РОЗРАХУНКОВА РОБОТА
З ПРЕДМЕТУ
"ТЕОРІЯ ФУНКЦІЇ КОМПЛЕКСНОЇ ЗМІННОЇ"
ВАРІАНТ №2

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1.

$$\sqrt[4]{\frac{-1+i\sqrt{3}}{2}}$$

$$\sqrt[4]{\frac{-1+i\sqrt{3}}{2}} = \sqrt[4]{\sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2}} \cdot \left(\cos\left(\frac{\arctan(-\sqrt{3})+\pi+2\pi k}{4}\right) + \right.$$

$$\left. + \, i\sin\left(\frac{\arctan(-\sqrt{3})+\pi+2\pi k}{4}\right)\right) = 1 \cdot \left(\cos\left(\frac{\pi}{6}+\frac{1}{2}\pi k\right) + i\sin\left(\frac{\pi}{6}+\frac{1}{2}\pi k\right)\right)$$

$$0 \le k \le 3$$

$$\begin{aligned}
k=0: &\quad \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i \\
k=1: &\quad \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\
k=2: &\quad \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i \\
k=3: &\quad \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i
\end{aligned}$$

2.

$$\cos\left(\frac{\pi}{6}+2i\right)$$

$$\cos\left(\frac{\pi}{6}+2i\right) = \cos\frac{\pi}{6}\cos 2i - \sin\frac{\pi}{6}\sin 2i = \frac{\sqrt{3}}{2}\cos 2i - \frac{1}{2}\sin 2i = \frac{\sqrt{3}}{2}\operatorname{ch} 2 - \frac{1}{2} i \operatorname{sh} 2$$

3.

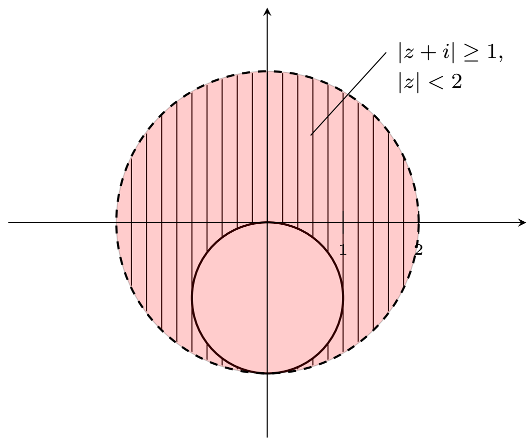
$$\operatorname{Arcsin} 4$$

$$\operatorname{Arcsin} 4 = -i\ \operatorname{Ln}\ \left(4i \pm \sqrt{1-16}\right) = -i\ \operatorname{Ln}\ \left(i\left(4\pm\sqrt{15}\right)\right) = \left\langle \begin{array}{c} z = 4\pm\sqrt{15}\ (x=0) \\ \arg z = \dfrac{\pi}{2} \end{array} \right\rangle =$$

$$= -i\left(\ln\left|4\pm\sqrt{15}\right| + i\left(\frac{\pi}{2}+2\pi k\right)\right) = -i\ln\left(4\pm\sqrt{15}\right) + \frac{\pi}{2} + 2\pi k, \quad k\in\mathbb{Z}$$

4.

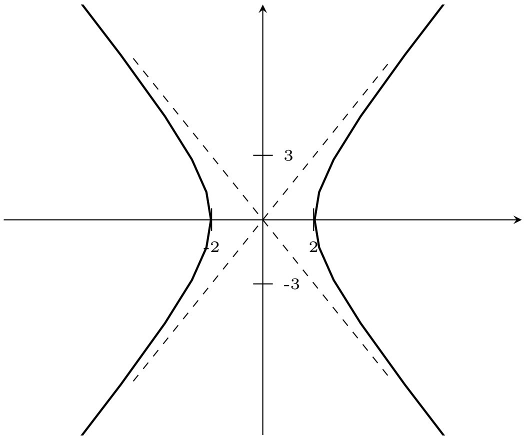
$$|z+i| \ge 1, \quad |z| < 2$$



5.

$$z = 2\sec t - 3i \operatorname{tg} t$$

$$\begin{cases} \dfrac{x}{2} = \dfrac{2}{\cos t}, \\ \dfrac{y}{3} = -\dfrac{\sin t}{\cos t} \end{cases} \Rightarrow \frac{x^2}{4} - \frac{y^2}{9} = -\frac{\sin^2 t}{\cos^2 t} + \frac{1}{\cos^2 t} = \frac{1 - \sin^2 t}{\cos^2 t} = 1$$



6.

$$u = x^3 - 3xy^2 + 1, \quad f(0) = 1$$

$u'_x = 3(x^2 - y^2), \quad u'_y = -6xy, \quad u'_x = v'_y, v'_x = -u'_y \Longrightarrow v = 3\displaystyle\int (x^2 - y^2)\ \mathrm{d}y\ =$
$= 3\left(x^2 y - \dfrac{1}{3}y^3\right), \quad v'_x = 6xy + C'(x), \quad v'_x = -u'_y \Longrightarrow 6xy + C'(x) = 6xy, \quad C'(x) = 0,$
$C(x) = C \in \mathbb{R}, \quad v = 3x^2 y - y^3 + C$
$f(z) = x^3 - 3xy^2 + 1 + 3x^2 yi - y^3 i + Ci, \quad f(0) = 1, \quad f(0) = 0 - 0 + 1 + 0i - 0i + Ci =$
$= Ci + 1 = 1 \Longrightarrow C = 0, \quad f(z)x^3 - 3xy^2 + 1 + 3x^2 yi - y^3 i = (x + iy)^3 + 1 = z^3 + 1$

7.

$$\int\limits_{\mathcal{L}} (z+1)e^z\ \mathrm{d}z\ , \quad \mathcal{L} : \{|z| = 1, \quad \mathfrak{Re} z \geq 0\}$$

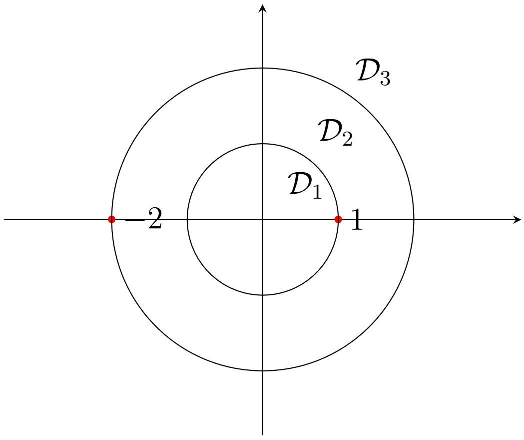
$z = x + iy, \quad (z+1)e^z = (x + iy + 1)e^{x+iy} = (z + iy + 1) \cdot (\cos(x + iy) + i\sin(x + iy))$

8.

$$f(z) = \frac{z - 4}{z^4 + z^3 - 2z^2}$$

$\left[\begin{array}{l} z_1 = 0 \\ z_2 = 1 \\ z_3 = -2 \end{array}\right. \left[\begin{array}{l} \mathcal{D}_1 :\ 0 < |z| < 1 \\ \mathcal{D}_2 :\ 1 < |z| < 2 \\ \mathcal{D}_3 :\ |z| > 2 \end{array}\right. f(z) = \dfrac{z - 4}{z^2(z-1)(z+2)} = \dfrac{1}{z^2}\left(\dfrac{A}{z-1} + \dfrac{B}{z+2}\right) =$
$= \left\langle A = \dfrac{z-4}{z+2}\Big|_{z=1} = -\dfrac{3}{3} = -1, \quad B = \dfrac{z-4}{z-1}\Big|_{z=-2} = \dfrac{-6}{-3} = 2 \right\rangle = \dfrac{1}{z^2}\left(\dfrac{2}{z+2} - \dfrac{1}{z-1}\right)$
$\dfrac{1}{z-1} = -\dfrac{1}{1-z} = -\sum\limits_{n=0}^{\infty} z^n \in \mathcal{D}_1$
$\dfrac{1}{z-1} = \dfrac{1}{z} \cdot \dfrac{1}{1 - \frac{1}{z}} = \dfrac{1}{z}\sum\limits_{n=0}^{\infty}\left(\dfrac{1}{z}\right)^n = \sum\limits_{n=0}^{\infty} \dfrac{1}{z^{n+1}} \in \mathcal{D}_2, \mathcal{D}_3$
$\dfrac{2}{z+2} = \dfrac{1}{1 - \left(-\frac{z}{2}\right)} = \sum\limits_{n=0}^{\infty}\left(-\dfrac{z}{2}\right)^n = \sum\limits_{n=0}^{\infty}(-1)^n \dfrac{z^n}{2^n} \in \mathcal{D}_1, \mathcal{D}_2$
$\dfrac{2}{z+2} = \dfrac{2}{z} \cdot \dfrac{1}{1 - \left(-\frac{2}{z}\right)} = \dfrac{2}{z}\sum\limits_{n=0}^{\infty}\left(-\dfrac{2}{z}\right)^n = \sum\limits_{n=0}^{\infty}(-1)^n \dfrac{2^{n+1}}{z^{n+1}} \in \mathcal{D}_3$

$$\mathcal{D}_1: \quad f(z) = \frac{1}{z^2}\left(\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n} - \left(-\sum_{n=0}^{\infty} z^n\right)\right) = \sum_{n=0}^{\infty} z^{n-2}\left(\frac{(-1)^n}{2^n}+1\right)$$
$$\mathcal{D}_2: \quad f(z) = \frac{1}{z^2}\left(\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{z^{n-2}}{2^n} - \sum_{n=0}^{\infty}\frac{1}{z^{n+3}}$$
$$\mathcal{D}_3: \quad f(z) = \frac{1}{z^2}\left(\sum_{n=0}^{\infty}(-1)^n\frac{2^{n+1}}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right) = \sum_{n=0}^{\infty}\frac{((-1)^n 2^{n+1}-1)}{z^{n+3}}$$



10.

$$f(z) = \sin\left(\frac{z}{z-1}\right), \qquad z_0 = 1$$

$$f(z) = \sin\left(\frac{z+1-1}{z-1}\right) = \sin\left(1+\frac{1}{z-1}\right) = \sin 1\cos\left(\frac{1}{z-1}\right) + \cos 1\sin\left(\frac{1}{z-1}\right) =$$
$$= \sin(1)\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{z-1}\right)^{2n} + \cos(1)\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z-1}\right)^{2n+1} =$$
$$= \sin(1)\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!(z-1)^{2n}} + \cos(1)\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(z-1)^{2n+1}}$$

11.

$$f(z) = z^3 e^{\frac{7}{z^2}}$$

$$f(z) = z^3\left(1+\frac{7}{z^2}+\frac{7^2}{2!z^4}+\frac{7^3}{3!z^6}+\dots\right) = z^3+7z+\underbrace{\frac{7^2}{2!z}+\frac{7^3}{3!z^3}+\dots,}_{\text{головна частина}} \qquad z_0 = 0$$

енскінчена кількість доданків в головній частині $\Longrightarrow z_0$ - істотно особлива.

12.

$$f(z) = \frac{1}{\cos z}$$

$\cos z = 0 \Longrightarrow z_k = \dfrac{\pi}{2}+\pi k, k\in\mathbb{Z}, \quad (\cos z)'\big|_{z_k} = -\sin(z_k) = \pm 1 \neq 0$

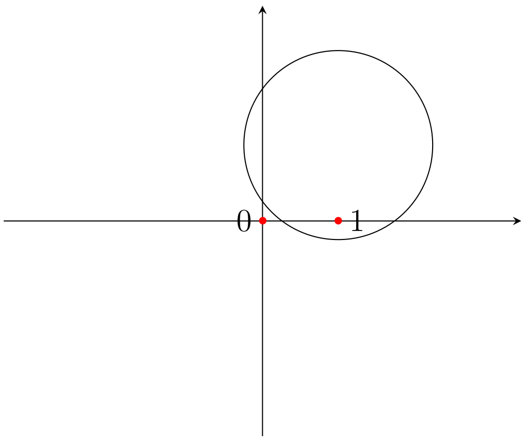
Нулі знаменика не є нулями чисельника. $m_1 = 0, \quad m_2 = 1 \Longrightarrow z_k$ - полюс I порядку.

13.

$$\mathcal{I} = \oint_{\mathcal{L}}\frac{2\ \mathrm{d}z}{z^2(z-1)}, \qquad \mathcal{L}: |z-1-i| = \frac{5}{4}$$

$z_1 = 0 \notin \mathcal{L}, \quad z_2 = 1: \quad m_1 = 0, \quad m_2 = 1$ - полюс I порядку.

$\mathcal{I} = 2\pi i \operatorname*{Res}_{z=1}\dfrac{2\ \mathrm{d}z}{z^2(z-1)} = 2\pi i\cdot\dfrac{2}{3z^2-2z}\Bigg|_{z=1} = 2\pi i\cdot\dfrac{2}{3-2} = 4\pi i$



14.

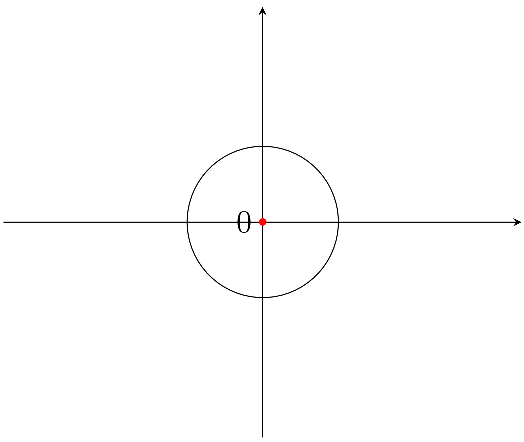
$$\mathcal{I} = \oint\limits_{\mathcal{L}} \frac{2 - z^2 + 3z^3}{4z^3} \ \mathrm{d}z \ , \quad \mathcal{L} : |z| = \frac{1}{2}$$

$z_0 = 0 : \quad m_1 = 0, \quad m_2 = 3$ - полюс III порядку.

$\mathcal{I} = 2\pi i \ \operatorname*{Res}_{z=0} \dfrac{2 - z^2 + 3z^3}{4z^3} = \pi i \lim\limits_{z \to 0} \dfrac{\mathrm{d}^{\,2}}{\mathrm{d}z^{\,2}} \left(\dfrac{2 - z^2 + 3z^3}{4z^3} \cdot (z - 0)^3 \right) =$

$= \pi i \lim\limits_{z \to 0} \dfrac{\mathrm{d}}{\mathrm{d}z} \left(\dfrac{\mathrm{d}}{\mathrm{d}z} \left(\dfrac{2}{4} - \dfrac{z^2}{4} + \dfrac{3z^3}{4} \right) \right) = \pi i \lim\limits_{z \to 0} \dfrac{\mathrm{d}}{\mathrm{d}z} \left(-\dfrac{1}{2}z + \dfrac{9}{4}z^2 \right) = \pi i \lim\limits_{z \to 0} \left(-\dfrac{1}{2} + \dfrac{9}{2}z \right) =$

$= \pi i \left(-\dfrac{1}{2} + 0 \right) = -\dfrac{\pi}{2} i$



15.

$$\mathcal{I} = \oint\limits_{\mathcal{L}} \frac{\cos 3z - 1 + \frac{9}{2}z^2}{z^4 \operatorname{sh} \frac{9}{4}z} \ \mathrm{d}z \ , \quad \mathcal{L} : |z| = 1$$

$z^4 \operatorname{sh} \dfrac{9}{4}z = 0, \quad z_1 = 0, \quad \sin \dfrac{9}{4}iz = 0, \quad z_k = -\dfrac{4}{9}i\pi k, \quad k \in \mathbb{Z}, \quad \cos 3z_k - 1 + \dfrac{9}{2}z_k^2 \neq 0$

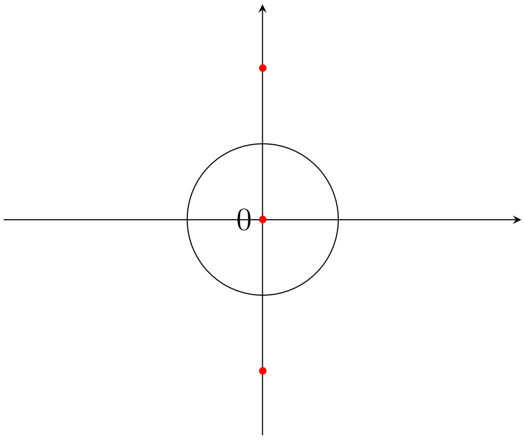
$z_1 = 0$ - нуль V порядку, $z_k = -\dfrac{4}{9}i\pi k \setminus \{0\}$ - не нуль.

$f(z) = \dfrac{\cos 3z - 1 + \frac{9}{2}z^2}{z^4 \operatorname{sh} \frac{9}{4}z} = \dfrac{\left(1 - \dfrac{3^2 z^2}{2!} + \dfrac{3^4 z^4}{4!} + \dfrac{3^6 z^6}{6!} + \dots \right) - 1 + \dfrac{9z^2}{2}}{z^4 \left(\dfrac{9z}{4} + \dfrac{9^3 z^3}{4^3 3!} + \dfrac{9^5 z^5}{4^5 5!} + \dots \right)} =$

$$= \frac{z^4\left(\frac{3^4}{4!}+\frac{3^6z^2}{6!}+\dots\right)}{z^5\left(\frac{9}{4}+\frac{9^3z^2}{4^33!}+\frac{9^5z^4}{4^55!}+\dots\right)} = \frac{z^4g_1(z)}{z^5g_2(z)}, \quad g_i(0)\neq 0$$

$z_0=0: \quad m_1=4, \quad m_2=5 \Longrightarrow z_0=0$ - полюс I порядку.

$$\mathcal{I} = 2\pi i \operatorname*{Res}_{z=0} f(z) = 2\pi i \lim_{z\to 0}\frac{z^4g_1(z)}{z^5g_2(z)}\cdot z = 2\pi i \lim_{z\to 0}\frac{\frac{3^4}{4!}+\frac{3^6z^2}{6!}+\dots}{\frac{9}{4}+\frac{9^3z^2}{4^33!}+\frac{9^5z^4}{4^55!}+\dots} = 2\pi i\frac{\frac{3^4}{4!}}{\frac{9}{4}} = 3\pi i$$



16.

$$\mathcal{I} = \oint\limits_{\mathcal{L}}\left(\underbrace{ze^{\frac{1}{z+6}}}_{\mathcal{I}_1}+\underbrace{\frac{2\cos\left(\frac{\pi z}{5}\right)}{(z+5)^2(z+3)}}_{\mathcal{I}_2}\right)\,\mathrm{d}z = \mathcal{I}_1+2\mathcal{I}_2, \quad \mathcal{L}: |z+6|=2$$

$(\mathcal{I}_1)$ $\mathcal{I}_1 = 2\pi i \operatorname*{Res}_{z=-6}\left[ze^{\frac{1}{z+6}}\right] = 2\pi i \operatorname*{Res}_{z=-6}\left[(z+6-6)\left(1+\frac{1}{z+6}+\frac{1}{2!(z+6)^2}+\dots\right)\right] =$

$$= 2\pi i \operatorname*{Res}_{z=-6}\left[\left((z+6)+1+\frac{1}{2!(z+6)}+\dots\right)-\left(6+\frac{6}{z+6}+\frac{6}{2!(z+6)^2}+\dots\right)\right] =$$

$$= 2\pi i \operatorname*{Res}_{z=-6}\left[(z+6)-5-\underbrace{\frac{11}{2}\cdot\frac{1}{z+6}+\dots}_{\text{головна частина}}\right] = \langle z_0=-6 \text{ - істотно особлива точка}\rangle =$$

$$= 2\pi i\cdot\left(-\frac{11}{2}\right) = -11\pi i$$

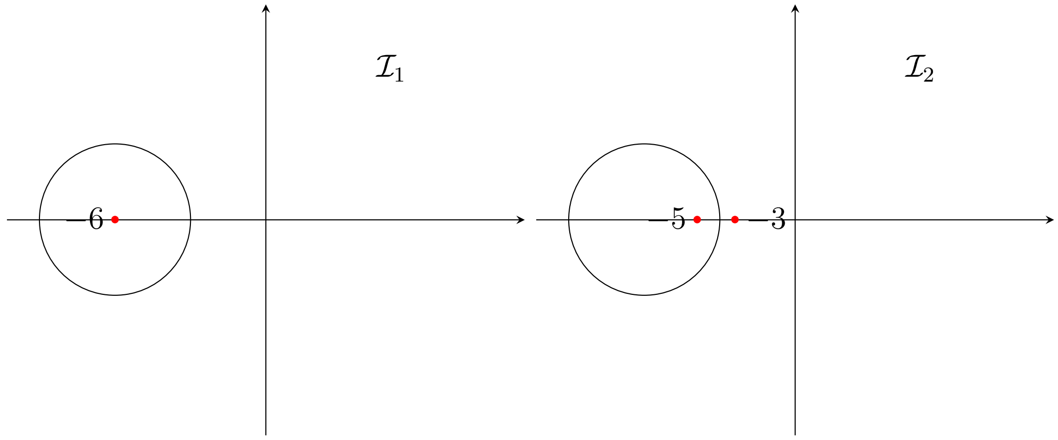
$(\mathcal{I}_2)$ $z_0=-5: \quad \cos\left(\frac{5\pi}{5}\right)=1\neq 0, \quad m_1=0, \quad m_2=2, \quad z_0$ - полюс II порядку.

$$\mathcal{I}_2 = 2\pi i \operatorname*{Res}_{z=-5}\left[\frac{\cos\left(\frac{\pi z}{5}\right)}{(z+5)^2(z+3)}\right] = 2\pi i \lim_{z\to -5}\left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\cos\left(\frac{\pi z}{5}\right)}{(z+5)^2(z+3)}\cdot(z+5)^2\right)\right] =$$

$$= 2\pi i \lim_{z\to -5}\left[\frac{-\frac{\pi}{5}\cdot(z+3)\cdot\sin\left(\frac{\pi z}{5}\right)-\cos\left(\frac{\pi z}{5}\right)}{(z+3)^2}\right] = 2\pi i\frac{-\frac{\pi}{5}\cdot(-2)\cdot\sin(-\pi)-\cos(-\pi)}{(-2)^2} =$$

$$= 2\pi i\cdot\frac{\frac{2\pi}{5}\cdot 0-(-1)}{4} = \frac{\pi i}{2}$$

$$\mathcal{I} = \mathcal{I}_1 + 2\mathcal{I}_2 - 11\pi + 2 \cdot \frac{\pi i}{2} = -10\pi i$$



17.

$$\mathcal{I} = \int\limits_0^{2\pi} \frac{\mathrm{d}t}{4 + \sqrt{15}\sin t}$$

$$\mathcal{I} = \oint\limits_{|z|=1} \frac{\mathrm{d}z}{iz \cdot \left(4 + \sqrt{15} \cdot \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)} = \oint\limits_{|z|=1} \frac{\mathrm{d}z}{4iz + \frac{\sqrt{15}}{2}z^2 - \frac{\sqrt{15}}{2}} =$$

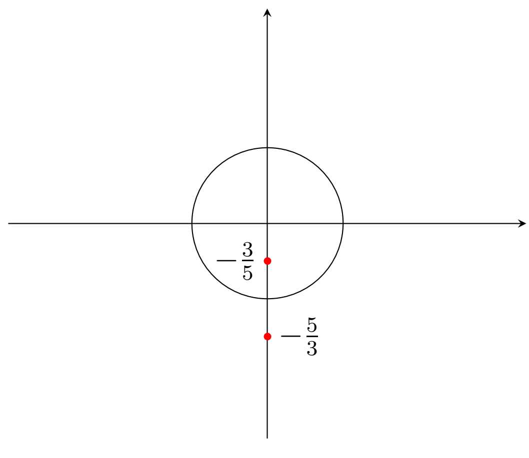
$$4iz + \frac{\sqrt{15}}{2}z^2 - \frac{\sqrt{15}}{2} = 0$$

$$= \left\langle \frac{\sqrt{15}}{2} \cdot \left(z + i\sqrt{\frac{3}{5}}\right)\left(z + i\sqrt{\frac{5}{3}}\right) \right\rangle = \oint\limits_{|z|=1} \frac{\mathrm{d}z}{\frac{\sqrt{15}}{2} \cdot \left(z + i\sqrt{\frac{3}{5}}\right)\left(z + i\sqrt{\frac{5}{3}}\right)} =$$

$z_0 = i\sqrt{\frac{3}{5}}$ - полюс I порядку.

$$= 2\pi i \operatorname*{Res}_{z=-i\sqrt{\frac{3}{5}}} \left[\frac{1}{\frac{\sqrt{15}}{2} \cdot \left(z + i\sqrt{\frac{3}{5}}\right)\left(z + i\sqrt{\frac{5}{3}}\right)}\right] = 2\pi i \cdot \left(\frac{1}{4i + \sqrt{15}z}\right)\Bigg|_{z=-i\sqrt{\frac{3}{5}}} = 2\pi i \cdot$$

$$\cdot \frac{1}{4i - i\sqrt{15}\sqrt{\frac{3}{5}}} = \frac{2\pi i}{4i - 3i} = 2\pi$$



19.

$$\mathcal{I} = \int\limits_{-\infty}^{+\infty} \frac{x-1}{(x^2+4)^2}\ \mathrm{d}x$$

$$\mathcal{I} = \int\limits_{-\infty}^{+\infty} \frac{x-1}{(x+2i)^2(x-2i)^2}\ \mathrm{d}x\ = \left\langle \begin{matrix} m_1 = 0, \quad m_2 = 2 \\ z_0 = 2i \text{ - полюс II порядку.} \end{matrix} \right\rangle = 2\pi i\ \underset{2i}{\mathrm{Res}}\ \left[\frac{z-1}{(z^2+4)^2}\right] =$$

$$= 2\pi i \lim_{z\to 2i}\left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{(z-1)(z-2i)^2}{(z+2i)^2(z-2i)^2}\right)\right] = 2\pi i \lim_{z\to 2i}\left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{(z-1)}{(z+2i)^2}\right)\right] =$$

$$= 2\pi i \lim_{z\to 2i}\left[\frac{(z+2i)^2 - 2(z+2i)(z-1)}{(z+2i)^4}\right] = 2\pi i\cdot\frac{(4i)^2 - 2\cdot(4i)\cdot(2i-1)}{(4i)^4} = 2\pi i\cdot\frac{-16+16+8i}{256} =$$

$$= 2\pi i \cdot \frac{i}{32} = -\frac{\pi}{16}$$

20.

$$\mathcal{I} = \int\limits_{-\infty}^{+\infty} \frac{(x-1)\sin x}{(x^2+9)^2}\ \mathrm{d}x\ = \underbrace{\int\limits_{-\infty}^{+\infty} \frac{x\sin x}{(x^2+9)^2}\ \mathrm{d}x}_{\mathcal{I}_1} - \underbrace{\int\limits_{-\infty}^{+\infty} \frac{\sin x}{(x^2+9)^2}\ \mathrm{d}x}_{\mathcal{I}_2} = \mathcal{I}_1 - \mathcal{I}_2$$

$$(\mathcal{I}_1)\ \int\limits_{-\infty}^{+\infty} \frac{x\sin x}{(x^2+9)^2}\ \mathrm{d}x\ = \mathfrak{Im}\left[2\pi i\ \underset{z=3i}{\mathrm{Res}}\ \frac{ze^{iz}}{(z-3i)^2(z+3i)^2}\right] = \mathfrak{Im}\left[2\pi i \lim_{z\to 3i}\left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{ze^{iz}}{(z+3i)^2}\right)\right]\right] =$$

$$= \mathfrak{Im}\left[2\pi i \lim_{z\to 3i}\left[\frac{(e^{iz}+zie^{iz})(z+3i) - 2z(z+3i)e^{iz}}{(z+3i)^4}\right]\right] =$$

$$= \mathfrak{Im}\left[2\pi i\cdot\frac{(e^{-3}-3e^{-3})\cdot(-36) + 36e^{-3})}{6^4}\right] = \mathfrak{Im}\left[\frac{72\pi i e^{-3}\cdot 2}{1296}\right] = \frac{\pi}{6e^3}$$

$$(\mathcal{I}_2)\ \int\limits_{-\infty}^{+\infty} \frac{\sin x}{(x^2+9)^2}\ \mathrm{d}x\ = \mathfrak{Im}\left[2\pi i\ \underset{z=3i}{\mathrm{Res}}\ \frac{e^{iz}}{(z-3i)^2(z+3i)^2}\right] = \mathfrak{Im}\left[2\pi i \lim_{z\to 3i}\left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{e^{iz}}{(z+3i)^2}\right)\right]\right] =$$

$$= \mathfrak{Im}\left[2\pi i \lim_{z\to 3i}\left[\frac{ie^{iz}(z+3i)^2 - 2(z+3i)e^{iz}}{(z+3i)^4}\right]\right] = \mathfrak{Im}\left[2\pi i\cdot\frac{ie^{-3}\cdot(-36) - 12ie^{-3}}{6^4}\right] =$$

$$= \mathfrak{Im}\left[2\pi\cdot\frac{36e^{-3}+12e^{-3}}{1296}\right] = 0$$

$$\mathcal{I} = \mathcal{I}_1 - \mathcal{I}_2 = \frac{\pi}{6e^3} - 0 = \frac{\pi}{6e^3}$$