Упражнение 4.1

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x^{(i)}$$

$H_k = v_0 + L(v_1 \dots v_k)$ – k-мерное лин. многообразие

Будем искать v_0 как решение задачи minimize H_0:

$$v_0 = \underset{a_0 \in \mathbb{R}^n}{\operatorname{argmin}} \left(\sum_{i=1}^{N} dist^2(x_i, H_0)\right) = \underset{a_0 \in \mathbb{R}^n}{\operatorname{argmin}} \left(\sum_{i=1}^{N} \| x^{(i)} - a_0 \|^2\right) =$$

это и есть выборочное среднее

$$= \frac{1}{N}\sum_{i=1}^{N} x^{(i)} = \bar{x}.$$

Задание 1

а) $a \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ $\quad \frac{\partial (a^T x)}{\partial x} = a$

$a^T = (a_1, a_2 \dots a_n)$

$x = (x_1, x_2 \dots x_n)^T$

$a^T x = (a_1 x_1, a_2 x_2 \dots a_n x_n)^T$

$$\frac{\partial (a^T x)}{\partial x} = (a_1, a_2 \dots a_n)^T = (a^T)^T = a$$

б) $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$ $\quad \frac{\partial (Ax)}{\partial x} = A$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & \dots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} \longrightarrow Ax = \begin{pmatrix} a_{11}x_1 & a_{12}x_2 & \dots & a_{1n}x_n \\ a_{n1}x_1 & \dots & & a_{nn}x_n \end{pmatrix}$$

$$\frac{\partial (Ax)}{\partial x} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = A$$

в) $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$ $\quad \frac{\partial (x^T A x)}{\partial x} = (A + A^T)x$

из б) $x^T A = \begin{pmatrix} a_{11}x_1 & \dots & a_{1n}x_1 \\ a_{n1}x_n & \dots & a_{nn}x_n \end{pmatrix} = T$

$$Tx = \begin{pmatrix} a_{11}x_1^2 & \dots & a_{1n}x_1x_n \\ & \dots & \\ a_{n1}x_nx_1 & \dots & a_{nn}x_n^2 \end{pmatrix} \to \frac{\partial (Tx)}{\partial x} = \begin{pmatrix} 2a_{11}x_1 & \dots & (a_{1n}+a_{n1})x_n \\ & \dots & \\ (a_{n1}+a_{1n})x_1 & \dots & 2a_{nn}x_n \end{pmatrix} = (A^T + A)x$$

Если $A = A^T$, то, очевидно, $A + A^T = 2A$.

г) $x \in \mathbb{R}^n$ $\quad \frac{\partial \|x\|^2}{\partial x} = 2x$

$\|x\|^2 = (x_1^2 + \dots + x_n^2)^T$

$$\frac{\partial \|x\|^2}{\partial x} = (2x_1 \;\; 2x_2 \dots 2x_n)^T = 2x$$

д) $g(x)$ - скалярн, $x \in \mathbb{R}^n$ $\quad \frac{\partial g(x)}{\partial x} = diag(g'(x))$

$$g(x) = (g(x_1) \dots g(x_n))^T \quad \frac{\partial g(x)}{\partial x} = \begin{pmatrix} g_{x_1}'(x_1) & \underbrace{g_{x_2}'(x_2)}_{=0} & \dots & \overbrace{g_{x_1}'(x_n)}^{=0} \\ & \dots & & \\ \underbrace{g_{x_n}'(x_1)}_{=0} & \dots & & g_{x_n}'(x_n) \end{pmatrix} =$$

$$= \begin{pmatrix} g_{x_1}'(x_1) & 0 & \dots & 0 \\ 0 & g_{x_2}'(x_2) & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & g_{x_n}'(x_n) \end{pmatrix} = diag\, g'(x)$$

е) $h: \mathbb{R}^n \to \mathbb{R}^m$, $g: \mathbb{R}^m \to \mathbb{R}^n$, $x \in \mathbb{R}^n$

$h(x) = (h(x_1) \dots h(x_n))^T \quad g(h(x)) = (g(h(x_1)) \dots g(h(x_n)))^T$

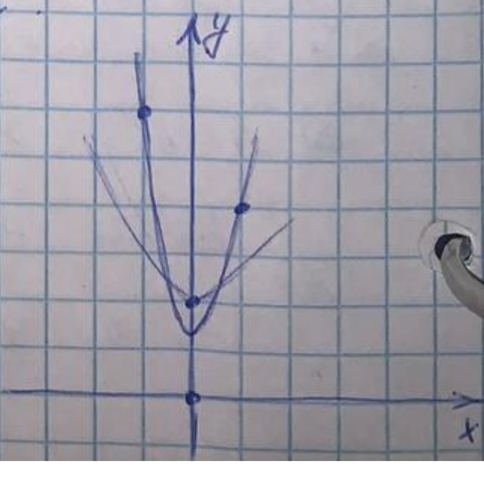
$$\frac{\partial g(h(x))}{\partial x} = \frac{\partial (g(h(x_i)))^T_{i=\overline{1,n}}}{\partial x}$$

произв. сложн. ф-ции $(f(g(x)))'_x = \frac{\partial f}{\partial g} \frac{\partial g}{\partial x} \to$

$\to$ тогда из д) $\frac{\partial g}{\partial h} \cdot \frac{\partial h}{\partial x}$.

Задание 3

x	1	1	0	0	-1
y	4	4	0	2	6



$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2$

$$x^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad y = \begin{pmatrix} 4 \\ 4 \\ 0 \\ 2 \\ 6 \end{pmatrix}$$

$$X^T X = \begin{pmatrix}5 & 1 & 3\\ 1 & 3 & 1\\ 3 & 1 & 3\end{pmatrix} \qquad X^T y = \begin{pmatrix}16\\ 2\\ 14\end{pmatrix}$$

$$\to \left(\begin{array}{ccc|c}0 & 0 & -1 & -4\\ 1 & 0 & 0 & 1\\ 0 & -1 & 0 & 1\end{array}\right) \to \beta = (1\ \ -1\ \ 4)^T$$

Т.о. $f(x) = 1 - x + 4x^2$

$\lambda = 1 \to$ ридж регрессия

$$X^T X + \lambda I = \begin{pmatrix}6 & 1 & 3\\ 1 & 4 & 1\\ 3 & 1 & 4\end{pmatrix} \to \left(\begin{array}{ccc|c}0 & 1 & 0 & -1/2\\ 2 & 0 & 0 & 3\\ 0 & 0 & 1 & 5/2\end{array}\right) \to$$

$$\to \beta = (3/2\ \ -1/2\ \ 5/2)$$

$$g(x) = 3/2 - 1/2\,x + 5/2\,x^2$$

Задание 9

x_1	0	1	0	2	2	2	4	3
x_2	-1	0	0	0	1	0	1	2
y	0	0	0	0	0	1	1	1

$\hat{Pr}(y=0) = 5/8 \qquad \hat{Pr}(y=1) = 3/8$

$$\hat{\mu}_0 = \begin{pmatrix}1\\ 0\end{pmatrix} \qquad \hat{\mu}_1 = \begin{pmatrix}3\\ 1\end{pmatrix}$$

$$\hat{\Sigma}_0 = \frac{1}{N_0 - 1} \sum_{y^{(i)}=0} (x^{(i)} - \hat{\mu}_0)(x^{(i)} - \hat{\mu}_0)^T = \frac{1}{4}\begin{pmatrix}4 & 2\\ 2 & 2\end{pmatrix}$$

$$\hat{\Sigma}_1 = \frac{1}{2}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$

Оцениваем матр. ковар.:

$$\hat{\Sigma} = \frac{1}{N-K} \sum_k \sum_{y_i=k} (x^{(i)} - \hat{\mu}_k)(x^{(i)} - \hat{\mu}_k)^T = \frac{1}{8}\begin{pmatrix}6 & 3\\ 3 & 4\end{pmatrix} = \begin{pmatrix}3/4 & 3/8\\ 3/8 & 1/2\end{pmatrix}$$

$$\hat{\Sigma}_0^{-1} = \begin{pmatrix}1/2 & -1/2\\ -1/2 & 1\end{pmatrix}, \quad \hat{\Sigma}_1^{-1} = \begin{pmatrix}1/6 & -1/3\\ -1/3 & 2/3\end{pmatrix},$$

$$\hat{\Sigma}^{-1} = \begin{pmatrix}32/15 & -16/15\\ -8/5 & 16/15\end{pmatrix}.$$

Линейн. дискрим. ф-ции:

$$\delta_0^1(x) = \begin{pmatrix}x_1\\x_2\end{pmatrix}^T \begin{pmatrix}32/15 & -16/15\\ -8/5 & 16/15\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}^T \begin{pmatrix}32/15 & -16/15\\ -8/5 & 16/15\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \ln 5 - \ln 8 =$$

$$= \begin{pmatrix}32/15\, x_1 - 8/5\, x_2 \\ -16/15\, x_1 + 16/15\, x_2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}\frac{32}{15} & -\frac{16}{15}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \ln 5 - \ln 8 =$$

$$= \frac{32}{15}x_1 - \frac{8}{5}x_2 - \frac{16}{15} + \ln 5 - \ln 8.$$

$$\delta_1^1(x) = \frac{32}{5}x_1 - \frac{24}{5}x_2 - \frac{16}{15}x_1 + \frac{16}{15}x_2 - \frac{1}{2}\begin{pmatrix}32/5 + 8/5 \\ -16/5 + 16/5\end{pmatrix}^T\begin{pmatrix}1\\0\end{pmatrix} + \ln 3 - \ln 8 =$$

$$= \frac{16}{3}x_1 - \frac{56}{15}x_2 - 4 + \ln 3 - \ln 8$$

Разд. пов-ть $\delta_0(x) = \delta_1(x)$:

$$\frac{32}{15}x_1 - \frac{8}{5}x_2 - \frac{16}{15} + \ln 5 = \frac{16}{3}x_1 - \frac{56}{15}x_2 - 4 + \ln 3$$

$$\frac{48}{15}x_1 - \frac{32}{15}x_2 - \frac{44}{15} + \ln\frac{3}{5} = 0$$

$$48x_1 - 32x_2 - 44 + 15\ln 3/5 = 0$$

Квадрат. дискр. ф-ции:

$$\delta_0^2(x) = -1/2 \ln \cancel{\frac{128}{225}} - \frac{56}{15}x_2 + \frac{4}{3}x_1 + \ln 5/8 - 4x_1^2 + \frac{16}{15}x_2^2,$$

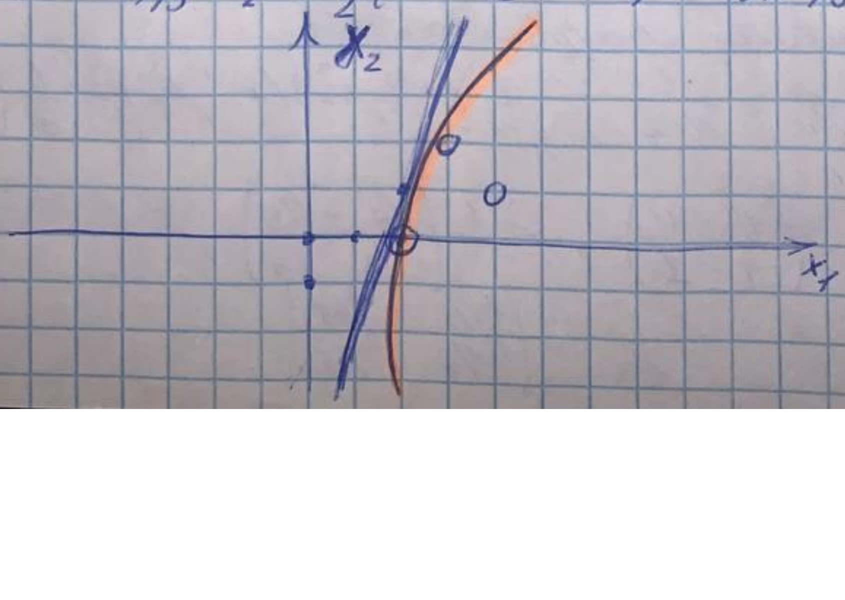
$$\delta_1^2(x) = -x_1^2 + 4/3\, x_1 + 16/15\, x_2^2 + \frac{32}{15}x_2 - 1/2 \ln 3/2$$

Разделяющая поверхность:

$$-\frac{1}{2} - \frac{56}{15}x_2 + \cancel{\frac{4}{3}x_1} + \ln\frac{5}{8} - 4x_1^2 + \cancel{\frac{16}{15}x_2^2} =$$

$$= -x_1^2 + \cancel{4/3\, x_1} + \cancel{16/15\, x_2^2} + 32/15\, x_2 - \frac{1}{2}\ln\frac{3}{2}$$

$$3x_1^2 + 88/15\, x_2 - \frac{1}{2}(\ln 3/2 - 1) - \ln 5/8 = 0.$$



Задание 15

$\hat{Pr}(y=0) = 1/2$ $\hat{Pr}(y=1) = 1/2$

$\hat{Pr}(x_1=0 \mid y=0) = 3/5$ $\hat{Pr}(x_1=1 \mid y=0) = 2/5$

$\hat{Pr}(x_2=0 \mid y=0) = 2/5$ $\hat{Pr}(x_2=1 \mid y=0) = 3/5$

$\hat{Pr}(x_1=0 \mid y=1) = 3/5$ $\hat{Pr}(x_1=1 \mid y=1) = 2/5$

$\hat{Pr}(x_2=0 \mid y=1) = 0$ $\hat{Pr}(x_2=1 \mid y=1) = 1$

$$Pr(y=0 \mid x_1=1, x_2=1) = \frac{2/5 \cdot 3/5 \cdot 1/2}{3/25 + 1/5} = \frac{6/50}{8/25} = \frac{3 \cdot \cancel{25}}{\cancel{25} \cdot 8} = \frac{3}{8}$$

$$Pr(y=1 \mid x_1=1, x_2=1) = \frac{2/5 \cdot 1 \cdot 1/2}{3/25 + 1/5} = \frac{1/5}{8/25} = \frac{1 \cdot \cancel{25}^{5}}{\cancel{5} \cdot 8} = \frac{5}{8}$$

Ответ: 3/8 и 5/8.