AST 1420: Galactic Structure and Dynamics
Problem Set 1

Due date: February 11, 2026, by 5pm

Most of the exercises in this problem set must be solved on a computer and a good way to hand in the problem set is as a `Jupyter notebook` (label the number of each problem). Please *re-run the entire notebook (with Cell > Run All) after re-starting the notebook kernel before sending it in*; this will make sure that the input and output are fully consistent. You are also welcome to send in an additional write-up in LaTeX as a PDF to supplement the discussions/comments for some of the problems, or include them in the Jupyter notebook with the markdown feature.

1. **Warming-up with Timescales:** Calculate the dynamical time, direct collisional time, and relaxation time for the stellar systems below. You should assume a size for a typical star or galaxy. Comment on the dynamical state of each system, in particular whether the system can be considered collisionless or not and your reasoning behind this. Dark matter is ignored in this question.

Object Type	**Radius (kpc)**	**Velocity (km/s)**	**N**	**Particle**
globular cluster	0.01	5	10^5	star
dwarf galaxy	0.1	10	10^7	star
Milky Way	10	200	10^{11}	star
galaxy group	1000	200	50	galaxy
galaxy cluster	5000	1000	10^3	galaxy

2. **The Milky Way Potential:** Assume the mass of the Milky Way Galaxy is dominated by dark matter having a spherical radial density profile which is isothermal (v_c = constant) and cuts off at 10 $R_{\rm sun}$. Assume the Sun rotation velocity is 220 km s^{-1} at a distance 8 kpc from the Galaxy center.

   a. What is the rotation period at the Sun. How many orbits (roughly) since the Galaxy formed?

   b. What is the total mass within the Sun's orbit?

   c. What is the escape velocity from the Solar neighborhood? Find this by deriving the form of the potential versus radius between $R_{\rm sun}$ and the outermost radius, and then the form of the potential beyond this. Set the potential at infinity to zero. Find the other constant in the potential law by matching the rotation speed between $R_{\rm sun}$ and 10 $R_{\rm sun}$. Then use the potential at $R_{\rm sun}$ to find $v_{\rm esc}$.

   d. What is the orbital period of a typical dwarf satellite galaxy at 100 kpc (assume circular orbits)? How many orbits since this galaxy formed?

e. What is the escape velocity for this satellite?

3. Orbits in spherical potentials:

Consider a test particle (unit mass $m = 1$) moving in the xy-plane under the spherical potential:

$$\Phi(r) = -\frac{b}{r} + (1-b)r^2 \tag{1}$$

where b is a constant and we use units with $G = 1$. You will analyze orbits for $b = 0$, 0.25, 0.75, and 1.

Use the following initial conditions for all cases:

- Position: $(x_0, y_0) = (1, 0)$
- Velocity: $(v_{x,0}, v_{y,0}) = (0, 0.8)$

Complete the following steps:

(a) **Energy and angular momentum:** For each value of b, compute the total energy E and angular momentum L_z for the given initial conditions.
Recall: $E = \frac{1}{2}v^2 + \Phi(r)$ and $L_z = xv_y - yv_x$.

(b) **Visualize the potential:** Plot $\Phi(r)$ vs r for all four values of b on the same figure. On a separate figure, plot the effective potential:

$$\Phi_{\text{eff}}(r) = \Phi(r) + \frac{L_z^2}{2r^2} \tag{2}$$

for all four values of b (using the L_z computed in part a).

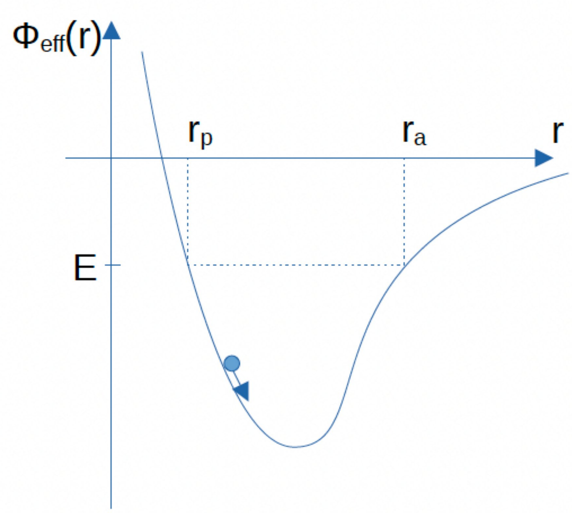

Figure 1: Example illustration of the effective potential with energy level E and turning points r_p (pericenter) and r_a (apocenter), for Problem 3(c).

(c) **Determine if orbits are bound; compute turning points:** For each b, determine whether the orbit is bound.

For bound orbits, find the pericenter r_p and apocenter r_a by solving $E = \Phi_{\text{eff}}(r)$ (the turning points where radial velocity is zero).

Create a figure showing $\Phi_{\text{eff}}(r)$ for each b, with a horizontal line at the energy E, and mark the pericenter and apocenter, as illustrated in Figure 1.

(d) **Orbit integration:** Integrate the orbit for each value of b for approximately 5 radial oscillations using the **Velocity Verlet** algorithm.

Here we will write our own orbital integration code. An example Jupyter notebook (`example_orbit_integration.ipynb`) is provided that demonstrates how to implement and use the Velocity Verlet integrator for a circular orbit in a Keplerian potential. Use this as a reference for your implementation.

Given position $\mathbf{x}_n$, velocity $\mathbf{v}_n$, and acceleration $\mathbf{a}_n$ at time step n, the next time step is computed as:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{v}_n \Delta t + \frac{1}{2}\mathbf{a}_n \Delta t^2 \tag{3}$$

$$\mathbf{a}_{n+1} = \mathbf{a}(\mathbf{x}_{n+1}) \quad \text{(recompute acceleration at new position)} \tag{4}$$

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \frac{1}{2}(\mathbf{a}_n + \mathbf{a}_{n+1})\Delta t \tag{5}$$

Note that the velocity update uses the **average** of the old and new accelerations. This is what makes the integrator symplectic and gives it excellent energy conservation.

Create the following plots:

- A 2×2 grid showing the orbit (x, y) for each b
- A 2×2 grid showing $r(t)$ vs t for each b, with horizontal dashed lines indicating the analytically computed r_p and r_a from part (c). Verify that the numerical turning points match the analytical predictions.

(e) **Assess orbit closure:** Based on your orbit integration results, determine which orbits are closed (i.e., the particle returns to exactly the same position after some number of radial oscillations) and which are not closed (rosette patterns).

Relate your findings to **Bertrand's theorem**, which states that only two potentials have all bound orbits closed: the Keplerian potential ($\Phi \propto -1/r$) and the harmonic oscillator ($\Phi \propto r^2$).