Testare nr 2
la obiectul de studiu
ecuații diferențiale și algebra
a studentei grupei 3MI
Razloga Anastasia.

(5) Să se găsească comutatorul operatorilor

a) $X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, $X_2 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

$$[X_1, X_2] = X_1 X_2 - X_2 X_1 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) - \left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) =$$

$$= \left[x\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial x}\right) + x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}\right) + y\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial x}\right) + y\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial y}\right)\right] -$$

$$- \left[y\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right) + y\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial y}\right) + x\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial x}\right) + x\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y}\right)\right] =$$

$$= x\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial x}\right) + x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}\right) + y\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial x}\right) + y\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial y}\right) -$$

$$- y\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right) - y\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial y}\right) - x\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial x}\right) - x\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y}\right) =$$

$$= \cancel{xy\frac{\partial^2}{\partial x^2}} + \cancel{x^2\frac{\partial^2}{\partial x\partial y}} + \cancel{y^2\frac{\partial^2}{\partial y\partial x}} + \cancel{yx\frac{\partial^2}{\partial y^2}} - y\frac{\partial}{\partial x} - \cancel{xy\frac{\partial^2}{\partial x^2}} -$$

$$- \cancel{y^2\frac{\partial^2}{\partial y\partial x}} - \cancel{x^2\frac{\partial^2}{\partial y\partial x}} - x\frac{\partial}{\partial y} - \cancel{xy\frac{\partial^2}{\partial y}} = \boxed{-y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}} \Rightarrow -X_2.$$

в) $X = 2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, $Y = (1+x)\frac{\partial}{\partial x} + (2+y)\frac{\partial}{\partial y}$

$$[X\,Y] = XY - YX = \left(2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} + y\frac{\partial}{\partial y}\right) -$$

$$- \left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} + y\frac{\partial}{\partial y}\right)\left(2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) = 2\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = X$$

1. Să se scrie forma ecuației $\dot{x} = a + cx + dy$; $\dot{y} = b + ex + fy$ ce admite operatorul

a) $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

$\xi^1(x,y) = y$; $\xi^2(x,y) = -x$

$\xi^1_x P + \xi^1_y Q = \xi^1 P_x + \xi^2 P_y + D(P)$

$\xi^2_x P + \xi^2_y Q = \xi^1 Q_x + \xi^2 Q_y + D(Q)$

$D = 0$

$P_x = c$; $P_y = d$; $Q_x = e$; $Q_y = f$

Scriem ecuațiile determinante

$-b - ex - fy = y \cdot c + (-x)d$

$-1(a + cx + dy) = ye - xf$

Examinăm prima ecuație determinantă

$x: -e = -d \quad y: -f = c$

Examinăm a doua ecuație determinantă

$x: -c = -f \quad y: -d = e$

Obținem

$a = 0$; $b = 0$; $c = f$; $d = e$

$$\begin{cases} \dot{x} = fx + ey \\ \dot{y} = dx + cy \end{cases}$$

b) $X = xy\frac{\partial}{\partial y}$

$$\xi^1_x P + \xi^1_y Q = \xi^1 P_x + \xi^2 P_y + D(P)$$

$$\xi^2_x P + \xi^2_y Q = \xi^1 Q_x + \xi^2 Q_y + D(Q)$$

$$D = 0$$

$\xi^1 = 0;\ \xi^2 = xy;$

$P_x = c\ ;\ P_y = d$

$Q_x = e\ ;\ Q_y = f$

Scriem ecuațiile determinante:

$0\cdot(a + cx + dy) + 0\cdot(b + ex + fy) = 0\cdot c + xy\cdot d \Rightarrow xyd = 0$

$y(a + cx + dy) + x(b + ex + fy) = 0\cdot e + xy\cdot f.$

Examinăm prima ecuație determinantă

$xy : d = 0$

Examinăm a doua ecuație determinantă

$x : b = 0\ ;\ xy : c + f = f\ ;\ x^2 : e = 0\ ;\ y : a = 0\ ;\ y^2 : d = 0$

Obținem:

$a = 0;\ b = 0;\ d = 0;\ e = 0;\ f = f;\ c = 0$

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = fy \end{cases}$$

4. Să se scrie ecuațiile de structură a algebrei Lie $h_3 = \{X_1, X_2, X_3\}$ dacă constantele de structură sunt $C_{12}^{1} = C_{23}^{3} = 2$, $C_{13}^{2} = 1$.

$$[X_1, X_2] = C_{12}^{1} X_1 + C_{12}^{2} X_2 + C_{12}^{3} X_3 = 2X_1.$$

$$[X_1, X_3] = C_{13}^{1} X_1 + C_{13}^{2} X_2 + C_{13}^{3} X_3 = X_2.$$

$$[X_2, X_3] = C_{23}^{1} X_1 + C_{23}^{2} X_2 + C_{23}^{3} X_3 = 2X_3.$$

6. Să se scrie ecuațiile și constantele de structură precum și forma lui Killing pentru algebra Lie h_3 cu baza

$$X_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad X_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Rezolvare:

1. $[X_1, X_2] = [X_1, X_2] - [X_2, X_1]$

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -X_3$$

2. $[X_1, X_3] = [X_1, X_3] - [X_3, X_1]$

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

$$= 2X_1.$$

3. $[X_2, X_3] = [X_2, X_3] - [X_3, X_2]$

$$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$$

$$= -2X_2$$

După ce am scris ecuațiile de structură, aflăm și constantele de structură

- $[X_1, X_2] = C_{12}^1 X_1 + C_{12}^2 X_2 + C_{12}^3 X_3 = -X_3 \Rightarrow \boxed{C_{12}^3 = -1.}$
- $[X_1, X_3] = C_{13}^1 X_1 + C_{13}^2 X_2 + C_{13}^3 X_3 = 2X_1 \Rightarrow \boxed{C_{13}^1 = 2.}$
- $[X_2, X_3] = C_{23}^1 X_1 + C_{23}^2 X_2 + C_{23}^3 X_3 = -2X_2 \Rightarrow \boxed{C_{23}^2 = -2}$

$$C_{12}^3 = -1, \quad C_{13}^1 = 2, \quad C_{23}^2 = -2.$$

$a_{11} = 2x^3$; $a_{12} = 0$; $a_{13} = -2x^1$

$a_{21} = 0$; $a_{22} = -2x^3$; $a_{23} = 2x^2$

$a_{31} = 0$; $a_{32} = x^1$; $a_{33} = 0$

$$ad(X) = \begin{pmatrix} 2x^3 & 0 & -2x^1 \\ 0 & -2x^3 & 2x^2 \\ 0 & x^1 & 0 \end{pmatrix}$$

$$ad(y) = \begin{pmatrix} 2y^3 & 0 & -2y^1 \\ 0 & -2y^3 & 2y^2 \\ 0 & y^1 & 0 \end{pmatrix}$$

Cu ajutorul ultimelor matrici obținem matricea produs cu elementele de pe diagonala principală în forma:

$$\left((ad(x))(ad(y))\right) = \begin{pmatrix} 4x^3y^3 & -2x^1y^1 & -4y^1x^3 \\ 0 & 2y^1x^2+4x^3y^3 & -4x^3y^2 \\ 0 & -2x^1y^3 & 2x^1y^2 \end{pmatrix}$$

De aici, utilizând formula $K\langle x,y\rangle = tr\left((ad(x))(ad(y))\right)$ găsim

$$K\langle X, Y\rangle = 4x^3y^3 + 2y^1x^2 + 4x^3y^3 + 2x^1y^2.$$

Sau în forma desfășurată se va scrie:

$$K\langle X, Y\rangle = 8x^3y^3 + 2x^2y^1 + 2x^1y^2$$

De aici, luând în considerație (16.3), avem:

$K_{11} = 0 \quad K_{12} = 2 \quad K_{13} = 0$

$K_{21} = 2 \quad K_{22} = 0 \quad K_{23} = 0$

$K_{31} = 0 \quad K_{32} = 0 \quad K_{33} = 8$

Obținem:

$$\det(K_{\alpha,\beta}) = \det\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} = 0.$$

Conform definiției 16.2 am demonstrat că are loc Forma lui Killing K pentru algebra Lie L_3 este degenerată, adică $\det(K_{\alpha,\beta}) = 0$.