L03-GaussianQuad

December 9, 2024

Supporting textbook chapters: 5.4-5.9

0.1 Gaussian quadrature

0.1.1 General Idea

$\int_a^b f(x)dx \approx \sum_{k=1}^{N+1} w_k f(x_k)$, with w_k "weights" and x_k "sample points"

Newton-Cotes: * have to use N equally-spaced sampled points. * N^{th}-order N-C exact for polynomial of degree N. * A N^{th}-order polynomial approximates a well-behaved function better than a $(N-1)^{\text{th}}$-order polynomial, because of the added degree of freedom.

Gaussian quadrature: * N unequally-spaced points $\Rightarrow$ N more degrees of freedom, * exact for $(2N-1)^{th}$-order polynomial. * other way to look at it: it will give the same level of accuracy as an approximation by a $(2N-1)^{th}$-order polynomial.

Remarkably, there is a universal rule to choose the w_k and x_k: * $x_k = $ roots of N^{th} Legendre polynomial $P_N(x)$. * $w_k = \left[\frac{2}{1-x^2}\left(\frac{dP_N}{dx}\right)^{-2}\right]_{x=x_k}$, while $P_N(x_k) = 0$.

Pros: * complicated error formula, but in general: approximation error improves by a factor c/N^2 when you increase # of sample points by 1 * e.g., going from $N = 10$ to $N = 11$ sample points improves your estimate by a factor of $\sim 100 \Rightarrow$ converge very quickly to true value of the integral.

Cons: * only works well if function is reasonably smooth (since sample points are farther apart), * really hard to get an accurate estimate of the error, if needed.

0.1.2 About Legendre Polynomials

- Defined to be mutually orthonormal:

$$\forall (M, N) \in \mathbb{N}^2, \quad \int_{-1}^{1} P_N(x) P_M(x) dx = \frac{2\delta_{MN}}{2N+1}.$$

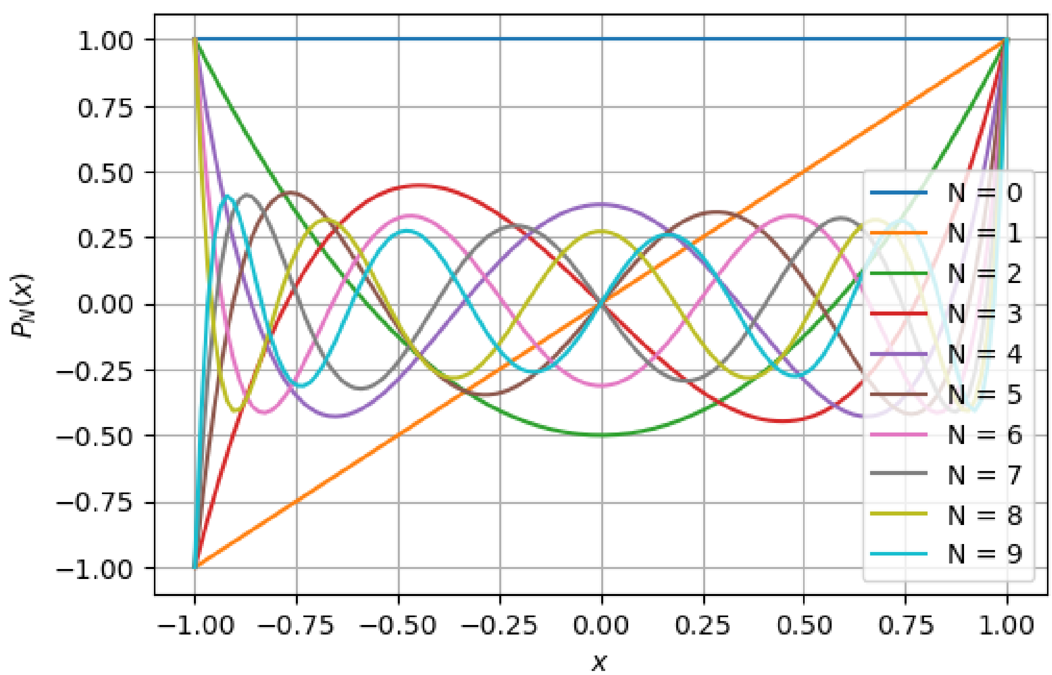
- Defined recursively:

$$P_0(x) = 1 \Rightarrow P_1(x) = x \Rightarrow P_2(x) = \dots, \text{ or}$$
$$(N+1)P_{N+1}(x) = (2N+1)xP_N(x) - NP_{N-1}(x), \text{ or}$$
$$\frac{d}{dx}\left[(1-x^2)\frac{dP_{N+1}}{dx}(x)\right] = -N(N+1)P_N(x), \text{ or}$$
$$P_N(x) = \frac{1}{2^N N!}\frac{d^N}{dx^N}\left[(x^2-1)^N\right] \dots$$

[43]:
```python
from scipy.special import legendre
from matplotlib.pyplot import plot, grid, xlabel, ylabel, figure, legend
from numpy import linspace

x = linspace(-1, 1, 128)
figure(dpi=100)
for N in range(10):
    plot(x, legendre(N)(x), label='N = {}'.format(N))
grid()
xlabel("$x$")
ylabel("$P_N(x)$")
legend()
```

[43]: <matplotlib.legend.Legend at 0x7f8766abae50>



0.1.3 Finding the weights and points

$\int_a^b f(x)dx \approx \sum_{k=1}^{N+1} w_k f(x_k)$, with w_k “weights” and x_k “sample points”

- It’s beautiful that there is a universal rule for choosing the w_k and x_k; in the context of this course, we’ll just accept that it works rather than doing the derivation (Appendix C of textbook if you’re curious)
- Old-fashioned tables exist. Textbook mentions Abramowitz and Stegun; was replaced long ago by NIST’s Digital Library for Mathematical Functions. For Gauss quadrature: https://dlmf.nist.gov/3.5#v

- Don't even write your own program to find sample points and weights – use given subroutines.
  - `gaussxw.py` for integration limits from -1 to $+1$,
  - `gaussxwab.py` for integration limits from a to b.
- The calculation of weights and points is expensive. Use `gaussxw.py` if you are going to change the limits repeatedly (and see end of §5.6.1, pp. 167-168, for how to do).

[ ]:

```

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