Computational Statistics and Data Analysis 64 (2013) 20–33

Contents lists available at SciVerse ScienceDirect

Computational Statistics and Data Analysis

journal homepage: www.elsevier.com/locate/csda

Power Lindley distribution and associated inference

M.E. Ghitany[a], D.K. Al-Mutairi[a], N. Balakrishnan[b,*], L.J. Al-Enezi[a]

[a] Department of Statistics and Operations Research, Faculty of Science, Kuwait University, P.O. Box 5969 Safat, 13060, Kuwait

[b] Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

ARTICLE INFO

Article history:
Received 28 April 2012
Received in revised form 25 November 2012
Accepted 20 February 2013
Available online 4 March 2013

Keywords:
Power Lindley distribution
Maximum likelihood estimator
Simulation algorithms
Lambert function

ABSTRACT

A new two-parameter power Lindley distribution is introduced and its properties are discussed. These include the shapes of the density and hazard rate functions, the moments, skewness and kurtosis measures, the quantile function, and the limiting distributions of order statistics. Maximum likelihood estimation of the parameters and their estimated asymptotic standard errors are derived. Three algorithms are proposed for generating random data from the proposed distribution. A simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators of the parameters as well as the coverage probability and the width of the confidence interval for each parameter. An application of the model to a real data set is presented finally and compared with the fit attained by some other well-known two-parameter distributions.

1. Introduction

The Lindley distribution was originally proposed by Lindley (1958) in the context of Fiducial and Bayesian Statistics. It has the probability density function (PDF)

$$\begin{aligned} f_1(t) &= \frac{\beta^2}{\beta+1}\,(1+t)\,e^{-\beta t}, \quad t>0,\ \beta>0, \\ &= p\,\xi_1(t) + (1-p)\,\xi_2(t), \end{aligned} \tag{1}$$

where

$$\begin{aligned} p &= \frac{\beta}{\beta+1}, \\ \xi_1(t) &= \beta e^{-\beta t}, \quad t>0, \\ \xi_2(t) &= \beta^2 t e^{-\beta t}, \quad t>0. \end{aligned}$$

The density in (1) readily reveals that the Lindley distribution is a two-component mixture of an exponential distribution (with scale β) and a gamma distribution (with shape 2 and scale β), with mixing proportion $p = \beta/(\beta+1)$.

In the context of reliability studies, Ghitany et al. (2008) studied in great detail the Lindley distribution and its application. However, there are situations in which the Lindley distribution may not be suitable from a theoretical or applied point of view. So, to obtain a more flexible family of distributions, we introduce here a new extension of the Lindley distribution by

* Corresponding author.
E-mail address: bala@mcmaster.ca (N. Balakrishnan).

0167-9473/$ – see front matter
http://dx.doi.org/10.1016/j.csda.2013.02.026

considering the power transformation $X = T^{1/\alpha}$. The PDF of the X is readily obtained to be

$$f(x) = \frac{\alpha\beta^2}{\beta+1}(1+x^\alpha)\,x^{\alpha-1}e^{-\beta x^\alpha}, \quad x>0,\ \alpha,\beta>0,$$
$$= p\,g_1(t) + (1-p)\,g_2(t), \tag{2}$$

where

$$p = \frac{\beta}{\beta+1},$$
$$g_1(x) = \alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}, \quad x>0,$$
$$g_2(x) = \alpha\beta^2 x^{2\alpha-1}e^{-\beta x^\alpha}, \quad x>0.$$

We call the distribution in (2) a power Lindley distribution. From (2), we see that the power Lindley distribution is a two-component mixture of Weibull distribution (with shape α and scale β), and a generalized gamma distribution (with shape parameters 2, α and scale β), with mixing proportion $p = \beta/(\beta+1)$.

The survival function (SF) of the power Lindley distribution is given by

$$S(x) = P(X>x) = \left(1+\frac{\beta}{\beta+1}x^\alpha\right)e^{-\beta x^\alpha}, \quad x>0,\ \beta>0. \tag{3}$$

We use $X \sim PL(\alpha,\beta)$ to denote the random variable having a power Lindley distribution with parameters α, β and with PDF and SF as in (2) and (3), respectively.

The aim of this paper is to discuss some properties of the power Lindley distribution. These include the shapes of the density and hazard rate functions, the moments and some associated measures, the quantile function, and the limiting distributions of order statistics. Maximum likelihood estimation of the model parameters and their asymptotic standard errors are derived. Three different algorithms are proposed for generating random data from the proposed distribution. A Monte Carlo simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators as well as the coverage probability and width of the confidence interval for each parameter. Application of the model to a real data set is finally presented and compared to the fit attained by some other well-known two-parameter distributions.

2. Probability density function

In this section, we discuss the shape characteristics of the PDF $f(x)$ in (2) of the $PL(\alpha,\beta)$ distribution.

The behavior of $f(x)$ at $x=0$ and $x=\infty$, respectively, are given by

$$f(0) = \begin{cases} \infty, & \text{if } \alpha<1, \\ \dfrac{\beta^2}{\beta+1}, & \text{if } \alpha=1, \\ 0, & \text{if } \alpha>1, \end{cases} \qquad f(\infty)=0.$$

The following theorem states that there are three shapes for the PDF of the power Lindley distribution, depending on the range of the parameters α and β.

Theorem 1. *The PDF $f(x)$ in (2) of the $PL(\alpha,\beta)$ distribution is*

(a) *decreasing if* (i) $\left\{0<\alpha\le\frac{1}{2},\ \beta>0\right\}$ *or* (ii) $\left\{\frac{1}{2}<\alpha<1,\ \beta\ge\eta_1(\alpha)\right\}$, *where*

$$\eta_1(\alpha) = \frac{1-2\sqrt{\alpha(1-\alpha)}}{\alpha},$$

or (iii) $\{\alpha=1,\ \beta\ge1\}$;

(b) *unimodal if* (i) $\{\alpha=1,\ 0<\beta<1\}$ *or* (ii) $\{\alpha>1,\ \beta>0\}$;

(c) *decreasing–increasing–decreasing if* $\left\{\frac{1}{2}<\alpha<1,\ 0<\beta<\eta_1(\alpha)\right\}$.

Proof. The derivative of $f(x)$ is obtained from (2) as

$$f'(x) = \frac{\alpha\beta^2}{\beta+1}\,x^{\alpha-2}e^{-\beta x^\alpha}\,\psi_1(x^\alpha), \quad x>0,$$

where

$$\psi_1(y) = a_1y^2+b_1y+c_1, \quad y=x^\alpha>0,$$

with

$$a_1=-\alpha\beta, \qquad b_1 = 2\alpha-1-\alpha\beta, \qquad c_1=\alpha-1.$$

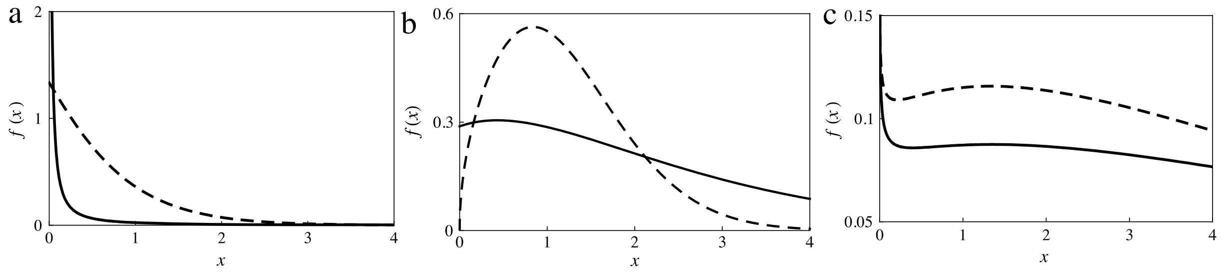

Fig. 1. Probability density function of power Lindley distribution for the cases (a) (α, β): $(0.2, 4)$ (solid), $(1, 2)$ (dashed), (b) (α, β): $(1, 0.7)$ (solid), $(1.5, 1)$ (dashed), (c) (α, β): $(0.85, 0.3)$ (solid), $(0.9, 0.35)$ (dashed).

Clearly, $f'(x)$ and $\psi_1(y)$ have the same sign. The quadratic function $\psi_1(y)$ is decreasing $\big($unimodal with maximum value at the point $y_1 = -\frac{b_1}{2a_1}\big)$ if $b_1 \le 0$ $(b_1 > 0)$ with $\psi_1(0) = c_1$ and $\psi_1(\infty) = -\infty$.

Under the stated conditions in (a), (b), and (c), the function $\psi_1(y)$ is negative, changes sign from positive to negative, and changes sign from negative to positive to negative, respectively. This completes the proof of the theorem. □

From Theorem 1, it can be seen that when $\alpha = 1$, i.e., in the case of Lindley distribution, $f(x)$ is decreasing (unimodal) if $\beta \ge 1$ $(0 < \beta < 1)$ with $f(0) = \frac{\beta^2}{\beta+1}$ and $f(\infty) = 0$.

Fig. 1 shows the PDF of the power Lindley distribution for some selected choices of α and β, displaying all the shapes established in Theorem 1.

3. Hazard rate function

The hazard rate function (HRF) of the power Lindley distribution is given by

$$h(x) = \frac{f(x)}{S(x)} = \alpha\beta^2 \, \frac{(1 + x^\alpha)\, x^{\alpha-1}}{\beta + 1 + \beta x^\alpha}, \quad x > 0,\ \alpha, \beta > 0. \tag{4}$$

The behavior of $h(x)$ at $x = 0$ and $x = \infty$, respectively, are given by

$$h(0) = f(0) = \begin{cases} \infty, & \text{if } \alpha < 1, \\ \dfrac{\beta^2}{\beta+1}, & \text{if } \alpha = 1, \\ 0, & \text{if } \alpha > 1, \end{cases} \qquad h(\infty) = \begin{cases} 0, & \text{if } \alpha < 1, \\ \beta, & \text{if } \alpha = 1, \\ \infty, & \text{if } \alpha > 1. \end{cases}$$

The following theorem shows that there are three shapes for the HRF of the power Lindley distribution.

Theorem 2. *The HRF $h(x)$ in (4) of the PL(α, β) distribution is*

(a) *decreasing if* (i) $\left\{0 < \alpha \le \frac{1}{2}, \beta > 0\right\}$ *or* (ii) $\left\{\frac{1}{2} < \alpha < 1, \beta \ge \eta_2(\alpha)\right\}$, *where*

$$\eta_2(\alpha) = \frac{(2\alpha - 1)^2}{4\alpha(1-\alpha)};$$

(b) *increasing if* $\{\alpha \ge 1, \beta > 0\}$;

(c) *decreasing–increasing–decreasing if* $\left\{\frac{1}{2} < \alpha < 1, 0 < \beta < \eta_2(\alpha)\right\}$.

Proof. From (4), the derivative of $h(x)$ is obtained as

$$h'(x) = \frac{\alpha\beta^2\, x^{\alpha-1}}{(\beta + 1 + \beta x^\alpha)^2}\, \psi_2(x^\alpha), \quad x > 0,$$

where

$$\psi_2(y) = a_2 y^2 + b_2 y + c_2, \quad y = x^\alpha > 0,$$

with

$$a_2 = (\alpha - 1)\beta, \qquad b_2 = 2(\alpha - 1)(\beta + 1) + 1, \qquad c_2 = (\alpha - 1)(\beta + 1).$$

The rest of the proof of this theorem follows similarly to that of Theorem 1. □

From Theorem 2, it can be seen that when $\alpha = 1$, i.e., in the case of Lindley distribution, $h(x)$ is increasing for all $\beta > 0$ with $h(0) = \frac{\beta^2}{\beta+1}$ and $h(\infty) = \beta$.

Fig. 2 shows the HRF $h(x)$ of the power Lindley distribution for some choices of α and β, displaying all the shapes established in Theorem 2.

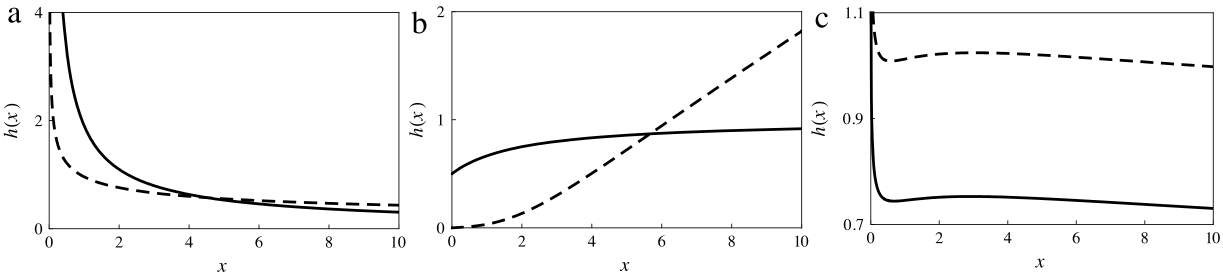

Fig. 2. Hazard rate function of power Lindley distribution for the cases (a) (α, β): (0.2, 10) (solid), (0.6, 2) (dashed), (b) (α, β): (1, 1) (solid), (2, 0.1) (dashed), (c) (α, β): (0.88, 1.2) (solid), (0.9, 1.5) (dashed).

4. Moments and associated measures

The r^{th} raw moment (about the origin) of the power Lindley distribution is given by

$$\begin{aligned}\mu_r' = E(X^r) &= p\,\frac{\Gamma\left(\frac{r}{\alpha}+1\right)}{\beta^{r/\alpha}} + (1-p)\,\frac{\Gamma\left(\frac{r}{\alpha}+2\right)}{\beta^{r/\alpha}} \\ &= \frac{r\,\Gamma\left(\frac{r}{\alpha}\right)\,[\alpha(\beta+1)+r]}{\alpha^2\,\beta^{r/\alpha}\,(\beta+1)}.\end{aligned}$$

Note that when $\alpha = 1$, i.e., in the case of Lindley distribution, the above expression simply reduces to

$$\mu_r' = \frac{r!\,(\beta+1+r)}{\beta^r\,(\beta+1)}.$$

Therefore, the mean and variance of the power Lindley distribution, respectively, are

$$\mu = \frac{\Gamma\left(\frac{1}{\alpha}\right)[\alpha(\beta+1)+1]}{\alpha^2\,\beta^{1/\alpha}\,(\beta+1)}, \qquad \sigma^2 = \frac{2\Gamma\left(\frac{2}{\alpha}\right)[\alpha(\beta+1)+2]\alpha^2(\beta+1) - \Gamma^2\left(\frac{1}{\alpha}\right)[\alpha(\beta+1)+1]^2}{\alpha^4\,\beta^{2/\alpha}\,(\beta+1)^2}.$$

The skewness and kurtosis measures can be obtained from the expressions

$$\begin{aligned}Skewness &= \frac{\mu_3' - 3\mu_2'\,\mu + 2\mu^3}{\sigma^3}, \\ Kurtosis &= \frac{\mu_4' - 4\mu_3'\,\mu + 6\mu_2'\,\mu^2 - 3\mu^4}{\sigma^4},\end{aligned}$$

upon substituting for the raw moments.

Fig. 3 shows the mean, variance, skewness, and kurtosis of the power Lindley distribution as a function of α for $\beta = 1$. This figure shows that each of these measures of $PL(\alpha, \beta)$ can be smaller/larger than that of the Lindley distribution. Moreover, we observe that the skewness can be negative which is not possible for the Lindley distribution, and this means that for modeling negatively skewed data, the power Lindley distribution will be a useful model.

5. Quantile function

Recently, Jodra (2010) showed that the quantile function of the Lindley distribution is given by

$$F_T^{-1}(u) = -1 - \frac{1}{\beta} - \frac{1}{\beta}\,W_{-1}\left(-\frac{\beta+1}{e^{\beta+1}}(1-u)\right),$$

where $W_{-1}(\cdot)$ denotes the negative branch of the Lambert W function (i.e., the solution of the equation $W(z)\,e^{W(z)} = z$). Note that $-\frac{1}{e} < -\frac{\beta+1}{e^{\beta+1}}(1-u) < 0$ and so $W_{-1}(\cdot)$ is unique which implies that $F_T^{-1}(u)$ is also unique; see Chapeau-Blondeau and Monir (2002).

From the transformation $X = T^{1/\alpha}$, it readily follows that the quantile function of the power Lindley distribution is given by

$$\begin{aligned}F^{-1}(u) &= [F_T^{-1}(u)]^{1/\alpha} \\ &= \left[-1 - \frac{1}{\beta} - \frac{1}{\beta}\,W_{-1}\left(-\frac{\beta+1}{e^{\beta+1}}(1-u)\right)\right]^{1/\alpha}.\end{aligned} \tag{5}$$

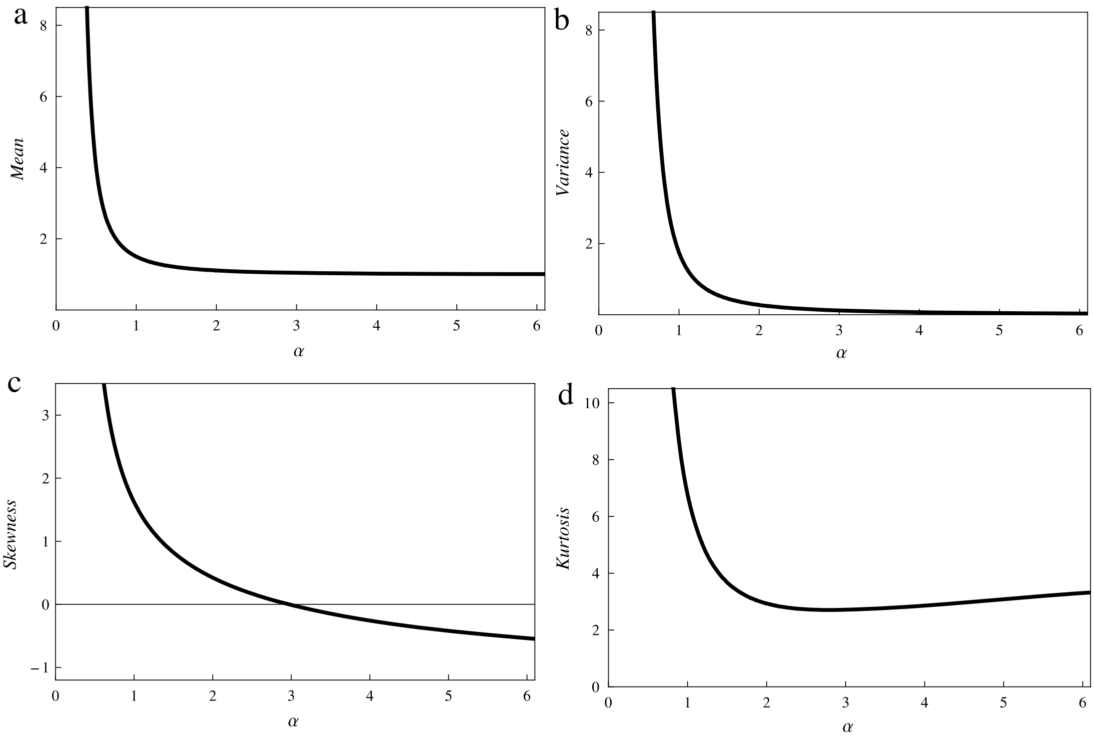

Fig. 3. Mean, variance, skewness, and kurtosis of power Lindley distribution as a function of α for $\beta = 1$.

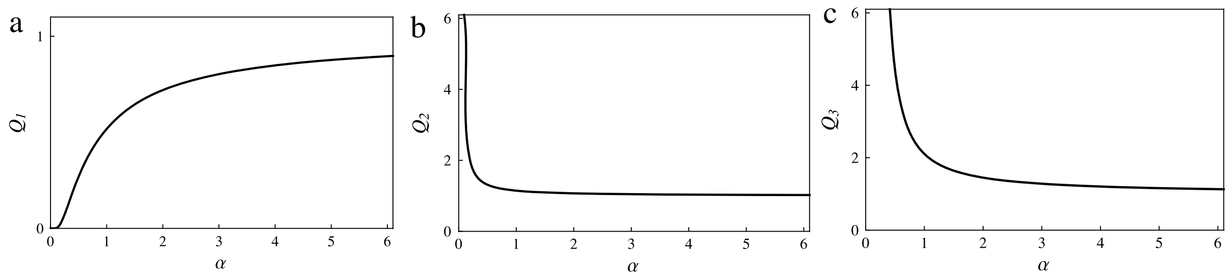

Fig. 4. Quartiles of power Lindley distribution as a function of α for $\beta = 1$.

Consequently, the quartiles of the power Lindley distribution, respectively, are given by

$$Q_1 = F^{-1}\left(\frac{1}{4}\right) = \left[-1 - \frac{1}{\beta} - \frac{1}{\beta} W_{-1}\left(-\frac{3(\beta+1)}{4e^{\beta+1}}\right)\right]^{1/\alpha},$$

$$Q_2 = F^{-1}\left(\frac{1}{2}\right) = \left[-1 - \frac{1}{\beta} - \frac{1}{\beta} W_{-1}\left(-\frac{\beta+1}{2e^{\beta+1}}\right)\right]^{1/\alpha},$$

$$Q_3 = F^{-1}\left(\frac{3}{4}\right) = \left[-1 - \frac{1}{\beta} - \frac{1}{\beta} W_{-1}\left(-\frac{\beta+1}{4e^{\beta+1}}\right)\right]^{1/\alpha}.$$

Fig. 4 presents a plot of these quartiles as a function of α for $\beta = 1$. This figure shows that each of these measures of $PL(\alpha, \beta)$ can be smaller/larger than that of the Lindley distribution.

6. Limiting distributions of order statistics

The distribution of minimum and maximum of random variables play an important role in statistical applications. For example, $X_{1:n} = \min\{X_1, \ldots, X_n\}$ and $X_{n:n} = \max\{X_1, \ldots, X_n\}$ are the lifetimes of series and parallel systems, respectively.

In the following theorem, we provide the limiting distributions of $X_{1:n}$ and $X_{n:n}$ arising from the power Lindley model.

Theorem 3. *Let $X_{1:n}$ and $X_{n:n}$ be the minimum and maximum of a random sample $X_1, X_2, \ldots, X_n$ from $PL(\alpha, \beta)$, respectively. Then:*

$$\text{(a)}\ \lim_{n\to\infty} P\left\{\frac{X_{1:n}-a_n^*}{b_n^*} \le x\right\} = 1 - e^{-x^\alpha}, \quad x > 0,$$

$$\text{(b)}\ \lim_{n\to\infty} P\left\{\frac{X_{n:n}-a_n}{b_n} \le t\right\} = \exp(-e^{-t}), \quad -\infty < t < \infty,$$

where

$$a_n^* = 0, \qquad b_n^* = F^{-1}(1/n), \qquad a_n = F^{-1}(1-1/n), \qquad b_n = \frac{1}{nf(a_n)}.$$

Proof. For the $PL(\alpha, \beta)$, we have, by using L'Hospital rule,

$$\lim_{\epsilon\to 0+} \frac{F(\epsilon x)}{F(\epsilon)} = \lim_{\epsilon\to 0+} \frac{x f(\epsilon x)}{f(\epsilon)} = x^\alpha \lim_{\epsilon\to 0+} \frac{1+(\epsilon x)^\alpha}{1+\epsilon^\alpha} = x^\alpha.$$

Therefore, by Theorem 8.3.6(ii) of Arnold et al. (1992), the minimal domain of attraction of the $PL(\alpha, \beta)$ distribution is the standard Weibull distribution, proving Part (a).

For the $PL(\alpha, \beta)$, we have

$$\lim_{x\to\infty} \frac{d}{dx}\left(\frac{1}{h(x)}\right) = \frac{1}{\alpha\beta^2} \lim_{x\to\infty} \frac{d}{dx}\left(\frac{\beta+1+\beta x^\alpha}{(1+x^\alpha)x^{\alpha-1}}\right) = \frac{1}{\alpha\beta^2} \lim_{x\to\infty} \frac{(\alpha-1)x^\alpha - 1}{(1+x^\alpha)^2} - \frac{(\alpha-1)(\beta+1)}{x^\alpha} = 0.$$

Therefore, by Theorem 8.3.3 of Arnold et al. (1992), the maximal domain of attraction of the $PL(\alpha, \beta)$ distribution is the standard Gumbel distribution, proving Part (b). □

Now, we use Theorem 3 to find the limiting distribution of any order statistic.

Theorem 4. *Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from $PL(\alpha, \beta)$. Then, for $i = 1, 2, \ldots, n$,*

$$\text{(a)}\ \lim_{n\to\infty} P\left\{\frac{X_{i:n}-a_n^*}{b_n^*} \le x\right\} = 1 - \sum_{r=0}^{i-1} e^{-x^\alpha} \frac{x^{r\alpha}}{r!}, \quad x > 0,$$

$$\text{(b)}\ \lim_{n\to\infty} P\left\{\frac{X_{n-i+1:n}-a_n}{b_n} \le t\right\} = \sum_{r=0}^{i-1} e^{-e^{-t}} \frac{e^{-rt}}{r!}, \quad -\infty < t < \infty,$$

where

$$a_n^* = 0, \qquad b_n^* = F^{-1}(1/n), \qquad a_n = F^{-1}(1-1/n), \qquad b_n = \frac{1}{nf(a_n)}.$$

Proof. The theorem follows from Eqs. (8.4.2) and (8.4.3) of Arnold et al. (1992). □

Remarks. (i) The limiting distribution of $\frac{X_{i:n}-a_n^*}{b_n^*}$ has a PDF

$$u(x) = \frac{\alpha}{\Gamma(i)} x^{i\alpha-1} e^{-x^\alpha}, \quad x > 0,$$

which is the PDF of a generalized gamma distribution with shape parameters i, α and unit scale parameter;

(ii) The limiting distribution of $\frac{X_{n-i+1:n}-a_n}{b_n}$ has a PDF

$$v(t) = \frac{1}{\Gamma(i)} e^{-it} \exp(-e^{-t}), \quad -\infty < t < \infty,$$

which is the PDF of a weighted standard Gumbel distribution with weight function $w(t) = e^{-(i-1)t}$.

7. Maximum likelihood estimation of parameters

Let $x_1, \ldots, x_n$ be a random sample of size n from $PL(\alpha, \beta)$. Then, the log-likelihood function is given by

$$\begin{aligned}\ln L &= \sum_{i=1}^{n} \ln f(x_i)\\ &= n[\ln(\alpha) + 2\ln(\beta) - \ln(\beta+1)] + \sum_{i=1}^{n} \ln(1+x_i^{\alpha}) + (\alpha-1)\sum_{i=1}^{n} \ln(x_i) - \beta \sum_{i=1}^{n} x_i^{\alpha}.\end{aligned}$$

The MLEs $\hat{\alpha}, \hat{\beta}$ of α, β are then the solutions of the non-linear equations

$$\begin{aligned}\frac{\partial}{\partial\alpha}\ln L &= \frac{n}{\alpha} + \sum_{i=1}^{n} \frac{x_i^{\alpha}\ln(x_i)}{1+x_i^{\alpha}} + \sum_{i=1}^{n}\ln(x_i) - \beta\sum_{i=1}^{n} x_i^{\alpha}\ln(x_i) = 0,\\ \frac{\partial}{\partial\beta}\ln L &= \frac{n(\beta+2)}{\beta(\beta+1)} - \sum_{i=1}^{n} x_i^{\alpha} = 0.\end{aligned}$$

The last equation is equivalent to the quadratic equation

$$\left(\sum_{i=1}^{n} x_i^{\alpha}\right)\beta^2 + \left(\sum_{i=1}^{n} x_i^{\alpha} - n\right)\beta - 2n = 0$$

which has a unique solution in β given by

$$\hat{\beta}(\hat{\alpha}) = \frac{-\left(\sum_{j=1}^{n} x_j^{\hat{\alpha}} - n\right) + \sqrt{\left(\sum_{j=1}^{n} x_j^{\hat{\alpha}} - n\right)^2 + 8n\sum_{j=1}^{n} x_j^{\hat{\alpha}}}}{2\sum_{j=1}^{n} x_j^{\hat{\alpha}}},$$

where $\hat{\alpha}$ is the solution of the non-linear equation

$$G(\alpha) = \frac{n}{\alpha} + \sum_{i=1}^{n} \frac{x_i^{\alpha}\ln(x_i)}{1+x_i^{\alpha}} + \sum_{i=1}^{n}\ln(x_i) - \hat{\beta}(\alpha)\sum_{i=1}^{n} x_i^{\alpha}\ln(x_i) = 0.$$

We also find

$$\begin{aligned}\frac{\partial^2}{\partial\alpha^2}\ln L &= -\frac{n}{\alpha^2} + \sum_{i=1}^{n} \frac{x_i^{\alpha}\ln^2(x_i)}{(1+x_i^{\alpha})^2} - \beta\sum_{i=1}^{n} x_i^{\alpha}\ln^2(x_i),\\ \frac{\partial^2}{\partial\beta^2}\ln L &= -\frac{n(\beta^2+4\beta+2)}{\beta^2(\beta+1)^2},\\ \frac{\partial^2}{\partial\alpha\partial\beta}\ln L &= -\sum_{i=1}^{n} x_i^{\alpha}\ln(x_i).\end{aligned}$$

From Gradshteyn and Ryzhik (2007, p. 573), we have

$$\int_0^{\infty} t^{\nu-1}\ln t\, e^{-\beta t}\, dt = \frac{\Gamma(\nu)}{\beta^{\nu}}[\psi(\nu) - \ln\beta], \quad \nu, \beta > 0,$$

and similarly, we have

$$\int_0^{\infty} t^{\nu-1}(\ln t)^2\, e^{-\beta t}\, dt = \frac{\Gamma(\nu)}{\beta^{\nu}}\left\{[\psi(\nu) - \ln\beta]^2 + \zeta(2,\nu)\right\}, \quad \nu, \beta > 0,$$

where $\Gamma(t)$ is the gamma function, $\psi(t) = \frac{d}{dt}\ln\Gamma(t)$ is the digamma function, and $\zeta(z,\nu)$ is the Riemann's zeta function defined by

$$\zeta(z,\nu) = \sum_{m=0}^{\infty} \frac{1}{(\nu+m)^z}, \quad z > 1,\ \nu \neq 0, -1, -2, \ldots.$$

By using these expressions, we find the following expectations:

$$\begin{aligned}
E[X^\alpha \ln(X)] &= \frac{\beta^2}{\alpha(\beta+1)} \int_0^\infty t\ln(t)(1+t)e^{-\beta t}\, dt \\
&= \frac{\beta[\psi(2)-\ln\beta]+2[\psi(3)-\ln\beta]}{\alpha\beta(\beta+1)}, \\
E[X^\alpha (\ln X)^2] &= \frac{\beta^2}{\alpha^2(\beta+1)} \int_0^\infty t(\ln t)^2(1+t)e^{-\beta t}\, dt \\
&= \frac{\beta\{[\psi(2)-\ln\beta]^2+\zeta(2,2)\}+2\{[\psi(3)-\ln\beta]^2+\zeta(2,3)\}}{\alpha^2\beta(\beta+1)}, \\
E\left[\frac{X^\alpha(\ln X)^2}{(1+X^\alpha)^2}\right] &= \frac{\beta^2}{\alpha^2(\beta+1)} \int_0^\infty \frac{t(\ln t)^2}{1+t}\, e^{-\beta t}\, dt \\
&= \frac{\beta^2}{\alpha^2(\beta+1)} \int_0^\infty \left(1-\frac{1}{1+t}\right)(\ln t)^2\, e^{-\beta t}\, dt \\
&= \frac{\beta}{\alpha^2(\beta+1)} \left\{[\psi(1)-\ln\beta]^2+\zeta(2,1)-\beta J(\beta)\right\},
\end{aligned}$$

where

$$J(\beta) = \int_0^\infty \frac{(\ln t)^2}{1+t}\, e^{-\beta t}\, dt.$$

So, the expected Fisher information matrix of $\boldsymbol{\theta}$ based on a single observation is given by

$$\mathbf{I}(\boldsymbol{\theta}) = [I_{ij}(\boldsymbol{\theta})] = E\left[-\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln f(X;\boldsymbol{\theta})\right],$$

where

$$\begin{aligned}
I_{11} &= \frac{1}{\alpha^2} - E\left[\frac{X^\alpha \ln^2(X)}{(1+X^\alpha)^2}\right] + \beta E\left[X^\alpha \ln^2(X)\right] \\
&= \frac{1}{\alpha^2} + \frac{1}{\alpha^2(\beta+1)}\left\{\beta\left[(\psi(2)-\ln\beta)^2+\zeta(2,2)\right] + 2\left[(\psi(3)-\ln\beta)^2+\zeta(2,3)\right]\right. \\
&\quad \left. -\beta\left[(\psi(1)-\ln\beta)^2+\zeta(2,1)\right]+\beta^2 J(\beta)\right\},
\end{aligned}$$

$$I_{22} = \frac{\beta^2+4\beta+2}{\beta^2(\beta+1)^2},$$

$$\begin{aligned}
I_{12} &= E[X^\alpha \ln(X)] \\
&= \frac{\beta[\psi(2)-\ln\beta]+2[\psi(3)-\ln\beta]}{\alpha\beta(\beta+1)}.
\end{aligned}$$

The following values that are required are well known (see, for example Gradshteyn and Ryzhik, 2007):

(i) $\psi(1) = -0.577216$, $\psi(2) = 0.422784$, $\psi(3) = 0.922784$;
(ii) $\zeta(2,1) = 1.644934$, $\zeta(2,2) = 0.644934$, $\zeta(2,3) = 0.394934$.

Under mild regularity conditions (see Lehmann and Casella (1998), pp. 461–463), the asymptotic distribution of the MLE $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is such that

$$\sqrt{n}\,(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}) \xrightarrow{D} N_2(\mathbf{0}, \mathbf{I}^{-1}(\boldsymbol{\theta})),$$

where $\xrightarrow{D}$ denotes convergence in distribution and $\mathbf{I}^{-1}(\boldsymbol{\theta})$ is the inverse of the matrix $\mathbf{I}(\boldsymbol{\theta})$, with

$$\frac{1}{n}\,\mathbf{I}^{-1}(\boldsymbol{\theta}) = \frac{1}{n}\begin{pmatrix} I_{11} & I_{12} \\ I_{12} & I_{22}\end{pmatrix}^{-1} = \begin{pmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\beta}) \\ Cov(\hat{\alpha},\hat{\beta}) & Var(\hat{\beta})\end{pmatrix}.$$

Thus, the asymptotic $100(1-\delta)\%$ confidence intervals of α and β, respectively, are

$$\hat{\alpha} \pm z_{\frac{\delta}{2}} \sqrt{\widehat{Var(\hat{\alpha})}},$$

$$\hat{\beta} \pm z_{\frac{\delta}{2}} \sqrt{\widehat{Var(\hat{\beta})}},$$

where z_q is the upper q-th quantile of the standard normal distribution.

The above forms of the confidence intervals may lead to negative lower bounds. For this reason, we apply the logarithmic transformation and use the delta method to arrive at the asymptotic normality distribution of $\ln(\hat{\alpha})$ and $\ln(\hat{\beta})$, respectively, as

$$\sqrt{n}(\ln(\hat{\alpha}) - \ln(\alpha)) \xrightarrow{D} N\left(0, \frac{\sigma_1^2}{\alpha^2}\right),$$

$$\sqrt{n}(\ln(\hat{\beta}) - \ln(\beta)) \xrightarrow{D} N\left(0, \frac{\sigma_1^2}{\beta^2}\right),$$

where $\sigma_1^2 = n\,Var(\hat{\alpha})$ and $\sigma_2^2 = n\,Var(\hat{\beta})$.

Now, the asymptotic $100(1-\delta)\%$ confidence intervals of $\ln(\alpha)$ and $\ln(\beta)$, respectively, are

$$\ln(\hat{\alpha}) \pm z_{\delta/2} \frac{\sqrt{\widehat{Var(\hat{\alpha})}}}{\hat{\alpha}} \equiv (L_1, U_1),$$

$$\ln(\hat{\beta}) \pm z_{\delta/2} \frac{\sqrt{\widehat{Var(\hat{\beta})}}}{\hat{\beta}} \equiv (L_2, U_2).$$

Finally, using the inverse logarithmic transformation, the asymptotic $100(1-\delta)\%$ confidence intervals of α and β, respectively, can be obtained as

$$(e^{L_1},\ e^{U_1}),$$
$$(e^{L_2},\ e^{U_2}).$$

8. Generation algorithms and Monte Carlo study

In this section, we propose three different algorithms for generating random data from the power Lindley distribution:

(a) The first algorithm is based on generating random data from the Lindley distribution using the exponential–gamma mixture form in (1) and then raise such data to the power $1/\alpha$;
(b) The second algorithm is based on generating random data from the gamma–generalized gamma mixture form in (2) of the power Lindley distribution;
(c) The third algorithm is based on generating random data from the inverse CDF in (4) of the power Lindley distribution.

Algorithm 1 (*Mixture Form of the Lindley Distribution*).

1. Generate $U_i \sim Uniform(0, 1), i = 1, \dots, n$;
2. Generate $V_i \sim Exponential(\beta), i = 1, \dots, n$;
3. Generate $W_i \sim Gamma(2, \beta), i = 1, \dots, n$;
4. If $U_i \le p = \frac{\beta}{\beta+1}$, then set $X_i = V_i^{1/\alpha}$, otherwise, set $X_i = W_i^{1/\alpha}, i = 1, \dots, n$.

Algorithm 2 (*Mixture Form of the Power Lindley Distribution*).

1. Generate $U_i \sim Uniform(0, 1), i = 1, \dots, n$;
2. Generate $Y_i \sim Weibull(\alpha, \beta), i = 1, \dots, n$;
3. Generate $Z_i \sim GG(2, \alpha, \beta), i = 1, \dots, n$;
4. If $U_i \le p = \frac{\beta}{\beta+1}$, then set $X_i = Y_i$, otherwise, set $X_i = Z_i, i = 1, \dots, n$.

Algorithm 3 (*Inverse CDF*).

1. Generate $U_i \sim Uniform(0, 1), i = 1, \dots, n$;
2. Set

$$X_i = \left[-1 - \frac{1}{\beta} - \frac{1}{\beta} W_{-1}\left(-\frac{\beta+1}{e^{\beta+1}}(1-U_i)\right)\right]^{1/\alpha}, \quad i = 1, \dots, n,$$

where $W_{-1}(\cdot)$ denotes the negative branch of the Lambert W function.

In the following simulation study, we used Algorithm 3 to generate data from the power Lindley distribution. The simulation experiment was repeated $N = 10{,}000$ times each with sample size $n = 25, 50, 75, 100, 200$ and $(\alpha, \beta) = (0.2, 4)$, $(1.5, 1)$, $(0.9, 0.35)$. Note that the selected values of (α, β) give decreasing, unimodal and decreasing–increasing–decreasing shapes, respectively, as displayed in Fig. 1.

Four quantities were examined in this Monte Carlo study:

(a) Average bias of the MLE $\hat{\nu}$ of the parameter $\nu = \alpha, \beta$:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\nu} - \nu);$$

Table 1
Bias (MSE) and coverage probability (average width) for the parameter α.

α	β	n	Bias (MSE)	CP (AW)
0.2	4	25	0.0121 (0.0014)	0.9328 (0.1351)
		50	0.0058 (0.0006)	0.9384 (0.0920)
		75	0.0044 (0.0004)	0.9421 (0.0744)
		100	0.0028 (0.0003)	0.9502 (0.0638)
		200	0.0015 (0.0001)	0.9441 (0.0448)
1.5	1	25	0.0824 (0.0699)	0.9228 (0.9434)
		50	0.0392 (0.0298)	0.9417 (0.6442)
		75	0.0291 (0.0194)	0.9438 (0.5199)
		100	0.0185 (0.0135)	0.9455 (0.4483)
		200	0.0103 (0.0067)	0.9458 (0.3144)
0.9	0.35	25	0.0445 (0.0205)	0.9339 (0.5087)
		50	0.0219 (0.0089)	0.9408 (0.3491)
		75	0.0139 (0.0055)	0.9457 (0.2828)
		100	0.0102 (0.0041)	0.9454 (0.2436)
		200	0.0039 (0.0019)	0.9493 (0.1709)

Table 2
Bias (MSE) and coverage probability (average width) for the parameter β.

α	β	n	Bias (MSE)	CP (AW)
0.2	4	25	0.4064 (1.4708)	0.9479 (3.9412)
		50	0.1932 (0.5064)	0.9475 (2.5048)
		75	0.1243 (0.2953)	0.9474 (1.9766)
		100	0.0851 (0.1988)	0.9510 (1.6797)
		200	0.0444 (0.0937)	0.9489 (1.1629)
1.5	1	25	0.0021 (0.0355)	0.9475 (0.7213)
		50	0.0020 (0.0169)	0.9539 (0.5039)
		75	−0.0020 (0.0110)	0.9494 (0.4103)
		100	0.0000 (0.0082)	0.9490 (0.3545)
		200	−0.0002 (0.0040)	0.9510 (0.2500)
0.9	0.35	25	−0.0094 (0.0091)	0.9480 (0.3837)
		50	−0.0050 (0.0047)	0.9479 (0.2688)
		75	−0.0033 (0.0030)	0.9518 (0.2185)
		100	−0.0021 (0.0023)	0.9508 (0.1890)
		200	−0.0001 (0.0011)	0.9503 (0.1334)

(b) Mean square error (MSE) of the MLE $\hat{\nu}$ of the parameter $\nu = \alpha, \beta$:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\nu}-\nu)^2;$$

(c) Coverage probability (CP) of 95% confidence intervals of the parameter $\nu = \alpha, \beta$, i.e., the percentage of intervals containing the true value of ν;
(d) Average width (AW) of 95% confidence intervals of the parameter $\nu = \alpha, \beta$.

Tables 1 and 2 present the average bias of the estimates which are seen to be small and it is also seen that the MSEs of the estimates decrease as the sample size increases. Also, these tables show that the coverage probabilities of the confidence intervals are quite close to the nominal level of 95% and that the average confidence widths decrease as the sample size increases, as one would expect.

Tables 1 and 2 suggest that the bias of $\hat{\alpha}$ is positive, while the bias of $\hat{\beta}$ may be positive or negative. To get some insight about this issue, let $\ln L(\boldsymbol{\theta}) = \ln L(\theta_1, \theta_2, \dots, \theta_p)$ be the log-likelihood function based on a sample of size n from an arbitrary parametric distribution. It is assumed that $\ln L(\boldsymbol{\theta})$ is regular w.r.t. all derivatives up to and including the third order.

The joint cumulants of the derivatives of $\ln L(\boldsymbol{\theta})$ are given by

$$\kappa_{ij} = E\left(\frac{\partial^2 \ln L}{\partial\theta_i\partial\theta_j}\right), \qquad \kappa_{ijl} = E\left(\frac{\partial^3 \ln L}{\partial\theta_i\partial\theta_j\partial\theta_l}\right), \qquad \kappa_{ij,l} = E\left(\frac{\partial^2 \ln L}{\partial\theta_i\partial\theta_j}\cdot\frac{\partial \ln L}{\partial\theta_l}\right), \quad i, j, l = 1, 2, \dots, p.$$

It is assumed that each of $\kappa_{ij},\ \kappa_{ijl},\ \kappa_{ij,l}$ is $O(n)$. Then, Cox and Snell (1968) showed that when the sample data are independent (but not necessarily identically distributed) the bias of the s-th element of the MLE $\hat{\boldsymbol{\theta}}$ is given by

$$\mathrm{Bias}(\hat{\theta}_s) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p}\kappa^{si}\kappa^{jl}\left(0.5\kappa_{ijl} + \kappa_{ij,l}\right) + O(n^{-2}), \quad s = 1, 2, \dots, p,$$

where κ^{ij} is the (i,j)-th element of the inverse of the expected information matrix $[-\kappa_{ij}]$, i.e., κ^{ij} is the (i,j)-th element of the variance–covariance matrix of $\hat{\boldsymbol{\theta}}$.

Cordeiro and Klein (1994) noted that Cox–Snell bias expression also holds if the data are dependent, and that it can re-written as

$$\text{Bias}(\hat{\theta}_s) = \sum_{i=1}^{p} \kappa^{si} \sum_{j=1}^{p} \sum_{l=1}^{p} \left(\kappa_{ij}^{(l)} - 0.5\kappa_{ijl} \right) \kappa^{jl} + O(n^{-2}), \quad s = 1, 2, \ldots, p,$$

where $\kappa_{ij}^{(l)} = \frac{\partial \kappa_{ij}}{\partial \theta_l}$, $i, j, l = 1, 2, \ldots, p$, is assumed to be $O(n)$. Note that the $O(n^{-1})$ bias formula introduced by Cox and Snell (1968), and re-expressed by Cordeiro and Klein (1994) is appealing, since it enables us to obtain analytic expression for the bias even though the score equations do not admit closed-form solution.

For the power Lindley model, we have

$$\kappa^{11} = Var(\hat{\alpha}), \qquad \kappa^{22} = Var(\hat{\beta}), \qquad \kappa^{12} = \kappa^{21} = Cov(\hat{\alpha}, \hat{\beta}).$$

Also, we have

$$\kappa_{111} = \frac{2n}{\alpha^3} + \frac{n\beta^2}{\alpha^3(\beta+1)} \int_0^\infty \left[\frac{1-t}{(1+t)^3} - \beta \right] t(\ln t)^3 (1+t) e^{-\beta t} dt,$$

$$\kappa_{222} = \frac{2n(\beta^3 + 6\beta^2 + 6\beta + 2)}{\beta^3(\beta+1)^3},$$

$$\kappa_{112} = \kappa_{121} = \kappa_{211} = -n \frac{\beta\{[\psi(2) - \ln\beta]^2 + \zeta(2,2)\} + 2\{[\psi(3) - \ln\beta]^2 + \zeta(2,3)\}}{\alpha^2 \beta(\beta+1)},$$

$$\kappa_{221} = \kappa_{122} = \kappa_{212} = 0.$$

Finally, since $\kappa_{11} = -nI_{11},\ \kappa_{22} = -nI_{22},\ \kappa_{12} = \kappa_{21} = -nI_{12}$, we obtain

$$\kappa_{11}^{(1)} = \frac{2n}{\alpha} I_{11},$$

$$\kappa_{11}^{(2)} = \frac{n\beta}{\alpha^2(\beta+1)^2} \int_0^\infty \left[\frac{-\beta^2 t + \beta(1-t) + 2}{(1+t)^2} - \beta(-\beta^2 t + \beta(2-t) + 3) \right] t(\ln t)^2 (1+t) e^{-\beta t} dt,$$

$$\kappa_{22}^{(1)} = 0,$$

$$\kappa_{22}^{(2)} = \frac{2n(\beta^3 + 6\beta^2 + 6\beta + 2)}{\beta^3(\beta+1)^3},$$

$$\kappa_{12}^{(1)} = \kappa_{21}^{(1)} = \frac{n}{\alpha} I_{12},$$

$$\kappa_{12}^{(2)} = \kappa_{21}^{(2)} = n \frac{\beta^2 + 5\beta + 3 + (\beta^2 + 4\beta + 2)[\psi(2) - \ln\beta]}{\alpha\beta^2(\beta+1)^2}.$$

Table 3 gives the values of $Bias(\hat{\alpha})$ and $Bias(\hat{\beta})$ to $O(n^{-1})$ for the same combinations of α, β and n considered in Tables 1 and 2. Table 3 shows that the bias decreases when the sample size increases. Also, this table suggests that $Bias(\hat{\alpha})$ is positive and $Bias(\hat{\beta})$ may be positive or negative depending on the values of α and β. Also, the biases given in this table agree, to large extent, with the corresponding simulation results in Tables 1 and 2. The cases $\alpha = 1.5$, $\beta = 1$, $n = 75, 200$ in the simulations show very small negative bias which may be due to the level of tolerance set for convergence in the simulated MLEs.

9. Illustrative example

The following data represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20 mm (Bader and Priest, 1982):

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

For these data, we fit the proposed power Lindley distribution as well as the following three well-known two-parameter distributions:

(i) $Gompertz(\alpha_1, \lambda_1)$

$$f_{Go}(x) = \lambda_1 \exp\left\{ \alpha_1 x - \frac{\lambda_1}{\alpha_1}(e^{\alpha_1 x} - 1) \right\}, \quad x > 0,\ \alpha_1, \lambda_1 > 0;$$

Table 3
Bias of the MLEs $\hat{\alpha}$ and $\hat{\beta}$ to $O(n^{-1})$.

α	β	n	Bias ($\hat{\alpha}$)	Bias ($\hat{\beta}$)
0.2	4	25	0.0114	0.3420
		50	0.0057	0.1710
		75	0.0038	0.1140
		100	0.0028	0.0855
		200	0.0014	0.0427
1.5	1	25	0.0777	0.0014
		50	0.0389	0.0007
		75	0.0259	0.0005
		100	0.0194	0.0003
		200	0.0097	0.0002
0.9	0.35	25	0.0401	−0.0085
		50	0.0201	−0.0042
		75	0.0134	−0.0028
		100	0.0100	−0.0021
		200	0.0050	−0.0011

Table 4
Summary of fitted distributions.

Model	MLEs	S.E.	ln L
Exponential	$\tilde{\lambda} = 0.408$	0.049	−130.869
Lindley	$\tilde{\beta} = 0.659$	0.058	−119.190
Gompertz	$\hat{\alpha}_1 = 2.042$	0.185	−53.625
	$\hat{\lambda}_1 = 0.008$	0.004	
Gamma	$\hat{\alpha}_2 = 23.382$	3.953	−50.037
	$\hat{\lambda}_2 = 9.538$	1.630	
Weibull	$\hat{\alpha}_3 = 5.505$	0.500	−49.596
	$\hat{\lambda}_3 = 0.005$	0.002	
Power Lindley	$\hat{\alpha} = 3.868$	0.319	−49.059
	$\hat{\beta} = 0.050$	0.016	

(ii) *Gamma*(α_2, λ_2)

$$f_{Ga}(x) = \frac{\lambda_2^{\alpha_2}}{\Gamma(\alpha_2)}\, x^{\alpha_2 - 1}\, e^{-\lambda_2 x}, \quad x > 0,\ \alpha_2, \lambda_2 > 0;$$

(iii) *Weibull*(α_3, λ_3)

$$f_W(x) = \alpha_3 \lambda_3\, x^{\alpha_3 - 1}\, e^{-\lambda_3 x^{\alpha_3}}, \quad x > 0,\ \alpha_3, \lambda_3 > 0.$$

Since the exponential distribution is a special case of the above three distributions (as $\alpha_1 \to 0$ and $\alpha_2 = \alpha_3 = 1$) and that the Lindley distribution is a special case of the power Lindley distribution, we fit the exponential and Lindley distributions for these data as well. Summary of all these fitted distributions is provided in Table 4. Table 4 shows that the power Lindley model has the largest log-likelihood among these models which indicates that the power Lindley model provides the best fit for the given data among all these models. Since the better fitting models are all two-parameter models, selection by largest log-likelihood will also be equivalent to selecting the Lindley distribution as the best model by Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) as well. Table 5 presents the results of the goodness-of-fit tests of all these models for the considered data. These results show once again that the power Lindley model has the smallest (largest) test-statistic (p-value) of both Kolmogorov–Smirnov and Anderson–Darling goodness-of-fit tests, thus revealing that the best fit is provided by the power Lindley model.

Finally, Figs. 5 and 6 show the probability–probability (P–P) and quantile–quantile (Q–Q) plots for all the fitted models. Once again, these two plots support the conclusion that the power Lindley distribution provides the best fit among all the considered models.

10. Concluding remarks

A new two-parameter distribution, called the power Lindley distribution, is introduced and studied in detail. This model provides more flexibility than the Lindley distribution in terms of the shape of the density and hazard rate functions as well as its skewness and kurtosis. Maximum likelihood estimates of the parameters and their standard errors are then derived. Three algorithms are proposed for generating data from the proposed distribution. A simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators of the model parameters as well as the coverage

Table 5
Goodness-of-fit tests.

Model	K–S statistic (p-value)	A–D statistic (p-value)
Exponential	0.448 (0.000)	20.413 (0.000)
Lindley	0.401 (0.000)	17.151 (0.000)
Gompertz	0.085 (0.673)	0.928 (0.397)
Gamma	0.058 (0.962)	0.334 (0.910)
Weibull	0.056 (0.973)	0.274 (0.956)
Power Lindley	0.044 (0.998)	0.160 (0.998)

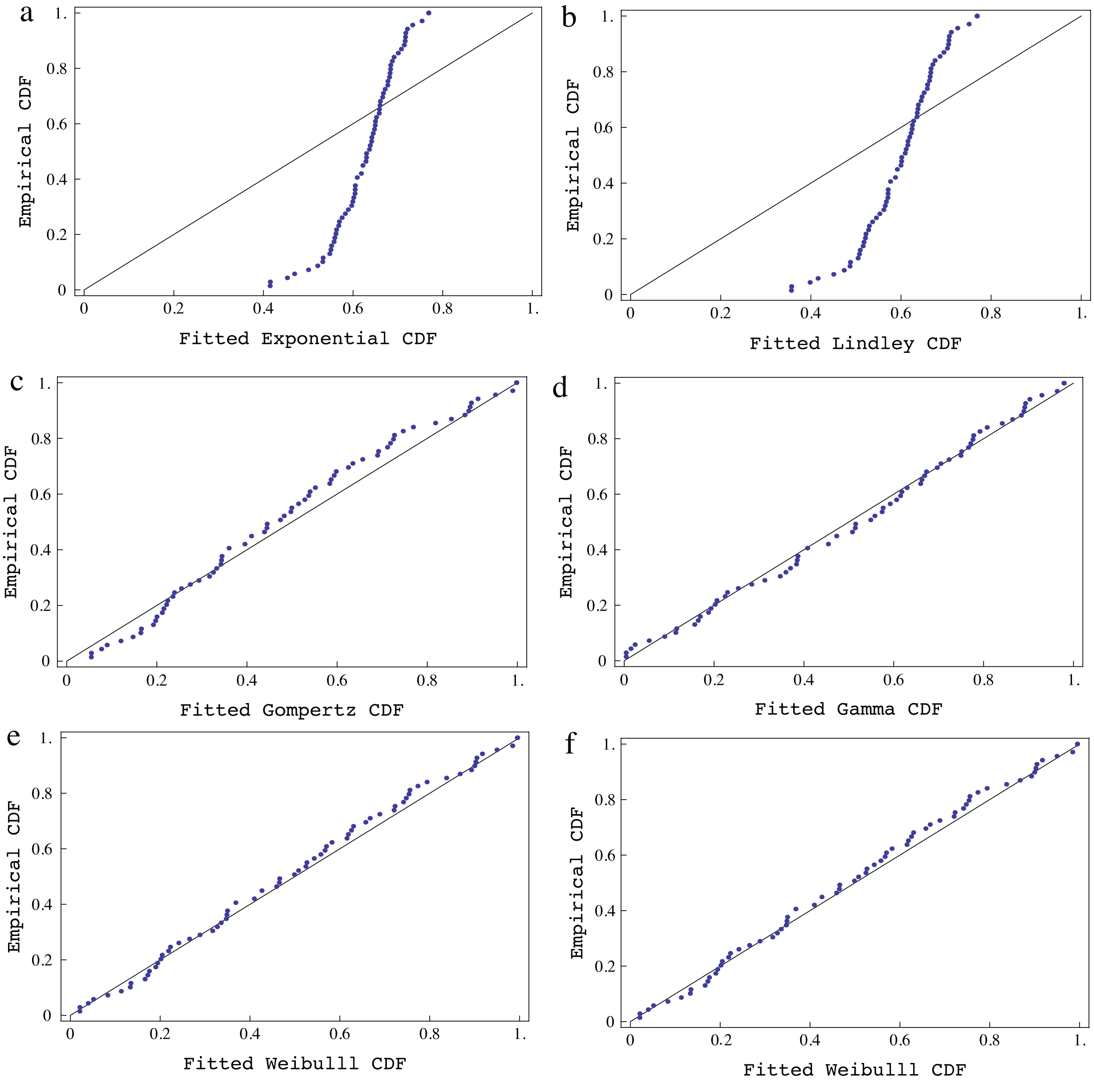

Fig. 5. P–P plots for the fitted distributions.

probability and average width of the confidence intervals for the parameters. Application of the proposed distribution to a real data shows better fit than many other well-known two-parameter distributions, such as gamma, Weibull and Gompertz.

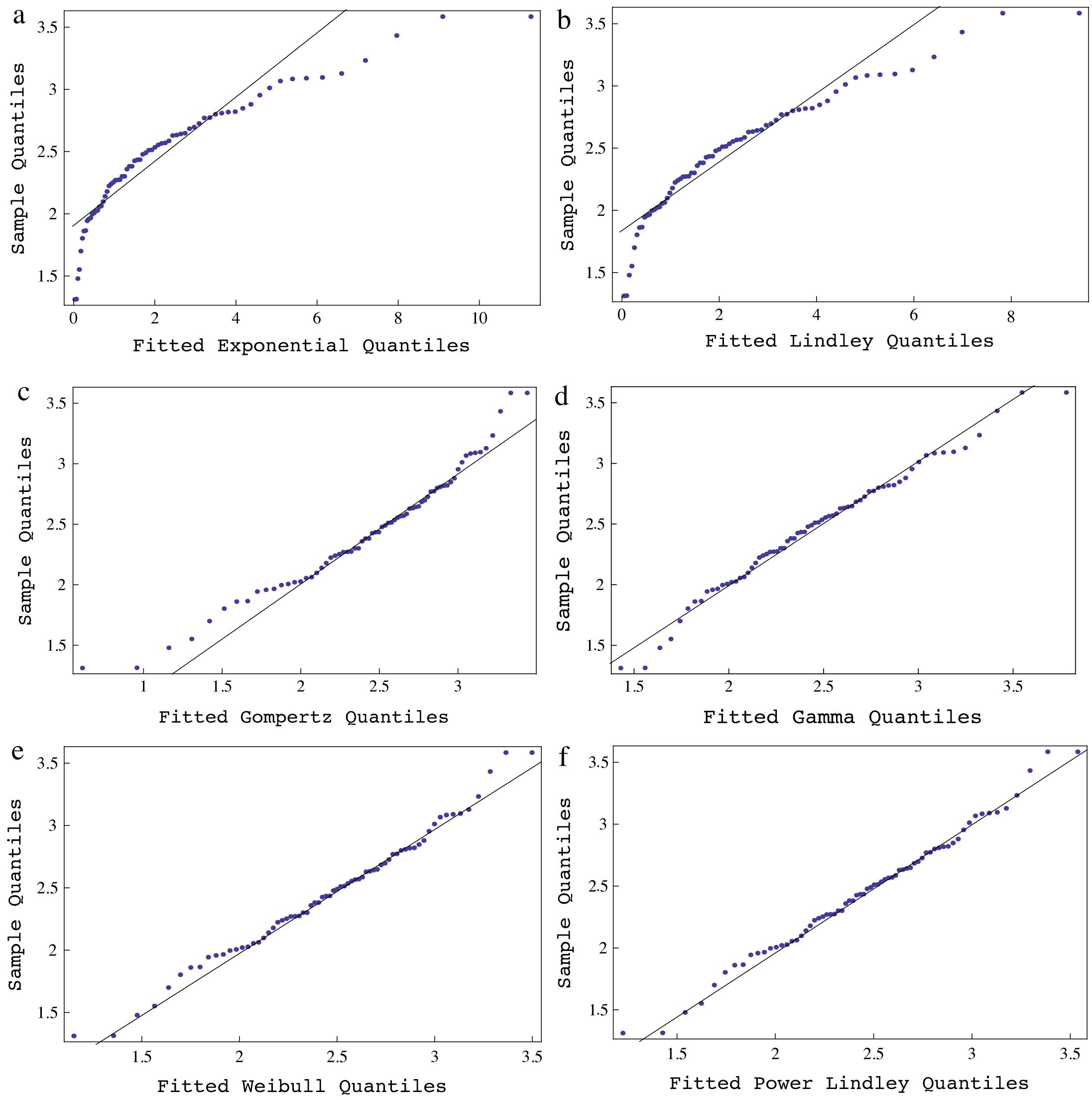

Fig. 6. Q–Q plots for the fitted distributions.

Acknowledgments

The authors are thankful to the referees for some useful comments on an earlier version of this manuscript which led to this improved version.

References

Arnold, B.C., Balakrishnan, N., Nagaraja, H.N., 1992. A First Course in Order Statistics. John Wiley & Sons, New York.
Bader, M.G., Priest, A.M., 1982. Statistical aspects of fiber and bundle strength in hybrid composites. in: Hayashi, T., Kawata, K., Umekawa, S. (Eds.), Progress in Science and Engineering Composites. ICCM-IV, Tokyo, pp. 1129–1136.
Chapeau-Blondeau, F., Monir, A., 2002. Evaluation of the Lambert W function and application to generation of generalized Gaussian noise with exponent 1/2. IEEE Transactions on Signal Processing 50 (9), 2160–2165.
Cordeiro, G.M., Klein, R., 1994. Bias correction in ARMA models. Statistics & Probability Letters 19, 169–176.
Cox, D.R., Snell, E.J., 1968. A general definition of residuals. Journal of the Royal Statistical Society, Series B 30, 248–275.
Ghitany, M.E., Atieh, B., Nadarajah, S., 2008. Lindley distribution and its application. Mathematics and Computers in Simulation 78, 493–506.
Gradshteyn, I.S., Ryzhik, I.M., 2007. Tables of Integrals, Series, and Products, seventh ed. Academic Press, New York.
Jodra, P., 2010. Computer generation of random variables with Lindley or Poisson–Lindley distribution via the Lambert W function. Mathematics and Computers in Simulation 81, 851–859.
Lehmann, L.E., Casella, G., 1998. Theory of Point Estimation, Second ed. Springer, New York.
Lindley, D.V., 1958. Fiducial distributions and Bayes' theorem. Journal of the Royal Statistical Society, Series A 20, 102–107.