

Test Flight Problem Set Solutions

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- A1. The statement is false. If $n \geq 2$, then for any m , we have that $3m + 5n \geq 13$ (since $3m + 5n \geq 3m + 10 \geq 13$). Thus, the only way to find such a solution for n in the natural numbers would be when $n = 1$. Substituting, we have $3m + 5 \cdot 1 = 3m + 5 = 12$, or $3m = 7$. But since there is no natural number m satisfying this equation, we have proved the result. QED
- A2. The statement is true. Without loss of generality, we may assume the consecutive five integers may be written in the form: $n - 2, n - 1, n, n + 1, n + 2$. If we sum these integers, we have $5n$, which is divisible by 5. Hence, we have proved the result. QED
- A3. The statement is true. We may rewrite $n^2 + n + 1$, as $n \cdot (n + 1) + 1$. If n is even, then $n + 1$ is odd. If n is odd, then $n + 1$ is even. In either case, $n \cdot (n + 1)$ is even because the product of an even and odd number is even. Hence, we may write $n \cdot (n + 1) + 1$ as $2k + 1$, which is odd. Hence, we have proved the result. QED
- A4. Recall from the remainder theorem: if a, b are integers with $b > 0$, then there exist unique integers q, r such that $a = bq + r$ and $0 \leq r < b$. If we let $b = 4$ (and $n = q$), then we have the statement that $a = 4n + r$ with $0 \leq r < 4$. If $r = 0$ or 2 , then we have $a = 4n$ or $a = 4n + 2$, which are even natural numbers. If $r = 1$ or 3 , we have that $a = 4n + 1$ or $a = 4n + 3$, which are odd natural numbers. Since a is any odd natural number, satisfying the antecedent, we have that it must be of one of the following forms $a = 4n + 1$ or $a = 4n + 3$. Hence, we have proved the result. QED
- A5. Recall from the remainder theorem: if a, b are integers with $b > 0$, then there exist unique integers q, r such that $a = bq + r$ and $0 \leq r < b$. If we take $b = 3$, then we have the statement that $a = 3q + r$ with $0 \leq r < 3$. Expanding out (and letting $n = a$), we have that $n = 3q$, or $n = 3q + 1$, or $n = 3q + 2$. Let's now write $n, n + 2$, and $n + 4$ in these forms: n is either $3q, 3q + 1$, or $3q + 2$. $n + 2$ is either $3q + 2, 3q + 3$, or $3q + 4$. $n + 4$ is either $3q + 4, 3q + 5$, or $3q + 6$. But we see that in each of the forms, there exists an element which

is divisible by 3 i.e. if n , $3|3q$ and if $n+2$, $3|(3q+3)$, and if $n+4$, $3|(3q+6)$. Hence, we have proved the result. QED

- A6. We prove this by contradiction i.e. assume there exists $n > 3$, such that n , $n+2$, and $n+4$ are prime. But from the proof of #5, we have just shown that one of n , $n+2$, $n+4$ must be divisible by 3. And since $n > 3$, 3 is not one of the primes. Thus, one of n , $n+2$, $n+4$ is not prime. Hence, we have proved the result. QED
- A7. Let the sum, $2+2^2+2^3+\dots+2^n$, be denoted by S . Multiplying by 2, we have that $2S = 2^2+2^3+\dots+2^{n+1}$. Subtracting S from $2S$, we have that $S = 2^{n+1} - 2$, which was to be proved. QED
- A8. By the assumption, we have that for any given $\epsilon > 0$, there exists an n where for all $m \geq n$, $|a_m - L| < \epsilon$. The statement that Ma_n tends to ML as n tends to infinity is equivalent to saying that for any given $\epsilon_1 > 0$, there exists an n where all $m \geq n$, $|Ma_m - ML| < \epsilon_1$. This simplifies to $|M|a_m - L| \iff M|a_m - L| < \epsilon_1 \iff |a_m - L| < \frac{\epsilon_1}{M}$. This will be true if we take $\epsilon_1/M = \epsilon$, and find such an n . Since we can always do this, we have proved the result. QED
- A9. Let $A_n = (0, 1/n)$. We have that A_n is a subset of A_1 since $(0, 1/n)$ is a subset of $(0, 1)$. Suppose that x is an element of $(0, 1)$. We can always find a natural number m such that $1/m < x$. But that means that x is not an element of A_m . Hence, x is not an element of the intersection of A_n where n is a natural number. Since we can always find this number m , we must necessarily have that intersection of A_n is empty. Hence, we have proved the result. QED
- A10. Let $A_n = [0, 1/n)$. We may write this set as $0 \cup B_n$, where $B_n = (0, 1/n)$. The intersection of A_n for n in the natural numbers may thus be written as $0 \cup (\cap B_n)$. But since we have proved from #9 that intersection of all B_n is the empty set, we have that $0 \cup \emptyset = \{0\}$. Hence, we have proved the result. QED