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ДОМАШНЯ РОБОТА №12  
З ПРЕДМЕТУ  
"ТЕОРІЯ ФУНКЦІЇ КОМПЛЕКСНОЇ ЗМІННОЇ"  
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$$\begin{aligned}
1. \quad (a) \quad & \int_0^{2\pi} \frac{dt}{\sqrt{21} \sin t + 5} = \int_{|z|=1} \frac{dz}{iz \left( \sqrt{21} \frac{1}{2i} \left( z - \frac{1}{z} \right) + 5 \right)} = \left\langle \begin{array}{l} \frac{1}{2} \sqrt{21} z^2 + 5iz - \frac{1}{2} \sqrt{21} = 0 \\ z = -i\sqrt{\frac{3}{7}}, z = -i\sqrt{\frac{7}{3}} \\ z = -i\sqrt{\frac{3}{7}} \text{ полюс I порядка} \end{array} \right\rangle = \\
& = \int_{|z|=1} \frac{dz}{\frac{\sqrt{21}}{2} \left( z + i\sqrt{\frac{3}{7}} \right) \left( z + i\sqrt{\frac{7}{3}} \right)} = 2\pi i \operatorname{Res}_{z=-i\sqrt{\frac{3}{7}}} \frac{1}{\frac{\sqrt{21}}{2} \left( z + i\sqrt{\frac{3}{7}} \right) \left( z + i\sqrt{\frac{7}{3}} \right)} = \\
& = 2\pi i \lim_{z \rightarrow -i\sqrt{\frac{3}{7}}} \frac{1}{\frac{\sqrt{21}}{2} \left( z + i\sqrt{\frac{7}{3}} \right)} = -\frac{i\sqrt{21}}{4} \cdot \frac{2}{\sqrt{21}} \cdot 2\pi i = \pi
\end{aligned}$$

$$\begin{aligned}
2. \quad (a) \quad & \int_{-\infty}^{\infty} \frac{x^2 + 10}{(x^2 + 4)^2} dx = \int_{-\infty}^{\infty} \frac{(x - i\sqrt{5})(x + i\sqrt{5})}{(x - 2i)^2(x + 2i)^2} dx = 2\pi i \operatorname{Res}_{z=2i} \frac{z^2 + 10}{(z^2 + 4)^2} = \\
& = 2\pi i \lim_{z \rightarrow 2i} \frac{d}{dz} \left( \frac{z^2 + 10}{(z + 2i)^2} \right) = 2\pi i \lim_{z \rightarrow 2i} \frac{2z(z + 2i)^2 - 2(z^2 + 10)(z + 2i)}{(z + 2i)^3} = \\
& = 2\pi i \lim_{z \rightarrow 2i} \frac{4iz - 20}{(z + 2i)^3} = 2\pi i \frac{-28}{-64i} = \frac{7}{8}\pi, \langle 2i \text{ полюс II порядка} \rangle
\end{aligned}$$

$$\begin{aligned}
(b) \quad & \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^4} = 2\pi i \operatorname{Res}_{z=i} \frac{1}{(z - i)^4(z + i)^4} = \frac{1}{3}\pi i \lim_{z \rightarrow i} \frac{d^3}{dz^3} \left( \frac{1}{(z + i)^4} \right) = \\
& = \left\langle \frac{d^3}{dz^3} \left( \frac{1}{(z + i)^4} \right) = \frac{d^2}{dz^2} \left( -\frac{4}{(z + i)^5} \right) = \frac{d}{dz} \left( \frac{20}{(z + i)^6} \right) = -\frac{120}{(z + i)^7} \right\rangle = \\
& = \frac{1}{3}\pi i \lim_{z \rightarrow i} \frac{-120}{(z + i)^7} = \pi \frac{120}{3 \cdot 2^7} = \frac{5}{16}\pi, \langle i \text{ полюс IV порядка} \rangle
\end{aligned}$$

$$\begin{aligned}
3. \quad (a) \quad & \int_0^{\infty} \frac{\cos 2x}{\left( x^2 + \frac{1}{4} \right)^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos 2x}{\left( x^2 + \frac{1}{4} \right)^2} dx = \frac{1}{2} \Re \left[ 2\pi i \operatorname{Res}_{z=\frac{i}{2}} \frac{e^{2iz}}{\left( z^2 + \frac{1}{4} \right)^2} \right] = \\
& = \frac{1}{2} \Re \left[ 2\pi i \lim_{z \rightarrow \frac{i}{2}} \frac{d}{dz} \left( \frac{e^{2iz}}{\left( z + \frac{i}{2} \right)^2} \right) \right] = \frac{1}{2} \Re \left[ 2\pi i \lim_{z \rightarrow \frac{i}{2}} \frac{e^{2iz} 2i \left( z + \frac{i}{2} \right)^2 - e^{2iz} (2z + i)}{\left( z + \frac{i}{2} \right)^4} \right] = \\
& = \frac{1}{2} \Re \left[ 2\pi i \frac{e^{-1} 2i^3 - e^{-1} 2i}{i^4} \right] = \frac{1}{2} \Re [4\pi e^{-1} + 4\pi e^{-1}] = \frac{4\pi}{e}
\end{aligned}$$

$$\begin{aligned}
(b) \quad & \int_{-\infty}^{\infty} \frac{(x^3 + 1) \sin x}{x^4 + 5x^2 + 4} dx = \Im \left[ 2\pi i \operatorname{Res}_{z=2i} \frac{(z^3 + 1)e^{iz}}{(z^2 + 4)(z^2 + 1)} + 2\pi i \operatorname{Res}_{z=i} \frac{(z^3 + 1)e^{iz}}{(z^2 + 4)(z^2 + 1)} \right] = \\
& = \Im \left[ 2\pi i \frac{(z^3 + 1)e^{iz}}{4z^3 + 10z} \right]_{2i} + 2\pi i \frac{(z^3 + 1)e^{iz}}{4z^3 + 10z} \Big|_i = \Im \left[ \frac{2\pi i \cdot (-8i + 1) \cdot e^{-2}}{-32i + 20i} + \right. \\
& \left. + \frac{2\pi i \cdot (-i + 1) \cdot e^{-1}}{-4i + 10i} \right] = \Im \left[ -\frac{\pi}{6e^2} + \frac{4\pi i}{3e^2} - \frac{\pi}{3e} - \frac{\pi i}{3e} \right] = \left( \frac{4\pi - \pi e}{3e^2} \right) = \frac{\pi(4 - e)}{3e^2}
\end{aligned}$$