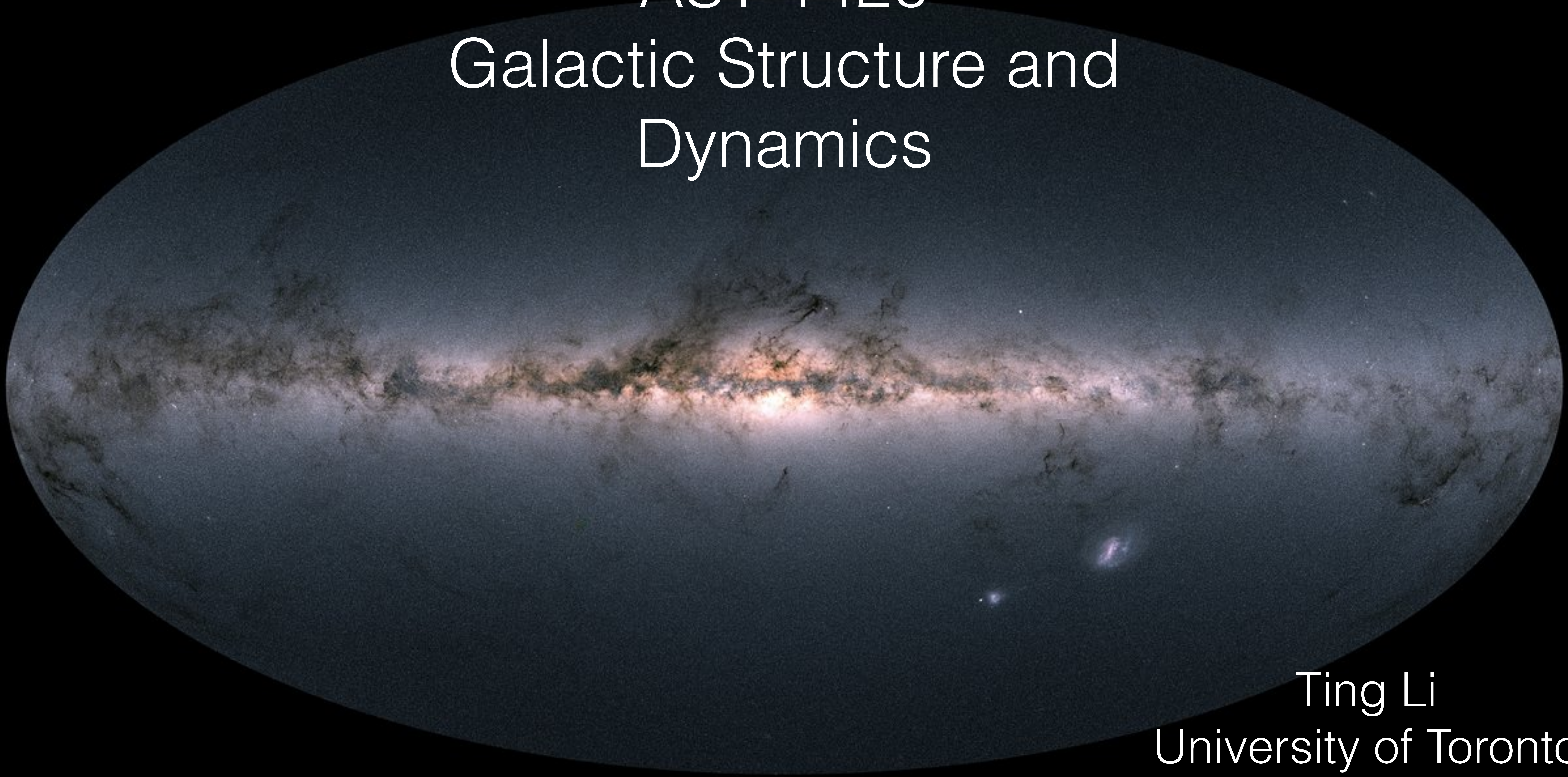
AST 1420
Galactic Structure and Dynamics



Gaia's all-sky view

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Winter 2026

Participation

- Participation consists of two parts:
  - In-class participation

    Title include: AST1420 — Week X Reading — First Name Last Name (This Sunday, X=3)
  - Each student should send one email to ting.li@astro.utoronto.ca by the end of Sunday. The email should contain one of the two following, or both:
    (1) at least one question about the weeks' reading material; or
    (2) a re-plot or re-calcuate of one interactive code in the chapters, with some minor changes.
    For the question you have, try to do a bit research yourself, or ask ChatGPT/GoogleGemini, and is the question get solved? (It is ok to specify the remaining puzzles.)
    For the coding, specify the changes you made, and what you learned from the plots or the results.

To understand the dynamical state of a stellar system, we need to solve the equations of motion:

For particle *i*, the equations of motion are:

$$\frac{dv_{k,i}}{dt} = G \sum_{j=1, i\neq j}^{N} \frac{m_j}{(\vec{x_i} - \vec{x_j})^2}$$

$$\frac{dx_{k,i}}{dt} = v_{k,i} \qquad (k = 1,2,3)$$

This corresponds to a closed set of 6N equations, and a total of 6N unknowns (x, y, z, v_x, v_y, v_z)

<u>This set of equations can explain many types of systems:</u>

Exoplanet systems	$N = 2$
Globular clusters	$N \sim 10^4 - 10^6$
Galaxies	$N \sim 10^7 - 10^{11}$
Galaxy groups/clusters	$N \sim 10^3 - 10^4$

However for N > 2.5, no analytical solution exists !!

for N > 10^9, no numerical solution exists !!

Galaxies as smooth mass distributions

Gravitational potential theory

Newtonian gravity

- Bodies move under the influence of forces, force set by mass distribution

- 3D force ~ gradient of a scalar potential

$$\mathbf{F} = m\mathbf{a} = -m\nabla\Phi$$

- Poisson equation: relation between potential and mass density

$$\nabla^2\Phi = 4\pi G\rho$$

Reminder that ∇F is a vector with components $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$, while $\nabla^2 F$ is a scalar $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$—this is the divergence of the gradient, also called the Laplacian.

Newtonian gravity

- Bodies move under the influence of forces, force set by mass distribution

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$$\mathbf{F} = m\mathbf{a} = -m\nabla\Phi$$

- Poisson equation: relation between potential and mass density

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Potential Density Pairs (Φ, ρ)

Potential —> Density: derivative

Density —> Potential: solving the differential equation

only a few families of analytical functions for both potential and density

Reminder that ∇F is a vector with components $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$, while $\nabla^2 F$ is a scalar $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$—this is the divergence of the gradient, also called the Laplacian.

Some important properties of the Poisson equation

- Linear: if (Φ_1, ρ_1) and (Φ_2, ρ_2) each individually solve the Poisson equation, then $(\Phi_1 + \Phi_2, \rho_1 + \rho_2)$ does too
  [not typically the case for modified gravity theories]

- Both force and density only depend on at least one derivative of the potential
  —> can add or subtract constant to potential and physics stays the same
  —> typically try to normalize potentials such that

$$\Phi_\infty = 0$$

Gravitational force and gravitational field

- Typically deal with the *gravitational field* rather than the force

- Gravitational field = force per unit mass

$$\mathbf{F}(\mathbf{x}) = m\mathbf{g}(\mathbf{x})$$

- Gravity on earth: $g = GM/R_e^2 \sim 9.8$ m/s^2 pointing down

- So typically do not really distinguish between *force*, *field*, and *acceleration*

Units!

- Always important to make sure units are correct!
  - Units of both sides of an equation have to match
  - Non-linear functions (like exp[.]) only make sense to perform on unitless quantities
- Field/accelaration: $[a] = [x][t]^{-2} = m/s^2$
- Force: from Newton’s second law units are $[m][a] = [m][x][t]^{-2}$
- Potential: spatial integral of field —> $[a][x] = [v]^2 = [x]^2[t]^{-2} = (m/s)^2$
- G = [2nd derivative potential]/[density] = $[t]^{-2}\ [m]^{-1}\ [x]^3$
- For galaxies: $G = 4.302\,\mathrm{kpc}\,(100\,\mathrm{km/s})^2\,\left(10^{10}\,M_\odot\right)^{-1}$


Credit: Gemini Observatory / AURA
Gemini Observatory Legacy Image

$\phi(\mathbf{x})$???

The 'spherical cow' treatment of galaxies

Many of the galaxies that we are interested in look like this:

(Credit: M101: European Space Agency & NASA; NGC 660: Gemini Observatory, AURA)

The 'spherical cow' treatment of galaxies

- Spherical approximation still useful:
  - To 'zero-th order': radial behavior of a mass distribution most important for its dynamics
  - Dark-matter distribution ~ spherical, so much research on DM dynamics uses spherical potentials and orbits in spherical potentials
- *Far* easier to work with spherical potentials (force-law, orbits, equilibria) than at the next order of approximation

Spherical mass distributions

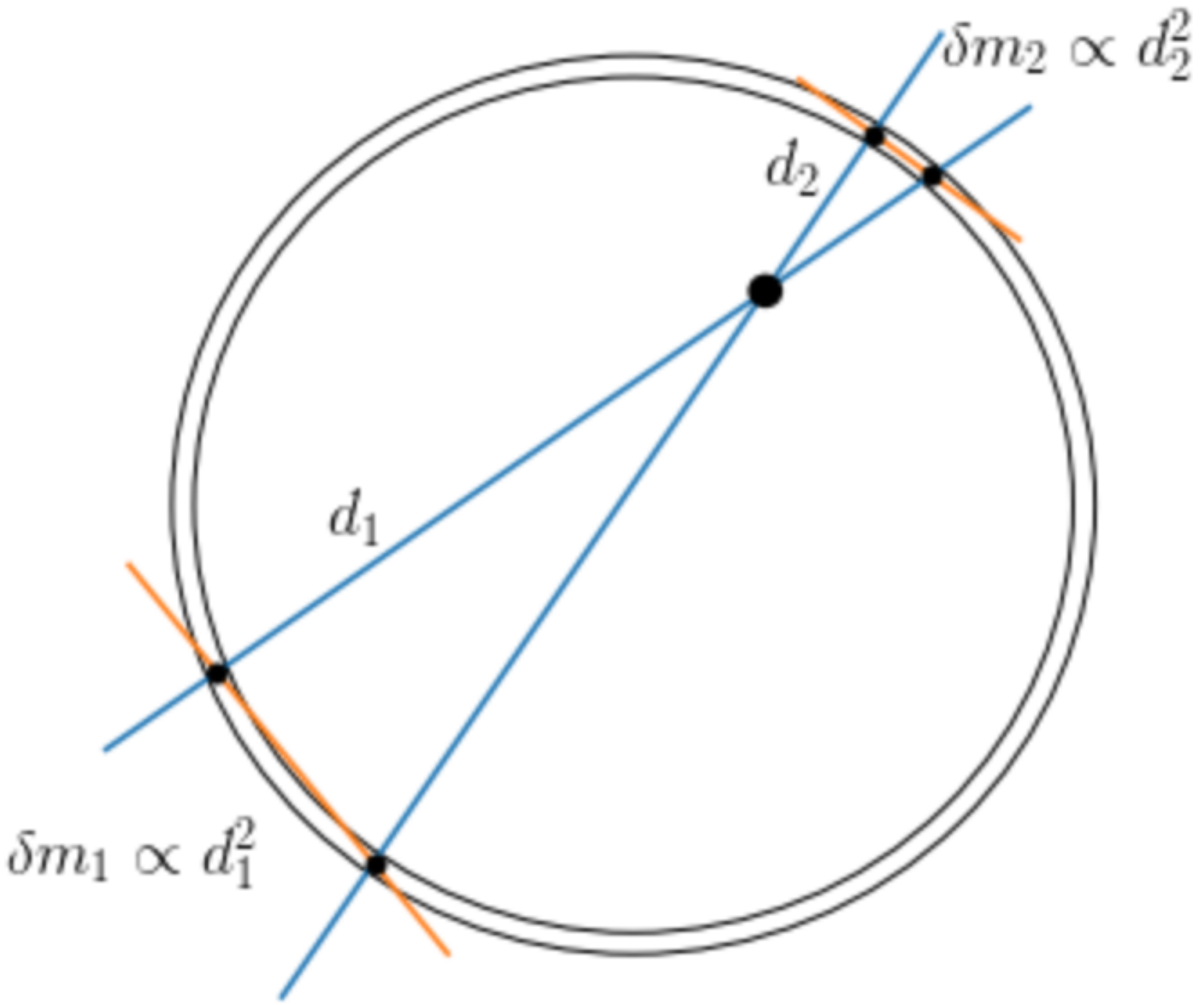
Potential of a spherical mass distribution

Spherical Symmetry: $\nabla^2\Phi = 4\pi G\rho \longrightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho(r)$

- Problem: given density $\rho(r)$, what is $\phi(r)$?
- Solve Poisson equation?
- Two theorems by Newton (!) significantly simplify finding $\phi(r)$

Potential of a spherical mass distribution

Newton's first shell theorem: *A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.*



Potential of a spherical mass distribution

Newton's first shell theorem: *A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.*

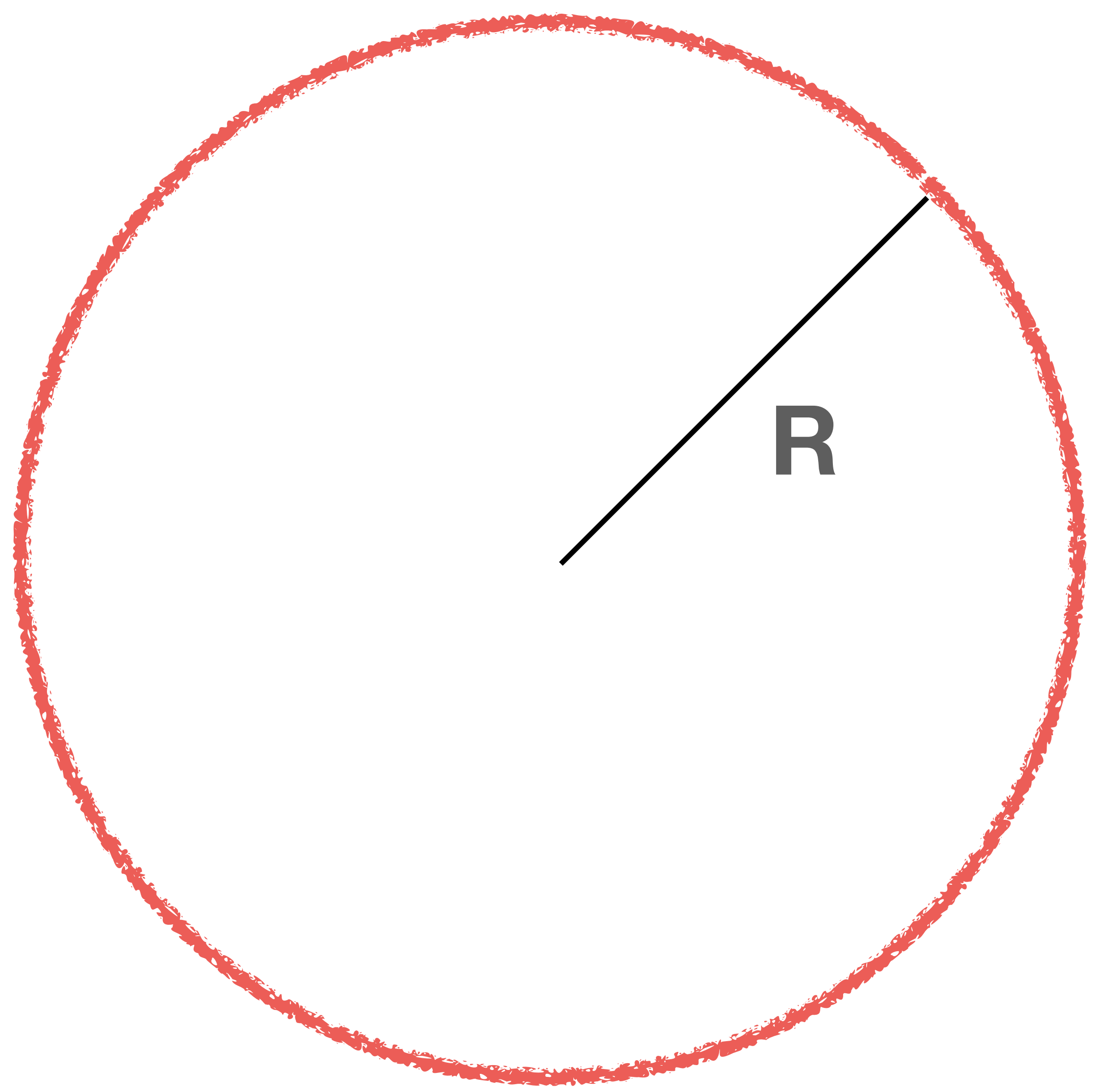
Newton's second shell theorem: *The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all of the shell's matter were concentrated into a point at its center.*

Potential of a spherical shell with mass M_{shell} and radius R

- Newton's second theorem: force the same as if all mass were concentrated in a point —> potential the same
- Potential of a point-mass is ϕ = -GM/r

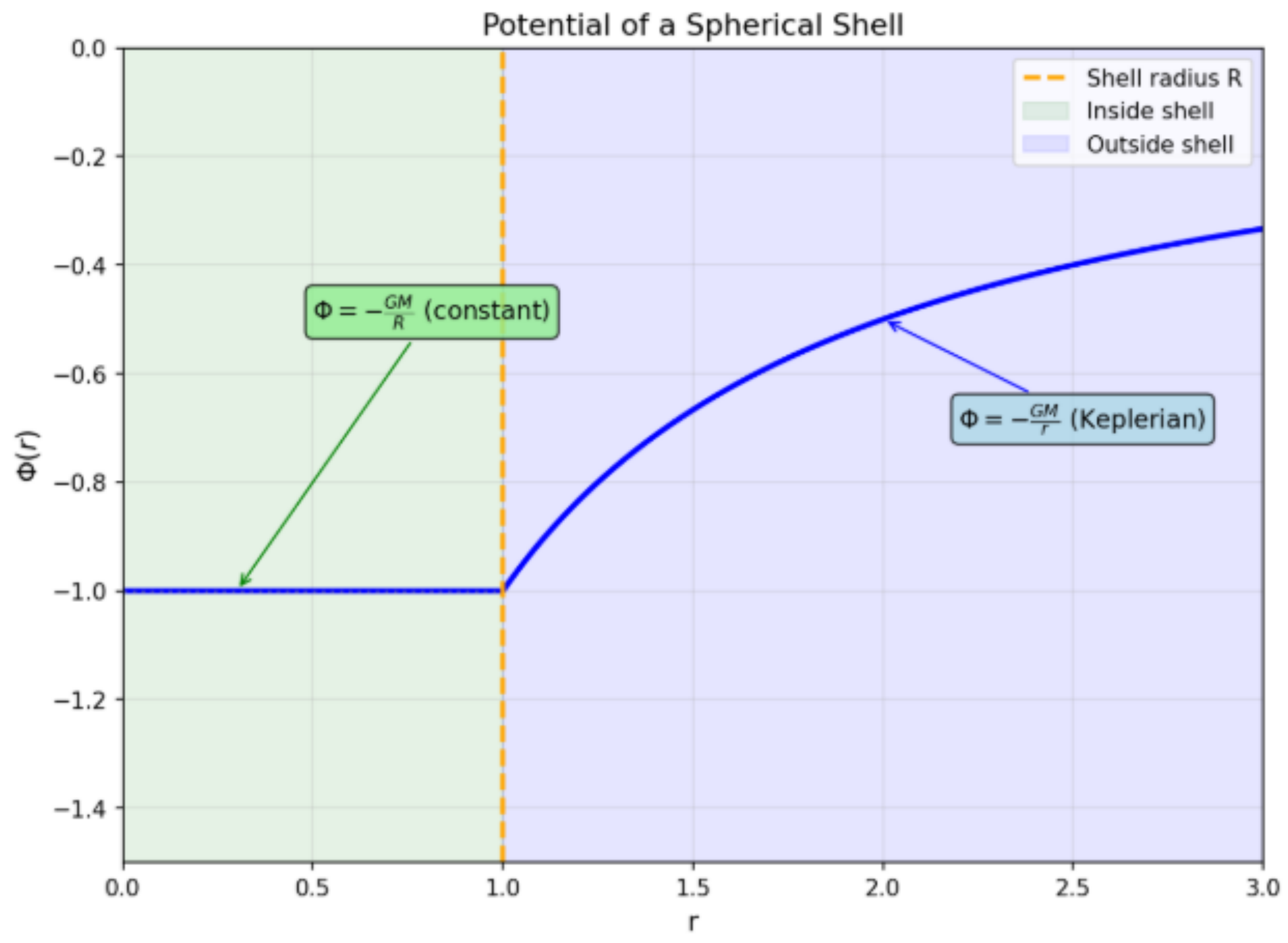
$$\Phi_{\text{shell}}(r > R) = -\frac{GM_{\text{shell}}}{r}$$

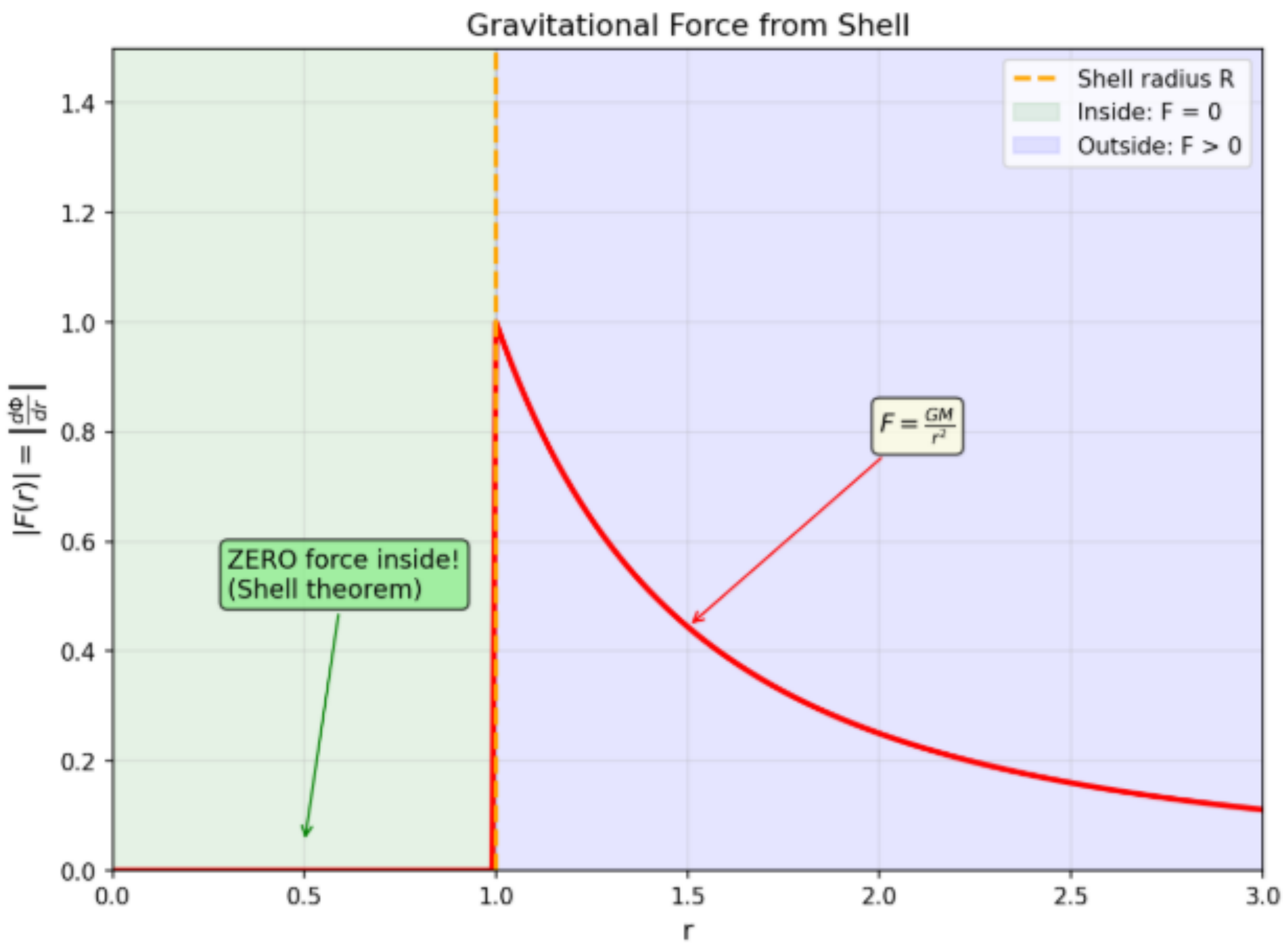
Potential of a spherical shell with mass M_{shell} and radius R



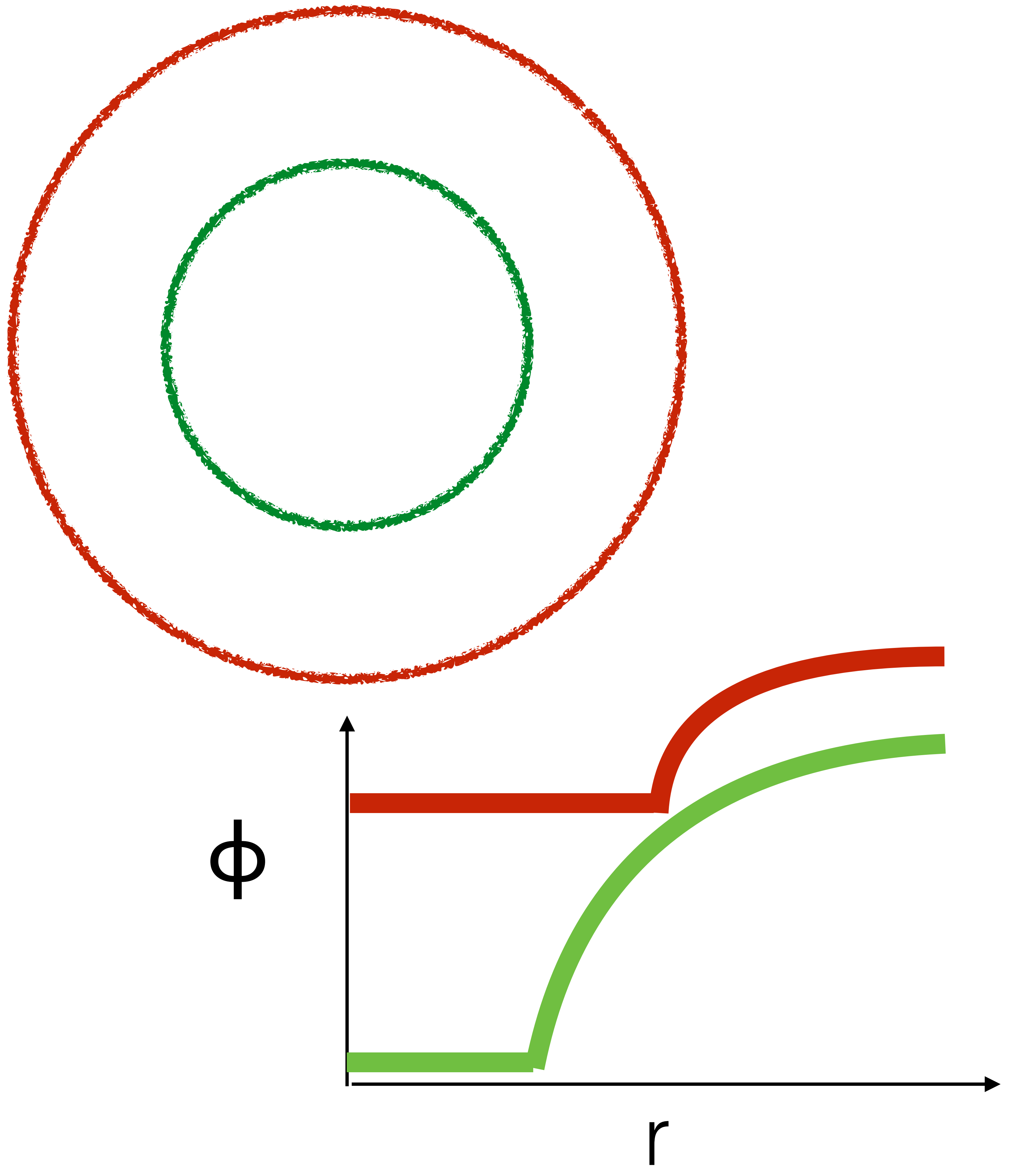
$$\Phi_{\text{shell}}(r < R) = -\frac{GM_{\text{shell}}}{R}$$

$$\Phi_{\text{shell}}(r > R) = -\frac{GM_{\text{shell}}}{r}$$





Potential of any spherical mass distribution



Potential of any spherical mass distribution

- $\rho(r)$ can be composed into shells with mass $4\pi r^2 \rho(r) dr$
- Potential of each shell as in previous slide

$$\Phi(r) = -4\pi G \left[\frac{1}{r} \int_0^r \mathrm{d}r' \, \rho(r') \, r'^2 + \int_r^\infty \mathrm{d}r' \, \rho(r') \, r' \right] .$$

$$\rho(r) \rightarrow \Phi(r)$$

Force of any spherical mass distribution

- Force must be radial, because potential only depends on r
- Newton's first law: no force from shells outside current radius r
- Newton's second law: each shell M_{shell} inside r gives $-GM_{shell}/r^2$
- Total force: $-GM(<r)/r^2$
- Mass outside current r **has no influence on the acceleration**

Circular velocity

Circular velocity

- The circular velocity is the velocity of a body on a circular orbit
- Circular orbit acceleration is centripetal: v_c^2/r
- Equal to radial force (both point inwards): $v_c^2/r = GM(<r) / r^2$

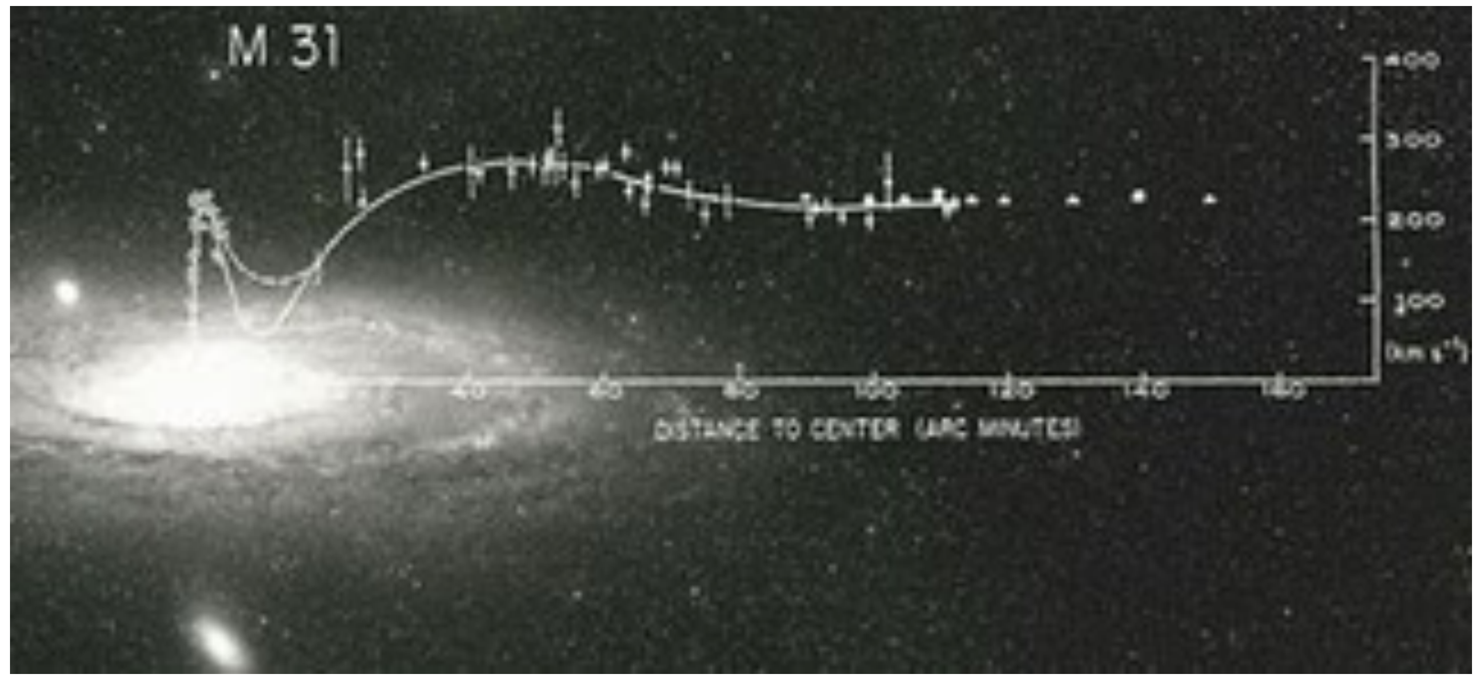
$$v_c(r)^2 = -\, r\, a_r(r) = \frac{G\, M(<r)}{r}$$

Circular velocity measures enclosed mass (for spherical potentials)

$$M(< r) = \frac{v_c^2\, r}{G}$$

- Galaxies are observed to have v_c ~ constant

$$M(< r) = \frac{v_c^2\, r}{G} \propto r$$



- E.g., in the Milky Way near the Sun: r ~ 8 kpc, v_c = 220 km/s

$$M(r < 8\,\mathrm{kpc}) \approx 9 \times 10^{10}\, M_\odot$$

Dynamical time

- Typical dynamical time: period of circular orbit at radius r

$$t_{\rm dyn} = \frac{2\pi\, r}{v_c}\,.$$

$$t_{\rm dyn} = 2\pi\, r\sqrt{\frac{r}{G\,M(\,<r)}} = \sqrt{\frac{3\pi}{G\,\bar{\rho}}}\,,$$

Dynamical time

$$t_{\rm dyn} \propto (G\bar{\rho})^{-1/2}$$

- *Very important relation!*
- Given estimate of average density —> typical orbital time

Dynamical time examples

In [1]:
```
import numpy
import astropy.units as u
import astropy.constants as apyconst
```

The Sun

$m = M_\odot \quad r = R_\odot$

Dynamical time examples

The Milky Way disk $m = \sim 10^{11}\,M_\odot$ $\quad r = \sim 8\,\mathrm{kpc}$

Some examples of spherical potentials

Point mass / *Keplerian*

$$\Phi(r) = -\frac{GM}{r},$$

$$v_c(r) = \sqrt{\frac{GM}{r}}.$$

Homogeneous density sphere

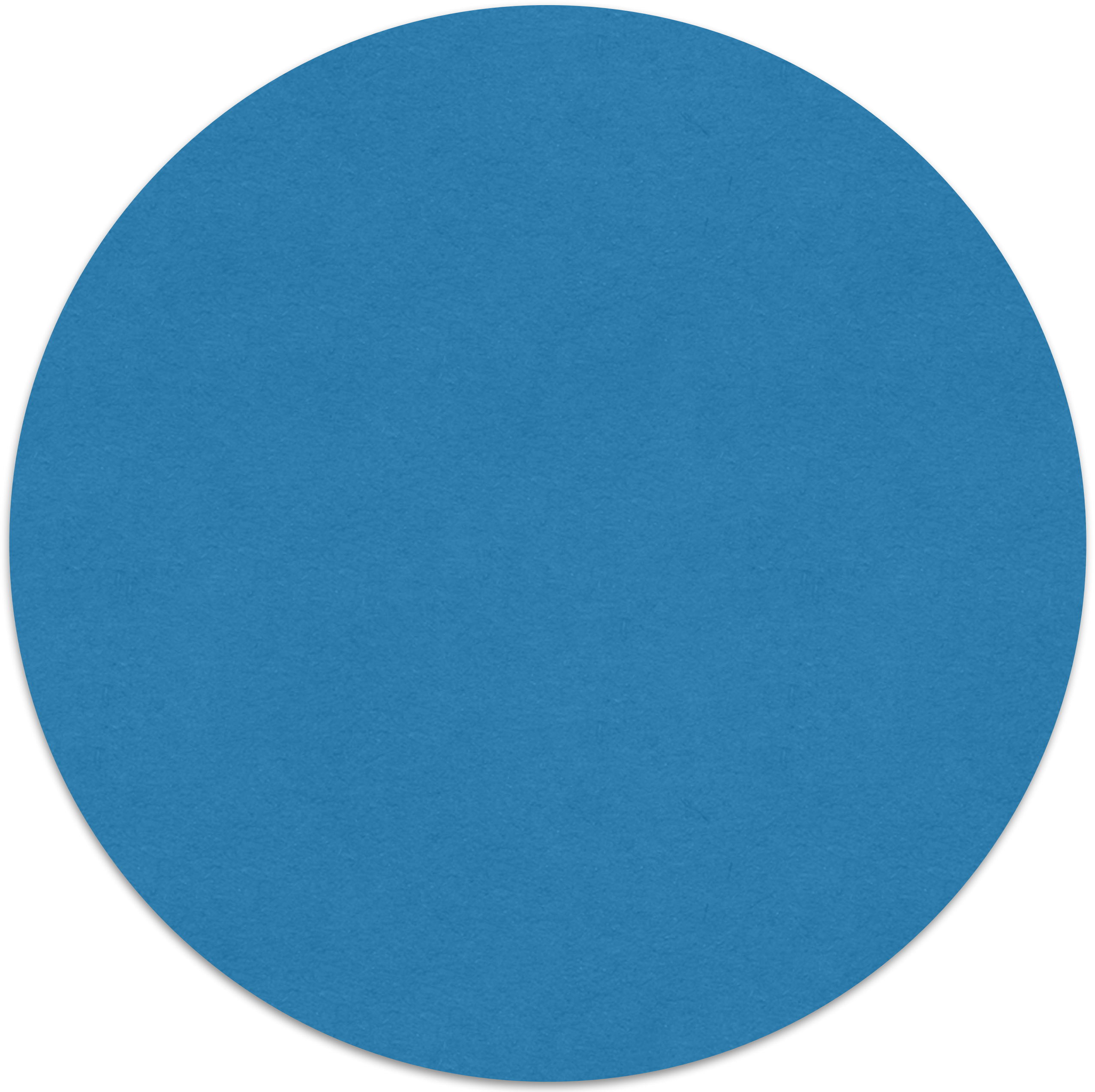
$$\rho(r) = \rho_0 = \text{ constant}$$

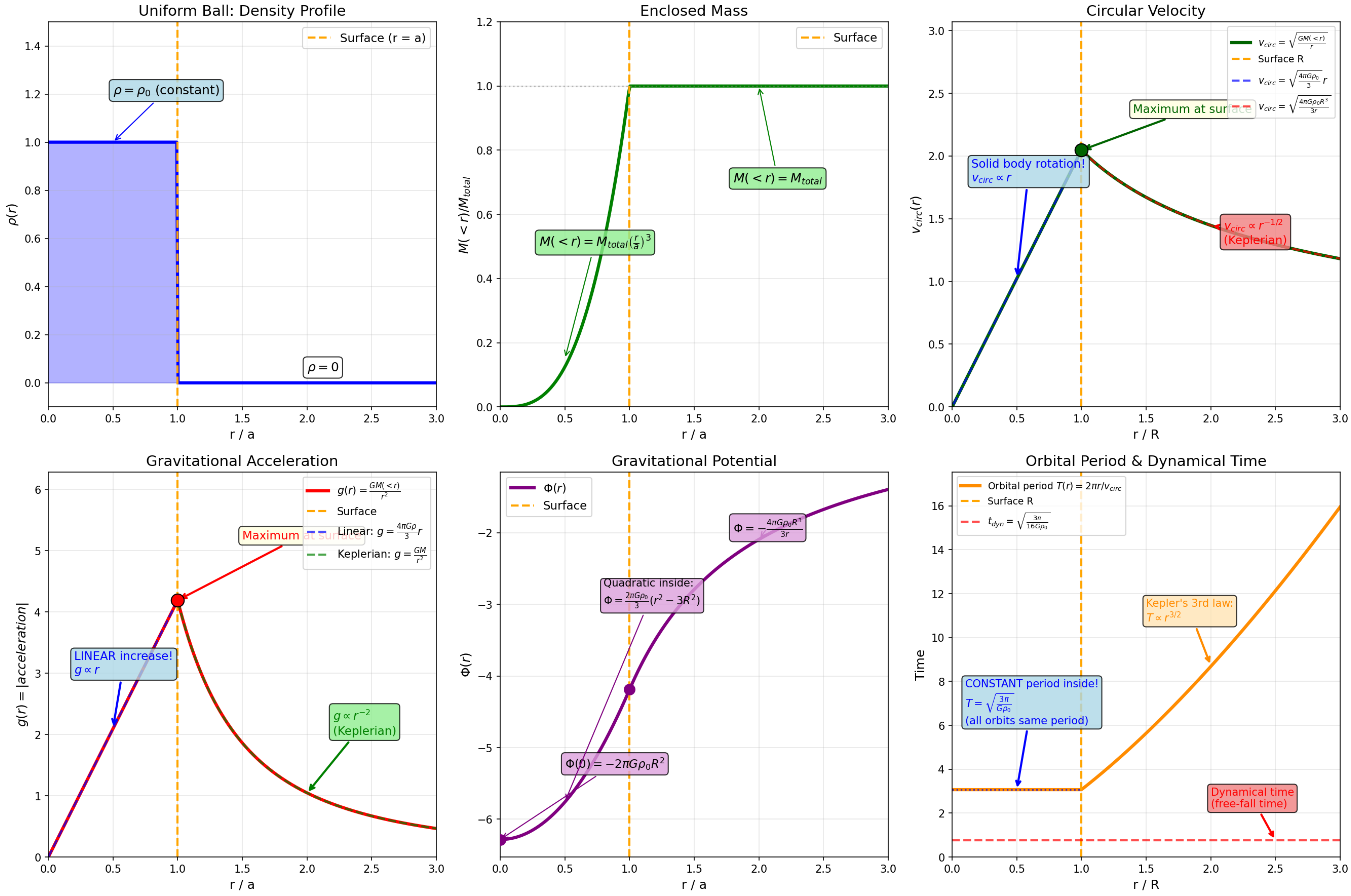
$$\Phi(r) = \frac{2\pi G \rho_0}{3} r^2$$

$$v_c(r) = \sqrt{\frac{4\pi G \rho_0}{3}}\, r\,,$$

$$\Omega(r) = v_c(r)/r = \text{ constant.}$$

$$t_{\rm dyn} = \sqrt{\frac{3\pi}{G\rho_0}}\,.$$

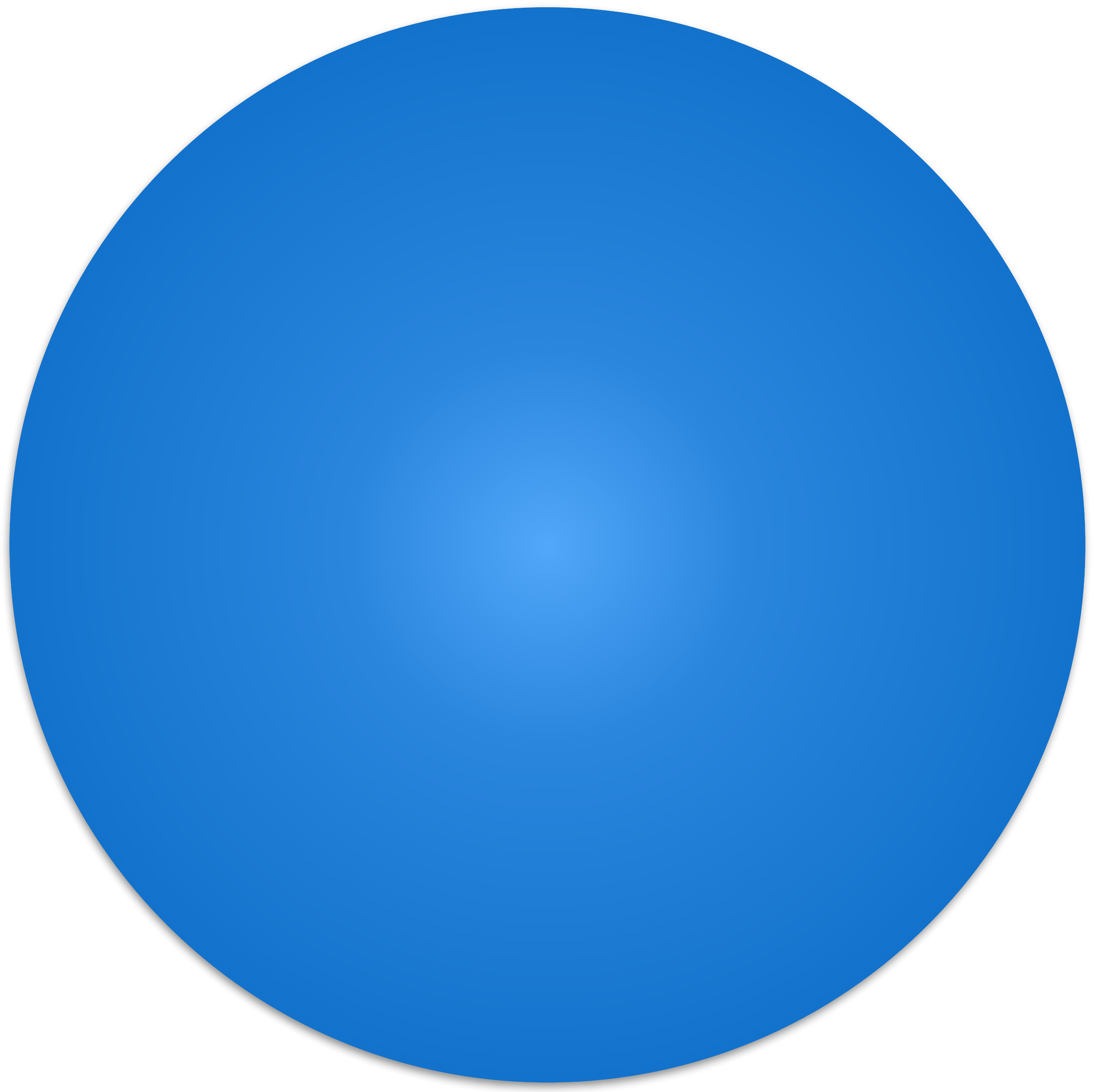


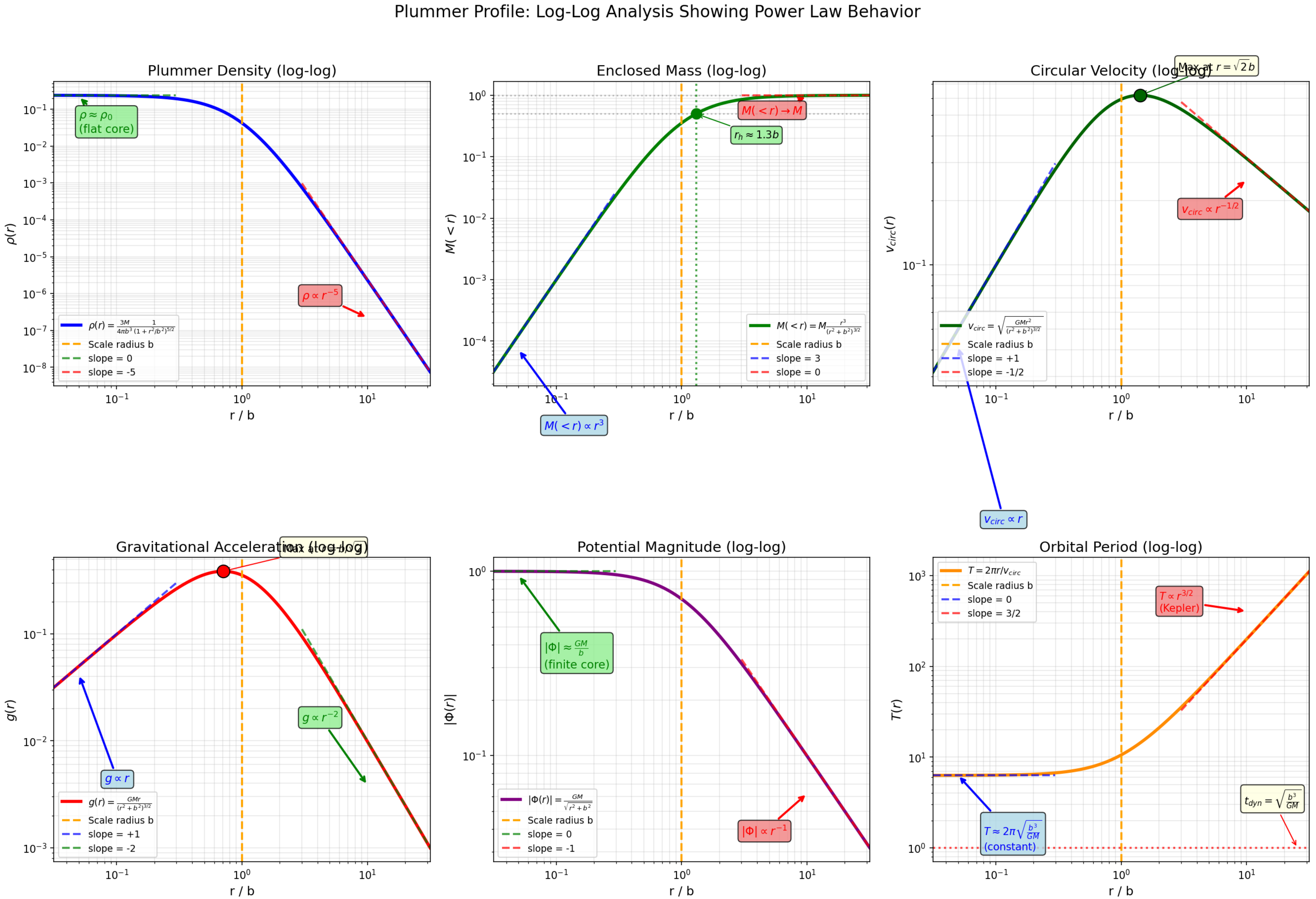


Uniform Ball: Density Profile
Surface (r = a)
$\rho = \rho_0$ (constant)
$\rho = 0$
$\rho(r)$
r / a
Enclosed Mass
Surface
$M(<r) = M_{total}$
$M(<r) = M_{total}(\frac{r}{a})^3$
$M(<r)/M_{total}$
r / a
Circular Velocity
$v_{circ} = \sqrt{\frac{GM(<r)}{r}}$
Surface R
$v_{circ} = \sqrt{\frac{4\pi G\rho_0}{3}}\, r$
$v_{circ} = \sqrt{\frac{4\pi G\rho_0 R^3}{3r}}$
Maximum at surface
Solid body rotation!
$v_{circ} \propto r$
$v_{circ} \propto r^{-1/2}$
(Keplerian)
$v_{circ}(r)$
r / R
Gravitational Acceleration
$g(r) = \frac{GM(<r)}{r^2}$
Surface
Linear: $g = \frac{4\pi G\rho}{3} r$
Keplerian: $g = \frac{GM}{r^2}$
Maximum at surface
LINEAR increase!
$g \propto r$
$g \propto r^{-2}$
(Keplerian)
$g(r) = |acceleration|$
r / a
Gravitational Potential
$\Phi(r)$
Surface
$\Phi = -\frac{4\pi G\rho_0 R^3}{3r}$
Quadratic inside:
$\Phi = \frac{2\pi G\rho_0}{3}(r^2 - 3R^2)$
$\Phi(0) = -2\pi G\rho_0 R^2$
$\Phi(r)$
r / a
Orbital Period & Dynamical Time
Orbital period $T(r) = 2\pi r/v_{circ}$
Surface R
$t_{dyn} = \sqrt{\frac{3\pi}{16 G\rho_0}}$
Kepler's 3rd law:
$T \propto r^{3/2}$
CONSTANT period inside!
$T = \sqrt{\frac{3\pi}{G\rho_0}}$
(all orbits same period)
Dynamical time
(free-fall time)
Time
r / R

Plummer sphere

$$\Phi(r) = -\frac{GM}{r} \quad \rightarrow \quad \Phi(r) = -\frac{GM}{\sqrt{r^2 + b^2}} .$$

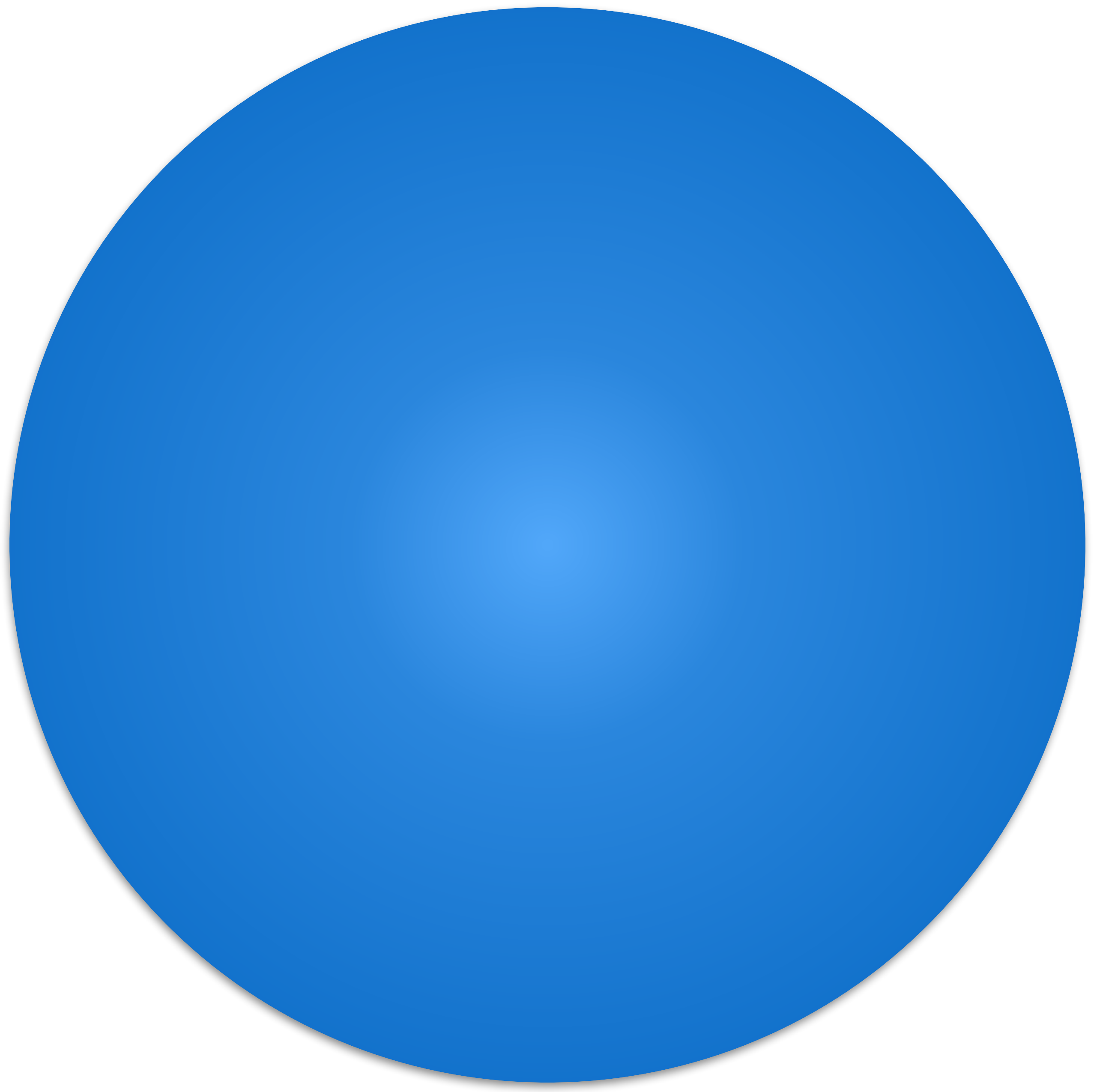




Plummer Profile: Log-Log Analysis Showing Power Law Behavior
Plummer Density (log-log)
$\rho \approx \rho_0$ (flat core)
$\rho \propto r^{-5}$
$\rho(r) = \frac{3M}{4\pi b^3}\frac{1}{(1+r^2/b^2)^{5/2}}$
Scale radius b
slope = 0
slope = -5
$\rho(r)$
r / b
Enclosed Mass (log-log)
$M(<r) \to M$
$r_h \approx 1.3b$
$M(<r) \propto r^3$
$M(<r) = M\frac{r^3}{(r^2+b^2)^{3/2}}$
Scale radius b
slope = 3
slope = 0
$M(<r)$
r / b
Circular Velocity (log-log)
Max at $r = \sqrt{2}b$
$v_{circ} \propto r^{-1/2}$
$v_{circ} \propto r$
$v_{circ} = \sqrt{\frac{GMr^2}{(r^2+b^2)^{3/2}}}$
Scale radius b
slope = +1
slope = -1/2
$v_{circ}(r)$
r / b
Gravitational Acceleration (log-log)
$g \propto r$
$g \propto r^{-2}$
$g(r) = \frac{GMr}{(r^2+b^2)^{3/2}}$
Scale radius b
slope = +1
slope = -2
$g(r)$
r / b
Potential Magnitude (log-log)
$|\Phi| \approx \frac{GM}{b}$ (finite core)
$|\Phi| \propto r^{-1}$
$|\Phi(r)| = \frac{GM}{\sqrt{r^2+b^2}}$
Scale radius b
slope = 0
slope = -1
$|\Phi(r)|$
r / b
Orbital Period (log-log)
$T = 2\pi r/v_{circ}$
Scale radius b
slope = 0
slope = 3/2
$T \propto r^{3/2}$ (Kepler)
$T \approx 2\pi\sqrt{\frac{b^3}{GM}}$ (constant)
$t_{dyn} = \sqrt{\frac{b^3}{GM}}$
$T(r)$
r / b

Isochrone potential

$$\Phi(r) = -\frac{GM}{r} \quad \rightarrow \quad \Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}}$$



Power-law models

always a convenient model in astrophysics!

$$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^{\alpha},$$

$$M(< r) = \frac{4\pi\, \rho_0\, r_0^3}{3-\alpha} \left(\frac{r_0}{r}\right)^{\alpha-3}$$

$$\Phi(r) = -\frac{4\pi\, G\, \rho_0\, r_0^2}{(3-\alpha)(2-\alpha)} \left(\frac{r_0}{r}\right)^{\alpha-2}, \qquad (\alpha \neq 2),$$

$$\Phi(r) = \frac{4\pi\, G\, \rho_0\, r_0^2}{(3-\alpha)} \ln r, \qquad (\alpha = 2).$$

Power-law models

always a convenient model in astrophysics!

$$\Phi(r) = \frac{4\pi\, G\, \rho_0\, r_0^2}{(3-\alpha)} \ln r\,, \qquad (\alpha = 2)\,.$$

$$v_c^2(r) = \frac{4\pi\, \rho_0\, r_0^2}{3-\alpha} \left(\frac{r_0}{r}\right)^{\alpha-2}\,.$$

logarithmic potential $$\Phi(r) = v_c^2\, \ln r\,, \qquad (\alpha = 2)\,.$$

v_c **—> potential —> mass**

Two-Power-law models

if one power-law doesn't fit, try two!

$$\rho(r) = \frac{\rho_0 \, a^\alpha}{r^\alpha \, (1 + r/a)^{\beta-\alpha}} \, .$$

$$\rho(r) = \begin{cases} \rho_0 \left(\frac{a}{r}\right)^\alpha, & r \ll a \, , \\ \rho_0 \left(\frac{a}{r}\right)^\beta, & r \gg a \, . \end{cases}$$

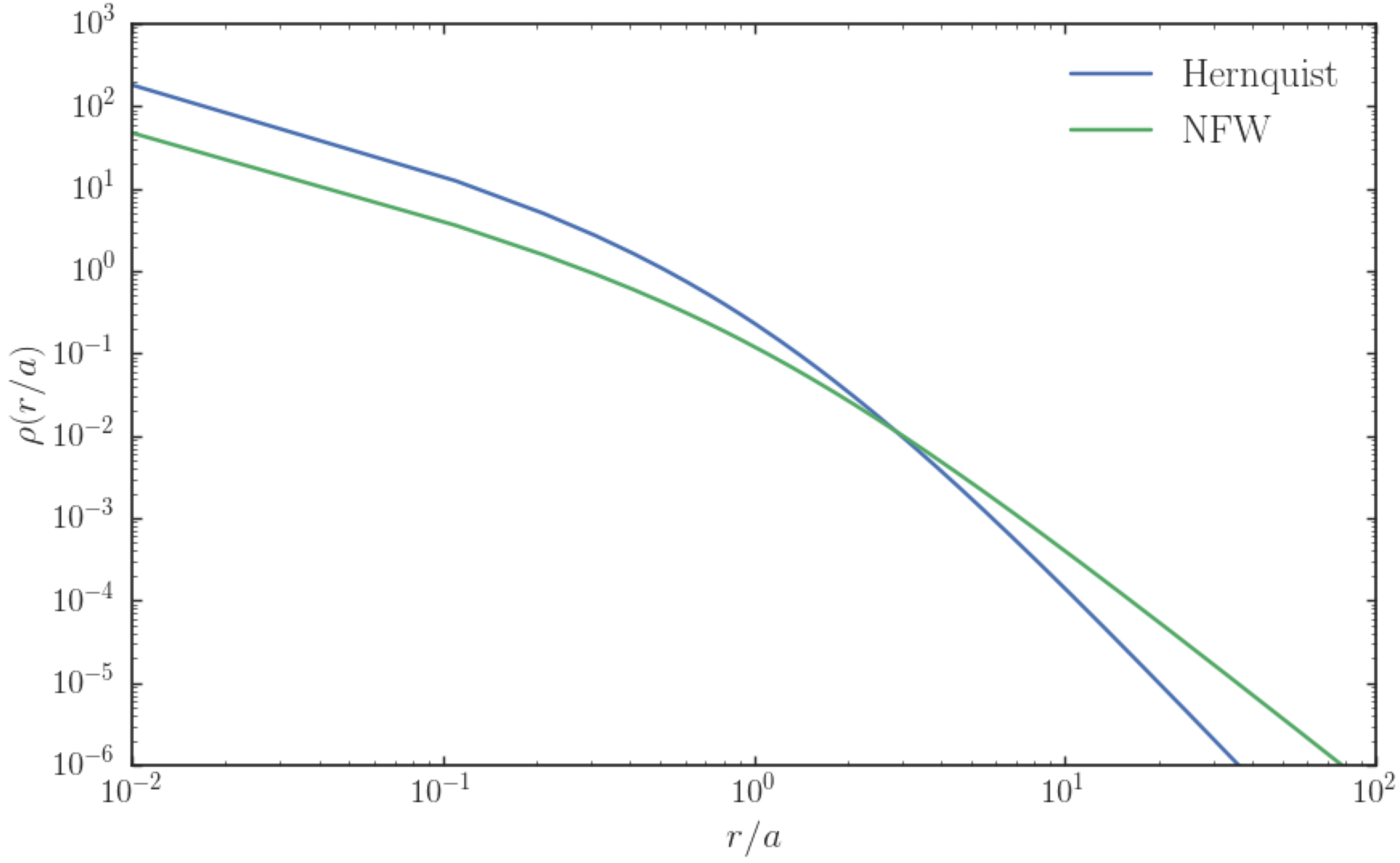
- Two important parameter settings:
  - **Hernquist**: alpha=1, beta=4 —> elliptical galaxies, bulges, DM halos
  - **NFW**: alpha=1, beta=3 —> dark-matter halos

Hernquist profile

- Simple model for bulges, elliptical galaxies, and DM halos
- More tractable than NFW: has finite mass and simple form of potential

$$\Phi(r) = -\frac{GM}{r+a} \ .$$

Two-Power-law models



NFW profile

- “Standard” model for dark-matter halos: resulting DM profile in cosmological simulations of structure formation, origin not particularly well understood

- Parameterized in many different ways:

    - ρ_0 and a

    - Mass, concentration: mass to ‘virial radius’, concentration = (virial radius)/a

    - Vmax, rmax: peak of the rotation curve and radius of the peak

Break

Equation of Motions and Recap of Classical Mechanics

Assume test particle mass m = 1

$$\ddot{x} = F_x = -(\nabla\Phi)_x = -\frac{\partial \Phi}{\partial x}$$
$$\ddot{y} = F_y = -(\nabla\Phi)_y = -\frac{\partial \Phi}{\partial y}$$
$$\ddot{z} = F_z = -(\nabla\Phi)_z = -\frac{\partial \Phi}{\partial z}$$

$$\dot{v}_x = -(\nabla\Phi)_x \quad \dot{x} = v_x$$
$$\dot{v}_y = -(\nabla\Phi)_y \quad \dot{y} = v_y$$
$$\dot{v}_z = -(\nabla\Phi)_z \quad \dot{z} = v_z$$

Lagrangian in cartesian coordinates

kinetic energy potential energy

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \Phi(x, y, z)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial \mathcal{L}}{\partial y}$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} = \frac{\partial \mathcal{L}}{\partial z}$$

Lagrangian in cylindrical coordinates

$$\mathcal{L} = \frac{1}{2}(\dot{R}^2 + (R\dot{\phi})^2 + \dot{z}^2) - \Phi(R, \phi, z)$$

$$\ddot{R} = -\frac{\partial \Phi}{\partial R} + R\dot{\phi}^2$$
$$\frac{d}{dt}(R^2\dot{\phi}) = -\frac{\partial \Phi}{\partial \phi}$$
$$\ddot{z} = -\frac{\partial \Phi}{\partial z}$$

Orbits in spherical potentials

$$\Phi = \Phi(r)$$

Equations of motion in a spherical potential

- Only non-zero component of the force in spherical coordinates is the radial component and Newton's second law becomes

$$\ddot{\mathbf{r}} = g_r(r)\,\hat{\mathbf{r}}$$

- This implies that the total specific angular momentum $L = r \times \dot{r}$ is conserved

$$\begin{aligned}
\dot{\mathbf{L}} &= \mathbf{N} \\
&= \mathbf{r} \times \mathbf{g} \\
&= g_r(r)\, r\, \hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0\,,
\end{aligned}$$

The orbital plane

- Because the angular momentum vector is conserved, position **r** and velocity **rdot** are always perpendicular to constant **L** —> motion is confined to a plane perpendicular to the angular momentum vector

- Thus, we can focus on the motion within the *orbital plane*

Motion in the orbital plane

Lagrangian: T-V $$\mathcal{L} = \frac{1}{2}((r\dot{\phi})^2 + \dot{r}^2) - \Phi(r)$$

Energy: T+V $$E = \frac{1}{2}((r\dot{\phi})^2 + \dot{r}^2) + \Phi(r)$$

Assume test particle mass m = 1

$$\ddot{x} = F_x = -(\nabla\Phi)_x = -\frac{\partial\Phi}{\partial x}$$
$$\ddot{y} = F_y = -(\nabla\Phi)_y = -\frac{\partial\Phi}{\partial y}$$
$$\ddot{z} = F_z = -(\nabla\Phi)_z = -\frac{\partial\Phi}{\partial z}$$

$$\dot{v}_x = -(\nabla\Phi)_x \quad \dot{x} = v_x$$
$$\dot{v}_y = -(\nabla\Phi)_y \quad \dot{y} = v_y$$
$$\dot{v}_z = -(\nabla\Phi)_z \quad \dot{z} = v_z$$

Lagrangian in cartesian coordinates

kinetic energy potential energy

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \Phi(x, y, z)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial\mathcal{L}}{\partial x}$$
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}} = \frac{\partial\mathcal{L}}{\partial y}$$
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{z}} = \frac{\partial\mathcal{L}}{\partial z}$$

Lagrangian in cylindrical coordinates

$$\mathcal{L} = \frac{1}{2}(\dot{R}^2 + (R\dot{\phi})^2 + \dot{z}^2) - \Phi(R, \phi, z)$$

$$\ddot{R} = -\frac{\partial\Phi}{\partial R} + R\dot{\phi}^2$$
$$\frac{d}{dt}(R^2\dot{\phi}) = -\frac{\partial\Phi}{\partial\phi}$$
$$\ddot{z} = -\frac{\partial\Phi}{\partial z}$$

Motion in the orbital plane

Lagrangian: T-V $\mathcal{L} = \frac{1}{2}((r\dot{\phi})^2 + \dot{r}^2) - \Phi(r)$

Energy: T+V $E = \frac{1}{2}((r\dot{\phi})^2 + \dot{r}^2) + \Phi(r)$

Equations of Motion

$$\ddot{r} = r\dot{\phi}^2 - \frac{d\Phi}{dr}$$

$$\frac{d}{dt}(r^2\dot{\phi}) = 0$$

- The last equation is just another manifestation of the conservation of angular momentum as $L = r^2\dot{\phi}$

Motion in the orbital plane

$$\ddot{r} = r\dot{\phi}^2 - \frac{d\Phi}{dr}$$
$$L = r^2\dot{\phi}$$

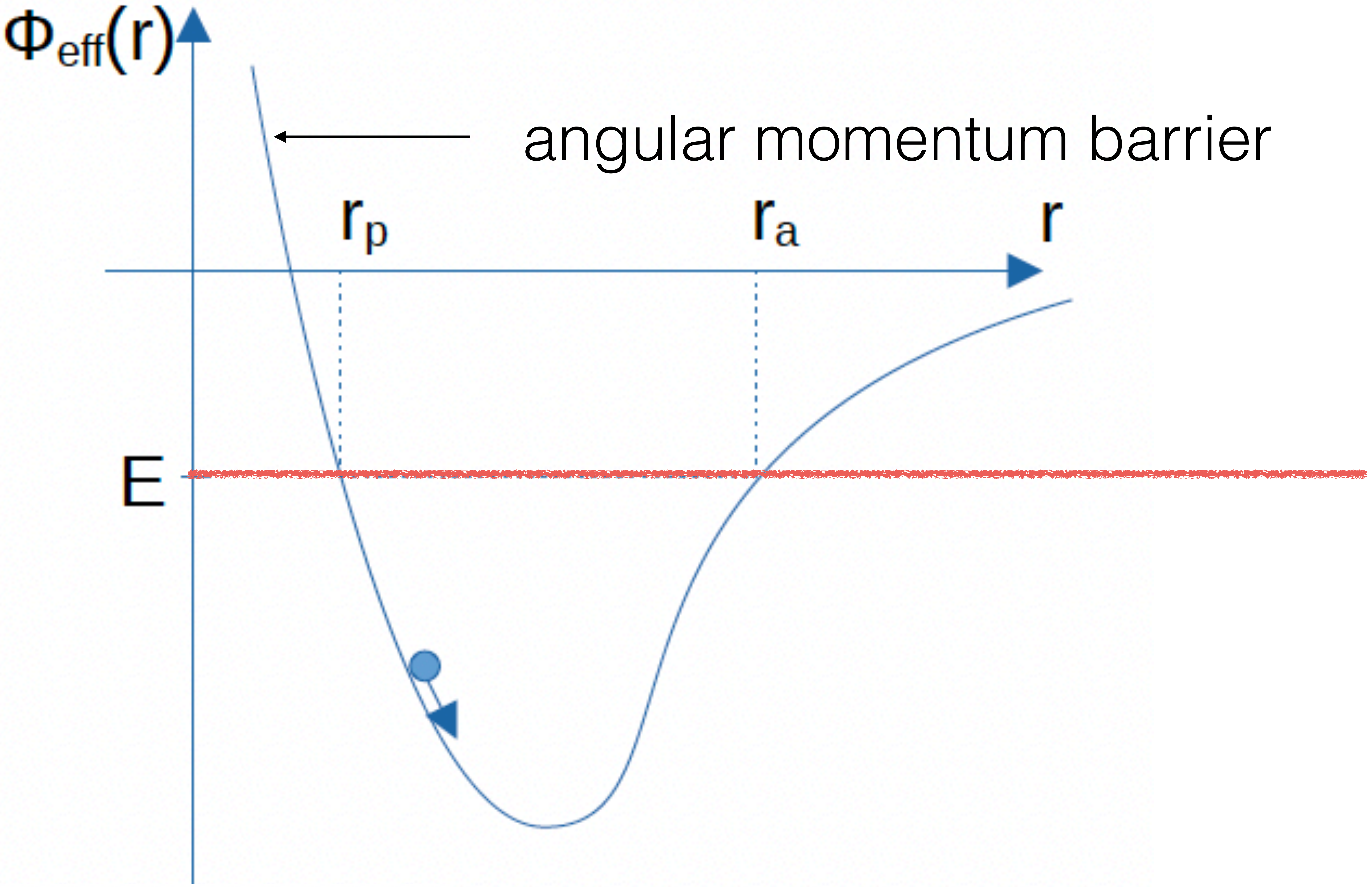
$$\longrightarrow \quad \ddot{r} = -\frac{d\Phi}{dr} + L^2/r^3 = -\frac{d\Phi_{eff}(r)}{dr}$$

effective potential $$\boxed{\Phi_{eff}(r) = \Phi(r) + L^2/2r^2}$$

energy $$E = \frac{1}{2}((r\dot{\phi})^2 + \dot{r}^2) + \Phi(r)$$

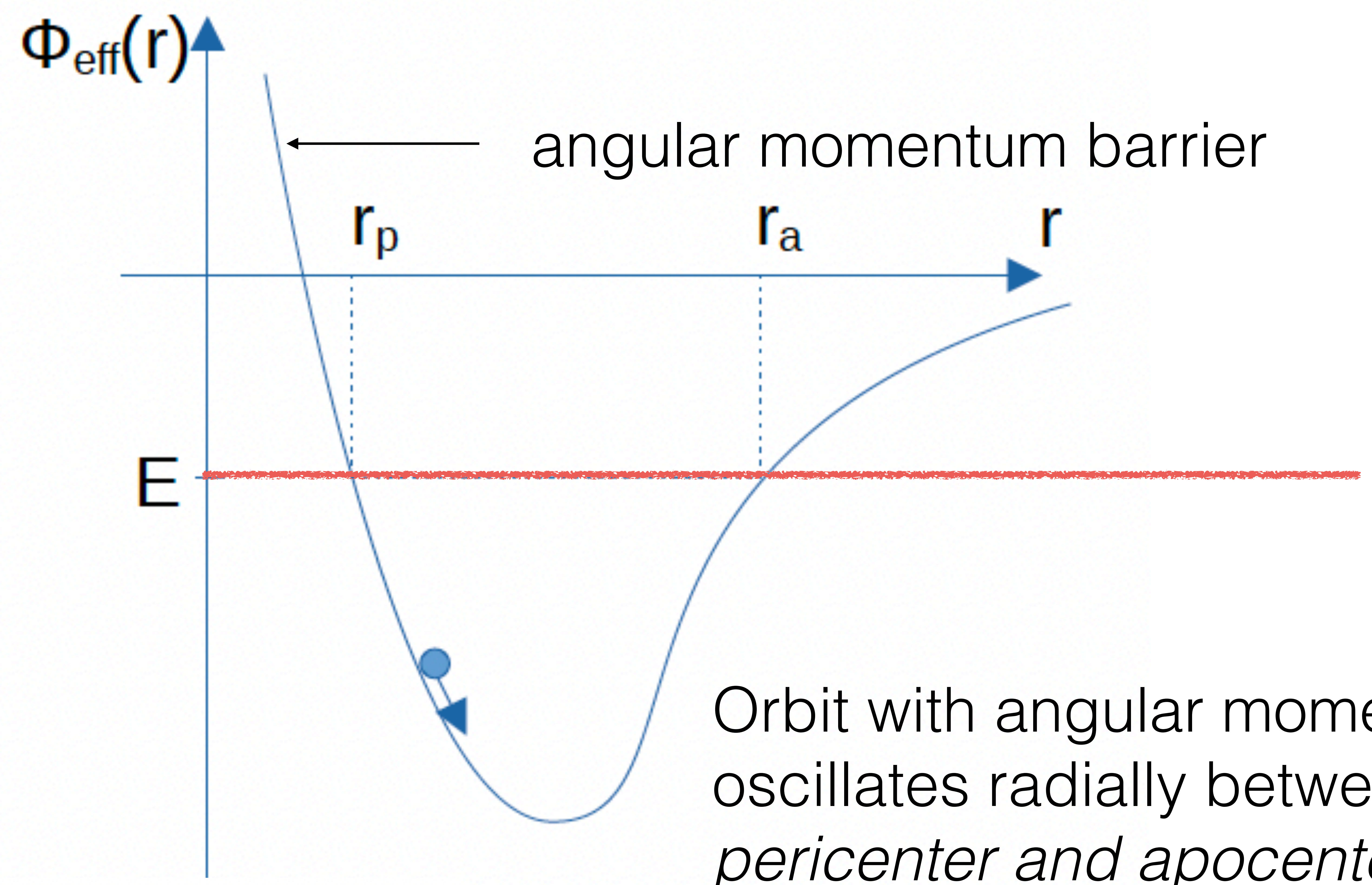
$$\longrightarrow \quad E = \Phi_{eff}(r) + \frac{1}{2}\dot{r}^2 = \Phi(r) + \frac{L^2}{2r^2} + \frac{1}{2}\dot{r}^2$$

Effective potential



$$E = \Phi_{eff}(r) + \frac{1}{2}\dot{r}^2 = \Phi(r) + \frac{L^2}{2r^2} + \frac{1}{2}\dot{r}^2$$

Effective potential

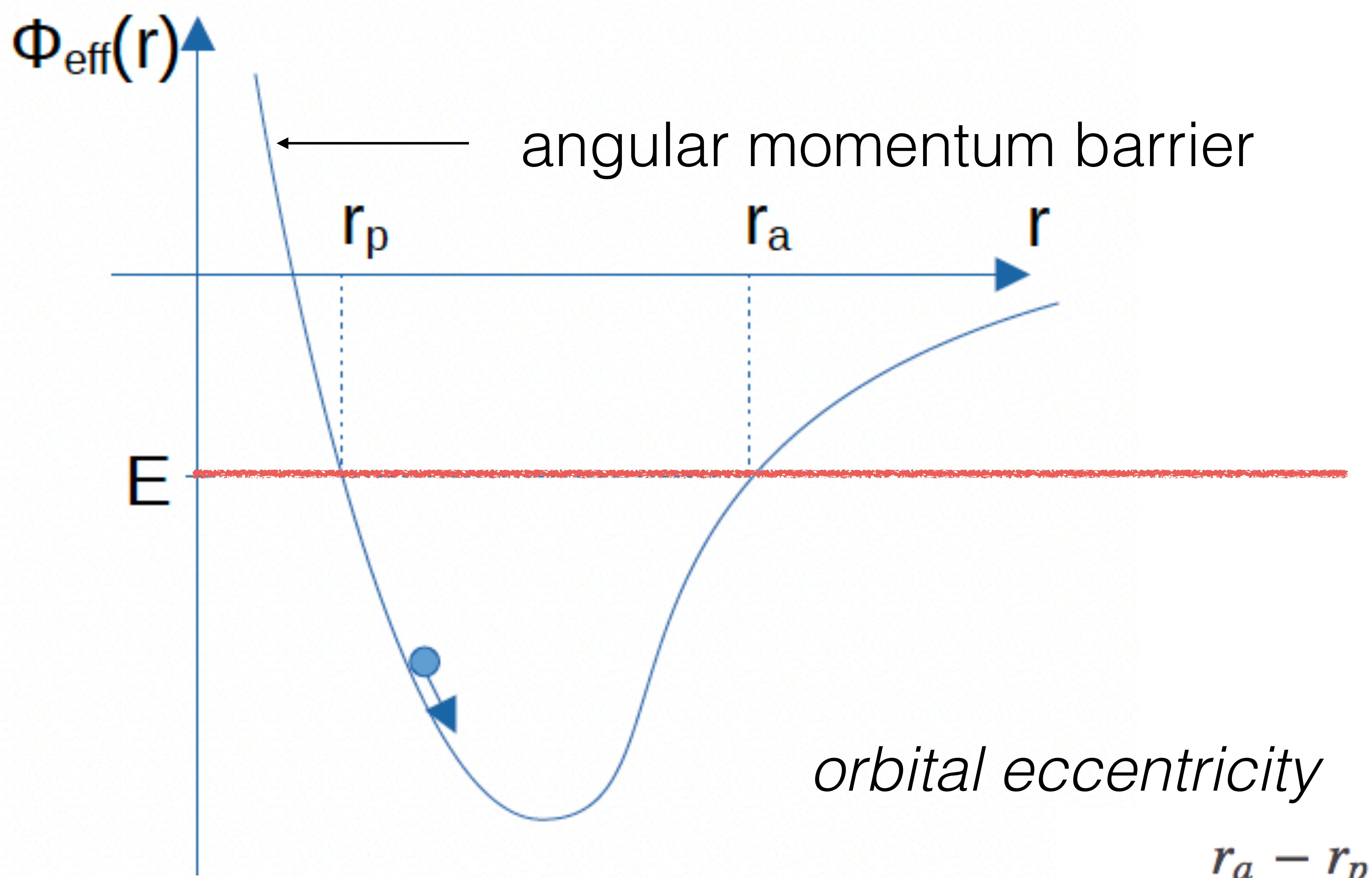


Orbit with angular momentum L oscillates radially between r_p and r_a, *pericenter and apocenter*

$$E = \Phi_{eff}(r) + \frac{1}{2}\dot{r}^2 = \Phi(r) + \frac{L^2}{2r^2} + \frac{1}{2}\dot{r}^2$$

$$E - \Phi(r_a) = \frac{L^2}{2r_a^2}$$

Effective potential



orbital eccentricity

$$e = \frac{r_a - r_p}{r_a + r_p} \, .$$

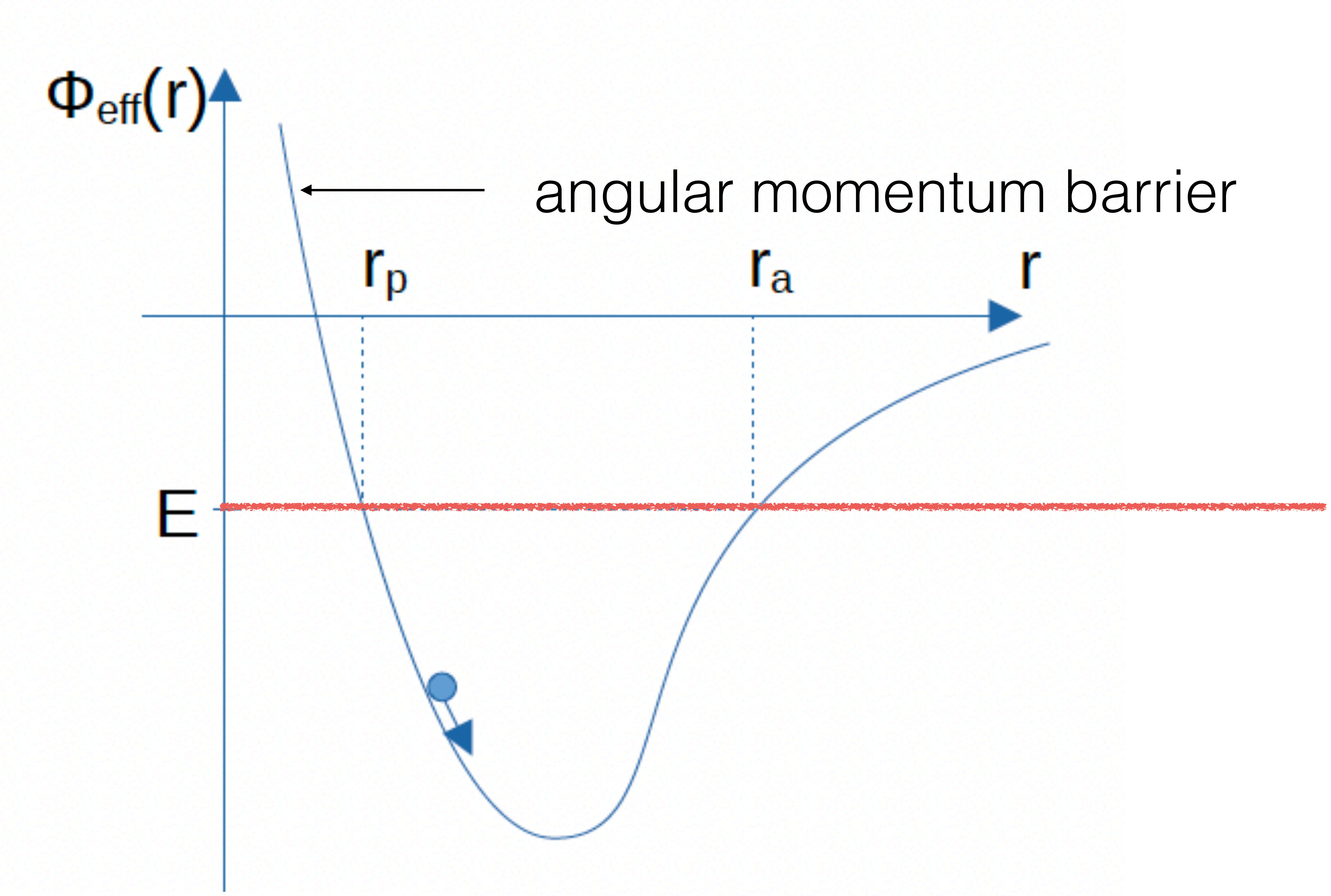
$$E = \Phi_{eff}(r) + \frac{1}{2}\dot{r}^2 = \Phi(r) + \frac{L^2}{2r^2} + \frac{1}{2}\dot{r}^2$$

$$E - \Phi(r_a) = \frac{L^2}{2r_a^2}$$

For a given potential, the orbital shape is determined by angular momentum L and energy E.

But will the orbits be closed or not?

Effective potential



$$E = \Phi_{eff}(r) + \frac{1}{2}\dot{r}^2 = \Phi(r) + \frac{L^2}{2r^2} + \frac{1}{2}\dot{r}^2$$

Radial and azimuthal period

- Radial oscillation has a period T_r: the *radial period*

$$T_r = 2 \int_{r_p}^{r_a} \mathrm{d}r \, \frac{1}{\sqrt{2[E - \Phi(r)] - L^2/r^2}} \, ,$$

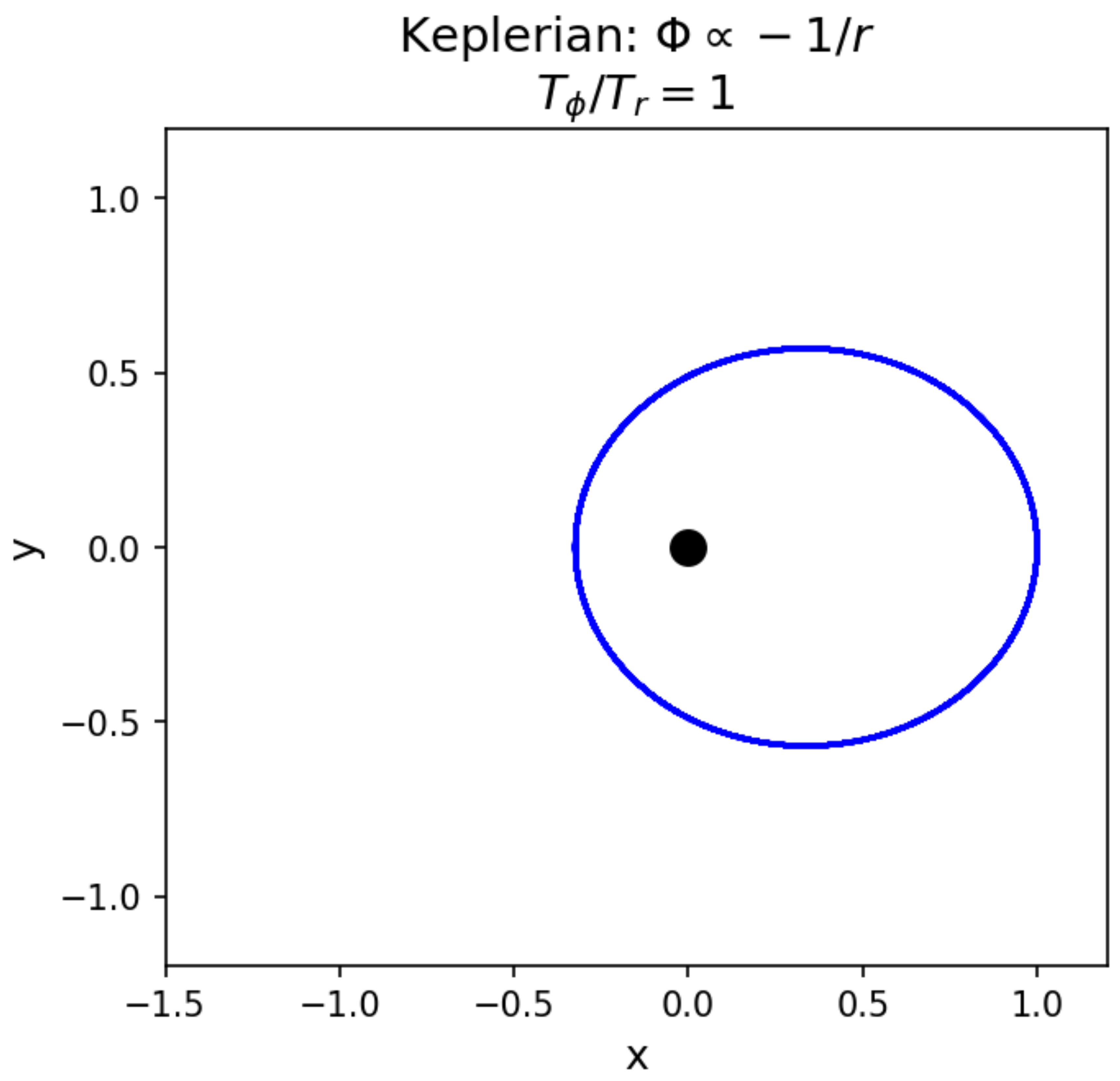
- In one radial period go through the following range in azimuth

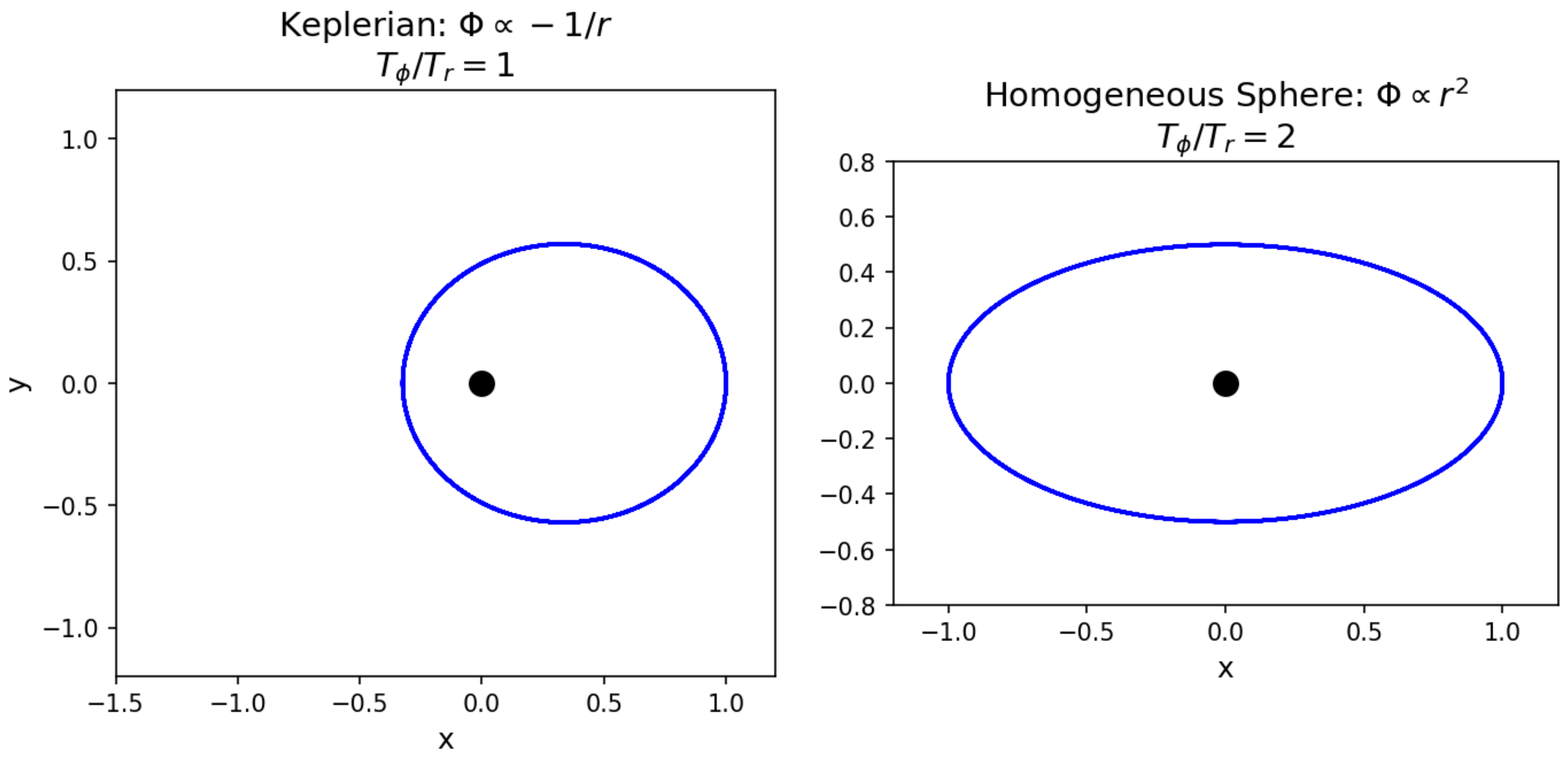
$$\Delta\phi = 2\,L \int_{r_p}^{r_a} \mathrm{d}r \, \frac{1}{r^2\sqrt{2[E - \Phi(r)] - L^2/r^2}} \, ,$$

- And the azimuthal period is

$$T_\phi = \frac{2\pi}{|\,\Delta\phi\,|}\, T_r$$

Closed orbit when T_ϕ/T_r = integer


Keplerian: $\Phi \propto -1/r$
$T_\phi/T_r = 1$
1.0
0.5
0.0
−0.5
−1.0
y
−1.5
−1.0
−0.5
0.0
0.5
1.0
x


Keplerian: $\Phi \propto -1/r$
$T_\phi/T_r = 1$
y
x
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
Homogeneous Sphere: $\Phi \propto r^2$
$T_\phi/T_r = 2$
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
−0.5
0.0
0.5
1.0
x

Orbits in the homogeneous sphere vs. in the Kepler potential

- Homogeneous sphere to Kepler potential spans the range of plausible spherical potentials

- Orbits are ellipses in both!

- Homogeneous sphere: radial period = azimuthal period / 2

- Kepler potential: radial period = azimuthal period

- Thus, for any spherical potential the radial period is somewhere between half and once the azimuthal period —> stars oscillate radially more rapidly than azimuthally

Keplerian: $\Phi \propto -1/r$
$T_\phi/T_r = 1$
y
x
Homogeneous Sphere: $\Phi \propto r^2$
$T_\phi/T_r = 2$
x
Isochrone: $\Phi = -GM/(b + \sqrt{b^2 + r^2})$
$1 < T_\phi/T_r < 2$
x

Keplerian: $\Phi \propto -1/r$
$T_\phi/T_r = 1$
Homogeneous Sphere: $\Phi \propto r^2$
$T_\phi/T_r = 2$
Isochrone: $\Phi = -GM/(b+\sqrt{b^2+r^2})$
$1 < T_\phi/T_r < 2$
x
y