РОЗРАХУНКОВА РОБОТА
З ПРЕДМЕТУ
”ТЕОРІЯ ФУНКЦІЇ КОМПЛЕКСНОЇ ЗМІННОЇ”
ВАРІАНТ №2

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1.
$$\sqrt[4]{\frac{-1+i\sqrt{3}}{2}}$$

$$\sqrt[4]{\frac{-1+i\sqrt{3}}{2}} = \sqrt[4]{\sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2}} \cdot \left(\cos\left(\frac{\arctan(-\sqrt{3})+\pi+2\pi k}{4}\right) + \right.$$
$$\left. + \, i\sin\left(\frac{\arctan(-\sqrt{3})+\pi+2\pi k}{4}\right)\right) = 1\cdot\left(\cos\left(\frac{\pi}{6}+\frac{1}{2}\pi k\right) + i\sin\left(\frac{\pi}{6}+\frac{1}{2}\pi k\right)\right)$$

$$0 \le k \le 3$$

$$k=0: \quad \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$
$$k=1: \quad \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$k=2: \quad \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$
$$k=3: \quad \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

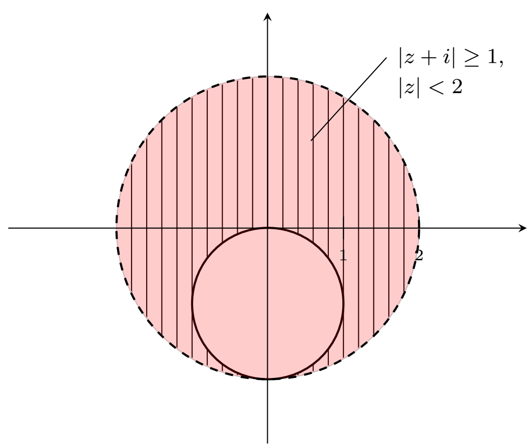
2.
$$\cos\left(\frac{\pi}{6}+2i\right)$$

$$\cos\left(\frac{\pi}{6}+2i\right) = \cos\frac{\pi}{6}\cos 2i - \sin\frac{\pi}{6}\sin 2i = \frac{\sqrt{3}}{2}\cos 2i - \frac{1}{2}\sin 2i = \frac{\sqrt{3}}{2}\operatorname{ch} 2 - \frac{1}{2}i\operatorname{sh} 2$$

3.
$$\operatorname{Arcsin} 4$$

$$\operatorname{Arcsin} 4 = -i\ \operatorname{Ln}\ \left(4i \pm \sqrt{1-16}\right) = -i\ \operatorname{Ln}\ \left(i\left(4\pm\sqrt{15}\right)\right) = \left\langle \begin{array}{c} z = 4\pm\sqrt{15}\ (x=0) \\ \arg z = \dfrac{\pi}{2} \end{array} \right\rangle =$$
$$= -i\left(\ln\left|4\pm\sqrt{15}\right| + i\left(\frac{\pi}{2}+2\pi k\right)\right) = -i\ln\left(4\pm\sqrt{15}\right) + \frac{\pi}{2} + 2\pi k, \quad k\in\mathbb{Z}$$

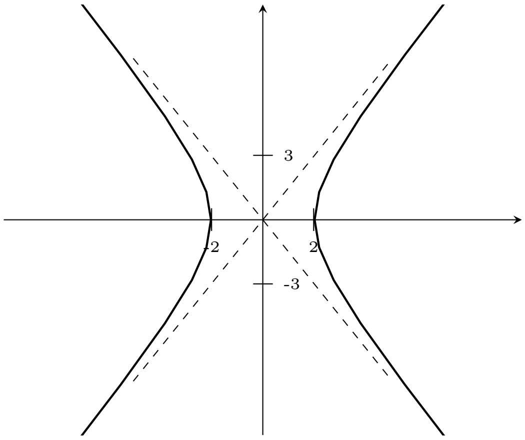
4.
$$|z+i| \ge 1, \quad |z| < 2$$



5.

$$z = 2\sec t - 3i\,\mathrm{tg}\,t$$

$$\begin{cases} \dfrac{x}{2} = \dfrac{2}{\cos t}, \\ \dfrac{y}{3} = -\dfrac{\sin t}{\cos t} \end{cases} \Rightarrow \frac{x^2}{4} - \frac{y^2}{9} = -\frac{\sin^2 t}{\cos^2 t} + \frac{1}{\cos^2 t} = \frac{1-\sin^2 t}{\cos^2 t} = 1$$



6.

$$u = x^3 - 3xy^2 + 1, \quad f(0) = 1$$

$$u'_x = 3(x^2 - y^2), \quad u'_y = -6xy, \quad u'_x = v'_y, v'_x = -u'_y \Longrightarrow v = 3\int (x^2 - y^2)\ \mathrm{d}y\ =$$
$$= 3\left(x^2 y - \frac{1}{3}y^3\right), \quad v'_x = 6xy + C'(x), \quad v'_x = -u'_y \Longrightarrow 6xy + C'(x) = 6xy, \quad C'(x) = 0,$$
$$C(x) = C \in \mathbb{R}, \quad v = 3x^2 y - y^3 + C$$
$$f(z) = x^3 - 3xy^2 + 1 + 3x^2 yi - y^3 i + Ci, \quad f(0) = 1, \quad f(0) = 0 - 0 + 1 + 0i - 0i + Ci =$$
$$= Ci + 1 = 1 \Longrightarrow C = 0, \quad f(z)x^3 - 3xy^2 + 1 + 3x^2 yi - y^3 i = (x+iy)^3 + 1 = z^3 + 1$$

7.

$$\int_{\mathcal{L}} (z+1)e^z\ \mathrm{d}z\ , \quad \mathcal{L} : \{|z| = 1, \quad \mathfrak{Re} z \geq 0\}$$

$$z = x + iy, \quad (z+1)e^z = (x + iy + 1)e^{x+iy} = (z + iy + 1)\cdot(\cos(x+iy) + i\sin(x+iy))$$

8.

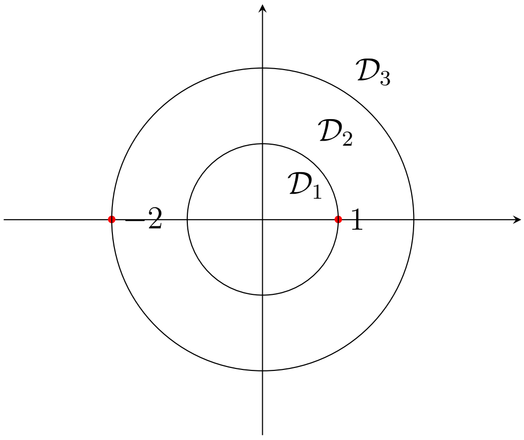
$$f(z) = \frac{z-4}{z^4 + z^3 - 2z^2}$$

$$\left[\begin{array}{l} z_1 = 0 \\ z_2 = 1 \\ z_3 = -2 \end{array}\right. \left[\begin{array}{l} \mathcal{D}_1 :\ 0 < |z| < 1 \\ \mathcal{D}_2 :\ 1 < |z| < 2 \\ \mathcal{D}_3 :\ |z| > 2 \end{array}\right. f(z) = \frac{z-4}{z^2(z-1)(z+2)} = \frac{1}{z^2}\left(\frac{A}{z-1} + \frac{B}{z+2}\right) =$$
$$= \left\langle A = \left.\frac{z-4}{z+2}\right|_{z=1} = -\frac{3}{3} = -1, \quad B = \left.\frac{z-4}{z-1}\right|_{z=-2} = \frac{-6}{-3} = 2\right\rangle = \frac{1}{z^2}\left(\frac{2}{z+2} - \frac{1}{z-1}\right)$$
$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n \in \mathcal{D}_1$$
$$\frac{1}{z-1} = \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} \in \mathcal{D}_2, \mathcal{D}_3$$
$$\frac{2}{z+2} = \frac{1}{1-\left(-\frac{z}{2}\right)} = \sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n} \in \mathcal{D}_1, \mathcal{D}_2$$
$$\frac{2}{z+2} = \frac{2}{z}\cdot\frac{1}{1-\left(-\frac{2}{z}\right)} = \frac{2}{z}\sum_{n=0}^{\infty}\left(-\frac{2}{z}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{2^{n+1}}{z^{n+1}} \in \mathcal{D}_3$$

$$\mathcal{D}_1: \quad f(z)=\frac{1}{z^2}\left(\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n}-\left(-\sum_{n=0}^{\infty}z^n\right)\right)=\sum_{n=0}^{\infty}z^{n-2}\left(\frac{(-1)^n}{2^n}+1\right)$$

$$\mathcal{D}_2: \quad f(z)=\frac{1}{z^2}\left(\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right)=\sum_{n=0}^{\infty}(-1)^n\frac{z^{n-2}}{2^n}-\sum_{n=0}^{\infty}\frac{1}{z^{n+3}}$$

$$\mathcal{D}_3: \quad f(z)=\frac{1}{z^2}\left(\sum_{n=0}^{\infty}(-1)^n\frac{2^{n+1}}{z^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right)=\sum_{n=0}^{\infty}\frac{((-1)^n2^{n+1}-1)}{z^{n+3}}$$



10.

$$f(z)=\sin\left(\frac{z}{z-1}\right), \quad z_0=1$$

$$f(z)=\sin\left(\frac{z+1-1}{z-1}\right)=\sin\left(1+\frac{1}{z-1}\right)=\sin 1\cos\left(\frac{1}{z-1}\right)+\cos 1\sin\left(\frac{1}{z-1}\right)=$$
$$=\sin(1)\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{z-1}\right)^{2n}+\cos(1)\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z-1}\right)^{2n+1}=$$
$$=\sin(1)\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!(z-1)^{2n}}+\cos(1)\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(z-1)^{2n+1}}$$

11.

$$f(z)=z^3e^{\frac{7}{z^2}}$$

$$f(z)=z^3\left(1+\frac{7}{z^2}+\frac{7^2}{2!z^4}+\frac{7^3}{3!z^6}+\dots\right)=z^3+7z+\underbrace{\frac{7^2}{2!z}+\frac{7^3}{3!z^3}+\dots,}_{\text{головна частина}} \quad z_0=0$$

енскінчена кількість доданків в головній частині $\Longrightarrow z_0$ - істотно особлива.

12.

$$f(z)=\frac{1}{\cos z}$$

$\cos z=0 \Longrightarrow z_k=\dfrac{\pi}{2}+\pi k, k\in\mathbb{Z}, \quad (\cos z)'\big|_{z_k}=-\sin(z_k)=\pm1\neq0$

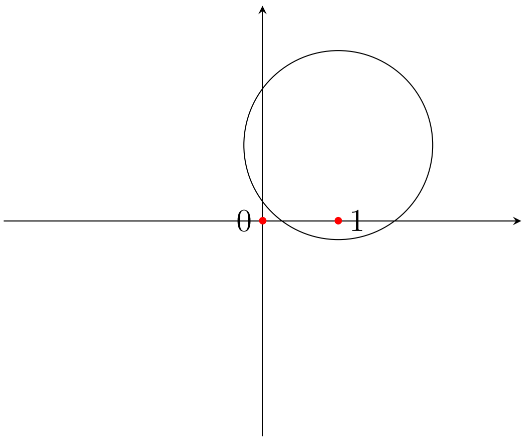
Нулі знаменика не є нулями чисельника. $m_1=0, \quad m_2=1\Longrightarrow z_k$ - полюс I порядку.

13.

$$\mathcal{I}=\oint\limits_{\mathcal{L}}\frac{2\ \mathrm{d}z}{z^2(z-1)}, \quad \mathcal{L}:|z-1-i|=\frac{5}{4}$$

$z_1=0\notin\mathcal{L}, \quad z_2=1: \quad m_1=0, \quad m_2=1$ - полюс I порядку.

$\mathcal{I}=2\pi i\ \operatorname*{Res}\limits_{z=1}\dfrac{2\ \mathrm{d}z}{z^2(z-1)}=2\pi i\cdot\dfrac{2}{3z^2-2z}\bigg|_{z=1}=2\pi i\cdot\dfrac{2}{3-2}=4\pi i$



14.

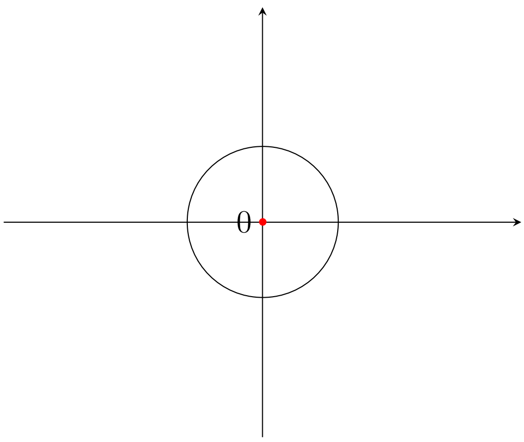
$$\mathcal{I} = \oint\limits_{\mathcal{L}} \frac{2 - z^2 + 3z^3}{4z^3}\ \mathrm{d}z\ , \quad \mathcal{L} : |z| = \frac{1}{2}$$

$z_0 = 0 : \quad m_1 = 0, \quad m_2 = 3$ - полюс III порядку.

$\mathcal{I} = 2\pi i\ \operatorname*{Res}_{z=0} \dfrac{2 - z^2 + 3z^3}{4z^3} = \pi i \lim\limits_{z\to 0} \dfrac{\mathrm{d}^{\ 2}}{\mathrm{d}z^{\ 2}} \left(\dfrac{2 - z^2 + 3z^3}{4z^3} \cdot (z-0)^3 \right) =$

$= \pi i \lim\limits_{z\to 0} \dfrac{\mathrm{d}}{\mathrm{d}z} \left(\dfrac{\mathrm{d}}{\mathrm{d}z} \left(\dfrac{2}{4} - \dfrac{z^2}{4} + \dfrac{3z^3}{4} \right) \right) = \pi i \lim\limits_{z\to 0} \dfrac{\mathrm{d}}{\mathrm{d}z} \left(-\dfrac{1}{2}z + \dfrac{9}{4}z^2 \right) = \pi i \lim\limits_{z\to 0} \left(-\dfrac{1}{2} + \dfrac{9}{2}z \right) =$

$= \pi i \left(-\dfrac{1}{2} + 0 \right) = -\dfrac{\pi}{2} i$



15.

$$\mathcal{I} = \oint\limits_{\mathcal{L}} \frac{\cos 3z - 1 + \frac{9}{2}z^2}{z^4 \operatorname{sh} \frac{9}{4}z}\ \mathrm{d}z\ , \quad \mathcal{L} : |z| = 1$$

$z^4 \operatorname{sh} \dfrac{9}{4}z = 0, \quad z_1 = 0, \quad \sin \dfrac{9}{4}iz = 0, \quad z_k = -\dfrac{4}{9}i\pi k, \quad k \in \mathbb{Z}, \quad \cos 3z_k - 1 + \dfrac{9}{2}z_k^2 \neq 0$

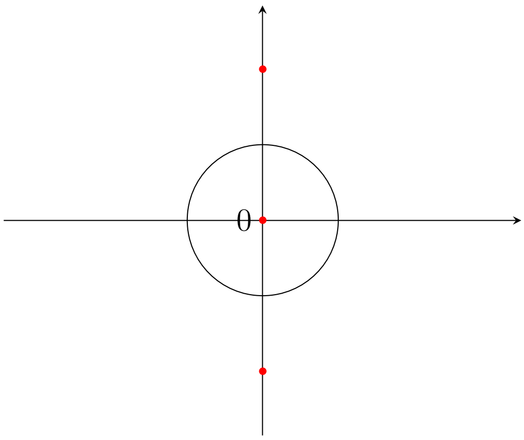
$z_1 = 0$ - нуль V порядку, $z_k = -\dfrac{4}{9}i\pi k \setminus \{0\}$ - не нуль.

$f(z) = \dfrac{\cos 3z - 1 + \frac{9}{2}z^2}{z^4 \operatorname{sh} \frac{9}{4}z} = \dfrac{\left(1 - \dfrac{3^2z^2}{2!} + \dfrac{3^4z^4}{4!} + \dfrac{3^6z^6}{6!} + \dots\right) - 1 + \dfrac{9z^2}{2}}{z^4 \left(\dfrac{9z}{4} + \dfrac{9^3z^3}{4^3 3!} + \dfrac{9^5z^5}{4^5 5!} + \dots \right)} =$

$$= \frac{z^4\left(\frac{3^4}{4!}+\frac{3^6z^2}{6!}+\dots\right)}{z^5\left(\frac{9}{4}+\frac{9^3z^2}{4^33!}+\frac{9^5z^4}{4^55!}+\dots\right)} = \frac{z^4g_1(z)}{z^5g_2(z)}, \quad g_i(0)\neq 0$$

$z_0 = 0: \quad m_1 = 4, \quad m_2 = 5 \Longrightarrow z_0 = 0$ - полюс I порядку.

$$\mathcal{I} = 2\pi i \operatorname*{Res}_{z=0} f(z) = 2\pi i \lim_{z\to 0}\frac{z^4g_1(z)}{z^5g_2(z)}\cdot z = 2\pi i \lim_{z\to 0}\frac{\frac{3^4}{4!}+\frac{3^6z^2}{6!}+\dots}{\frac{9}{4}+\frac{9^3z^2}{4^33!}+\frac{9^5z^4}{4^55!}+\dots} = 2\pi i\frac{\frac{3^4}{4!}}{\frac{9}{4}} = 3\pi i$$



16.