An Introduction to Minimum Principle

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Minimum Principle and Lagrange Multipliers

Introduction

Optimal control involves minimizing a functional similar to ordinary optimization in finite dimensions. For unconstrained optimization, we find stationary points where the derivative of the function is zero. This concept extends to infinite dimensions in the calculus of variations.

Lagrange Multipliers in Finite Dimensions

Consider an optimization problem with constraint $g(x) = \mathbf{0}$, where $g : \mathbb{R}^m \to \mathbb{R}^d$:

$$\begin{aligned} &\min_x V(x) \\ &\text{s.t. } g(x) = \mathbf{0} \end{aligned}$$

In constrained optimization in $\mathbb{R}^m$, we use Lagrange multipliers:

$$\hat{V}(x, p) = V(x) + p^T g(x)$$

where p is a Lagrange multiplier. If x^0, p^0 is a stationary point, then:

$$\begin{aligned} \nabla_x \hat{V}(x^0, p^0) &= \nabla V(x^0) + \nabla g(x^0) p^0 = \mathbf{0} \\ \nabla_p \hat{V}(x^0, p^0) &= g(x^0) = \mathbf{0} \end{aligned}$$

Lagrange Multipliers in Finite Dimensions

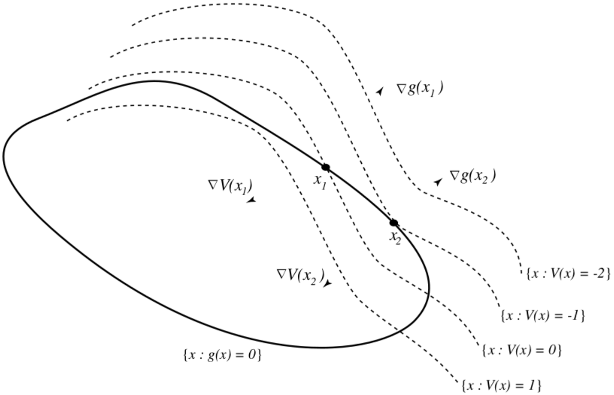

Figure 1: An optimization problem in $\mathbb{R}^2$ with a single constraint, $g(x) = 0$.

$$\nabla_x \hat{V}(x^0, p^0) = \nabla V(x^0) + \nabla g(x^0)p^0 = \mathbf{0}$$
$$\nabla_p \hat{V}(x^0, p^0) = g(x^0) = \mathbf{0}$$

Generalizing to Infinite Dimensions

To generalize this to functional minimization, suppose that F is a functional on $D^r[t_0, t_1]$, $F(z)$ is a real number, and $z \in D^r[t_0, t_1]$. We define the directional derivative:

$$D_\eta F(z) = \lim_{\epsilon \to 0} \frac{F(z + \epsilon\eta) - F(z)}{\epsilon}$$

A function z_0 is a stationary point of F if $D_\eta F(z_0) = 0$ for any η.

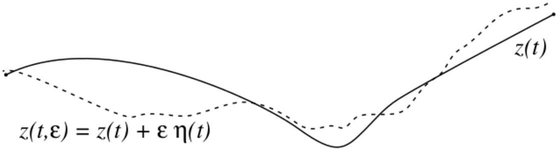

Figure 2: A Perturbation of function $z \in D[t_0, t_1]$

Steps to Extend the Lagrange Multiplier Method

1. **Append state equations:** Define a new cost functional including the state dynamics:

$$\hat{V}(x, u) = \int_{t_0}^{t_1} \ell \, dt + m(x(t_1)) + \int_{t_0}^{t_1} p^T (f - \dot{x}) \, dt$$

2. **Integration by parts:** Eliminate the derivative of x:

$$\int_{t_0}^{t_1} p^T \dot{x} \, dt = \left[p^T x \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{p}^T x \, dt$$

3. **Define the Hamiltonian:**

$$H(x, p, u, t) = \ell(x, u, t) + p^T f(x, u, t)$$

Steps to Extend the Lagrange Multiplier Method (cont.)

4. **Compute variations:** For perturbations $x(t, \epsilon) = x^\circ(t) + \epsilon\eta(t)$ and $u(t, \delta) = u^\circ(t) + \delta\psi(t)$, set $\frac{d}{d\epsilon}\hat{V}(\epsilon) = 0$ and $\frac{d}{d\delta}\hat{V}(\delta) = 0$.
5. **Derive necessary conditions:** From the variations, derive the equations:
$$\dot{p} = -\frac{\partial H}{\partial x}, \quad p(t_1) = \frac{\partial m}{\partial x}(x(t_1))$$
$$\frac{\partial H}{\partial u} = 0$$

Deriving the Minimum Principle

Combine the steps to derive the Minimum Principle. If u° is optimal, there exists a costate $p(t)$ such that:

$$u^\circ(t) = \arg\min_u H(x^\circ(t), p(t), u, t)$$

The state and costate satisfy:

$$\dot{x}^\circ = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}$$

with boundary conditions:

$$x(t_0) = x_0, \quad p(t_1) = \frac{\partial m}{\partial x}(x(t_1))$$

The Penalty Approach

The Penalty Approach

A third heuristic approach to the Minimum Principle involves relaxing the hard constraint $\dot{x} - f = 0$, and instead imposing a large, yet "soft" constraint.

$$\hat{V}(x,u) = \int_{t_0}^{t_1} \ell(x(t),u(t),t)\,dt + \frac{k}{2}\int_{t_0}^{t_1} |\dot{x}(t) - f(x(t),u(t),t)|^2\,dt + m(x(t_1))$$

Perturbation and Optimality

If (x_k, u_k) minimizes $\hat{V}_k$, and letting (x°, u°) denote a solution to the original problem:

$$\hat{V}_k(x_k, u_k) \leq \hat{V}_k(x^\circ, u^\circ) = V^\circ$$

Assuming ℓ and m are positive, this gives the uniform bound:

$$\int_{t_0}^{t_1} |\dot{x}(t) - f(x(t), u(t), t)|^2 \, dt \leq \frac{2}{k} V^\circ$$

Application to LQR

Application to LQR

The LQR problem tests the Minimum Principle's utility for constructing optimal policies. Consider the general LTI model with quadratic cost:

$$\dot{x} = Ax + Bu$$

$$V = \int_{t_0}^{t_1} (x^T Q x + u^T R u)\, dt + x^T(t_1) M x(t_1)$$

Solving the LQR Problem

The Hamiltonian:

$$H = x^T Q x + u^T R u + p^T (Ax + Bu)$$

Control can be computed through:

$$\nabla_u H = 0 \Rightarrow u = -\frac{1}{2} R^{-1} B^T p$$

Resulting in:

$$\dot{x} = Ax - \frac{1}{2} B R^{-1} B^T p$$

$$\dot{p} = -\nabla_x H = -2Qx - A^T p$$

Nonlinear Examples

Nonlinear Examples

We solve some nonlinear problems where it is possible to obtain an explicit solution to the coupled state-costate equations given in the Minimum Principle. At the same time, we also give several extensions of this result.

Example 11.5.1: Bilinear System

Consider the control of a bilinear system, defined by the differential equation:

$$\dot{x} = ux, \quad x(0) = x_0 > 0, \quad 0 \leq u(t) \leq 1$$

Suppose the goal is to make x large while keeping the derivative of x small on average. The cost criterion is:

$$V(u) = \int_0^{t_1} [u(\tau) - 1]x(\tau)\, d\tau$$

Optimal Control of Bilinear System

The Hamiltonian becomes:

$$H(x, p, u, t) = x\{u(p+1) - 1\}$$

By the Minimum Principle, the optimal control is:

$$u^{\circ}(t) = \begin{cases} 1 & \text{if } p(t) + 1 < 0 \\ 0 & \text{if } p(t) + 1 > 0 \\ \text{unknown} & \text{if } p(t) = -1 \end{cases}$$

Costate and State Equations

The costate and state variables satisfy:

$$\dot{p} = -(p+1)u^\circ + 1$$

$$\dot{x}^\circ = u^\circ x^\circ$$

with the boundary conditions:

$$x^\circ(0) = x_0, \quad p(t_1) = 0$$

Solving the Differential Equations

If $t = t_1$, then $p(t) + 1 = 1 > 0$ so $u^\circ(t) = 0$. By continuity, if $t \approx t_1$, then $p(t) > 0$, so $u^\circ(t) = 0$ and:

$$\dot{p}(t) = 1 \quad \text{for } t \approx t_1 \implies p(t) = t - t_1$$

For $t < t_1 - 1$, we have $p(t) + 1 < 0$ so $u^\circ(t) = 1$ and:

$$\dot{p} = -p \quad \text{for } t < t_1 - 1 \implies p(t) = -e^{-t+t_1-1}$$

Bang-Bang Control

The optimal control is bang-bang:

$$u(t) = \begin{cases} 1 & \text{if } t < t_1 - 1 \\ 0 & \text{if } t > t_1 - 1 \end{cases}$$

The optimal state trajectory is:

$$\dot{x}^\circ(t) = \begin{cases} x^\circ(t) & \text{if } t < t_1 - 1 \\ 0 & \text{if } t > t_1 - 1 \end{cases}$$

$$x^\circ(t) = \begin{cases} x_0 e^t & \text{if } t < t_1 - 1 \\ x_0 e^{t_1 - 1} & \text{if } t > t_1 - 1 \end{cases}$$

Minimum Principle with Constraints

Suppose $x(t_1)$ is free, t_1 is fixed, and u° is a solution to the optimal control problem (11.1) under the constraint $u \in U$. The Minimum Principle with constraints states:

1. There exists a costate vector $p(t)$ such that $u^\circ(t) = \arg\min_{u \in U} H(x^\circ(t), p(t), u, t)$.
2. The pair (p, x°) satisfy the 2-point boundary value problem.

Example 11.5.1 with Constraints

Consider the bilinear system:

$$\dot{x} = ux, \quad x(0) = x_0 > 0, \quad 0 \leq u(t) \leq 1$$

with the cost criterion:

$$V(u) = \int_0^{t_1} [u(\tau) - 1]x(\tau)\, d\tau$$

The Hamiltonian:

$$H(x, p, u, t) = x\{u(p+1) - 1\}$$

Optimal control:

$$u^\circ(t) = \begin{cases} 1 & \text{if } p(t) + 1 < 0 \\ 0 & \text{if } p(t) + 1 > 0 \end{cases}$$

Costate and State Variables for Constrained Problem

The costate and state variables satisfy:

$$\dot{p} = -(p+1)u^\circ + 1$$

$$\dot{x} = u^\circ x$$

with boundary conditions $x^\circ(0) = x_0$, $p(t_1) = 0$.

Solving the Constrained Problem

If $t \approx t_1$:

$$p(t) = t - t_1$$

For $t < t_1 - 1$:

$$p(t) = -e^{-t+t_1-1}$$

The optimal control is bang-bang:

$$u(t) = \begin{cases} 1 & t < t_1 - 1 \\ 0 & t > t_1 - 1 \end{cases}$$

Optimal State Trajectory for Constrained Problem

The optimal state trajectory:

$$\dot{x}^\circ(t) = \begin{cases} x^\circ(t) & t < t_1 - 1 \\ 0 & t > t_1 - 1 \end{cases}$$

$$x^\circ(t) = \begin{cases} x_0 e^{t} & t < t_1 - 1 \\ x_0 e^{t_1 - 1} & t > t_1 - 1 \end{cases}$$

Minimum Principle with Final Value Constraints

Suppose $t_0, t_1, x(t_0) = x_0$, and $x(t_1) = x_1$ are prespecified. The Minimum Principle with final value constraints states:

1. There exists a costate vector $p(t)$ such that $u^\circ(t) = \arg\min_u H(x^\circ(t), p(t), u, t)$.
2. The pair (p, x°) satisfy the 2-point boundary value problem.

Example 11.5.2: LQR Problem with Terminal State

Consider the LQR problem with the terminal state specified. The cost criterion is:

$$V = \frac{1}{2}\int_0^{t_1} (x^T Q x + u^T R u)\, dt, \quad R > 0, Q \geq 0$$

The Hamiltonian:

$$H = p(Ax + Bu) + \frac{1}{2}(x^T Q x + u^T R u)$$

The optimal control:

$$u^\circ(t) = -R^{-1} B^T p(t)$$

Costate Equation for LQR Problem

The costate vector satisfies the differential equation:

$$\dot{p}(t) = -\frac{\partial H}{\partial x} = -p^T A - x^T Q$$

$$(x, p) = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

with boundary conditions:

$$x(t_0) = x_0, \quad x(t_1) = x_1$$

State Transition Matrix

Let $\psi(t, \tau)$ denote the state transition matrix:

$$\frac{d}{dt}\psi(t, \tau) = H(t)\psi(t, \tau), \quad \psi(t, t) = I$$

Decompose $\psi(t, \tau)$ as:

$$\psi(t, \tau) = \begin{pmatrix} \psi_{11}(t, \tau) & \psi_{12}(t, \tau) \\ \psi_{21}(t, \tau) & \psi_{22}(t, \tau) \end{pmatrix}$$

Solving for Optimal Control

Compute the initial condition $p(t_0) = p_0$ to find $p(t)$ for all t:

$$x_1 = x(t_1) = \psi_{11}(t_1, t_0)x_0 + \psi_{12}(t_1, t_0)p_0$$

Assuming $\psi_{12}(t_1, t_0)$ is invertible, we have:

$$p_0 = \psi_{12}(t_1, t_0)^{-1}(x_1 - \psi_{11}(t_1, t_0)x_0)$$

Optimal Control and State Trajectory

For all t:

$$p(t) = \psi_{21}(t, t_0)x_0 + \psi_{22}(t, t_0)p_0$$

The optimal control is:

$$u^\circ(t) = -R^{-1}B^T p(t)$$

The optimal state trajectory is:

$$x^\circ(t) = \psi_{11}(t, t_0)x_0 + \psi_{12}(t, t_0)p_0, \quad t \geq t_0$$

Minimum Principle with Free Terminal Time

Suppose $t_0, x(t_0) = x_0$ are fixed, and some components of $x(t_1)$ are specified. The Minimum Principle with free terminal time states:

1. There exists a costate vector $p(t)$ such that $u^\circ(t) = \arg\min_u H(x^\circ(t), p(t), u, t)$.
2. The pair (p, x°) satisfy the 2-point boundary value problem with modified boundary conditions.
3. The unspecified terminal time t_1 satisfies:

$$\frac{\partial m}{\partial t}(x^\circ(t_1), t) + H(x^\circ(t_1), p(t_1), u^\circ(t_1), t_1) = 0$$

Example 11.5.3: Minimum Time Problem

Consider a single input linear state space model:

$$\dot{x} = Ax + bu$$

We wish to find u° which drives x from $x(0) = x_0$ to $x(t_1) = x_1$ in minimum time, under the constraint $|u(t)| \leq 1$. The cost criterion is:

$$V(u) = t_1 = \int_0^{t_1} 1 \, dt$$

The Hamiltonian is:

$$H = 1 + p^T(Ax + bu)$$

Bang-Bang Control Law

Minimizing H gives a bang-bang control law:

$$u^\circ(t) = \begin{cases} 1 & \text{if } p(t)^T b < 0 \\ -1 & \text{if } p(t)^T b > 0 \end{cases}$$

If A, b are time-invariant and the eigenvalues of A are real, distinct, and negative, then $p(t)^T b$ changes sign at most $n-1$ times, bounding the number of switching times for the optimal control.

Costate Equation for Minimum Time Problem

The costate equation is:

$$\dot{p} = -\nabla_x H = -A^T p$$

Using the final time boundary condition $H|_{t=t_1} = 0$:

$$1 + p(t_1)^T A x^\circ(t_1) + p(t_1)^T b u^\circ(t_1) = 1 + p(t_1)^T A x^\circ(t_1) - |p(t_1)^T b| = 0$$

Example: Double Integrator

Consider the double integrator $\ddot{y} = u$:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The optimal control is:

$$u^{\circ}(t) = -\mathrm{sgn}(p(t)^T b) = -\mathrm{sgn}(p_2(t))$$

Solving Costate Equations

From the costate equations, for constants c_1, c_2:

$$p_1(t) = c_1, \quad p_2(t) = -c_1 t + c_2$$

The optimal control is bang-bang:

$$u^\circ(t) = \begin{cases} 1 & \text{if } t < t_1 - 1 \\ 0 & \text{if } t > t_1 - 1 \end{cases}$$

Optimal Trajectories

The optimal state trajectories follow quadratic paths:

$$x_1(t) = \begin{cases} \frac{1}{2}(x_2(t))^2 + K_1 & \text{if } u^\circ(t) = 1 \\ -\frac{1}{2}(x_2(t))^2 + K_2 & \text{if } u^\circ(t) = -1 \end{cases}$$