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A Log-Linear Model for the Birnbaum–Saunders Distribution

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The Birnbaum–Saunders distribution was derived to model times to failure for metals subject to fatigue. In this article, we formulate and develop a log-linear model for the Birnbaum–Saunders distribution. The model may be used for accelerated life testing or to compare the median lives of several populations. Methods of analyzing data for this log-linear model are discussed, with maximum likelihood and least squares methods being compared. It is found that, for commonly occurring conditions, the notorious intractability of the Birnbaum–Saunders distribution is not a serious problem because least squares and normal-theory procedures provide a reasonable alternative.

KEY WORDS: Bias approximation; Fatigue life; Iterative least squares; Least squares; Maximum likelihood.

Fatigue is the structural damage that results when a material is subjected to fluctuating stresses and strains. It was first recognized as a cause of failure in carriage axles in the early 1800s (Stulen and Schulte 1986). It was blamed in the 1950s for the crashes of two British Comet airliners (Bishop 1986). In the latter tragedies, fatigue resulted from stresses on the cabin wall over repeated take-offs and landings; in both accidents, the cabins exploded during the initial climb through 30,000–35,000 feet (Bishop 1986).

Statistical models of the fatigue process yield descriptions of randomly varying failure times of fatiguing materials. The Birnbaum–Saunders (1969a,b) distribution, for example, was derived from a model showing that failure is due to the development and growth of a dominant crack. Desmond (1985) demonstrated that the Birnbaum–Saunders distribution describes quite generally the failure time T that occurs when some kind of accumulating damage $D(t)$ exceeds a threshold: $T = \inf\{t\colon D(t) > \text{threshold}\}$.

If we let T be a random variable from the Birnbaum–Saunders distribution, then the cdf may be written as $F(t) = \Phi\{\alpha^{-1}[(t/\beta)^{1/2} - (\beta/t)^{1/2}]\}(0 < t < \infty)$, where $\Phi(x)$ is the standard normal cdf. The distribution has two positive parameters—α, the shape parameter, and β, which is both the scale parameter and the median value of the distribution.

Statistical analysis for this distribution has been limited to considering single samples. Birnbaum and Saunders (1969b) derived the maximum likelihood estimators (MLE's) for α and β. In the same article, they also considered another estimator for β, which they called the mean mean estimator. Ahmad (1988) derived a jackknife estimator for β based on the mean mean estimator. Engelhardt, Bain, and Wright (1981) provided tests of hypothesis and confidence intervals, based on the MLE's, using Monte Carlo techniques and asymptotic results.

In this article, we consider a log-linear model for the Birnbaum–Saunders distribution. Such a model may be motivated by applications in the characterization of materials. Engineers have devised empirical laws describing the time to fatigue failure of materials subjected to different patterns of cycling forces. Materials can then be characterized by the values of parameters in these empirical laws. Such characterizations are important for predicting the performance of the materials under different conditions. For example, one is often interested in predicting fatigue life at low stress levels. To conduct testing at low stress levels is very time consuming, however. To avoid this problem, failures of specimens at higher stress levels are observed, and then the failure times at lower stress levels are predicted from the empirical characterization of the material's properties. This type of testing is known as accelerated life testing.

Suppose a metal specimen is subjected to cyclic stretching and compressing such that either the stress range per cycle, S, or the strain range per cycle, E, or the work per cycle, W_c, is constant. Let N be the number of cycles to failure of the specimen. Several empirical laws that have been found useful are of the form

$$\ln(N) = a + bx, \tag{1}$$

where x is either S, E, $\ln(S)$, or $\ln(E)$ (American Society for Testing Materials 1981) or $\ln(W_c)$ (Ellyin 1988; Garud 1981). The choice of x depends on the metal and the test conditions. Table 1 and Figure 1 (Sec. 6) exhibit data from Brown and Miller (1978) in which $\ln(W_c)$ is an appropriate choice. The values of W_c were computed by Bhargava (1984). These data represent results of fatigue tests on 1% Cr-Mo-V steel (Brown and Miller 1978, 1981; Kanazawa, Miller, and Brown 1977). Cylindrical specimens were subjected to combined torsional and axial loads over constant-amplitude cycles until failure. These data are analyzed further in Section 6.

In practice, (1) holds in some average sense only, so that estimation of a and b is a statistical problem. To provide an appropriate statistical framework for the estimation problem, we will make the following assumptions:

1. The number of cycles to failure has a Birnbaum–Saunders distribution, and N in (1) represents the median of that distribution.
2. The shape parameter α is independent of the work per cycle.

Assumption 1 is justified by Desmond's (1985) arguments. We may posit, for example, that the ultimate breakdown of the specimen is due to exceeding some critical value of a crack whose extension is driven by the cyclic forces. Assumption 2 may not always be justified; the data in Table 1 provide an example in which the assumption does appear justified, as we will see in Section 6.

If T has a Birnbaum–Saunders distribution with median β and shape parameter α and if $c > 0$ is a constant, then cT has a Birnbaum–Saunders distribution with median $c\beta$. So, if the assumptions are valid, then Equation (1) may appropriately be revised as follows:

$$N = e^{a+bx}\delta, \tag{2}$$

where δ is distributed as a Birnbaum–Saunders distribution with a shape parameter α and a scale parameter 1. In (2), N is now a random variable. Taking logarithms, we see that

$$\ln(N) = a + bx + \ln(\delta). \tag{3}$$

We refer to such a model as log-linear. Note that the random noise is now additive.

The error term $\ln(\delta)$ in (3) has what we call a sinh-normal distribution. That distribution is described in Section 1. Section 2 generalizes the preceding example of a log-linear model. Inference for the unknown parameters of such models is then discussed—point estimation in Section 3, hypothesis testing and interval estimation in Section 4. Section 5 summarizes a Monte Carlo study of the ideas presented in Sections 3 and 4, and Section 6 analyzes the example introduced previously.

1. THE SINH-NORMAL DISTRIBUTION

In the example of the previous section, a linear model was found for the logarithm of a Birnbaum–Saunders variate. This section summarizes the distribution of a logarithmically transformed Birnbaum–Saunders variate.

Let Y be a random variable with a cumulative distribution given by $F(y) = \Phi\{(2/\alpha)\sinh[(y - \gamma)/\sigma]\}(-\infty < y < \infty)$. Then Y is said to have a sinh-normal distribution, which we will denote by SN(α, γ, σ), and γ is a location parameter, σ is a scale parameter, and α is a shape parameter; α and σ must be positive, while γ is unrestricted.

The following relationship between the sinh-normal and the Birnbaum–Saunders distributions can be easily established (see Rieck 1989).

Table 1. Brown and Miller's Biaxial Fatigue Data

Work MJ/m3	*Life cycles*	*Work MJ/m3*	*Life cycles*	*Work MJ/m3*	*Life cycles*	*Work MJ/m3*	*Life cycles*
11.5	3280	24.0	804	40.1	750	60.3	283
13.0	5046	24.6	1093	40.1	316	60.5	212
14.3	1563	25.2	1125	43.0	456	62.1	327
15.6	4707	25.5	884	44.1	552	62.8	373
16.0	977	26.3	1300	46.5	355	66.5	125
17.3	2834	27.9	852	47.3	242	67.0	187
19.3	2266	28.3	580	48.7	190	67.1	135
21.1	2208	28.4	1066	52.9	127	67.9	245
21.5	1040	28.6	1114	56.6	185	68.8	137
22.6	700	30.9	386	59.9	255	75.4	200
22.6	1583	31.9	745	60.2	195	100.5	190
24.0	482	34.5	736				

Theorem 1.1. Let T be a random variable from a Birnbaum–Saunders distribution with shape parameter α and scale parameter β. Then the distribution of $Y = \ln(T)$ is a sinh-normal distribution with shape, location, and scale parameter given by α, $\gamma = \ln(\beta)$, and $\sigma = 2$, respectively.

The density function for a sinh-normal distribution is given by

$$f(y) = [2/(\alpha\sigma\sqrt{2\pi})]\cosh[(y - \gamma)/\sigma] \times \exp\{(-2/\alpha^2)\sinh^2[(y - \gamma)/\sigma]\}.$$

This distribution has some interesting properties, among which are the following (see Rieck 1989):

1. The distribution is symmetric about the location parameter γ.
2. The distribution is strongly unimodal for $\alpha \leq 2$ and bimodal for $\alpha > 2$.
3. The mean and variance are given by $E(Y) = \gamma$ and $\text{var}(Y) = \sigma^2\omega(\alpha)$, where $\omega(\alpha)$ is the variance when $\sigma = 1$. There is no closed-form expression for $\omega(\alpha)$, but Rieck (1989) provided asymptotic approximations to it for small and large α.
4. If $Y_\alpha \sim \text{SN}(\alpha, \gamma, \sigma)$, then $S_\alpha = (Y_\alpha - \gamma)/(.5\alpha\sigma)$ converges in distribution to the standard normal distribution as α approaches 0.

2. THE LOG-LINEAR MODEL

In this section, we generalize the log-linear model (1)–(3) by allowing for an arbitrary number of explanatory variables.

Let $T_1, T_2, \ldots, T_n$ be independent random variables in which the distribution of T_i is a Birnbaum–Saunders distribution with shape parameter α_i and scale parameter β_i. The distribution of T_i is assumed to depend on a set of p explanatory variables, denoted by $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})$, as follows:

1. $\beta_i = \exp(\mathbf{x}_i'\boldsymbol{\theta})$ for $i = 1, 2, \ldots, n$, where $\boldsymbol{\theta}' = (\theta_1, \theta_2, \ldots, \theta_p)$ is a vector of unknown parameters to be estimated.
2. The shape parameter is independent of the explanatory vector $\mathbf{x}_i$; that is, $\alpha_i = \alpha$ for $i = 1, 2, \ldots, n$.

Birnbaum and Saunders (1969b) showed that if $c > 0$, then cT_i has a Birnbaum–Saunders distribution with shape parameter α and scale parameter $c\beta_i$. Using this fact and the assumptions mentioned previously, we see that T_i may be expressed as $T_i = \exp(\mathbf{x}_i'\boldsymbol{\theta})\delta_i$, where δ_i is distributed according to a Birnbaum–Saunders distribution with shape parameter α and scale parameter 1. If we assume that $Y_i = \ln(T_i)$, then we have that $Y_i = \mathbf{x}_i'\boldsymbol{\theta} + \ln(\delta_i)$. This gives us a log-linear model for the Birnbaum–Saunders distribution in which $\ln(\delta_i)$ is the error term for the model. Using Theorem 1.1, we see that this error term is distributed as $\text{SN}(\alpha, 0, 2)$. In the sections that follow, we will let ε_i denote $\ln(\delta_i)$. We will refer to the log-linear model

$$Y_i = \mathbf{x}_i'\boldsymbol{\theta} + \varepsilon_i. \tag{4}$$

3. POINT ESTIMATION

In this section, we discuss maximum likelihood and least squares estimation for our log-linear model.

Maximum Likelihood: Principles

Let $y_1, y_2, \ldots, y_n$ be n independent observations from Model (4). Then the log-likelihood function may be written as

$$\ln(L) = \sum_i \ln(W_i) - \sum_i Z_i^2/2 + \text{constant},$$

where

$$W_i = (2/\alpha)\cosh[(y_i - \mathbf{x}_i'\boldsymbol{\theta})/2] \tag{5}$$

and

$$Z_i = (2/\alpha)\sinh[(y_i - \mathbf{x}_i'\boldsymbol{\theta})/2]. \tag{6}$$

Differentiating $\ln(L)$ with respect to θ_j and α, we obtain the likelihood equations

$$\partial \ln(L)/\partial\theta_j = \left(\frac{1}{2}\right) \sum_i x_{ij}\{Z_iW_i - Z_i/W_i\} = 0 \quad \text{for } j = 1, \ldots, p \tag{7}$$

and

$$\partial \ln(L)/\partial\alpha = -n/\alpha + (1/\alpha) \sum_i Z_i^2 = 0. \tag{8}$$

An expression for the MLE of α^2 in terms of the MLE for $\boldsymbol{\theta}$ may be found by solving Equation (8) for α^2 and is given by

$$\hat{\alpha}^2 = (4/n) \sum_i \sinh^2[(y_i - \mathbf{x}_i'\hat{\boldsymbol{\theta}})/2], \tag{9}$$

where $\hat{\boldsymbol{\theta}}$ is the MLE for the vector $\boldsymbol{\theta}$. The MLE for the vector $\boldsymbol{\theta}$, however, has no closed-form expression, so a numerical procedure must be used to solve Equations (7). An iterative ordinary least squares procedure is discussed at the end of this section.

The asymptotic distribution of the MLE may be found by adapting methods described by Chanda (1954) and Bradley and Gart (1962). Suppose that data are collected so that as the sample size n increases (a) $|x_{ij}| \leq B$ for some bound B for all i and j, and (b) for some positive definite matrix M,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i'/n = M.$$

Then (a) there exists at least one solution of (7)–(8) that is a consistent estimate of the true parameter $\boldsymbol{\eta} = (\boldsymbol{\theta}', \alpha)'$; (b) of all possible solutions of (7)–(8), one and only one tends in probability to $\boldsymbol{\eta}$; and (c) $n^{1/2}(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta})$ converges in distribution to a multivariate normal random variable with mean $\mathbf{0}$ and covariance matrix equal to the inverse of

$$\begin{bmatrix} C(\alpha)M/4 & \mathbf{0} \\ \mathbf{0}' & 2/\alpha^2 \end{bmatrix},$$

where $\mathbf{0}$ is a column of p zeros and $C(\alpha)$ is a function discussed in the Appendix. For proofs of these results, see Rieck and Nedelman (1990). If $\hat{\alpha}$ is the MLE of α, then the asymptotic covariance matrix of $\hat{\boldsymbol{\eta}}$ may be estimated by

$$\widehat{\text{cov}}(\hat{\boldsymbol{\eta}}) = \begin{bmatrix} 4(X'X)^{-1}/C(\hat{\alpha}) & \mathbf{0} \\ \mathbf{0}' & \hat{\alpha}^2/(2n) \end{bmatrix}, \quad (10)$$

where $X' = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is assumed to have rank p.

Multiple Maxima of the Likelihood

The bimodality of the sinh-normal density when $\alpha > 2$ can cause multiple maxima of the likelihood. Rieck (1989) provided an example in which there are three solutions to the likelihood equations; two solutions are maxima, but the third solution, which is also the least squares solution, is a saddle point.

The first author's experience with many data sets at an aircraft-engine company suggests that $\alpha > 2$ is unusual in practice. Moreover, recognizing that α is within 2% of the coefficient of variation of a Birnbaum–Saunders variate whenever $0 < \alpha < 1$, a referee has pointed out that in much of engineering practice a coefficient of variation exceeding 1 would not be tolerated (new measurements under more controlled conditions would likely be obtained).

It is easily shown that if $\alpha < 2$ is known, then the MLE of $\boldsymbol{\theta}$ is unique, provided that X has rank p (Rieck 1989). Stronger results about uniqueness elude us. Nonetheless, we believe that in most cases when Model (4) is appropriate the likelihood will have a unique maximum.

Finite Sample Bias of the MLE's

In this subsection, we investigate a reduced-bias estimator of α^2. The MLE's are asymptotically unbiased and efficient, but for small samples the MLE's may not be unbiased and may not have optimal mean squared error (MSE). Here, we obtain approximations to the biases of the MLE's using an approach similar to that used by Young and Bakir (1987) for the log-gamma regression model. Although $\hat{\boldsymbol{\theta}}$ is found to be unbiased to order $1/n$, $\hat{\alpha}^2$ is not. We seek a reduced-bias estimator of α^2 to improve bias and, we might hope, also improve MSE performance in small samples.

Let $\boldsymbol{\tau}' = (\boldsymbol{\theta}', \alpha^2)$, $\hat{\boldsymbol{\tau}}$ as the MLE of $\boldsymbol{\tau}$, and $b_j = \mathrm{E}(\hat{\tau}_j) - \tau_j$ represent the bias of τ_j for $j = 1, \ldots, p + 1$. Let $f(y_i)$ denote the sinh-normal density with shape parameter α, location parameter $\mathbf{x}_i'\boldsymbol{\theta}$, and scale parameter 2. To obtain expressions for the approximate biases, we define the following quantities:

$$U_{r(i)} = \partial \ln[f(y_i)]/\partial\tau_r,$$

$$V_{rs(i)} = \partial^2 \ln[f(y_i)]/\partial\tau_r\partial\tau_s,$$

$$W_{rst(i)} = \partial^3 \ln[f(y_i)]/\partial\tau_r\partial\tau_s\partial\tau_t,$$

$$I_{rs} = \mathrm{E}\left\{-\sum_i V_{rs(i)}\right\},$$

$$K_{rst} = \mathrm{E}\left\{\sum_i W_{rst(i)}\right\},$$

and

$$J_{r,st} = \mathrm{E}\left\{\sum_i U_{r(i)}V_{st(i)}\right\}$$

for $r, s, t = 1, 2, \ldots, p + 1$. These quantities were evaluated by Rieck and Nedelman (1990).

Using the quantities mentioned previously, the bias of τ_r is given, to order $1/n$, by

$$b_r = \left(\frac{1}{2}\right) \sum_s \sum_t \sum_u I^{rs} I^{tu} (K_{stu} + 2J_{t,su}),$$

where I^{rs} is the rs element of the inverse of the Fisher information matrix and all of the preceding summations are over the values $1, 2, \ldots, p + 1$ (e.g., see Young and Bakir 1987). The resulting first-order approximations for the biases are

$$b_r = 0 \quad \text{for } r = 1, 2, \ldots, p$$

and

$$b_{p+1} = -\{[2 + (4/\alpha^2)]/C(\alpha)\}\{p\alpha^2/n\}.$$

These results indicate that the MLE's for the coefficients of the log-linear model are unbiased to order $1/n$. We see, however, that the bias for the MLE of the square of the shape parameter is negative and that the bias increases as the number of explanatory variables increases. Since $\mathrm{var}(\hat{\alpha}) = \mathrm{E}(\hat{\alpha}^2) - \mathrm{E}(\hat{\alpha})^2 > 0$ and $\hat{\alpha}^2$ has a negative bias, $\hat{\alpha}$ is also negatively biased. The bias is also affected by the value of the shape parameter itself.

To find a reduced-bias estimator of α^2, note that the expected value of the MLE of α^2 to order $1/n$ is

$$\mathrm{E}(\hat{\alpha}^2) = \alpha^2 - pA(\alpha)\alpha^2/n$$
$$= \{[n - pA(\alpha)]/n\}\alpha^2,$$

where $A(\alpha) = \{[2 + (4/\alpha^2)]/C(\alpha)\}$. This suggests that a reduced-bias estimator of α^2 would be

$$\ddot{\alpha}^2 = \{n/[n - pA(\hat{\alpha})]\}\hat{\alpha}^2. \tag{11}$$

The performance of this estimator is investigated by Monte Carlo in Section 5.

Least Squares

Least squares estimation yields a closed-form estimate for $\boldsymbol{\theta}$ in Model (4). Although, as we show later, the least squares estimator (LSE) of $\boldsymbol{\theta}$ is not as efficient as the MLE, nevertheless it provides an unbiased estimator for $\boldsymbol{\theta}$ and is highly efficient for small values of α.

Property 3 of Section 1 implies that, for Model (4), $E(\varepsilon_i) = 0$ and $\text{var}(\varepsilon_i) = 4\omega(\alpha)$ $(i = 1, \ldots, n)$. If the observations $y_1, \ldots, y_n$ are independent, then $\text{cov}(\varepsilon_i, \varepsilon_j) = 0$ $(i, j = 1, \ldots, n)$. With these conditions on the ε_i's, the best linear unbiased estimator of $\boldsymbol{\theta}$ may be found by using ordinary least squares and is given by

$$\tilde{\boldsymbol{\theta}} = (X'X)^{-1}X'\mathbf{Y}, \tag{12}$$

where $\mathbf{Y}$ is the column vector of the y_i's and X is as previously defined. The covariance matrix of $\tilde{\boldsymbol{\theta}}$ is $\text{cov}(\tilde{\boldsymbol{\theta}}) = 4\omega(\alpha)(X'X)^{-1}$, and an unbiased estimator of $\omega(\alpha)$ is given by

$$\tilde{\omega}(\alpha) = \sum_i (y_i - \mathbf{x}_i'\tilde{\boldsymbol{\theta}})^2/[4(n - p)].$$

The efficiency of $\tilde{\boldsymbol{\theta}}$ may be found by using the covariance matrix (10). Based on this information, the efficiency of $\tilde{\boldsymbol{\theta}}$ is a function of α and is $1/[C(\alpha)\omega(\alpha)]$. In Table 2, we present the efficiency for the LSE of $\boldsymbol{\theta}$ for various values of α. From this table we see that the LSE of $\boldsymbol{\theta}$ is highly efficient for small values of α and that the efficiency decreases as α increases. The last four columns of Table 2 are estimates of the small-sample relative efficiency of the LSE compared to MLE. These estimated relative efficiencies are based on simulation results, which are discussed in Section 5. The result that the LSE for $\boldsymbol{\theta}$ is highly efficient for small values of α is to be expected by Property 4 of Section 1.

An Iterative Ordinary Least Squares Procedure to Calculate the MLE's

No closed-form solution for the MLE of θ is available, so a numerical procedure must be used. Two popular procedures are the Newton–Raphson method and Fisher's method of scoring. The former is a generic method discussed in most books on numerical methods (e.g., Kennedy and Gentle 1980). In this section, we will concentrate on Fisher's method of scoring. It will be shown that the method of scoring is equivalent to an iterative ordinary least squares procedure.

Table 2. Least Squares Efficiency With Respect to Alpha

		Monte Carlo Results			
		θ_1		θ_2	
Alpha	Efficiency	N = 10	N = 20	N = 10	N = 20
.5	.99	1.00	1.00	.99	1.00
1.5	.81	.88	.84	.89	.84

Using the same approach as Dobson (1983), we see that the iterative procedure for the method of scoring is given by $\boldsymbol{\eta}^{(m)} = \boldsymbol{\eta}^{(m-1)} + [I(\boldsymbol{\eta}^{(m-1)})]^{-1}\mathbf{U}^{(m-1)}$, where $\boldsymbol{\eta}^{(m)\prime} = (\boldsymbol{\theta}^{(m)\prime}, \alpha^{(m)})$ is the vector of parameter estimates at the mth iteration, $I(\boldsymbol{\eta}^{(m-1)})$ is the information matrix evaluated at $\boldsymbol{\eta}^{(m-1)}$, and $\mathbf{U}^{(m-1)}$ is the vector of first derivatives of the log-likelihood function evaluated at $\boldsymbol{\eta}^{(m-1)}$. Substituting in the appropriate values for the quantities mentioned previously, we obtain the following system of equations (Rieck 1989):

$$X'X\boldsymbol{\theta}^{(m)} = X'\mathbf{Y}^* \tag{13}$$

and

$$\alpha^{(m)} = [\alpha^{(m-1)}/2]\left\{1 + \sum_i Z_i^2/n\right\}, \tag{14}$$

where $\mathbf{Y}^* = X\boldsymbol{\theta}^{(m-1)} + [2/C(\alpha^{(m-1)})]\mathbf{R}$ and $\mathbf{R}$ is a column vector with the ith component given by $[Z_iW_i - (Z_i/W_i)]$, where W_i and Z_i are defined in (5)–(6) and are evaluated at $(\boldsymbol{\theta}^{(m-1)}, \alpha^{(m-1)})$. The system of equations given in (13) has the same form as the normal equations of ordinary least squares, except that these equations must be solved iteratively, since $\mathbf{Y}^*$ depends on the parameters. Therefore, our iterative least squares procedure would be as follows:

1. Find $\boldsymbol{\theta}^{(0)}$ using ordinary least squares (12). Find $\alpha^{(0)}$ from $\boldsymbol{\theta}^{(0)}$ using the square root of the closed-form solution (3.5).
2. Put $m - 1 = 0$.
3. Calculate $\mathbf{R}$ and then $\mathbf{Y}^*$ using $\alpha^{(m-1)}$ and $\boldsymbol{\theta}^{(m-1)}$.
4. Calculate $\boldsymbol{\theta}^{(m)}$ using ordinary least squares with X and $\mathbf{Y}^*$, as in (3.9). Calculate $\alpha^{(m)}$ from (3.10) using $\boldsymbol{\theta}^{(m-1)}$ and $\alpha^{(m-1)}$ in Z_i.
5. If $|\boldsymbol{\theta}^{(m)} - \boldsymbol{\theta}^{(m-1)}|$ and $|\alpha^{(m)} - \alpha^{(m-1)}|$ are sufficiently small, then go to step 7.
6. Add 1 to $m - 1$ and go to step 3.
7. Stop and use $\boldsymbol{\eta}^{(m)\prime} = (\boldsymbol{\theta}^{(m)\prime}, \alpha^{(m)})$ as the MLE.

Rieck (1989) compared the Newton–Raphson and scoring methods. For Newton–Raphson, the number of computations per iteration increases with the sample size, which is not the case for the method of scoring. But Newton–Raphson generally requires fewer iterations. Newton–Raphson was found superior in a simulation study with one or two parameters and sample sizes 10 or 20.

4. HYPOTHESIS TESTING AND INTERVAL ESTIMATION

The linear model developed in the previous sections can be used in a wide variety of applications—for example, to compare the location parameters of several populations or to assess environmental and control-variable effects in life-testing situations. Hypothesis testing and interval estimation are useful in such applications. The MLE of θ is mathematically intractable, and the exact distribution for the LSE of θ is difficult to obtain. Therefore, we rely on large-sample asymptotic results and/or the robustness of normal-theory procedures to make inferences concerning the linear-model coefficients.

The sinh-normal distribution is symmetric, and for small α it is almost normal. Moreover, since the shape parameter of ε_i is independent of i, Model (4) exhibits homoscedasticity. Normal-theory procedures for means are known to be robust to nonnormality and even more so if symmetry and homoscedasticity are retained (e.g., Kendall and Stuart 1974, 1976). Hence, for example, t or F tests may be used to compare location parameters in two-sample or multisample situations. Or interval estimates of linear-model coefficients may be constructed around LSE's using the t distribution and the MSE estimate of variance.

If the sample size is large, likelihood procedures may be applied. The generalized likelihood ratio (LR) test may be used to test hypotheses about subsets of parameters (e.g., see Dobson 1983). Or interval estimates of linear-model coefficients may be constructed around MLE's using the z distribution and the information-matrix estimate of variance.

The validity of the assertions in this section will be explored via Monte Carlo simulation in Section 5.

5. A MONTE CARLO STUDY

This section summarizes a Monte Carlo experiment (see Rieck 1989) designed to answer the following three questons:

1. Is the reduced-bias estimator for α^2 (11) better than the MLE for α^2 (9)?
2. How do LSE and MLE compare in finite samples?
3. How well do normal-theory and asymptotic procedures for hypothesis testing work with small samples?

These questions were investigated for the special case of comparing two location parameters, which corresponds to a model of the form $y = \theta_1 + \theta_2 x + \varepsilon$, where $x = 0$ for the first population and $x = 1$ for the second population.

In the simulation, the values of α used were .5, 1.0, 1.5, and 2.0; θ_1 was set to 0; and the values of θ_2 used were 0, $.2s(\alpha), \ldots, .8s(\alpha)$, and $s(\alpha)$, where $s(\alpha)$ is the standard deviation for a sinh-normal distribution with shape, location, and scale parameters α, 0, and 1, respectively. Sample sizes were either $N = 10$ or $N = 20$ for both samples. The errors ε were generated from $\mathrm{SN}(\alpha, 0, 1)$ by first sampling z from $N(0, 1)$, using the algorithm of Chen (1971) and then putting $\varepsilon = \sinh^{-1}(\alpha z/2)$.

For each combination of α, θ_2, and N, 5,000 replications were simulated. The three preceding questions were addressed as follows:

1.–2. For each replication, the vector $\boldsymbol{\theta}$ was estimated by MLE and LSE. The iterative-ordinary-least-squares procedure was used for the MLE, and α^2 was estimated by its MLE (9) and its adjusted MLE (11). Means and variances of the estimates were accumulated over the 5,000 replications.

3. For each replication, the hypotheses that $\theta_2 = 0$ versus $\theta_2 \neq 0$ were tested using two tests, a standard t test (t) based on a t distribution with $2N - 2$ df and an LR test, based on a chi-squared distribution with 1 df. Three levels of significance were used—.1, .05, and .01. The frequencies of rejection were accumulated over the 5,000 replications.

Results of the simulations will be discussed and summarized in Tables 2–5. To conserve space, results for $\alpha = .5$ and $\alpha = 1.5$ only are reported. For more complete results, see Rieck (1989).

Comparison of Estimators for α and α^2

In Table 3, we compare the MLE of α^2 with the reduced-bias (adjusted) estimator. Also included in the table is a comparison of the MLE of α and the square root of the adjusted estimator of α^2. The values in the table are based on 5,000 runs when θ_2 was set to 0. As can be seen in the table, there is a negative bias for the MLE's of α^2 and α. It can also be seen that the adjusted estimator of α^2 is virtually unbiased and its square root is almost unbiased for α. In terms of MSE, the adjusted estimator of α is slightly better than the MLE, but the adjusted estimator of α^2 does worse than the MLE.

LSE Versus MLE

The last four columns of Table 2 show relative efficiencies of the LSE's for θ_1 and θ_2 compared to the MLE's for the case when $\theta_1 = \theta_2 = 0$. The LSE's are known to be unbiased. The MLE's appeared unbiased in the Monte Carlo experiments, an observation consistent with the approximation to the bias

Table 3. Comparison of Estimates for Alpha

		α				α^2			
		MLE		Adjusted		MLE		Adjusted	
N	True alpha	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
10	.5	.47	.007	.50	.007	.23	.006	.25	.007
	1.5	1.39	.072	1.47	.068	1.96	.547	2.24	.617
20	.5	.48	.004	.50	.003	.24	.003	.25	.004
	1.5	1.44	.031	1.49	.030	2.11	.251	2.25	.264

found in Section 3. The relative efficiencies are less than or equal to 1, suggesting the superiority of MLE over LSE even in small samples. The nearness of the relative efficiencies to 1, especially for small α, adds attractiveness to the computationally more tractable LSE, however.

Hypothesis Testing

Two questions are of interest when comparing the two tests of hypothesis: (a) How do observed Type I error rates compare to nominal levels of significance? (b) What are the power functions of the tests?

To answer (a), the simulations runs with $\theta_2 = 0$ are relevant. Table 4 displays observed Type I error rates for the two tests when the nominal levels of significance were set at .1, .05, and .01. The values with asterisks in the table are observations beyond two-sigma limits from the nominal values. Over the three nominal values combined, the t test exceeded the limits 10.4% of the time and the LR test 79.2%.

The simulated power functions for the two tests are shown in Table 5. The nominal level of significance there was .05; results were similar for nominal levels of .1 and .01. The power function is defined so that with argument $\theta_2/s(\alpha)$ it expresses the probability of rejecting the null hypothesis when the sample was generated using that value of θ_2. Recall that θ_2 took on values 0, $.2s(\alpha)$, . . . , $s(\alpha)$. The LR test has consistently higher power than the t test, and the difference between the two power curves becomes more striking as the value of α increases. The price for this advantage is a larger risk of a Type I error, however.

6. AN EXAMPLE OF ACCELERATED LIFE TESTING

We now analyze the data presented in the Introduction.

Applying the methods of estimation discussed in Section 3, we obtained the maximum likelihood and least squares estimates for the regression coefficients. The maximum likelihood and least squares estimates (with estimated standard errors) were, respectively, 12.280 (.403) and 12.289 (.406) for a and -1.671 (.112) and -1.673 (.113) for b. The MLE for α was .41, so the close agreement between the maximum likelihood and least squares estimates is to be expected. A plot of the data and the least squares fitted line are shown in Figure 1. An examination of the plot indicates that the log-linear relationship between W_c and N seems reasonable. Since α is small, an F test for lack of fit was also conducted, taking advantage of replicates at several values of the independent variable. The p value for this test was .82, which indicates again that the log-linear relationship is reasonable. From the uniform scatter of the data about the regression line, we find no evidence to contradict the assumption that α is independent of W_c. A quantile–quantile plot of the residuals versus the sinh-normal quantiles exhibits no striking departures from linearity (nor does a normal probability plot, which is not surprising since α is small).

Table 4. Type I Error Rates Versus Nominal Levels of Significance

		Nominal significance level and test					
		.10		.05		.01	
N	Alpha	t	LR	t	LR	t	LR
10	.5	.095	.118*	.044	.061*	.007	.012
	1.5	.109	.134*	.054	.070*	.013	.019*
20	.5	.102	.115*	.050	.061*	.012	.015*
	1.5	.105	.118*	.051	.057	.012	.013*

* Observed Type I error rates are beyond two-sigma limits from the nominal levels of significance.

Table 5. Power Comparison of Test Statistics

			Simulated power function with argument equal to					
N	*Alpha*	*Test*	*.0*	*.2*	*.4*	*.6*	*.8*	*1.0*
10	.5	*t*	.044	.074	.136	.259	.404	.551
		LR	.061	.097	.165	.301	.455	.603
	1.5	*t*	.054	.072	.137	.234	.366	.554
		LR	.070	.092	.179	.303	.474	.656
20	.5	*t*	.050	.094	.236	.460	.685	.872
		LR	.061	.106	.255	.486	.713	.888
	1.5	*t*	.051	.098	.236	.441	.687	.869
		LR	.057	.122	.294	.535	.784	.927

7. DISCUSSION

Taking logarithms of a variate having the Birnbaum–Saunders distribution with parameters α and β results in a variate with a sinh-normal distribution that is symmetric about $\ln \beta$ and nearly normal for small values of α. If

$$\ln \beta = \sum_{i=1}^{p} x_i \theta_i$$

is linear in the unknown parameters $\theta_1, \ldots, \theta_p$, then the sinh-normal distribution can be beneficially used. This article presents an example of such use drawn from the engineering study of fatigue life and develops the theory for such log-linear models with additive, sinh-normal errors.

Maximum likelihood estimation for log-linear models of fatigue life encounters the common problems of likelihood inference—the possibility of multiple maxima, biases in finite samples, the absence of closed-form solutions. The first problem is not thought to be important in practical applications. As for the second problem, there is no bias, to order $1/n$, for the log-linear model coefficients, and a re-

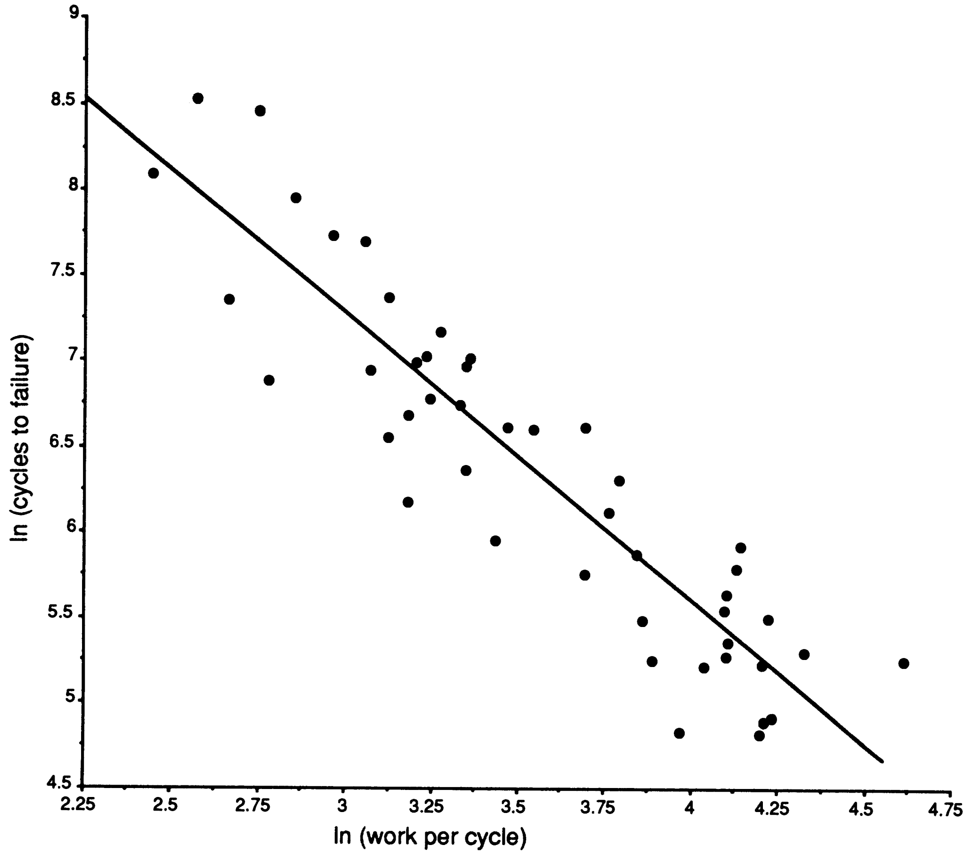

Figure 1. Scatterplot and Least Squares Fitted Line for the Data of Brown and Miller (1978).

duced-bias estimator is provided for the shape parameter. The third problem is mitigated by an iterative ordinary least squares algorithm. On the other hand, least squares estimates themselves are highly efficient, especially when α is small.

Several authors have argued that the distribution of fatigue life is lognormal and that least squares regression is appropriate for linear models with response $\log(N)$ (e.g., American Society for Testing Materials 1981; Freudenthal 1952). Desmond (1985) showed that the arguments for the lognormal, when correctly interpreted, actually imply the Birnbaum–Saunders distribution for fatigue life. Our results herein show that the success of least squares regression is due to the approximate normality of the sinh-normal when α is small.

Statistical folklore promises the robustness of normal-theory interval estimation and hypothesis testing for the log-linear models, especially when α is small. A small Monte Carlo study suggests the validity of the folklore, at least for Type I error. The LR test appears to have greater power when the alternative is true, but it has an inflated Type I error rate when the null is true. More such Monte Carlo studies are needed to elucidate further the small-sample properties of the procedures.

Under many commonly occurring conditions, it is felt that the value for α will be less than 1. An example in which such an assumption is reasonable is provided in the Introduction and Section 6. For such situations, least squares and normal-theory procedures are adequate in analyzing the data. There may occasionally occur cases in which α may be greater than 1, however. Under these circumstances, the likelihood procedures will provide better results if the sample sizes are sufficiently large.

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APPENDIX: PROPERTIES OF $C(\alpha)$

The value of $-\partial^2 \ln(L)/\partial\theta_j\partial\theta_k$ may be expressed as

$$\left(\frac{1}{4}\right) \sum_i x_{ij}x_{ik}\{2Z_i^2 + (4/\alpha^2) - 1 + Z_i^2/[Z_i^2 + (4/\alpha^2)]\}, \qquad j, k = 1, \ldots, p.$$

We define $C(\alpha)$ as the expectation of the quantity in the braces; that is,

$$C(\alpha) = \mathrm{E}\{2Z_i^2 + (4/\alpha^2) - 1 + Z_i^2/[Z_i^2 + (4/\alpha^2)]\}.$$

The value $Z_i = (2/\alpha)\sinh[(y_i - \mathbf{x}_i'\boldsymbol{\theta})/2]$ has a standard normal distribution. Therefore, we have that

$$C(\alpha) = 1 + (4/\alpha^2) + \mathrm{E}\{Z_i^2/[Z_i^2 + (4/\alpha^2)]\}.$$

We see that

$$\begin{aligned} \mathrm{E}\{Z_i^2/[Z_i^2 + (4/\alpha^2)]\} &= \mathrm{E}\{1 - 1 + Z_i^2/[Z_i^2 + (4/\alpha^2)]\} \\ &= 1 - (4/\alpha^2)\mathrm{E}\{[Z_i^2 + (4/\alpha^2)]^{-1}\}. \end{aligned}$$

Using a table of integrals (e.g., Ryshik and Gradstein 1963), we have that

$$\mathrm{E}\{[Z_i^2 + (4/\alpha^2)]^{-1}\} = (\pi\alpha^2/8)^{1/2}\{1 - \mathrm{erf}[(2/\alpha^2)^{1/2}]\}\exp(2/\alpha^2),$$

where $\mathrm{erf}(x)$ is the error function (e.g., see Ryshik and Gradstein 1963). So we have that

$$C(\alpha) = 2 + (4/\alpha^2) - (2\pi/\alpha^2)^{1/2}\{1 - \mathrm{erf}[(2/\alpha^2)^{1/2}]\}\exp(2/\alpha^2).$$

For small values of α, it is easy to see that $C(\alpha) \approx 1 + (4/\alpha^2)$, where the relative error for values of α less than .5 is less than .3%. As α aproaches infinity, we see that $C(\alpha)$ approaches 2.

In most instances, we use $C(\alpha)$ as a divisor, and $1/C(\alpha)$ is an increasing function of α up to approximately $\alpha = 5.526$ and a decreasing function for values of α above 5.526.

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