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1 Spectral Methods for Hyperbolic Equations

1.1 Review from Last Week: Wave Equation

Recall from last week the 1D wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

We considered a piano string of length L, initially at rest. At time $t = 0$ the string is struck by the piano hammer a distance d from the end of from the string. The string vibrates as a result of being struck, except at the ends, $x = 0$, and $x = L$, where it is held fixed.

We considered the case $v = 100\text{ms}^{-1}$, with the initial condition that $\phi(x) = 0$ everywhere but the velocity $\psi(x)$ is nonzero, with profile

$$\psi(x) = C\frac{x(L-x)}{L^2} \exp\left[-\frac{(x-d)^2}{2\sigma^2}\right], \tag{1}$$

where $L = 1\text{m}$, $d = 10\text{cm}$, $C = 1\text{ms}^{-1}$, and $\sigma = 0.3\text{m}$.

1.2 New This Week

Now, we can expand the initial conditions in terms of a Fourier sine series:

$$\phi_0(x) = \sum_{k=1}^{\infty} \tilde{\phi}_{0,k} \sin\left(\frac{k\pi x}{L}\right)$$

$$\psi_0(x) = \sum_{k=1}^{\infty} \tilde{\psi}_{0,k} \sin\left(\frac{k\pi x}{L}\right)$$

Note that all the terms in this series vanish at $x = 0$ and $x = L$, automatically satisfying the boundary conditions for ϕ, and similarly for ψ.

We can show, the general solution in terms of the Fourier series is:

$$\phi(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi x}{L}\right) \left[\tilde{\phi}_{0,k} \cos(\omega_k t) + \frac{\tilde{\psi}_{0,k}}{\omega_k} \sin(\omega_k t)\right]$$

* By substitution:

$$v^2 \frac{\partial^2 \phi(x,t)}{\partial x^2} = -\sum_{k=1}^{\infty} (vk\pi/L)^2 \sin(k\pi x/L) \left[\tilde{\phi}_{0,k} \cos(\omega_k t) + \frac{\tilde{\psi}_{0,k}}{\omega_k} \sin(\omega_k t)\right]$$

and

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} = -\sum_{k=1}^{\infty} (\omega_k)^2 \sin(k\pi x/L) \left[\tilde{\phi}_{0,k} \cos(\omega_k t) + \frac{\tilde{\psi}_{0,k}}{\omega_k} \sin(\omega_k t) \right]$$

* This requires, to satisfy the wave equation, that for each k:

$$\omega_k^2 = (vk\pi/L)^2.$$

* At $t = 0$:

$$\phi_0(x) = \sum_{k=1}^{\infty} \sin(k\pi x/L) \tilde{\phi}_{0,k}, \;\; \psi_0(x) = \sum_{k=1}^{\infty} \sin(k\pi x/L) \tilde{\psi}_{0,k}$$

as required, thus the solution satisfies the initial conditions. Furthermore the solution automatically satisfies the boundary conditions since $sin(k\pi x/L)$ vanishes at the $x = 0$ and $x = L$ for all integer k.

1.3 Exercise 1

Use a series solution like this, but truncated at a finite N, to repeat Exercise 6 from last week. In case you forgot: **Solve from times 0 to 0.1s, and make a plot of ϕ vs x over the entire length of string at each of the following times:** * 0.006 s * 0.004 s * 0.002 s * 0.012 s * 0.100 s

You can afford to use a large N, since you should only have to do one Fourier transform. You may use packaged Fourier functions such as numpy.fft.rfft, numpy.fft.irfft . You can even use functions from the textbook's dcst.py file.

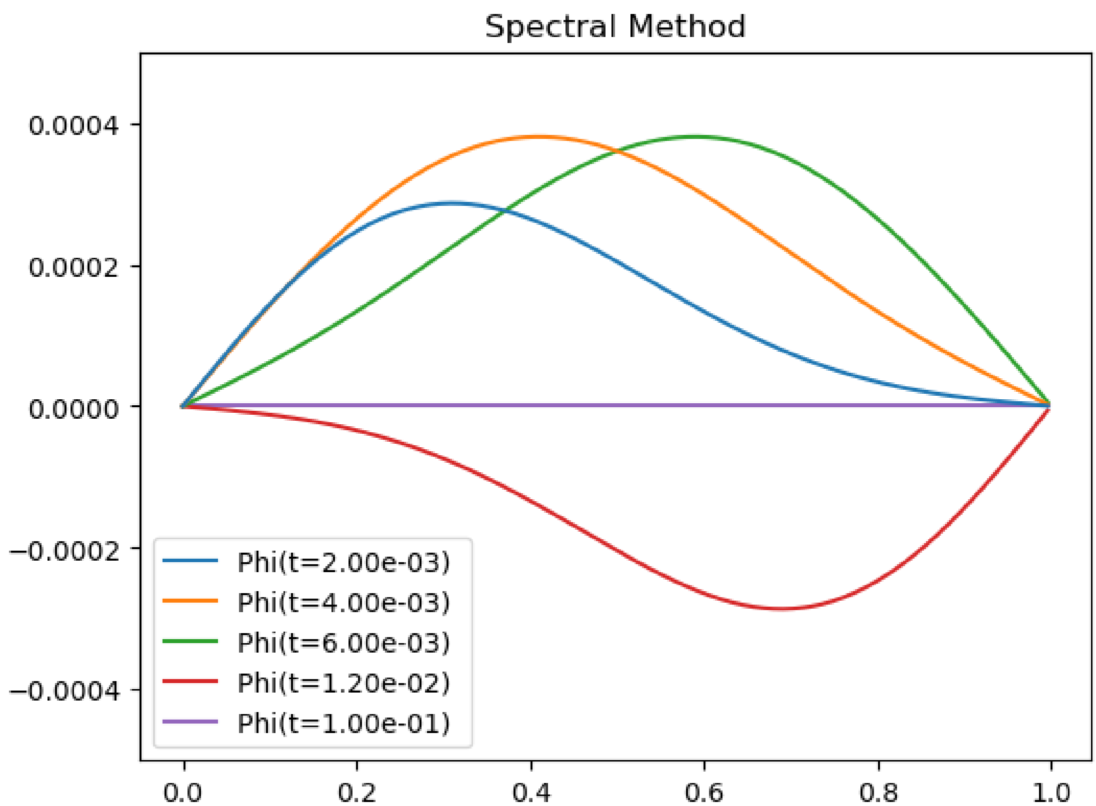
[21]:
```
import matplotlib.pyplot as plt
_ = 1.0 #m
d = 0.1 #m
v = 100.0 #m/s
C = 1.0 #m/s
sigma = 0.3 #m/s
SnapshotTimes = np.array ([2e-3, 4e-3, 6e-3, 12e-3, 100e-3])

def setup_grid(N = 2**8):
    a=L/N
    x = np.arange(0,L,a)
    psi = C*x*(L-x)*np.exp(-(x-d)**2/(2*sigma**2))/L**2
    phi = np. zeros_like(x)
    return x, phi, psi
```

[22]:
```
def plot_results(positions, values, title):
    LegendLabel = []
    for c,t in enumerate(values) :
        time, phi, psi = t
        plt.plot(positions, phi, zorder=-c)
        plt.ylim ((-5e-4,5e-4))
        LegendLabel.append ('Phi(t=%3.2e) '%time)
        plt.legend (LegendLabel)
```

```
        plt.title(title)
```

[23]:
```
def makeFourierArrays (v=100.0,L=1.0,N=2**8):
    karr = np.arange(N)
    omarr = v*karr*np.pi/L
    return karr, omarr
```

[26]:
```
from dcst import dst, idst

[x, phi, psi] = setup_grid()
[karr, omarr] = makeFourierArrays()

phiDST = np.empty((len(SnapshotTimes), len(x)), float)
FTpsi = dst(psi)
FTphi = np.zeros_like(FTpsi)
FTphi[1:] = FTpsi[1:] / omarr[1:]

for ii, tt in enumerate(SnapshotTimes):
    phiDST[ii, :] = idst(FTphi * np.sin(omarr * tt))

results = []
for ii, tt in enumerate(SnapshotTimes):
    results.append([tt, phiDST[ii, :], None])

plot_results(x, results, 'Spectral Method')
```



2 Boundary Value Problem Example: Shooting Method for Projectile Motion

Suppose we want to choose an initial velocity v_0 for a projectile, thrown straight upward, to land after a certain elapsed time $t_L = 10$ s, where the projectile obeys Newton's 2nd Law:

$$\frac{d^2y}{dt^2} = \frac{F(y)}{m} \quad \Rightarrow \quad \frac{dy}{dt} = v \quad \text{and} \quad \frac{dv}{dt} = \frac{F(y)}{m}.$$

Take the gravitational constant to be $g = 9.81 m/s^2$

2.1 Exercise 2

Write code to integrate the equations, using RK4, from time $t = 0$ to $t = t_L$ to find y of the projectile at $t = t_L$. It should be a function of v_0. Assume the projectile starts at $y = 0$ at $t = 0$.

Hint: keep track of both y and v. At each timestep of the RK4 algorithm, you need to invoke a function that encodes the change in y and the change in v. The number of timesteps should be a parameter that you can set.

[32]:
```
t0 = 0.  # seconds
tL = 10.  # seconds
y0 = 0
```

```python
g = 9.81 # m/s^2
N = 1000
h = (tL - t0)/N
```

[34]:
```python
def f(r): # used in RK4 calculation
    return np.array([r[1], -g], float)

def height(v):
    r = np. array([0.0, v], float)
    for t in np.arange(t0, tL, h):
        k1 = h*f(r)
        k2 = h*f(r+0.5*k1)
        k3 = h*f(r+0.5*k2)
        k4 = h*f(r+k3)
        r += (k1+2*k2+2*k3+k4)/6.
    return r[0]
```

2.2 Exercise 3

Write code that performs a binary search to find the initial value of velocity, v_0, that gives us $y = 0$ (with target accuracy 1e-8) at $t = t_L$. Invoke the code from the previous exercise (try 1000 RK4 steps) at each step of the search.

[38]:
```python
target = 1e-8 # Target accuracy
v1 = 0.01
v2 = 1000.0
h1 = height(v1)
h2 = height(v2)

while np.abs(h2-h1) > target:
    vp = (v1+v2)/2.
    hp = height(vp)
    if h1*hp > 0:
        v1 = vp
        h1 = hp
    else:
        v2 = vp
        h2 = hp

v = (v1+v2)/2
print("The required initial velocity is", v, "m/s")
```

```
The required initial velocity is 49.05000000042803 m/s
```

3 Verlet Method Example: Orbit Calculation

The position $r(x, y)$ of Earth in its orbital plane can be approximately described by:

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{-GM\mathbf{r}}{r^3}$$

where $M = 1.9891 \times 10^{30}$ (mass of Sun in kg). You can import G from scipy.constants

[1]:
```
from scipy.constants import G
```

Note, when Earth is at its closest point to the Sun (perihilion): * its direction of motion is perpendicular to the line between itself and the Sun * $r = 1.471 \times 10^{11}$ m * linear velocity $v = 3.0287 \times 10^4$ m/s

3.1 Exercise 4

Write a program to calculate Earth's position using Verlet method, with timestep $h = 1$ hour, over 5 years.

You probably want to keep track of x and y separately (similarly v_x and v_y), and calculate $r = \sqrt{x^2 + y^2}$.

Define the x, y axes as: Earth starts at perihelion, along x>0, with positive velocity along y axis

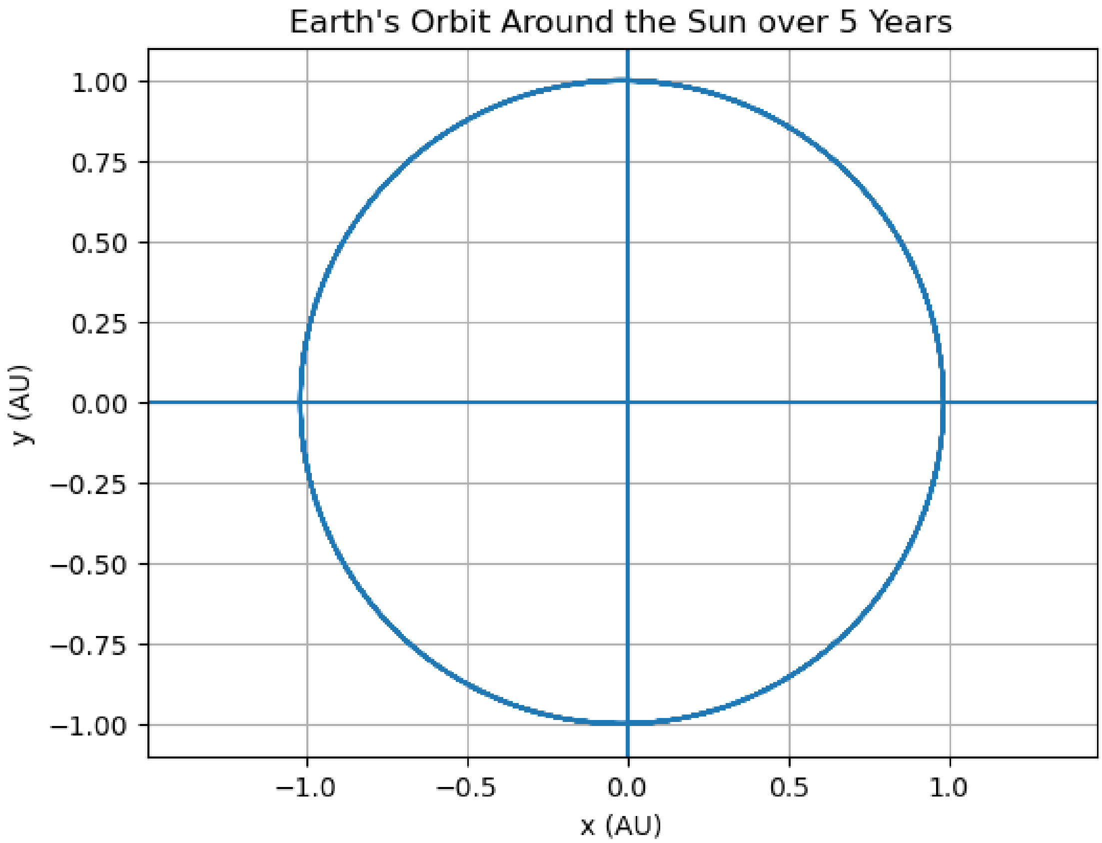
[42]:
```
import numpy as np
from scipy.constants import G

M = 1.989e30
PH = 1.471e11
VP = 3.0287e4
h = 3600.0
Nrevs = 5
year = 365.25 * 24 * 3600.0
T = Nrevs * year
Nsteps = int(T / h)

def f(r_):
    x, y, vx, vy = r_[0], r_[1], r_[2], r_[3]
    r = (x**2 + y**2)**0.5
    prefac = -G * M / r**3
    return np.array([vx, vy, x * prefac, y * prefac], float)

pos = np.empty((2, Nsteps), float)
vel = np.empty((2, Nsteps), float)

v0 = np.array([0.0, VP])
pos[:, 0] = np.array([PH, 0.0])
r = (pos[0, 0]**2 + pos[1, 0]**2)**0.5
vel[:, 0] = v0 - h * G * M * pos[:, 0] / r**3
```

```
for tt in range(1, Nsteps):
    pos[:, tt] = pos[:, tt - 1] + h * vel[:, tt - 1]
    r = (pos[0, tt]**2 + pos[1, tt]**2)**0.5
    vel[:, tt] = vel[:, tt - 1] - h * G * M * pos[:, tt] / r**3
```

3.2 Exercise 5

Plot y vs x over the 5 years. This should draw over an orbit 5 times; the orbit should be slightly, but visibly, non-circular. Make the plot in units of AU = Astronomical Unit = average Earth-Sun distance.

[49]:
```
import matplotlib.pyplot as plt
from scipy.constants import au

# Convert position to AU
x_au = pos[0, :] / au
y_au = pos[1, :] / au

plt.plot(x_au, y_au)
plt.xlabel("x (AU)")
plt.axhline(0.0)
plt.axvline(0.0)
plt.ylabel("y (AU)")
plt.title("Earth's Orbit Around the Sun over 5 Years")
plt.grid()
plt.axis('equal')
plt.show()
```


Earth's Orbit Around the Sun over 5 Years
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y (AU)
−1.0
−0.5
0.0
0.5
1.0
x (AU)