^{(k-1)+})$. Write $\alpha \in I^{(k-1)+}(x_1, \ldots, x_m; x)$. Then for each $1 \leq i \leq m$ there is a morphism

$$s_i : x_i \longrightarrow \alpha \in \mathcal{O}$$

and there is also a morphism

$$t : x \longrightarrow \alpha \in \mathcal{O}.$$

We write G_k for the set of all generating morphisms of this kind.

Before giving the relations on these morphisms we make the following observation about morphisms in $\mathcal{O}_k$. Consider

$$\begin{aligned} \alpha &\in I^{(k-1)+}(x_1, \ldots, x_m; x) \\ \beta &\in I^{(k-1)+}(y_1, \ldots, y_m; y) \end{aligned}$$

A morphism $\alpha \xrightarrow{g} \beta \in \mathcal{O}_k$ is given by a permutation σ and morphisms

$$\begin{aligned} x_i &\xrightarrow{f_i} y_{\sigma(i)} \\ x &\xrightarrow{f} y \quad \in \mathcal{O}_{k-1} \end{aligned}$$

So for each face map γ there is a unique 'restriction' of g to the specified face, giving a morphism γg of $(k-1)$-opetopes.

Note that, to specify a morphism in the category $\mathcal{FO}_{k-1}{}^{\text{op}} \times \mathcal{O}_{k-1}$ the morphisms f_i above should be in the direction $y_{\sigma(i)} \longrightarrow x_i$, but since these are all unique isomorphisms the direction does not matter; the convention above helps the notation. We now give the relations on the above generating morphisms.

7

- Relations

1) For any morphism
$$\alpha \xrightarrow{g} \beta \ \in \mathcal{O}_k$$

and face map
$$x_i \xrightarrow{s_i} \alpha$$

the following diagrams commute

$$
\begin{array}{ccc}
x_i & \xrightarrow{s_i} & \alpha \\
\downarrow{\scriptstyle s_i(g)} & & \downarrow{\scriptstyle g} \\
y_{\sigma(i)} & \xrightarrow{s_{\sigma(i)}} & \beta
\end{array}
\qquad
\begin{array}{ccc}
x & \xrightarrow{t} & \alpha \\
\downarrow{\scriptstyle t(g)} & & \downarrow{\scriptstyle g} \\
y & \xrightarrow{t} & \beta
\end{array}
$$

We write these generally as

$$
\begin{array}{ccc}
x & \xrightarrow{\gamma} & \alpha \\
\downarrow{\scriptstyle \gamma g} & & \downarrow{\scriptstyle g} \\
y & \xrightarrow{\gamma'} & \beta
\end{array}
$$

2) Faces are identified where composition occurs: consider $\theta \in \mathcal{O}_k$ where $k \geq 2$. Recall that θ is constructed as an arrow of a slice multicategory, so is given by a labelled tree, with nodes labelled by its $(k-1)$-faces, and edges labelled by object-morphisms, that is, morphisms of $\mathcal{O}_{k-2}$.

So by the formal description of trees (Section 1.2), θ is a certain bijection, and the elements that are in bijection with each other are the $(k-2)$-faces of the $(k-1)$-faces of θ; they are given by composable pairs of face maps of the second kind above. That is, the node labels are given by face maps $\alpha \xrightarrow{\gamma} \theta$ and then the inputs and outputs of those are given by pairs

$$x \xrightarrow{\gamma_1} \alpha \xrightarrow{\gamma_2} \theta$$

where $\gamma_2 \in G_k$ and $\gamma_1 \in G_{k-1}$. Now, if

$$
\begin{array}{llllll}
& x & \xrightarrow{\gamma_1} & \alpha & \xrightarrow{\gamma_2} & \theta \\
\text{and} & y & \xrightarrow{\gamma_3} & \beta & \xrightarrow{\gamma_4} & \theta
\end{array}
$$

8

correspond under the bijection, there must be a unique object-morphism

$$f : x \longrightarrow y$$

labelling the relevant edge of the tree. Then for the composites in $\mathcal{O}$ we have the relation: the following diagram commutes

3) Composition in $\mathcal{O}_k$ is respected, that is, if $g \circ f = h \in \mathcal{O}_k$ then $g \circ f = h \in \mathcal{O}$.

4) Identities in $\mathcal{O}_k$ are respected, that is, given any morphism $x \xrightarrow{\gamma} \alpha \in \mathcal{O}$ we have $\gamma \circ 1_x = \gamma$.

Note that only the relation (2) is concerned with the identification of faces with one another; the other relations are merely dealing with isomorphic copies of opetopes.

We immediately check that the above relations have not identified any morphisms of $\mathcal{O}_k$.

Lemma 1.6 *Each $\mathcal{O}_k$ is a full subcategory of $\mathcal{O}$.*

Proof.   Clear from definitions.   □

We now check that the above relations have not identified any $(k-1)$-faces of k-opetopes.

Proposition 1.7 *Let $x \in \mathcal{O}_{k-1}$, $\alpha \in \mathcal{O}_k$ and $\gamma_1, \gamma_2 \in G_k$ with*

$$\gamma_1, \ \gamma_2 : x \longrightarrow \alpha$$

Then $\gamma_1 = \gamma_2 \in \mathcal{O} \implies \gamma_1 = \gamma_2 \in G_k$.

We prove this by expressing all morphisms from $(k-1)$-opetopes to k-opetopes in the following "normal form"; this is a simple exercise in term rewriting (see [11]).

Lemma 1.8 *Let $x \in \mathcal{O}_{k-1}$, $\alpha \in \mathcal{O}$. Then a morphism*

$$x \longrightarrow \alpha \ \in \mathcal{O}$$

is uniquely represented by

$$x \xrightarrow{\ \gamma\ } \alpha$$

or a pair

$$x \xrightarrow{\ f\ } y \xrightarrow{\ \gamma\ } \alpha$$

where $f \in \mathcal{O}_{k-1}$ and $\gamma \in G_k$.

Proof. Any map $x \longrightarrow \alpha$ is represented by terms of the form

$$x \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} x_m \xrightarrow{\ \gamma\ } \alpha_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{j-1}} \alpha_j \xrightarrow{g_j} \alpha$$

where each $f_i \in \mathcal{O}_{k-1}$ and each $g_r \in \mathcal{O}_k$. Equalities are generated by equalities in components of the following forms:

1) $\quad \xrightarrow{\quad \gamma \quad} \xrightarrow{\quad g \quad} = \xrightarrow{\quad \gamma g \quad} \xrightarrow{\quad \gamma' \quad}$

2) $\quad \xrightarrow{\quad f \quad} \xrightarrow{\quad f' \quad} = \xrightarrow{\quad f' \circ f \quad} \qquad \in \mathcal{O}_{k-1}$

3) $\quad \xrightarrow{\quad g \quad} \xrightarrow{\quad g' \quad} = \xrightarrow{\quad g' \circ g \quad} \qquad \in \mathcal{O}_k$

4) $\quad \xrightarrow{\quad 1 \quad} \xrightarrow{\quad \gamma \quad} = \xrightarrow{\quad \gamma \quad}$

where $\gamma \in G_k$ and γg and γ' are as defined above. That is, equalities in terms are generated by equations $t = t'$ where t' is obtained from t by replacing a component of t of a left hand form above, with the form in the right hand side, or vice versa.

We now orient the equations in the term rewriting style in the direction
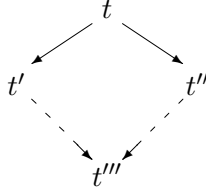
$$\Longrightarrow$$

from left to right in the above equations. We then show two obvious properties:

1) Any reduction of t by $\Longrightarrow$ terminates in at most $2j + m$ steps.

2) If we have
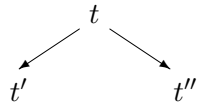
then there exists t''' with

$$
\begin{array}{ccc}
 & t & \\
\swarrow & & \searrow \\
t' & & t'' \\
\searrow & & \swarrow \\
 & t''' &
\end{array}
$$

where the dotted arrows indicate a chain of equations (in this case of length at most 2).

The first part is clear from the definitions; for the second part the only non-trivial case is for a component of the form

$$\xrightarrow{\quad\gamma\quad}\xrightarrow{\quad g_1\quad}\xrightarrow{\quad g_2\quad}.$$

This reduces uniquely to

$$\xrightarrow{\quad\gamma(g_2\circ g_1)\quad}\xrightarrow{\quad\gamma'\quad}$$

since 'restriction' is unique, as discussed earlier.

It follows that, for any terms t and s, $t = s$ if and only if t and s reduce to the same normal form as above. $\square$

Proof of Proposition 1.7 . γ_1 and γ_2 are in normal form. $\square$

2 Opetopic Sets

In this section we examine the theory of opetopic sets. We begin by following through our modifications to the opetopic theory to include the theory of opetopic sets. We then use results of [12] to prove that the category of opetopic sets is indeed equivalent to the category of presheaves on $\mathcal{O}$, the category of opetopes defined in Section 1.

Recall that, by the equivalences proved in the [6] and [5], we have equivalent categories of opetopes, multitopes and Leinster opetopes. So we may define equivalent categories of opetopic sets by taking presheaves on any of these three categories. In the following definitions, although the opetopes we consider are the 'symmetric multicategory' kind, the concrete description of an opetopic set is not *precisely* as a presheaf on the category of these opetopes. The sets given in the data are indexed not by opetopes themselves but by *isomorphism classes* of opetopes; so at first sight this

resembles a presheaf on the category of Leinster opetopes. However, we do not pursue this matter here, since the equivalences proved in our earlier work are sufficient for the purposes of this article.

We adopt this presentation in order to avoid naming the same cells repeatedly according to the symmetries; that is, we do not keep copies of cells that are isomorphic by the symmetries.

2.1   Definitions

In [3], weak n-categories are defined as opetopic sets satisfying certain universality conditions. However, opetopic sets are defined using only symmetric multicategories with a *set* of objects; in the light of the results of our earlier work, we seek a definition using symmetric multicategories with a *category* of objects. The definitions we give here are those given in [3] but with modifications as demanded by the results of our previous work.

The underlying data for an opetopic n-category are given by an opetopic set. Recall that, in [3], given a symmetric multicategory Q a Q-opetopic set X is given by, for each $k \geq 0$, a symmetric multicategory $Q(k)$ and a set $X(k)$ over $o(Q(k))$, where

$$Q(0) = Q$$
$$\text{and} \quad Q(k+1) = Q(k)_{X(k)}{}^+.$$

An opetopic set is then an I-opetopic set, where I is the symmetric multicategory with one object and one (identity) arrow.

The idea is that the category of opetopic sets should be equivalent to the presheaf category

$$[\mathbf{Opetope}^{\mathrm{op}}, \mathbf{Set}]$$

and we use this to motivate our generalisation of the Baez-Dolan definitions.

Recall that we have for each $k \geq 0$ a category $\mathbb{C}(k)$ of k-opetopes, and each $\mathbb{C}(k)$ is a full subcategory of **Opetope**. A functor

$$\mathbf{Opetope}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

may be considered as assigning to each opetope a set of 'labels'.

Recall that for each k, $\mathbb{C}(k)$ is equivalent to a discrete category. So it is sufficient to specify 'labels' for each isomorphism class of opetopes.

Recall ([6]) that we call a symmetric multicategory Q *tidy* if it is freely symmetric with a category of objects $\mathbb{C}$ equivalent to a discrete category. Throughout this section we say 'Q has object-category $\mathbb{C}$ equivalent to S discrete' to mean that S is the set of isomorphism classes of $\mathbb{C}$, so $\mathbb{C}$ is equipped with a morphism $\mathbb{C} \xrightarrow{\sim} S$. We begin by defining the construction used for 'labelling' as discussed above. The idea is to give a set of labels as a set over the isomorphism classes of objects of Q, and then to 'attach' the labels using the following pullback construction.

12

Definition 2.1 *Let Q be a tidy symmetric multicategory with category of objects $\mathbb{C}$ equivalent to S discrete. Given a set X over S, that is, equipped with a function $f : X \longrightarrow S$, we define the* pullback multicategory Q_X *as follows.*

- *Objects: $o(Q_X)$ is given by the pullback*

$$
\begin{array}{ccc}
\cdot & \longrightarrow & \mathbb{C} \\
\downarrow & & \downarrow {\scriptstyle \sim} \\
X & \xrightarrow{\ f\ } & S
\end{array}
$$

  Observe that the morphism on the left is an equivalence, so $o(Q_X)$ is equivalent to X discrete. Write h for this morphism.

- *Arrows: given objects $a_1, \ldots a_k, a \in o(Q_X)$ we have*

$$Q_X(a_1, \ldots, a_k; a) \cong Q(fh(a_1), \ldots, fh(a_k); fh(a)).$$

- *Composition, identities and symmetric action are then inherited from Q.*

We observe immediately that since Q is tidy, Q_X is tidy. Also note that if Q is object-discrete this definition corresponds to the definition of pullback symmetric multicategory given in [3].

We are now ready to describe the construction of opetopic sets.

Definition 2.2 *Let Q be a tidy symmetric multicategory with object-category $\mathbb{C}$ equivalent to S discrete. A Q-opetopic set X is defined recursively as a set $X(0)$ over S together with a $Q_X{}^+$-opetopic set X_1.*

So a Q-opetopic set consists of, for each $k \geq 0$:

- a tidy symmetric multicategory $Q(k)$ with object-category $\mathbb{C}(k)$ equivalent to $S(k)$ discrete

- a set $X(k)$ and function $X(k) \xrightarrow{f_k} S(k)$

where

$$
\begin{aligned}
Q(0) &= Q \\
\text{and} \quad Q(k+1) &= Q(k)_{X(k)}{}^+.
\end{aligned}
$$

We refer to X_1 as the underlying $Q(k)_{X(k)}{}^+$-opetopic set of X.

We now define morphisms of opetopic sets. Suppose we have opetopic sets X and X' with notation as above, together with a morphism of symmetric multicategories

$$F : Q \longrightarrow Q'$$

and a function

$$F_0 : X(0) \longrightarrow X'(0)$$

such that the following diagram commutes

$$
\begin{array}{ccc}
X(0) & \xrightarrow{\ f_0\ } & S(0) \\
{\scriptstyle F_0}\downarrow & & \downarrow{\scriptstyle F} \\
X'(0) & \xrightarrow[\ f'_0\ ]{} & S'(0)
\end{array}
$$

where the morphism on the right is given by the action of F on objects. This induces a morphism

$$Q_{X(0)} \longrightarrow Q'_{X'(0)}$$

and so a morphism

$$Q_{X(0)}{}^+ \longrightarrow Q'_{X'(0)}{}^+.$$

We make the following definition.

Definition 2.3 *A morphism of Q-opetopic sets*

$$F : X \longrightarrow X'$$

is given by:

- *an underlying morphism of symmetric multicategories and function F_0 as above*

- *a morphism $X_1 \longrightarrow X'_1$ of their underlying opetopic sets, whose underlying morphism is induced as above.*

So F consists of

- a morphism $Q \longrightarrow Q'$

- for each $k \geq 0$ a function $F_k : X(k) \longrightarrow X'(k)$ such that the following diagram commutes

14

$$X(k) \xrightarrow{\;f_k\;} S(k)$$
$$\Big\downarrow F_k \qquad\qquad \Big\downarrow$$
$$X'(k) \xrightarrow[\;f'_k\;]{} S'(k)$$

where the map on the right hand side is induced as appropriate.

Note that the above notation for a Q-opetopic set X and morphism F will be used throughout this section, unless otherwise specified.

Definition 2.4 *An* opetopic set *is an I-opetopic set. A morphism of opetopic sets is a morphism of I-opetopic sets. We write* **OSet** *for the category of opetopic sets and their morphisms.*

Eventually, a weak n-category is defined as an opetopic set with certain properties. The idea is that k-cells have underlying shapes given by the objects of I^{k+}. These are 'unlabelled' cells. To make these into fully labelled k-cells, we first give labels to the 0-cells, via the function $X(0) \longrightarrow S(0)$, and then to 1-cells via $X(1) \longrightarrow S(1)$, and so on. This idea may be captured in the following 'schematic' diagram.

$$
\begin{array}{ccccc}
 & & Q(0)_{X(0)} \longrightarrow & Q(0) = I & \\
 & & \Big\downarrow + & \Big\downarrow + & \\
 & Q(1)_{X(1)} \longrightarrow Q(1) = Q(0)_{X(0)}{}^{+} & & I^{+} & \\
 & \Big\downarrow + & & \Big\downarrow + & \\
Q(2)_{X(2)} \longrightarrow Q(2) = Q(1)_{X(1)}{}^{+} & & & I^{2+} & \\
\Big\downarrow + & & & \Big\downarrow + & \\
Q(2)_{X(2)}{}^{+} & & & I^{3+} & \\
 & & & \Big\downarrow + & \\
 & & \vdots & &
\end{array}
$$

Bearing in mind our modified definitions, we use the Baez-Dolan terminology as follows.

Definitions 2.5

- *A k-dimensional cell (or k-cell) is an element of $X(k)$*
  (i.e. an isomorphism class of objects of $Q(k)_{X(k)}$).

- *A k-frame is an isomorphism class of objects of $Q(k)$*
  (i.e. an isomorphism class of arrows of $Q(k-1)_{X(k-1)}$).

- *A k-opening is an isomorphism class of arrows of $Q(k-1)$, for $k \geq 1$.*
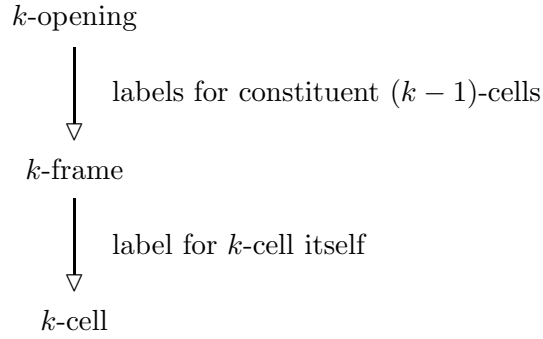
So a k-opening may acquire $(k-1)$-cell labels and become a k-frame, which may itself acquire a label and become a k-cell. We refer to such a cell and frame as being *in* the original k-opening.

On objects, the above schematic diagram becomes:
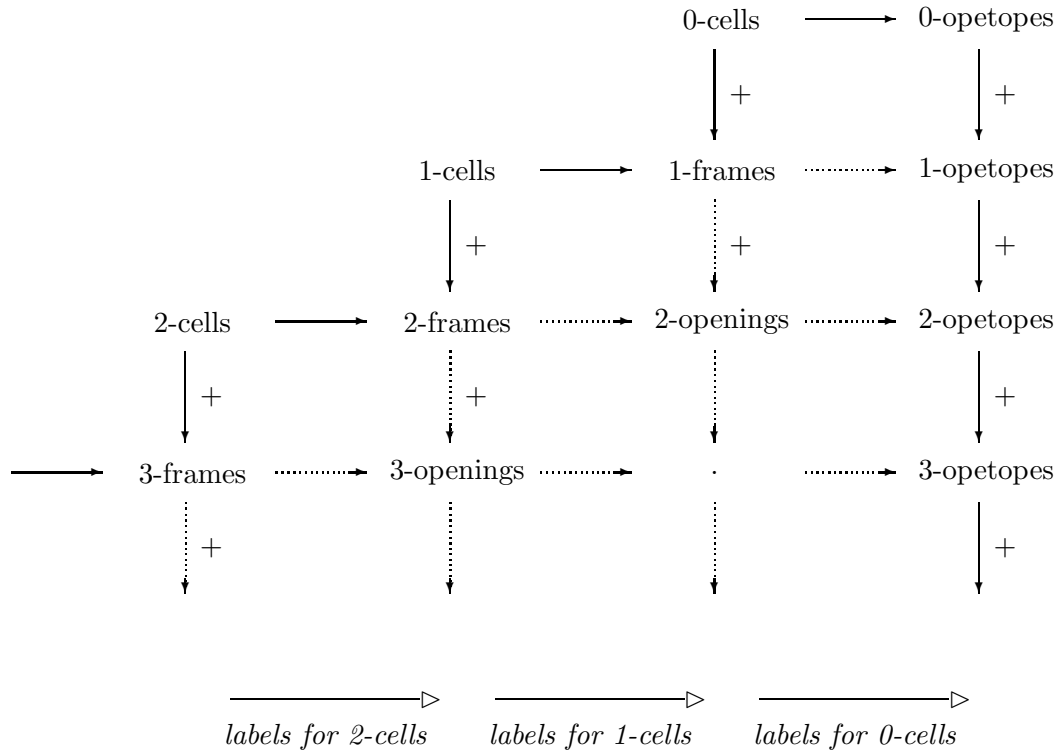
Horizontal arrows represent the process of labelling, as shown; vertical arrows represent the process of 'moving up' dimensions. Starting with a k-opetope, we have from right to left the progressive labelling of 0-cells, 1-cells, and so on, to form a k-cell at the far left, the final stages being:

16

k-opening

|
↓ labels for constituent $(k-1)$-cells

k-frame

|
↓ label for k-cell itself

k-cell

A k-opening acquires labels as an arrow of $Q(k-1)$, becoming a k-frame as an arrow of $Q(k-1)_{X(k-1)}$. That is, it has $(k-1)$-cells as its source and a $(k-1)$-cell as its target.

Definition 2.6 *A k-niche is a k-opening (i.e. arrow of $Q(k-1)$) together with labels for its source only.*
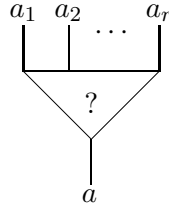
We may represent these notions as follows. Let f be an arrow of $Q(k-1)$, so f specifies a k-opening which we might represent as
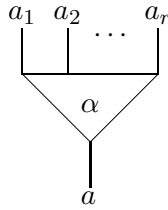

.
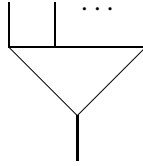
Then a niche in f is represented by
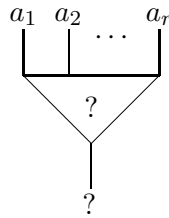
where $a_1, \ldots a_r$ are 'valid' labels for the source elements of f; a k-frame is represented by

17

where a is a 'valid' label for the target of f. Finally a k-cell is represented
by

Since all symmetric multicategories in question are tidy, we may in each
case represent the same isomorphism class by any symmetric variant of the
above diagrams. Also, we refer to k-cells as labelling k-opetopes, rather
than isomorphism classes of k-opetopes.

2.2 OSet is a presheaf category

In this section we prove the main result of this work, that the category of
opetopic sets is a presheaf category, and moreover, that it is equivalent to
the presheaf category

$$[\mathcal{O}^{\mathrm{op}}, \mathbf{Set}].$$

To prove this we use [12], Theorem 5.26, in the case $\mathcal{V} = \mathbf{Set}$. This theorem
is as follows.

Theorem 2.7 *Let $\mathcal{C}$ be a $\mathcal{V}$-category. In order that $\mathcal{C}$ be equivalent to
$[\mathcal{E}^{op}, \mathcal{V}]$ for some small category $\mathcal{E}$ it is necessary and sufficient that $\mathcal{C}$ be co-
complete, and that there be a set of small-projective objects in $\mathcal{C}$ constituting
a strong generator for $\mathcal{C}$.*

We see from the proof of this theorem that if E is such a set and $\mathcal{E}$ is
the full subcategory of $\mathcal{C}$ whose objects are the elements of E, then

$$\mathcal{C} \simeq [\mathcal{E}^{\mathrm{op}}, \mathcal{V}].$$

We prove the following propositions; the idea is to "realise" each isomor-
phism class of opetopes as an opetopic set; the set of these opetopic sets
constitutes a strong generator as required.

Proposition 2.8 **OSet** *is cocomplete.*

Proposition 2.9 *There is a full and faithful functor*

$$G : \mathcal{O} \longrightarrow \mathbf{OSet}.$$

Proposition 2.10 *Let $\alpha \in \mathcal{O}$. Then $G(\alpha)$ is small-projective in* **OSet***.*

Proposition 2.11 *Let*

$$E = \coprod \{G(\alpha) \mid \alpha \in \mathcal{O}\} \subseteq \mathbf{OSet}.$$

Then E is a strongly generating set for **OSet***.*

Corollary 2.12 **OSet** *is a presheaf category.*

Corollary 2.13

$$\mathbf{OSet} \simeq [\mathcal{O}^{op}, \mathbf{Set}].$$

Proof of Proposition 2.8. Consider a diagram

$$D : \mathbb{I} \longrightarrow \mathbf{OSet}$$

where $\mathbb{I}$ is a small category. We seek to construct a limit Z for D; the set of cells of Z of shape α is given by a colimit of the sets of cells of shape α in each $D(I)$.

We construct an opetopic set Z as follows. For each $k \geq 0$, $Z(k)$ is a colimit in **Set**:

$$Z(k) = \int^{I \in \mathbb{I}} D(I)(k).$$

Now for each k we need to give a function

$$F(k) : Z(k) \longrightarrow o(Q(k))$$

where

$$Q(k) = Q(k-1)_{Z(k-1)}^{\;+}$$
$$Q(0) = I.$$

That is, for each $\alpha \in Z(k)$ we need to give its frame. Now

$$Z(k) = \coprod_{I \in \mathbb{I}} D(I)(k) \Big/ {\sim}$$

where $\sim$ is the equivalence relation generated by

$$D(u)(\alpha_{I'}) \sim \alpha_I \quad \text{for all } u : I \longrightarrow I' \in \mathbb{I}$$
$$\text{and } \alpha_I \in D(I)(k).$$

19

So $\alpha \in Z(k)$ is of the form $[\alpha_I]$ for some $\alpha_I \in D(I)(k)$ where $[\alpha_I]$ denotes the equivalence class of α_I with respect to $\sim$.

Now suppose the frame of α_I in $D(I)$ is

$$(\beta_1, \ldots, \beta_j) \xrightarrow{\ ?\ } \beta$$

where $\beta_i, \beta \in D(I)(k-1)$ label some k-opetope x. We set the frame of $[\alpha_I]$ to be

$$(\ [\beta_1], \ldots, [\beta_j]\ ) \xrightarrow{\ ?\ } [\beta]$$

labelling the same opetope x. This is well-defined since a morphism of opetopic sets preserves frames of cells, so the frame of $D(u)(\alpha_I)$ is

$$(\ D(u)(\beta_1)\ , \ldots,\ D(u)(\beta_j)\ ) \xrightarrow{\ ?\ } D(u)(\beta)$$

also labelling k-opetope x. It follows from the universal properties of the colimits in **Set** that Z is a colimit for D, with coprojections induced from those in **Set**. Then, since **Set** is cocomplete, **OSet** is cocomplete. $\qquad \square$

Proof of Proposition 2.9. Let α be a k-opetope. We express α as an opetopic set $G(\alpha) = \hat{\alpha}$ as follows, using the usual notation for an opetopic set. The idea is that the m-cells are given by the m-faces of α.

For each $m \geq 0$ set

$$X(m) = \{\ [(x, f)]\ \mid\quad x \in \mathcal{O}_m\ \text{ and } x \xrightarrow{\ f\ } \alpha \in \mathcal{O}$$
$$\text{where } [\ ] \text{ denotes isomorphism class in } \mathcal{O}/\alpha\}.$$

So in particular we have
$$X(k) = \{[(\alpha, 1)]\}$$

and for all $m > k$, $X(m) = \emptyset$. It remains to specify the frame of $[(x, f)]$. The frame is an object of

$$Q(m) = Q(m-1)_{X(m-1)}{}^+$$

so an arrow of

$$Q(m-2)_{X(m-2)}{}^+$$

labelled with elements of $X(m-1)$. Now such an arrow is a configuration for composing arrows of $Q(m-2)_{X(m-2)}$; for the frame as above, this is given by the opetope x as a labelled tree. Then the $(m-1)$-cell labels are given as follows. Write

$$x : y_1, \ldots, y_j \longrightarrow y$$

say, and so we have for each i a morphism

$$y_i \longrightarrow x$$

and a morphism

$$y \longrightarrow x \in \mathcal{O}.$$

Then the labels in $X(m-1)$ are given by

$$[y_i \longrightarrow x \xrightarrow{f} \alpha] \in X(m-1)$$

and

$$[y \longrightarrow x \xrightarrow{f} \alpha] \in X(m-1).$$

Now, given a morphism

$$h : \alpha \longrightarrow \beta \in \mathcal{O}$$

we define

$$\hat{h} : \hat{\alpha} \longrightarrow \hat{\beta} \in \mathbf{OSet}$$

by

$$[(x, f)] \mapsto [(x, h \circ f)]$$

which is well-defined since if $(x, f) \cong (x', f')$ then $(x, hf) \cong (x', hf')$ in $\mathcal{O}/\alpha$. This is clearly a morphism of opetopic sets.

Observe that any morphism $\hat{\alpha} \longrightarrow \hat{\beta}$ must be of this form since the faces of α must be preserved. Moreover, if $\hat{h} = \hat{g}$ then certainly $[(\alpha, h)] = [(\alpha, g)]$. But this gives $(\alpha, h) = (\alpha, g)$ since there is a unique morphism $\alpha \longrightarrow \alpha \in \mathcal{O}$ namely the identity. So G is full and faithful as required. $\square$

Proof of Proposition 2.10. For any $\alpha \in \mathcal{O}_k$ we show that $\hat{\alpha}$ is small-projective, that is that the functor

$$\psi = \mathbf{OSet}(\hat{\alpha}, -) : \mathbf{OSet} \longrightarrow \mathbf{Set}$$

preserves small colimits. First observe that for any opetopic set X

$$\psi(X) = \mathbf{OSet}(\hat{\alpha}, X) \;\; \cong \;\; \{k\text{-cells in } X \text{ whose underlying } k\text{-opetope is } \alpha\}$$
$$\subseteq \;\; X(k)$$

and the action on a morphism $F : X \longrightarrow Y$ is given by

$$\psi(F) = \mathbf{OSet}(\hat{\alpha}, F) : \;\; \begin{matrix} \mathbf{OSet}(\hat{\alpha}, X) & \longrightarrow & \mathbf{OSet}(\hat{\alpha}, Y) \\ x & \mapsto & F(x). \end{matrix}$$

So ψ is the 'restriction' to the set of cells of shape α. This clearly preserves colimits since the cells of shape α in the colimit are given by a colimit of the sets cells of shape α in the original diagram. $\square$

Proof of Proposition 2.11. First note that

$$\hat{\alpha} = \hat{\beta} \iff \alpha \cong \beta \in \mathcal{O}$$

so

$$E \cong \coprod_k S_k$$

where for each k, S_k is the set of k-dimensional Leinster opetopes. Since each S_k is a set it follows that E is a set.

We need to show that, given a morphism of opetopic sets $F : X \longrightarrow Y$, we have

$\mathbf{OSet}(\hat\alpha, F)$ is an isomorphism for all $\hat\alpha \implies F$ is an isomorphism.

Now, we have seen above that

$$\mathbf{OSet}(\hat\alpha, X) \cong \{\text{cells of } X \text{ of shape } \alpha\}$$

so

$\mathbf{OSet}(\hat\alpha, F) = F|_\alpha = F$ restricted to cells of shape α.

So

$\mathbf{OSet}(\hat\alpha, F)$ is an isomorphism for all $\hat\alpha$
$\iff F|_\alpha$ is an isomorphism for all $\alpha \in \mathcal{O}$
$\iff F$ is an isomorphism.

$\square$

Proof of Corollary 2.12. Follows from Propositions 2.8, 2.9, 2.10, 2.11 and [12] Theorem 5.26. $\square$

Proof of Corollary 2.13. Let $\mathcal{E}$ be the full subcategory of $\mathbf{OSet}$ whose objects are those of E. Since G is full and faithful, $\mathcal{E}$ is the image of G and we have

$$\mathcal{O} \simeq \mathcal{E}$$

and hence

$$\mathbf{OSet} \simeq [\mathcal{E}^{\mathrm{op}}, \mathbf{Set}] \simeq [\mathcal{O}^{\mathrm{op}}, \mathbf{Set}].$$

$\square$

References

[1] John Baez. An introduction to n-categories. In *7th Conference on Category Theory and Computer Science*, volume 1290 of *Springer Lecture Notes in Computer Science*, 1997. Also available via `http://math.ucr.edu/home/baez`.

[2] John Baez and James Dolan. Letter to R. Street, Nov. 30, 1995. Corrected version as of Dec. 3, 1995. Available via `http://math.ucr.edu/home/baez`.

[3] John Baez and James Dolan. Higher-dimensional algebra III: n-categories and the algebra of opetopes. *Adv. Math.*, 135(2):145–206, 1998. Also available via `http://math.ucr.edu/home/baez`.

[4] Eugenia Cheng. A relationship between trees and Kelly-Mac Lane graphs, October 2002. E-print `math.CT/0304287`.

[5] Eugenia Cheng. Weak n-categories: comparing opetopic foundations, October 2002. E-print `math.CT/0304279`.

[6] Eugenia Cheng. Weak n-categories: opetopic and multitopic foundations, October 2002. E-print `math.CT/0304277`.

[7] Claudio Hermida, Michael Makkai, and John Power. On weak higher dimensional categories, 1997. Available via `http://triples.math.mcgill.ca`.

[8] Claudio Hermida, Michael Makkai, and John Power. Higher dimensional multigraphs. *Thirteenth Annual IEEE Symposium on Logic in Computer Science*, pages 199–206, 1998. IEEE Computer Soc., Los Alamitos, CA.

[9] Claudio Hermida, Michael Makkai, and John Power. On weak higher-dimensional categories I: Part 1. *Journal of Pure and Applied Algebra*, 154:221–246, 2000.

[10] Claudio Hermida, Michael Makkai, and John Power. On weak higher-dimensional categories I – 2. *Journal of Pure and Applied Algebra*, 157:247–277, 2001.

[11] R. C. De Vrijer J. W. Klop. *Term Rewriting Systems*. Number 25 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1999.

[12] G. M. Kelly. *Basic Concepts of Enriched Category Theory*. Number 64 in London Mathematical Society Lecture Note Series. Cambridge University Press, 1982.

[13] Tom Leinster. *Operads in Higher Dimensional Category Theory*. PhD thesis, University of Cambridge, July 2000. E-print `math.CT/0011106`.