Домашня робота 2

1.11 $(ab+cd) \vdots (a+c)$. Prove $(ad+cb) \vdots (a+c)$.

$(a+c) \vdots (a+c) \Rightarrow (b+d)(a+c) \vdots (a+c) \Rightarrow (ab+cd+ad+cd) \vdots (a+c)$,

$(ab+cd) \vdots (a+c) \Rightarrow (ad+cb) \vdots (a+c)$

1.12 (b) Prove that if $2a-b \vdots 11$ then $51a-9b \vdots 11$.

$(2a-b) \vdots 11 \Rightarrow (18a-9b) \vdots 11 \Rightarrow (51a-9b-33a) \vdots 11 \Rightarrow (51a-9b) \vdots 11 - 33a \vdots 11$

1.13 Prove that if $x^y \vdots y^x$ and $y^x \vdots z^y$ then $x^z \vdots z^x$.

$x^y \vdots y^x \Rightarrow (x^y)^z \vdots (y^x)^z \Rightarrow x^{yz} \vdots y^{xz} \Rightarrow (x^z)^y \vdots (y^z)^x \Rightarrow$

$y^z \vdots z^y \Rightarrow (y^z)^x \vdots (z^y)^x$

$\Rightarrow (x^z)^y \vdots (z^y)^x \Rightarrow x^{zy} \vdots z^{yx} \Rightarrow (x^z)^y \vdots (z^x)^y \Rightarrow x^z \vdots z^x$

1.15 (b) Let δ be any common divisor of a and b. Prove $\gcd\left(\frac{a}{\delta}, \frac{b}{\delta}\right) = \frac{\gcd(a,\, b)}{\delta}$.

$$\gcd\left(\frac{a}{\delta}, \frac{b}{\delta}\right) = \max\left\{d \mid \frac{a}{\delta} \vdots d,\ \frac{b}{\delta} \vdots d\right\} = \frac{1}{\delta}\max\{d | a \vdots d,\ b \vdots d,\ d \vdots \delta\} = \frac{1}{\delta}\max\{d | a \vdots d,\ b \vdots d\} =$$
$$= \frac{\gcd(a,\, b)}{\delta}$$

1.19 Prove that if $a = b+2 \vdots 2$ then $\gcd(a,\, b) = 2$ and if $a = b + 2 \overline{\vdots}\, 2$ then $\gcd(a,\, b) = 1$

Let $a = 2k+3,\ b = 2k+1$

$2k+3 = (2k+1)\cdot 1 + 2$

$2k+1 = 2\cdot k + 1$

$2 = 1 \cdot 2$

$\gcd(a,\, b) = \gcd(2k+3,\ 2k+1) = 1$

Let $a = 2k+4,\ b = 2k+2$

$2k+4 = (2k+2)\cdot 1 + 2$

$2k+2 = 2\cdot(k+1)$

$\gcd(a,\, b) = \gcd(2k+4,\ 2k+2) = 2$

1.20 Prove $\gcd(ab,\, bc,\, ac) \vdots \gcd^2(a,\, b,\, c)$

$\gcd^2(a,\, b,\, c) = \gcd(a,\, b,\, c)\cdot\gcd(a,\, b,\, c) = \gcd(a\cdot\gcd(a,\, b,\, c),\ b\cdot\gcd(a,\, b,\, c),\ c\cdot\gcd(a,\, b,\, c))$
$= \gcd(\gcd(aa,\, ab,\, ac),\ \gcd(ba,\, bb,\, bc),\ \gcd(ca,\, cb,\, cc)) = \gcd(aa,\, ab,\, ac,\, bb,\, bc,\, cc)$

$\Rightarrow \gcd(ab,\, bc,\, ac) \vdots \gcd(aa,\, ab,\, ac,\, bb,\, bc,\, cc) \Rightarrow \gcd(ab,\, bc,\, ac) \vdots \gcd^2(a,\, b,\, c)$

1.21 Let $a,\ b$ be coprime.

$\gcd(a+b,\ a-b) = \gcd(a-b,\ 2b)$

If $a \overline{\vdots}\, 2$ and $b \overline{\vdots}\, 2 \Rightarrow a-b \vdots 2 \Rightarrow \gcd(a-b,\ 2b)$

If $a \overline{\vdots}\, 2$ and $b \vdots 2 \Rightarrow a-b \overline{\vdots}\, 2 \Rightarrow \gcd(a,\, b) = 1 = \gcd(b-a,\ b) \Rightarrow \gcd(a-b,\ 2b) = 1$

1.22 (b) Let $a,\ b$ be coprime. Prove that $\gcd(a^2+b^2,\ ab) = 1$

$1 = \gcd(a, b) = \gcd(a+b, a) = \gcd(a+b, ab) = \gcd((a+b)^2, ab) = \gcd(a^2+2ab+b^2, ab) =$
$= \gcd(a^2+b^2,\ ab)$

(c) $\gcd(a+b,\ a^2+b^2) = ?$

If $a \vdots 2$ and $b \vdots 2 \Rightarrow a+b \vdots 2 \Rightarrow a^2+b^2 \vdots 2 \Rightarrow \gcd(a+b,\ a^2+b^2) = 2$

If $a \vdots 2$ and $b \vdots 2 \Rightarrow a-b \vdots 2 \Rightarrow a^2+b^2 \vdots 2 \Rightarrow \gcd(a+b,\ a^2+b^2) = 1$

1.23 Let $a,\ b$ be odd and $a-b=2^n$. Prove that a and b are coprime.
$\gcd(a,\ b) = \gcd(b,\ a) = \gcd(a,\ a-b) = \gcd(a,\ 2^n) = 1$

1.28 Prove that $\gcd(a,\ b) = \gcd(13a+8b,\ 5a+3b)$
$\gcd(13a+8b,\ 8a+5b) = \gcd(8a+5b,\ 5a+3b) = \gcd(5a+3b,\ 3a+2b) =$
$= \gcd(3a+2b,\ 2a+b) = \gcd(2a+b,\ a+b) = \gcd(a+b,\ -a) = \gcd(-a,\ -b) = \gcd(a,\ b)$