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Z boson decay into photons

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Abstract. We perform a complete $\mathcal{O}(\alpha^4)$ calculation of the coupling of the Z boson to three photons via both fermion and W boson loops and keeping the full dependence on the quark and W masses. To evaluate the W boson contribution to the fourth rank polarization tensor we use the unitary gauge. We find that the contributions from fermion and boson loops are remarkably similar. Expressions for the helicity amplitudes are presented. The results are applied to the decay $Z \to \gamma\gamma\gamma$ where we find a partial width of about 1.35 eV for $m_{\text{top}} > 91$ GeV and $\sin^2\theta_W = 0.23$, of which the W boson loops account for approximately 0.3 eV mainly through their interference with the fermion loops.

1 Introduction

One interesting aspect of quantum field theories is the generation of interactions that are not present at the classical level. Such interactions occur when a virtual pair of particles is emitted, radiate further particles and are then reabsorbed. The most famous example of this is the scattering of light by light which was first studied in the context of quantum electrodynamics [1]. This process occurs at $\mathcal{O}(\alpha^4)$ when four photons are attached to a charged fermion loop. Within the $SU(2) \times U(1)$ model of electroweak interactions, there is also a contribution at the same order from charged boson loops. Although it has been shown that the boson contribution is finite for photon-photon scattering [2, 3], as indeed it must be in a renormalizable theory, the effects of the W loops have not been widely studied.

Recently, prompted by experiments at LEP, attention has focussed on the decay of the Z boson into photons [4–6]. The two photon decay is forbidden by Yang's theorem, however the three photon decay is allowed and the fermion contribution is well known [7–9]. As dictated by the Appelquist-Carrazone decoupling theorem [10], the top quark contribution rapidly decouples for $m_{\text{top}} > M_Z/2$ and can essentially be ignored. The remaining light leptons and quarks give a contribution to the amplitude proportional to the vector coupling with the Z boson, v_f and the cube of the electric charge, e_f. A closed form for the light fermion contribution to the partial width can be found [11, 9],

$$\Gamma(Z \to \gamma\gamma\gamma) = \alpha^3 \alpha_Z \left(3 \sum_q e_q^3 v_q + \sum_l e_l^3 v_l\right)^2 \frac{1}{6} \frac{M_Z}{12\pi^3} X, \tag{1}$$

where,

$$X = 200\,\zeta_5 - 8\pi^2 \zeta_3 + \frac{7}{15}\pi^4 - 128\zeta_3 + \frac{41}{3}\pi^2 - 124 \sim 14.954, \tag{2}$$

and ζ_n is the nth Riemann zeta function. The factor 3 multiplying the quark contribution is due to colour. For five flavours of light quarks, $\alpha(M_Z) = 1/128$, $\alpha_Z = \alpha(M_Z)/\sin^2\theta_W \cos^2\theta_W$ and $\sin^2\theta_W = 0.23$, we find,

$$\Gamma(Z \to \gamma\gamma\gamma) = 1.05 \text{ eV}. \tag{3}$$

In addition to the fermion loop contribution, there is also a contribution from $W^\pm$ boson loops which probes the non-abelian nature of the electroweak model. Both trilinear ($WW\gamma$ and WWZ) and quartic ($WW\gamma\gamma$ and $WW\gamma Z$) vertices contribute and in principle this provides a test of these couplings. In practice, however, if these couplings deviate from the structure dictated by the $SU(2) \times U(1)$ gauge theory, the W boson loop contribution is incalculable. Three calculations exist in the literature. Baillargeon and Boudjema [4] use a non-linear R_ξ gauge [12], while Pham [5] and Dong et al. [6] use a linear R_ξ gauge. By making an approximation where M_W is large compared to all other scales in the problem, [4, 6] find that the W loop alone contributes about 0.02 eV to the Z boson width, approximately 50 times smaller than the fermion contribution. Pham, estimates that the total $Z \to \gamma\gamma\gamma$ width for both fermion and boson loops is about 2 eV. No estimate exists where the exact dependence on both M_W and the unknown top quark mass is kept.

It is worth noting that in all three cases, the 't Hooft-Feynman gauge is chosen so that $\xi = 1$. This gauge has the particular advantage that the $k^\mu k^\nu$ part of the W boson propagator is zero so that individual diagrams do not contain superficial divergences. On the other hand, these gauges do contain many more diagrams than the unitary gauge due to the propagation of the unphysical goldstone boson and ghost fields. For the purposes of our paper, we choose to minimize the number of Feynman diagrams and use the unitary gauge. It is straightforward to cancel the superficial divergences before reducing the tensor integral to scalar integrals in the usual way [13].

The organization of our paper is as follows. In Sect. 2, we construct the polarization tensor for the $Z\gamma\gamma\gamma$ coupling. Due to gauge invariance and the possibility of exchanging identical photons, this tensor be described by three independent scalar amplitudes (rather than four [6, 14]). We find that the boson amplitudes are (surprisingly) closely related to the fermion amplitudes. In order to translate the polarization tensor into a physical decay width, we introduce helicity amplitudes in Sect. 3. Numerical results for the $Z\to\gamma\gamma\gamma$ partial width are given in Sect. 4, while the main results are summarized in Sect. 5. Appendix A contains some definitions of integrals appearing in the calculation.

2 The $Z\gamma\gamma\gamma$ polarization tensor

The matrix element, $\mathscr{T}$, for the scattering of an on-shell Z boson with three on-shell photons can be written as,

$$\mathscr{T}=\varepsilon_\alpha(p_4)\varepsilon_\mu(p_1)\varepsilon_\nu(p_2)\varepsilon_\rho(p_3)\mathscr{T}^{\alpha\mu\nu\rho}(p_1,p_2,p_3), \tag{4}$$

where we denote the ingoing momenta and Lorentz indices of the photons by p_1^μ, p_2^ν, p_3^ρ while the momentum and Lorentz index for the Z boson is p_4^α. Using momentum conservation, the momentum of the Z boson is related to the photon momenta by,

$$p_4^\alpha=-p_1^\alpha-p_2^\alpha-p_3^\alpha, \tag{5}$$

so that the polarization tensor for the $Z\gamma\gamma\gamma$ coupling, $\mathscr{T}^{\alpha\mu\nu\rho}(p_1,p_2,p_3)$, depends only on the three photon momenta.

The most general fourth rank tensor contains 81 terms of the type $p_i^\mu p_j^\nu p_k^\rho p_l^\alpha$, 54 terms of the type $p_i^\mu p_j^\nu g^{\rho\alpha}$ and 3 terms of the form $g^{\mu\nu}g^{\rho\alpha}$. However, the general tensor must have certain properties which provide relations between the different terms and allow for a somewhat simpler expression. For example, using the fact that in the matrix element, the tensor is always contracted with physical polarizations for on-shell photons allows us to make the identification,

$$p_1^\mu=p_2^\nu=p_3^\rho=0. \tag{6}$$

This drastically reduces the number of terms to 24 terms of the type $p_i^\mu p_j^\nu p_k^\rho p_l^\alpha$ and 30 terms of the type $p_i^\mu p_j^\nu g^{\rho\alpha}$ while the number of $g^{\mu\nu}g^{\rho\alpha}$ terms remains 3.

We also note that the tensor must be completely symmetric under interchange of the photon momenta and indices,

$$p_1^\mu\leftrightarrow p_2^\nu\leftrightarrow p_3^\rho\leftrightarrow p_1^\mu. \tag{7}$$

It is therefore useful to write the tensor in a manifestly symmetric way,

$$\mathscr{T}^{\alpha\mu\nu\rho}(p_1,p_2,p_3)=\sum_{perm}\mathscr{M}^{\alpha\mu\nu\rho}(p_1,p_2,p_3), \tag{8}$$

where the sum is over the six possible permutations of the photon four momenta. Gauge invariance requires that the tensor is transverse,

$$\begin{aligned}p_{1\mu}\mathscr{T}^{\alpha\mu\nu\rho}(p_1,p_2,p_3)&=p_{2\nu}\mathscr{T}^{\alpha\mu\nu\rho}(p_1,p_2,p_3)\\&=p_{3\rho}\mathscr{T}^{\alpha\mu\nu\rho}(p_1,p_2,p_3)=0.\end{aligned} \tag{9}$$

This reduces the number of independent terms to 3 so that,

$$\begin{aligned}&\mathscr{M}^{\alpha\mu\nu\rho}(p_1,p_2,p_3)\\&=A_1(p_1,p_2,p_3)\frac{1}{p_1\cdot p_3}\left(\frac{p_3^\mu p_1^\rho}{p_1\cdot p_3}-g^{\mu\rho}\right)p_1^\alpha\left(\frac{p_3^\nu}{p_2\cdot p_3}-\frac{p_1^\nu}{p_1\cdot p_2}\right)\\&\quad+A_2(p_1,p_2,p_3)\left\{\frac{1}{p_2\cdot p_3}\left(\frac{p_1^\alpha p_3^\mu}{p_1\cdot p_3}-g^{\alpha\mu}\right)\left(\frac{p_1^\nu p_2^\rho}{p_1\cdot p_2}-g^{\nu\rho}\right)\right.\\&\quad\left.+\frac{1}{p_1\cdot p_3}\left(\frac{p_1^\nu}{p_1\cdot p_2}-\frac{p_3^\nu}{p_2\cdot p_3}\right)(p_1^\rho g^{\alpha\mu}-p_1^\alpha g^{\mu\rho})\right\}\\&\quad+A_3(p_1,p_2,p_3)\frac{1}{p_1\cdot p_3}\left(\frac{p_1^\alpha p_3^\mu}{p_1\cdot p_3}-g^{\alpha\mu}\right)\left(\frac{p_3^\nu p_2^\rho}{p_2\cdot p_3}-g^{\nu\rho}\right).\end{aligned} \tag{10}$$

Finally, we note that the tensor is also transversal with respect to the Z boson momentum,

$$p_{4\alpha}\mathscr{T}^{\alpha\mu\nu\rho}(p_1,p_2,p_3)=0. \tag{11}$$

The three functions, A_i, receive contributions from the fermion and boson loops such that,

$$\begin{aligned}&A_i(p_1,p_2,p_3)=A_i(s,t)\\&=\frac{ie^3g_Z}{16\pi^2}\left(\sum_f e_f^3 v_f A_i^f(s,t,m_f)+\cos^2\theta_W A_i^b(s,t,M_W)\right),\end{aligned} \tag{12}$$

where the sum runs over fermions with mass m_f, electric charge e_f and vector coupling to the Z boson v_f. We have also introduced the Mandelstam variables s, t and u,

$$\begin{aligned}&s=(p_1+p_2)^2=2p_1\cdot p_2,\quad s_1=s-M_Z^2,\\&t=(p_2+p_3)^2=2p_2\cdot p_3,\quad t_1=t-M_Z^2,\\&u=(p_3+p_1)^2=2p_3\cdot p_1,\quad u_1=u-M_Z^2.\end{aligned} \tag{13}$$

The different photon permutations are obtained by exchanging the Mandelstam variables in the obvious way,

$$A_i(p_3,p_1,p_2)=A_i(u,s),\quad A_i(p_1,p_3,p_2)=A_i(u,t), \tag{14}$$

and so on.

It is straightforward to obtain the scalar functions A_i^f and A_i^b using the standard techniques for one loop integrals of reducing the tensor integral to a combination of scalar integrals [13]. There are three types of boson loop as shown in Fig. 1 containing triple and quartic boson couplings. As mentioned earlier, we use the unitary gauge so that only physical particles can propagate – there are no diagrams containing ghosts or unphysical Goldstone bosons. As a penalty, each diagram is superficially divergent. However, all graphs taken together are finite and we find that,

$$A_1^b(s,t,M_W)=\frac{1}{4}\left(\frac{M_Z^2}{M_W^2}-6\right)A_1^f(s,t,M_W),$$

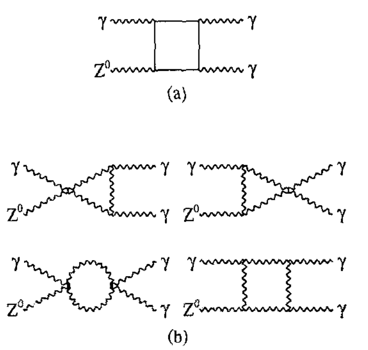

Fig. 1. Feynman diagrams in the unitary gauge for Z boson decay into three photons **a** via fermion loops and **b** via W boson loops

$$\begin{aligned}A_2^b(s,t,M_W)=&\frac{1}{4}\left(\frac{M_Z^2}{M_W^2}-6\right)A_2^f(s,t,M_W)\\&+\frac{1}{4}\left(\frac{M_Z^2}{M_W^2}+10\right)\Big(2M_W^2ut\,F(M_W)\\&-\frac{ut}{s}E(t,u,M_W)\Big)-2sut\,F(M_W),\\A_3^b(s,t,M_W)=&\frac{1}{4}\left(\frac{M_Z^2}{M_W^2}-6\right)A_3^f(s,t,M_W)\\&+\frac{1}{4t}\left(\frac{M_Z^2}{M_W^2}+10\right)\{t^2E(s,t,M_W)\\&-u^2E(u,s,M_W)+2M_W^2ut(u-t)F(M_W)\},\end{aligned}\tag{15}$$

where the functions E and F are combinations of scalar integrals and are defined in the Appendix.

As a consistency check, we have constructed the complete tensor to make sure that reinserting the scalar functions into (8) and (10) does indeed regenerate the full tensor which is therefore automatically gauge invariant with respect to the external photons.

It is quite remarkable that the boson contribution is so similar to that of a fermion loop, for which we find,

$$\begin{aligned}&A_1^f(s,t,m_f)\\&=\frac{4st}{t_1}+\frac{8t}{u}(sB_1(s,m_f)-s_1B_1(t,m_f))\\&-\frac{4M_Z^2(s+2u)t}{t_1^2}B_1(t,m_f)+\frac{2st(2t+u)}{u^2}E(s,t,m_f)\\&+\frac{8m_f^2t}{u}E(s,t,m_f)+\frac{4m_f^2t}{s}E(t,u,m_f)\\&+4m_f^2(sC(s,m_f)+tC(t,m_f)+u_1C_1(u,m_f))\\&-\frac{8m_f^2(s+2u)t}{t_1}C_1(t,m_f)-\frac{4m_f^2st(u+2t)}{u}D(s,t,m_f)\\&-2m_f^2(ut\,D(t,u,m_f)+st\,D(s,t,m_f)\\&+usD(u,s,m_f))-8m_f^4t\,F(m_f),\end{aligned}\tag{16}$$

$$\begin{aligned}&A_2^f(s,t,m_f)\\&=\frac{4(2s-u)t}{3t_1}-\frac{2s(u-8t)}{3u}B_1(s,m_f)\\&-\frac{2u(s+4t)}{3s}B_1(u,m_f)+\frac{2t(3su-4ut+8ts)}{3su}B_1(t,m_f)\\&-\frac{4M_Z^2ut}{t_1^2}B_1(t,m_f)-\frac{4ut^2}{3s^2}E(t,u,m_f)-\frac{su}{3t}E(s,u,m_f)\\&+\frac{(3stu+8st^2)}{3u^2}E(s,t,m_f)-\frac{8m_f^2ut}{t_1}C_1(t,m_f)\\&+\frac{2m_f^2t(ut_1-4st)}{u}D(s,t,m_f)+2m_f^2uu_1D(u,s,m_f)\\&+\frac{8m_f^2sut^2}{3}\left(\frac{1}{s^2}D(t,u,m_f)+\frac{1}{t^2}D(u,s,m_f)\right.\\&\left.+\frac{1}{u^2}D(s,t,m_f)\right)+\frac{4m_f^2t}{u}E(s,t,m_f)-\frac{8m_f^4t}{3}F(m_f)\end{aligned}\tag{17}$$

and,

$$\begin{aligned}&A_3^f(s,t,m_f)\\&=\frac{4u}{3}-\frac{4s}{3}\left(\frac{2t-u}{t}B_1(s,m_f)+\frac{(2s-u)t}{s^2}B_1(t,m_f)\right.\\&\left.+\frac{u_1u^2}{s^2t}B_1(u,m_f)\right)-\frac{(s-3t_1)t}{3u}E(s,t,m_f)\\&+\frac{(t-2u_1)u^2}{3t^2}E(u,s,m_f)+\frac{2u^2t}{3s^2}E(t,u,m_f)\\&+\frac{4m_f^2u}{t}(sC(s,m_f)+tC(t,m_f)+u_1C_1(u,m_f))\\&+\frac{2m_f^2(4st+6tu-3M_Z^2u)}{3}D(s,t,m_f)\\&+\frac{2m_f^2ut(3s-2u)}{3s}D(t,u,m_f)\\&+\frac{2m_f^2u(3t^2-3tu-5su)}{3t}D(u,s,m_f)-\frac{8m_f^4u}{3}F(m_f),\end{aligned}\tag{18}$$

where the scalar integrals B_1, C, C_1 and D are defined in the Appendix. We have checked that this reproduces the result for the vector coupling of the Z boson with three gluons given in [8].

3 The $Z\gamma\gamma\gamma$ helicity amplitudes

Using the $Z\gamma\gamma\gamma$ polarization tensor given in the previous section, we can construct the corresponding helicity amplitudes. For simplicity, we work in the rest frame of the p_1 and p_2 system where the momenta are given in (E, p_x, p_y, p_z) notation by,

$$\begin{aligned}p_1^\mu&=(-p,0,0,-p),\\p_2^\mu&=(-p,0,0,p),\\p_3^\mu&=(q,q\sin\theta,0,q\cos\theta),\\p_4^\mu&=(E,-q\sin\theta,0,-q\cos\theta),\end{aligned}\tag{19}$$

where all momenta are ingoing and where $E=\sqrt{q^2+M_Z^2}$.

The appropriate helicity vectors are then,

$$e_1^{+\mu}=e_2^{-\mu}=\frac{1}{\sqrt{2}}(0,\,-i,\,1,\,0),$$

$$e_1^{-\mu}=e_2^{+\mu}=\frac{1}{\sqrt{2}}(0,\,i,\,1,\,0),$$

$$e_3^{+\mu}=e_4^{-\mu}=\frac{1}{\sqrt{2}}(0,\,i\cos\theta,\,1,\,-i\sin\theta),$$

$$e_3^{-\mu}=e_4^{+\mu}=\frac{1}{\sqrt{2}}(0,\,-i\cos\theta,\,1,\,i\sin\theta),$$

$$e_0=\frac{1}{M_Z}(q,\,-E\sin\theta,\,0,\,-E\cos\theta). \tag{20}$$

Here $e_i^{\pm}$ represents the $\lambda_i=\pm$ polarization vector of particle i while e_0 represents a Z boson that is longitudinally polarized. It is useful to define the quantity,

$$\Delta=\sqrt{\frac{-M_Z^2}{2stu}}, \tag{21}$$

which occurs in all helicity amplitudes associated with a longitudinally polarized Z boson.

In terms of the three independent functions given in (10) and (12), we find that there are nine independent helicity amplitudes, $\mathscr{T}_{\lambda_1\lambda_2\lambda_3\lambda_Z}$, with $\lambda_1=+$ which are given by,

$$\mathscr{T}_{++++}=2\left(\frac{A_1(t,u)}{s_1}+\frac{A_2(s,t)+A_2(u,t)+A_3(u,s)}{t}+(t\leftrightarrow u)\right),$$

$$\mathscr{T}_{+++-}=-2\left(\frac{A_1(t,u)+A_1(u,t)}{s_1}\right),$$

$$\mathscr{T}_{++-+}=2\left(\frac{A_1(s,t)-A_1(u,t)-A_2(s,t)+A_2(u,t)}{s_1}\right.$$
$$\left.+\frac{A_3(s,t)}{u}-\frac{uA_3(u,t)}{ss_1}+(t\leftrightarrow u)\right),$$

$$\mathscr{T}_{++--}=2\left(\frac{-A_1(s,t)+A_1(u,t)+A_2(s,t)-A_2(u,t)}{s_1}\right.$$
$$\left.+\frac{A_3(t,s)}{u}-\frac{tA_3(u,t)}{ss_1}+(t\leftrightarrow u)\right),$$

$$\mathscr{T}_{+-++}=2\left(\frac{A_1(s,u)-A_2(s,u)-A_2(t,u)}{s_1}\right.$$
$$\left.+\frac{A_2(t,s)+A_2(u,s)}{s}+\frac{A_3(s,t)}{u}-\frac{uA_3(u,t)}{ss_1}\right),$$

$$\mathscr{T}_{+-+-}=-2\left(\frac{A_1(s,u)}{s_1}+\frac{t(A_2(s,u)+A_2(t,u))}{us_1}\right.$$
$$\left.+\frac{tA_3(u,t)}{ss_1}\right), \tag{22}$$

while for the amplitudes where the Z boson is longitudinally polarized,

$$\mathscr{T}_{+++0}=\frac{2\Delta}{M_Z^2}\left(\frac{su-tM_Z^2}{s_1}A_1(t,u)\right.$$
$$\left.+s(A_2(s,u)+A_2(t,u))+\frac{suA_3(u,s)}{t}-(t\leftrightarrow u)\right)$$

$$\mathscr{T}_{++-0}=\frac{2\Delta}{M_Z^2}\left(\frac{su-tM_Z^2}{s_1}(A_1(s,u)-A_1(t,u)+A_2(t,u)\right.$$
$$-A_2(s,u))+s_1(A_1(t,s)-A_2(t,s))-\frac{st}{u}(A_3(s,t)$$
$$\left.-A_3(t,s))+\frac{ut(s+M_Z^2)}{ss_1}A_3(u,t)-(t\leftrightarrow u)\right),$$

$$\mathscr{T}_{+-+0}=\frac{2\Delta}{M_Z^2}\left(\frac{su-tM_Z^2}{s_1}A_1(s,u)-s_1A_1(u,s)\right.$$
$$+\frac{t(s+M_Z^2)}{s_1}(A_2(s,u)+A_2(t,u))$$
$$-t(A_2(t,s)+A_2(u,s))-\frac{st}{u}A_3(s,t)$$
$$\left.+\frac{ut(s+M_Z^2)}{ss_1}A_3(u,t)\right). \tag{23}$$

The other three helicity amplitudes with $\lambda_1=+$ are obtained by exchanging u and t,*

$$\mathscr{T}_{+--+}=\mathscr{T}_{+-+-}(t\leftrightarrow u),$$

$$\mathscr{T}_{+---}=\mathscr{T}_{+-++}(t\leftrightarrow u),$$

$$\mathscr{T}_{+--0}=\mathscr{T}_{+-+0}(t\leftrightarrow u), \tag{24}$$

while the amplitudes with $\lambda_1=-$ are obtained by the parity relations,

$$\mathscr{T}_{-\lambda_2\lambda_3\lambda_Z}=\mathscr{T}_{+-\lambda_2-\lambda_3-\lambda_Z}, \tag{25}$$

for $\lambda_Z=\pm$, and,

$$\mathscr{T}_{-\lambda_2\lambda_3 0}=-\mathscr{T}_{+-\lambda_2-\lambda_3 0}, \tag{26}$$

for longitudinally polarized Z bosons.

4 Numerical Results

Using the helicity amplitudes, from the previous section and the explicit forms for the scalar integrals given in the Appendix, it is straightforward to calculate the $Z\to\gamma\gamma\gamma$ decay rate,

$$\Gamma(Z\to\gamma\gamma\gamma)=\frac{1}{3!\cdot 384\pi^3M_Z^3}\int\sum_{\lambda_1\ldots\lambda_Z}|\mathscr{T}_{\lambda_1\lambda_2\lambda_3\lambda_4}|^2 ds\,dt\,du$$
$$\times\delta(M_Z^2-s-t-u). \tag{27}$$

The factor 1/3! is the identical particle factor for the photons. The coupling constants and the masses of the particles in the loop are contained in the definition of the A_i in (12).

For the numerical results, we take $M_Z=91.175$ GeV, $\alpha(M_Z)=e^2/(4\pi)=1/128$, $\alpha_Z=g_Z^2/(4\pi)=\alpha(M_Z)/\sin^2\theta_W\cos^2\theta_W$ and $\sin^2\theta_W=0.23$ while the W boson mass is given by the relation, $M_W=M_Z\cos\theta_W=80.0$ GeV. The vector coup-

* Note that in Appendix B of [8], the amplitude V_{+---} is incorrectly given and should read $V_{+---}(s,t,u)=V_{+-++}(s,u,t)$

ling v_f is given by,

$$v_f = \frac{T_3^f}{2} - e_f \sin^2\theta_W, \tag{28}$$

with $T_3^f = +1/2$ for up-type quarks ($e_f = +2/3$) and $T_3^f = -1/2$ for down-type quarks ($e_f = -1/3$) and charged leptons ($e_f = -1$). For very light fermions, the amplitudes are essentially independent of the precise value of the fermion mass and we choose $m_u = m_d = m_s = m_c = m_e = m_\mu = m_\tau = 100\,\text{eV}$. Varying this mass between 100 eV and 2 GeV does not change the results by more than the Monte Carlo error on the integration, which is less than 1%. For the bottom quark, we take $m_b = 5\,\text{GeV}$.

The only unknown parameter is the top quark mass, m_{top}, and in Fig. 2, we show $\Gamma(Z \to \gamma\gamma\gamma)$ in the interval $0 < m_{\text{top}} < 200\,\text{GeV}$. The direct experimental lower limit from CDF on the top quark is 91 GeV [15], while radiative corrections suggest that m_{top} lies in the range $m_{\text{top}} = 124^{+26}_{-28}\,\text{GeV}$ [16]. Nevertheless, we show the whole range of m_{top} in order to exhibit the threshold behaviour around $m_{\text{top}} = M_Z/2$ where the possibility of making two on-shell top quarks vanishes. Beyond the threshold, the top quark rapidly decouples as is well known [11].

At small m_{top}, the top quark makes a sizeable contribution. This is because each quarks contribution is proportional to the cube of the quark charge and therefore quarks with $e_f = +2/3$ dominate. The lepton contribution is suppressed both by colour and by the fact that v_f is small for charged leptons. The rate at small m_{top} is therefore approximately 50% larger than at large m_{top} where the up and charm quark contribution dominates. The Monte Carlo estimate of the partial width at small m_{top} agrees with the analytic formula of (1).

Figure 2 shows both the total $Z \to \gamma\gamma\gamma$ partial width (including both fermion and boson loops) and the fermion contribution alone. On its own, the W loop contribution is 0.026 eV, in rough agreement with the estimates of [4, 6], however, the interference with the larger fermion loop contribution is significant and increases the width by about 27% or 0.3 eV over the whole range of m_{top}.

We note that the decay rate does depend quite sensitively on the precise choice of $\sin^2\theta_W$ through the vector coupling of the Z boson with the quarks. Allowing $\sin^2\theta_W = 1 - M_W^2/M_Z^2$ to vary between 0.20 and 0.25 (i.e. $81.5\,\text{GeV} > M_W > 78.9\,\text{GeV}$) causes the total three photon decay width to vary up or downwards by a factor of about 2. The W loop contribution alone remains almost constant at 0.026 eV.

5 Conclusions

In summary, we have computed the polarization tensor for the $Z\gamma\gamma\gamma$ coupling including both fermion and W boson loops using the unitary gauge. The three independent amplitudes describing this tensor have not appeared in the literature before. It is remarkable that the boson and fermion contributions are so similar, bearing in mind the non-supersymmetric nature of the standard model. By projecting out the different helicity amplitudes we have obtained the $Z \to \gamma\gamma\gamma$ decay width keeping the full dependence on the top quark and W boson masses. For the allowed range of m_{top}, the entire standard model contribution to the $Z \to \gamma\gamma\gamma$ decay is 1.35 eV. In other words, one such decay would occur for every 10^9 hadronic Z boson events. Such a rate is clearly beyond even the high luminosity option at LEP.

In fact, three photon events have already been observed at LEP [19], however, they are completely consistent with the purely QED process,

$$e^+e^- \to \gamma\gamma\gamma. \tag{29}$$

This tree level process provides an irreducible background corresponding to a partial width of about 10 keV [20] and makes the observation of the standard model process described here completely unlikely. Nevertheless, many extensions to the standard model such as models where the Z boson is a bound state of charged constituents do allow an anomalously large three photon decay rate [21, 20] analogous to the decay of the J/ψ into photons. Any disagreement with the QED prediction cannot be due to standard model processes and must indicate some new physics.

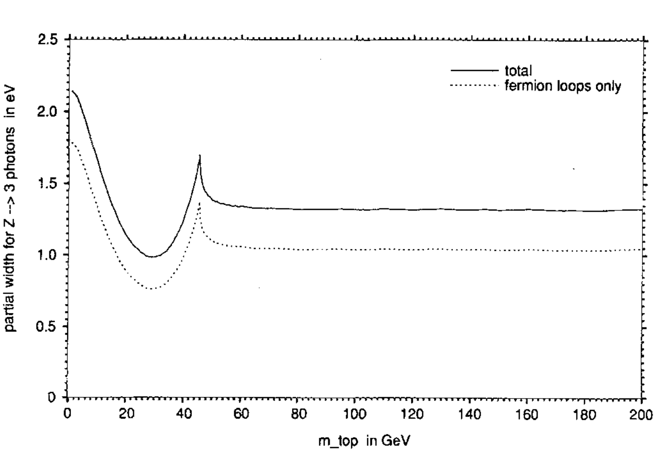

Fig. 2. The partial width for $Z \to \gamma\gamma\gamma$ in eV as a function of the top quark mass m_{top}. The total (fermion + boson) width is shown as a solid line, while the fermion contribution alone is shown dotted

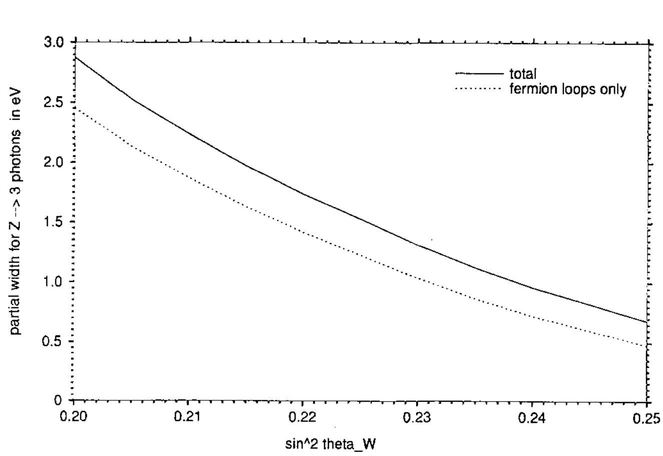

Fig. 3. The partial width for $Z \to \gamma\gamma\gamma$ in eV as a function of $\sin^2\theta_W$ for $m_{\text{top}} = 120\,\text{GeV}$. The total (fermion + boson) width is shown as a solid line, while the fermion contribution alone is shown dotted

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Appendix A. Scalar integrals

In this appendix we define the integrals appearing in the calculation. First, there is the finite part of the two-point function $B(s, m)$ with momentum $p^2 = s$ and internal mass m,

$$B(s, m) = -\int_0^1 dx \log(m^2 - i\varepsilon - sx(1-x)) = -\left\{\log(m^2) - 2 + (2z-1)\log\left(\frac{-z}{1-z}\right)\right\}, \quad (30)$$

with

$$z = \tfrac{1}{2}(1 + \sqrt{1 - 4(m^2 - i\varepsilon)/s}). \quad (31)$$

Only the following combination is present,

$$B_1(s, m) = B(s, m) - B(M_Z^2, m). \quad (32)$$

The next integral that appears is the three-point function $C(s, m)$ with two external massless lines $p_1^2 = p_2^2 = 0$, $(p_1 + p_2)^2 = s$ and an internal mass m,

$$C(s, m) = \frac{1}{i\pi^2}\int \frac{d^4q}{(q^2 - m^2)((q+p_1)^2 - m^2)((q+p_1+p_2)^2 - m^2)} = \int_0^1 \frac{dx}{sx}\log\left(1 - i\varepsilon - \frac{s}{m^2}x(1-x)\right) = \frac{1}{2s}\left[\log\left(\frac{-z}{1-z}\right)\right]^2, \quad (33)$$

with z given by (31).

The three-point function $C_1(s, m)$ with one external massless line, $p_1^2 = 0$, $p_2^2 = M_Z^2$, $(p_1 + p_2)^2 = s$ and an internal mass m is given by,

$$s_1 C_1(s, m) = s\,C(s, m) - M_Z^2 C(M_Z^2, m). \quad (34)$$

Finally there is the four-point function with three massless and one massive external line, $p_1^2 = p_2^2 = p_3^2 = 0$, $p_4^2 = M_Z^2$ and an internal mass m,

$$D(s, t, m) = \frac{1}{i\pi^2}\int \frac{d^4q}{(q^2-m^2)((q+p_1)^2-m^2)((q+p_1+p_2)^2-m^2)((q-p_4)^2-m^2)}$$
$$= \frac{1}{st}\int_0^1 \frac{dx}{x(1-x) + m^2u/ts}\left\{-\log\left(1 - i\varepsilon - \frac{M_Z^2}{m^2}x(1-x)\right) + \log\left(1 - i\varepsilon - \frac{s}{m^2}x(1-x)\right) + \log\left(1 - i\varepsilon - \frac{t}{m^2}x(1-x)\right)\right\}. \quad (35)$$

This result can be expressed in terms of Spence functions via the relation,

$$\int_0^1 \frac{dx}{x(1-x) + m^2u/ts}\log\left(1 - i\varepsilon - \frac{v}{m_f^2}x(1-x)\right) = \frac{2}{\sqrt{1 + 4m^2u/ts}}\left\{Sp\left(\frac{x_-}{x_- - y}\right) - Sp\left(\frac{x_+}{x_+ - y}\right) + Sp\left(\frac{x_-}{y - x_+}\right) - Sp\left(\frac{x_+}{y - x_-}\right) - \log\left(\frac{-x_-}{x_+}\right)\log\left(1 - i\varepsilon + \frac{vu}{st}\right)\right\}, \quad (36)$$

where,

$$x_\pm = \tfrac{1}{2}(1 \pm \sqrt{1 + 4m^2u/ts}), \quad (37)$$

and,

$$y = \tfrac{1}{2}(1 + \sqrt{1 - 4(m^2 - i\varepsilon)/v}). \quad (38)$$

As auxiliary functions we also define,

$$E(s, t, m) = s\,C(s, m) + t\,C(t, m) + s_1 C_1(s, m) + t_1 C_1(t, m) - st\,D(s, t, m), \quad (39)$$

and,

$$F(m) = D(s, t, m) + D(t, u, m) + D(u, s, m). \quad (40)$$

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