A First Course in Probability and Statistics

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Contents

Chapter 1

About

This is a collection of notes intended for students studying probability and statistics at the advanced undergraduate level at Iowa State University. Specifically, these notes are modeled after course notes for STAT 588, 341, and 342. This site is a work in progress.

Some relevant textbook references include John E. Freund's Mathematical Statistics with Applications by Miller and Miller, although many boks on this material are available.

Chapter 2

Experiments and the role of probability

In this chapter we introduce concepts related to scientific studies, data collection, and random sampling.

2.1 Experiments

There are many settings in which data is collected and studied. In this course we will limit our focus to *random sampling experiments* and *random sampling, randomized intervention experiments* defined below.

Both of these settings start with a *research question* of interest in the the context of a *population* or set of individuals. For example, in a political poll the population is the set of eligible voters and the research question is "which of two candidates has majority support?"

Experiments like polls necessarily only measure an outcome or *response* (like voter preference) for a limited number of individuals from the population. The collection of observed individuals is the *sample*. For many reasons relating to, e.g., cost, time, and access, the size of the sample is small compared to the size of the population. On the other hand, there are rare instances in which a whole population is observed, and we call this a *census*. In a census, all possible information is gathered, whereas in an experiment only a limited subset of information is obtained. We will be concerned only with experiments and how best to use the limited information gathered in order to best answer research questions.

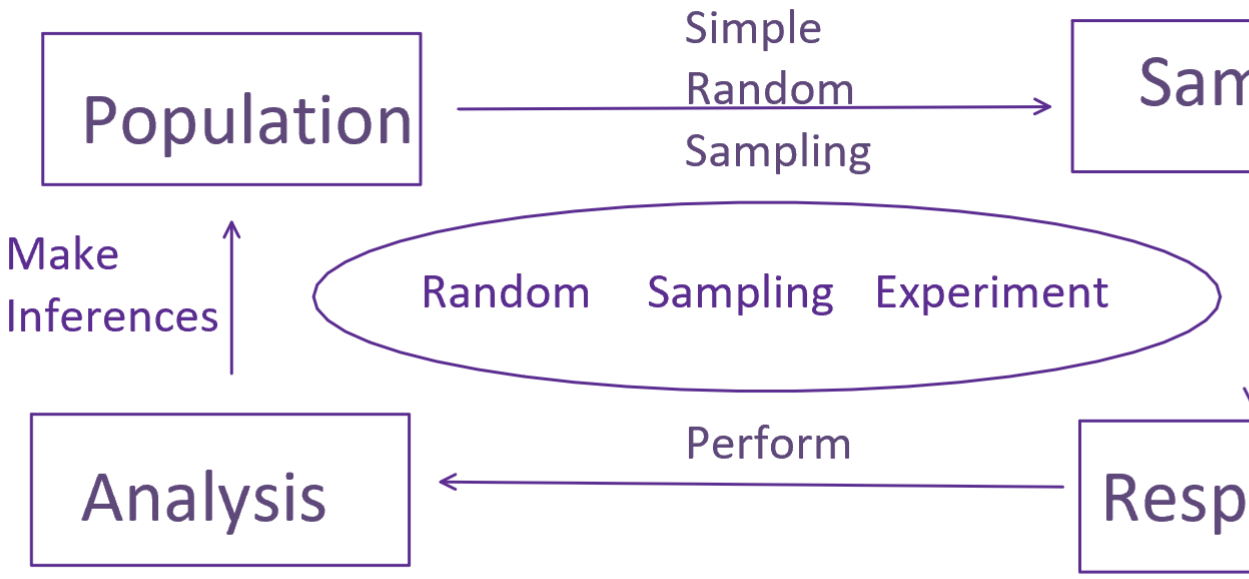
When only a sample of a population is available a natural question is "how is the sample determined?" If there is freedom in the choice of sample then "how

should the sample be chosen?" Intuitively, if we can only observe some of the individuals belonging to the population then we prefer to observe a *representative* sample of them. Representative means that the heterogeneities/differences between individuals that exist in the population ought to be more or less accurately reflected in the sample. For example, a poll of eligible voters in Iowa is not representative if it includes only registered Democrats, and we would not expect such a poll to accurately reflect the population preference between two candidates. If we have a representative sample from the population then we assume the results of our experiment are *generalizable* to the population. In other words, the observations we make are characteristic of the population and we would not expect vastly different observations had we obtained a different, similarly representative sample, or if we had observed the whole population. There are different ways to obtain representative samples. One way is via *stratified sampling*. For example, suppose we knew or could measure the relative population proportions of all factors relevant to voting preference, say, party affiliation, sex, age, income level, education level, and religious affiliation. Then, we could construct a sample of individuals with values of these demographic variables matching the levels present in the population—a representative sample—by randomly selecting the right number of individuals from each of these subgroupings or *strata*. However, this takes a lot of information to pull off. Instead, researchers typically attempt to construct a *simple random sample* of individuals from the population. A simple random sample (SRS) of size n from a finite population means every subset of n individuals is equally likely to be chosen. You can imagine a SRS as pulling numbers out of a hat, so to speak. An SRS of eligible voters could be constructed, for instance, by randomly choosing 100 social security numbers from the list of SSNs of all eligible voters. SRSs are not always trivial to construct—as in the polling example they require a list of population individuals, which may be difficult to obtain.

We are at the point where we can define a *random sampling experiment* as an endeavor that selects a SRS from a population and records a response from sampled individuals for the purpose of answering a research question. As noted, an important example is a poll.

Random sampling experiments are valuable, but limited to addressing questions about one variable at a time. In many cases researchers are interested in how one variable affects the value of another, and these questions can often be answered using interventional experiments. An *intervention* is an act by a researcher intended to affect the value of the response variable in sampled individuals. When an intervention is applied to samples the samples are often referred to as *experimental units*. An important example of an interventional experiment is a clinical trial. In a basic clinical trial a random sample of patients is recruited from a population of patients with a certain medical condition. Then, a subset of the patients is given a treatment of interest — the intervention — while the other patients are given a different treatment or perhaps no treatment at all (maybe a placebo). The response, probably related to the health of patients

post-treatment, is then compared between the intervention and non-intervention groups. This experiment is like a random sampling experiment with an added intervention step. The key question is "how do we determine which experimental units receive the intervention?" Recall that when we obtain a sample of individuals from a population we wish to do so in such a way as to obtain a representative sample. The same idea applies to interventions. The group of samples receiving the intervention should be similar to the group not receiving the intervention; that is, both groups should be representative of the total set of sampled individuals. Therefore, it makes sense to *randomize* the application of the intervention over sampled individuals.



A *random sampling, randomized intervention experiment* consists of obtaining a SRS from a population, randomizing an intervention over those samples, and recording a response variable relating to a research question. This type of experiment is often explained by conceptualizing two (or more) different populations. Suppose the experiment is a clinical trial and the intervention consists of receiving a treatment versus no treatment. Then, we can interpret the randomization step as defining two SRSs from two (abstract or hypothetical) populations: a sample from the set of treated patients and a sample from

the set of untreated patients. The research question typically concerns the relationship between these two populations, such as, "are treated patients, on average, healthier than untreated patients, with respect to the response variable?"

We will only consider randomized interventional experiments. Briefly, consider what can go wrong if we do not randomize intervention. Suppose the patients in our clinical trial consist of both old and young people and that old people are generally more in need of treatment. If we give the treatment to only young(old) people, then we will underestimate(overestimate) the effect of the treatment compared to what its effect would be, on average, over the whole population. In effect, we have *confounded* the treatment/intervention effect with the effect of age on the response. This may seem like a silly example because we could easily avoid assigning the intervention to only young or only old people. But, not all potential confounding variables are obvious/visible. Unknown confounders, also called *lurking variables*, can only be systematically accounted for via randomizing the intervention. Of course it is possible for randomized groups not to be representative in a particular occurrence, but, systematically, randomized groups will tend to be representative, so it's a good practice.

When the intervention is randomized we cautiously assume substantial differences in the observed responses between intervention groups can be attributed to the intervention. In other words, we believe in *causation*—that the intervention caused the observed differences. When confounding variables are present we never know which variable is responsible for observed differences in responses and we cannot support claims about cause and effect relationships using the data alone.

In some studies researchers analyze the relationships between variables without performing randomized interventions; these are usually *observational studies*. For instance, in the Framingham Heart Study, researchers recorded the health and lifestyle choices of Massachusetts residents over several years. By collecting a large amount of data they were able to establish a strong relationship or *association* between cigarette smoking and cancer. Their data alone is not enough to imply causation, but their study inspired many follow-up experiments like lab experiments on animals, and the development of biochemical theories about tumor development. The combination of these various works leaves little doubt today that tobacco use dramatically increases the likelihood of developing cancer.

2.2 The role of probability

Consider the example of a political poll on the preferences of voters between two candidates—this is a random sampling experiment. The population consists of all eligible voters in the upcoming election. Each voter can be associated, say,

with a 1 or a 0, indicating their preference between the two candidates. Let these 0-1 preferences be denoted $x_1, \ldots, x_N$ where N is the population size. There exists a population level value θ equal to

$$\theta := N^{-1} \sum_{j=1}^{N} x_j$$

denoting the population voter preference. Most research questions of interest concern the unknown value of θ. In the polling experiment we observe a random subset of $x_1, \ldots, x_N$ values, say, $X_1, \ldots, X_n$ for $n < N$—keep in mind these are not the first n x's, but a random subset of n x's. Correspondingly, we can compute the sample voter preference

$$\hat{\theta}_n := n^{-1} \sum_{i=1}^{n} X_i.$$

For every possible sample of n voters, there is a corresponding value of $\hat{\theta}_n$. We can think of the polling experiment as randomly choosing one of these $\hat{\theta}_n$ values out of a hat containing all of them. Randomly sampling voters causes randomness in the observed $\hat{\theta}_n$ preference value. And, the goal of *probability* is to quantify this randomness, e.g., to be able to compute the chance of $\hat{\theta}_n$ taking on any particular value, given knowledge of the population.

That last phrase "given knowledge of the population" is very important, and illustrates the difference between probability and statistics (inference) quite succintly. Probability characterizes the chance of observing a particular random sample given the population, whereas statistics seeks to explain some characteristic of the unknown/unobserved population given only one sample (subset of population individuals).

Probability plays a key role in statistics problems, and the first part of our course is devoted to developing methods for computing probabilities of samples in a variety of useful special cases.

2.3 Exercises

1. Google James Lind's Scurvy experiments. What was James' research question? What was the population? Did he obtain a random sample from the population? If not, does that make you suspicious of his findings? Why or why not? Did James use any interventions? If so did he randomize them? Do you think his randomization scheme is reliable? What were his conclusions?

2. Find a recent example of an experiment in the news or a scientific publication. Describe the research question, population, intervention (if there

Table 2.1: A table of o rings at risk, o ring failires, and launch temperatures of space shuttle flights.

at risk	failed	launch temp.
6	1	70
6	0	69
6	0	68
6	0	67
6	0	72
6	0	73
6	0	70
6	1	57
6	1	63
6	1	70
6	0	78
6	0	67
6	2	53
6	0	67
6	0	75
6	0	70
6	0	81
6	0	76
6	0	79
6	0	75
6	0	76
6	1	58

is one), and response. Is it a random sampling experiment, a random sample randomized intervention experiment, or something else, like an observational study?

3. The Challenger space shuttle exploded when it experienced o-ring failures thought to be caused by low launch temperature. The launch temperature was 31 degrees Fahrenheit. The following data can be interpreted as a random sample of counts of o-ring failures from a population of launches. Given this data do you think we can make reliable conclusions about launch safety at 31 degrees launch temperature? What concepts discussed in this section are relevant here?

Chapter 3

Probability and Counting

Probability is a tricky subject. We have an intuitive sense about probability, but we use it in different ways, which can sometimes lead to confusion. Probability is used in at least two ways: -to describe the relative frequency of events, e.g., what is the chance of observing 5 heads in the next ten flips of a fair coin; and, -to communicate degrees of belief, e.g., the Packers have a 30% chance of winning the Super Bowl.

We will use probability exclusively in the sense of the first interpretation—to characterize the chances of different outcomes in repeatable trials. This sense of probability corresponds to characterizing the possible outcomes of random sampling. Although probability is often used to communicate degrees of belief there are good reasons not to use it for this purpose, but a formal, nuanced discussion of quantification of beliefs is outside our present purview.

3.1 Terminology

In our discussion of probability we will think of an *experiment* as the act of measuring/observing a variable one one or more samples from a population. The *sample space* is the set of possible realizations of the experiment. For example, if the experiment is to flip one coin and record whether it is heads or tails, then we can think of this as a random sampling from sample space $\{H, T\}$ where the outcome may be either $\{H\}$ or $\{T\}$. Any subset of the sample space of an experiment is an *event*; for example, $\{H\}$ and $\{H, T\}$ are events, and so is $\emptyset$ which denotes the "empty set", the set of nothing.

3.2 Set relations

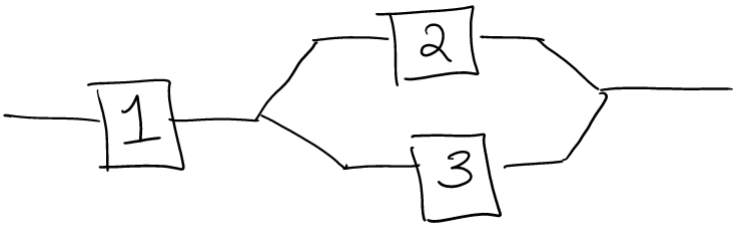
Events and sample spaces are sets, and we will make use of relations between sets. -Set union: for sets/events A and B, $A \cup B$ denotes the set of elements in at least one of A or B. For example, $\{1,2,3\} \cup \{3,4,5\} = \{1,2,3,4,5\}$. -Set intersection: $A \cap B$ denotes the set of elements in both A and B. For example, $\{1,2,3\} \cap \{3,4,5\} = \{3\}$. -Set complement: A^c denotes the set of elements in the sample space $\mathcal{S}$ but not in A. For example, if $\mathcal{S} = \{1,2,3,4,5\}$ then $A^c = \{4,5\}$. -Set subtraction: $A \backslash B$ of $A - B$ means $A \cap B^c$ which is the set of elements in A but not in B. For example, $\{1,2,3\} - \{3,4,5\} = \{1,2\}$.

3.2.1 Sample space example

Here's an example to illustrate sample spaces. A gas station has six pumps, A, B, C, D, E, F. -What is the sample space of the number of pumps in use? $\mathcal{S} = \{0,1,2,3,4,5,6\}$. -What is the sample space of pumps in use? $\mathcal{S} = \{\{A,B,C,D,E,F\},\{A,B,C,D,E\},\ldots,\{F\},\emptyset\}$. This is the *power set* of $\{A,B,C,D,E,F\}$, the set of all subsets of those pumps. -Suppose you test pump A every day until it fails to function. What is the sample space of this experiment? $\mathcal{S} = \{F, SF, SSF, \ldots\}$. This is a *countably infinite* sample space. -You measure the amount of gas pumped by the next customer. $\mathcal{S} = (0, ?)$, an interval with some upper bound equal to however much gas the station has available. This is an *uncountably infinite* sample space.

3.2.2 Set relations example

This next example illustrates set relations.



Consider this system of series and parallel components. Each component either functions/succeeds (S) or fails (F). The experiment simply observes if the system functions/succeeds or fails. -What i the sample space in terms of the three components? $\mathcal{S} = \{SSS, SSF, SFS, FSS, SFF, FFS, FSF, FFF\}$. -Find the even two components succeed. $A = \{SSF, SFS, FSS\}$. -Find the even at leas two components succeed. $B = \{SSF, SFS, FSS, SSS\} = A \cup \{SSS\}$. -Find the event the system functions. $C = \{SSS, SFS, SSF\}$.

3.3 Probability Axioms

Andrey Kolmogorov formalized rules/axioms of probability we are mostly, intuitively familiar with.

1. The probability of the sample space is 1, $P(\mathcal{S}) = 1$.
2. Probabilities are non-negative. If $A \subset \mathcal{S}$ then $P(A) \geq 0$.
3. Countable additivity. This last one is a bit tricky. Let's start with an intuitive, simpler case. Suppose $\mathcal{S} = \{1, 2, 3, 4, 5\}$. The important part is $\mathcal{S}$ is finite and includes only different things that don't "overlap". Then, we know that, for example, $P(\{1\} \cup \{2\}) = P(\{1\}) + P(\{2\})$, which says that the probability of the union of *disjoint* events equals the sum of probabilities of each event. Kolmogorov requires this extends to countably infinite $\mathcal{S}$, hence the name countable additivity. Specifically, let events A_1, A_2, ... be a sequence of mutually disjoint events (none overlap, like pizza slices) so that $A_i \cap A_j = \emptyset$ for every $i \neq j$. Then,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Chapter 4

Parts

You can add parts to organize one or more book chapters together. Parts can be inserted at the top of an .Rmd file, before the first-level chapter heading in that same file.

Add a numbered part: `# (PART) Act one {-}` (followed by `# A chapter`)

Add an unnumbered part: `# (PART\*) Act one {-}` (followed by `# A chapter`)

Add an appendix as a special kind of un-numbered part: `# (APPENDIX) Other stuff {-}` (followed by `# A chapter`). Chapters in an appendix are prepended with letters instead of numbers.

Chapter 5

Footnotes and citations

5.1 Footnotes

Footnotes are put inside the square brackets after a caret `^[]`. Like this one [1].

5.2 Citations

Reference items in your bibliography file(s) using `@key`.

For example, we are using the **bookdown** package [Xie, 2021] (check out the last code chunk in index.Rmd to see how this citation key was added) in this sample book, which was built on top of R Markdown and **knitr** [Xie, 2015] (this citation was added manually in an external file book.bib). Note that the `.bib` files need to be listed in the index.Rmd with the YAML `bibliography` key.

The RStudio Visual Markdown Editor can also make it easier to insert citations: https://rstudio.github.io/visual-markdown-editing/#/citations

[1] This is a footnote.

Chapter 6

Blocks

6.1 Equations

Here is an equation.

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{6.1}$$

You may refer to using `\@ref(eq:binom)`, like see Equation (6.1).

6.2 Theorems and proofs

Labeled theorems can be referenced in text using `\@ref(thm:tri)`, for example, check out this smart theorem **??**.

::: {.theorem #tri} For a right triangle, if c denotes the *length* of the hypotenuse and a and b denote the lengths of the **other** two sides, we have

$$a^2 + b^2 = c^2$$

:::

Read more here https://bookdown.org/yihui/bookdown/markdown-extensions-by-bookdown.html.

6.3 Callout blocks

The R Markdown Cookbook provides more help on how to use custom blocks to design your own callouts: https://bookdown.org/yihui/rmarkdown-cookbook/custom-blocks.html

Chapter 7

Sharing your book

7.1 Publishing

HTML books can be published online, see: https://bookdown.org/yihui/bookdown/publishing.html

7.2 404 pages

By default, users will be directed to a 404 page if they try to access a webpage that cannot be found. If you'd like to customize your 404 page instead of using the default, you may add either a `_404.Rmd` or `_404.md` file to your project root and use code and/or Markdown syntax.

7.3 Metadata for sharing

Bookdown HTML books will provide HTML metadata for social sharing on platforms like Twitter, Facebook, and LinkedIn, using information you provide in the `index.Rmd` YAML. To setup, set the `url` for your book and the path to your `cover-image` file. Your book's `title` and `description` are also used.

This `gitbook` uses the same social sharing data across all chapters in your book- all links shared will look the same.

Specify your book's source repository on GitHub using the `edit` key under the configuration options in the `_output.yml` file, which allows users to suggest an edit by linking to a chapter's source file.

Read more about the features of this output format here:

https://pkgs.rstudio.com/bookdown/reference/gitbook.html

Or use:

```
?bookdown::gitbook
```

Bibliography

Yihui Xie. *Dynamic Documents with R and knitr*. Chapman and Hall/CRC, Boca Raton, Florida, 2nd edition, 2015. URL http://yihui.org/knitr/. ISBN 978-1498716963.

Yihui Xie. *bookdown: Authoring Books and Technical Documents with R Markdown*, 2021. URL https://CRAN.R-project.org/package=bookdown. R package version 0.24.