LOCAL GORENSTEIN DUALITY

(with T. Barthel, D. Heard, G. Valenzuela)

My comfort zone : algebraic topology

$$X \text{ topological space} \overset{\text{functorial}}{\rightsquigarrow} \text{algebraic construction } A$$

$$\begin{array}{l} f, g : X \to Y \\ f \simeq g \end{array} \rightsquigarrow A(f) = A(g)$$

<u>Examples</u> : fundamental group π_1, homotopy groups π_n, (co)homology with coefficients in a ring, K-theory, ...

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$\left.\begin{array}{l} X \text{ connected space} \\ k \text{ commutative ring} \end{array}\right\} \xrightarrow{\text{functor}} H^*(X;k)$ graded commutative ring

(graded commutative algebra if k is a field)

TOPOLOGY / HOMOTOPY THEORY $\longrightarrow$ COMMUTATIVE ALGEBRA

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TOPOLOGY / HOMOTOPY THEORY $\longrightarrow$ COMMUTATIVE ALGEBRA

For exaple, $H^*(S^n;k) \cong \Lambda(x)$ $\quad |x| = n$

$H^*(\mathbb{C}P^n;k) \cong k[x]/(x^{n+1})$ $\quad |x| = 2$

$H^*(\mathbb{C}P^\infty;k) \cong k[x]$ $\quad |x| = 2$

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TOPOLOGY / HOMOTOPY THEORY $\longrightarrow$ COMMUTATIVE ALGEBRA

<u>STRATEGY</u> : use invariants / techniques from commutative algebra

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TOPOLOGY / HOMOTOPY THEORY $\longrightarrow$ COMMUTATIVE ALGEBRA

STRATEGY: use invariants / techniques from commutative algebra

(R, m, k) graded commutative local ring (e.g. $H^*(X;k)$ when k is a field)

Noetherian $\supset$ Cohen-Macaulay $\supset$ Gorenstein $\supset$ regular

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- $\mathfrak{p} \subset R$ prime ideal

  $ht(\mathfrak{p}) = \sup(t \mid \mathfrak{p} = \mathfrak{p}_0 \supsetneq \cdots \supsetneq \mathfrak{p}_t)$
  $\mathfrak{p}_i$ prime ideals

  $\dim(R) = \sup(ht(\mathfrak{p}) \mid \mathfrak{p} \subset R \text{ prime ideal})$

  $\operatorname{depth}(R) \leq \dim(R)$

- $\operatorname{depth}(R)$ = maximal length of a regular sequence $\{x_1, \ldots, x_r\}$ $x_i \in R$.

  $\operatorname{depth}(R) = \min\{i \mid \operatorname{Ext}^i_R(k, R) \neq 0\}$

Noetherian $\supset$ Cohen-Macaulay $\supset$ Gorenstein $\supset$ regular

(depth $R = \dim R$)

What is a Gorenstein ring? It is a ring with finite injective dimension.

$(R, \mathfrak{m}, k)$ local commutative ring, $d = \dim R$

<u>TFAE</u>: (a) R Gorenstein

(b) $\operatorname{injdim} R = d$

(c) R Cohen-Macaulay and $\operatorname{Ext}^d_R(k, R) \cong k$.

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- Another way,
  - if $\dim R = 0$, $\operatorname{Hom}_R(k, R) \cong k$ as k-v.s.
  - if $\dim R = d$, $\{x_1, \dots, x_d\}$ regular sequence then $R/(x_1, \dots, x_d)$ Gorenstein of dimension zero.

- What is a Gorenstein ring? Let's see basic examples

(0) R k-algebra which is finite dimensional as a k-v.s.

R Gorenstein iff R Poincaré duality algebra.

(e.g. $\Lambda(x) \cong H^*(S^n; k)$)
$|x| = n.$

(1) Polynomial algebras

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(1) Polynomial algebras

QUESTION : Can we detect these properties on $H^*(X;k)$ without explicit computations?

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Examples k field

(0) X finite CW-complex,

$H^\bullet(X;k)$ Noetherian, $\dim H^\bullet(X,k) = 0$, Cohen-Macaulay

$H^\bullet(X;k)$ Gorenstein iff $H^*(X;k)$ Poincaré duality algebra.

(1) $(\mathbb{C}P^\infty)^n$, $H^*((\mathbb{C}P^\infty)^n; k) = k[x_1, \dots, x_n]$ $|x_i| = 2$

Noetherian, $\dim H^\bullet((\mathbb{C}P^\infty)^n, k) = n$, Gorenstein.

(2) My favourite example BG, G finite group.

Let G be a finite group, BG is a topological space built out of the group structure of G.

- It is the classifying space for principal G-bundles
- It is EG/G where EG is a free G-space, contractible.
- $\pi_1(BG) \cong G$, $\pi_i(BG) = 0$ $i \neq 1$.

Eg. $\mathbb{R}P^\infty \simeq B\mathbb{Z}/2\mathbb{Z}$; $S^\infty/\mathbb{Z}/2\mathbb{Z}$

$k = \mathbb{F}_p$ Group $G \rightsquigarrow BG$ topological space $\rightsquigarrow H^*(BG; \mathbb{F}_p)$ 'algebra'

$$H^*(BG; k) \cong \underbrace{H^*(G; k)}_{\text{group cohomology}} = \operatorname{Ext}^*_{kG}(k, k)$$

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What do we know?

__Example__ : $H^*(B(\mathbb{Z}/p)^r; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[x_1,\dots,x_r] & |x_i|=1 \quad p=2 \\ \mathbb{F}_p[x_1,\dots,x_r] \otimes \Lambda(y_1,\dots,y_r) & p \text{ odd} \\ \quad |y_i|=1,\ |x_i|=2 & \end{cases}$

Noetherian, dim $= r$ (rank), Gorenstein.

But, we do not know explicit computations in general.

- $H^*(BG;k)$ is __Noetherian__ (ascent technique) [Venkov]

Let $G \xrightarrow[\rho]{} U(n)$ be a faithful unitary representation (e.g the regular representation).

Consider the induced continuous map $B\rho : BG \longrightarrow BU(n)$ with fiber homotopy equivalent to $U(n)/G$.

$$U(n)/G \longrightarrow BG \longrightarrow BU(n)$$

A Serre spectral sequence argument : H^*BG is a f.g. $H^*BU(n)$-module.

then H^*BG Noetherian

[__Rk__: $S \xrightarrow[f]{} R$ ring hom., S Noetherian, R fg S-module $\Rightarrow$ R Noetherian]

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What about $\dim H^*(BG)$? Descent technique [Quillen]

Let $E \leq G$, $\mathrm{res}_E : H^\bullet BG \longrightarrow H^\bullet BE$

$$H^* BG \xrightarrow[\rho]{} \lim_{\substack{\longleftarrow \\ E \leq G}} H^* BE \subset \prod H^\bullet BE$$

$E \cong (\mathbb{Z}/p)^r$ elementary abelian p-group

conjugacy relations.

Quillen showed that ρ is an F-isomorphism, that is, $\mathrm{Ker}\,\rho$, $\mathrm{Coker}\,\rho$ are nilpotent, and $\mathrm{Spec}^h(H^\bullet BG)$ is determined by $\mathrm{colim}\,\mathrm{Spec}^h(H^\bullet BE)$.

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<u>What about Cohen-Macaulay property?</u> The cohomology of semi-dihedral 2-groups is not Cohen-Macaulay.

But ...

THEOREM (Benson–Carlson '94) G finite group

H^*G Cohen–Macaulay iff H^*G Gorenstein.

To understand the proof, we need other characterizations of the Gorenstein property.

- **Local cohomology**: R commutative Noetherian ring, $\mathfrak{a} \subset R$ ideal, M R-module.

$$\Gamma_{\mathfrak{a}} M = \{ m \in M \mid \mathfrak{a}^t m = 0 \text{ for some } t \in \mathbb{N} \}$$

The functor $\Gamma_{\mathfrak{a}} : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$ is left exact. Then, the local cohomology of ${}^R M$ with support in $\mathfrak{a}$ are the i-th right derived functors

$$H^i_{\mathfrak{a}}(M)$$

$(R, \mathfrak{m}, k)$ Noetherian commutative local ring of dimension d

R Gorenstein iff R Cohen–Macaulay iff $H^i_{\mathfrak{m}}(R) \cong \begin{cases} 0 & i \neq d \\ E_R(k) & i = d \end{cases}$

$+$

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injective hull of k as R-module.

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THEOREM (Benson–Carlson '94) G finite group

H^*G Cohen–Macaulay iff H^*G Gorenstein.

- Benson–Carlson constructed a SS $H^*_m(H^*G) \Rightarrow H_*G = \mathrm{Hom}_{\mathbb{F}_p}(H^*G, \mathbb{F}_p)$
- If H^*G is Cohen–Macaulay then SS degenerates and

$$H^d_m(H^*G) \cong H_*G = \mathrm{Hom}_{\mathbb{F}_p}(H^*G, \mathbb{F}_p) \text{ injective hull}$$

Remark: $\left(H^d_m(H^*G)\right)^* \cong$ shift of H^*G by d.

THEOREM (Benson–Carlson '94) G finite group

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- Benson–Carlson constructed a SS $H^*_{\mathfrak{m}}(H^*G) \Rightarrow H_*G = \mathrm{Hom}_{\mathbb{F}_p}(H^*G, \mathbb{F}_p)$

QUESTIONS:

- What is special about BG? That is, under which hypothesis on X such a spectral sequence exists?
  (then H^*X Cohen–Macaulay iff H^*X Gorenstein)
- What about other primes? $H^*_{\mathfrak{p}}(H^*X)$?

$$
\begin{tikzcd}
\left.\begin{array}{l} X \text{ connected space} \\ k \text{ commutative ring} \end{array}\right\} \arrow[rr, squiggly, "\text{functor}"] \arrow[dr, squiggly] & & H^*(X;k) \text{ graded commutative ring} \\
& C^*(X;k) \arrow[ur, squiggly, "\pi_*"'] &
\end{tikzcd}
$$

(stable homotopy theory)

$C^*(X;k) = \mathrm{Map}(\Sigma^\infty_+ X, Hk)$ is a ring spectrum

Hk = Eilenberg MacLane spectrum

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k commutative ring

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$$\begin{tikzcd}
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\end{tikzcd}$$

$C^*(X;k)$ (stable homotopy theory)
$= \mathrm{Map}(\Sigma^\infty_+ X, Hk)$ is a ring spectrum

Hk = Eilenberg MacLane spectrum

We work in the context of stable homotopy category and more precisely ring spectra R.

There is a spectral version of $H^*_{\mathfrak{p}}$, $\Gamma_{\mathfrak{p}} : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$.

- Dwyer-Greenlees-Iyengar introduced spectral version of Gorenstein, and Gorenstein duality in spectra.

__Definition__ (BHV) A ring spectrum satisfies local Gorenstein duality of shift a if for every $\mathfrak{p} \in \mathrm{Spec}^h(\pi_* R)$ of dim d

$$\pi_* \Gamma_{\mathfrak{p}} R \cong \begin{cases} I_{\mathfrak{p}} & * = a+d \\ 0 & \text{otherwise} \end{cases}$$

where $I_{\mathfrak{p}}$ injective hull of $\pi_* R/\mathfrak{p}$.

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Proposition If R satisfies local Gorenstein duality there is a SS

$$H^*_{\mathfrak{p}}(\pi_* R)_{\mathfrak{p}} \Longrightarrow \Sigma^{a+d} I_{\mathfrak{p}}$$

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$$H_{\mathfrak{p}}^*(\pi_* R)_{\mathfrak{p}} \Rightarrow \Sigma^{a+d} I_{\mathfrak{p}}$$

Results (1) Ascent techniques along fibrations of spaces

$$F \longrightarrow Y \longrightarrow X$$

(2) Apply to obtain new examples.

- p-compact groups
- compact Lie groups under assumption (Greenlees)
- Hopf spaces with fg mod p cohomology
- p-local finite groups and fusion systems.

THANK YOU FOR
YOUR ATTENTION!