ДОМАШНЯ РОБОТА №12
З ПРЕДМЕТУ
"ТЕОРІЯ ФУНКЦІЇ КОМПЛЕКСНОЇ ЗМІННОЇ"

ФІ-12 Бекешева Анастасія

1. (a) $$\int\limits_0^{2\pi} \frac{\mathrm{d}t}{\sqrt{21}\sin t + 5} = \int\limits_{|z|=1} \frac{\mathrm{d}z}{iz\left(\sqrt{21}\frac{1}{2i}\left(z-\frac{1}{z}\right)+5\right)} = \left\langle \begin{array}{c} \frac{1}{2}\sqrt{21}z^2 + 5iz - \frac{1}{2}\sqrt{21} = 0 \\ z = -i\sqrt{\frac{3}{7}}, z = -i\sqrt{\frac{7}{3}} \\ z = -i\sqrt{\frac{3}{7}} \text{ полюс I порядку} \end{array} \right\rangle =$$

$$= \int\limits_{|z|=1} \frac{\mathrm{d}z}{\frac{\sqrt{21}}{2}\left(z+i\sqrt{\frac{3}{7}}\right)\left(z+i\sqrt{\frac{7}{3}}\right)} = 2\pi i \operatorname*{Res}_{z=-i\sqrt{\frac{3}{7}}} \frac{1}{\frac{\sqrt{21}}{2}\left(z+i\sqrt{\frac{3}{7}}\right)\left(z+i\sqrt{\frac{7}{3}}\right)} =$$

$$= 2\pi i \lim_{z\to -i\sqrt{\frac{3}{7}}} \frac{1}{\frac{\sqrt{21}}{2}\left(z+i\sqrt{\frac{7}{3}}\right)} = -\frac{i\sqrt{21}}{4}\cdot\frac{2}{\sqrt{21}}\cdot 2\pi i = \pi$$

2. (a) $$\int\limits_{-\infty}^{\infty} \frac{x^2+10}{(x^2+4)^2}\ \mathrm{d}x\ = \int\limits_{-\infty}^{\infty} \frac{(x-i\sqrt{5})(x+i\sqrt{5})}{(x-2i)^2(x+2i)^2}\ \mathrm{d}x\ = 2\pi i \operatorname*{Res}_{z=2i} \frac{z^2+10}{(z^2+4)^2} =$$

$$= 2\pi i \lim_{z\to 2i} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^2+10}{(z+2i)^2}\right) = 2\pi i \lim_{z\to 2i} \frac{2z(z+2i)^2 - 2(z^2+10)(z+2i)}{(z+2i)^3} =$$

$$= 2\pi i \lim_{z\to 2i} \frac{4iz-20}{(z+2i)^3} = 2\pi i\frac{-28}{-64i} = \frac{7}{8}\pi, \langle 2i \text{ полюс II порядку}\rangle$$

(b) $$\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)^4} = 2\pi i \operatorname*{Res}_{z\to i} \frac{1}{(z-i)^4(z+i)^4} = \frac{1}{3}\pi i \lim_{z\to i} \frac{\mathrm{d}^{\,3}}{\mathrm{d}z^{\,3}}\left(\frac{1}{(z+i)^4}\right) =$$

$$= \left\langle \frac{\mathrm{d}^{\,3}}{\mathrm{d}z^{\,3}}\left(\frac{1}{(z+i)^4}\right) = \frac{\mathrm{d}^{\,2}}{\mathrm{d}z^{\,2}}\left(-\frac{4}{(z+i)^5}\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{20}{(z+i)^6}\right) = -\frac{120}{(z+i)^7}\right\rangle =$$

$$= \frac{1}{3}\pi i \lim_{z\to i} \frac{-120}{(z+1)^7} = \pi\frac{120}{3\cdot 2^7} = \frac{5}{16}\pi, \langle i \text{ полюс IV порядку}\rangle$$

3. (a) $$\int\limits_0^{\infty} \frac{\cos 2x}{\left(x^2+\frac{1}{4}\right)^2}\ \mathrm{d}x\ = \frac{1}{2}\int\limits_{-\infty}^{\infty} \frac{\cos 2x}{\left(x^2+\frac{1}{4}\right)^2}\ \mathrm{d}x\ = \frac{1}{2}\mathfrak{Re}\left[2\pi i \operatorname*{Res}_{z=\frac{i}{2}} \frac{e^{2iz}}{\left(z^2+\frac{1}{4}\right)^2}\right] =$$

$$= \frac{1}{2}\mathfrak{Re}\left[2\pi i \lim_{z\to\frac{i}{2}} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{e^{2iz}}{\left(z+\frac{i}{2}\right)^2}\right)\right] = \frac{1}{2}\mathfrak{Re}\left[2\pi i \lim_{z\to\frac{i}{2}} \frac{e^{2iz}2i(z+\frac{i}{2})^2 - e^{2iz}(2z+i)}{\left(z+\frac{i}{2}\right)^4}\right] =$$

$$= \frac{1}{2}\mathfrak{Re}\left[2\pi i\frac{e^{-1}2i^3 - e^{-1}2i}{i^4}\right] = \frac{1}{2}\mathfrak{Re}\left[4\pi e^{-1} + 4\pi e^{-1}\right] = \frac{4\pi}{e}$$

(b) $$\int\limits_{-\infty}^{\infty} \frac{(x^3+1)\sin x}{x^4+5x^2+4}\ \mathrm{d}x\ = \mathfrak{Im}\left[2\pi i \operatorname*{Res}_{z=2i} \frac{(z^3+1)e^{iz}}{(z^2+4)(z^2+1)} + 2\pi i \operatorname*{Res}_{z=i} \frac{(z^3+1)e^{iz}}{(z^2+4)(z^2+1)}\right] =$$

$$= \mathfrak{Im}\left[2\pi i\frac{(z^3+1)e^{iz}}{4z^3+10z}\bigg|_{2i} + 2\pi i\frac{(z^3+1)e^{iz}}{4z^3+10z}\bigg|_{i}\right] = \mathfrak{Im}\left[\frac{2\pi i\cdot(-8i+1)\cdot e^{-2}}{-32i+20i} +\right.$$

$$\left. + \frac{2\pi i\cdot(-i+1)\cdot e^{-1}}{-4i+10i}\right] = \mathfrak{Im}\left[-\frac{\pi}{6e^2} + \frac{4\pi i}{3e^2} - \frac{\pi}{3e} - \frac{\pi i}{3e}\right] = \left(\frac{4\pi-\pi e}{3e^2}\right) = \frac{\pi(4-e)}{3e^2}$$