# An Introduction to Minimum Principle

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Consider an optimization problem with constraint $g(x) = \mathbf{0}$, where $g : \mathbb{R}^m \to \mathbb{R}^d$:

$$\min_x V(x)$$
$$\text{s.t. } g(x) = \mathbf{0}$$

In constrained optimization in $\mathbb{R}^m$, we use Lagrange multipliers:

$$\hat{V}(x, p) = V(x) + p^T g(x)$$

where $p$ is a Lagrange multiplier. If $x^0, p^0$ is a stationary point, then:

$$\nabla_x \hat{V}(x^0, p^0) = \nabla_x V(x^0) + \nabla_x g(x^0) p^0 = \mathbf{0}$$
$$\nabla_p \hat{V}(x^0, p^0) = g(x^0) = \mathbf{0}$$

# Lagrange Multipliers in Finite Dimensions

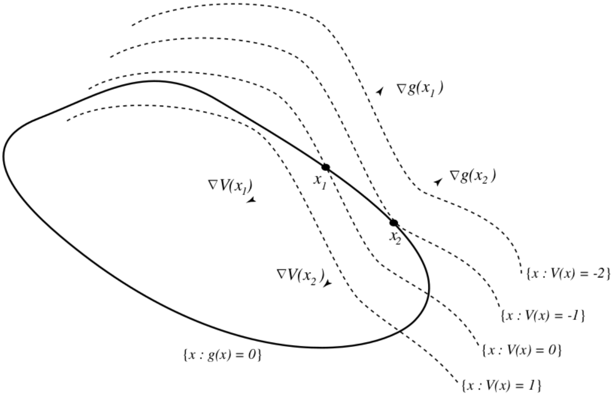


**Figure 1:** An optimization problem in $\mathbb{R}^2$ with a single constraint, $g(x) = 0$.

$$\nabla_x \hat{V}(x^0, p^0) = \nabla_x V(x^0) + \nabla_x g(x^0) p^0 = \mathbf{0}$$
$$\nabla_p \hat{V}(x^0, p^0) = g(x^0) = \mathbf{0}$$

# Generalizing to Infinite Dimensions

To generalize this to functional minimization, suppose that $F$ is a functional on $D^r[t_0, t_1]$, $F(z)$ is a real number, and $z \in D^r[t_0, t_1]$. We define the directional derivative:

$$D_\eta F(z) = \lim_{\epsilon \to 0} \frac{F(z + \epsilon\eta) - F(z)}{\epsilon}$$

A function $z_0$ is a stationary point of $F$ if $D_\eta F(z_0) = 0$ for any $\eta$.

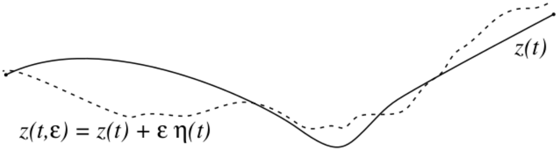


**Figure 2:** A Perturbation of function $z \in D[t_0, t_1]$

# Steps to Extend the Lagrange Multiplier Method

Let us consider a constrained functional optimization problem:

$$\min_u V(x,u) = \int \ell d\tau + m$$

$$\text{s.t. } \dot{x} - f = \mathbf{0}, \quad x \in D^n[t_0,t_1], u \in D^m[t_0,t_1].$$

1. **Append state equations:** Define a new cost functional including the state dynamics:

$$\hat{V}(x,u) = \int_{t_0}^{t_1} \ell \, dt + m(x(t_1)) + \int_{t_0}^{t_1} p^T (f - \dot{x}) \, dt$$

2. **Integration by parts:** Eliminate the derivative of $x$:

$$\int_{t_0}^{t_1} p^T \dot{x} \, dt = p^T x|_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{p}^T x \, dt$$

## Steps to Extend the Lagrange Multiplier Method (cont.)

3. **Define the Hamiltonian:**

$$H(x, p, u, t) := \ell(x, u, t) + p^T f(x, u, t)$$

$$\hat{V}(x, u) = \int_{t_0}^{t_1} H(x, p, u, t)\, dt + \int_{t_0}^{t_1} \dot{p}^\top x dt$$
$$+ p^\top(t_0)x(t_0) - p^\top(t_1)x(t_1) + m(x(t_1))$$

4. **Compute variations:** For perturbations $x(t, \epsilon) = x^\circ(t) + \epsilon\eta(t)$ and $u(t, \delta) = u^\circ(t) + \delta\psi(t)$, let $\hat{V}(\epsilon) = \hat{V}(x^\circ + \epsilon\eta, u^\circ)$, and $\hat{V}(\delta)=\hat{V}(x^\circ, u^\circ + \delta\psi)$.

$$\frac{d}{d\epsilon}\hat{V}(\epsilon) = \int_{t_0}^{t_1} \frac{\partial}{\partial x} H(x^\circ, p, u^\circ, t)\eta(t)\, dt + \int_{t_0}^{t_1} \dot{p}^T \eta(t)\, dt$$
$$- p^T(t_1)\eta(t_1) + p^T(t_0)\eta(t_0) + \frac{\partial}{\partial x} m(x^\circ(t_1))\eta(t_1) = 0.$$

Similarly, by considering perturbations in $u^\circ$:

$$\frac{d}{d\delta}\hat{V}(\delta) = \int_{t_0}^{t_1} \frac{\partial}{\partial u} H(x^\circ, p, u^\circ, t)\psi(t) dt = 0$$

# Steps to Extend the Lagrange Multiplier Method (cont.)

4. **Compute variations:** Determine perturbations in $x^\circ$:

$$\int_{t_0}^{t_1} \frac{\partial}{\partial x} H(x^\circ, p, u^\circ, t)\eta(t)\,dt + \int_{t_0}^{t_1} \dot{p}^T \eta(t)\,dt$$
$$- p^T(t_1)\eta(t_1) + p^T(t_0)\eta(t_0) + \frac{\partial}{\partial x} m(x^\circ(t_1))\eta(t_1) = 0.$$

   Similarly, by considering perturbations in $u^\circ$:

$$\int_{t_0}^{t_1} \frac{\partial}{\partial u} H(x^\circ, p, u^\circ, t)\psi(t)dt = 0$$

5. **Derive necessary conditions:** From the variations, derive the equations:

$$\dot{p} = -\nabla_x H, \quad p(t_1) = \nabla_x m(x(t_1))$$
$$\nabla_u H = \mathbf{0}$$

# Deriving the Minimum Principle

Combine the steps to derive the Minimum Principle. If $u^\circ$ is optimal, there exists a costate $p(t)$ such that:

$$u^\circ(t) \in \arg\min_u H(x^\circ(t), p(t), u, t)$$

The state and costate satisfy:

$$\dot{x}^\circ = \nabla_p H, \quad \dot{p} = -\nabla_x H$$

with boundary conditions:

$$x(t_0) = x_0, \quad p(t_1) = \nabla_x m(x(t_1))$$

# The Penalty Approach

# The Penalty Approach

Another approach to the Minimum Principle involves relaxing the hard constraint $\dot{x} - f = 0$, and instead imposing a large, yet "soft" constraint by definiong the cost function:

$$\hat{V}(x,u) = \int_{t_0}^{t_1} \ell(x(t),u(t),t)\,dt + \frac{k}{2}\int_{t_0}^{t_1} |\dot{x}(t) - f(x(t),u(t),t)|^2\,dt + m(x(t_1))$$

If $(x_k, u_k)$ minimizes $\hat{V}_k$, and letting $(x^\circ, u^\circ)$ denote a solution to the original problem:

$$\hat{V}_k(x_k,u_k) \le \hat{V}_k(x^\circ,u^\circ) = V^\circ$$

Assuming $\ell$ and $m$ are positive, subtracting the left side of the above inequality gives the uniform bound:

$$\int_{t_0}^{t_1} |\dot{x}(t) - f(x(t),u(t),t)|^2\,dt \le \frac{2}{k} V^\circ$$

# Implications for Large $k$

As $k$ becomes large, the term $\frac{k}{2}\int_{t_0}^{t_1} |\dot{x}(t) - f(x(t), u(t), t)|^2\, dt$ ensures that $\dot{x}(t) \approx f(x(t), u(t), t)$:

$$\int_{t_0}^{t_1} |\dot{x}(t) - f(x(t), u(t), t)|^2\, dt \to 0 \text{ as } k \to \infty$$

Thus, for large $k$, the pair $(x_k, u_k)$ will approximately satisfy the differential equation $\dot{x} = f$.

# Perturbation Analysis

If we perturb $x_k$ to form $x_k + \epsilon\eta$ and define $\hat{V}(\epsilon) = \hat{V}(x_k + \epsilon\eta, u_k)$, then we must have:

$$\frac{d}{d\epsilon}\hat{V}(\epsilon)\bigg|_{\epsilon=0} = 0$$

Using the definition of $\hat{V}$, we get:

$$\begin{aligned}\hat{V}(\epsilon) = &\int_{t_0}^{t_1} \ell(x_k(t) + \epsilon\eta(t), u_k(t), t)\, dt + m(x_k(t_1) + \epsilon\eta(t_1)) \\ &+ \frac{k}{2}\int_{t_0}^{t_1} |\dot{x}_k(t) + \epsilon\dot{\eta}(t) - f(x_k(t) + \epsilon\eta(t), u_k(t), t)|^2\, dt\end{aligned}$$

# Computing the Derivative

The derivative of this expression with respect to $\epsilon$ can be computed as follows:

$$\begin{aligned}
\frac{d}{d\epsilon}\hat{V}(0) &= \int_{t_0}^{t_1} \frac{\partial \ell}{\partial x}(x_k(t), u_k(t), t)\eta(t)\, dt \\
&+ k\int_{t_0}^{t_1} (\dot{x}_k(t) - f(x_k(t), u_k(t), t))^T [\dot{\eta}(t) - \frac{\partial f}{\partial x}(x_k(t), u_k(t), t)\eta(t)]\, dt \\
&+ \frac{\partial m}{\partial x}(x_k(t_1))\eta(t_1) \\
&= 0
\end{aligned}$$

# Computing the Derivative (cont.)

To eliminate the derivative term $\dot{\eta}$, we integrate by parts:

$$\int_{t_0}^{t_1} \left\{ \frac{\partial \ell}{\partial x}(x_k(t), u_k(t), t) + p_k(t)^T \frac{\partial f}{\partial x}(x_k(t), u_k(t), t) + \frac{d}{dt}\left(p_k(t)^T\right) \right\} \eta(t)\, dt$$

$$- p_k(t_1)^T \eta(t_1) + \frac{\partial m}{\partial x}(x_k(t_1))\eta(t_1)$$

$$= 0$$

where we have set $p_k(t) = -k(\dot{x}_k(t) - f(x_k(t), u_k(t), t))$.

# Resulting Equations

Since $\eta$ is arbitrary, we have:

$$\begin{aligned}
\mathbf{0} &= \frac{d}{dt}\left(p_k(t)^T\right) + \frac{\partial \ell}{\partial x}(x_k(t), u_k(t), t) + p_k(t)^T \frac{\partial f}{\partial x}(x_k(t), u_k(t), t) \\
\mathbf{0} &= \frac{d}{dt}\left(p_k(t)^T\right) + \frac{\partial H}{\partial x}(x_k(t), p_k(t), u_k(t), t) \\
\dot{p}_k &= -\nabla_x H
\end{aligned}$$

with the boundary condition:

$$p_k(t_1) = \nabla_x m(x_k(t_1))$$

Considering perturbations in $u$ gives the equation:

$$\nabla_u H(x_k(t), p_k(t), u_k(t), t) = \mathbf{0}$$

This is a weak form of the Minimum Principle for the perturbed problem.

# Application to LQR

# Application to LQR

The LQR problem tests the Minimum Principle's utility for constructing optimal policies. Consider the general LTI model with quadratic cost:

$$\dot{x} = Ax + Bu$$

$$V = \int_{t_0}^{t_1} (x^T Q x + u^T R u)\, dt + x^T(t_1) M x(t_1)$$

# Solving the LQR Problem

The Hamiltonian:

$$H = x^T Qx + u^T Ru + p^T (Ax + Bu)$$

Control can be computed through:

$$\nabla_u H = 0 \Rightarrow u = -\frac{1}{2} R^{-1} B^T p$$

Resulting in:

$$\dot{x} = Ax + Bu = Ax - \frac{1}{2} BR^{-1} B^T p$$

Through the expression $\nabla_x H = 2Qx + A^T p$, we find that $\dot{p}$ is:

$$\dot{p} = -\nabla_x H = -2Qx - A^T p$$

# Solving the LQR Problem (cont.)

The equations form the coupled set of differential equations:

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & -\frac{1}{2}BR^{-1}B^T \\ -2Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

with boundary conditions:

$$x(t_0) = x_0$$

$$p(t_1) = \nabla_x m(x_k(t_1)) = 2Mx(t_1)$$

If we scale $p$ by defining $\lambda = \frac{1}{2}p$, we get:

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \mathcal{H} \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

with the optimal control:

$$u^\circ(t) = -R^{-1}B^T\lambda(t)$$

# Solving the ODE

This ODE can be solved using the *sweep method*. Suppose that $\lambda(t) = P(t)x(t)$. Then:

$$\dot{\lambda} = \dot{P}x + P(Ax + Bu)$$

Substituting $u^\circ = -R^{-1}B^\top \lambda = -R^{-1}B^\top Px$ gives:

$$\dot{\lambda} = \dot{P}x + P(Ax - BR^{-1}B^\top Px)$$

From the coupled ODE with $\mathcal{H}$ above, we also have:

$$\dot{\lambda} = -Qx - A^\top \lambda = -Qx - A^\top Px$$

Equating the two expressions for $\dot{\lambda}$ gives the Riccati Differential Equation:

$$\dot{P}x + P(Ax - BR^{-1}B^\top Px) = -Qx - A^\top Px$$

# Boundary Condition

The boundary condition for $\lambda$ is:

$$\lambda = \frac{1}{2}p = \frac{1}{2}\nabla_x m(x(t_1)) \implies P(t_1) = M$$

Solving for $P(t)$ gives $\lambda(t)$, which in turn gives $p(t)$.

# Nonlinear Examples

# Minimum Principle with Input Constraints

**Theorem 11.2**. Suppose that $x(t_1)$ is free, $t_1$ is fixed, and suppose that $u^\circ$ is a solution to the optimal control problem:

$$V^\circ = \min_{u \in \mathcal{U}} V(u)$$

We then have

(a) There exists a costate vector $p(t)$ such that

$$u^\circ(t) = \arg\min_{u \in \mathcal{U}} H(x^\circ(t), p(t), u, t).$$

(b) The pair $(p, x^\circ)$ satisfy the 2-point boundary value problem:

$$\begin{aligned} \dot{x}^\circ(t) &= \nabla_p H(x^\circ(t), p(t), u^\circ(t), t) \quad (= f(x^\circ(t), u^\circ(t), t)) \\ \dot{p}(t) &= -\nabla_x H(x^\circ(t), p(t), u^\circ(t), t), \end{aligned}$$

with the two boundary conditions

$$x(t_0) = x_0; \quad p(t_1) = \nabla_x m(x(t_1), t).$$

# Example 11.5.1: Bilinear System

Consider the control of a bilinear system, defined by the differential equation:

$$\dot{x} = ux, \quad x(0) = x_0 > 0, \quad 0 \leq u(t) \leq 1$$

Suppose the goal is to make $x$ large while keeping the derivative of $x$ small on average. The cost criterion is:

$$V(u) = \int_0^{t_1} \dot{x}(\tau) - x(\tau) d\tau = \int_0^{t_1} [u(\tau) - 1] x(\tau)\, d\tau$$

The Hamiltonian becomes:

$$H(x, p, u, t) = pf + \ell = x\{u(p+1) - 1\}$$

# Example 11.5.1: Bilinear System (cont.)

By the Minimum Principle:

$$H(x, p, u, t) = x\{u(p+1) - 1\}$$
$$\nabla_u H = x(p+1)$$

Since $x^\circ(t) > 0$ for all $t$, the minimization leads to:

$$u^\circ(t) = \begin{cases} 1 & \text{if } p(t) + 1 < 0 \\ 0 & \text{if } p(t) + 1 > 0 \\ \text{unknown} & \text{if } p(t) + 1 = 0 \end{cases}$$

# Example 11.5.1: Bilinear System (cont.)

The costate and state variables satisfy the differential equations:

$$\dot{p} = -\frac{\partial H}{\partial x} = -(p+1)u^\circ + 1$$
$$\dot{x} = \frac{\partial H}{\partial p} = u^\circ x^\circ$$

with boundary conditions:

$$x^\circ(0) = x_0, \quad p(t_1) = \frac{\partial m}{\partial x} = 0$$

# Example 11.5.1: Bilinear System (cont.)

**Solving the Differential Equations**

If $t = t_1$, then $p(t) + 1 = 1 > 0$ so $u^\circ(t) = 0$. By continuity, if $t \approx t_1$, then $p(t) > 0$, so $u^\circ(t) = 0$ and:

$$\dot{p}(t) = 1 \quad \text{for } t \approx t_1 \implies p(t) = t - t_1$$

For $t < t_1 - 1$, we have $p(t) + 1 < 0$ so $u^\circ(t) = 1$. Since $p(t_1 - 1) = -1$, we have:

$$\dot{p} = -p \quad \text{for } t < t_1 - 1 \implies p(t) = -e^{-t+t_1-1}$$

# Example 11.5.1: Bilinear System (cont.)

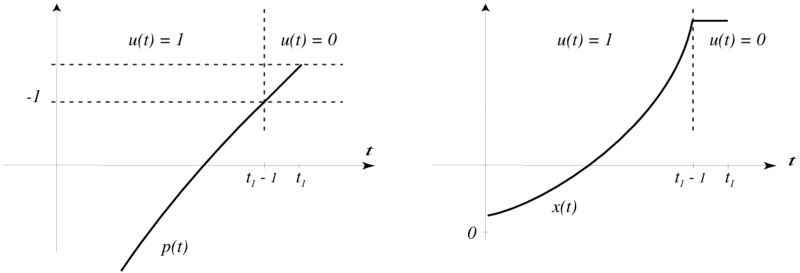


**Figure 3:** The costate trajectory, optimal control, and optimal state trajectory for the bilinear model.

The optimal state trajectory is:

$$x^{\circ}(t) = \begin{cases} x_0 e^t & \text{if } t < t_1 - 1 \\ x_0 e^{t_1 - 1} & \text{if } t > t_1 - 1 \end{cases}$$

# Minimum Principle with Final Value Constraints

**Theorem 11.3**. Suppose that $t_0, t_1, x(t_0) = x_0$, and $x(t_1) = x_1$ are prespecified, and suppose that $u^\circ$ is a solution to the optimal control

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0 \in \mathbb{R}^n$$

$$V(u) = \int_{t_0}^{t_1} \ell(x(t), u(t), t)\, dt + m(x(t_1)),$$

subject to those constraints. Then

(a) There exists a costate vector $p(t)$ such that

$$u^\circ(t) = \arg\min_u H(x^\circ(t), p(t), u, t).$$

(b) The pair $(p, x^\circ)$ satisfy the 2-point boundary value problem

$$\dot{x}^\circ = \nabla_p H, \quad \dot{p} = -\nabla_x H,$$

with the two boundary conditions

$$x(t_0) = x_0; \quad x(t_1) = x_1.$$

# Example 11.5.2: LQR Problem with Terminal State

Consider the LQR problem with the terminal state specified, $x(t_1) = x_1$.
The cost criterion is:

$$V = \frac{1}{2}\int_0^{t_1} (x^T Qx + u^T Ru)\, dt, \quad R \succ 0, Q \succeq 0$$

The Hamiltonian:

$$H = p(Ax + Bu) + \frac{1}{2}(x^T Qx + u^T Ru)$$

The optimal control:

$$u^\circ(t) = -R^{-1}B^T p(t)$$

# Example 11.5.2: LQR Problem with Terminal State (cont.)

The costate vector satisfies the differential equation:

$$\dot{p}(t) = -\frac{\partial H}{\partial x} = -p^T A - x^T Q$$

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

with boundary conditions:

$$x(t_0) = x_0, \quad x(t_1) = x_1$$

# Example 11.5.2: LQR Problem with Terminal State (cont.)

Let $\psi(t,\tau)$ denote the state transition matrix:

$$\frac{d}{dt}\psi(t,\tau) = \mathcal{H}(t)\psi(t,\tau), \quad \psi(t,t) = I$$

Decompose $\psi(t,\tau)$ as:

$$\psi(t,\tau) = \begin{pmatrix} \psi_{11}(t,\tau) & \psi_{12}(t,\tau) \\ \psi_{21}(t,\tau) & \psi_{22}(t,\tau) \end{pmatrix}$$

# Example 11.5.2: LQR Problem with Terminal State (cont.)

Compute the initial condition $p(t_0) = p_0$ to find $p(t)$ for all $t$:

$$x_1 = x(t_1) = \psi_{11}(t_1, t_0)x_0 + \psi_{12}(t_1, t_0)p_0$$

Assuming $\psi_{12}(t_1, t_0)$ is invertible, we have:

$$p_0 = \psi_{12}(t_1, t_0)^{-1}(x_1 - \psi_{11}(t_1, t_0)x_0)$$

# Example 11.5.2: LQR Problem with Terminal State (cont.)

For all $t$:

$$p(t) = \psi_{21}(t, t_0)x_0 + \psi_{22}(t, t_0)p_0$$

The optimal control is:

$$u^\circ(t) = -R^{-1}B^T p(t)$$

The optimal state trajectory is:

$$x^\circ(t) = \psi_{11}(t, t_0)x_0 + \psi_{12}(t, t_0)p_0, \quad t \geq t_0$$

# Minimum Principle with Free Terminal Time

Suppose $t_0, x(t_0) = x_0$ are fixed, and some components of $x(t_1)$ are specified. The Minimum Principle with free terminal time states:

1. There exists a costate vector $p(t)$ such that $u^\circ(t) = \arg\min_u H(x^\circ(t), p(t), u, t)$.
2. The pair $(p, x^\circ)$ satisfy the 2-point boundary value problem with modified boundary conditions.
3. The unspecified terminal time $t_1$ satisfies:

$$\frac{\partial m}{\partial t}(x^\circ(t_1), t) + H(x^\circ(t_1), p(t_1), u^\circ(t_1), t_1) = 0$$

# Example 11.5.3: Minimum Time Problem

Consider a single input linear state space model:

$$\dot{x} = Ax + bu$$

We wish to find $u^\circ$ which drives $x$ from $x(0) = x_0$ to $x(t_1) = x_1$ in minimum time, under the constraint $|u(t)| \leq 1$. The cost criterion is:

$$V(u) = t_1 = \int_0^{t_1} 1\, dt$$

The Hamiltonian is:

$$H = 1 + p^T(Ax + bu)$$

# Bang-Bang Control Law

Minimizing $H$ gives a bang-bang control law:

$$u^\circ(t) = \begin{cases} 1 & \text{if } p(t)^T b < 0 \\ -1 & \text{if } p(t)^T b > 0 \end{cases}$$

If $A, b$ are time-invariant and the eigenvalues of $A$ are real, distinct, and negative, then $p(t)^T b$ changes sign at most $n-1$ times, bounding the number of switching times for the optimal control.

# Costate Equation for Minimum Time Problem

The costate equation is:

$$\dot{p} = -\nabla_x H = -A^T p$$

Using the final time boundary condition $H|_{t=t_1} = 0$:

$$1 + p(t_1)^T A x^\circ(t_1) + p(t_1)^T b u^\circ(t_1) = 1 + p(t_1)^T A x^\circ(t_1) - |p(t_1)^T b| = 0$$

# Example: Double Integrator

Consider the double integrator $\ddot{y} = u$:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The optimal control is:

$$u^{\circ}(t) = -\mathrm{sgn}(p(t)^T b) = -\mathrm{sgn}(p_2(t))$$

# Solving Costate Equations

From the costate equations, for constants $c_1, c_2$:

$$p_1(t) = c_1, \quad p_2(t) = -c_1 t + c_2$$

The optimal control is bang-bang:

$$u^\circ(t) = \begin{cases} 1 & \text{if } t < t_1 - 1 \\ 0 & \text{if } t > t_1 - 1 \end{cases}$$

# Optimal Trajectories

The optimal state trajectories follow quadratic paths:

$$x_1(t) = \begin{cases} \frac{1}{2}(x_2(t))^2 + K_1 & \text{if } u^\circ(t) = 1 \\ -\frac{1}{2}(x_2(t))^2 + K_2 & \text{if } u^\circ(t) = -1 \end{cases}$$