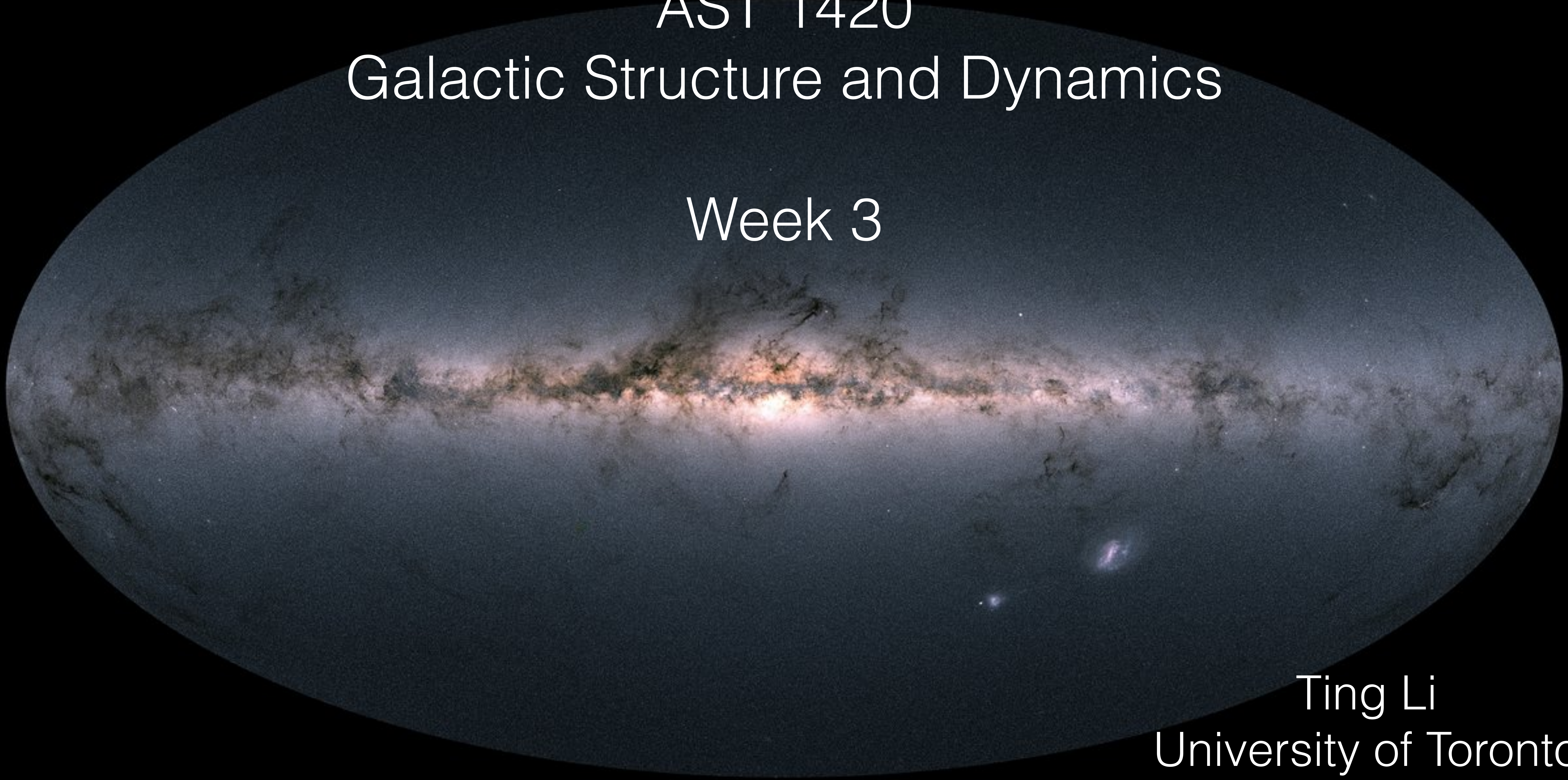
# AST 1420
# Galactic Structure and Dynamics

## Week 3



Ting Li
University of Toronto
Winter 2026

Gaia's all-sky view

# Next two weeks

Week 3 — today

Week 4 — Gravity, orbits and equilibria in disks; actions (guest lecturer: Jo Bovy)

Week 4 reading assignment — chapter 3.4.3, 4.4, 7, 9, 10 (skip 10.2) — email due Feb 1

Week 5 — mass of dwarf spheroidal galaxies, project 1 (guest lecturer: Tri Nguyen)

Week 5 reading assignment — email me your choice of papers for end-of-semester presentations (email due Feb 8)

No class on Feb 17

Week 6 — Feb 24.

# Final presentation topics

mass measurements / mass profile measurements

IoM/action-angle

hydro/N-body galaxy simulations

merger, accretion, tides

galactic chemical evolution

dark matter halo formation and halo mass function

gravitational lensing

stellar mass - halo mass relation

stellar-population synthesis modeling

formation and evolution of bar and/or spiral structure

# Assignment 1

Due Feb 11, 5pm.

Submit via Quercus

Jupyter notebook(s) and also export to PDF files. Upload both.

Can submit multiple files

Include your name in the submitted files.

## Recap last week: potential theory

- Gravitational potential and gravitational force $$\mathbf{g}(\mathbf{x}) = -\boldsymbol{\nabla}\Phi,$$

- Poisson equation, relate density and potential $$\nabla^2\Phi = 4\pi G\rho.$$

**Recap last week:**
**spherical mass**

- Newton's theorem 1: inside spherical shell —> no force
- Newton's theorem 2: outside spherical shell —> as if point mass

$$\Phi(r) = -4\pi G \left[ \frac{1}{r} \int_0^r \mathrm{d}r' \, \rho(r') \, r'^2 + \int_r^\infty \mathrm{d}r' \, \rho(r') \, r' \right] .$$

$$\rho(r) \rightarrow \Phi(r)$$

**Recap last week:**
**spherical mass**

- Circular velocity —> mass *inside*

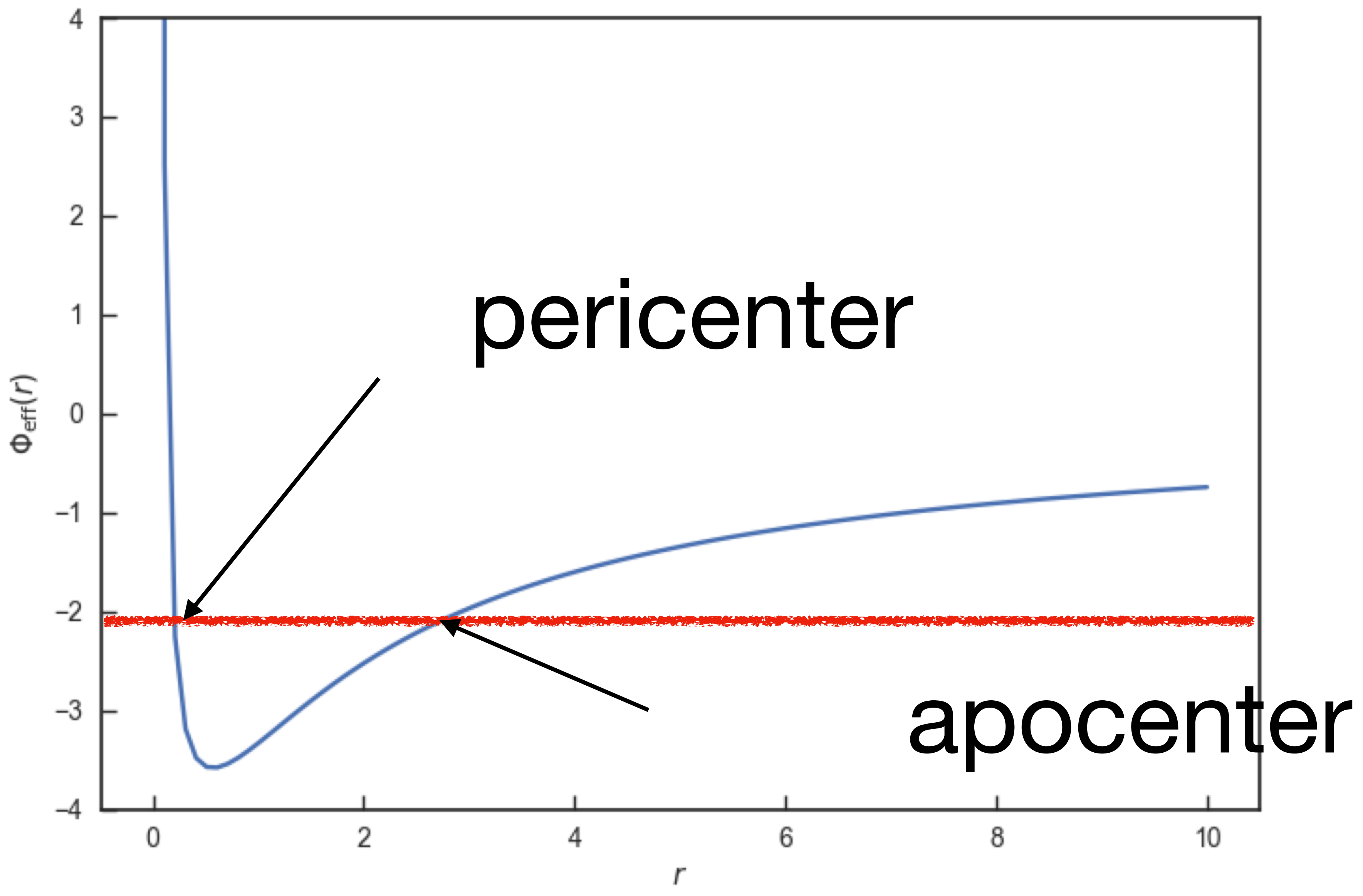
$$v^2 = -r\,g_r(r) = \frac{G\,M(<r)}{r}$$

- Dynamical time —> mean density *inside*

$$t_{\rm dyn} = \sqrt{\frac{3\pi}{G\,\bar{\rho}}}\,,$$

## Recap last week: orbits in spherical potentials

- Angular momentum conserved:
    - Direction: motion confined to *orbital plane*
    - Magnitude: can reduce problem to 1D in *effective potential*
- Energy, angular momentum, potential —> shape and period of an orbit.




$$\Phi_{\rm eff}(r) = \Phi(r) + \frac{L^2}{2\,r^2}\,,$$

# Galaxies as collisionless systems

# Galaxies as collisionless systems

- A collisionless system is one where the approximation of stars as a smooth distribution of mass holds rather than a collection of point masses
- Collisions don’t matter much to the orbits of stars
- To more quantitively determine whether collisions matter, we can compute the time necessary for close encounters to change the velocity by order unity

# (Two body) Relaxation Time

- Timescale on which individual encounters are important — time over which the combined effect of many close encounters has changed a star's velocity by 100%

$$t_{\rm relax} = n_{\rm relax} \times t_{\rm cross} \approx \frac{N}{8\,\ln N}\,\frac{2\pi R}{v} = \frac{N}{8\,\ln N}\,t_{\rm dyn}\,.$$

- The relaxation time is the timescale on which:

  many small gravitational encounters
  cumulatively change stellar orbits
  erase memory of initial conditions

## Galaxies as collisionless systems

For galaxies, $N \approx 10^{11}$ and $t_{\rm cross} \approx 100$ Myr. Therefore,

```
trelax= 10.**11./8./numpy.log(10.**11)*100*u.Myr
print(trelax.to(10**10*u.Myr))
```

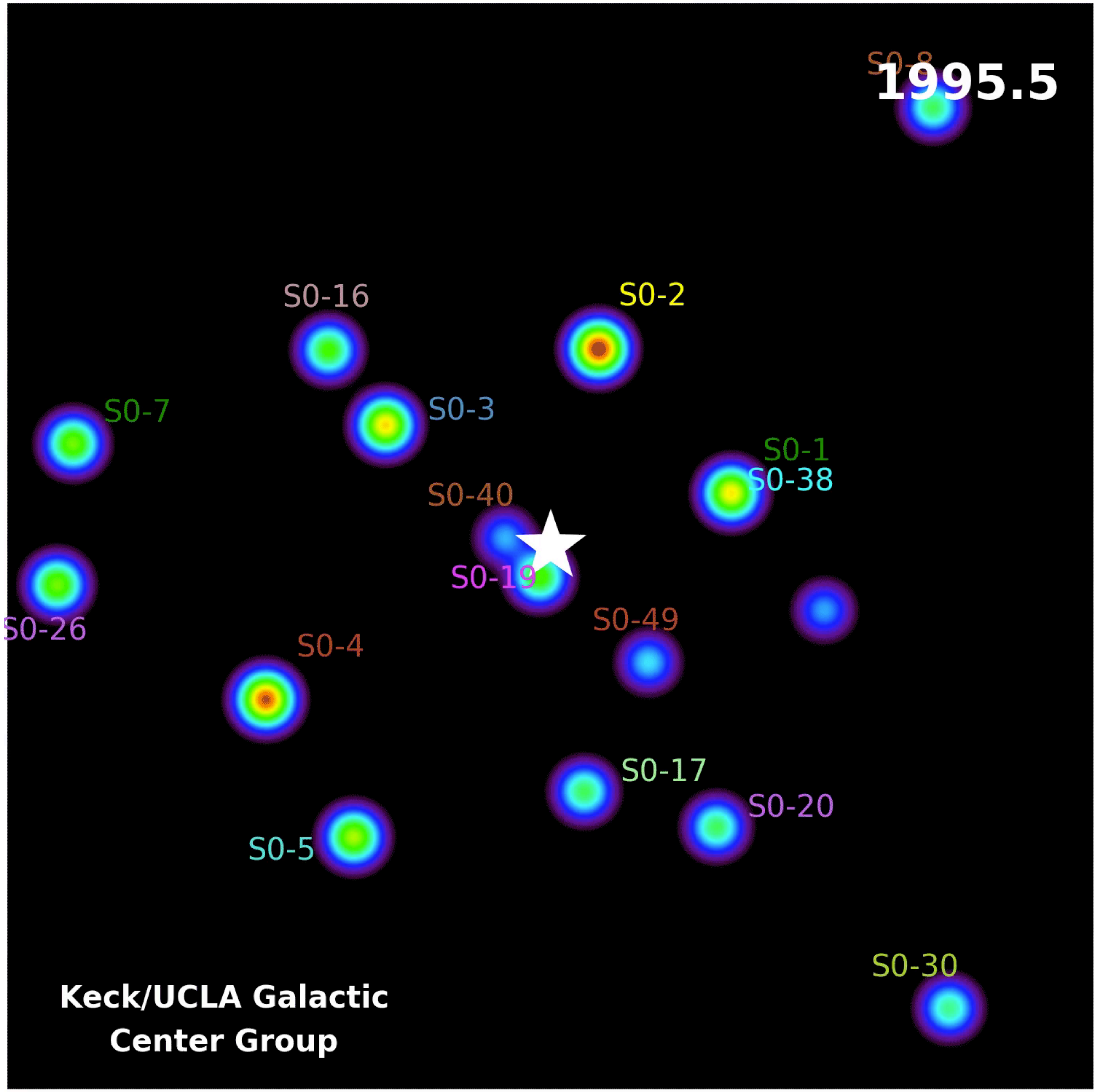
```
4.935164567082407 1e+10 Myr
```

- Therefore, collisions are only important on timescales >> the age of the Universe
- We can therefore usefully treat galaxies as smooth mass distributions

# Dynamical equilibrium

# Why dynamical equilibrium?

- We would like to understand:
    - Mass-to-light ratio of galaxies
    - Orbital structure in galaxies —> formation and evolution
    - Detect black holes at the centers of galaxies
    - Distribution of dark matter in galaxies
- Gravitational force —> mass density
- Newton’s second law: force ~ acceleration
- We cannot measure accelerations of stars.
- We observe position and velocity (phase space)


1995.5
S0-8
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S0-38
S0-40
S0-19
S0-26
S0-49
S0-4
S0-17
S0-20
S0-5
S0-30
Keck/UCLA Galactic
Center Group

# Why dynamical equilibrium?

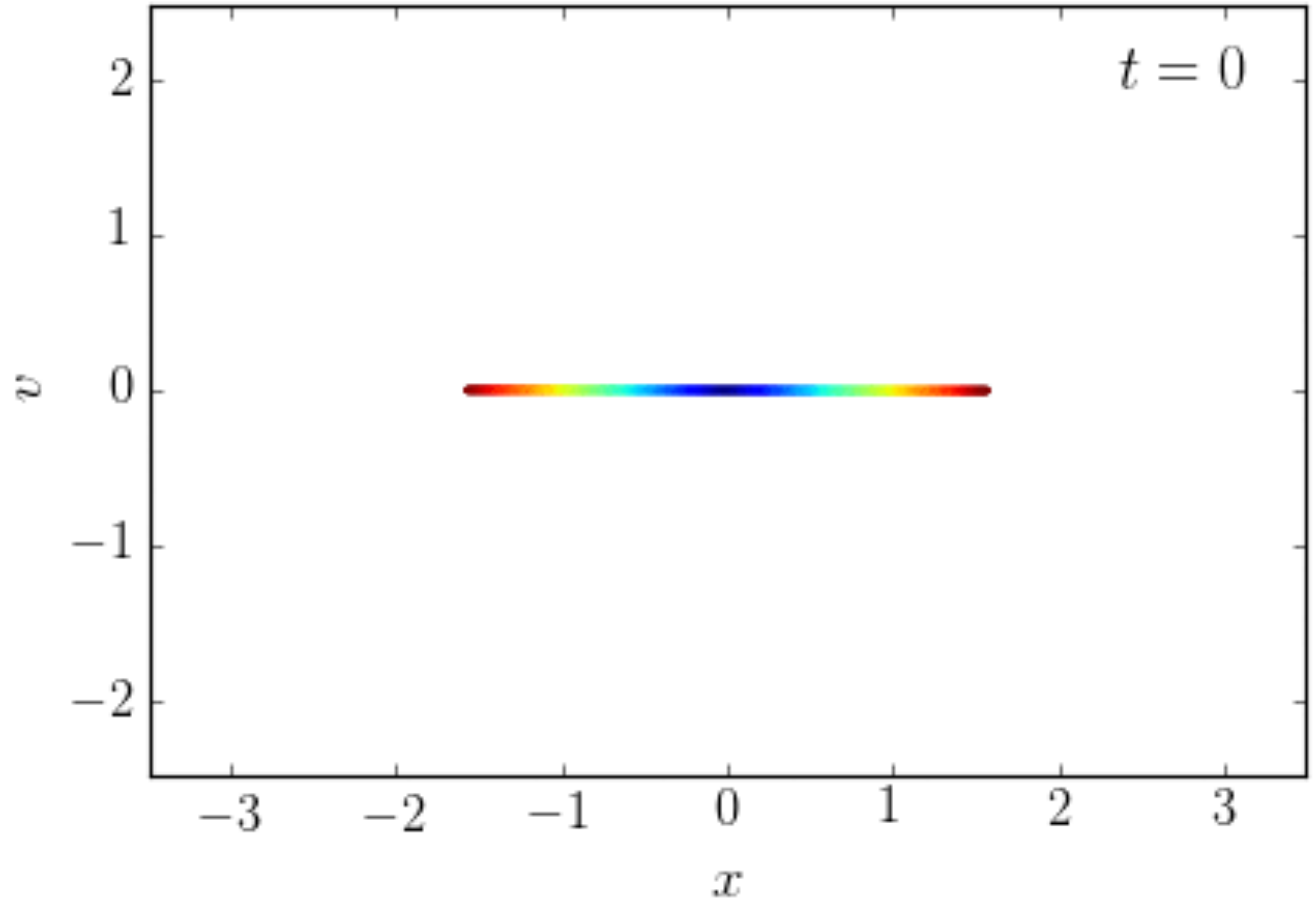
- Because we cannot measure accelerations on galactic scales, Newton's 2nd law implies that we cannot learn anything from (x,v) about the gravitational potential and mass distribution
- Thus, we need to make additional assumptions about (the distribution of) stellar orbits to relate $\rho(x)$ to (x,v)
- That a stellar system is in equilibrium is one of the most powerful and common additional assumptions
- But also many other add'l assumptions: e.g., objects at the same place in the past, objects move on circular orbits

# Do we expect galaxies to be in dynamical equilibrium?

- Relaxation time >> age of the Universe: two-body relaxation can therefore not be responsible for generating equilibrium
- But dynamical systems typically reach equilibrium condition quickly through non-collisional processes: e.g., violent relaxation and phase-mixing
- These happen on few-dozen *dynamical* times << relaxation time

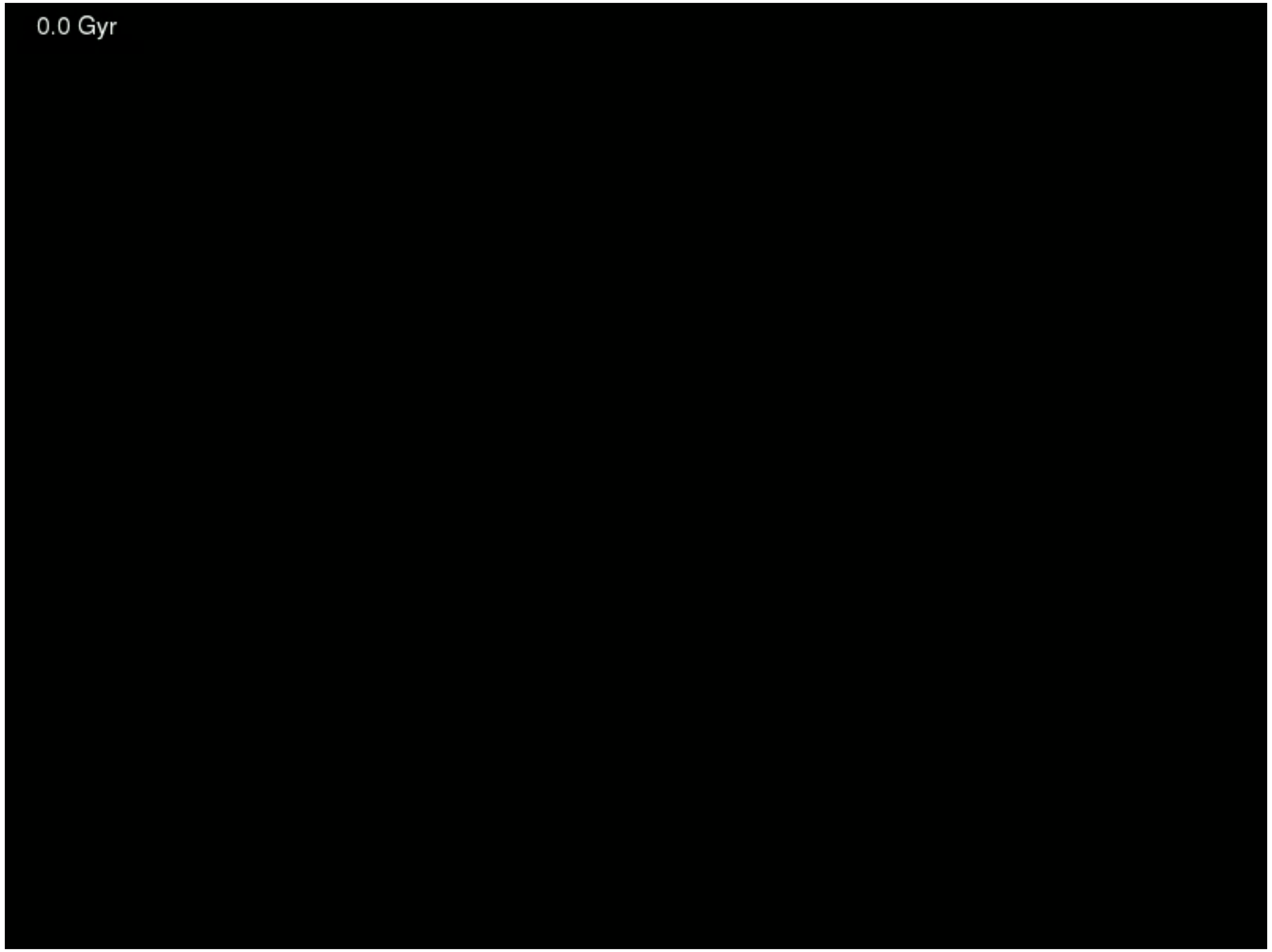
# Phase-mixing

stars on regular orbits with different frequencies drift out of phase, producing a smooth distribution in phase space even in a time-independent potential.



# Violent relaxation

Rapid evolution of the stellar distribution function caused by a rapidly time-varying gravitational potential, typically during formation or major mergers.




Credit: Greg Stinson, MUGS (http://mugs.mcmaster.ca/)

# Galaxies are in a quasi-equilibrium state

- Galaxies reach quasi-steady-state on $O(t_{dyn})$ time scale
- Much happens, but quasi-equilibrium quickly restored
- Because dynamical times increases with increasing r, central regions much closer to equilibrium than outer regions
- Outer halo has longer dynamical time: few Gyr —> equilibrium suspect
- Equilibrium: phase space distribution function f(x,v,t) = f(x,v), independent of time

# Basic understanding of galactic equilibria

- Will derive mathematical expressions relating equilibrium phase-space distribution f(x,v) to mass distribution
- All equilibrium statements basically balance kinetic energy (~velocity dispersion) with potential energy (~density distribution)
    - For given density distribution, velocities too high for balance —> system will expand, not in equilibrium
    - Velocities too low for balance —> system will collapse, not in equilibrium
- Because we can directly measure kinetic energy (~velocity dispersion), but potential energy depends on mass distribution —> requiring balance constrains mass

# Virial theorem

- Virial theorem one of the most basic expressions of dynamical equilibrium
- Relates *overall* kinetic and potential energy, so no fine-grained constraints on mass distribution
- Many different versions, can be derived for different types of forces, but focus on simple form here

# Virial theorem

- Can derive virial theorem by considering the *viral quantity G* and stating that it is conserved

$$G = \sum_{i=1}^{N} w_i \, \mathbf{x}_i \cdot \mathbf{v}_i \, ,$$

- d G / d t = 0 —>

$$\sum_{i=1}^{N} w_i \dot{\mathbf{x}}_i \cdot \mathbf{v}_i = - \sum_{i=1}^{N} w_i \, \mathbf{x}_i \cdot \dot{\mathbf{v}}_i \, ,$$

- or

$$\sum_{i=1}^{N} w_i \mathbf{v}_i \cdot \mathbf{v}_i = - \sum_{i=1}^{N} w_i \, \mathbf{x}_i \cdot \mathbf{g}(\mathbf{x}_i)$$

# Virial theorem

- For a point-mass potential: $\Phi(r) = -GM/r$

$$\sum_{i=1}^{N} w_i |\mathbf{v}_i|^2 = \sum_{i=1}^{N} w_i \frac{GM}{r_i}$$

- If $w_i = m_i$ —> LHS is twice kinetic energy, RHS = -potential energy:

$$2K = -W \quad \text{or} \quad E = -K = W/2$$

- But $w_i$ can be whatever you want!
- Mass estimator

$$M = \frac{1}{G} \frac{\sum_{i=1}^{N} w_i |\mathbf{v}_i|^2}{\sum_{i=1}^{N} w_i \frac{1}{r_i}} .$$

# Virial theorem for self-gravitating system

- Previous result was for external force/potential
- For a self-gravitating system, force is the force from all other points

$$\mathbf{F}(\mathbf{x}_i) = \sum_{j \neq i} G\, m_j\, (\mathbf{x}_i - \mathbf{x}_j)/|\mathbf{x}_i - \mathbf{x}_j|^3 .$$

- Self-gravitating system of N bodies, mass estimator

$$M = \frac{1}{G} \frac{\sum_i |\mathbf{v}_i^2|}{\boxed{\frac{1}{N} \sum_{i=1}^{N} \sum_{j<i} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}}} .$$

average inverse pairwise separation $\displaystyle\sum_{i=1}^{4} \sum_{j<i} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} = \frac{1}{r_{12}} + \frac{1}{r_{13}} + \frac{1}{r_{14}} + \frac{1}{r_{23}} + \frac{1}{r_{24}} + \frac{1}{r_{34}}$

# Mass estimators from the viral theorem

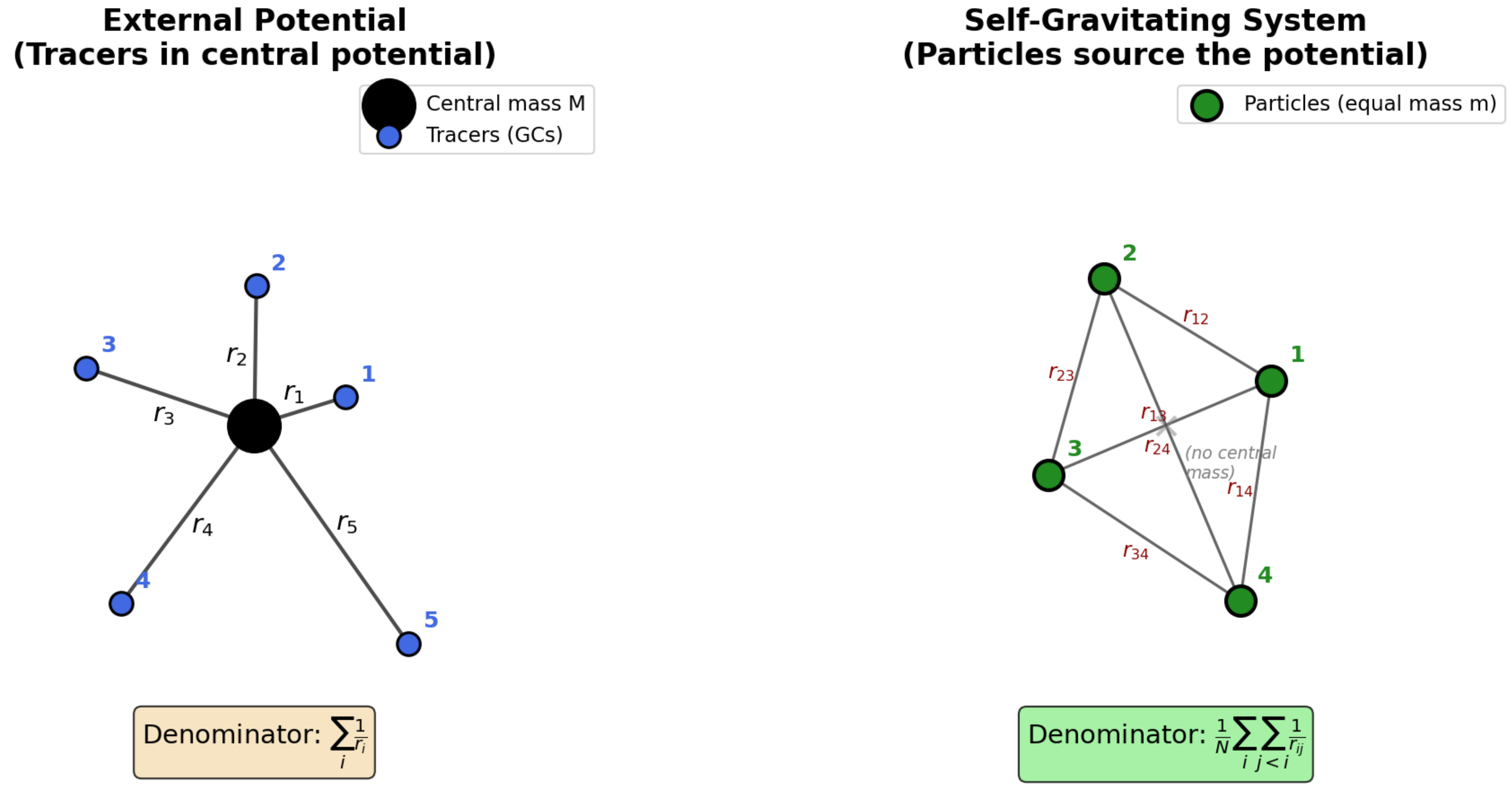
- First estimator: N tracers in an external potential

$$M = \frac{1}{G} \frac{\sum_{i=1}^{N} w_i |\mathbf{v}_i|^2}{\sum_{i=1}^{N} w_i \frac{1}{r_i}} .$$

- Second estimator: Self-gravitating system of N bodies

$$M = \frac{1}{G} \frac{\sum_i |\mathbf{v}_i^2|}{\frac{1}{N} \sum_{i=1}^{N} \sum_{j<i} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}} .$$

- Both of these can give a useful estimate of (i) the mass of a stellar system and (ii) whether or not it is self-gravitating




e.g. GCs orbiting around Milky Way

e.g. stars in a GC

# Virial theorem application 1: Mass of the Milky Way to r ~ 40 kpc

- Milky Way has ~150 globular clusters: dense stellar clusters orbiting mostly in the halo
- Can use those in the region beyond the disk to get a rough estimate of the total mass
- Select globular clusters with r > 20 kpc
- Assuming globular clusters are test particles in an external potential
- Cannot measure full **v** for all clusters, but assume vx = vy = vz

$$\sum_i |\mathbf{v}_i|^2 = 3 \sum_i v_{\rm LSR}^2.$$

# Read the data

Hide code

```
import astropy.units as u
import astropy.constants as const
import astropy.coordinates as apycoords
gcdata= harris_read()
c= apycoords.SkyCoord(ra=gcdata['RA'],
                      dec=gcdata['DEC'],
                      distance=gcdata['R_Sun'],
                      unit=(u.deg,u.deg,u.kpc),
                      frame='icrs')
gc_frame= apycoords.Galactocentric(galcen_distance=8.1*u.kpc,
                                   z_sun=25.*u.pc)
gc = c.transform_to(gc_frame)
gc.representation_type = 'spherical'
```

# Tracer estimate external potential

- With $w_i = 1$

$$M = \frac{1}{G} \frac{\sum_{i=1}^{N} w_i |\mathbf{v}_i|^2}{\sum_{i=1}^{N} w_i \frac{1}{r_i}} . \longrightarrow M = \frac{3 \sum_i v_{\mathrm{LOS},i}^2}{G \sum_i \frac{1}{r_i}}$$

```
indx= (gc.distance > 20.*u.kpc)*(True^numpy.isnan(gcdata['v_LSR']))
M_est= (3.*numpy.sum(gcdata['v_LSR'][indx]**2.)\
        *(u.km/u.s)**2\
       /numpy.sum(1./gc.distance[indx])/const.G)
print("""For a point-mass central potential,
    with %i GCs with median distance %.0f kpc
    the mass is %.2f 10^12 Msun""" % \
    (numpy.sum(indx),
     numpy.median(gc.distance[indx]).to(u.kpc).value,
     M_est.to(10**12*u.Msun).value))
```

Hide code

```
For a point-mass central potential,
    with 19 GCs with median distance 35 kpc
    the mass is 0.45 10^12 Msun
```

# Self-gravitating estimate

$$M = \frac{1}{G} \frac{\sum_i |\mathbf{v}_i|^2}{\frac{1}{N} \sum_{i=1}^{N} \sum_{j<i} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}}$$

$$\sum_i |\mathbf{v}_i|^2 = 3 \sum_i v_{\mathrm{LSR}}^2$$

Hide code

```
mutual_dist= \
 (numpy.atleast_2d(gc.distance[indx]).T-gc.distance[indx])
mutual_dist[numpy.diag_indices(numpy.sum(indx))]= 0.
mutual_dist= numpy.triu(mutual_dist)
M_est= (3.*numpy.sum(gcdata['v_LSR'][indx]**2.)*(u.km/u.s)**2\
        /numpy.sum(1./numpy.fabs(mutual_dist)[mutual_dist > 0])\
        *numpy.sum(indx)/const.G)
print("""For a self-gravitating system,
    with %i GCs with median distance %.0f kpc
    the mass is %.2f 10^12 Msun""" % \
    (numpy.sum(indx),
     numpy.median(gc.distance[indx]).to(u.kpc).value,
     M_est.to(10**12*u.Msun).value))
```

```
For a self-gravitating system,
    with 19 GCs with median distance 35 kpc
    the mass is 0.17 10^12 Msun
```

# Virial theorem application 1: Mass of the Milky Way to r ~ 40 kpc

- Both estimates give $>10^{11}$ $M_{sun}$ out to ~40 kpc
- GCs clearly not self-gravitating: each cluster would have $\sim 10^{10}$ $M_{sun}$!
- Tracer estimate: $4 \times 10^{11}$ $M_{sun}$ >> mass in disk+bulge —> dark matter
- Better estimates ~ $3 \times 10^{11}$ $M_{sun}$ so virial estimator not so bad!

# Virial theorem

- Relates *overall* kinetic and potential energy when system in equilibrium

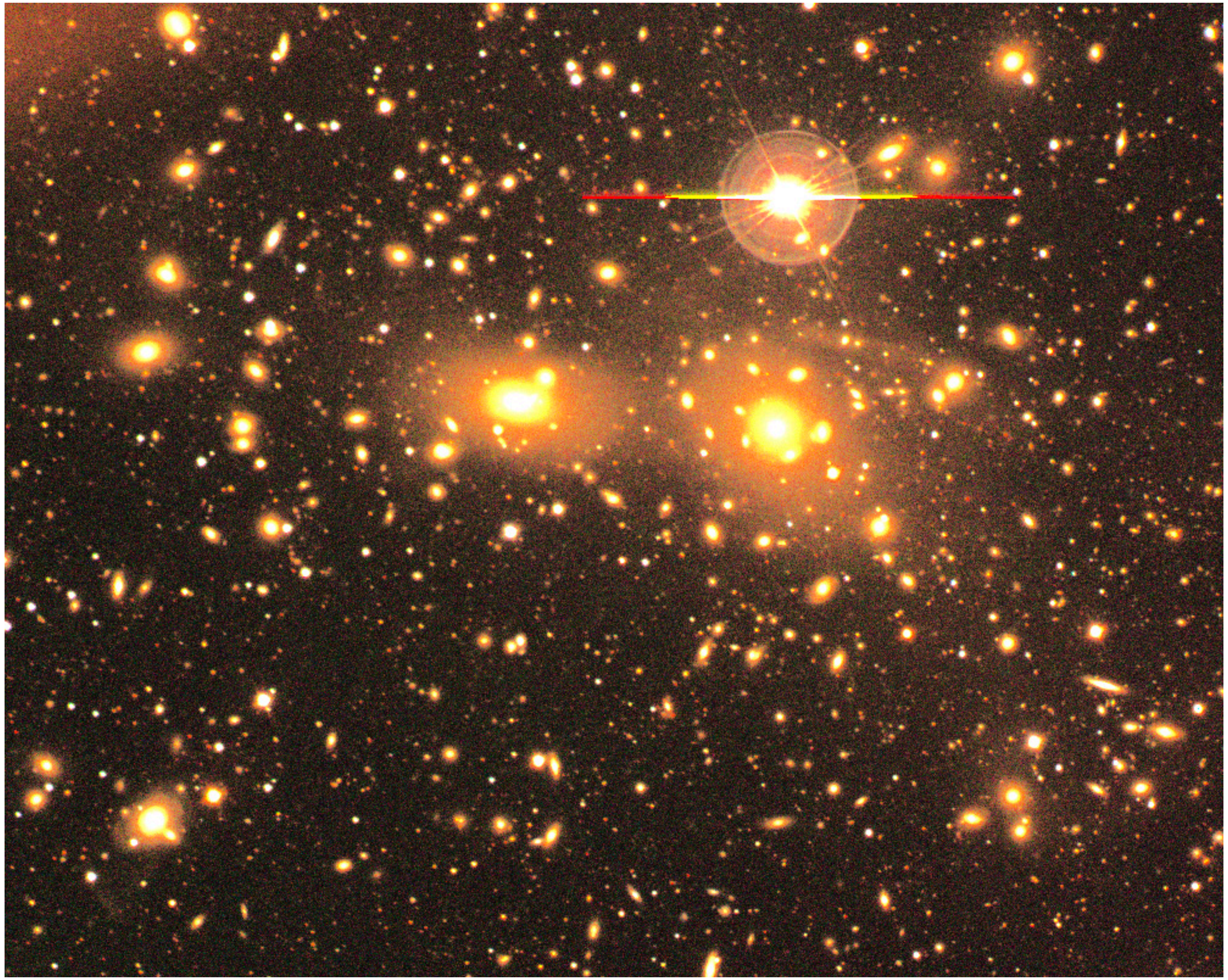
K: kinetic energy

W: potential energy

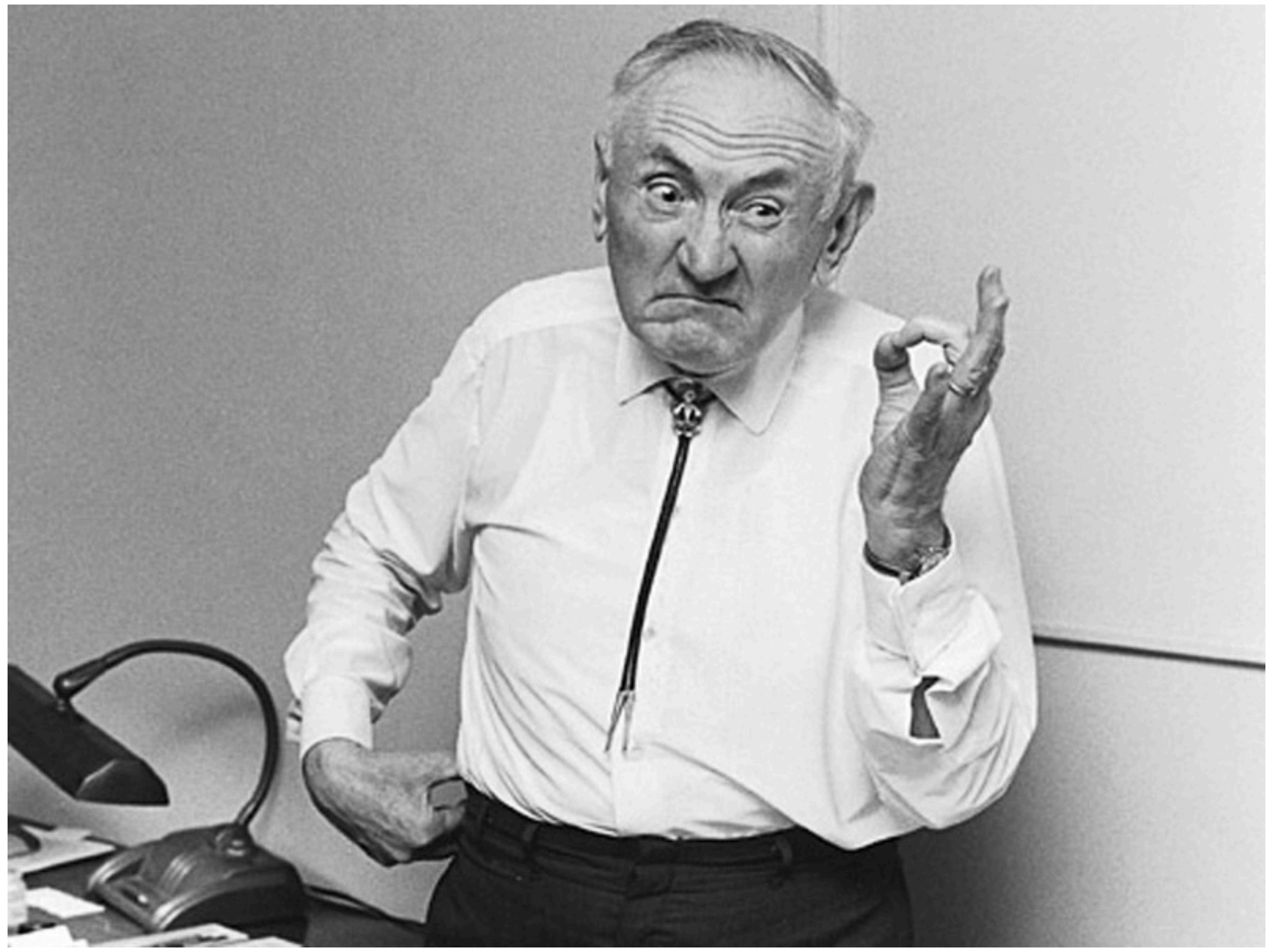
Energy $E = K+W$

$2K = -W$   or   $E = -K = W/2$

# Virial theorem application 2: Coma Cluster



Fritz Zwicky



# Virial theorem application 2: Coma Cluster

THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

VOLUME 86 OCTOBER 1937 NUMBER 3

ON THE MASSES OF NEBULAE AND OF CLUSTERS OF NEBULAE

F. ZWICKY

ABSTRACT

Present estimates of the masses of nebulae are based on observations of the *luminosities* and *internal rotations* of nebulae. It is shown that both these methods are unreliable; that from the observed luminosities of extragalactic systems only lower

homogeneous density sphere $\langle v^2 \rangle = \frac{3}{5}\frac{GM}{R}$,

Isotropic $\langle v^2 \rangle = 3\sigma_{\rm los}^2$

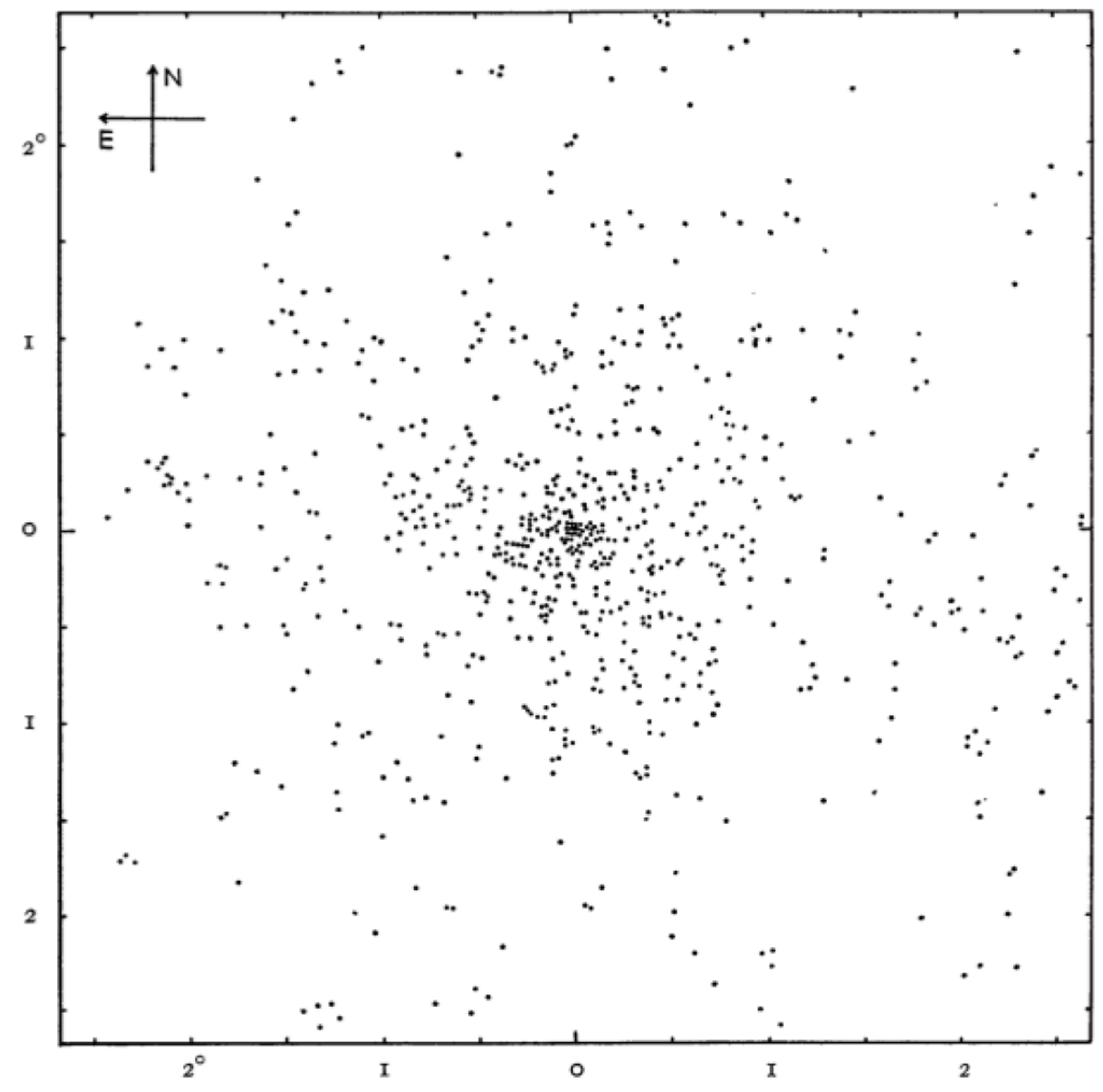



FIG. 3.—The Coma cluster of nebulae

# Virial theorem application 2: Coma Cluster

as yet unknown masses. The mass $\mathscr{M}$, as obtained from the virial theorem, can therefore be regarded as correct only in order of magnitude.

Combining (33) and (34), we find

$$\mathscr{M} > 9 \times 10^{46}\text{gr} . \tag{35}$$

The Coma cluster contains about one thousand nebulae. The average mass of one of these nebulae is therefore

$$\overline{M} > 9 \times 10^{43}\ \text{gr} = 4.5 \times 10^{10}\ M_{\odot} . \tag{36}$$

Inasmuch as we have introduced at every step of our argument inequalities which tend to depress the final value of the mass $\mathscr{M}$, the foregoing value (36) should be considered as the lowest estimate for the average mass of nebulae in the Coma cluster. This result is somewhat unexpected, in view of the fact that the luminosity of an average nebula is equal to that of about $8.5 \times 10^7$ suns. According to (36), the conversion factor $\gamma$ from luminosity to mass for nebulae in the Coma cluster would be of the order

$$\gamma = 500 , \tag{37}$$

as compared with about $\gamma' = 3$ for the local Kapteyn stellar system.

homogeneous density sphere $\quad \langle v^2 \rangle = \dfrac{3}{5} \dfrac{GM}{R} ,$

Isotropic $\quad \langle v^2 \rangle = 3\sigma_{\text{los}}^2$

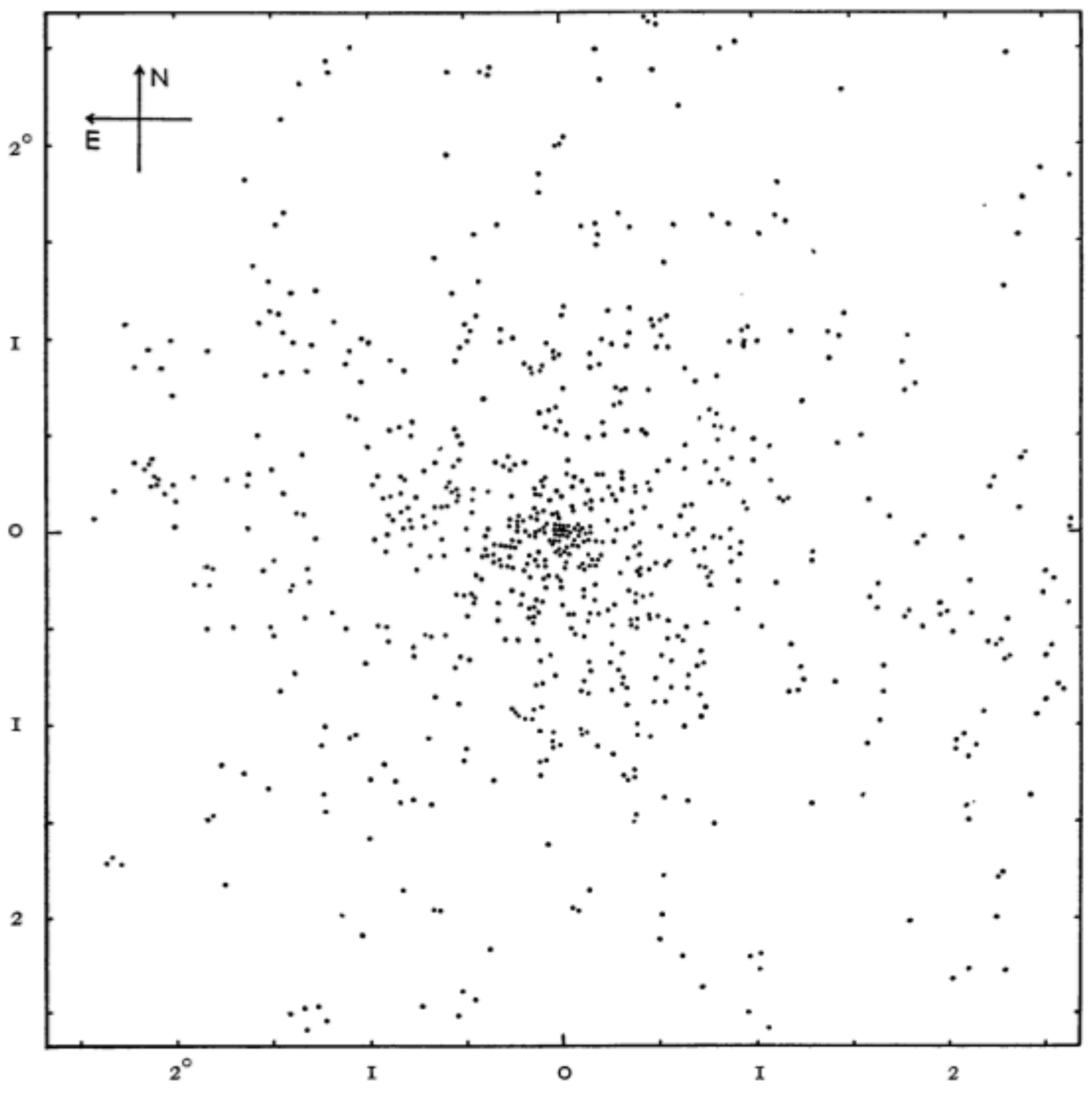



FIG. 3.—The Coma cluster of nebulae

# Break?

# The collisionless Boltzmann equation

# Phase-space distribution function

- So far have mostly dealt with individual stars and their orbits (**x**,**v**)[t]
- Often more interested in dynamical evolution of a population of stars
- Populations are described by *distribution functions* f(**x**,**v**,t), or DF
- Density of stars in phase space: $dN = f(\mathbf{x}, \mathbf{v}, t)\, d^3\mathbf{x}\, d^3\mathbf{v}$

"How crowded is it at point (x, v)?"

# Phase-space distribution function for a collisionless system

- For system of N point masses, distribution function is technically the *joint* distribution of all N phase-space points

$$f^{(N)}(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_N, t) \qquad\qquad \mathbf{w} = (\mathbf{x},\mathbf{v})$$

- For a collisionless system, presence of star at $\mathbf{w}_1$ does not affect whether or not a star is present at $\mathbf{w}_2$

- Moreover, individual identity of stars is unimportant and the distribution is invariant under $\mathbf{w}_1$ <—> $\mathbf{w}_2$

$$f^{(N)}(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_N, t) = \prod_{i=1}^{N} f(\mathbf{w}_i, t)\,,$$

# The Goal: How Does f Evolve?
## e.g. follow a star along its orbit

Conservation of Stars (Continuity Equation)

$$\frac{d}{dt}(\text{number of stars in } \mathscr{V}) = -\,(\text{flow of stars out through surface})$$

$$\int_{\mathscr{V}} \frac{\partial f}{\partial t}\, d\mathbf{w} = -\oint_S f\, \dot{\mathbf{w}}_\perp\, dS$$

### The collisionless Boltzmann equation (CBE)

Cartesian coordinates $$\frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \dot{\mathbf{x}}\, \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}}\, \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{v}} = 0\,,$$

Canonical coordinates $$\frac{\partial f(\mathbf{q}, \mathbf{p}, t)}{\partial t} + \dot{\mathbf{q}}\, \frac{\partial f(\mathbf{q}, \mathbf{p}, t)}{\partial \mathbf{q}} + \dot{\mathbf{p}}\, \frac{\partial f(\mathbf{q}, \mathbf{p}, t)}{\partial \mathbf{p}} = 0\,,$$

# The *equilibrium* collisionless Boltzmann equation

- Collisionless Boltzmann equation (CBE) holds for any collisionless distribution function

- Equilibrium: f(**x**,**v**,t) = f(**x**,**v**), DF is independent of time, CBE —>

$$\dot{\mathbf{q}}\,\frac{\partial f(\mathbf{q},\mathbf{p})}{\partial \mathbf{q}} + \dot{\mathbf{p}}\,\frac{\partial f(\mathbf{q},\mathbf{p})}{\partial \mathbf{p}} = 0\,.$$

- Fundamental equation of dynamical equilibria of galaxies

# Liouville theorem

- Consider how the phase-space distribution f(**x**,**v**,t) changes along an orbit in the gravitational potential

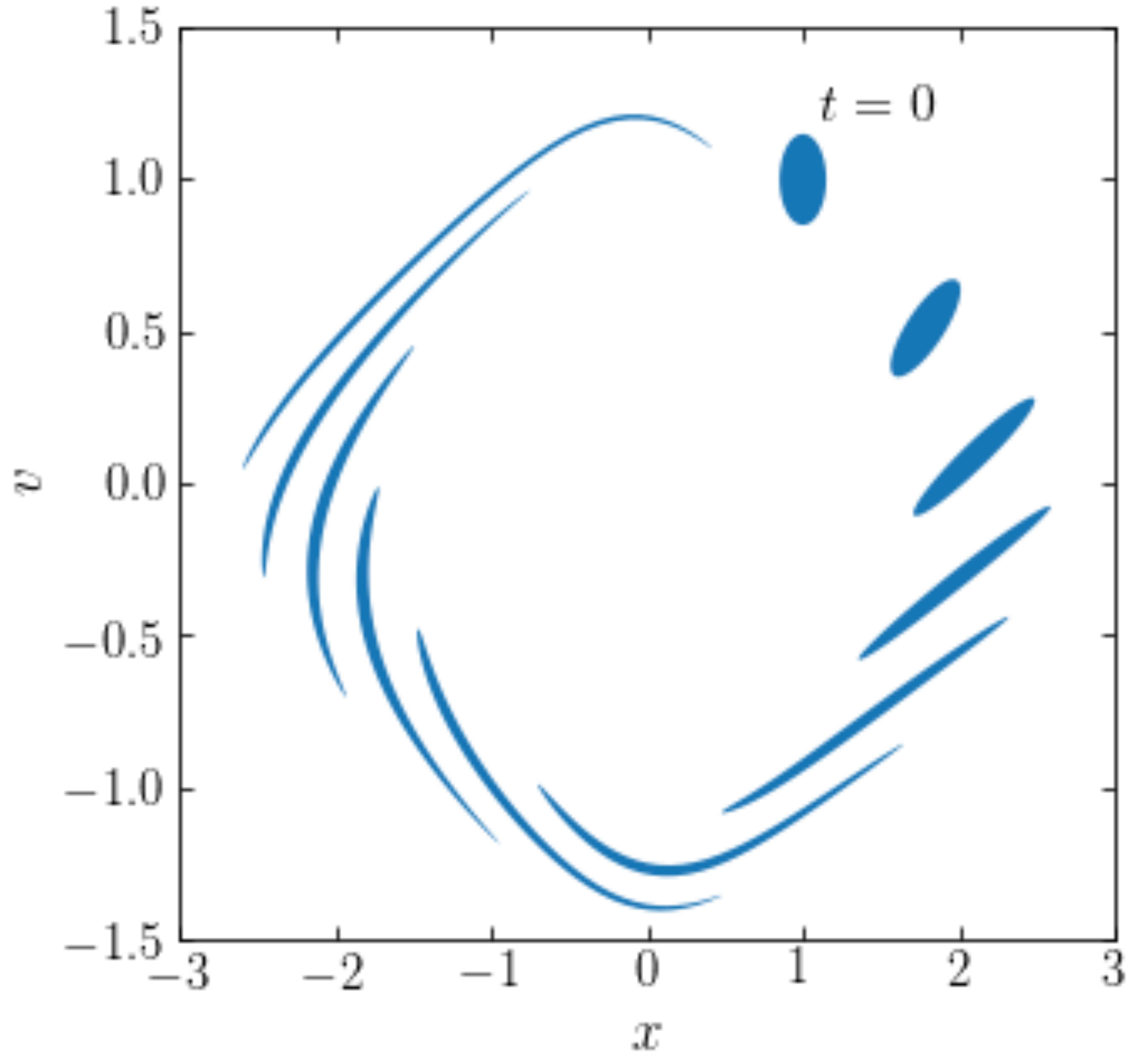
$$\frac{\mathrm{d}f(\mathbf{x},\mathbf{v})}{\mathrm{d}t} = \frac{\partial f(\mathbf{x},\mathbf{v})}{\partial t} + \dot{\mathbf{x}}\,\frac{\partial f(\mathbf{x},\mathbf{v})}{\partial \mathbf{x}} + \dot{\mathbf{v}}\,\frac{\partial f(\mathbf{x},\mathbf{v})}{\partial \mathbf{v}}$$
$$= 0\,,$$

- Liouville’s theorem: phase-space density is conserved along orbits.

- Start with little patch of stars in Δ(**x**,**v**) —> *phase-space* density conserved along orbit. If you ride along with a star, the local phase-space density around you stays constant.

- Note: density in **x** and **v** separately is *not* conserved: system can, e.g., contract radially, but needs to cover wider range of velocities to make up for this

# Liouville theorem



# Liouville theorem

## What IS conserved

- 6D phase-space volume
- Phase-space density along each orbit
- Total area of blob (in 2D)

$$\Delta^3 x \cdot \Delta^3 v = \text{constant}$$

## What is NOT conserved

- Shape of the blob
- Spatial extent $\Delta x$ alone
- Velocity extent $\Delta v$ alone
- Spatial density $\rho(x)$

A blob can stretch into a thin spiral — both $\Delta x$ and $\Delta v$ increase — while keeping the same total area.

# Two Ways to Measure Change

## $\partial f/\partial t$

**Partial Derivative**

Stand at a fixed point (x, v) in phase space. Watch stars flow past.

**Equilibrium CBE (when in equilibrium)**

$$v \cdot \partial f/\partial x - \nabla\Phi \cdot \partial f/\partial v = 0$$

(when $\partial f/\partial t = 0$)

## $df/dt$

**Total Derivative**

Ride along with a star. See how density around you changes.

**Liouville / CBE (always)**

$$\partial f/\partial t + v \cdot \partial f/\partial x - \nabla\Phi \cdot \partial f/\partial v = 0$$

(equivalent to $df/dt = 0$)

# Key Takeaways

- Liouville's theorem ($df/dt = 0$) is always true for collisionless systems
- The CBE is just Liouville written out using the chain rule
- Equilibrium additionally requires $\partial f/\partial t = 0$ (no explicit time dependence)
- Phase-space volume is conserved, but shape can change dramatically

$$\frac{\mathrm{d}f(\mathbf{x}, \mathbf{v})}{\mathrm{d}t} = \frac{\partial f(\mathbf{x}, \mathbf{v})}{\partial t} + \dot{\mathbf{x}}\,\frac{\partial f(\mathbf{x}, \mathbf{v})}{\partial \mathbf{x}} + \dot{\mathbf{v}}\,\frac{\partial f(\mathbf{x}, \mathbf{v})}{\partial \mathbf{v}}$$
$$= 0\,,$$

# Jeans equations

*James Hopwood Jeans*

# Moments of the distribution function

- Observationally often easier to measure moments of the distribution function
- Tracer number density (zeroth-moment of distribution function), unit: #/pc$^3$

$$\nu(\mathbf{x}) = \int \mathrm{d}\mathbf{v}\, f(\mathbf{x}, \mathbf{v})\,,$$

- Mean velocity (first-moment) — 3 velocities, unit: km/s

$$\bar{\mathbf{v}}(\mathbf{x}) = \frac{1}{\nu(\mathbf{x})} \int \mathrm{d}\mathbf{v}\, \mathbf{v} f(\mathbf{x}, \mathbf{v})\,,$$

- Velocity dispersion (second-moment), unit: (km/s)$^2$ — 6 velocity dispersion

$$\sigma_{ij}(\mathbf{x}) = \frac{1}{\nu(\mathbf{x})} \int \mathrm{d}\mathbf{v}\, (v_i - \bar{v}_i)(v_j - \bar{v}_j) f(\mathbf{x}, \mathbf{v})\,.$$

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix}$$

# What Do We Want to Measure?

**GIVEN**

Gravitational potential $\Phi(x)$
(from mass distribution $\rho$)

Tracer density $\nu(x)$
(from star counts, surface brightness)

**SOLVE FOR Kinematics**

Mean velocity $\bar{v}(x)$
3 components

Velocity dispersion $\sigma_{ij}(x)$
6 independent components

**Key Insight: $\nu \neq \rho$**

The tracer density $\nu$ (e.g., K-giants) need not equal the mass density $\rho$ (stars + dark matter). The Jeans equation relates $\nu$, kinematics, and $\Phi$ — letting you infer $M(<r)$ from visible tracers probing invisible mass.

# Jeans equations: moments of the CBE

- We can derive relations between the moments of the distribution function and the mass distribution by taking moments of the CBE

- First Jeans equation: Integrate CBE over all velocities

$$\int \mathrm{d}^3\mathbf{v}\,\frac{\partial f}{\partial t} + \int \mathrm{d}^3\mathbf{v}\,v_i\frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\int \mathrm{d}^3\mathbf{v}\,\frac{\partial f}{\partial v_i} = 0,$$

- Gives continuity equation of density

$$\frac{\partial \nu}{\partial t} + \nabla \cdot (\nu\bar{\mathbf{v}}) = 0 \qquad \text{or} \qquad \frac{\partial \nu}{\partial t} + \frac{\partial(\nu\bar{v}_x)}{\partial x} + \frac{\partial(\nu\bar{v}_y)}{\partial y} + \frac{\partial(\nu\bar{v}_z)}{\partial z} = 0$$

1 equation, 3 unknowns (mean velocities)

# Jeans equations: moments of the CBE

- 2nd Jeans equation: multiplying CBE by a component $v_j$ (j=x, y, z) of the velocity and integrate over all velocities gives

$$\nu \frac{\partial \overline{v}_j}{\partial t} + \nu \overline{v}_i \frac{\partial \overline{v}_j}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial (\nu \sigma_{ij}^2)}{\partial x_i}.$$

1+3 equations, 3+6 unknowns (6 velocity dispersions)

# Jeans equations: closure

- Jeans equations do not close in the following sense:
- Given potential $\phi$ and number density of population $\nu$, the first Jeans equation gives a single constraint/equation on the 3D mean velocity
- The second Jeans equation gives 3 more constraints/equations, but introduces the 6D dispersion tensor, and we thus only have 1+3 equations for 3+6 unknowns (mean velocity and dispersion)
- We could derive higher-order Jeans equations, but these always involve unknown higher-order moments
- Because the number of unknowns grows faster than the number of equations, the Jeans equations do not close
- Thus, potential $\phi(\mathbf{x})$ and density of population $\nu(\mathbf{x})$ do not suffice to derive a unique equilibrium distribution function $f(\mathbf{x},\mathbf{v})$

# Jeans equations: closure

- Modeling of observational data using the Jeans equations requires some “closure assumption”
- For example, if we assume that the system is isotropic, the dispersion tensor = $\sigma^2$ **I**, and we have 4 equations for 4 unknowns —> unique distribution function!
- In the Milky Way not necessarily required because we can measure all moments of the distribution function directly

# Spherical Jeans equations

- Now look at Jeans equations for spherical system

- Derived from collisionless Boltzmann equation in spherical coordinates

$$\frac{\mathrm{d}(\nu\,\overline{v_r^2})}{\mathrm{d}r} + \nu\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \frac{2\overline{v_r^2} - \overline{v_\theta^2} - \overline{v_\phi^2}}{r}\right) = 0\,.$$

# Orbital Anisotropy β

- To better appreciate the role of *orbital anisotropy*, we introduce a parameter to give the anisotropy

$$\beta \equiv 1 - \frac{\sigma_\theta^2 + \sigma_\phi^2}{2\,\sigma_r^2}$$
$$= 1 - \frac{\overline{v_\theta^2} + \overline{v_\phi^2}}{2\,\overline{v_r^2}}$$

$$\sigma_r^2 = \overline{v_r^2} - \bar{v}_r^2$$

$\sigma_r^2 = \overline{v_r^2}$ when $\bar{v}_r^2 = 0$

- β quantifies balance of radial motions and tangential motions
    - Radially biased: $\sigma_r >> \sigma_\theta, \sigma_\phi$: $\beta \rightarrow 1$
    - Isotropic: $\sigma_r = \sigma_\theta = \sigma_\phi$: $\beta = 0$
    - Tangentially biased: $\sigma_r << \sigma_\theta, \sigma_\phi$: $\beta \rightarrow -\infty$
- Typically depends on radius $\beta = \beta(r)$

# Spherical Jeans equations

$$\frac{\mathrm{d}(\nu \overline{v_r^2})}{\mathrm{d}r} + \nu \left( \frac{\mathrm{d}\Phi}{\mathrm{d}r} + \frac{2\overline{v_r^2} - \overline{v_\theta^2} - \overline{v_\phi^2}}{r} \right) = 0 \,.$$

- in terms of β

$$\frac{\mathrm{d}(\nu \overline{v_r^2})}{\mathrm{d}r} + 2\frac{\beta}{r}\,\nu \overline{v_r^2} = -\nu \frac{\mathrm{d}\Phi}{\mathrm{d}r} \,.$$

**Still not closed: 1 equation, 2 unknowns ($\sigma_r$ and β)**

- In terms of enclosed mass

$$\frac{d\Phi}{dr} = \frac{GM(<r)}{r^2} \longrightarrow M(<r) = -\frac{r\,\sigma_r^2}{G}\left(\frac{\mathrm{d}\ln(\nu\,\sigma_r^2)}{\mathrm{d}\ln r} + 2\beta\right)$$

# Spherical Jeans equations —> mass

$$M(<r) = -\frac{r\,\sigma_r^2}{G}\left(\frac{\mathrm{d}\ln(\nu\,\sigma_r^2)}{\mathrm{d}\ln r} + 2\,\beta\right)$$

- If we can measure the velocity dispersion $\sigma_r^2$ and stellar density $\nu$ as a function of r —> M(r)

- However, if we cannot measure β then assumed β has large effect on inferred

Scenario A

**Larger mass**

Isotropic orbits (β = 0)

Scenario B

**Smaller mass**

Radially biased (β > 0)

*mass-anisotropy degeneracy*

# How to Close the System

## Option 1: Assume β

- $\beta = 0$ (isotropy)
- $\beta =$ constant
- $\beta(r)$ parametric form

## Option 2: Measure β

- Full 6D phase space!
- Measure v in 3D directly
- Compute β from data

### Summary: When is the system closed?

| Setup | Eqns | Unknowns | Closed? |
|---|---|---|---|
| General 3D Jeans | 4 | 9 | No |
| Spherical Jeans | 1 | 2 ($\sigma_r$, $\beta$) | No |
| Spherical + assumed β | 1 | 1 | Yes ✓ |
| Spherical + 3D velocities | 1 | 1 | Yes ✓ |

# Key Takeaways

Jeans equations are moments of the CBE — they lose information about f(x,v)

The hierarchy never closes: each moment adds more unknowns than equations

Spherical Jeans still has 2 unknowns ($\sigma_r$ and β) — not closed!

Mass-anisotropy degeneracy: same $\sigma_r$ from different (M, β) combinations

$\nu \neq \rho$: visible tracers probe invisible mass → powerful for dark matter

Full 6D phase space (Gaia + spectroscopy) breaks the degeneracy

# The mass of the Milky Way’s dark matter halo

## Mass of the Milky Way: spherical Jeans equation

- We have that:

$$M(<r) = -\frac{r\,\sigma_r^2}{G}\left(\frac{\mathrm{d}\ln(\nu\,\sigma_r^2)}{\mathrm{d}\ln r} + 2\,\beta\right)$$

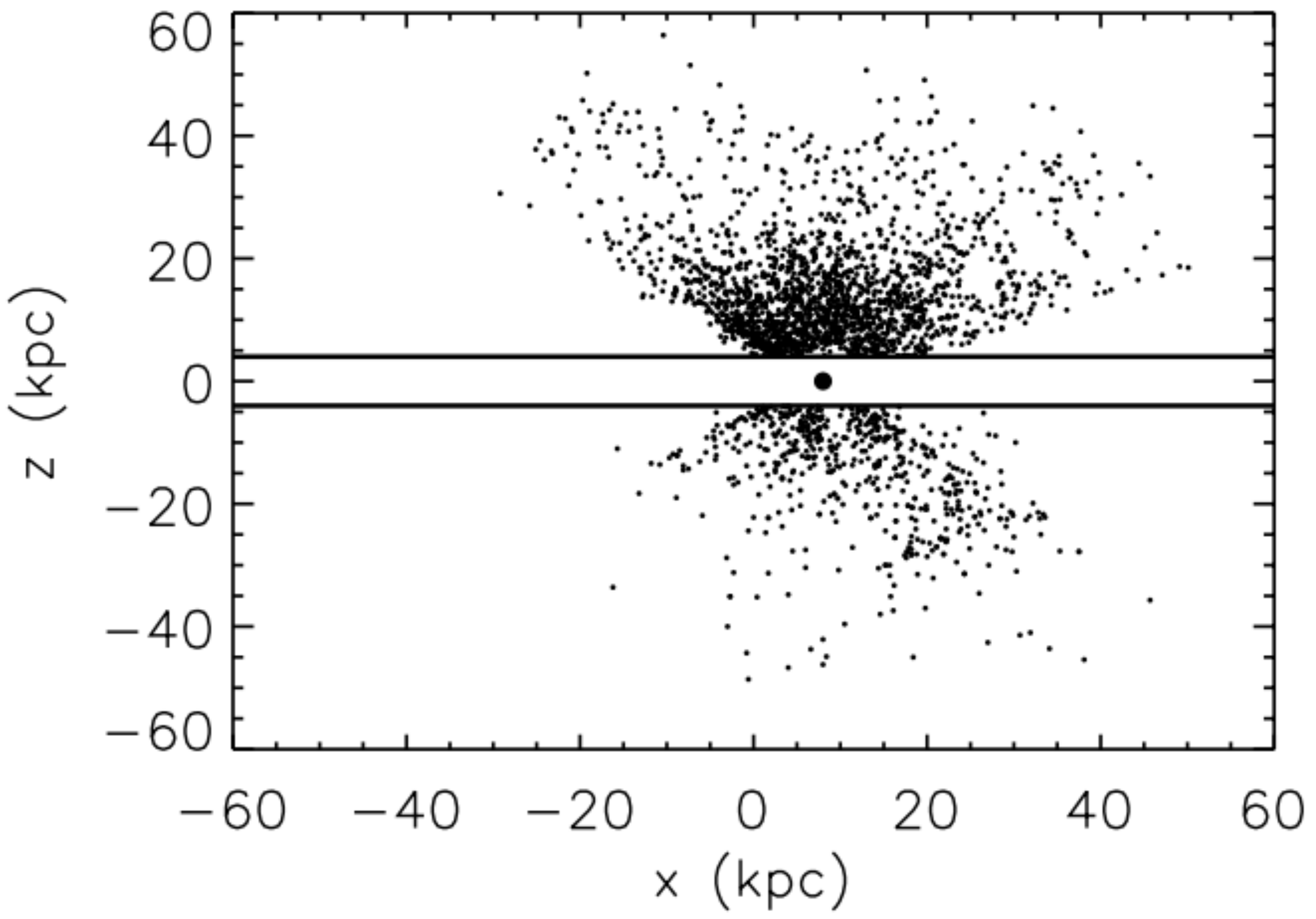
- In terms of the circular velocity

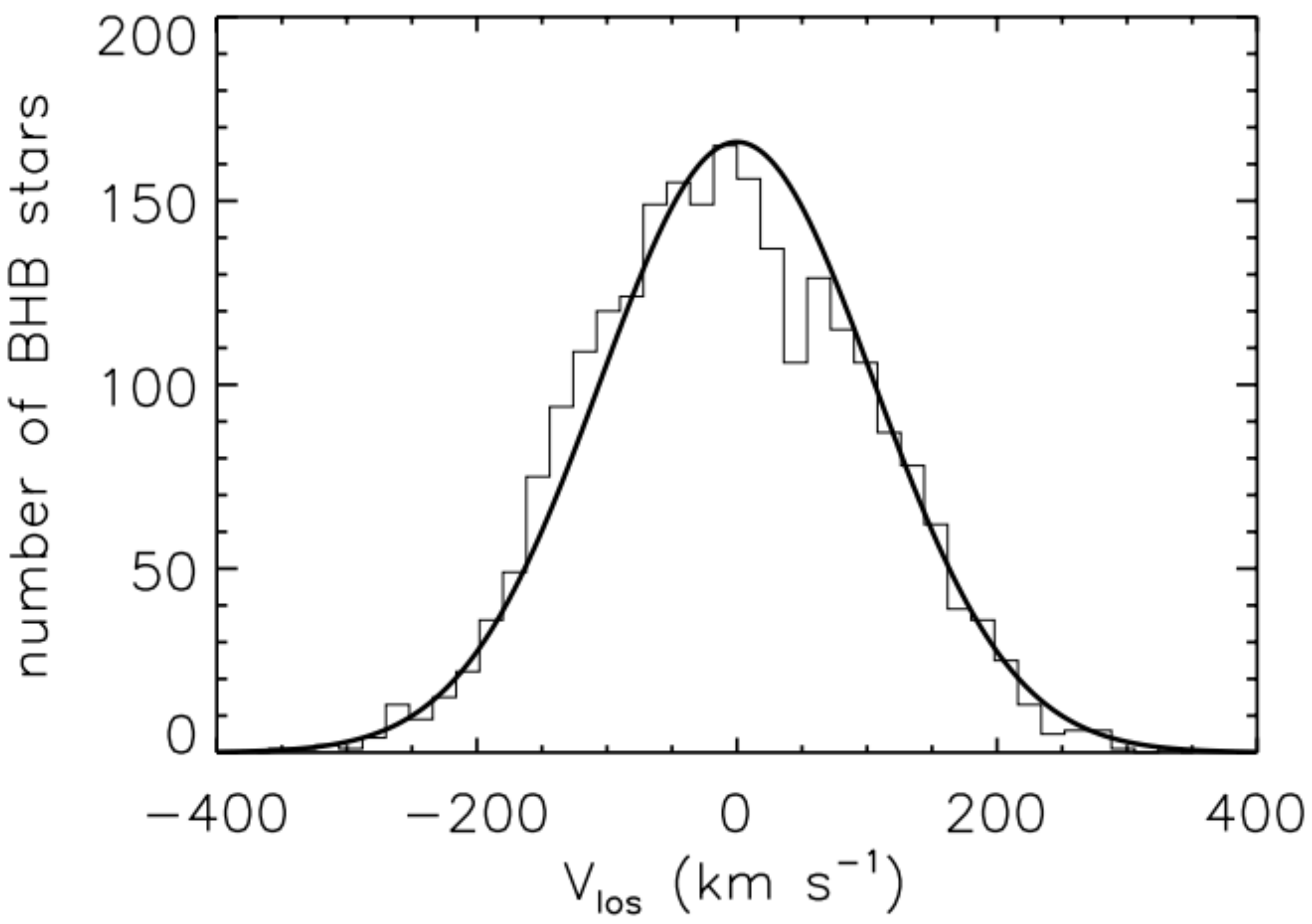
$$\frac{d\Phi}{dr} = \frac{GM(<r)}{r^2} = \frac{v_c^2}{r}$$

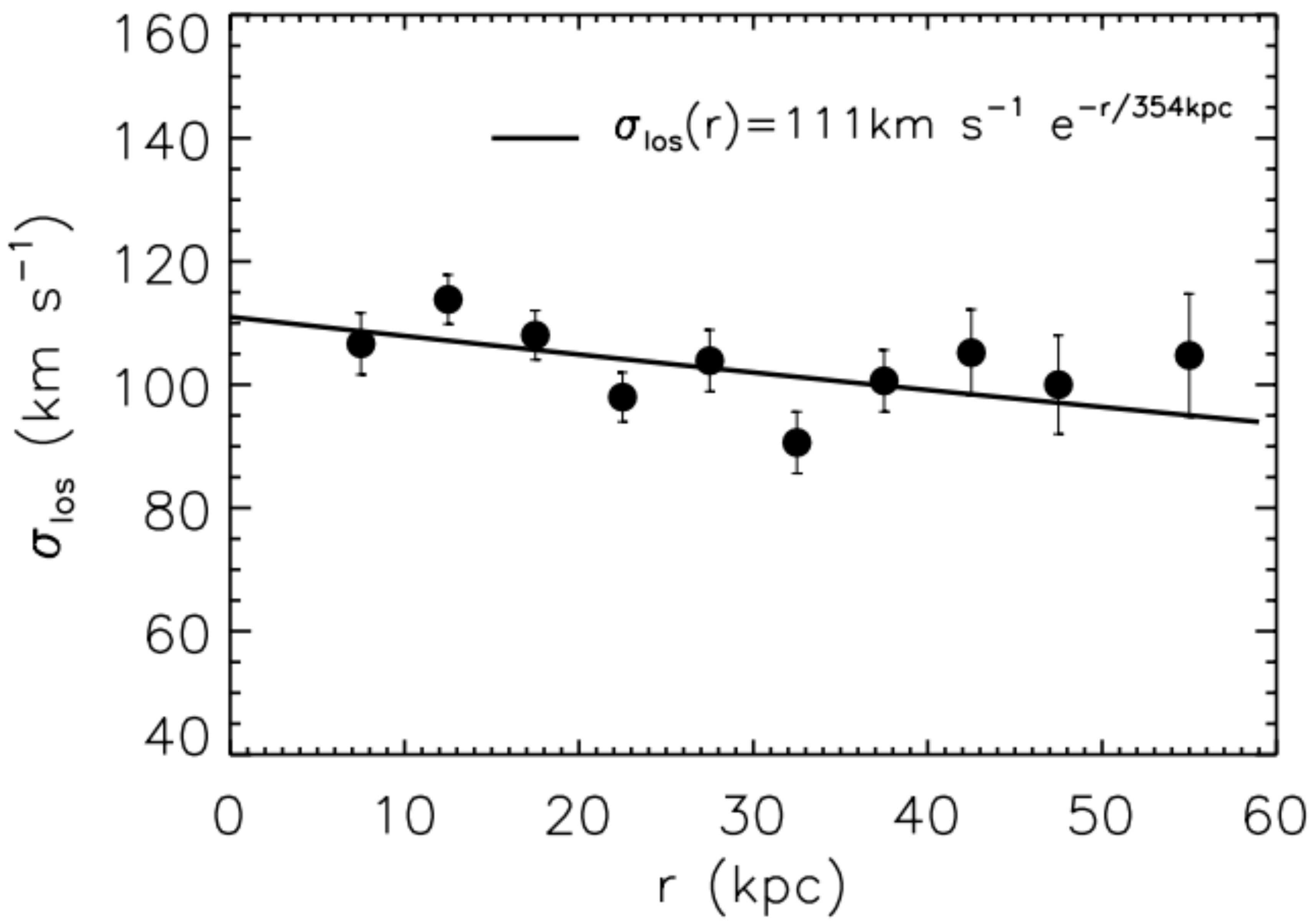
$$v_c^2(r) = -\sigma_r^2\left(\frac{\mathrm{d}\ln(\nu\,\sigma_r^2)}{\mathrm{d}\ln r} + 2\,\beta\right)$$

- Thus, can use measurements of density $\nu$ and velocity dispersion $\sigma_r^2$ to measure the "rotation curve"

Xue et al. (2008)



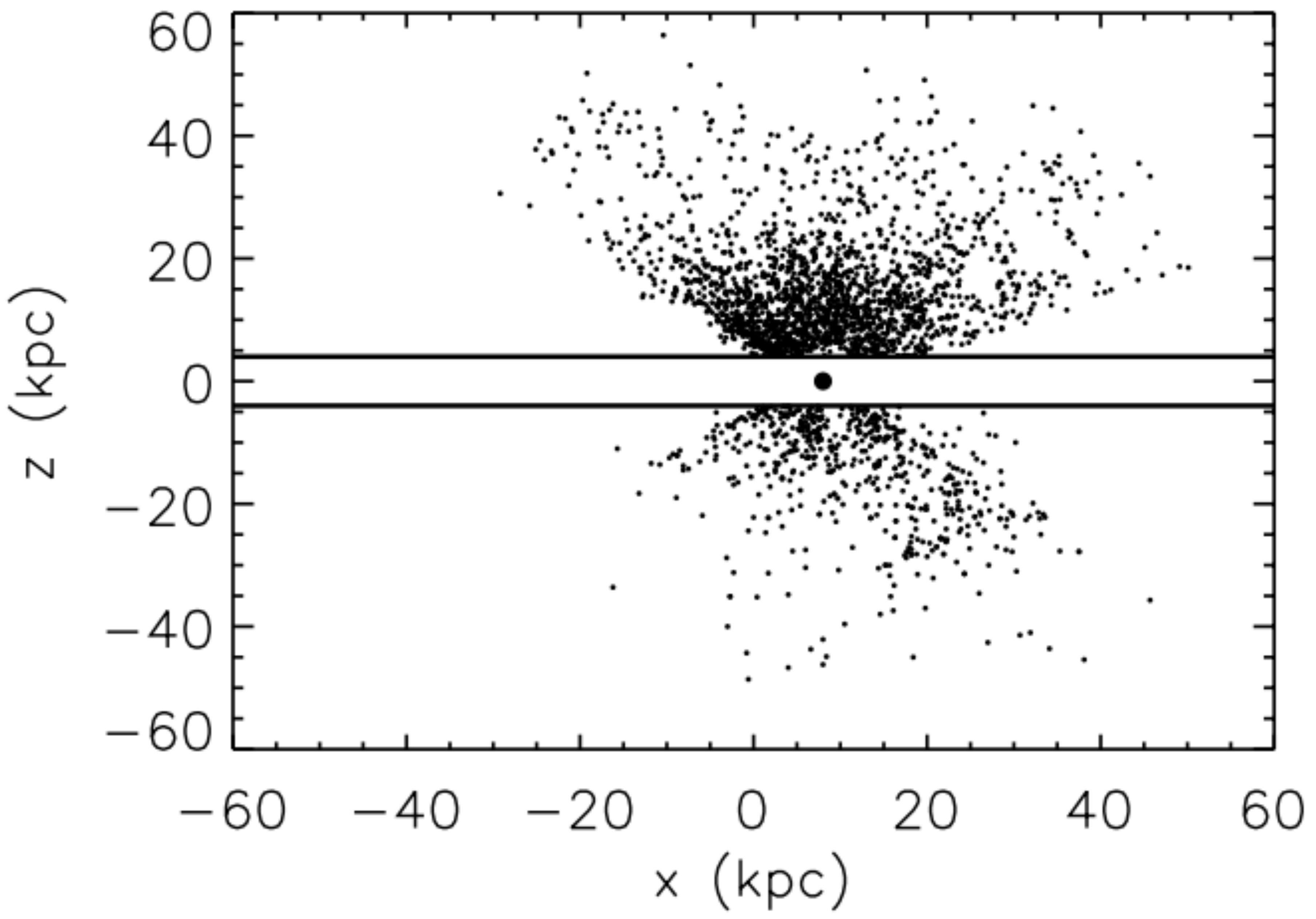
# Spherical Jeans ingredients

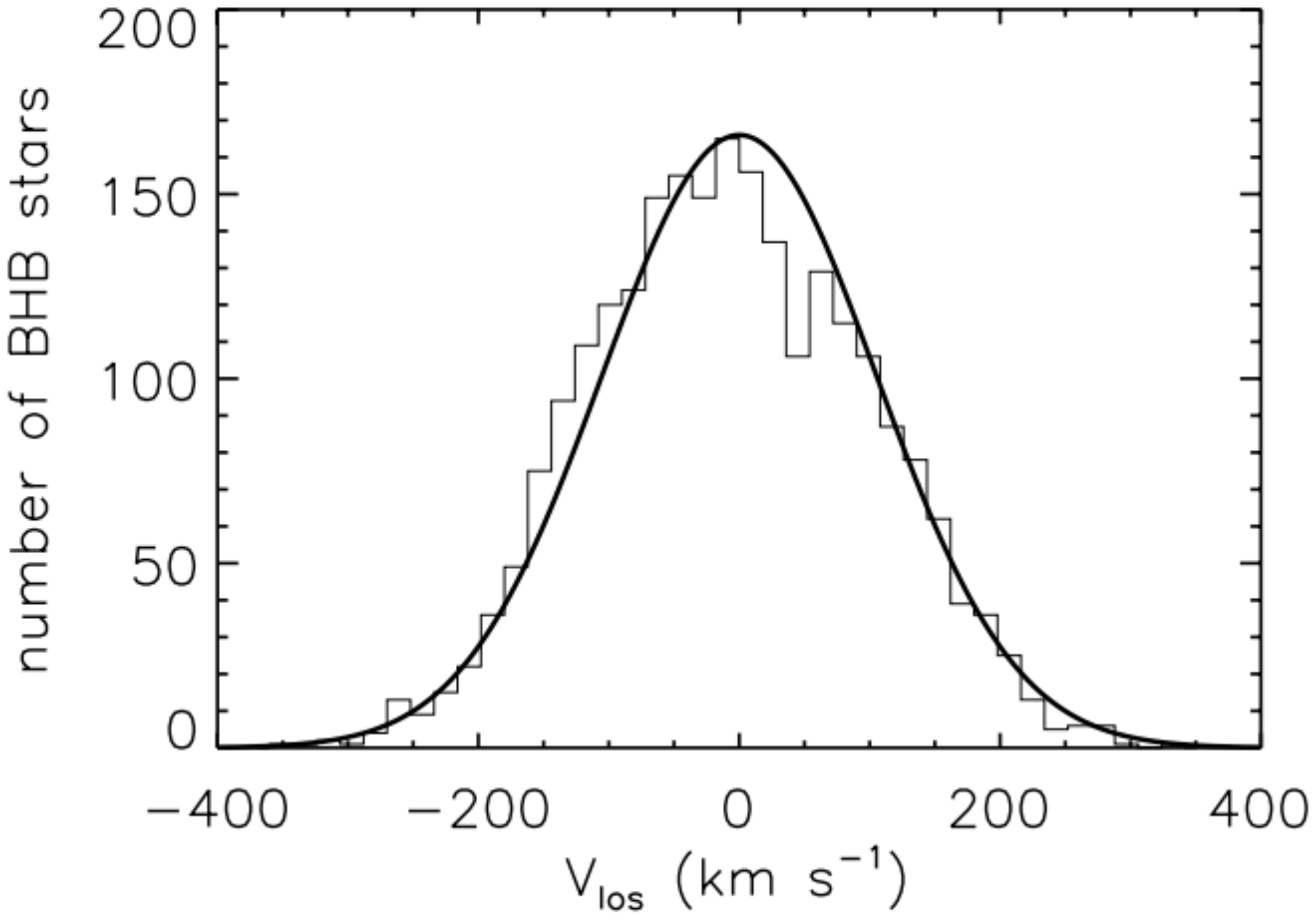
- Stellar density: $\nu \sim 1/r^{3.5}$ (e.g., Bell et al. 2008)

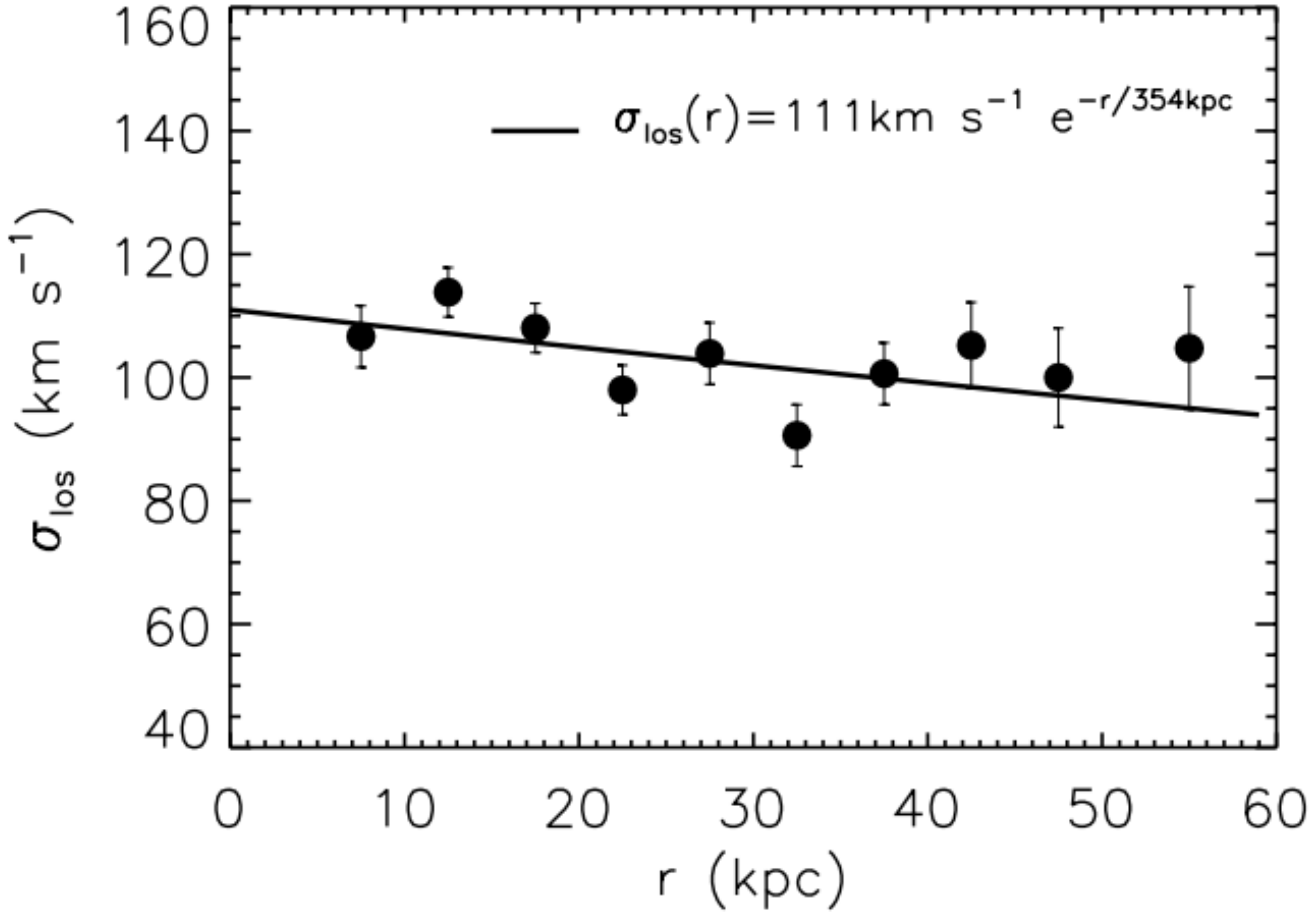
- Velocity dispersion $\sigma_r(r) \approx \sigma_{\rm los}(r).$

$$\sigma_{\rm los}(r) = \left(111 \pm 1\,{\rm km\,s^{-1}}\right)\,\exp\left(-\frac{r}{354^{+91}_{-60}\,{\rm kpc}}\right)$$

Xue et al. (2008)



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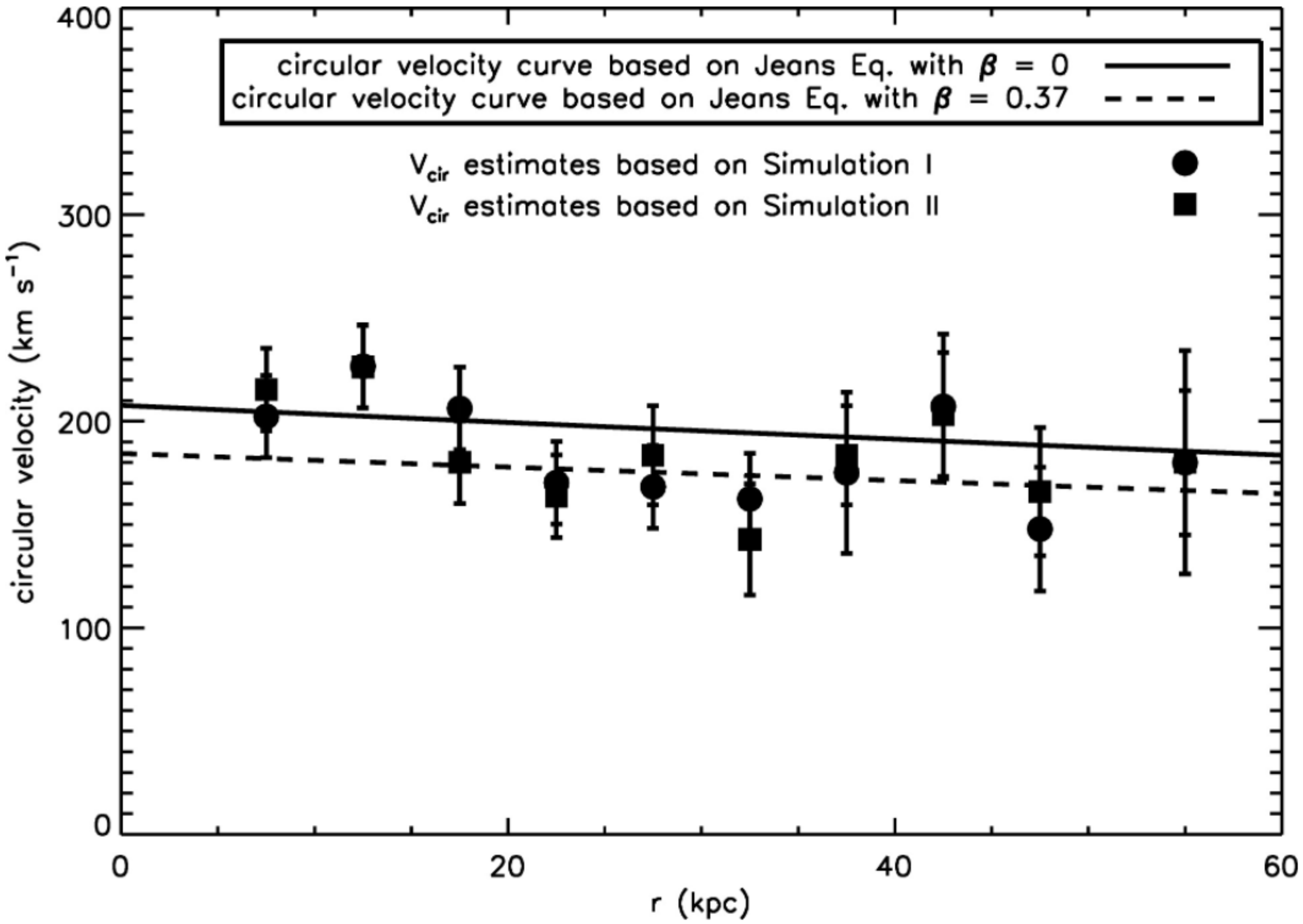
$$\sigma_{\rm los}(r) = \left(111 \pm 1\,{\rm km\,s^{-1}}\right)\,\exp\left(-\frac{r}{354^{+91}_{-60}\,{\rm kpc}}\right)$$

- Anisotropy β: unknown

$$v_c^2(r) = -\sigma_r^2\left(\frac{{\rm d}\ln(\nu\,\sigma_r^2)}{{\rm d}\ln r} + 2\,\beta\right) \longrightarrow \Delta v_c \approx -\frac{\sigma_r^2}{v_c}\,\beta \approx -62\,\beta\,{\rm km\,s^{-1}}\,,$$

# Inferred "rotation curve"




Xue et al. (2008)

Higher anisotropy —> more radial—> lower mass with same dispersion

# Jeans theorem

# Jeans theorem

- A different method for modeling galaxies as equilibrium systems is to explicitly write down distributions f(**x**,**v**) that satisfy the equilibrium CBE

$$\dot{\mathbf{x}}\frac{\partial f(\mathbf{x},\mathbf{v})}{\partial \mathbf{x}} + \dot{\mathbf{v}}\frac{\partial f(\mathbf{x},\mathbf{v})}{\partial \mathbf{v}} = 0 \qquad \dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y} + \dot{z}\frac{\partial f}{\partial z} + \dot{v}_x\frac{\partial f}{\partial v_x} + \dot{v}_y\frac{\partial f}{\partial v_y} + \dot{v}_z\frac{\partial f}{\partial v_z} = 0$$

- Hard to solve the 6 variable (x,y,z,vx,vy,vz) partial differential equation!
- The Jeans theorem tells us how we can easily construct models that satisfy this equation
- In spherical systems, 6 —> 2

$$f(\mathbf{x},\mathbf{v}) = f(E, L)$$

# Integrals of the motion

- Any quantity that is conserved along the orbit and only depends on the phase-space coordinates (**x**,**v**) (and not on time):

$$I(\mathbf{x}_1, \mathbf{v}_1) = I(\mathbf{x}_2, \mathbf{v}_2)$$

- Noether's theorem: each of a system's continuous symmetry properties has a corresponding quantity that is conserved
    - Time-independence potential —> energy
    - Spherical symmetry —> angular momentum in 3 components
    - Axisymmetric symmetry —> angular momentum along perpendicular axis

# Integrals of the motion satisfy the equilibrium CBE

- Conservation means

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 0\,.$$

- Write out in terms of phase-space derivatives

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \dot{\mathbf{x}}\,\frac{\partial I}{\partial \mathbf{x}} + \dot{\mathbf{v}}\,\frac{\partial I}{\partial \mathbf{v}} = 0\,.$$

- Compare to

$$\dot{\mathbf{x}}\frac{\partial f(\mathbf{x},\mathbf{v})}{\partial \mathbf{x}} + \dot{\mathbf{v}}\frac{\partial f(\mathbf{x},\mathbf{v})}{\partial \mathbf{v}} = 0$$

# Jeans theorem

**Part 1**

**Any function of integrals of motion is a solution of the equilibrium CBE.**

$f = f(I_1, I_2, \ldots, I_n) \implies \frac{\mathrm{d}f}{\mathrm{d}t} = \sum_i \frac{\partial f}{\partial I_i} \frac{\mathrm{d}I_i}{\mathrm{d}t} = 0$

**Part 2**

**Any solution of the equilibrium CBE depends on (x,v) only through integrals of motion.**

If f satisfies the CBE, then f itself is an integral of motion.

**Bottom line: To find equilibrium DFs, just write f as a function of the integrals!**

# Jeans theorem

- The Jeans theorem implies that we can build equilibrium models by writing down *any* function of the integrals of the motion
- Clearly large amount of freedom
- Next we discuss unique solutions under simplifying assumptions (same as "closure assumptions" for Jeans equations)

# The Power of Jeans theorem

**Bottom line: To find equilibrium DFs, just write f as a function of the integrals!**

$$f(\mathbf{x}, \mathbf{v}) = f(E, L)$$

| System | Integrals | Distribution function |
|---|---|---|
| Spherical | $E, \mathbf{L}$ | $f(E, \mathbf{L})$ |
| Spherical + symmetric | $E, L$ | $f(E, L)$ |
| Spherical + isotropic | $E$ only | $f(E)$ |
| Axisymmetric | $E, L_z$, (+ third integral) | $f(E, L_z, I_3)$ |

# Spherical distribution functions

$$f(\mathbf{x}, \mathbf{v}) = f(E, L)$$

- From Jeans theorem, we know that spherical distribution functions can only be a function of E and **L**
- If population being modeled is spherically symmetric, can only be a function of E and $|\mathbf{L}| = L$
- Simplest is ergodic: f(E), isotropic, $\beta = 0$

# Eddington formula — section 5.6.1

- Given: A density profile ρ(r) and potential $\Psi(r)$
  Find: The unique isotropic DF f($\mathcal{E}$) that produces this density

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2} \frac{\mathrm{d}}{\mathrm{d}\mathcal{E}} \int_0^{\mathcal{E}} \mathrm{d}\Psi \frac{1}{\sqrt{\mathcal{E} - \Psi}} \frac{\mathrm{d}\rho}{\mathrm{d}\Psi}$$

- Eddington formula gives the unique ergodic distribution function for a given density
- Doesn’t use self-consistency: work for tracers
- Not guaranteed to yield positive density everywhere!
- (but there is *always* a f(E,L) that produces a given density)

# Ergodic distribution functions from simple *guess*

- Rather than solving for the ergodic distribution function for a given density (hard!), we can just make a guess at what a good form would be $f=f(E)$
- and then see what density and velocity distribution that implies
- Simple!

# The singular isothermal sphere — section 5.6.2

- Starting with

$$f(\mathcal{E}) = \frac{\rho_1}{(2\pi\,\sigma^2)^{3/2}} \exp\left(\frac{\Psi - \frac{v^2}{2}}{\sigma^2}\right),$$

- solved by

$$\rho(r) = \frac{\sigma^2}{2\pi\,G\,r^2}$$

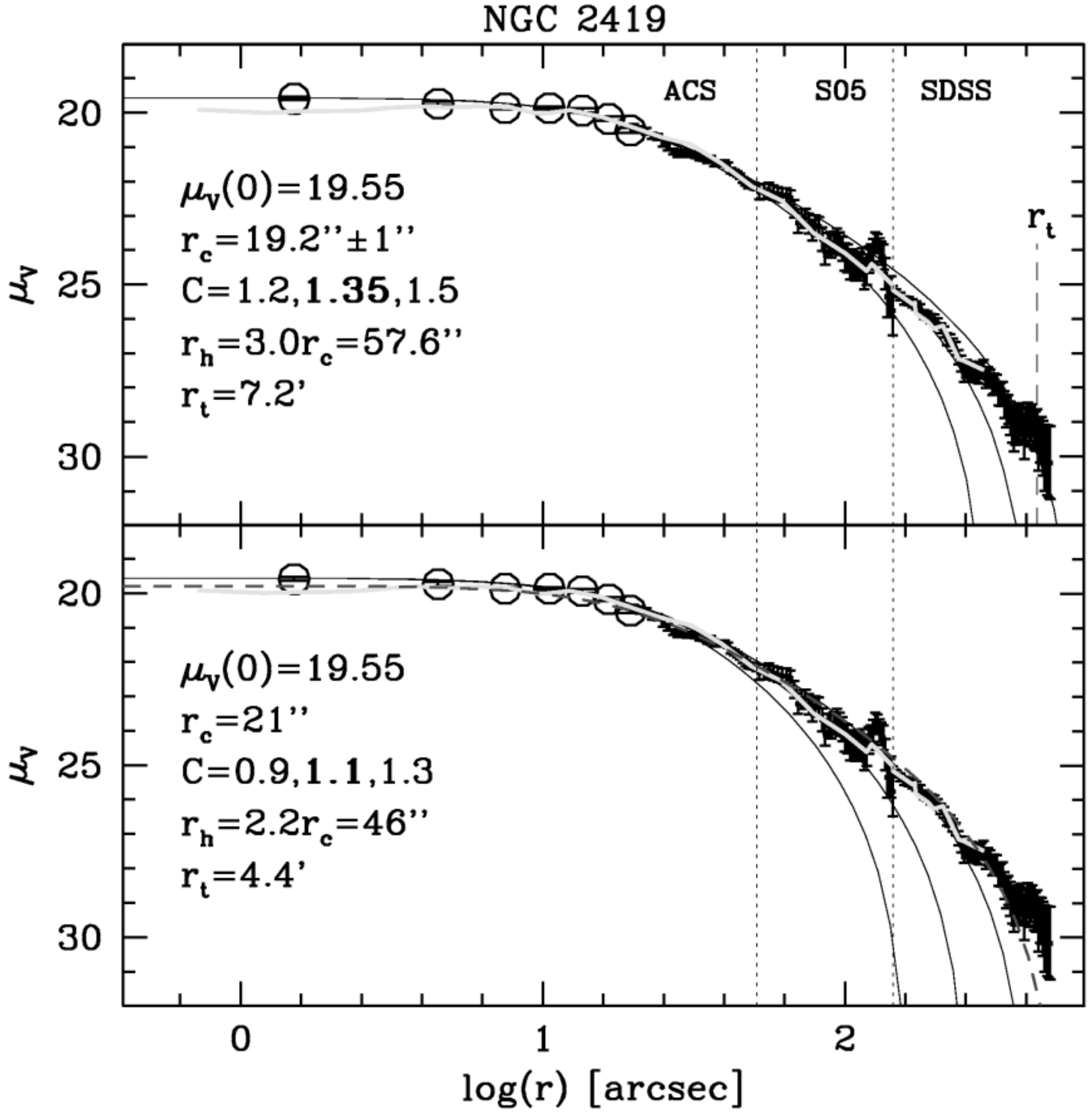
- which is the density of the logarithmic potential that we have seen before

$$\Phi(r) = 2\,\sigma^2\,\ln r + \text{constant}, \qquad v_c(r) = \sqrt{2}\,\sigma.$$

# King profile: NGC 2419

Urban, Shawn. (2015). Intergalactic Vagrant: NGC 2419

# King profile: Globular cluster NGC 2419




NGC 2419
(Bellazzini 2007)

# King profile — section 5.6.3

- Problem with the singular isothermal sphere: extends to infinity
- Can cut-off at finite radius by adjusting:

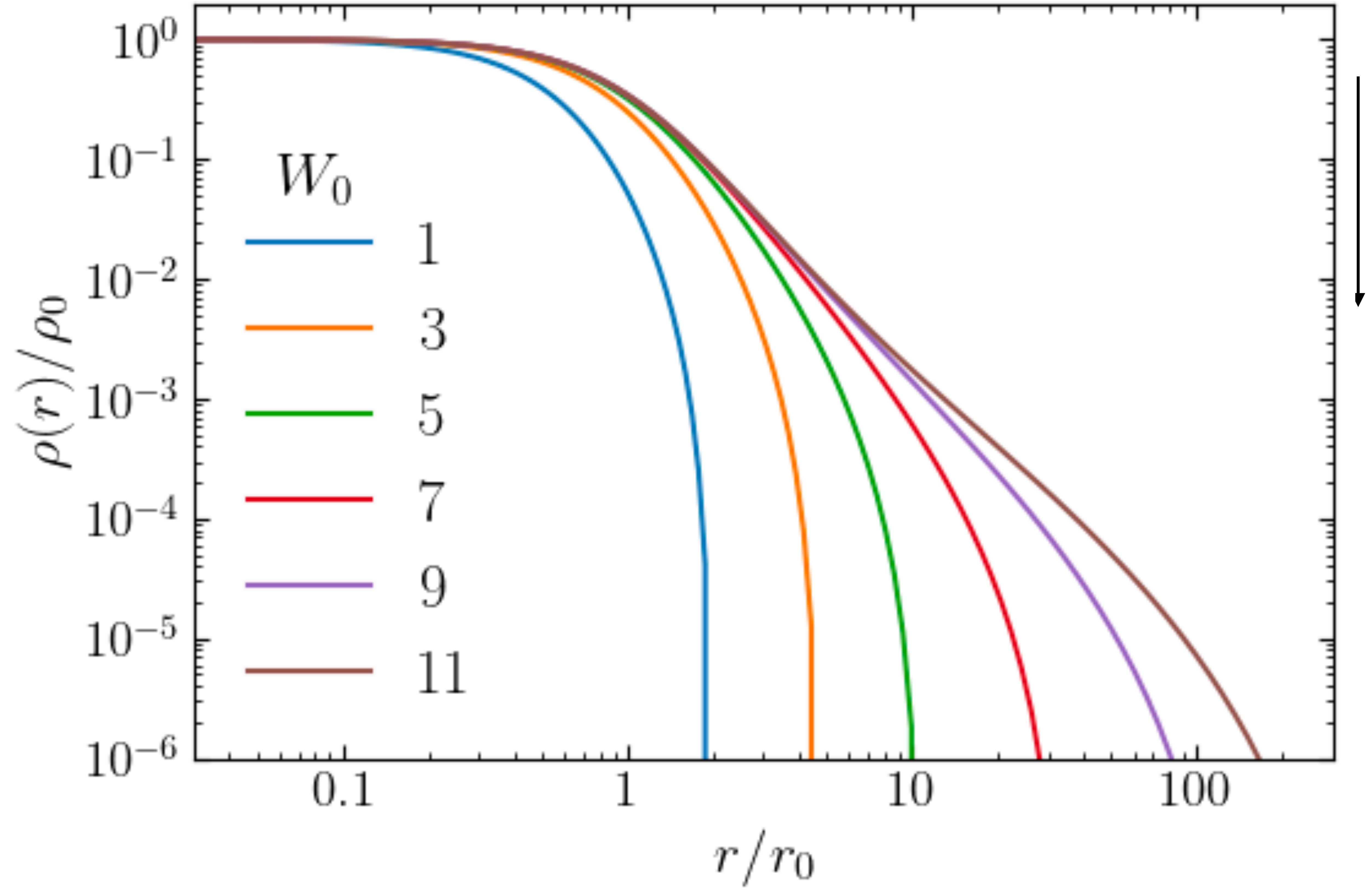
$$f(\mathcal{E}) = \frac{\rho_1}{(2\pi\,\sigma^2)^{3/2}}\left[\exp\left(\frac{\Psi - \frac{v^2}{2}}{\sigma^2}\right) - 1\right]$$

- Gives density

$$\rho(\Psi) = \rho_1\left[e^{\tilde{\Psi}}\,\mathrm{erf}\sqrt{\tilde{\Psi}} - \sqrt{\frac{4\,\tilde{\Psi}}{\pi}}\left(1 + \frac{2\,\tilde{\Psi}}{3}\right)\right] \qquad \tilde{\Psi} = \Psi/\sigma^2$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

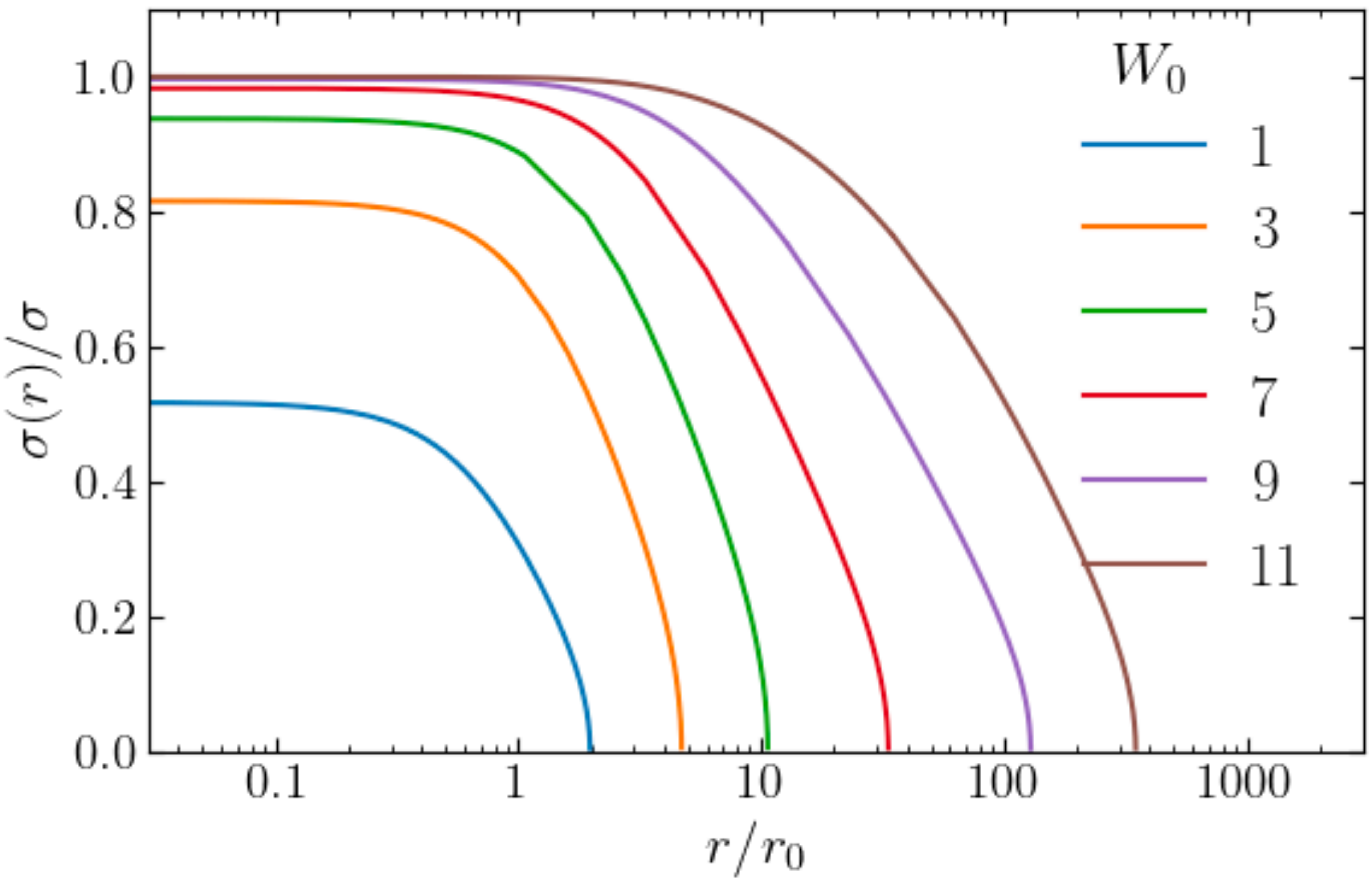
# King profile — section 5.6.3




$W_0 = \Psi(0)/\sigma^2$

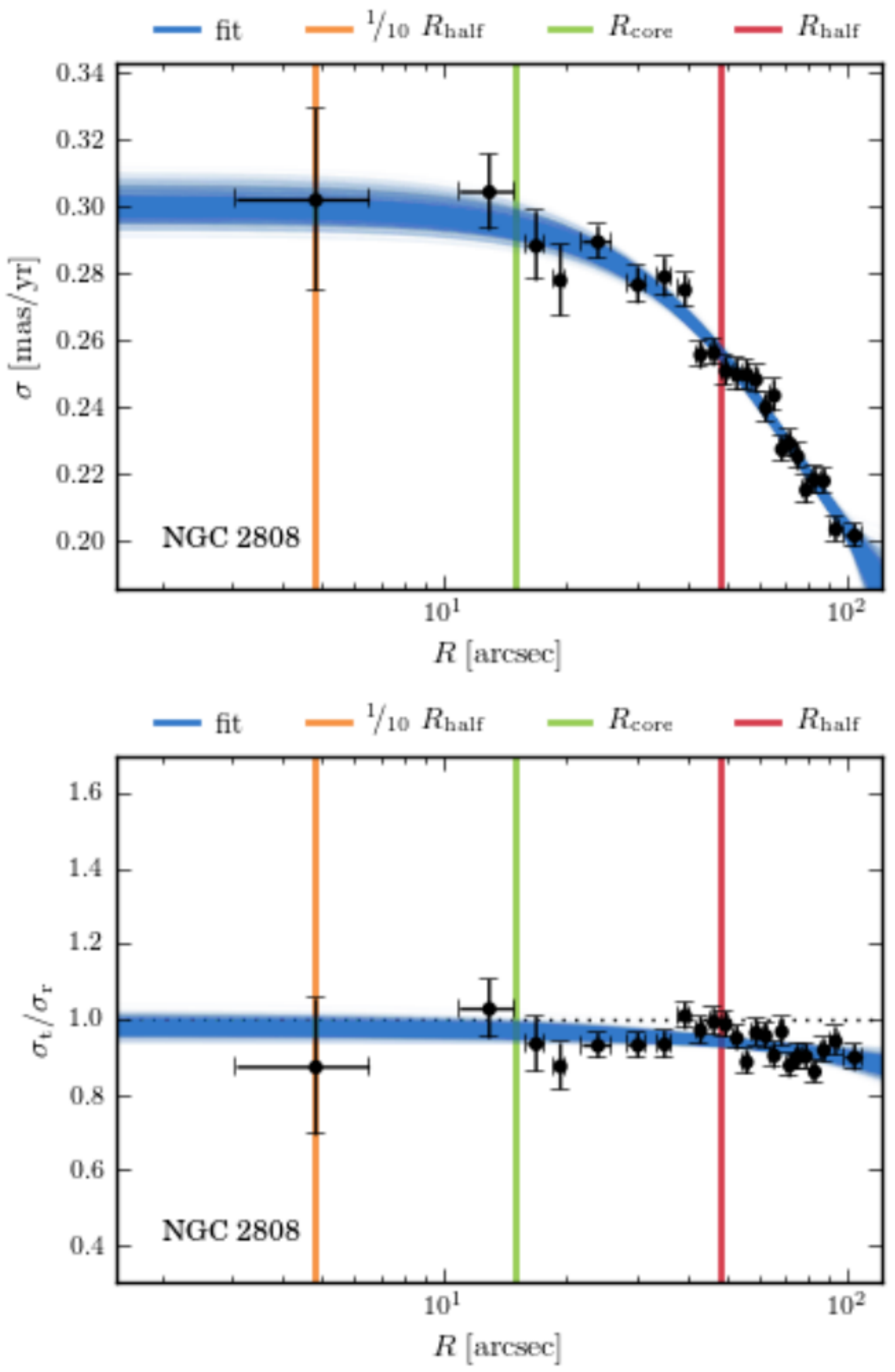
- Popular model for globular clusters, can also represent elliptical galaxies

# King profile: velocity dispersion

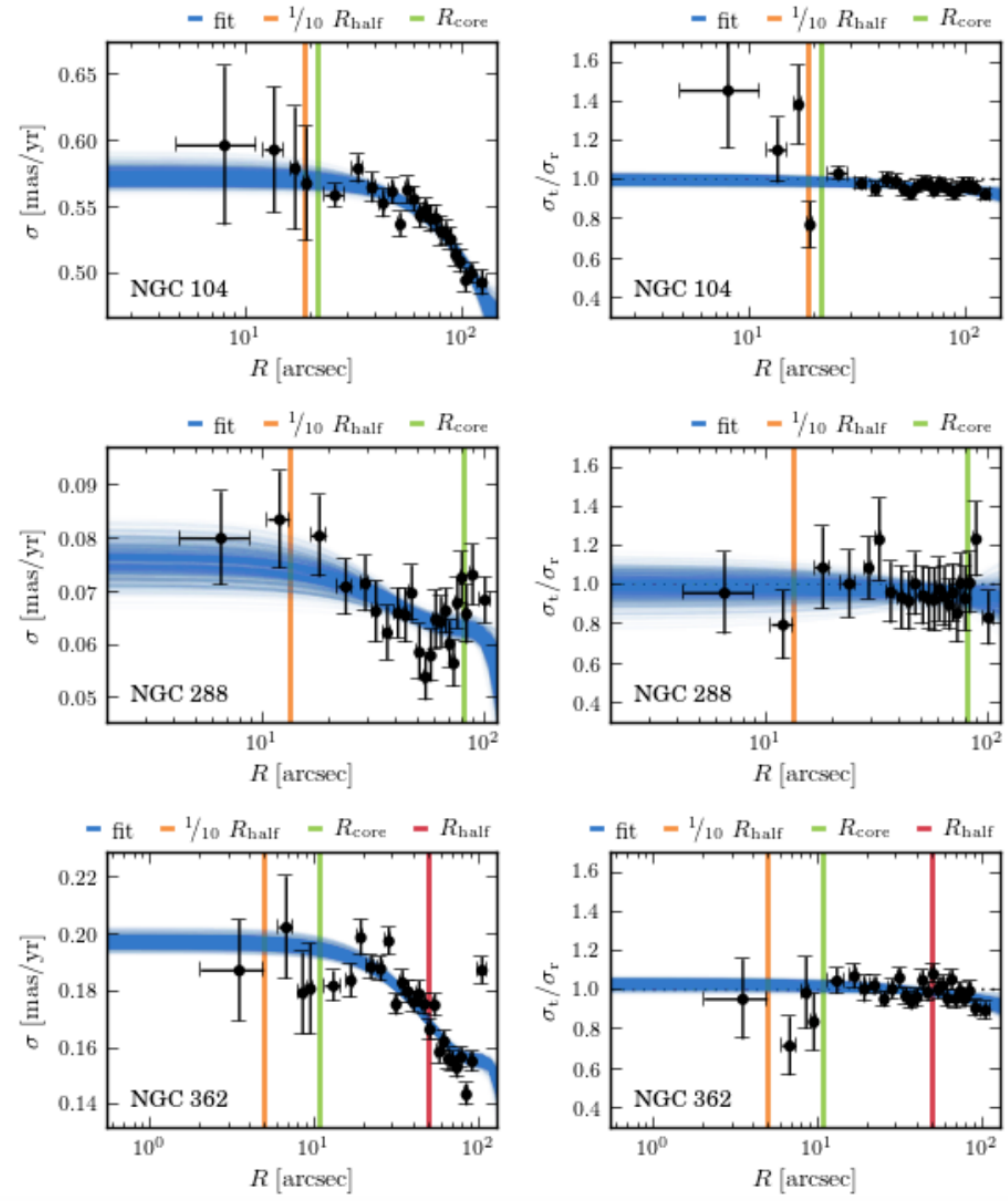



- ~isothermal in inner region, goes to zero at larger radii

# King profile: velocity dispersion




NGC 2808
(Watkins et al. 2015)




(Watkins et al. 2015)