ДОМАШНЯ РОБОТА №2
З ПРЕДМЕТУ
”ТЕОРІЯ ФУНКЦІЇ КОМПЛЕКСНОЇ ЗМІННОЇ”

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1. (a) $(1+i\sqrt{3})^3 = (i\sqrt{3})^3 + 9i^2 + 3\sqrt{3}i + 1 = -3\sqrt{3}i + 3 - 3 + 3\sqrt{3}i + 1 = 1 + i\sqrt{3}i -$
$- 9 - 3\sqrt{3}i = -8$

   (b) $(\sqrt{3}-i)^5 = (\sqrt{3}-i)^2(\sqrt{3}-i)^3 = (3\sqrt{3} - 9i + 3\sqrt{3}i^2 - i^3)(3 - 2\sqrt{3}i + i^2) =$
$= (3\sqrt{3} - 9i - 3\sqrt{3} + i)(2 - 2\sqrt{3}i) = -8i(2 - 2\sqrt{3}i) = -16i - 16\sqrt{3}$

   (c) $$\left(\frac{\sqrt{3}+i}{\sqrt{2}}\right)^{12} = \left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i\right)^{12}, \qquad \arg z = \arctan\sqrt{3} = \frac{\pi}{6},$$
$$\left(\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right)^{12} = 2^6\left(\cos 2\pi + i\sin 2\pi\right) = 64\left(\cos 2\pi + i\sin 2\pi\right) = 64$$

2. (a) $\sqrt[3]{i}, \quad \sqrt[3]{|i|} = 1, \quad \arg z = \dfrac{\pi}{2},$
$$\sqrt[3]{i} = \sqrt[3]{|i|}\left(\cos\frac{\frac{\pi}{2}+2\pi k}{3} + i\sin\frac{\frac{\pi}{2}+2\pi k}{3}\right) = \left(\cos\frac{\frac{\pi}{2}+2\pi k}{3} + i\sin\frac{\frac{\pi}{2}+2\pi k}{3}\right)$$
$$\sqrt[3]{i} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \qquad \sqrt[3]{i} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$
$$\sqrt[3]{i} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 - i$$

   (b) $\sqrt[4]{-1}, \quad \sqrt[4]{|-1|} = 1, \quad \arg z = \pi,$
$$\sqrt[4]{-1} = \sqrt[4]{|-1|}\left(\cos\frac{\pi+2\pi k}{4} + i\sin\frac{\pi+2\pi k}{4}\right) = \left(\cos\frac{\pi+2\pi k}{4} + i\sin\frac{\pi+2\pi k}{4}\right)$$
$$\sqrt[4]{-1} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \qquad \sqrt[4]{-1} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$
$$\sqrt[4]{-1} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \qquad \sqrt[4]{-1} = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

   (c) $\sqrt{2 - i2\sqrt{3}}, \quad |2 - i2\sqrt{3}| = 4, \quad \arg z = -\dfrac{\pi}{3},$
$$\sqrt{2 - i2\sqrt{3}} = 2\left(\cos\frac{-\frac{\pi}{3}+2\pi k}{2} + i\sin\frac{-\frac{\pi}{3}+2\pi k}{2}\right)$$
$$\sqrt{2 - i2\sqrt{3}} = 2\left(\cos -\frac{\pi}{6} + i\sin -\frac{\pi}{6}\right) = \sqrt{3} - i,$$
$$\sqrt{2 - i2\sqrt{3}} = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i$$

3. $z_1 = \ln(1 + x\cos y) + i\cdot 4^x, \quad z_2 = \sin y + i(2^{x+1} - 3)$
$$\begin{cases} z_1 = \bar{z}_2 \\ z_2 = \bar{z}_1 \end{cases} \quad \begin{cases} \ln(1 + x\cos y) + i\cdot 4^x = \sin y - i(2^{x+1} - 3) \\ \sin y + i(2^{x+1} - 3) = \ln(1 + x\cos y) - i\cdot 4^x \end{cases} \quad \begin{matrix} 2i\cdot 4^x = -2i(2^{x+1} - 3) \\ (2^x)^2 = -(2^x\cdot 2 - 3) \end{matrix}$$
$t^2 = -(2t - 3), \quad t = -3, t = 1, \quad 2^x = -3, 2^x = 1 \Longrightarrow x = 0, \quad \ln(1) + i = \sin y - i(2-3),$
$0 = \sin y \Longrightarrow y = \pi k, k \in z$

4. Нехай $z_1 = a + ib, \quad z_2 = c + id$

   (a) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2, \quad \overline{(a+ib) - (c+id)} = \overline{(a+ib)} - \overline{(c+id)}, \quad \overline{(a-c) + i(b-d)} =$
$= (a - ib) - (c - id), \quad (a-c) - i(b-d) = (a-c) - i(b-d)$

   (b) $$\overline{\frac{z_1}{z_2}} = \frac{\bar{z}_1}{\bar{z}_2}, \quad \overline{\frac{a+ib}{c+id}} = \frac{a-ib}{c-id}, \quad \frac{\overline{(ac+bd) - i(-ad+bc)}}{c^2+d^2} = \frac{(ac+bd) + i(-ad+bc)}{c^2+d^2},$$
$$\frac{(ac+bd) + i(-ad+bc)}{c^2+d^2} = \frac{(ac+bd) + i(-ad+bc)}{c^2+d^2}$$

5. (a) $$\frac{z-1}{z+1} = \hat{z}, \quad \bar{\hat{z}} = \overline{\frac{z-1}{z+1}} = \frac{\bar{z}-1}{\bar{z}+1}$$
$$\mathfrak{Re}\,\hat{z} = \frac{\hat{z} + \bar{\hat{z}}}{2} = \frac{1}{2}\cdot\left(\frac{z-1}{z+1} + \frac{\bar{z}-1}{\bar{z}+1}\right) = \frac{1}{2}\cdot\left(\frac{z\bar{z} + z - \bar{z} - 1 + \bar{z}z + \bar{z} - z - 1}{z\bar{z} + z + \bar{z} + 1}\right) =$$

$$= \frac{1}{2} \cdot \left(\frac{2z\bar{z} - 2}{z\bar{z} + z + \bar{z} + 1} \right) = \frac{z\bar{z} - 1}{z\bar{z} + z + \bar{z} + 1}, \qquad \frac{z\bar{z} - 1}{z\bar{z} + z + \bar{z} + 1} = 0,$$

$$\left[\begin{array}{l} z\bar{z} - 1 = 0 \\ z\bar{z} + z + \bar{z} + 1 \neq 0 \end{array} \right., \qquad \left[\begin{array}{l} z\bar{z} = 1 \\ z\bar{z} + z + \bar{z} + 1 \neq 0 \end{array} \right., \qquad |z| = |\bar{z}|, \qquad z\bar{z} = 1 \Leftrightarrow |z| = 1$$

(b) $\frac{z-1}{z+1} = \hat{z}, \qquad \bar{\hat{z}} = \overline{\frac{z-1}{z+1}} = \frac{\bar{z}-1}{\bar{z}+1}$

$$\mathfrak{Im}\hat{z} = \frac{\hat{z} - \bar{\hat{z}}}{2i} = \frac{1}{2i} \cdot \left(\frac{z-1}{z+1} + \frac{\bar{z}-1}{\bar{z}+1} \right) = \frac{1}{2i} \cdot \left(\frac{z\bar{z} + z - \bar{z} - 1 - \bar{z}z - \bar{z} + z + 1}{z\bar{z} + z + \bar{z} + 1} \right) =$$

$$= \frac{1}{2i} \cdot \left(\frac{-2\bar{z} + 2z}{z\bar{z} + z + \bar{z} + 1} \right) = \frac{1}{i} \cdot \left(\frac{z - \bar{z}}{z\bar{z} + z + \bar{z} + 1} \right), \qquad \frac{1}{i} \cdot \left(\frac{z - \bar{z}}{z\bar{z} + z + \bar{z} + 1} \right) = 0$$

$$\left[\begin{array}{l} z - \bar{z} = 0 \\ i(z\bar{z} + z + \bar{z} + 1) \neq 0 \end{array} \right., \qquad \left[\begin{array}{l} z = \bar{z} \\ z\bar{z} + z + \bar{z} + 1 \neq 0 \end{array} \right., \qquad z = \bar{z} \Leftrightarrow \mathfrak{Im}z = 0$$

6. $z_1\mathfrak{Im}(\bar{z}_2 z_3) + z_2\mathfrak{Im}(\bar{z}_3 z_1) + z_3\mathfrak{Im}(\bar{z}_1 z_2) = z_1\frac{\bar{z}_2 z_3 - z_2\bar{z}_3}{2i} + z_2\frac{\bar{z}_3 z_1 - z_3\bar{z}_1}{2i} + z_3\frac{\bar{z}_1 z_2 - z_1\bar{z}_2}{2i} =$

$$= \frac{1}{2i} \cdot \left(\underline{z_1\bar{z}_2 z_3} - z_1 z_2\bar{z}_3 + z_2\bar{z}_3 z_1 - z_2 z_3\bar{z}_1 + z_3\bar{z}_1 z_2 - \underline{z_3 z_1\bar{z}_2} \right) = \frac{1}{2i} \cdot \left(\underline{-z_1 z_2\bar{z}_3} + \underline{z_2\bar{z}_3 z_1} - \right.$$

$$\left. -z_2 z_3\bar{z}_1 + z_3\bar{z}_1 z_2 \right) = \frac{1}{2i} \cdot (-z_2 z_3\bar{z}_1 + z_3\bar{z}_1 z_2) = 0$$

7. (a) $\bar{z} = z^2, \qquad z = x + iy, \qquad x - iy = (x^2 - y^2) + 2xyi,$

$$\begin{cases} x = x^2 - y^2 \\ -iy = 2xyi \end{cases} \quad 2x = -1, \qquad x = -\frac{1}{2}, -\frac{1}{2} = \frac{1}{4} - y^2, \qquad y^2 = \frac{3}{4}, \qquad y = \pm\frac{\sqrt{3}}{2}$$

$$y = 0: \qquad x = x^2 \qquad \Longrightarrow \qquad x = 1, \qquad x = 0$$

$$z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \qquad z = 0, \qquad z = 1$$

(b) $z = |z|, \qquad z = x+iy, \qquad |z| = \sqrt{x^2 + y^2}, \qquad x+iy = \sqrt{x^2 + y^2}, \qquad x^2+y^2 = (x+iy)^2,$
$2y^2 - 2xyi = 0, \qquad y(y-xi) = 0, \qquad y = 0, y = ix, \qquad z = x, z = x-x = 0, x > 0, x \in R$