Calculate exact formula for $\int_\Omega J_3 v_{\mathrm{CR}}$

Definitions:

General definition from dissertation

$(J_1 v_{\mathrm{CR}})(z) := |\mathcal{T}(z)|^{-1} \sum_{T\in\mathcal{T}(z)} v_{\mathrm{CR}}|_T\ (z)$ and linear interpolation

$\flat_F|_T := \frac{(2n-1)!}{(n-1)!} \prod_{P\in\mathcal{V}(F)} \lambda_P$

$J_2 v_{\mathrm{CR}} := J_1 v_{\mathrm{CR}} + \sum_{F\in\mathcal{F}(\Omega)} \left(⨍_F (v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\,\mathrm{d}s\right)\flat_F$

$\Phi_{T,P} := \alpha - \beta\lambda_P$

$\flat_T := (n+1)^{n+1} \prod_{P\in\mathcal{V}(T)} \lambda_P$

$J_3 v_{\mathrm{CR}} := J_2 v_{\mathrm{CR}} + \sum_{T\in\mathcal{T}} \sum(P\in\mathcal{V}(T))\left(\int_T (v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\Phi_{T,P}\,\mathrm{d}x\right)\Phi_{T,P}\flat_T$

With n=2 and restricted to a triangle

On a given triangle T with barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$ and given the averaging coefficients $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, the bubble coefficients $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and the volume coefficients (c_1, c_2, c_3) the following functions are defined:

$$J_1 v_{\mathrm{CR}} := a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 \in P_1$$

$$\flat_F := 6 \prod_{P\in\mathcal{V}(F)} \lambda_P \in P_2$$

$$J_2 v_{\mathrm{CR}} := J_1 v_{\mathrm{CR}} + 6b_1\lambda_2\lambda_3 + 6b_2\lambda_1\lambda_3 + 6b_3\lambda_1\lambda_2 \in P_2$$

$$\Phi_P := \frac{1}{|T|}(\alpha - \beta\lambda_P) \in P_1$$

$$J_3 v_{\mathrm{CR}} := J_2 v_{\mathrm{CR}} + 27\lambda_1\lambda_2\lambda_3(c_1\Phi_1 + c_2\Phi_2 + c_3\Phi_3) \in P_4$$

Calculate $J_3 v_{\mathrm{CR}} = J_3 v_{\mathrm{CR}}(\lambda_1, \lambda_2, \lambda_3)$:

$J_3 v_{\mathrm{CR}} = J_2 v_{\mathrm{CR}} + 27\lambda_1\lambda_2\lambda_3(c_1\Phi_1 + c_2\Phi_2 + c_3\Phi_3)$

$= J_1 v_{\mathrm{CR}} + 6b_1\lambda_2\lambda_3 + 6b_2\lambda_1\lambda_3 + 6b_3\lambda_1\lambda_2 + \frac{27}{|T|}\lambda_1\lambda_2\lambda_3[c_1(\alpha - \beta\lambda_1) + c_2(\alpha - \beta\lambda_2) + c_3(\alpha - \beta\lambda_3)]$

$= a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + 6(b_1\lambda_2\lambda_3 + b_2\lambda_1\lambda_3 + b_3\lambda_1\lambda_2) + \frac{27}{|T|}\lambda_1\lambda_2\lambda_3[\alpha(c_1 + c_2 + c_3) - \beta(c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3)]$

Integrate $J_3 v_{\mathrm{CR}}$:

Compute integrating coefficients:

using $\int_T \lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma = 2|T| \frac{\alpha!\beta!\gamma!}{(2+\alpha+\beta+\gamma)!}$ we get the following coefficients:

$\int_T 1 = |T|$

$\int_T \lambda_j = \frac{1}{3}|T|$

$\int_T \lambda_i \lambda_j = \frac{1}{12}|T|$

$\int_T \lambda_i^2 = \frac{1}{6}|T|$

$\int_T \lambda_i \lambda_j \lambda_k = \frac{1}{60}|T|$

$\int_T \lambda_i^2 \lambda_j = \frac{1}{30}|T|$

$\int_T \lambda_i^3 = \frac{1}{10}|T|$

$\int_T \lambda_i^2 \lambda_j \lambda_k = \frac{1}{180}|T|$

Exact integration on one triangle T:

$$\begin{aligned}
\int_T J_3 v_{\mathrm{CR}} &= \int_T a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + 6(b_1\lambda_2\lambda_3 + b_2\lambda_1\lambda_3 + b_3\lambda_1\lambda_2) + \frac{27}{|T|}\lambda_1\lambda_2\lambda_3[\alpha(c_1+c_2+c_3) - \beta(c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3)] \\
&= \frac{1}{3}|T|(a_1+a_2+a_3) + \frac{6}{12}|T|(b_1+b_2+b_3) + \frac{27}{60}\alpha(c_1+c_2+c_3) - \frac{27}{180}\beta(c_1+c_2+c_3) \\
&= |T|\left(\frac{1}{3}a_{\mathrm{sum}} + \frac{1}{2}b_{\mathrm{sum}}\right) + \frac{3}{20}(3\alpha - \beta)c_{\mathrm{sum}}
\end{aligned}$$

Exact integral on Ω:

$\int_\Omega J_3 v_{\mathrm{CR}} = \sum_{T\in\mathcal{T}} \int_T J_3 v_{\mathrm{CR}}$