Домашня робота 4

2.3 Let $f(n)$, $g(n)$ be multiplicative.

(b) Prove $l(n) = f^a(n)$ - multiplicative.
$l(n) = f^a(n) = \underbrace{f(n)f(n)\dots f(n)}_{a}$. Let $n = ab$, $\gcd(a,\ b) = 1$

$\Rightarrow l(ab) = \underbrace{l(a)l(b)l(a)l(b)\dots l(a)l(b)}_{a}$

(c) Prove $y(n) = \dfrac{f(n)}{g(n)}$ - multiplicative.

Let $n = ab$, $\gcd(a,\ b) = 1 \Rightarrow \dfrac{f}{g}(ab) = \dfrac{f}{g}(a) \cdot \dfrac{f}{g}(b) = \dfrac{f(a)f(b)}{g(a)g(b)}$

2.4 $g(n) = \sum\limits_{d \mid n} f(n)$.

(c) Prove $g(p^\alpha) = \sum\limits_{\beta=0}^{\alpha} f(p^\beta)$.

$n = p^\alpha$

2.9 Let $\gcd(a,\ b) > 1$

(a) Let $a = r_1^{\alpha_1} r_2^{\alpha_2} \dots r_t^{\alpha_t} q_1^{\alpha_{t+1}} q_2^{\alpha_{t+2}} \dots q_t^{\alpha_{t+k}}$, $b = r_1^{\beta_1} r_2^{\beta_2} \dots r_t^{\beta_t} p_1^{\beta_{t+1}} p_2^{\beta_{t+2}} \dots p_t^{\beta_{t+n}}$
$\tau(ab) = \tau\left(r_1^{\alpha_1} r_2^{\alpha_2} \dots r_t^{\alpha_t} q_1^{\alpha_{t+1}} q_2^{\alpha_{t+2}} \dots q_t^{\alpha_{t+k}} r_1^{\beta_1} r_2^{\beta_2} \dots r_t^{\beta_t} p_1^{\beta_{t+1}} p_2^{\beta_{t+2}} \dots p_t^{\beta_{t+n}}\right) =$
$= (1+\alpha_1+\beta_1)(1+\alpha_2+\beta_2)\dots(1+\alpha_t+\beta_t)(1+\beta_{t+1})(1+\beta_{t+2})\dots$
$\dots(1+\beta_{t+n})(1+\beta_{t+1})(1+\alpha_{t+2})\dots(1+\alpha_{t+n})$
$\tau(a)\tau(b) = \tau(r_1^{\alpha_1} r_2^{\alpha_2} \dots r_t^{\alpha_t} q_1^{\alpha_{t+1}} q_2^{\alpha_{t+2}} \dots q_t^{\alpha_{t+k}})\tau(r_1^{\beta_1} r_2^{\beta_2} \dots r_t^{\beta_t} p_1^{\beta_{t+1}} p_2^{\beta_{t+2}} \dots p_t^{\beta_{t+n}}) =$
$= (1+\alpha_1)(1+\alpha_2)\dots(1+\alpha_t)\dots(1+\alpha_{t+k})(1+\beta_1)(1+\beta_2)\dots(1+\beta_k)\dots(1+\beta_{t+n})$
$\tau(ab) < \tau(a)\tau(b)$

(b) Let $a = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$, $b = p_1^{\beta_1} p_2^{\beta_2} \dots p_t^{\beta_k}$
$\sigma(ab) = \sigma\left(r_1^{\alpha_1} r_2^{\alpha_2} \dots r_t^{\alpha_t} q_1^{\alpha_{t+1}} q_2^{\alpha_{t+2}} \dots q_t^{\alpha_{t+k}} r_1^{\beta_1} r_2^{\beta_2} \dots r_t^{\beta_t} p_1^{\beta_{t+1}} p_2^{\beta_{t+2}} \dots p_t^{\beta_{t+n}}\right) =$
$= \prod\limits_{i=0}^{t} \dfrac{r_i^{\alpha_i+\beta_i+1}-1}{r_i-1} \prod\limits_{i=t}^{t+k} \dfrac{q_i^{\alpha_i+1}-1}{q_i-1} \prod\limits_{i=t}^{t+n} \dfrac{p_i^{\beta_i+1}-1}{p_i-1}$
$\sigma(a)\sigma(b) = \sigma(r_1^{\alpha_1} r_2^{\alpha_2} \dots r_t^{\alpha_t} q_1^{\alpha_{t+1}} q_2^{\alpha_{t+2}} \dots q_t^{\alpha_{t+k}})\sigma(r_1^{\beta_1} r_2^{\beta_2} \dots r_t^{\beta_t} p_1^{\beta_{t+1}} p_2^{\beta_{t+2}} \dots p_t^{\beta_{t+n}}) =$
$= \prod\limits_{i=0}^{t} \dfrac{r_i^{\alpha_i+1}-1}{r_i-1} \prod\limits_{i=0}^{t} \dfrac{r_i^{\beta_i+1}-1}{r_i-1} \prod\limits_{i=t}^{t+k} \dfrac{q_i^{\alpha_i+1}-1}{q_i-1} \prod\limits_{i=t}^{t+n} \dfrac{p_i^{\beta_i+1}-1}{p_i-1}$
$\sigma(ab) < \sigma(a)\sigma(b)$

2.10 Prove $\sum\limits_{k=1}^{n} \sigma(k) = \sum\limits_{k=1}^{n} k \left\lfloor \dfrac{n}{k} \right\rfloor$.

$$\sum_{k=1}^{n} \sigma(k) = \sum_{k=1}^{n} \sum_{d \mid k} d = \sum_{d=1}^{n} d \sum_{k \le n,\, d \mid k} 1 = \sum_{k=1}^{n} k \left\lfloor \frac{n}{k} \right\rfloor$$

2.16 Find $\min n$ that $n = 2^\alpha pq$, $\sigma(n) = 3n$.
$\sigma(n) = \sigma(2^\alpha)\sigma(p)\sigma(q) = (2^{\alpha+1}-1)\,(p+1)(q+1) =$

$(2^{\alpha+1}-1) \vdots 2$, $(p+1) \vdots 2$, $(q+1) \vdots 2$
$= 3 \cdot 2^\alpha pq \Rightarrow \alpha \ge 2$

Let $\alpha = 2$.
$12pq = 7(p+1)(q+1) \Rightarrow 5pq = 7(p+q+1) \Rightarrow p = \dfrac{7(q+1)}{5q-7} \Rightarrow q = 7,\ p = 2$ - contradiciton
Let $\alpha = 3$.
$24pq = 15(p+1)(q+1) \Rightarrow 9pq = 15(p+q+1) \Rightarrow p = \dfrac{15(q+1)}{9q-15} \Rightarrow q = 5,\ p = 3$
$n = 2^3 \cdot 5 \cdot 3 = 120 \Rightarrow \sigma(120) = 360 = 3 \cdot 120$

2.18 Let $\tau_m(n)$ be amount of all solutions of $x_1 x_2 \dots x_m = n$. Prove $\tau_m(n)$ - multiplicative.
$\tau_m(ab),\ \gcd(a,\ b) = 1 \Rightarrow \alpha_1\alpha_2\dots\alpha_m = a,\ \beta_1\beta_2\dots\beta_m = b,\ \forall\alpha_i,\ \beta_k :\ \alpha_i \neq \beta_k,\ i \leq m,\ k \leq m$
$\Rightarrow \tau_m(ab) = \tau_m(a)\tau_b(m)$

2.20 Find $\delta(n) = \prod\limits_{d\,|\,n} d$ for $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$.
$$\delta(n) = \delta(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}) = \prod_{d\,|\,p_1^{\alpha_1}\dots} d = p_1 p_1^2 \dots p_1^{\alpha_1} p_2 p_2^2 \dots p_2^{\alpha_2} p_t p_t^2 \dots p_t^{\alpha_t} =$$
$$= p_1^{1+2+\dots+\alpha_1} p_2^{1+2+\dots+\alpha_2} \dots p_t^{1+2+\dots+\alpha_t} = p_1^{\frac{1}{2}\alpha_1(\alpha+1)} p_2^{\frac{1}{2}\alpha_2(\alpha_2+1)} \dots p_t^{\frac{1}{2}\alpha_t(\alpha_t+1)} = \prod_{p_1^{\alpha_i}\,|\,n} p_i^{\frac{1}{2}\alpha_i(\alpha_i+1)}$$
$\delta(p_1^{\alpha_1})\delta(p_2^{\alpha_2})\dots\delta(p_t^{\alpha_t}) = p_1 p_1^2 \dots p_1^{\alpha_1} p_2 p_2^2 \dots p_2^{\alpha_2} p_t p_t^2 \dots p_t^{\alpha_t} = \delta(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t})$
$\Rightarrow \delta(n)$ - multiplicative

2.21 Prove that $\delta^2(n) = n^{\tau(n)}$.
$$\prod_{d\,|\,n} d = \prod_{d\,|\,n} \frac{n}{d} = n^{\tau(n)} \prod_{d\,|\,n} \frac{1}{d} \Rightarrow \left(\prod_{d\,|\,n} d\right)^2 = n^{\tau(n)} \Rightarrow \delta^2 = n^{\tau(n)}$$

2.24 Prove that $\sum\limits_{d\,|\,n} \mu(d) f(d) = \prod\limits_{p\,|\,n} (1 - f(p))$.
$$\sum_{d\,|\,p} \mu(d) f(d) = 1 - f(p),\ \sum_{d\,|\,p^\alpha} \mu(d) f(d) = 1 - f(p) + \underbrace{\mu(p^2) f(p^2) + \dots + \mu(p^\alpha) f(p^\alpha)}_{0} = 1 - f(p)$$
$$\sum_{d\,|\,n} \mu(d) f(d) = \prod_{p\,|\,n} (1 - f(p))$$