$$f(x,y,z) = x^3 + y^3 + z^3,$$
$$x^2 + y^2 + z^2 = 1.$$

1) $D(f) = \{(x,y,z) \in R^3\}$

2) $L(x,y,z) = f(x,y,z) + \lambda \cdot \varphi(x,y,z),$

$L(x,y,z) = x^3 + y^3 + z^3 + \lambda \cdot (x^2 + y^2 + z^2 - 1)$

3)

$$\frac{\partial L}{\partial x} = 3x^2 + \lambda(2x + 0) = 3x^2 + 2\lambda x$$
$$\frac{\partial L}{\partial y} = 3y^2 + \lambda(0 + 2y + 0) = 3y^2 + 2\lambda y$$
$$\frac{\partial L}{\partial z} = 3z^2 + \lambda(0 + 2z) = 3z^2 + 2\lambda z.$$

4)

$$\begin{cases} 3x^2 + 2\lambda x = 0 \\ 3y^2 + 2\lambda y = 0 \\ 3z^2 + 2\lambda z = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases} \qquad \begin{cases} x(3x + 2\lambda) = 0 \\ y(3y + 2\lambda) = 0 \\ z(3z + 2\lambda) = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

$$\begin{cases} x(3x + 2\lambda) = 0 \\ y(3y + 2\lambda) = 0 \\ z(3z + 2\lambda) = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

a) $\begin{cases} x = 0 \\ y = 0 \\ 3z + 2\lambda = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases}$ $\begin{cases} x = 0 \\ y = 0 \\ 3z + 2\lambda = 0 \\ z^2 = 1 \end{cases}$

$\begin{cases} x = 0 \\ y = 0 \\ \lambda = -3/2 \\ z = 1 \end{cases}$ sau $\begin{cases} x = 0 \\ y = 0 \\ \lambda = 3/2 \\ z = -1 \end{cases}$;

b) $\begin{cases} x = 0 \\ z = 0 \\ 3y + 2\lambda = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases}$ $\begin{cases} x = 0 \\ z = 0 \\ 3y + 2\lambda = 0 \\ y^2 = 1 \end{cases}$

$\begin{cases} x = 0 \\ z = 0 \\ \lambda = -3/2 \\ y = 1 \end{cases}$ sau $\begin{cases} x = 0 \\ z = 0 \\ \lambda = 3/2 \\ y = -1 \end{cases}$;

c) $\begin{cases} y = 0 \\ z = 0 \\ 3x + 2\lambda = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases}$ $\begin{cases} y = 0 \\ z = 0 \\ 3x + 2\lambda = 0 \\ x^2 = 1 \end{cases}$

$\begin{cases} x = 1 \\ z = 0 \\ \lambda = -3/2 \\ y = 0 \end{cases}$ sau $\begin{cases} x = -1 \\ z = 0 \\ \lambda = 3/2 \\ y = 0 \end{cases}$;

$$\begin{cases} x(3x+2\lambda)=0 \\ y(3y+2\lambda)=0 \\ z(3z+2\lambda)=0 \\ x^2+y^2+z^2=1 \end{cases}$$

d)

$$\begin{cases} x=0 \\ 3y+2\lambda=0 \\ 3z+2\lambda=0 \\ y^2+z^2=1 \end{cases} \quad \begin{cases} x=0 \\ y=-2\lambda/3 \\ z=-2\lambda/3 \\ y^2+z^2=1 \end{cases} \quad \begin{cases} x=0 \\ y=-2\lambda/3 \\ z=-2\lambda/3 \\ 4\lambda^2/9+4\lambda^2/9=1 \end{cases}$$

$$\begin{cases} x=0 \\ y=-2\lambda/3 \\ z=-2\lambda/3 \\ 8\lambda^2/9=1 \end{cases} \quad \begin{cases} x=0 \\ y=-2\lambda/3 \\ z=-2\lambda/3 \\ \lambda^2=9/8 \end{cases} \quad \begin{cases} x=0 \\ y=-2\lambda/3 \\ z=-2\lambda/3 \\ \lambda=\pm 3/2\sqrt{2} \end{cases}$$

$$\begin{cases} x=0 \\ y=-\frac{\sqrt{2}}{2} \\ z=-\frac{\sqrt{2}}{2} \\ \lambda=\frac{3}{2\sqrt{2}} \end{cases} \text{ sau } \begin{cases} x=0 \\ y=\frac{\sqrt{2}}{2} \\ z=\frac{\sqrt{2}}{2} \\ \lambda=-\frac{3}{2\sqrt{2}} \end{cases}$$

e)

$$\begin{cases} y = 0 \\ x = -\frac{\sqrt{2}}{2} \\ z = -\frac{\sqrt{2}}{2} \\ \lambda = \frac{3}{2\sqrt{2}} \end{cases} \text{ sau } \begin{cases} y = 0 \\ x = \frac{\sqrt{2}}{2} \\ z = \frac{\sqrt{2}}{2} \\ \lambda = -\frac{3}{2\sqrt{2}} \end{cases}$$

f)

$$\begin{cases} z = 0 \\ x = -\frac{\sqrt{2}}{2} \\ y = -\frac{\sqrt{2}}{2} \\ \lambda = \frac{3}{2\sqrt{2}} \end{cases} \text{ sau } \begin{cases} z = 0 \\ x = \frac{\sqrt{2}}{2} \\ y = \frac{\sqrt{2}}{2} \\ \lambda = -\frac{3}{2\sqrt{2}} \end{cases}$$

$$\begin{cases} x(3x + 2\lambda) = 0 \\ y(3y + 2\lambda) = 0 \\ z(3z + 2\lambda) = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

g)

$$\begin{cases} 3x + 2\lambda = 0 \\ 3y + 2\lambda = 0 \\ 3z + 2\lambda = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases} \qquad \begin{cases} x = -2\lambda/3 \\ y = -2\lambda/3 \\ z = -2\lambda/3 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

$\frac{4\lambda^2}{9}+\frac{4\lambda^2}{9}+\frac{4\lambda^2}{9}=1, \frac{12\lambda^2}{9}=1,\ \lambda^2=\frac{9}{12}=\frac{3}{4}$

$$\lambda=\pm\frac{\sqrt{3}}{2}$$

$\begin{cases} x=-\frac{\sqrt{3}}{3} \\ y=-\frac{\sqrt{3}}{3} \\ z=-\frac{\sqrt{3}}{3} \\ \lambda=\frac{\sqrt{3}}{2} \end{cases}\quad \begin{cases} x=\frac{\sqrt{3}}{3} \\ y=\frac{\sqrt{3}}{3} \\ z=\frac{\sqrt{3}}{3} \\ \lambda=-\frac{\sqrt{3}}{2} \end{cases}$

Concluzie: 14 puncte staționare (critice)

$$M_1\left(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right),\ \ \lambda=\frac{\sqrt{3}}{2};$$

$$M_2\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right),\ \ \lambda=-\frac{\sqrt{3}}{2};$$

$$M_3\left(0,-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right),\ \ \lambda=\frac{3}{2\sqrt{2}};$$

$$M_4\left(0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right),\ \ \lambda=-\frac{3}{2\sqrt{2}};$$

$$M_5\left(-\frac{\sqrt{2}}{2},0,-\frac{\sqrt{2}}{2}\right),\ \ \lambda=\frac{3}{2\sqrt{2}};$$

$$M_6\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right), \ \ \lambda = -\frac{3}{2\sqrt{2}};$$

$$M_7\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right), \ \ \lambda = \frac{3}{2\sqrt{2}};$$

$$M_8\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), \ \ \lambda = -\frac{3}{2\sqrt{2}};$$

$$M_9(0,0,1), \ \ \lambda = -\frac{3}{2};$$

$$M_{10}(0,0,-1), \ \ \lambda = \frac{3}{2};$$

$$M_{11}(0,1,0), \ \ \lambda = -\frac{3}{2};$$

$$M_{12}(0,-1,0), \ \ \lambda = \frac{3}{2};$$

$$M_{13}(1,0,0), \ \ \lambda = -\frac{3}{2};$$

$$M_{14}(-1,0,0), \ \ \lambda = \frac{3}{2};$$

5) Pentru a decide natura celor 14 puncte critice, se calculează diferențiala a doua a funcției $L(x, y, z)$:

$$d^2L = \frac{\partial^2 L}{\partial x^2}(dx)^2 + \frac{\partial^2 L}{\partial y^2}(dy)^2 + \frac{\partial^2 L}{\partial z^2}(dz)^2 +$$

$$+2\frac{\partial^2 L}{\partial x \partial y}dxdy + 2\frac{\partial^2 L}{\partial x \partial z}dxdz + 2\frac{\partial^2 L}{\partial y \partial z}dydz.$$

$$\frac{\partial^2 L}{\partial x^2} = L''_{xx} = (3x^2 + 2\lambda x)'_x = 6x + 2\lambda$$

$$\frac{\partial^2 L}{\partial x \partial y} = L''_{xy} = (3x^2 + 2\lambda x)'_y = 0$$

$$\frac{\partial^2 L}{\partial x \partial z} = L''_{xz} = (3x^2 + 2\lambda x)'_z = 0$$

$$\frac{\partial^2 L}{\partial y^2} = L''_{yy} = (3y^2 + 2\lambda y)'_y = 6y + 2\lambda$$

$$\frac{\partial^2 L}{\partial y \partial x} = L''_{yx} = (3y^2 + 2\lambda y)'_x = 0$$

$$\frac{\partial^2 L}{\partial y \partial z} = L''_{yz} = (3y^2 + 2\lambda y)'_z = 0$$

$$\frac{\partial^2 L}{\partial z^2} = L''_{zz} = (3z^2 + 2\lambda z)'_z = 6z + 2\lambda$$

$$\frac{\partial^2 L}{\partial z \partial x} = L''_{zx} = (3z^2 + 2\lambda z)'_x = 0$$

$$\frac{\partial^2 L}{\partial z \partial y} = L''_{zy} = (3z^2 + 2\lambda z)'_y = 0.$$

Avem:

$$d^2L = (6x + 2\lambda)(dx)^2 + (6y + 2\lambda)(dy)^2 + (6z + 2\lambda)(dz)^2 + 2 \cdot 0 + 2 \cdot 0 + 2 \cdot 0.$$

Pe ecuația de legătură $x^2 + y^2 + z^2 = 1$, avem $d\varphi = 0$, adică $2xdx + 2ydy + 2zdz = 0$ sau

$$xdx + ydy + zdz = 0.$$

Rezultă deci că restricția lui d^2L la legătura $\varphi = 0$ este:

$$zdz = -xdx - ydy.$$

$$d^2L = (6x + 2\lambda)(dx)^2 + (6y + 2\lambda)(dy)^2 + \\ + \frac{(6z + 2\lambda)(-xdx - ydy)^2}{z^2}.$$

$$\begin{cases} d^2L = (6x + 2\lambda)(dx)^2 + (6y + 2\lambda)(dy)^2 + \\ \qquad +(6z + 2\lambda)(dz)^2 \\ zdz = -xdx - ydy \end{cases}$$

Aflăm semnul lui d^2L în fiecare punct critic:

$$M_1\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right),\ \ \lambda = \frac{\sqrt{3}}{2};$$

$$\begin{cases} d^2L|_{M_1} = \left(-2\sqrt{3} + \sqrt{3}\right)(dx)^2 + (-2\sqrt{3} + \sqrt{3})(dy)^2 \\ \qquad +\left(-2\sqrt{3} + \sqrt{3}\right)(dz)^2 \\ zdz = -xdx - ydy \end{cases}$$

$$\begin{cases} d^2L|_{M_1} = -\sqrt{3}[(dx)^2 + (dy)^2 + (dz)^2] < 0 \\ \\ zdz = -xdx - ydy \end{cases}$$

Concluzie, în punctul critic M_1 avem maxim local condiționat.

$$M_3\left(0,-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right),\quad \lambda=\frac{3}{2\sqrt{2}};$$

$$\begin{cases} d^2L|_{M_3}=\left(0+\frac{3}{\sqrt{2}}\right)(dx)^2+(-3\sqrt{2}+\frac{3}{\sqrt{2}})(dy)^2 \\ \qquad +\left(-3\sqrt{2}+\frac{3}{\sqrt{2}}\right)(dz)^2 \\ -\frac{\sqrt{2}}{2}dz=0\cdot dx+\frac{\sqrt{2}}{2}dy \end{cases}$$

$$\begin{cases} d^2L|_{M_3}=\frac{3}{\sqrt{2}}(dx)^2-\frac{3}{\sqrt{2}}(dy)^2-\frac{3}{\sqrt{2}}(dz)^2 \\ dz=-dy \end{cases}$$

$$\begin{cases} d^2L|_{M_3}=\frac{3}{\sqrt{2}}[(dx)^2-(dy)^2-(dz)^2] \\ dz=-dy \end{cases}$$

$$\begin{cases} d^2L|_{M_3}=\frac{3}{\sqrt{2}}[(dx)^2-(dy)^2-(-dy)^2] \\ dz=-dy \end{cases}$$

$$d^2L|_{M_3}=\frac{3}{\sqrt{2}}[(dx)^2-2(dy)^2]$$

$$M_{13}(1,0,0),\quad \lambda = -\frac{3}{2};$$

$$\begin{cases} d^2L|_{M_{13}} = (6-3)(dx)^2 + (0-3)(dy)^2 \\ \qquad +(0-3)(dz)^2 \\ 0dz = -1dx - 0dy \end{cases}$$

$$\begin{cases} d^2L|_{M_{13}} = 3(dx)^2 - 3(dy)^2 - 3(dz)^2 \\ \\ dx = 0 \end{cases}$$

$$\begin{cases} d^2L|_{M_{13}} = 3\cdot 0 - 3(dy)^2 - 3(dz)^2 \\ \\ dx = 0 \end{cases}$$

$$d^2L|_{M_{13}} = -3(dy)^2 - 3(dz)^2 = -3[(dy)^2 + (dz)^2]$$

$$d^2L|_{M_{13}} < 0.$$

$$M_{14}(-1,0,0),\quad \lambda = \frac{3}{2}$$

$$\begin{cases} d^2L|_{M_{14}} = (-6+3)(dx)^2 + (0+3)(dy)^2 \\ \qquad +(0+3)(dz)^2 \\ 0dz = 1dx - 0dy \end{cases}$$

$$\begin{cases} d^2L|_{M_{14}} = -3(dx)^2 + 3(dy)^2 + 3(dz)^2 \\ \\ dx = 0 \end{cases}$$

$$d^2L|_{M_{14}} = 3(dy)^2 + 3(dz)^2 = 3[(dy)^2 + (dz)^2] > 0$$