Investment Analysis and Risk Management

Boris A. Lodiagin

Higher Schools of Economics - St Petersburg

blodyagin@hse.ru; @agsmv

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About the Course

Two parts:

1. Investments Analysis (Introduction to Investments, Portfolio Optimization, CAPM)
2. Risk Management (VaR, Financial Risk Metrics, Investment Decision Making)

Grading: Final grade = Class Participation (30%) + Cases (30%) + Exam/Project (40%)

Country risk

	2001–2023			2001–2008			2008–2010		
Industry	Mean (%)	StDev (%)	Sharpe	Mean (%)	StDev (%)	Sharpe (%)	Mean (%)	StDev (%)	Sharpe
Developed:									
World	0.3	4.5	0.07	0.0	3.9	−0.01	0.1	7.5	0.01
G7	0.3	4.5	0.07	−0.1	3.8	−0.03	0.1	7.2	0.01
US	0.6	4.5	0.13	0.0	3.9	−0.01	0.4	6.9	0.05
UK	0.3	4.9	0.07	0.3	3.9	0.06	0.2	8.1	0.03
Japan	0.3	4.5	0.06	0.0	4.6	0.00	0.2	6.3	0.03
Italy	0.3	7.0	0.05	0.3	5.1	0.06	−0.6	11.0	−0.05
France	0.5	6.1	0.08	0.3	5.3	0.07	0.0	10.0	0.00
Germany	0.5	6.8	0.07	0.6	6.7	0.08	0.3	10.3	0.03
Emerging:									
Emerging index	0.2	6.17	0.03	0.2	5.4	0.03	−0.4	10.3	−0.04
Chile	0.6	6.9	0.09	1.3	5.9	0.22	2.4	8.0	0.30
Brazil	1.1	10.3	0.11	2.4	10.8	0.22	1.3	11.8	0.11
Colombia	1.4	9.0	0.16	3.5	8.9	0.39	2.9	10.3	0.28
Mexico	0.9	6.8	0.13	1.6	6.1	0.26	1.3	10.3	0.13
Peru	1.4	8.2	0.17	2.7	7.8	0.35	3.2	13.0	0.24
China	0.7	7.5	0.10	1.4	8.3	0.16	1.1	9.2	0.12
UAE	0.3	9.0	0.04	-0.2	11.8	−0.02	−1.7	13.7	−0.13
South Africa	0.8	7.4	0.10	1.4	7.3	0.20	2.0	10.3	0.19
India	1.0	7.7	0.14	1.6	7.9	0.21	1.9	12.4	0.16
Malaysia	0.4	4.9	0.08	0.8	5.2	0.16	2.0	6.4	0.31
Saudi Arabia	0.5	6.1	0.08	–	–	–	–	–	–
Taiwan	0.8	7.0	0.12	0.6	7.9	0.08	1.6	10.1	0.16
Turkey	0.9	12.3	0.07	2.0	15.5	0.13	1.8	13.6	0.13

Country risk

Industry	Mean % 2010–2020	StDev % 2010–2020	Sharpe 2010–2020	Mean % 2020–2023	StDev % 2020–2023	Sharpe 2020–2023
Developed:						
World	0.5	3.6	0.15	0.9	5.6	0.16
G7	0.6	3.5	0.17	1.0	5.7	0.17
US	1.0	3.5	0.29	1.2	5.9	0.20
UK	0.3	4.3	0.08	0.7	5.8	0.12
Japan	0.5	3.8	0.12	0.5	4.8	0.11
Italy	0.4	6.6	0.06	1.1	8.3	0.13
France	0.6	5.0	0.12	1.1	7.1	0.16
Germany	0.5	5.5	0.09	0.7	7.4	0.10

Industry	Mean % 2010–2020	StDev % 2010–2020	Sharpe 2010–2020	Mean % 2020–2023	StDev % 2020–2023	Sharpe 2020–2023
Emerging:						
Emerging index	0.2	5.1	0.05	0.9	6.9	0.13
Chile	–0.5	6.3	–0.08	1.1	9.2	0.12
Brazil	0.1	9.0	0.01	0.8	11.2	0.07
Colombia	–0.1	7.0	–0.02	0.0	12.2	0.00
Mexico	0.0	5.6	–0.01	1.5	8.4	0.18
Peru	0.1	6.2	0.02	0.7	9.6	0.08
China	0.5	5.9	0.09	-0.4	8.4	–0.05
UAE	0.8	6.7	0.12	1.0	6.9	0.15
South Africa	0.1	6.4	0.02	0.6	8.3	0.07
India	0.3	6.1	0.05	1.2	6.9	0.18
Malaysia	0.0	4.2	–0.01	-0.3	4.9	–0.05
Saudi Arabia	0.0	6.2	0.01	1.2	5.9	0.21
Taiwan	0.7	4.6	0.16	1.4	7.9	0.18
Turkey	–0.4	9.0	–0.04	0.9	10.8	0.08

Country risk

Exhibit 1: Risk and Return for US Asset Classes by Decade (%)

		1930s	1940s	1950s	1960s	1970s	1980s	1990s	2000s	2010s*	1926–2017
Large company stocks	Return	–0.1	9.2	19.4	7.8	5.9	17.6	18.2	–1.0	13.9	10.2
	Risk	41.6	17.5	14.1	13.1	17.2	19.4	15.9	16.3	13.6	19.8
Small company stocks	Return	1.4	20.7	16.9	15.5	11.5	15.8	15.1	6.3	14.8	12.1
	Risk	78.6	34.5	14.4	21.5	30.8	22.5	20.2	26.1	19.4	31.7
Long-term corporate bonds	Return	6.9	2.7	1	1.7	6.2	13	8.4	7.7	8.3	6.1
	Risk	5.3	1.8	4.4	4.9	8.7	14.1	6.9	11.7	8.8	8.3
Long-term government bonds	Return	4.9	3.2	–0.1	1.4	5.5	12.6	8.8	7.7	6.8	5.5
	Risk	5.3	2.8	4.6	6	8.7	16	8.9	12.4	10.8	9.9
Treasury bills	Return	0.6	0.4	1.9	3.9	6.3	8.9	4.9	2.8	0.2	3.4
	Risk	0.2	0.1	0.2	0.4	0.6	0.9	0.4	0.6	0.1	3.1
Inflation	Return	–2.0	5.4	2.2	2.5	7.4	5.1	2.9	2.5	1.7	2.9
	Risk	2.5	3.1	1.2	0.7	1.2	1.3	0.7	1.6	1.1	4.0

Figure: Risk and Return for US Asset Classes by Decade (%)

Markowitz Portfolio Theory

1. Combining stocks into portfolios can reduce standard deviation, below the level obtained from a simple weighted average calculation
2. Correlation coefficients make this possible
3. The various weighted combinations of stocks that create this standard deviations constitute the set of **efficient portfolios**

Recap

- **Expected Return:**
  $E(R) = \sum_{i=1}^{n} p_i \cdot R_i$
  Where:
  - p_i *is the probability of state* i.
  - R_i *is the return in that state.*
- **Variance:**
  $\sigma^2 = Var(X) = \frac{\sum_{i=1}^{n}(R_i - E(R))^2}{n-1} = \frac{\sum_{i=1}^{n} R_i^2 - n\cdot(E(R))^2}{n-1}$
- **Standard Deviation:**
  $\sigma = \sqrt{Var(X)} = \sqrt{\sigma^2} = [\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij}]^{\frac{1}{2}} = \sqrt{E[(\tilde{R}_p - E[\tilde{R}_p])^2]}$
- **Covariance:**
  $Cov(X, Y) = E[(X - E[X])(Y - E[Y])] = \frac{\sum_{i=1}^{n}(X_i - E[X])(Y_i - E[Y])}{n-1}$
  Where:
  - R_i *is the return of the asset in state* i.
  - $E(R)$ *is the expected return.*
  - X *and* Y *are two random variables.*

Covariance

When combining several assets into a portfolio we have to consider the comovements between assets

- A portfolio is simply a combination of several single assets
- In a portfolio it is not enough to know the mean and variances of the single assets
- The assets move together, thus, total portfolio variance is not simply a sum of the single variances

- Covariance is a measure how random variables (assets) move together

$$\sigma_{i,j} = E\left[\left(\tilde{r}_i - E\left[\tilde{r}_i\right]\right)\left(\tilde{r}_j - E\left[\tilde{r}_j\right]\right)\right]$$

- Possible values:
  - positive: variables move together in same direction
  - zero: no relationship
  - negative: variables move opposite to each other

Efficient Frontier

Three Efficient Portfolios—Percentages Allocated to Each Stock (%)					
	Expected Return (%)	Standard Deviation (%)	A	B	T
United States Steel	6.0	76.4		0	0
Tesla	6.5	48.1		1	5
Newmont	5.0	36.7		7	9
Southwest Airlines	10.0	30.5	100	0	17
Amazon	8.0	28.3		1	10
Wells Fargo	6.8	21.6		21	23
ExxonMobil	5.3	19.4		0	0
Consolidated Edison	5.5	16.5		20	33
Johnson & Johnson	4.4	14.4		7	0
Coca-Cola	4.8	12.6		43	5
Expected portfolio return			10.0	5.4	6.8
Portfolio standard deviation			30.5	10.5	12.5

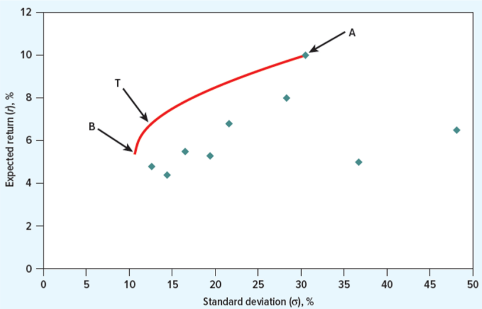
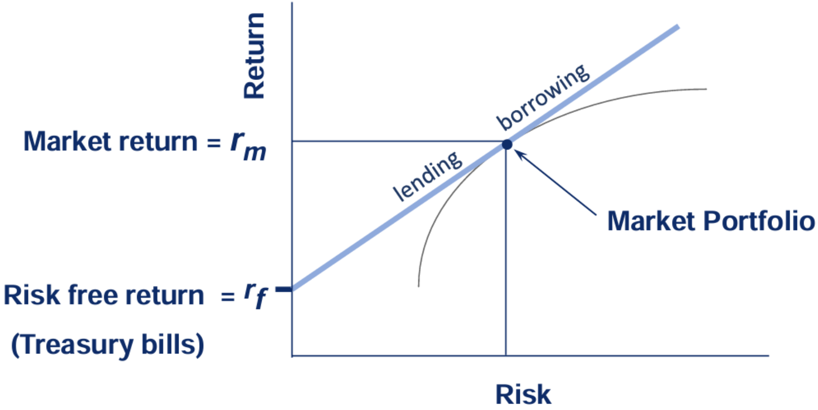


Figure: Three efficient portfolios all from the same 10 stocks.

Capital Market Line

Lending or borrowing at the risk free rate (R_F) allows us to exist outside the efficient frontier.



Capital Market Line

Portfolio of risky asset and risk-free asset

$$\text{Expected portfolio return} = (x_r r_r) + (x_f r_f)$$

$$\text{Portfolio variance} = x_r^2 \sigma_r^2 + x_f^2 \sigma_f^2 + 2(x_r x_f \rho_{rf} \sigma_r \sigma_f)$$

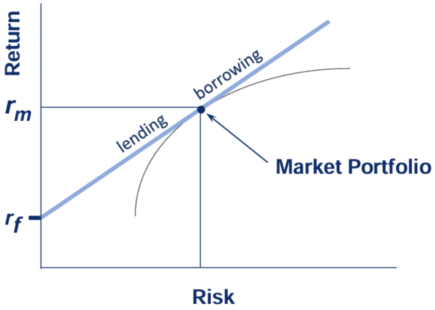
$$\sigma_f = 0 \Rightarrow \begin{array}{l} \text{Portfolio variance} = x_r^2 \sigma_r^2 \\ \text{Portfolio st. deviation} = x_r \sigma_r \end{array}$$

Capital Market Line

Lending or borrowing at the risk free rate (R_F) allows us to exist outside the efficient frontier.

	Return	St. Deviation
Efficient portfolio in tangency point	15%	16%
T-bills	5%	0%

	Fraction of risky assets in portfolio	Return	St. Deviation
A	0%	5%	0%
B	50%	10%	8%
C	100%	15%	16%
D	200% (borrowing at risk-free rate)	25% = (2*r-r$_f$)	32% = (2*σ)



Capital Market Line

The ratio of the risk premium to the standard deviation is called the Sharpe ratio. It characterizes the slope of CML:

$$\text{Sharpe ratio} = \frac{Risk\ premium}{St.\ deviation} = \frac{R_P - R_F}{\sigma_P}$$

Equation for the line itself:

$$\mathrm{R}_P = R_F + \frac{R_M - R_F}{\sigma_M}\sigma_P$$

We can see, that for all risk-efficient portfolios the Sharpe ratio will be the same.

Capital Market Line

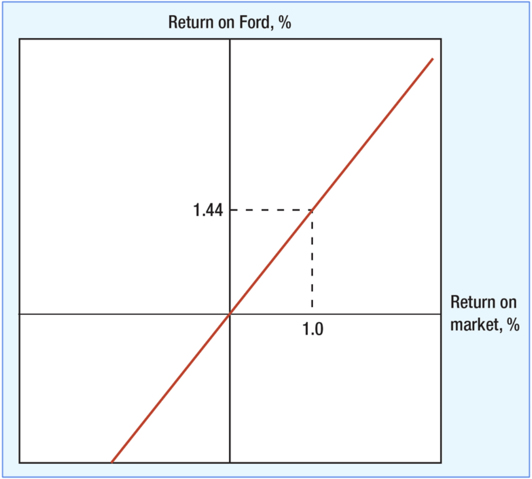
1. All rationale investors should have the same stock portfolio.
2. Investors should differ in how much of that portfolio they hold.
3. Depending on their risk preferences, they either lend (buy T-bills) or borrow money to buy more of the same portfolio

Portfolio Risk and Beta

1. **Market Portfolio** - portfolio of all assets in the economy.
2. The risk that a stock contributes to a well-diversified portfolio is its market risk.
3. **Beta** - sensitivity of a stock's return to the return of the market portfolio (movements of the overall market).

Portfolio Risk and Beta

The return on Ford stock changes on average by 1.44% for each additional 1% change in the market return. Beta is therefore 1.44.



Portfolio Risk and Beta

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$$

Covariance with the market

Variance of the market

- $\beta = 1$: Stock moves with the market.
- $\beta > 1$: Stock is more volatile than the market.
- $\beta < 1$: Stock is less volatile than the market.
- $\beta_i = \frac{\rho_{i,m}\sigma_i\sigma_m}{\sigma_m^2} = \frac{\rho_{i,m}\sigma_i}{\sigma_m}$

Exercise 1. Portfolio Risk and Beta

Calculation of Beta

Assuming that the risk (standard deviation) of the market is 25%, calculate the beta for the following assets:

1. A short-term US Treasury bill.
2. Gold, which has a standard deviation equal to the standard deviation of the market.
3. A new emerging market that is not currently included in the definition of "market"—the emerging market's standard deviation is 60%, and the correlation with the market is –0.1.
4. An initial public offering or new issue of stock with a standard deviation of 40% and a correlation with the market of 0.7.

Exercise 1. Portfolio Risk and Beta

Calculation of Beta

1. A short-term US Treasury bill.
   By definition, a short-term US Treasury bill has zero risk. Therefore, its beta is zero.
2. Gold, which has a standard deviation equal to the standard deviation of the market.
   Gold has a zero correlation with the market. Its beta is zero.
3. A new emerging market that is not currently included in the definition of "market"—the emerging market's standard deviation is 60%, and the correlation with the market is –0.1.
   Beta of the emerging market is $\frac{-0.1\times0.60}{0.25} = -0.24$.
4. An initial public offering or new issue of stock with a standard deviation of 40% and a correlation with the market of 0.7.
   IPOs are usually very risky but have a relatively low correlation with the market. Beta of the initial public offering is $\frac{0.7\times0.40}{0.25} = 1.12$.

Security Market Line

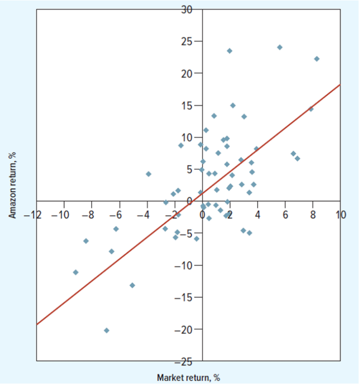
Risk metrics:

$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$

$\sigma_i = \rho_{iM}\sigma_i + (1 - \rho_{iM})\sigma_i$

$Total\ risk = Market\ risk + Specific\ risk$

$$Market\ risk = \rho_{iM}\sigma_i = \frac{\sigma_{iM}}{\sigma_i\sigma_M}\sigma_i$$
$$= \frac{\sigma_{iM}}{\sigma_M^2}\sigma_M = \beta_i\sigma_M$$

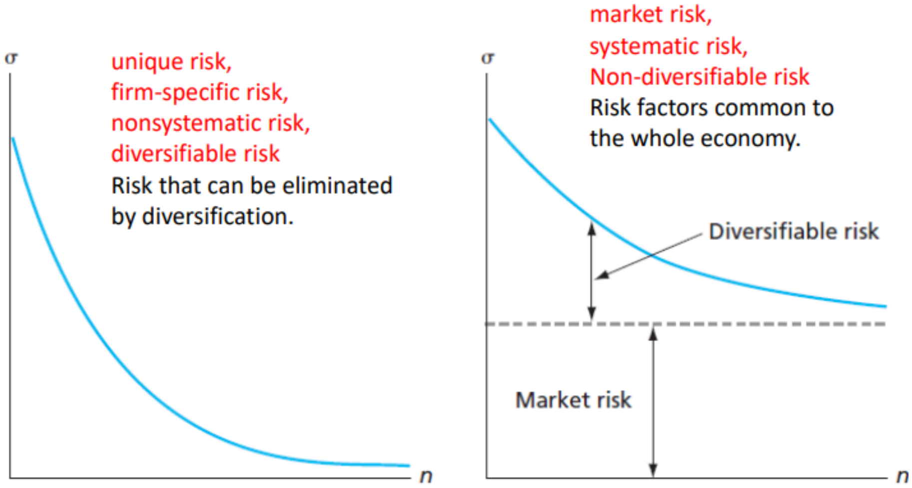


Portfolio Risk and Beta

Calculating the variance of the market returns and the covariance between the returns on the market and those of Anchovy Queen. Beta is the ratio of the variance to the covariance (i.e., $\beta = \sigma_{im} / \sigma^2_m$)

1	(1)	(2)	(3)	(4)	(5)	(6)	(7)
2							Product of
3				Deviation	Deviation	Squared	deviations
4				from	from average	deviation	from average
5		Market	Anchovy Q	average	Anchovy Q	from average	returns
6	Month	return	return	market return	return	market return	(cols 4 × 5)
7	1	−8%	−11%	−10	−13	100	130
8	2	4	8	2	6	4	12
9	3	12	19	10	17	100	170
10	4	−6	−13	−8	−15	64	120
11	5	2	3	0	1	0	0
12	6	8	6	6	4	36	24
13	Average	2	2		Total	304	456
14				Variance $= \sigma_m^2 = 304/6 = 50.67$			
15				Covariance $= \sigma_{im} = 456/6 = 76$			
16				Beta $(\beta) = \sigma_{im}/\sigma_m^2 = 76/50.67 = 1.5$			

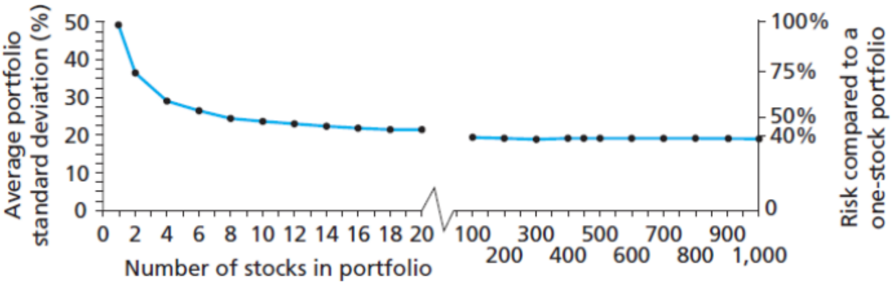
Diversification and portfolio risk



A: Firm-specific risk only

B: Market and unique risk

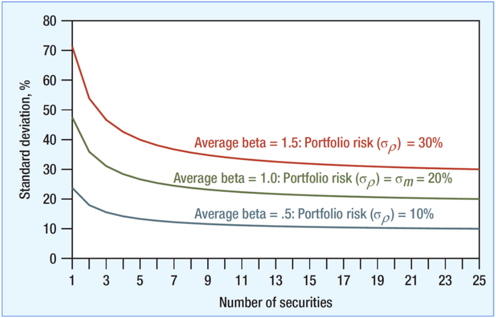
The effect of portfolio diversification



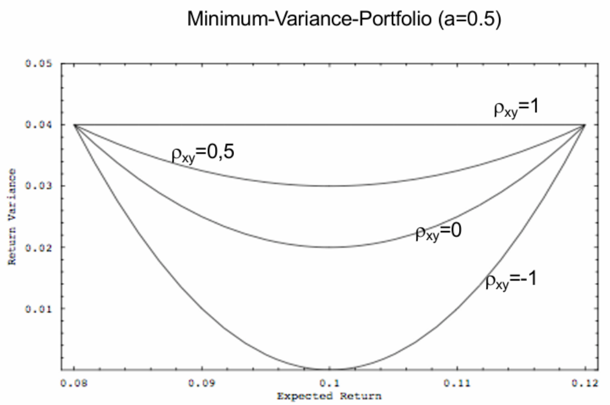
On average, portfolio risk does fall with diversification, but the power of diversification to reduce risk is limited by common sources of risk.

Security Market Line

The total risk of a portfolio is lower than the average of the total risks of the individual stocks. That's because some of the total risk of an individual stock is diversifiable and so goes away when you add it to a portfolio. But since beta measures undiversifiable risk, there's no diversification effect when adding a stock to a portfolio. So **the beta of a portfolio** is simply **the weighted-average beta** of the stocks in the portfolio.

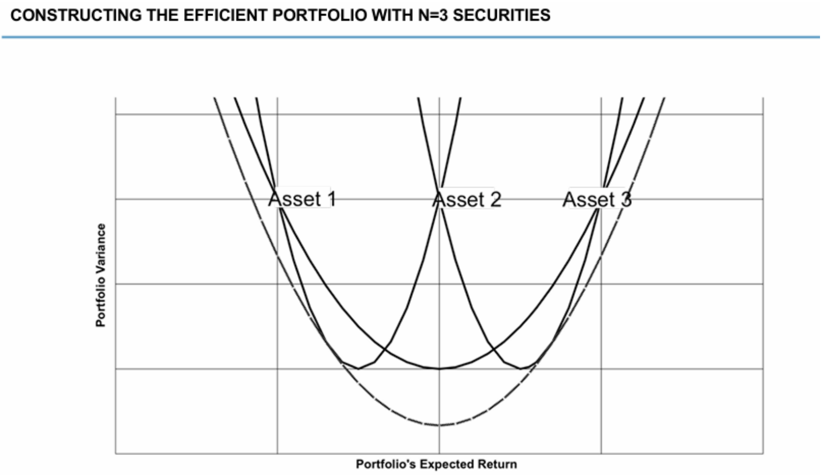


Diversification and portfolio risk



→ Depending on the correlation different μ/σ-combinations are possible

Diversification and portfolio risk



Diversification and portfolio risk

DIVERSIFICATION: Volatility of an equally weighted portfolio

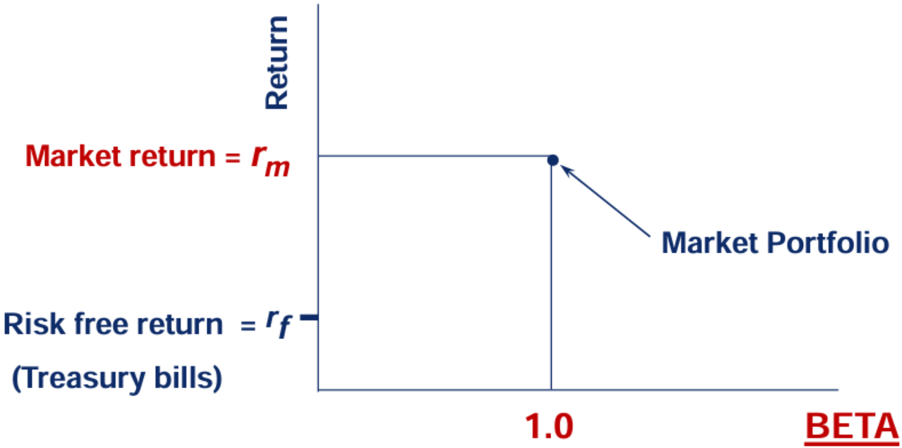
Average monthly tracking error of randomly selected portfolios of listed US-stocks over the period 1960-2001

Source: www.aaii.com

Table 1. Average Tracking Error for Different-Sized Portfolios (1960–2001)

No. of Stocks in Portfolio	Monthly Tracking Error of Portfolio Relative to: Value-Weighted Index (%)	Equal-Weighted Index (%)
1	5.49	9.23
3	3.34	5.84
5	2.61	4.66
7	2.22	4.01
10	1.88	3.4
15	1.54	2.8
20	1.34	2.44
30	1.10	2.02
45	0.90	1.65
65	0.75	1.39
100	0.60	1.13

Security Market Line

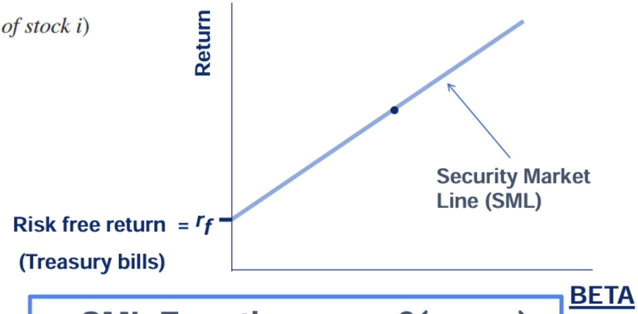


Security Market Line

$$r_i = r_f + \frac{r_M - r_f}{\sigma_M}(\textit{market risk of stock } i)$$

$$r_i = r_f + \frac{r_M - r_f}{\sigma_M}(\beta_i \sigma_M)$$

$$r_i = r_f + \beta_i (r_M - r_f)$$



SML Equation = $r_f + \beta(r_m - r_f)$

In a competitive market, the expected risk premium on any security or portfolio—not just efficient portfolios—is its beta multiplied by the market risk premium $(r_m - r_f)$.

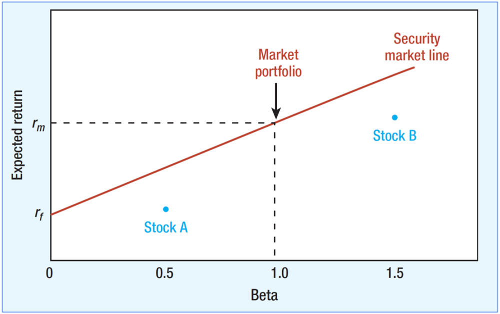
This means that all investments must plot along SML.

$r_i - r_f = \beta(r_m - r_f) =>$

$r_i = r_f + \beta(r_m - r_f)$

CAPM

$r_i = r_f + \beta(r_m - r_f)$



The Capital Asset Pricing Model

These estimates of the returns expected by investors in February 2020 are based on the capital asset pricing model. We assume 2% for the interest rate r_f and 7% for the expected risk premium $r_M - r_f$.

Stock	Beta (β)	Expected Return
United States Steel	2.98	22.9
Southwest Airlines	1.58	13.0
Amazon	1.55	12.8
Wells Fargo	1.14	10.0
ExxonMobil	1.14	10.0
Johnson & Johnson	0.75	7.3
Tesla	0.50	5.5
Coca-Cola	0.46	5.2
Consolidated Edison	0.31	4.1
Newmont	0.16	3.1

Principles of Portfolio Selection

1. Common stock portfolios that offer the highest expected return for a given st. dev. Are known as **efficient portfolios**.
2. If investor can lend and borrow money at the risk-free rate one efficient portfolio would be better than all other. Depending on his/her risk appetite investor can combine it with risk-free investments or borrow money for additional risky investment.
3. Contribution of the stock to the portfolio depends on its sensitivity to the changes in the value of portfolio
4. **Beta** measures marginal contribution of a stock to the risk of the market portfolio.
5. Risk premium demanded by investors is proportional to the **beta**.

The Capital Asset Pricing Model

The CAPM provides a precise prediction of the relationship we should observe between the risk of an asset and its expected return.

1. Provides a benchmark rate of return for evaluating possible investments.
2. The model helps us make an educated guess as to the expected return on assets that have not yet been traded in the marketplace.

The index model describes an empirical relationship between the return on an individual stock, R_i , and that of a broad market-index portfolio, R_M:

$$E(R_i) - R_F = \beta_i[E(R_M) - R_F] + \alpha_i + \epsilon_i$$

$E(R_M)$ - expected market return; α_i - the expected firm-specific return - Hunt for positive-alpha stocks; ϵ_i - "noise", or firm-specific risk; $E(R_M) - R_F$ - excess market return; $E(R_i) - R_F$ - excess individual stock return

The Capital Asset Pricing Model

Therefore, the expected excess return on a stock, given the market's excess return RM, is:

$$E(R_i - R_F | R_M - R_F) = \beta_i (R_M - R_F) + \alpha_i$$

1. if a stock has a positive beta, it will **"inherit"** some of the **market's risk premium** *(positive beta also means that the stock is* **exposed to systematic risk***)*
2. higher alphas would imply higher expected returns **without a corresponding increase in risk**

The Model: Assumptions and Implications

The conditions that lead to the CAPM ensure competitive security markets and investors who choose from identical efficient portfolios using the mean-variance criterion:

1. Markets for securities are perfectly competitive and equally profitable to all investors.
    1. No investor is sufficiently wealthy that his or her actions alone can affect market prices.
    2. All information relevant to security analysis is publicly available at no cost.
    3. All securities are publicly owned and traded, and investors may trade all of them. Thus, all risky assets are in the investment universe.
    4. There are no taxes on investment returns. Thus, all investors realize identical returns from securities.
    5. Investors confront no transaction costs that inhibit their trading.
    6. Lending and borrowing at a common risk-free rate are unlimited.

The Model: Assumptions and Implications

The conditions that lead to the CAPM ensure competitive security markets and investors who choose from identical efficient portfolios using the mean-variance criterion:

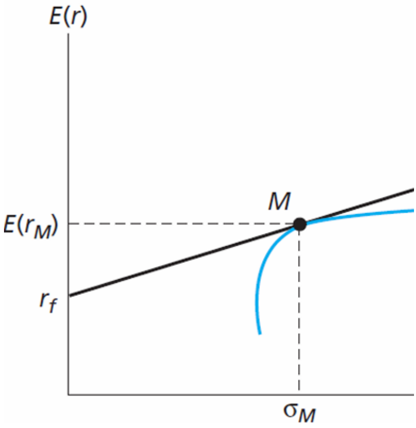
2. Investors are alike in every way except for initial wealth and risk aversion; hence, they all choose investment portfolios in the same manner.
    1. Investors plan for the same (single-period) horizon.
    2. Investors are rational, mean-variance optimizers.
    3. Investors are efficient users of analytical methods, and by assumption *1.B* they have access to all relevant information. Hence, they use the same inputs and consider identical portfolio opportunity sets. This assumption is often called *homogeneous expectations*.

The Model: Assumptions and Implications

Given these assumptions, we summarize the equilibrium that will prevail in this hypothetical world of securities and investors

1. All investors will choose to hold the market portfolio (M), which includes all assets of the security universe. For simplicity, we shall refer to all assets as stocks. The proportion of each stock in the market portfolio equals the market value of the stock divided by the total market value of all stocks.
2. The market portfolio will be on the efficient frontier. Moreover, it will be the optimal risky portfolio, the tangency point of the capital allocation line (CAL) to the efficient frontier.
3. The risk premium on the market portfolio will be proportional to the variance of the market portfolio and investors' typical degree of risk aversion: $E(R_M) - R_F = A\sigma^2 M$, where σ^2 is the standard deviation of the return on the market portfolio and A represents the degree of risk aversion of the average investor.
4. The risk premium on individual assets will be proportional to the risk premium on the market portfolio (M) and to the beta coefficient of the security on the market portfolio.

The Efficient Frontier and the Capital Market Line



1. The CAPM implies that a passive strategy, using the CML as the optimal CAL, is a powerful alternative to an active strategy.
2. A passive investor receives a "free ride" by simply investing in the market portfolio and benefiting from the efficiency of that portfolio.
3. Mutual fund theorem: States that all investors desire the same portfolio of risky assets and can be satisfied by a single mutual fund composed of that portfolio.

Exercise 1. CAPM Model

Market Risk, the Risk Premium, and Risk Aversion

Suppose the risk-free rate is 5%, the average investor has a risk-aversion coefficient of $A = 2$, and the standard deviation of the market portfolio is 20%.

1. Please, calculate the expected rate of return on the market.
2. If investors were more risk averse, what risk premium they would prefer to hold shares?

Exercise 1. CAPM Model

Market Risk, the Risk Premium, and Risk Aversion

We estimate the equilibrium value of the market risk premium as $2 \times .20^2 = .8$. So the expected rate of return on the market must be:

$$\mathrm{E(R}_M) = R_F + \textit{Equilibrium risk premium}$$
$$= .05 + .08 = .13 = 13\%$$

If investors more risk averse, it would take higher risk premium to induce them to hold shares. For example, if the average degree of risk aversion were 3, the market risk premium would be $3 \times .20^2 = .12$, or 12%, and the expected return would be 17%.

Exercise 2. CAPM Model

Expected returns and Risk Premiums

Suppose the risk premium of the market portfolio is 9%, and we estimate the beta of Dell as $\beta_D = 1.3$. T-bill rate is 5%.

1. Please, calculate the expected rate of return of Dell stocks.
2. If the beta of Dell were only 1.2, what would be the required risk premium for Dell?
3. If the market risk premium were 8% and $\beta_D = 1.3$, what would be the Dell's risk premium?

Exercise 2. CAPM Model

Market Risk, the Risk Premium, and Risk Aversion

The risk premium for the stock is therefore 1.3 times the market risk premium , or $1.3 \times 9\% = 11.7\%$. The expected rate of return on Dell is the risk-free rate plus the risk premium. If T-Bill rate were 5%, the expected rate of return would be $5\% + 11.7\% = 16.7\%$, or:

$$\mathrm{E}(\mathrm{R}_D) = R_F + \beta_D[Market\ risk\ premium]$$
$$= 5\% + 1.3 \times 9\% = 16.7\%$$

If the estimate of the beta of Dell were only 1.2, the required risk premium for Dell would fall to 10.8%. Similarly, if the market risk premium were only 8% and $\beta_D = 1.3$, Dell's risk premium would be only 10.4%.

Exercise 3. CAPM Model

Market Risk, the Risk Premium, and Risk Aversion

If the beta of Exxon stock is 0.65, the risk-free rate is 4%, and the expected return of the entire market is 14% per annum, estimate the expected return of Exxon stock.

Exercise 4. CAPM Model

Market Risk, the Risk Premium, and Risk Aversion

The portfolio consists of shares A, B and C, whose beta coefficients are equal to 0,5; 1,0 and 1,2, respectively.

Money invested in A equals 5 mn RUB, B - 6 mn, C - 9 mn. **What is the Beta of the portfolio?**

Exercise 5. CAPM Model

Market Risk, the Risk Premium, and Risk Aversion

The correlation of the returns between Asset X and the market portfolio is 0,8.

The standard deviation of returns of Asset X is 35%, and the standard deviation of the returns of market portfolio is 20%.

Calculate Beta for Asset X

Exercise 6. FF3F Model

Market Risk, the Risk Premium, and Risk Aversion

The following data is known for a stock: Market factor beta is 1.4; Size factor beta is 0.4; Value factor beta is -1.1.

The market risk premium is 7%, the size premium is 3.7%, the value premium is 5.2%. The risk-free rate is 5%.

Calculate the expected return of the stock according to the Fama-French three-factor model.

Exercise 8. WACC

Market Risk, the Risk Premium, and Risk Aversion

TechGrowth Inc. has a market capitalization of \$80 million, and \$40 million in outstanding debt. TechGrowth's equity cost of capital is 12%, and its debt cost of capital is 5%. If its corporate tax rate is 20%, **what is TechGrowth's weighted average cost of capital?**

Exercise 8. WACC

Market Risk, the Risk Premium, and Risk Aversion

TechGrowth Inc. is considering a major expansion project and needs to calculate its weighted average cost of capital (WACC). The company has 2 million common shares outstanding, with a current market price of $50 per share. It also has debt with a market value of $40 million and a cost of 6% per year. The risk-free rate is 2.5%, and the market risk premium is estimated to be 7%. The company's equity beta is 1.4, and its corporate tax rate is 20%. **Calculate the WACC for TechGrowth Inc.'s expansion project.**

CAPM Application

CAPM can deliver a value for the correct discount rate for risky cashflows

- The CAPM tells us the required rate of return for a risky asset, depending on its beta risk
- An asset delivers (ultimately) cash flows
- Thus, the CAPM can be used to find a correct discount rate for risky cash flows if we know the beta of this asset
- As cash flows are typically risky, we have to use risk-adjusted discount rates
- Companies will usually be financed by equity as well as bonds, then we have to use the so called WACC (Weighted Average Capital Costs) for discounting
  - WACC = weighted average of equity costs and debt costs
- The CAPM can tell us something about the equity portion

- However: Empirical tests have to tell us whether such a use of the model CAPM is warranted

CAPM Application

CAPM can be used for performance measurment

- By using the CAPM the risk-adjusted outperformance (alpha) can be measured
- For that purpose we write the CAPM as:

$$\tilde{r}_{it} - r_{ft} = \alpha_i + \left(\tilde{r}_{Mt} - r_{ft}\right)\beta_i + \tilde{\varepsilon}_{it}$$

- α can be estimated from a regression approach
- It is called Jensen 's Alpha and it gives the risk-adjusted outperformance

Empirical tests of the CAPM

- For empirical applications the CAPM is written in the following form:

$$
\begin{aligned}
&\tilde{r}_{it} = r_{ft} + \left(\tilde{r}_{Mt} - r_{ft}\right)\beta_i + \varepsilon_{it} \\
&E\left[\varepsilon_{it}\right] = 0 \\
&\operatorname{cov}\left[\varepsilon_{it}, r_{mt}\right] = 0 \\
&\operatorname{cov}\left[\varepsilon_{it}, \varepsilon_{it-1}\right] = 0 \\
&\operatorname{cov}\left[\varepsilon_{it}, \varepsilon_{j\neq i,t}\right] = 0 \\
&\operatorname{cov}\left[\varepsilon_{it}, \varepsilon_{it}\right] = \sigma_{\varepsilon}^2
\end{aligned}
$$

- Here, returns are assumed to be independent and stationary (iid-assumption)

Empirical tests of the CAPM

"All interesting models involve unrealistic simplifications, which is why they must be tested against data" (Fama and French, 2004, p. 30).

Empirical tests of the CAPM

- For empirical testing the CAPM in its ex-post form is usually written as follows:

$$\tilde{r}_{it} = \alpha_i + \tilde{r}_{Mt}\beta_i + \varepsilon_{it}$$

This corresponds to the market model, if instead of the market portfolio a market index is used.

- Two basic assumption have been introduced, here:
  - Stationarity (distribution parameters are stable over time)
  - Returns are statistically independent (no autocorrelation)

Performing a CAPM-test

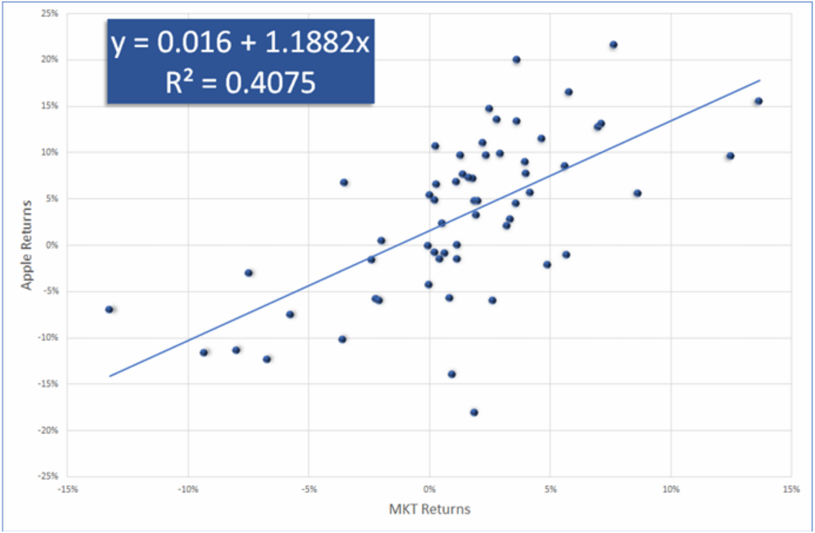
1) Select a data sample (set of stocks, time period, return periods)
2) Define a proxy for the market portfolio (stock index) and estimate on that basis the market model by carrying out an OLS-estimation (according to our assumption the OLS-estimation is efficient and unbiased)
3) Test the following hypothesis:

$$\tilde{r}_{it} = \gamma_0 + \gamma_1 \beta_i + \varepsilon_{it}$$

Fama and MacBeth (1973): Beta estimation

- As a first step, you need an estimate of beta.
  - Betas can be time-varying by using rolling window time-series regression until $t-1$. In this case, $\boldsymbol{\beta}_i$ becomes $\boldsymbol{\beta}_{i,t-1}$. However, this estimate is usually too noisy.
  - Alternatively, the betas can be estimated in a portfolio approach.
    - Fama and Macbeth (1973) first estimate the pre-ranking betas. Then, they assign the firms based on their pre-ranking betas into 20 portfolios. Finally, they estimate the betas for each of the portfolios and use it as a proxy for the beta from each firm within the portfolio.

Estimation of the market beta of Apple with monthly returns from 1/2016-12/2020



Fama and MacBeth (1973): regression

- Two-Step Regression
  - Step 1: Time-Series Regression to determine factor exposures
  - Step 2: Cross-Sectional Regression at each point in time t

    Time-Series Average of the output variables

Dependent Variable | Independent Variable 1 | Independent Variable 2 | Independent Variable j

$$t = 1, \quad R^e_{i,1} = \gamma_{0,1} + \beta_{i,1}\gamma_{1,1} + \beta_{i,2}\gamma_{2,1} + \cdots + \beta_{i,j}\gamma_{j,1} + \mu_{i,1}$$

$$t = 2, \quad R^e_{i,2} = \gamma_{0,2} + \beta_{i,1}\gamma_{1,2} + \beta_{i,2}\gamma_{2,2} + \cdots + \beta_{i,j}\gamma_{j,2} + \mu_{i,2}$$

$$\cdots$$

$$t = T, \quad R^e_{i,3} = \gamma_{0,T} + \beta_{i,1}\gamma_{1,T} + \beta_{i,2}\gamma_{2,T} + \cdots + \beta_{i,j}\gamma_{j,T} + \mu_{i,T}$$

Time-Series Average $\qquad \bar{\gamma}_0 \qquad \bar{\gamma}_1 \qquad \bar{\gamma}_2 \qquad \bar{\gamma}_j$ factor risk premium

$$\bar{\gamma}_j = \frac{1}{T}\sum_{t=1}^{T}\gamma_{j,t} \qquad t\text{–}stat(\bar{\gamma}_j) = \frac{\bar{\gamma}_j}{\sigma(\hat{\gamma}_j)/\sqrt{n}} \sim t$$

$$R^e_{i,t} = \gamma_{0,t} + \boldsymbol{\beta}_i{}'\boldsymbol{\gamma}_t + \mu_{i,t},$$ for each t

Fama and MacBeth (1973): results

SUMMARY RESULTS FOR THE REGRESSION

$$R_p = \hat{\gamma}_{0t} + \hat{\gamma}_{1t}\hat{\beta}_p + \hat{\gamma}_{2t}\hat{\beta}^2{}_p + \hat{\gamma}_{3t}\bar{s}_p(\hat{\epsilon}_i) + \hat{\eta}_{pt}$$

PERIOD	$\bar{\hat{\gamma}}_0$	$\bar{\hat{\gamma}}_1$	$\bar{\hat{\gamma}}_2$	$\bar{\hat{\gamma}}_3$	$t(\bar{\hat{\gamma}}_0)$	$t(\bar{\hat{\gamma}}_1)$	$t(\bar{\hat{\gamma}}_2)$	$t(\bar{\hat{\gamma}}_3)$
Panel A:								
1935–6/68 ..	.0061	.0085	...	...	3.24	2.57	...	...
Panel D:								
1935–6/68 ..	.0020	.0114	—.0026	.0516	.55	1.85	—.86	1.11

- The results from $\hat{\gamma}_1$ show that to some extent, the market beta has a linear relation with returns.
- The results from $\hat{\gamma}_2$ and $\hat{\gamma}_3$ show that other predictors used in this study cannot predict the cross-section of stocks returns.
- The results from Fama and Macbeth (1973) do not invalidate the CAPM.

Fama and MacBeth (1973): regression

- Fama-Macbeth regression can be used to examine the cross-sectional relation between a dependent variable and one or more independent variables in the average period.
- A statistically significant average slope coefficient indicates a cross-sectional relation between a dependent variable and a independent variable after controlling for the effect of other independent variables.
- Nowadays, Fama-Macbeth regression is widely used with lagged firm characteristics, deciles or percentile ranks.
- Benefit: control for a large set of potential variables

Fama/French-Approach: Time-Series regression

Dependent Variable

Independent Variable

$$R_{i,t} - R_{f,t} = \alpha_i + \beta_{i,RMRF}(R_{M,t} - R_{f,t}) + \varepsilon_{i,t}$$

- $\boldsymbol{\beta}_i = (\beta_{i,1}, \beta_{i,2}, \dots, \beta_{i,j})$ factor exposure / loading / sensitivity
- $\boldsymbol{R}_t = (R_{1,t}, R_{2,t}, \dots, R_{j,t})$ factor-mimicking portfolio return

Replicating Fama and French (2004)

1) We analyze a large sample of listed US-stocks

2) We start at winsorizing the monthly returns at 1% and 99% to avoid that the results being driven by outliers.

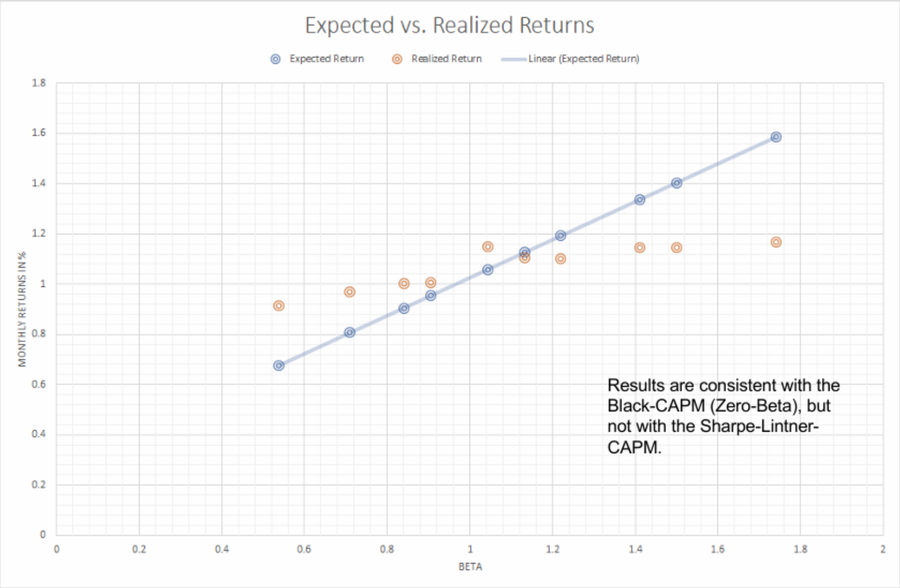
3) We calculate monthly pre-ranking betas based on rolling windows from 24-60 months. Market returns are taken from Kenneth French's library, and the rest of the data is based on CRSP.

4) Then, we form value-weighted decile portfolios based on the pre-ranking betas from 1/1932 until 12/2020.

5) We estimate the monthly (value-weighted) returns and the betas for each portfolio.

6) The expected return is based on the risk-free rate (T-Bill Rate) and market returns according to the CAPM equation.

Replicating Fama and French (2004)



Results from Fama and French (2004)

- CAPM predicts that variation in expected returns should only be explained by differences in betas.
- Former studies (e.g., Fama and McBeth (1973) have confirmed that prediction.
- However, Banz (1981) and Basu (1977) have asserted that returns seem to be correlated to firm size or firm's earnings-price ratios.
- Chan et al. (1991) find that book-to-market, BE/ME, has a strong predictability power on Japanese stocks.

The Fama/French Three-Factor-Model

$$\tilde{r}_{it} - \tilde{r}_t = \alpha_i + \beta_i(\tilde{r}_{it} - \tilde{r}_{mt}) + s_i\widetilde{SMB}_t + h_i\widetilde{HML}_t + \tilde{\varepsilon}_{it}$$

- SMB is the realized return on a diversified portfolio that is long in small stocks and short in large stocks
- HML is the realized return on a diversified portfolio that is long in stocks with high book-to-market ratios and short in low book-to-market stocks
- According to Fama/French (2004) standard deviation of all three factors is between 14 and 21%. Expected returns are 8.3, 3.6 and 5 percent.

The Fama/French Three-Factor-Model

Factor construction

- The size factor reflects the performance of a portfolio that is long in small and short in large stocks (small minus big). All stocks are divided in two groups; Fama/French are using the median market cap at the NYSE as the critical value.
- The value factor reflects the performance of a portfolio that is long in stocks with high book-to-market values and short in stocks with low book-to-market values (high minus low). For the long-portfolio only the 30% of the stocks with the highest book-to-market values are used, for the short portfolio the 30% of the stocks with the lowest book-to-market values

The Fama/French Five-Factor-Model

- Inspired by this research (most importantly Hou et al. 2015) Fama/French have recently suggested to adjust the three-factor model to a five-factor model by adding a profitability (RMW) and an investment (CMA) factor.
- In the context of a dividend discount model they argue that return expectations will be driven positively by the profitability of the firm and negatively by its investment activities.
- Profitability factor (robust minus weak, RMW): portfolio that is long in firms with high profitability (as measured by a ratio which is close to ROE) and short in firms with low profitability.
- Investment factor (conservative minus aggressive, CMA): portfolio that is long in firms with low investments (as measured by the change in balance sheet value) and short in firms with high investments.

The Fama/French Five-Factor-Model

The expected return of asset i in the Fama-French 5-factor model can be expressed as:

$E(R_{it}) = \alpha_i + \beta_{i,MKT}(R_{Mt} - R_f) + \beta_{i,SMB}SMB_t + \beta_{i,HML}HML_t + \beta_{i,RMW}RMW_t + \beta_{i,CMA}CMA_t + \epsilon_{it}$

Where:

- SMB_t = Small Minus Big factor at time t
- HML_t = High Minus Low factor at time t
- RMW_t = Robust Minus Weak factor at time t
- CMA_t = Conservative Minus Aggressive factor at time t
- $\beta_{i,MKT}, \beta_{i,SMB}, \beta_{i,HML}, \beta_{i,RMW}, \beta_{i,CMA}$ = Factor loadings (sensitivities) for asset i

The Fama/French Five-Factor-Model

- Cumulative monthly factor returns in the US
- Data is from Kenneth French's homepage [http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/)](http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/)

The standard *statistical* measures of outcome dispersion are **variance** and **standard deviation**.

Variance of return represents the expected squared deviation from the expected return.

$\sigma^2 = \text{Variance}(\tilde{r}_m) = \text{expected value of}(\tilde{r}_m - r_m)^2$,

Standard deviation is the square root of the variance:
$\sigma = \text{Standard deviation}(\tilde{r}_m) = \sqrt{\text{variance}(\tilde{r}_m)}$.

where $\tilde{r}_m$ is the actual return, and r_m is the expected return.

The Normal Distribution

The normal (Gaussian) distribution is one of the most widely used probability distributions. It has two parameters:

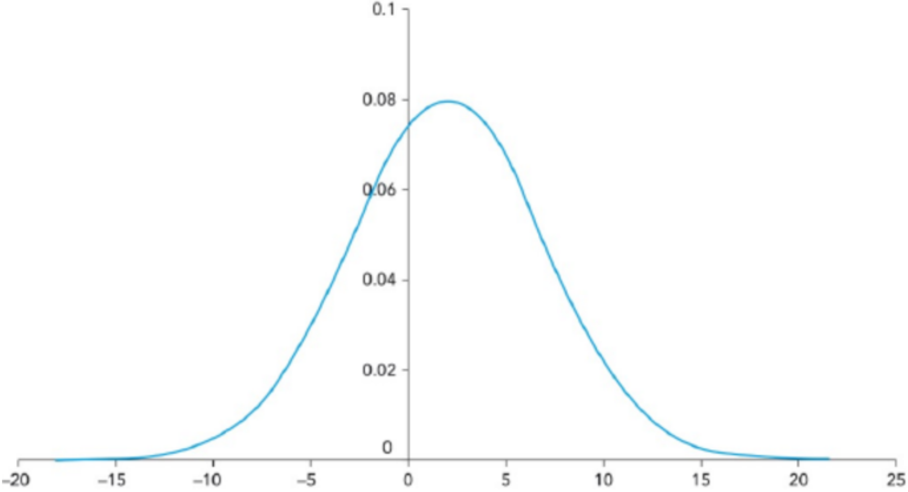
- the mean (μ);
- and standard deviation (σ).

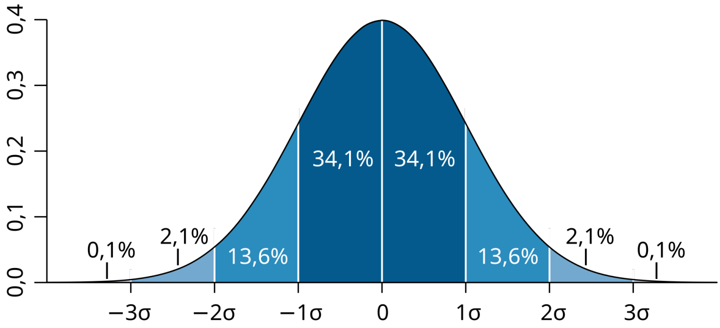
A normally distributed variable x, with mean μ and standard deviation σ, has a density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

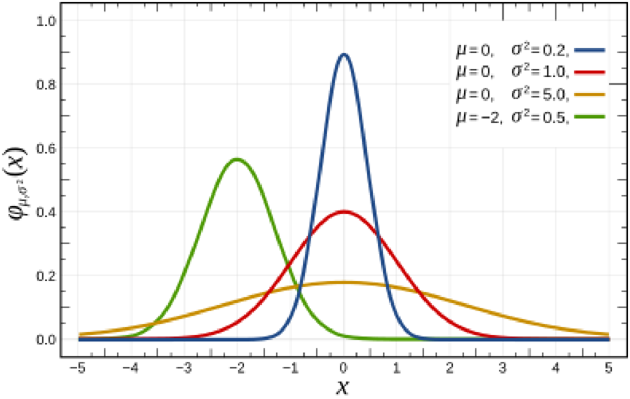
The Normal Distribution

Normal distribution with mean 2 and standard deviation 5.



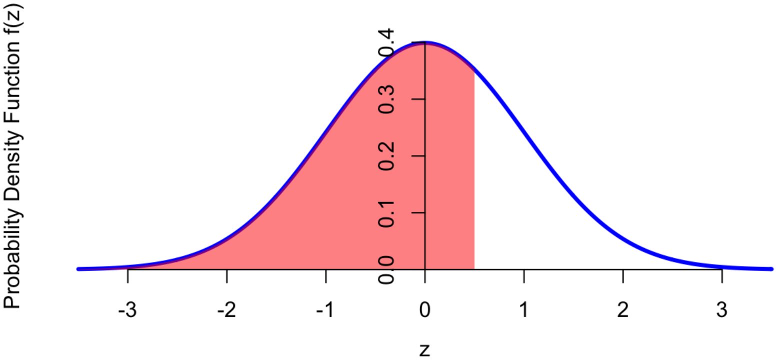


https://www.intmath.com/counting-probability/normal-distribution-graph-interactive.php

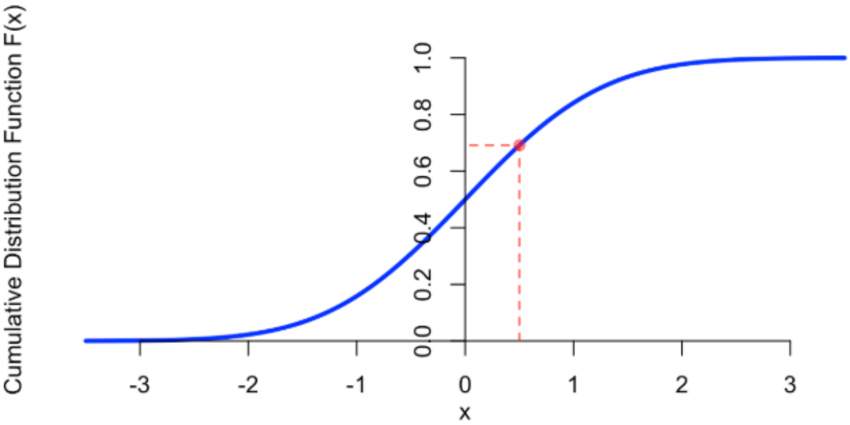


μ=0, σ²=0.2,
μ=0, σ²=1.0,
μ=0, σ²=5.0,
μ=−2, σ²=0.5,
1.0
0.8
0.6
0.4
0.2
0.0
−5
−4
−3
−2
−1
0
1
2
3
4
5
x
φ_{μ,σ²}(x)

The Normal Distribution



The Normal Distribution

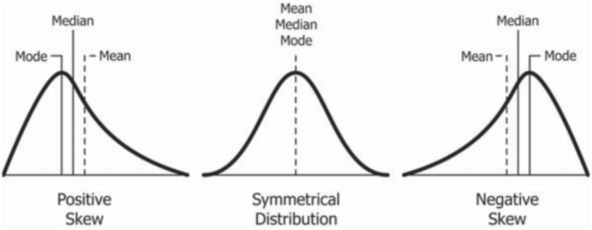


The Normal Distribution

The skewness is a measure of the asymmetry of the probability distribution.

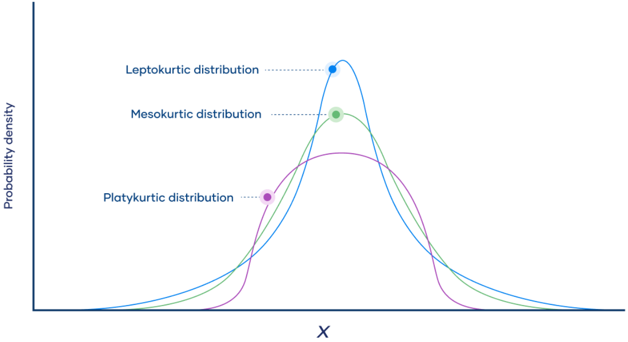
- **Negative skew:** $< \mathbf{0}$
  More extreme values lower than mean.
  Usual for stocks, as losses during crises are high.
- **Positive skew:** $> \mathbf{0}$
  More extreme values higher than mean.
  Stocks on bull market.

  Preferable for investor, as it's a sign on potential of grow.



The Normal Distribution

Kurtosis is a measure of the tailedness of a distribution. Tailedness is how often outliers occur. **Excess kurtosis** is the tailedness of a distribution relative to a normal distribution.



Value at Risk

Maximum expected loss of a portfolio at a given **confidence level** over a specified period

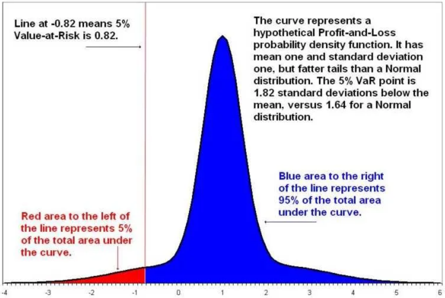
Or

Minimum guaranteed return of a portfolio at a given **confidence level** over a specified period

Why VaR matters: used by banks, investment funds, and corporations to assess risk.

Value at Risk

VaR modeling determines the potential for loss in the entity being assessed and the probability that the defined loss will occur. One measures VaR by assessing the amount of potential loss, the probability of occurrence for the amount of loss, and the time frame.



Definition of Value-at-Risk

Portfolio VaR is defined for a fixed forecasting horizon and confidence level.

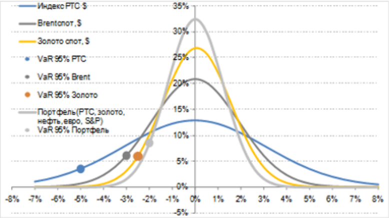
VaR is a monetary estimate of the amount that expected losses over a given time period will not exceed, with a specified probability.

Assuming that:

- the composition of the portfolio remains unchanged,
- there are no crisis phenomena in the market; the market is stable

Main VaR Calculation Methodologies:

- **Historical method:** Uses historical data to determine returns.
- **Analytical method (Variance-Covariance):** Assumes a normal distribution of returns; suitable for quick calculations but sensitive to assumptions.
- **Monte Carlo method:** Uses simulations to model multiple scenarios, accounting for nonlinearities.



VaR: Implementation Steps

Value at Risk: шаги расчета

1. Выбор актива или портфеля (например, набор акций или облигаций).
2. Определение временного горизонта (день, неделя, месяц) и уровня доверия (например, 95% или 99%).
3. Сбор исторических данных и расчет доходностей
   - $R_t = \frac{P_t}{P_{t-1}} - 1$
   - $R_t = \ln(P_t) - \ln(P_{t-1})$

   Где R_t - доходность, P_t - цена в период t
4. Применение метода расчета
5. Интерпретация: «VaR – это величина, при которой вероятность убытка, превышающего её, равна 5%».

Исторический

Параметрический

Монте-Карло

след. стр.

Формула для исторического VaR:

$$VaR_{\text{hist}} = -V \cdot q_{(1-\alpha)}$$

где:

- V — стоимость портфеля,
- $q_{(1-\alpha)}$ — квантиль распределения доходностей (например, 5-й процентиль для 95% доверия).

  Отрицательный знак отражает, что рассматривается убыток.

Общая формула параметрического метода для расчёта VaR в общем случае выглядит следующим образом:

$$VaR_\alpha = V_0 \cdot (\mu - z_\alpha \cdot \sigma)$$

где:

- V_0 — начальная стоимость портфеля,
- μ — средняя ожидаемая доходность (может быть опущена для коротких периодов),
- σ — стандартное отклонение (волатильность) доходности,
- z_α — квантиль стандартного нормального распределения для уровня доверия α.

VaR: Historical Method

Risk assessments of an asset and risk premia are based on analyzing a long history of transaction prices — this is known as the **historical approach**.

This approach relies on a strong but unfounded assumption:

> *The probabilistic distribution of asset returns does not change over time — "The future can only reflect what has occurred in the past."*

The future return distribution is inferred from past transaction price data. Each past observation is considered equally reliable and is assigned equal weight.

In practice, a historical time series of prices P_t is transformed into a return series in one of the following two ways:

$$R_t = \frac{P_t}{P_{t-1}} - 1$$

$$R_t = \ln(P_t) - \ln(P_{t-1})$$

This transformation is applied because price series are generally **non-stationary**, which distorts the resulting estimates. **Return series**, on the other hand, are typically **stationary**, which makes the estimates *consistent and efficient*.

VaR: Historical Method

The distribution of losses (or returns) is approximated by a sample distribution constructed from historical data.

Historical method (non-parametric method):
All scenarios of asset price changes in the portfolio based on observed historical changes are considered. Shocks from the past can be reused. All risk factors and their historical extremes must be taken into account.

VaR: Methods

Method	**Assumptions**	**Complexity**	**Computational Cost**
Historical Method	Uses actual **historical data**; assumes that **past** market **behavior will repeat**	**Low:** analysis and **sorting** of available **data**	**Low:** calculations are based on already **existing data**
Analytical Method (Variance-Covariance)	Assumes **normal distribution** of returns and **linearity** of the portfolio; uses a **covariance matrix**	**Medium:** requires calculation of statistical metrics: • *mean* • *standard deviation* • *covariances*	**Low to Medium:** formulas allow quick VaR calculation if assumptions hold
Monte Carlo Method	Flexible approach with **minimal assumptions** about distribution; allows modeling of **nonlinear dependencies** and complex scenarios	**High:** building the model and generating a large number of **random scenarios**	**High:** requires significant computational resources to run a large number of **simulations**

VaR: Monte Carlo Method

1. **A price movement trajectory is defined** (e.g., Wiener process, exponential, etc.) — this is a sequence of pseudo-randomly simulated prices starting from the current price and ending at the terminal price. The more steps used, the more accurate the estimates. Each trajectory is a **scenario** of movement toward the final price, which is determined based on the current price within that scenario.
2. **Random number generator**: a random number uniformly distributed over the interval [0;1][0; 1][0;1]. Possible values of the risk factor ξ are mapped to this interval — each realization draws a random value of the risk factor for each time point t.

- Suppose a random variable ξ is given (e.g., $\xi = h(t)$) with a known distribution and density function $f(\xi)$. The distribution $f(\xi)$ is hard to compute directly. We need to determine the probability that ξ falls into a specified interval $[a,b]$.
- We generate the random variable ξ N times. We count how many times $f(\xi)$ falls into the interval $[a,b]$ and divide by N. This gives us the estimated probability that $\xi \in [a,b]$. As N increases, the accuracy of this probability estimate improves.
- Knowing the probability of $f(\xi)$ falling into any interval allows us to reconstruct the entire distribution $f(\xi)$.

VaR: Monte Carlo Method

1. Based on historical data, estimate the expected return (mean) and variance.
2. Using a random number generator, simulate normally distributed random variables ξ with these parameters. Store the results in a matrix of size 500×1000 (500 scenarios $\times$ 1000 time intervals in T).
3. Compute the trajectory T of simulated prices up to P_{1000} using the formula:

$$P_t = P_{t-1} \cdot e^{\xi_{t-1}}$$

4. Revalue the portfolio using the formula:

$$\Delta V = Q(P_{1000} - P_0)$$

   where Q is the number of asset units.
5. Repeat steps 2–4 a total of 500 times. Sort the resulting ΔV values in descending order. To match the desired confidence level $(1-\alpha)$, take the maximum loss that is not exceeded in $500 \cdot (1-\alpha)$ cases.

$$\text{VaR} = \text{sorted value at position } 500 \cdot (1-\alpha)$$

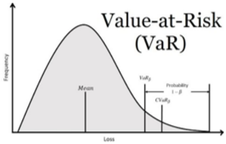
VaR: Addendum

Value at Risk: is it sufficient as a measure?

1. What are the advantages and disadvantages you see in using VaR?
2. Can VaR be considered a sufficient tool for risk management?
3. How can VaR be complemented with other risk measures?

Criticism:

1. VaR does not provide information about the magnitude of losses beyond the set threshold ("tail risk").
2. Sensitivity to model selection and historical data.
3. Problems with the assumption of normally distributed returns.



Other measures:

1. **Expected Shortfall (ES)** reflects the **average loss** conditional on losses exceeding the VaR level.
   In other words, if in the worst (1−α)% of cases the losses exceed the VaR value, ES shows **how large** these losses are **on average.**

$$ES_\alpha = \mu - \sigma \frac{\phi(Z_\alpha)}{1-\alpha}$$

ϕ(Zα) – the value of the standard normal density function at point Zα

2. **Marginal VaR** shows how much the total portfolio VaR would change **with a small change** in the position of a specific asset.
   This helps assess each asset's **contribution** to the overall portfolio risk.

$$\text{Marginal VaR}_i = \frac{\partial VaR}{\partial w_i}$$

w_i – weight of the i-th asset

3. **Component VaR** – the **contribution** of each asset to the total VaR of the portfolio.
   It is computed by multiplying the **marginal VaR** by the **asset's weight**:

$$\text{Component VaR}_i = w_i \times \text{Marginal VaR}_i$$

VaR: Exercise 1

We are interested in the normal VaR at a 95% confidence level and a 1-day holding period. The parameters μ and σ for this horizon are 0.005 and 0.02, respectively. The portfolio value is 1 million USD.

Then at the 99% confidence level.

Recalculate for a 10-day holding period.

VaR: Exercise 2

Find the 99.5% VaR. A total of 500 scenarios are calculated and sorted from worst to best.

Scenario Number	Loss (USD millions)
210	7.8
195	6.5
2	4.6
23	4.3
48	3.9
367	3.7
235	3.5
..	..
..	..
..	..

Calculate the 5-day VaR (Value at Risk) at a 97% confidence level for a portfolio consisting of two assets X and Y, with investments of \$10,000 and \$20,000 respectively, daily volatilities of 1% and 2%, and a correlation coefficient of 0.3 between their returns.

VaR: Exercise 4

The portfolio consists of assets X and Y.

- $\alpha = 0.01$ VaR 99.0%
- $\mu_X = 0.12$ (expected return of asset X)
- $\mu_Y = 0.15$ (expected return of asset Y)
- $\sigma_X = 0.24$ (standard deviation of asset X)
- $\sigma_Y = 0.30$ (standard deviation of asset Y)
- $\omega_X = 0.6$ (weight of asset X)
- $\rho_{XY} = 0.33$ (correlation between assets)
- Portfolio value = 5 million

Find the portfolio VaR for a 10-day horizon.

Downside Risk: Semi-Variance & Downside Deviation

- Volatility treats upside and downside deviations symmetrically. Many investors care primarily about losses.
- Let r_t be periodic returns and MAR be the minimum acceptable return (often 0 or r_f).

Downside deviation (DD):

$$DD(MAR) = \sqrt{\frac{1}{T}\sum_{t=1}^{T} \min\left(0,\ r_t - MAR\right)^2}$$

Semi-variance: $SV(MAR) = DD(MAR)^2$

Interpretation: DD measures the typical magnitude of underperformance below MAR (Minimum Acceptable Return, ignoring upside).

Sortino Ratio vs Sharpe Ratio

Sharpe (total risk):

$$SR = \frac{\mathbb{E}[r] - r_f}{\sigma(r)}$$

Sortino (downside risk):

$$SoR = \frac{\mathbb{E}[r] - MAR}{DD(MAR)}$$

- If returns are asymmetric (e.g., crash risk), SoR can rank assets/strategies differently than SR.
- Choosing MAR matters: typical choices are 0, r_f, or an investor target.
- Practical takeaway: *Sharpe penalizes "good volatility", Sortino does not.*

Path-Dependent Risk: Drawdowns

Let V_t be portfolio value and $H_t = \max_{0\le s\le t} V_s$ the running peak.

Drawdown (in %):

$$DD_t = \frac{V_t - H_t}{H_t} \le 0$$

Max Drawdown (MDD):

$$MDD = \min_t DD_t$$

- MDD captures the worst peak-to-trough decline (investor "pain").
- Two portfolios can have the same σ and even the same ES, but very different MDD (different paths).

Beyond MDD: Calmar, Ulcer Index (optional but useful)

Calmar ratio (common for funds/strategies):

$$Calmar = \frac{CAGR}{|MDD|}$$

Ulcer Index (penalizes persistent drawdowns):

$$UI = \sqrt{\frac{1}{T}\sum_{t=1}^{T} DD_t^2}$$

- Calmar: "return per unit of worst drawdown".
- Ulcer Index: sensitive to *depth and duration* of drawdowns.
- Good for comparing strategies with similar Sharpe/Sortino but different recovery behavior.

Benchmark-Relative Risk: Tracking Error & Information Ratio

Let active return be $a_t = r_{p,t} - r_{b,t}$ (portfolio minus benchmark).

Tracking error (TE):

$$TE = \sigma(a_t) = \sqrt{\mathrm{Var}(r_p - r_b)}$$

Information ratio (IR):

$$IR = \frac{\mathbb{E}[a_t]}{TE}$$

- TE measures how tightly you hug the benchmark (risk of active management).
- IR is the "Sharpe ratio of active returns".
- Useful whenever you have a benchmark: equity index, bond index, or strategic allocation.

CAPM Diagnostics: Alpha, Beta, R^2, Idiosyncratic Risk

CAPM regression (excess returns):

$$r_{p,t} - r_{f,t} = \alpha + \beta\,(r_{m,t} - r_{f,t}) + \varepsilon_t$$

- β: systematic exposure to the market.
- α (Jensen's alpha): average performance not explained by market exposure.
- R^2: fraction of variance explained by the market (how "index-like" the portfolio is).
- $\sigma(\varepsilon)$: idiosyncratic (diversifiable) risk.

Interpretation: High R^2 + low $\alpha \Rightarrow$ closet indexing; low $R^2 \Rightarrow$ diversified sources beyond market factor.

Risk Attribution: Who Drives Portfolio Risk? (Variance)

Portfolio variance:

$$\sigma_p^2 = \mathbf{w}^\top \Sigma\, \mathbf{w} \qquad \Rightarrow \qquad \sigma_p = \sqrt{\mathbf{w}^\top \Sigma\, \mathbf{w}}$$

Marginal contribution to volatility:

$$MC_i = \frac{\partial \sigma_p}{\partial w_i} = \frac{(\Sigma \mathbf{w})_i}{\sigma_p}$$

Component (total) contribution to volatility:

$$RC_i = w_i \cdot MC_i \qquad \text{and} \qquad \sum_i RC_i = \sigma_p$$

- Highlights concentration: small weights can contribute a lot if correlated or very volatile.
- Natural bridge to risk budgeting / risk parity ideas.

Bond Price Sensitivity: Duration (intuition & meaning)

- For a bond, the key market risk driver is the yield-to-maturity y (or relevant curve yield).
- **Duration** measures interest-rate risk: how sensitive the bond price is to changes in yield.

Modified duration (definition):

$$D_{\text{mod}} = -\frac{1}{P}\frac{\partial P}{\partial y}$$

First-order approximation (small yield changes):

$$\frac{\Delta P}{P} \approx -D_{\text{mod}} \cdot \Delta y$$

- Interpretation: if $D_{\text{mod}} = 5$, then a $+1\%$ (100 bps) increase in yield implies roughly a -5% price change.
- Longer maturity and lower coupon $\Rightarrow$ typically higher duration $\Rightarrow$ higher rate risk.

Convexity: improving the duration approximation

- Duration is a **linear** (first-order) approximation. For larger yield moves, curvature matters.
- **Convexity** captures the second-order sensitivity of price to yield.

Convexity (one common definition):

$$C = \frac{1}{P}\frac{\partial^2 P}{\partial y^2}$$

Duration + convexity approximation:

$$\frac{\Delta P}{P} \approx -D_{\text{mod}}\Delta y + \frac{1}{2}C(\Delta y)^2$$

- With positive convexity, price gains from yield decreases are larger (in magnitude) than price losses from equal yield increases.
- Practical use: convexity is crucial for long-maturity bonds and when Δy is not "small".

Exam Format: Project-Based Portfolio Defense

- **Common market, different clients:** all teams use the same asset universe and dataset.
- **Your task:** design and justify a portfolio that fits a specific *investment mandate* (constraints + objectives).
- **Core topics assessed:**
  - Modern Portfolio Theory (mean–variance, efficient frontier)
  - Risk-free asset and portfolio choice (risk/return trade-offs)
  - Risk measurement (volatility, drawdown, VaR/ES) and risk governance
  - Factor models (CAPM + one multi-factor specification) and interpretation
- **Output:** a short pitch + a transparent appendix with key calculations and assumptions.

Asset Universe and Data (Common for All Teams)

- **Universe:** 10-15 equities (A1–A15) + 1-3 diversifier (D) + 1 risk-free asset (RF).
- **Data:** historical return series (frequency and sample window as provided).
- **Portfolio constraints (baseline):**
    - Long-only: $w_i \geq 0$
    - Fully invested: $\sum_i w_i = 1$
    - Mandate-specific limits may add constraints on concentration, RF share, D share, etc.
- **Benchmark (if applicable):** either a provided market index return series or an equal-weight basket of A1–A15.

Required Analytics (Minimum Checklist)

- **Portfolio construction**
  - Estimate μ (expected returns) and Σ (covariance matrix)
  - Efficient frontier (at least: minimum-variance portfolio and your chosen final portfolio)
  - Explanation of the chosen point on the frontier (objective + constraints)
- **Risk report (portfolio level)**
  - Annualized return, volatility, Sharpe ratio (excess over RF)
  - Maximum drawdown (historical)
  - Tail risk: VaR(95%) *or* ES(95%) (use one approach consistently)
  - Concentration metrics (max weight, top-2 share, risk contribution)
- **Factor interpretation**
  - CAPM (beta, alpha, interpretation)
  - One multi-factor model (e.g., FF3/FF5, Carhart, or a course-approved factor set)

Five Mandates: Overview

Team	Client / Objective	Key Constraints (illustrative)
1	Pension / Capital preservation	Low volatility target; RF floor; Diversifier floor; concentration limits
2	Endowment / Balanced efficiency	Near-max Sharpe with robustness; modest RF buffer; diversification limits; turnover awareness
3	Active fund / Benchmark-relative	Outperform with controlled active risk: tracking error limit, beta range, active-weight limits
4	Insurance / Tail-risk first	ES (or VaR) limit; max drawdown cap; higher share of D and/or RF; strict concentration limits
5	ESG / Compliance-first	Exclusion of one asset; RF buffer; demonstrate *cost of constraint* vs unconstrained portfolio

Note: Numerical thresholds (volatility, TE, ES/VaR, drawdown) will be specified in the assignment sheet.

Mandate 1: Pension Fund (Capital Preservation)

- **Objective:** preserve capital, minimize downside risk, stable risk profile.
- **Typical constraints:**
  - Volatility cap (annualized) and/or drawdown cap
  - RF share $\geq$ a specified minimum
  - Diversifier share $w_D \geq$ a specified minimum
  - Concentration: $w_{A_i} \leq$ cap; top-2 weights $\leq$ cap
- **Defense focus:**
  - Why is your portfolio near the left side of the frontier?
  - Risk contributions: which positions dominate risk and why acceptable?
  - Clear monitoring and rebalancing rules (risk governance).

Mandate 2: Endowment (Balanced, Max Efficiency + Robustness)

- **Objective:** high risk-adjusted performance (Sharpe) with stable diversification.
- **Typical constraints:**
  - Choose a (near-)max Sharpe portfolio under diversification limits
  - RF buffer $\geq$ minimum for liquidity needs
  - Position caps to avoid "one-idea" portfolios
  - Optional: turnover awareness (transaction cost sensitivity)
- **Defense focus:**
  - Robustness: do weights change drastically across sample windows?
  - Interpretation: what factor exposures are you intentionally taking?
  - Trade-off: slight Sharpe loss vs improved stability (if applicable).

Mandate 3: Active Fund (Benchmark-Relative with TE Control)

- **Objective:** outperform the benchmark while controlling active risk.
- **Typical constraints:**
  - Tracking error (TE) $\leq$ limit
  - Beta to benchmark within a range (e.g., 0.9–1.1)
  - Active weight limits: $|w_i - w_i^{bench}| \leq$ cap
  - RF upper bound (to avoid "hiding in cash")
- **Defense focus:**
  - Clear "alpha hypothesis" (factor tilt, quality, momentum, etc.)
  - TE decomposition: where does active risk come from?
  - Consistency: does outperformance rely on one asset or broad structure?

Mandate 4: Insurance (Tail-Risk First)

- **Objective:** strict downside and tail-risk control (risk committee mindset).
- **Typical constraints:**
  - ES(95%) *or* VaR(95%) $\leq$ limit (state method clearly)
  - Max drawdown cap
  - Higher floors for w_D and/or w_{RF}
  - Tight concentration limits across equities
- **Defense focus:**
  - Why ES (or VaR) is appropriate for this client
  - Stress test logic and mitigation actions (without overfitting)
  - Governance: limits, reporting, escalation triggers.

Mandate 5: ESG / Compliance-First (Constraint Cost + Explainability)

- **Objective:** comply with exclusions and constraints; deliver a defensible portfolio.
- **Typical constraints:**
  - One asset is excluded (hard constraint)
  - RF buffer $\geq$ minimum
  - Remaining position caps + diversification requirements
- **Mandatory element: "Cost of Constraint"**
  - Compare constrained vs unconstrained efficient set (risk/return or Sharpe)
  - Explain what diversification/return opportunities are lost and how mitigated
- **Defense focus:** explainability (investor communication + risk reasoning).

Deliverables: What You Submit and Present

- **Presentation (recommended 2 slides + appendix):**
  - Slide 1: weights, mandate compliance, key performance/risk metrics
  - Slide 2: efficient frontier position + factor exposures + risk governance summary
- **Appendix (1–2 pages):**
  - Data summary, frequency and annualization rules
  - μ and Σ estimation approach, key formulas
  - Risk metrics definition and calculation steps (VaR/ES, drawdown)
  - Factor model outputs (coefficients, R^2, interpretation)
- **Clarity requirement:** every number should be traceable to a method and assumption.

Assessment Rubric (Suggested)

Criterion	Weight
Correct portfolio analytics (frontier, weights, metrics, consistency)	35%
Risk management quality (tail risk, drawdown, concentration, governance)	30%
Factor-model reasoning (CAPM + multi-factor, interpretation, robustness)	20%
Mandate fit and investor communication (constraints, narrative, trade-offs)	10%
Presentation quality (structure, transparency, answers to questions)	5%

Teamwork and Q&A Expectations

- **Recommended roles:** Portfolio Manager, Risk Manager, Quant/Analyst, Client/Compliance lead.
- **In the defense:**
  - Be explicit about trade-offs: return vs risk vs constraints
  - Show that constraints are satisfied (do not assume)
  - Explain factor exposures in plain investment language
  - Discuss how the portfolio is monitored and adjusted over time
- **Academic integrity:** cite data sources (if external) and keep methods reproducible.