An Introduction to Minimum Principle

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Consider an optimization problem with constraint $g(x) = \mathbf{0}$, where $g : \mathbb{R}^m \to \mathbb{R}^d$:

$$\min_x V(x)$$
$$\text{s.t. } g(x) = \mathbf{0}$$

In constrained optimization in $\mathbb{R}^m$, we use Lagrange multipliers:

$$\hat{V}(x, p) = V(x) + p^T g(x)$$

where p is a Lagrange multiplier. If x^0, p^0 is a stationary point, then:

$$\nabla_x \hat{V}(x^0, p^0) = \nabla_x V(x^0) + \nabla_x g(x^0) p^0 = \mathbf{0}$$
$$\nabla_p \hat{V}(x^0, p^0) = g(x^0) = \mathbf{0}$$

Lagrange Multipliers in Finite Dimensions

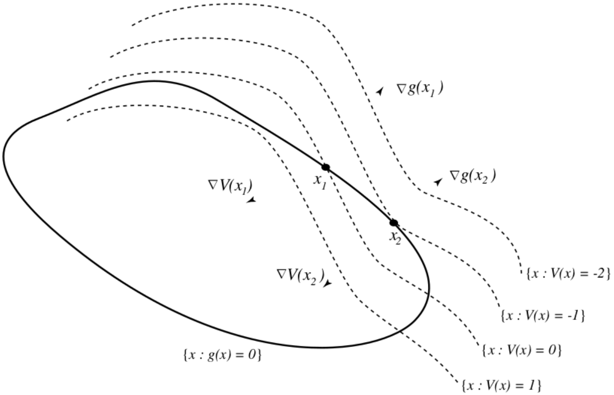

Figure 1: An optimization problem in $\mathbb{R}^2$ with a single constraint, $g(x) = 0$.

$$\nabla_x \hat{V}(x^0, p^0) = \nabla_x V(x^0) + \nabla_x g(x^0) p^0 = \mathbf{0}$$
$$\nabla_p \hat{V}(x^0, p^0) = g(x^0) = \mathbf{0}$$

Generalizing to Infinite Dimensions

To generalize this to functional minimization, suppose that F is a functional on $D^r[t_0, t_1]$, $F(z)$ is a real number, and $z \in D^r[t_0, t_1]$. We define the directional derivative:

$$D_\eta F(z) = \lim_{\epsilon \to 0} \frac{F(z + \epsilon\eta) - F(z)}{\epsilon}$$

A function z_0 is a stationary point of F if $D_\eta F(z_0) = 0$ for any η.

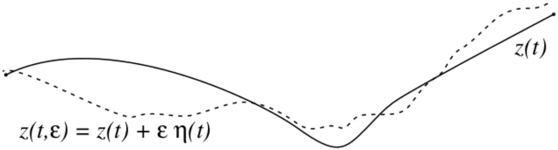

Figure 2: A Perturbation of function $z \in D[t_0, t_1]$

Steps to Extend the Lagrange Multiplier Method

Let us consider a constrained functional optimization problem:

$$\min_u V(x,u) = \int \ell d\tau + m$$

$$\text{s.t. } \dot{x} - f = \mathbf{0}, \quad x \in D^n[t_0, t_1], u \in D^m[t_0, t_1].$$

1. **Append state equations:** Define a new cost functional including the state dynamics:

$$\hat{V}(x,u) = \int_{t_0}^{t_1} \ell \, dt + m(x(t_1)) + \int_{t_0}^{t_1} p^T (f - \dot{x}) \, dt$$

2. **Integration by parts:** Eliminate the derivative of x:

$$\int_{t_0}^{t_1} p^T \dot{x} \, dt = p^T x|_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{p}^T x \, dt$$

Steps to Extend the Lagrange Multiplier Method (cont.)

3. **Define the Hamiltonian:**

$$H(x,p,u,t) := \ell(x,u,t) + p^T f(x,u,t)$$

$$\hat{V}(x,u) = \int_{t_0}^{t_1} H(x,p,u,t)\,dt + \int_{t_0}^{t_1} \dot{p}^\top x dt$$
$$+ p^\top(t_0)x(t_0) - p^\top(t_1)x(t_1) + m(x(t_1))$$

4. **Compute variations:** For perturbations $x(t,\epsilon) = x^\circ(t) + \epsilon\eta(t)$ and $u(t,\delta) = u^\circ(t) + \delta\psi(t)$, let $\hat{V}(\epsilon) = \hat{V}(x^\circ + \epsilon\eta, u^\circ)$, and $\hat{V}(\delta){=}\hat{V}(x^\circ, u^\circ + \delta\psi)$.

$$\frac{d}{d\epsilon}\hat{V}(\epsilon) = \int_{t_0}^{t_1} \frac{\partial}{\partial x} H(x^\circ, p, u^\circ, t)\eta(t)\,dt + \int_{t_0}^{t_1} \dot{p}^T\eta(t)\,dt$$
$$- p^T(t_1)\eta(t_1) + p^T(t_0)\eta(t_0) + \frac{\partial}{\partial x} m(x^\circ(t_1))\eta(t_1) = 0.$$

Similarly, by considering perturbations in u°:

$$\frac{d}{d\delta}\hat{V}(\delta) = \int_{t_0}^{t_1} \frac{\partial}{\partial u} H(x^\circ, p, u^\circ, t)\psi(t)dt = 0$$

Steps to Extend the Lagrange Multiplier Method (cont.)

4. **Compute variations:** Determine perturbations in x°:

$$\int_{t_0}^{t_1} \frac{\partial}{\partial x} H(x^\circ, p, u^\circ, t)\eta(t)\,dt + \int_{t_0}^{t_1} \dot{p}^T \eta(t)\,dt$$
$$- p^T(t_1)\eta(t_1) + p^T(t_0)\eta(t_0) + \frac{\partial}{\partial x} m(x^\circ(t_1))\eta(t_1) = 0.$$

   Similarly, by considering perturbations in u°:

$$\int_{t_0}^{t_1} \frac{\partial}{\partial u} H(x^\circ, p, u^\circ, t)\psi(t)dt = 0$$

5. **Derive necessary conditions:** From the variations, derive the equations:

$$\dot{p} = -\nabla_x H, \quad p(t_1) = \nabla_x m(x(t_1))$$
$$\nabla_u H = \mathbf{0}$$

Deriving the Minimum Principle

Combine the steps to derive the Minimum Principle. If u° is optimal, there exists a costate $p(t)$ such that:

$$u^\circ(t) \in \arg\min_u H(x^\circ(t), p(t), u, t)$$

The state and costate satisfy:

$$\dot{x}^\circ = \nabla_p H, \quad \dot{p} = -\nabla_x H$$

with boundary conditions:

$$x(t_0) = x_0, \quad p(t_1) = \nabla_x m(x(t_1))$$

The Penalty Approach

The Penalty Approach

Another approach to the Minimum Principle involves relaxing the hard constraint $\dot{x} - f = 0$, and instead imposing a large, yet "soft" constraint by definiong the cost function:

$$\hat{V}(x,u) = \int_{t_0}^{t_1} \ell(x(t),u(t),t)\,dt + \frac{k}{2}\int_{t_0}^{t_1} |\dot{x}(t) - f(x(t),u(t),t)|^2\,dt + m(x(t_1))$$

If (x_k, u_k) minimizes $\hat{V}_k$, and letting (x°, u°) denote a solution to the original problem:

$$\hat{V}_k(x_k,u_k) \le \hat{V}_k(x^\circ,u^\circ) = V^\circ$$

Assuming ℓ and m are positive, subtracting the left side of the above inequality gives the uniform bound:

$$\int_{t_0}^{t_1} |\dot{x}(t) - f(x(t),u(t),t)|^2\,dt \le \frac{2}{k} V^\circ$$

Implications for Large k

As k becomes large, the term $\frac{k}{2} \int_{t_0}^{t_1} |\dot{x}(t) - f(x(t), u(t), t)|^2 \, dt$ ensures that $\dot{x}(t) \approx f(x(t), u(t), t)$:

$$\int_{t_0}^{t_1} |\dot{x}(t) - f(x(t), u(t), t)|^2 \, dt \to 0 \text{ as } k \to \infty$$

Thus, for large k, the pair (x_k, u_k) will approximately satisfy the differential equation $\dot{x} = f$.

Perturbation Analysis

If we perturb x_k to form $x_k + \epsilon\eta$ and define $\hat{V}(\epsilon) = \hat{V}(x_k + \epsilon\eta, u_k)$, then we must have:

$$\frac{d}{d\epsilon}\hat{V}(\epsilon)\bigg|_{\epsilon=0} = 0$$

Using the definition of $\hat{V}$, we get:

$$\begin{aligned}\hat{V}(\epsilon) = &\int_{t_0}^{t_1} \ell(x_k(t) + \epsilon\eta(t), u_k(t), t)\, dt + m(x_k(t_1) + \epsilon\eta(t_1)) \\ &+ \frac{k}{2}\int_{t_0}^{t_1} |\dot{x}_k(t) + \epsilon\dot{\eta}(t) - f(x_k(t) + \epsilon\eta(t), u_k(t), t)|^2\, dt\end{aligned}$$

Computing the Derivative

The derivative of this expression with respect to ϵ can be computed as follows:

$$\begin{aligned}
\frac{d}{d\epsilon}\hat{V}(0) &= \int_{t_0}^{t_1} \frac{\partial \ell}{\partial x}(x_k(t), u_k(t), t)\eta(t)\, dt \\
&\quad + k\int_{t_0}^{t_1} (\dot{x}_k(t) - f(x_k(t), u_k(t), t))^T[\dot{\eta}(t) - \frac{\partial f}{\partial x}(x_k(t), u_k(t), t)\eta(t)]\, dt \\
&\quad + \frac{\partial m}{\partial x}(x_k(t_1))\eta(t_1) \\
&= 0
\end{aligned}$$

Computing the Derivative (cont.)

To eliminate the derivative term $\dot{\eta}$, we integrate by parts:

$$\int_{t_0}^{t_1} \left\{ \frac{\partial \ell}{\partial x}(x_k(t), u_k(t), t) + p_k(t)^T \frac{\partial f}{\partial x}(x_k(t), u_k(t), t) + \frac{d}{dt}\left(p_k(t)^T\right) \right\} \eta(t)\, dt$$

$$- p_k(t_1)^T \eta(t_1) + \frac{\partial m}{\partial x}(x_k(t_1))\eta(t_1)$$

$$= 0$$

where we have set $p_k(t) = -k(\dot{x}_k(t) - f(x_k(t), u_k(t), t))$.

Resulting Equations

Since η is arbitrary, we have:

$$\begin{aligned}
\mathbf{0} &= \frac{d}{dt}\left(p_k(t)^T\right) + \frac{\partial \ell}{\partial x}(x_k(t), u_k(t), t) + p_k(t)^T \frac{\partial f}{\partial x}(x_k(t), u_k(t), t) \\
\mathbf{0} &= \frac{d}{dt}\left(p_k(t)^T\right) + \frac{\partial H}{\partial x}(x_k(t), p_k(t), u_k(t), t) \\
\dot{p}_k &= -\nabla_x H
\end{aligned}$$

with the boundary condition:

$$p_k(t_1) = \nabla_x m(x_k(t_1))$$

Considering perturbations in u gives the equation:

$$\nabla_u H(x_k(t), p_k(t), u_k(t), t) = \mathbf{0}$$

This is a weak form of the Minimum Principle for the perturbed problem.

Application to LQR

Application to LQR

The LQR problem tests the Minimum Principle's utility for constructing optimal policies. Consider the general LTI model with quadratic cost:

$$\dot{x} = Ax + Bu$$

$$V = \int_{t_0}^{t_1} (x^T Q x + u^T R u)\, dt + x^T(t_1) M x(t_1)$$

Solving the LQR Problem

The Hamiltonian:

$$H = x^T Qx + u^T Ru + p^T (Ax + Bu)$$

Control can be computed through:

$$\nabla_u H = 0 \Rightarrow u = -\frac{1}{2} R^{-1} B^T p$$

Resulting in:

$$\dot{x} = Ax + Bu = Ax - \frac{1}{2} BR^{-1} B^T p$$

Through the expression $\nabla_x H = 2Qx + A^T p$, we find that $\dot{p}$ is:

$$\dot{p} = -\nabla_x H = -2Qx - A^T p$$

Solving the LQR Problem (cont.)

The equations form the coupled set of differential equations:

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & -\frac{1}{2}BR^{-1}B^T \\ -2Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

with boundary conditions:

$$x(t_0) = x_0$$

$$p(t_1) = \nabla_x m(x_k(t_1)) = 2Mx(t_1)$$

If we scale p by defining $\lambda = \frac{1}{2}p$, we get:

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \mathcal{H} \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

with the optimal control:

$$u^\circ(t) = -R^{-1}B^T\lambda(t)$$

Solving the ODE

This ODE can be solved using the *sweep method*. Suppose that $\lambda(t) = P(t)x(t)$. Then:

$$\dot{\lambda} = \dot{P}x + P(Ax + Bu)$$

Substituting $u^\circ = -R^{-1}B^\top \lambda = -R^{-1}B^\top Px$ gives:

$$\dot{\lambda} = \dot{P}x + P(Ax - BR^{-1}B^\top Px)$$

From the coupled ODE with $\mathcal{H}$ above, we also have:

$$\dot{\lambda} = -Qx - A^\top \lambda = -Qx - A^\top Px$$

Equating the two expressions for $\dot{\lambda}$ gives the Riccati Differential Equation:

$$\dot{P}x + P(Ax - BR^{-1}B^\top Px) = -Qx - A^\top Px$$

Boundary Condition

The boundary condition for λ is:

$$\lambda = \frac{1}{2}p = \frac{1}{2}\nabla_x m(x(t_1)) \implies P(t_1) = M$$

Solving for $P(t)$ gives $\lambda(t)$, which in turn gives $p(t)$.

Nonlinear Examples

Minimum Principle with constraints

Theorem 11.2 (Minimum Principle with constraints). Suppose that $x(t_1)$ is free, t_1 is fixed, and suppose that u° is a solution to the optimal control problem (11.1) under the constraint $u \in \mathcal{U}$. That is,

$$V^\circ = \min\{V(u) : u(t) \in \mathcal{U} \text{ for all } t\}.$$

We then have

(a) There exists a costate vector $p(t)$ such that

$$u^\circ(t) = \arg\min_{u \in \mathcal{U}} H(x^\circ(t), p(t), u, t).$$

(b) The pair (p, x°) satisfy the 2-point boundary value problem (11.4), with the two boundary conditions

$$x(t_0) = x_0; \quad p(t_1) = \frac{\partial}{\partial x} m(x(t_1), t).$$

Example 11.5.1: Bilinear System

Consider the control of a bilinear system, defined by the differential equation:

$$\dot{x} = ux, \quad x(0) = x_0 > 0, \quad 0 \leq u(t) \leq 1$$

Suppose the goal is to make x large while keeping the derivative of x small on average. The cost criterion is:

$$V(u) = \int_0^{t_1} [u(\tau) - 1]x(\tau)\, d\tau$$

Optimal Control of Bilinear System

The Hamiltonian becomes:

$$H(x, p, u, t) = x\{u(p + 1) - 1\}$$

By the Minimum Principle, the optimal control is:

$$u^\circ(t) = \begin{cases} 1 & \text{if } p(t) + 1 < 0 \\ 0 & \text{if } p(t) + 1 > 0 \\ \text{unknown} & \text{if } p(t) = -1 \end{cases}$$

Costate and State Equations

The costate and state variables satisfy:

$$\dot{p} = -(p+1)u^{\circ} + 1$$

$$\dot{x}^{\circ} = u^{\circ} x^{\circ}$$

with the boundary conditions:

$$x^{\circ}(0) = x_0, \quad p(t_1) = 0$$

If $t = t_1$, then $p(t) + 1 = 1 > 0$ so $u^\circ(t) = 0$. By continuity, if $t \approx t_1$, then $p(t) > 0$, so $u^\circ(t) = 0$ and:

$$\dot{p}(t) = 1 \quad \text{for } t \approx t_1 \implies p(t) = t - t_1$$

For $t < t_1 - 1$, we have $p(t) + 1 < 0$ so $u^\circ(t) = 1$ and:

$$\dot{p} = -p \quad \text{for } t < t_1 - 1 \implies p(t) = -e^{-t+t_1-1}$$

Bang-Bang Control

The optimal control is bang-bang:

$$u(t) = \begin{cases} 1 & \text{if } t < t_1 - 1 \\ 0 & \text{if } t > t_1 - 1 \end{cases}$$

The optimal state trajectory is:

$$\dot{x}^\circ(t) = \begin{cases} x^\circ(t) & \text{if } t < t_1 - 1 \\ 0 & \text{if } t > t_1 - 1 \end{cases}$$

$$x^\circ(t) = \begin{cases} x_0 e^{t} & \text{if } t < t_1 - 1 \\ x_0 e^{t_1 - 1} & \text{if } t > t_1 - 1 \end{cases}$$

Minimum Principle with Constraints

Suppose $x(t_1)$ is free, t_1 is fixed, and u° is a solution to the optimal control problem (11.1) under the constraint $u \in U$. The Minimum Principle with constraints states:

1. There exists a costate vector $p(t)$ such that $u^\circ(t) = \arg\min_{u \in U} H(x^\circ(t), p(t), u, t)$.
2. The pair (p, x°) satisfy the 2-point boundary value problem.

Example 11.5.1 with Constraints

Consider the bilinear system:

$$\dot{x} = ux, \quad x(0) = x_0 > 0, \quad 0 \leq u(t) \leq 1$$

with the cost criterion:

$$V(u) = \int_0^{t_1} [u(\tau) - 1]x(\tau)\, d\tau$$

The Hamiltonian:

$$H(x, p, u, t) = x\{u(p+1) - 1\}$$

Optimal control:

$$u^{\circ}(t) = \begin{cases} 1 & \text{if } p(t) + 1 < 0 \\ 0 & \text{if } p(t) + 1 > 0 \end{cases}$$

Costate and State Variables for Constrained Problem

The costate and state variables satisfy:

$$\dot{p} = -(p+1)u^\circ + 1$$

$$\dot{x} = u^\circ x$$

with boundary conditions $x^\circ(0) = x_0$, $p(t_1) = 0$.

Solving the Constrained Problem

If $t \approx t_1$:

$$p(t) = t - t_1$$

For $t < t_1 - 1$:

$$p(t) = -e^{-t+t_1-1}$$

The optimal control is bang-bang:

$$u(t) = \begin{cases} 1 & t < t_1 - 1 \\ 0 & t > t_1 - 1 \end{cases}$$

Optimal State Trajectory for Constrained Problem

The optimal state trajectory:

$$\dot{x}^{\circ}(t) = \begin{cases} x^{\circ}(t) & t < t_1 - 1 \\ 0 & t > t_1 - 1 \end{cases}$$

$$x^{\circ}(t) = \begin{cases} x_0 e^{t} & t < t_1 - 1 \\ x_0 e^{t_1 - 1} & t > t_1 - 1 \end{cases}$$

Minimum Principle with Final Value Constraints

Suppose $t_0, t_1, x(t_0) = x_0$, and $x(t_1) = x_1$ are prespecified. The Minimum Principle with final value constraints states:

1. There exists a costate vector $p(t)$ such that $u^\circ(t) = \arg\min_u H(x^\circ(t), p(t), u, t)$.
2. The pair (p, x°) satisfy the 2-point boundary value problem.

Example 11.5.2: LQR Problem with Terminal State

Consider the LQR problem with the terminal state specified. The cost criterion is:

$$V = \frac{1}{2}\int_0^{t_1} (x^T Q x + u^T R u)\, dt, \quad R > 0, Q \geq 0$$

The Hamiltonian:

$$H = p(Ax + Bu) + \frac{1}{2}(x^T Q x + u^T R u)$$

The optimal control:

$$u^\circ(t) = -R^{-1} B^T p(t)$$

Costate Equation for LQR Problem

The costate vector satisfies the differential equation:

$$\dot{p}(t) = -\frac{\partial H}{\partial x} = -p^T A - x^T Q$$

$$(x, p) = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

with boundary conditions:

$$x(t_0) = x_0, \quad x(t_1) = x_1$$

State Transition Matrix

Let $\psi(t, \tau)$ denote the state transition matrix:

$$\frac{d}{dt}\psi(t, \tau) = H(t)\psi(t, \tau), \quad \psi(t, t) = I$$

Decompose $\psi(t, \tau)$ as:

$$\psi(t, \tau) = \begin{pmatrix} \psi_{11}(t, \tau) & \psi_{12}(t, \tau) \\ \psi_{21}(t, \tau) & \psi_{22}(t, \tau) \end{pmatrix}$$

Solving for Optimal Control

Compute the initial condition $p(t_0) = p_0$ to find $p(t)$ for all t:

$$x_1 = x(t_1) = \psi_{11}(t_1, t_0)x_0 + \psi_{12}(t_1, t_0)p_0$$

Assuming $\psi_{12}(t_1, t_0)$ is invertible, we have:

$$p_0 = \psi_{12}(t_1, t_0)^{-1}(x_1 - \psi_{11}(t_1, t_0)x_0)$$

Optimal Control and State Trajectory

For all t:

$$p(t) = \psi_{21}(t, t_0)x_0 + \psi_{22}(t, t_0)p_0$$

The optimal control is:

$$u^\circ(t) = -R^{-1}B^T p(t)$$

The optimal state trajectory is:

$$x^\circ(t) = \psi_{11}(t, t_0)x_0 + \psi_{12}(t, t_0)p_0, \quad t \geq t_0$$

Minimum Principle with Free Terminal Time

Suppose $t_0, x(t_0) = x_0$ are fixed, and some components of $x(t_1)$ are specified. The Minimum Principle with free terminal time states:

1. There exists a costate vector $p(t)$ such that $u^\circ(t) = \arg\min_u H(x^\circ(t), p(t), u, t)$.
2. The pair (p, x°) satisfy the 2-point boundary value problem with modified boundary conditions.
3. The unspecified terminal time t_1 satisfies:

$$\frac{\partial m}{\partial t}(x^\circ(t_1), t) + H(x^\circ(t_1), p(t_1), u^\circ(t_1), t_1) = 0$$

Example 11.5.3: Minimum Time Problem

Consider a single input linear state space model:

$$\dot{x} = Ax + bu$$

We wish to find u° which drives x from $x(0) = x_0$ to $x(t_1) = x_1$ in minimum time, under the constraint $|u(t)| \leq 1$. The cost criterion is:

$$V(u) = t_1 = \int_0^{t_1} 1\, dt$$

The Hamiltonian is:

$$H = 1 + p^T(Ax + bu)$$

Bang-Bang Control Law

Minimizing H gives a bang-bang control law:

$$u^\circ(t) = \begin{cases} 1 & \text{if } p(t)^T b < 0 \\ -1 & \text{if } p(t)^T b > 0 \end{cases}$$

If A, b are time-invariant and the eigenvalues of A are real, distinct, and negative, then $p(t)^T b$ changes sign at most $n-1$ times, bounding the number of switching times for the optimal control.

Costate Equation for Minimum Time Problem

The costate equation is:

$$\dot{p} = -\nabla_x H = -A^T p$$

Using the final time boundary condition $H|_{t=t_1} = 0$:

$$1 + p(t_1)^T A x^\circ(t_1) + p(t_1)^T b u^\circ(t_1) = 1 + p(t_1)^T A x^\circ(t_1) - |p(t_1)^T b| = 0$$

Example: Double Integrator

Consider the double integrator $\ddot{y} = u$:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The optimal control is:

$$u^\circ(t) = -\mathrm{sgn}(p(t)^T b) = -\mathrm{sgn}(p_2(t))$$

Solving Costate Equations

From the costate equations, for constants c_1, c_2:

$$p_1(t) = c_1, \quad p_2(t) = -c_1 t + c_2$$

The optimal control is bang-bang:

$$u^\circ(t) = \begin{cases} 1 & \text{if } t < t_1 - 1 \\ 0 & \text{if } t > t_1 - 1 \end{cases}$$

Optimal Trajectories

The optimal state trajectories follow quadratic paths:

$$x_1(t) = \begin{cases} \frac{1}{2}(x_2(t))^2 + K_1 & \text{if } u^\circ(t) = 1 \\ -\frac{1}{2}(x_2(t))^2 + K_2 & \text{if } u^\circ(t) = -1 \end{cases}$$