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E. Mahmoudi [a] & H. Zakerzadeh [a]
[a] Department of Statistics, College of Mathematics, Yazd University, Yazd, Iran
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Generalized Poisson–Lindley Distribution

E. MAHMOUDI AND H. ZAKERZADEH

Department of Statistics, College of Mathematics,
Yazd University, Yazd, Iran

An extended version of the compound Poisson distribution is obtained by compounding the Poisson distribution with the generalized Lindley distribution. Estimation of the parameters is discussed using the method of moments and maximum likelihood estimators. Examples are given of the fitting of this distribution to data, and the fit is compared with that obtained using other distributions.

Keywords Generalized Lindley distribution; Quasi Newton–Raphson method; Truncated distributions; Weighted distributions.

Mathematics Subject Classification Primary 60E05; Secondary 62H05.

1. Introduction

The Poisson–Lindley distribution, with probability mass function

$$f(x;\theta) = \frac{\theta^2(x+\theta+2)}{(\theta+1)^{x+3}}, \quad x = 0, 1, \ldots, \ \theta > 0, \tag{1.1}$$

was introduced by Sankaran (1970) to model count data. The distribution arises from the Poisson distribution when its parameter λ follows a Lindley (1958) distribution with the probability density function:

$$g(\lambda;\theta) = \frac{\theta^2}{\theta+1}(1+\lambda)e^{-\theta\lambda}, \quad \lambda > 0, \ \theta > 0. \tag{1.2}$$

Ghitany et al. (2008) showed that in many ways (1.2) is a better model for some applications than one based on the exponential distribution. Also, Ghitany et al. (2008) introduced the zero-truncated Poisson–Lindley (ZTPL) distribution to model count data when the data to be modeled originate from a generating mechanism that structurally excludes zero counts.

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Address correspondence to E. Mahmoudi, Department of Statistics, College of Mathematics, Yazd University, Yazd 89195-741, Iran; E-mail: emahmoudi@yazduni.ac.ir

Zakerzadeh and Dolati (2010) introduced a family of distributions with the probability density function:

$$f(x; \alpha, \theta) = \frac{\theta^{\alpha+1}}{\theta + 1} \frac{x^{\alpha-1}}{\Gamma(\alpha + 1)} (\theta + x) e^{-\theta x}, \quad x > 0,\ \theta > 0,\ \alpha > 0, \tag{1.3}$$

which generalized the Lindley distribution. They mentioned that the model (1.3) can be used as an alternative to the two-parameter gamma distribution and the Weibull model which are commonly used for analyzing lifetime or skewed data.

This article offers a two-parameter generalized Poisson–Lindley (GPL) distribution, which generalizes Poisson–Lindley distribution (1.1), as a model for count data. We show that our distribution provide enough flexibility for analyzing different types of count data. The study examines various properties of this model. The article is organized as follows:

Section 2 introduces the two-parameter generalized Poisson–Lindley (GPL) distribution and presents its basic properties including: the behavior of the density function, the expressions for the moments, the distribution of the sums of random variables, and truncated and weighted GPL distributions. Section 3 discusses the different methods for parameters estimation. In Sec. 4, we present an algorithm for simulating random data from the GPL distribution. In this section, a simulation study is carried out to investigate the average bias and average mean square error (MSE) of the simulated estimates. An application of the GPL distribution, including two examples of the fitting this distribution to data, and comparing with other discrete distributions, are given in Sec. 5.

2. Definition and Some Properties

2.1. *The Generalization*

Suppose that the parameter λ of the Poisson distribution has a generalized Lindley distribution (1.3). Then, the compound Poisson distribution that results is

$$\begin{aligned} f(x; \alpha, \theta) &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} \frac{\theta^{\alpha+1}}{\theta + 1} \frac{\lambda^{\alpha-1}}{\Gamma(\alpha + 1)} (\theta + \lambda) e^{-\theta\lambda} d\lambda \\ &= \frac{\Gamma(x + \alpha)}{x!\Gamma(\alpha + 1)} \frac{\theta^{\alpha+1}}{\left(\theta + 1\right)^{x+\alpha+1}} \left(\alpha + \frac{x + \alpha}{\theta + 1}\right), \quad x = 0, 1, \ldots, \end{aligned} \tag{2.1}$$

where $\theta > 0$ and $\alpha > 0$. We see that (2.1) is a two-parameter generalization of the discrete Poisson–Lindley distribution. We call this distribution as the generalized Poisson–Lindley (GPL) distribution and denote it by $GPL(\alpha, \theta)$. Note that for $\alpha = 1$, the density function (2.1) reduces to the one-parameter Poisson–Lindley distribution (1.1). The cumulative distribution function of the GPL can be expressed in terms of an incomplete beta function ratio and hence as a sum of two binomial terms, when α is a positive integer. Let $I_q(a, b) = \frac{\beta_q(a,b)}{\beta(a,b)}$ with $\beta_q(x; a, b) = \int_0^q x^{a-1}(1 - x)^{b-1} dx$, denote the incomplete beta function ratio. Then we have the following proposition.

Proposition 2.1. *Let X is a random variable with $GPL(\alpha, \theta)$ distribution. Then for the positive integer values of α, the cumulative distribution function of X is given by*

$$F_X(x) = \sum_{j=0}^{1} p_j I_\eta(\alpha + j, x + 1) = \sum_{j=0}^{1} p_j \Pr(Y_j \geq \alpha + j),$$

where $p_j = \frac{\theta^{1-j}}{\theta+1}$, for $j = 0, 1$; $\eta = \frac{\theta}{\theta+1}$ and $Y_j \sim Bin(\alpha + j + x, \eta)$.

We will now describe the shape and density function.

2.2. *Shape and Density Function*

Let

$$g_1(\theta) = \frac{2\theta(\theta+1)(\theta+3) - 4\theta\sqrt{\theta^3 + 4\theta^2 + 5\theta + 2}}{2(\theta+1)^4},$$

and

$$g_2(\theta) = \frac{2\theta(\theta+1)(\theta+3) + 4\theta\sqrt{\theta^3 + 4\theta^2 + 5\theta + 2}}{2(\theta+1)^4}.$$

(i) If $0 < \alpha \leq g_1(\theta)$ or $g_2(\theta) \leq \alpha < \infty$, and $\alpha(\theta+2) + \theta(\theta+3) > 1$, then the GPL distribution is unimodal and has a unique mode at $[x^*] + 1$, where $[x^*]$ denotes the integer parts of x^* with

$$x^* = -\frac{\theta + \alpha(\theta(\theta+2) - 1) - \sqrt{\alpha^2(\theta+1)^4 - 2\alpha\theta(\theta+1)(\theta+3) + \theta^2}}{2\theta}.$$

(ii) If $0 < \alpha \leq g_1(\theta)$ or $g_2(\theta) \leq \alpha < \infty$, and $\alpha(\theta+2) + \theta(\theta+3) \leq 1$, then the GPL distribution is unimodal and has a unique mode at 0.

(iii) If $g_1(\theta) < \alpha < g_2(\theta)$, then the GPL distribution is unimodal and has a unique mode at 0.

For more details, see Johnson et al. (2005).

Figure 1 shows the density function of $GPL(\alpha, \theta)$ distribution for selected values of α and θ. We next describe the expressions for the moments of this distribution.

2.3. *Moments, Skewness, and Kurtosis*

If X has $GPL(\alpha, \theta)$ distribution, then direct calculation shows that the corresponding probability generating function (pgf) is given by

$$G(s) = \left(\frac{\theta}{\theta - s + 1}\right)^{\alpha+1} \left(\frac{\theta - s + 2}{\theta + 1}\right),$$

and the moment generating function (mgf) of X is defined by

$$M(t) = \left(\frac{\theta}{\theta - e^t + 1}\right)^{\alpha+1} \frac{\theta - e^t + 2}{\theta + 1}. \tag{2.2}$$

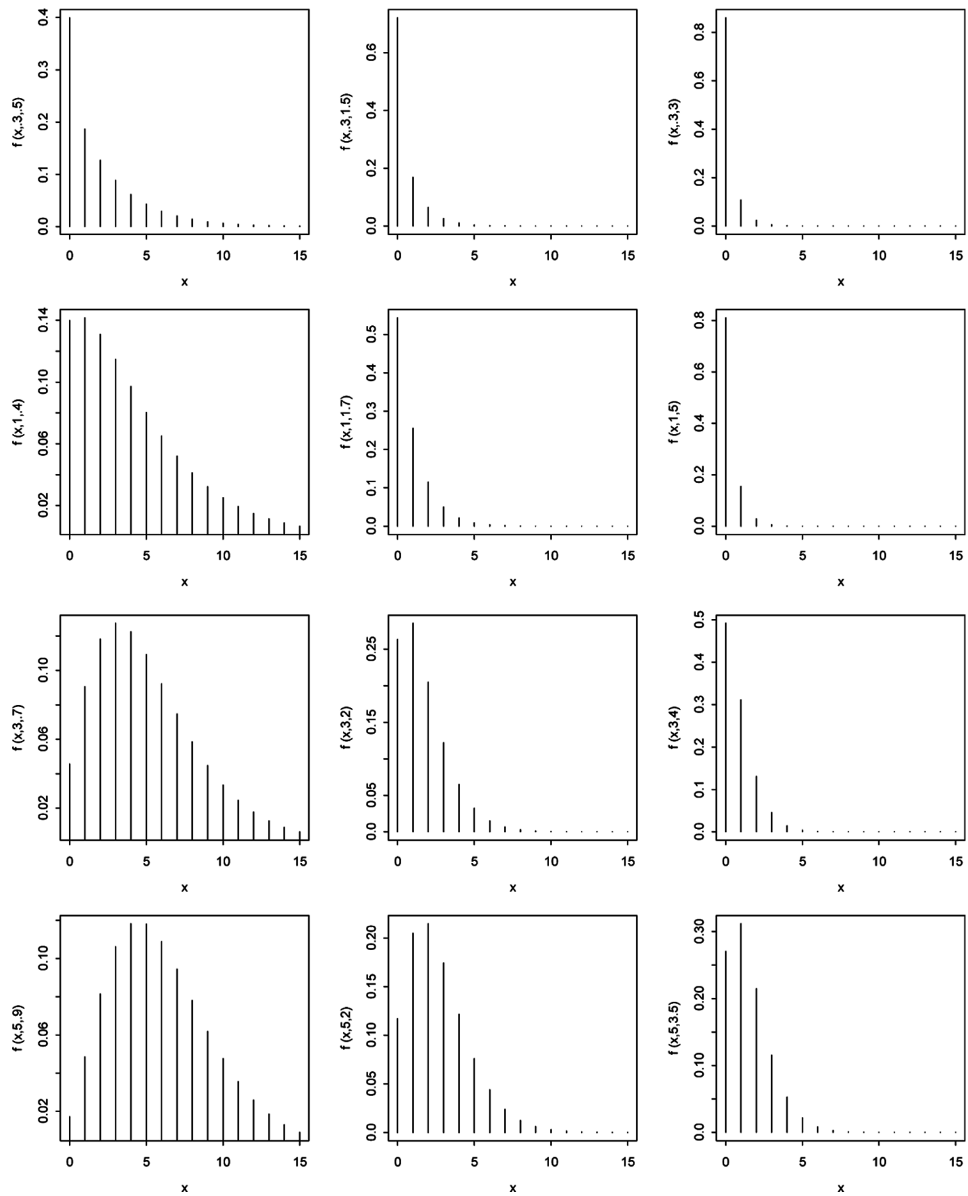

Figure 1. Graphs of the GPL distribution for different values (α, θ).

From (2.2), the first four moments of X about the origin are found to be

$$\mu_1' = \frac{\alpha(\theta+1)+1}{\theta(\theta+1)},$$

$$\mu_2' = \frac{\alpha^2(\theta+1)+\alpha(\theta^2+2\theta+3)+\theta+2}{\theta^2(\theta+1)},$$

$$\mu_3' = \frac{\alpha^3(\theta+1) + 3\alpha^2(\theta^2+2\theta+2) + \alpha(\theta^3+4\theta^2+11\theta+11) + \theta^2+6\theta+6}{\theta^3(\theta+1)},$$

$$\mu_4' = \frac{\begin{array}{c}\alpha^4 + \alpha^3(6\theta^2+\theta+10) + \alpha^2(7\theta^3+25\theta^2+6\theta+35)\\ +\,\alpha(\theta^4+8\theta^3+33\theta^2+11\theta+50) + \theta^3+14\theta^2+6\theta+24\end{array}}{\theta^4(\theta+1)}.$$

By using the identity $\mu_r = E(X-\mu)^r = \sum_{k=0}^{r}\binom{r}{k}\mu_k'(-\mu_1')^{r-k}$, the central moments can be obtained as follows:

$$\mu_2 = \frac{\alpha(\theta+1)^3+\theta^2+3\theta+1}{\theta^2(\theta+1)^2},$$

$$\mu_3 = \frac{\alpha(\theta+1)^4(\theta+2) + (\theta^3+6\theta^2+4\theta+1)(\theta+2)}{\theta^3(\theta+1)^3},$$

$$\mu_4 = \frac{\begin{array}{c}3\alpha^2(\theta+1)^6 + \alpha(\theta+1)^3(\theta^2+4\theta+12)(\theta^2+4\theta+9)\\ +\,(\theta^4+14\theta^3+34\theta^2+27\theta+9)(\theta^2+3\theta+1)\end{array}}{\theta^4(\theta+1)^4}.$$

Since the index of dispersion is given by

$$r = \frac{\sigma^2}{\mu} = 1 + \frac{\alpha(\theta+1)^2+2\theta+1}{\alpha\theta(\theta+1)^2+\theta(\theta+1)},$$

it follows that the $GPL(\alpha,\theta)$ is over-dispersed ($\sigma^2 > \mu$) for all values (α,θ), and equi-dispersed ($\sigma^2 = \mu$) for large enough amount of θ. Other important indices of the shape of the distribution are the skewness, $\sqrt{\beta_1} = \frac{\mu_3}{(\mu_2)^{3/2}}$, kurtosis, $\beta_2 = \frac{\mu_4}{(\mu_2)^2}$, and the coefficient of variation, $C.V. = \frac{\sigma}{\mu}$, respectively. For the GPL distribution these indices are given by

$$\sqrt{\beta_1} = \frac{(\alpha(\theta+1)^4+\theta^3+6\theta^2+4\theta+1)(\theta+2)}{(\alpha(\theta+1)^3+\theta^2+3\theta+1)^{3/2}},$$

$$\beta_2 = \frac{\begin{array}{c}3\alpha^2(\theta+1)^6 + \alpha(\theta+1)^3(\theta^2+4\theta+9)(\theta^2+4\theta+12)\\ +\,(\theta^2+3\theta+1)(\theta^4+14\theta^3+34\theta^2+27\theta+9)\end{array}}{(\alpha(\theta+1)^3+\theta^2+3\theta+1)^2}$$

and

$$C.V. = \frac{\sqrt{\alpha(\theta+1)^3+\theta^2+3\theta+1}}{\alpha(\theta+1)+1}.$$

Proposition 2.2.

(i) $\sqrt{\beta_1}$ *is an increasing function in* θ *for fixed* α, *and* $\frac{2}{\sqrt{1+\alpha}} < \sqrt{\beta_1} < \infty$.
(ii) β_2 *is an increasing function in* θ *for fixed* α, *and* $\frac{3(\alpha+3)}{\alpha+1} < \beta_2 < \infty$.
(iii) *C.V. is an increasing function in* θ *for fixed* α, *and* $\frac{1}{\sqrt{\alpha+1}} < C.V. < \infty$.

Figure 2 shows the dispersion, coefficient of variation, skewness and kurtosis of the GPL distribution as a function of (α,θ).

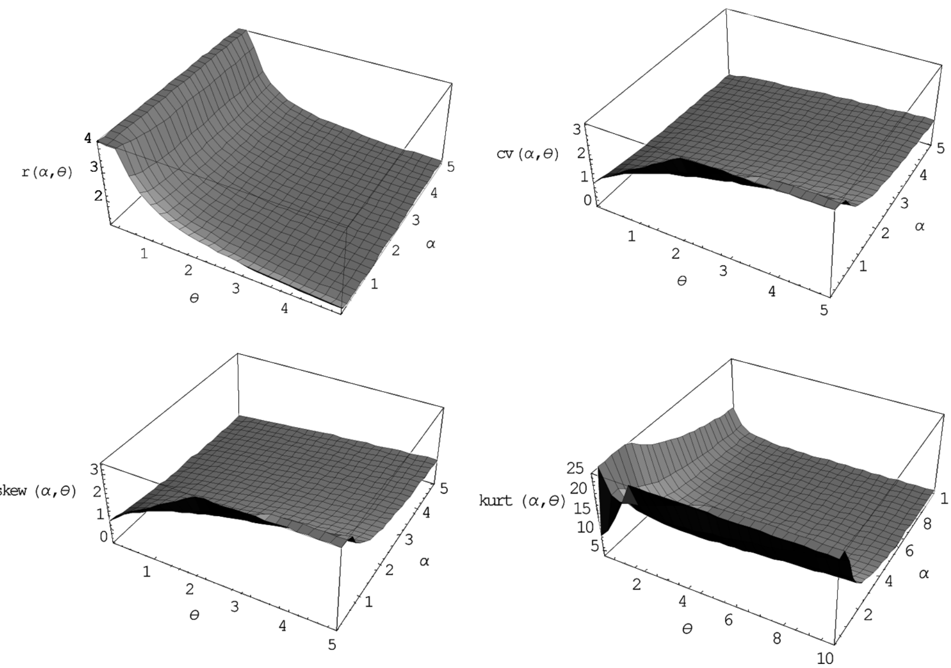

Figure 2. Graphs of dispersion, coefficient of variation, skewness, and kurtosis of the $GPL(\alpha, \theta)$.

Here, we give the distribution of sums of random variables. Truncated and weighted versions of this distribution is presented in the next.

2.4. *Distribution of Sums of Random Variables*

Let $f_{NB}(x; k, p)$ denote the density of the negative binomial distribution defined by, $f(x; k, p) = \binom{x+k-1}{x} p^k (1-p)^x$, $x = 0, 1, \ldots, k > 0$ and $0 < p < 1$.

Proposition 2.3. *Let X_1 and X_2 be two independent random variables from $GPL(\alpha_i, \theta)$, $i = 1, 2$, and let $U = X_1 + X_2$. Then the density function of the random variable U is given by*

$$f_U(u) = \sum_{i=0}^{1} \sum_{j=0}^{1} p_{ij} f_{NB}\left(u; \alpha_1 + \alpha_2 + i + j, \frac{\theta}{\theta+1}\right), \quad u = 0, 1, \ldots,$$

where $p_{ij} = \frac{\theta^{(2-(i+j))}}{(\theta+1)^2}$, $i, j = 0, 1$, $\theta > 0$.

It is well known that the distribution of sum of independent negative binomial random variables with the same probability of success is again a negative binomial distribution. The following result shows that the distribution of a sum of independent random variables from GPL distribution is a mixture of negative binomial distributions.

Proposition 2.4. *Let $X_1, \ldots, X_n$ denote independent random variables from $GPL(\alpha_i, \theta)$, for $i = 1, \ldots, n$. Then the density function of $X = \sum_{i=1}^{n} X_i$, is given by*

$$f_X(x) = \sum_{k=0}^{n} p_k f_{NB}\left(x; \alpha^* + k, \frac{\theta}{\theta+1}\right), \quad x = 0, 1, \ldots,$$

where

$$p_k = \frac{\binom{n}{k}\theta^{n-k}}{(\theta+1)^n}, \quad \textit{for } k = 0, \ldots, n; \ \textit{with} \ \sum_{k=0}^{n} p_k = 1, \ \textit{and } \alpha^* = \sum_{i=1}^{n} \alpha_i. \tag{2.3}$$

Proof. Using the mgf (2.2), the mgf of the random variable $X = \sum_{i=1}^{n} X_i$ is given by

$$M_X(t) = \left(\frac{\theta}{\theta - e^t + 1}\right)^{\alpha^*+n} \left(\frac{\theta - e^t + 2}{\theta + 1}\right)^n.$$

Now, let $M_{NB}(t; \alpha^* + k, \frac{\theta}{\theta+1}) = \left(\frac{\theta}{\theta - e^t + 1}\right)^{\alpha^*+n}$, $k = 0, \ldots, n$, be the mgf of the negative binomial with the shape parameter $\alpha^* + k$, and probability of success $\frac{\theta}{\theta+1}$. Then the mgf corresponding to the density $f_X(x)$, denoted by $M_f(t)$, could be obtained as

$$\begin{aligned}
M_f(t) &= \sum_{k=0}^{n} p_k M_{NB}\left(t; \alpha^* + k, \frac{\theta}{\theta+1}\right) \\
&= \frac{\theta^{n+\alpha^*}}{(\theta+1)^n}\left(\frac{1}{\theta - e^t + 1}\right)^{\alpha^*} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{\theta - e^t + 1}\right)^k \\
&= \frac{\theta^{n+\alpha^*}}{(\theta+1)^n}\left(\frac{1}{\theta - e^t + 1}\right)^{\alpha^*} \left(1 + \frac{1}{\theta - e^t + 1}\right)^n \\
&= \left(\frac{\theta}{\theta - e^t + 1}\right)^{\alpha^*+n} \left(\frac{\theta - e^t + 2}{\theta + 1}\right)^n,
\end{aligned}$$

which completes the proof. □

The cumulative distribution function of sum of the independent random variables from $GPL(\alpha_i, \theta)$, for $i = 1, \ldots, n$, can be expressed in terms of an incomplete beta function ratio and hence as a sum of binomial terms. Let us denote the incomplete beta function ratio by $I_q(a, b)$, and the hypergeometric function (Abramowitz and Stegun, 1972), as ${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$, where $(a)_n$ denotes the *Pochhammer's symbol*:

$$(a)_0 = 1, \quad (a)_n = a(a+1)\ldots(a+n-1), \quad n = 1, 2, \ldots$$

and $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ for $a > 0$, where $\Gamma(.)$ denotes the gamma function.

Proposition 2.5. *Let $X_1, \ldots, X_n$ is an independent random variables from $GPL(\alpha_i, \theta)$, for positive integer values of $\alpha_i s$, $i = 1, \ldots, n$. Then the cumulative distribution function* *of* $X = \sum_{i=1}^{n} X_i$, *is given by*

$$F_X(x) = \sum_{k=0}^{n} p_k I_\eta(\alpha^* + k, x + 1) = \sum_{k=0}^{n} p_k \Pr(Y_k \geq \alpha^* + k),$$

where p_k *and* α^* *are defined in* (2.3), $\eta = \frac{\theta}{\theta+1}$, $I_\eta(a, b)$ *is the incomplete beta function ratio and* $Y_k \sim Bin(\alpha^* + k + x, \eta)$.

2.5. *Truncated and Weighted Versions of the GPL Distribution*

In the most common form of truncation, the zeroes are not recorded. In this case, the zero-truncated distributions can be used as a distribution for the sizes of groups. This situation occurs in applications such as the number of offspring per family, the number of claims per claimant, and the number of occupants per car. The density function of the zero-truncated generalized Poisson–Lindley (ZTGPL) distribution is given by

$$f(x; \alpha, \theta) = \frac{\Gamma(x+\alpha)}{x!\Gamma(\alpha+1)} \frac{\theta^{\alpha+1}\left[1 - \left(\frac{\theta}{\theta+1}\right)^\alpha \left(\frac{\theta+2}{\theta+1}\right)\right]^{-1}}{(\theta+1)^{x+\alpha+1}} \left(\alpha + \frac{x+\alpha}{\theta+1}\right), \quad x = 1, 2, \ldots. \quad (2.4)$$

In Propositions 2.6–2.8, we will denote by $p_j = \frac{\theta^{1-j}}{\theta+1}$, for $j = 0, 1$, $1 - \eta = \frac{1}{\theta+1}$, and $I_\eta(a, b) = \frac{\beta_\eta(a,b)}{\beta(a,b)}$.

Proposition 2.6. *The ZTGPL distribution can be derived as a mixture of zero-truncated Poisson distribution as the form*

$$f(x; \alpha, \theta) = \sum_{j=0}^{1} p_j c_j \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!(1 - e^{-\lambda})} e^{-\lambda\theta} \lambda^{\alpha+j-1}(1 - e^{-\lambda}) d\lambda, \quad x = 1, 2, \ldots,$$

where $c_j^{-1} = \frac{(\theta+2)\Gamma(\alpha+j)}{\theta+1}\left(\theta^{-(\alpha+j)} - \frac{(\theta+1)^{-(\alpha+1)}}{\theta^{j-1}}\right)$, $j = 0, 1$.

Remark 2.1. Note that for $\alpha = 1$, the zero-truncated generalized Poisson–Lindley distribution (2.4) is the same as the zero-truncated Poisson–Lindley distribution, due to Ghitany et al. (2008).

The displaced generalized Poisson–Lindley (DGPL) distribution (left-truncated GPL distribution) is obtained by truncation of the first $r - 1$ probabilities of a $GPL(\alpha, \theta)$ distribution. It has the probability density function of the form

$$f(x; \alpha, \theta) = \frac{\Gamma(x+\alpha)(\alpha(\theta+1) + x + \alpha)(\theta+1)^{-x}/x!}{\sum_{x=r}^{\infty} \Gamma(x+\alpha)(\alpha(\theta+1) + x + \alpha)(\theta+1)^{-x}/x!}, \quad x = r, r+1, \ldots.$$

Proposition 2.7. *Let* X *is a random variable with* $GPL(\alpha, \theta)$ *distribution. Then, the left-truncated GPL distribution of* X *is given by*

$$f(x; \alpha, \theta) = \frac{\Gamma(x+\alpha)(\alpha(\theta+1) + x + \alpha)(\theta+1)^{-x}/x!}{\sum_{j=0}^{1} p_j I_{1-\eta}(r, \alpha + j)}, \quad x = r, r+1, \ldots.$$

The right-truncated generalized Poisson–Lindley (RTGPL) distribution has support $0, 1, 2, \ldots, m$ and probability density function

$$f(x; \alpha, \theta) = \frac{\Gamma(x+\alpha)(\alpha(\theta+1)+x+\alpha)(\theta+1)^{-x}/x!}{\sum_{x=0}^{m}\Gamma(x+\alpha)(\alpha(\theta+1)+x+\alpha)(\theta+1)^{-x}/x!}, \quad x = 0, 1, \ldots, m.$$

This distribution has the probability generating function of the form

$$G(s) = \sum_{j=0}^{1} p_j \frac{{}_2F_1\left(\alpha+j, -m, -m; \frac{\theta}{\theta+1}s\right)}{{}_2F_1\left(\alpha+j, -m, -m; \frac{\theta}{\theta+1}\right)},$$

where ${}_2F_1(a, b, c; z)$ is the hypergeometric function in Proposition 2.4.

Proposition 2.8. *The right-truncated GPL distribution can be expressed as*

$$f(x; \alpha, \theta) = \frac{(x+\alpha-1)!(\alpha(\theta+1)+x+\alpha)(\theta+1)^{-x}/x!}{\sum_{j=0}^{1} p_j I_\eta(\alpha+j, m+1)}, \quad x = 0, 1, \ldots, m.$$

Very often, the distribution of recorded observations produced by a certain model may differ from that under the hypothesis model, mainly because of the recording mechanism that may employ unequal probabilities of recording the observations. The observed distribution will thus have a density of the form $f_w(x) = \frac{w(x)f(x)}{E[w(X)]}$, where $f(x)$ is the original density anticipated under the hypothesized model and $w(x)$ is a function proportional to the probability with which an observation n is recorded. These model were introduced by Rao (1965) and are known as weighted models. When the weighted function $w(x)$ is equal to x, these models are known as size-biased models. The observed distribution is then termed size biased distribution.

Proposition 2.9. *Let X is a random variable with $GPL(\alpha, \theta)$ distribution. Then the size biased GPL distribution ($SBGPL$) of X is given by*

$$f(x; \alpha, \theta) = \frac{\Gamma(x+\alpha)}{(x-1)!\,\Gamma(\alpha+1)} \frac{\theta^{\alpha+2}}{(\theta+1)^{x+\alpha}(\alpha(\theta+1)+1)}\left(\alpha + \frac{x+\alpha}{\theta+1}\right), \quad x = 1, 2, \ldots,$$

with the probability generating function as

$$G_{SBGPL}(s) = \left(\frac{\theta}{\theta-s+1}\right)^{\alpha+2} \frac{\alpha(\theta-s+2)+1}{\alpha(\theta+1)+1}.$$

3. Different Methods of Estimation

3.1. *Method of Moments*

Given a random sample $x_1, \ldots, x_n$ of size n from the GPL distribution (2.1), the moment estimates (MOM), $\tilde{\theta}$ and $\tilde{\alpha}$, of θ and α are the solution of the equations:

$$\begin{cases} \theta^2\bar{x} + \theta(\bar{x}-\alpha) - (\alpha+1) = 0 \\ \theta^3\bar{x^2} + \theta^2(\bar{x^2}-\alpha) - \theta(\alpha+1)^2 - (\alpha^2+3\alpha+2) = 0. \end{cases}$$

These equations leads to the following answers of $\tilde{\theta}$ and $\tilde{\alpha}$,

$$\tilde{\theta} = \frac{(\tilde{\alpha} - \bar{x}) + \sqrt{(\bar{x} - \tilde{\alpha})^2 + 4\bar{x}(\tilde{\alpha} + 1)}}{2\bar{x}},$$

and

$$\tilde{\alpha} = \frac{-(\tilde{\theta}^2 + 2\tilde{\theta} + 3) + \sqrt{(\tilde{\theta}^2 + 2\tilde{\theta} + 3)^2 + 4(\tilde{\theta}^2(\tilde{\theta} + 1)\bar{x^2} - (\tilde{\theta} + 2))(\tilde{\theta} + 1)}}{2(\tilde{\theta} + 1)}.$$

Unfortunately, $\tilde{\theta}$ and $\tilde{\alpha}$ have no closed form, but using $\bar{x}$ and $\bar{x^2}$, we can compute $\tilde{\theta}$ and $\tilde{\alpha}$ numerically.

Theorem 3.1. *For the fixed values of α, the moment estimates $\tilde{\theta}$ of θ is consistent and asymptotically normal distribution as*:

$$\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{d} N(0, v^2(\theta)),$$

where

$$v^2(\theta) = \frac{\theta^2(\theta + 1)^2(\alpha(\theta + 1)^3 + \theta^2 + 3\theta + 1)\{2\alpha^2(\theta + 1)^2 + 4\alpha(\theta + 1) + 2\}^2}{4(\alpha(\theta + 1)^2 + 2\theta + 1)^2(\alpha(\theta + 1) + 1)}.$$

As a result of Theorem 3.1, the asymptotic $100(1 - \gamma)\%$ confidence interval for θ (for a given γ) is $\tilde{\theta} \pm Z_{\gamma/2} \frac{v(\tilde{\theta})}{\sqrt{n}}$.

3.2. *Maximum Likelihood*

In this part the maximum likelihood estimators of $GPL(\alpha, \theta)$ are considered, where both parameters are unknown. Given a random sample $x_1, x_2, \ldots, x_n$, of size n, from the GPL distribution (2.1), the log-likelihood function is of the form

$$\begin{aligned} l(\alpha, \theta) = \Big\{ & n(\alpha + 1)\ln\left(\frac{\theta}{\theta + 1}\right) + \sum_{i=1}^{n}[\ln\Gamma(x_i + \alpha) - \ln\Gamma(\alpha + 1)] \\ & + \sum_{i=1}^{n}\ln\left(\alpha + \frac{x_i + \alpha}{\theta + 1}\right) - \sum_{i=1}^{n}\ln x_i! - \sum_{i=1}^{n} x_i \ln(\theta + 1)\Big\}. \end{aligned}$$

Therefore, the normal equations are

$$\frac{\partial l}{\partial \theta} = \frac{n(\alpha + 1)}{\theta(\theta + 1)} - \sum_{i=1}^{n}\frac{x_i + \alpha}{\alpha(\theta + 1)^2 + (x_i + \alpha)(\theta + 1)} - \sum_{i=1}^{n}\frac{x_i}{\theta + 1} = 0,$$

and

$$\frac{\partial l}{\partial \alpha} = n\ln\left(\frac{\theta}{\theta + 1}\right) + \sum_{i=1}^{n}[\Psi(x_i + \alpha) - \Psi(\alpha + 1)] + \sum_{i=1}^{n}\frac{\theta + 2}{\alpha(\theta + 1) + x_i + \alpha} = 0,$$

where $\Psi(t) = \frac{\Gamma'(t)}{\Gamma(t)}$ denotes the digamma function. Unfortunately, these two nonlinear equations have no closed form of answers. Also, these equations can be solved using either Newton-Raphson, Quasi Newton-Raphson, Fisher–Scoring algorithm, or EM algorithm, by taking the moment estimates of α and θ as an initial values.

4. Random Data Generation

An algorithm, similar to the one given by Ghitany et al. (2008), can be used to generate random data from GPL distribution. Note that if V and W are two independent random variables having gamma distributions with respective parameters (α, θ) and $(\alpha + 1, \theta)$, and if Y is Poisson random variable, with parameter λ, independent of V and W, then the density function of the GPL can be written as a compound mixture of the form

$$f_{GPL}(x) = \frac{\theta}{\theta+1}\int_0^\infty f_Y(x)f_V(\lambda)d\lambda + \frac{1}{\theta+1}\int_0^\infty f_Y(x)f_W(\lambda)d\lambda.$$

To generate random data X_i, $i = 1, 2, \ldots, n$, from $GPL(\alpha, \theta)$ distribution, in regards of the above expression, one can use the following algorithm.

1. Generate U_i, $i = 1, 2, \ldots, n$, from $U(0, 1)$ distribution.
2. If $U_i \leq \frac{\theta}{\theta+1}$, generate $\lambda_i \sim Gamma(\alpha, \theta)$; otherwise, generate $\lambda_i \sim Gamma(\alpha + 1, \theta)$.
3. Generate X_i, $i = 1, 2, \ldots, n$, where $X_i \sim Poisson(\lambda)$.

4.1. *Simulation Study*

Using the above algorithm to generate random sample from the GPL distribution, a simulation study is carried out $N = 100,000$ times for each triple (α, θ, n) with $\alpha = .3, 1, 3$ and different values θ, for $n = 20$ (20) 100. The study calculates the following measures.

(i) Average bias of the moment estimators $\tilde{\mu}$ and $\tilde{\sigma^2}$ (the mean and variance of the GPL distribution), is given by

$$(1/N)\sum_{i=1}^{N}(\tilde{\mu}_i - \mu) \quad \text{and} \quad (1/N)\sum_{i=1}^{N}(\tilde{\sigma_i^2} - \sigma^2).$$

(ii) Average mean square error of the moment estimators $\tilde{\mu}$ and $\tilde{\sigma^2}$, is given by

$$(1/N)\sum_{i=1}^{N}(\tilde{\mu}_i - \mu)^2 \quad \text{and} \quad (1/N)\sum_{i=1}^{N}(\tilde{\sigma_i^2} - \sigma^2)^2.$$

Table 1 shows that the average bias, for each of the moment estimators $\tilde{\mu}$ and $\tilde{\sigma^2}$, is either positive and negative. The average bias is decreasing when θ is increased. Also, this table shows that the average mean square error of the moment estimators $\tilde{\mu}$ and $\tilde{\sigma^2}$. The average mean square error is decreasing when both θ and n are increased.

Table 1
The average bias and average mean square error (Ab, Ams) of the simulated moment estimators of the mean and variance of $GPL(\alpha, \theta)$.

	$\alpha = .3$					
	$\theta = .5$		$\theta = 1.5$		$\theta = 3$	
n	$Ab(\tilde{\mu})$	$Ab(\tilde{\sigma}^2)$	$Ab(\tilde{\mu})$	$Ab(\tilde{\sigma}^2)$	$Ab(\tilde{\mu})$	$Ab(\tilde{\sigma}^2)$
20	.001425	.001733	−.001371	−.003789	−.000186	−.001114
40	.001407	.004934	−.000649	−.003227	−.000470	−.001532
60	−.001558	−.019462	−.000683	−.002515	−.000423	−.000306
80	.002258	−.005119	.000659	.002089	−.000251	−.000349
100	−.000451	−.005119	−.000073	.000524	.000144	.000545
	$\alpha = 1$					
	$\theta = .4$		$\theta = 1.7$		$\theta = 5$	
20	−.000923	−.021111	−.000442	−.002997	.000066	−.001125
40	−.003864	.009872	.000450	.002836	−.000638	−.001256
60	−.001367	.007885	.001026	.002879	−.000051	.000941
80	−.004379	−.042473	.000810	.003560	.000188	.000322
100	.002823	.010161	−.000573	−.001115	.000026	.000765
	$\alpha = 3$					
	$\theta = .7$		$\theta = 2$		$\theta = 4$	
20	−.000127	.018451	.000640	.000153	.000654	.000967
40	.002108	.002935	−.000719	−.000351	.000032	−.000873
60	.005011	.026769	.002081	−.000938	−.000355	−.001540
80	−.002208	−.002269	−.001182	.001646	−.000543	.000456
100	.001853	.007695	−.000442	−.000606	.000254	−.0000315
	$\alpha = 3$					
	$\theta = .5$		$\theta = 1.5$		$\theta = 3$	
n	$Ams(\tilde{\mu})$	$Ams(\widetilde{\sigma}^2)$	$Ams(\tilde{\mu})$	$Ams(\widetilde{\sigma}^2)$	$Ams(\tilde{\mu})$	$Ams(\widetilde{\sigma}^2)$
20	.331890	18.327	.043865	.509168	.013233	.067458
40	.165794	9.2571	.022231	.257797	.006617	.033904
60	.112425	6.12697	.014473	.169670	.004512	.023283
80	.083083	4.5343	.011093	.128382	.003303	.017049
100	.067300	3.7117	.008809	.101622	.002680	.013840
	$\alpha = 1$					
	$\theta = .4$		$\theta = 1.7$		$\theta = 5$	
20	.814440	73.762	.067747	.697584	.014273	.045220
40	.409449	37.003	.034109	.354105	.007154	.022629
60	.269628	24.628	.022751	.235922	.004791	.015258
80	.204821	18.207	.017002	.178688	.003571	.011431
100	.164639	14.751	.013436	.139573	.002869	.009306

(continued)

Table 1
Continued

$\alpha = 3$						
	$\theta = .7$		$\theta = 2$		$\theta = 4$	
20	.643396	32.712	.127345	1.4387	.050290	.260725
40	.322223	15.884	.063537	.706423	.025101	.125923
60	.214870	10.703	.042744	.468639	.016950	.084504
80	.161135	7.8924	.032418	.350007	.012650	.064237
100	.128397	6.2807	.025688	.283804	.010070	.050458

Table 2
Expected frequencies, ML estimates, log-likelihood, χ^2-statistic, and p-values obtained by fitting Poisson, Hermite, negative binomial, Poisson–Lindley, and GPL distributions to data

Distribution of mistakes in copying groups of random digits. Data from Sankaran (1970).

		Expected frequencies			
No. errors per group	Observed frequencies	Poisson	Hermite	Poisson–Lindley	GPL
0	35	27.4	34.2	33.1	34.4
1	11	21.5	11.7	15.3	13.9
2	8	8.4	9.6	6.8	6.4
3	4	2.2	2.8	2.9	2.9
4	2	.4	1.3	1.2	1.3
$\hat{\alpha}_{MLE}$	–	–	–	–	.6703
$\hat{\theta}_{MLE}$	–	.783	–	1.7434	1.3875
Log-likelihood	–	−64.16	–	−73.351	−73.229
χ^2	–	8.05	.067	2.20	1.71
P-value	–	.0045	.7958	.1380	.1910

Accidents to 647 women working on high explosive shells. Data from Sankaran (1970).

		Expected frequencies			
No. of accidents	Observed frequencies	Poisson	Neg. binomial	Poisson–Lindley	GPL
0	447	406	442	439.5	446.4
1	132	189	140	142.8	133.7
2	42	45	45	45	44.5
3	21	7	14	13.9	14.9
4	3	1	5	4.2	5
≥ 5	2	.1	2	1.3	1.7
$\hat{\alpha}$	–	–	.8651	–	.7364
$\hat{\theta}$	–	.465	.6503	2.729	2.245
Log-likelihood	–	−541.79	−592.27	−592.71	−592.12
χ^2	–	61.08	4.44	4.82	3.09
P-value	–	0	.1097	.1855	.2133

5. Applications

Two sets of real data are considered. The first set of data represents the mistakes in copying groups of random digits and the second set are the number of accidents to 647 women working on high explosive shells in 5 weeks. Sankaran (1970) used these two sets of data for fitting the Poisson–Lindley distribution (1.1). His results were based on the moments estimates of the parameter. For better comparison, we find the maximum likelihood estimates of the parameter θ, in Poisson-Lindley distribution (1.1), for two sets of data. Also, the expected frequencies are based on the maximum likelihood estimates of the parameter. Note that the maximum likelihood estimates of the present two-parameter (GPL) distribution are computed by the quasi Newton method. Table 2 gives the comparison of observed and expected frequencies for the Poisson, Hermite, negative binomial, Poisson–Lindley, and the generalized Poisson–Lindley distributions. Also, the log-likelihood, χ^2-statistic, and p-value are presented in this table.

The present two-parameter (GPL) distribution appears to give a satisfactory fit in both cases, whereas the Poisson distribution does not. The p-values of the test for fitting the GPL distribution to the two sets of data shows that, the null hypothesis H_0 ("distribution of the data is GPL") cannot be rejected; indeed, the close agreement between the observed and expected frequencies and greater log-likelihood suggest that the $GPL(\hat{\alpha}, \hat{\theta})$ distribution provide a "good fit" to these two sets of data.

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