CSC384H Tutorial 8

Probability & Bayes Nets

Overview of Probability

Probability axioms

Let S be a sample space and A, B be events in S. A probability function P from the set of all events in S to the set of real numbers satisfies the following axioms:

1. $0 \leq P(A) \leq 1$.
2. $P(\varnothing) = 0$ and $P(S) = 1$.
3. If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$.

If A and B are not mutually exclusive, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ instead. Additionally, the probability of the complement of an event A is given by $P(\bar{A}) = 1 - P(A)$.

Linearity of expectation

Suppose a random process X has n possible outcomes, given by $a_1, a_2, \ldots, a_n$, which occur with probabilities $p_1, p_2, \ldots, p_n$ respectively. The **expected value** of the process is given by

$$E[X] = \sum_{k=1}^{n} a_k p_k \tag{1}$$

Linearity of expectation states that the expected value of the sum of random variables is equal to the sum of their individual expected values, regardless of whether they are independent. For random variables $X_1, X_2, \ldots, X_n$, we have

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E\left[X_i\right] \tag{2}$$

Conditional probability

Let $A_1, A_2, \ldots, A_n$ be events in a sample space S. If $P(\bigcap\limits_{i=2}^{n} A_i) \neq 0$, the **conditional probability of** A_1 **given** $A_2, \ldots, A_n$, denoted $P(A_1 \mid A_2, \ldots, A_n)$, is given by

$$P(A_1 \mid A_2, \ldots, A_n) = \frac{P(A_1, A_2, \ldots, A_n)}{P(A_2, \ldots, A_n)} \tag{3}$$

Note that equation 3 can also be rewritten as

$$P(A_1, A_2, \ldots, A_n) = P(A_1 \mid A_2, \ldots, A_n) P(A_2, \ldots, A_n)$$

Using the **chain rule**, we may expand the right hand side of the above equation as such:

$$\begin{aligned} P(A_1, A_2, \ldots, A_n) &= P(A_1 \mid A_2, \ldots, A_n)P(A_2, \ldots, A_n) \\ &= P(A_1 \mid A_2, \ldots, A_n)P(A_2 \mid A_3, \ldots, A_n)P(A_3, \ldots, A_n) \\ &\vdots \\ &= P(A_1 \mid A_2, \ldots, A_n) \ldots P(A_{n-1} \mid A_n)P(A_n) \end{aligned} \tag{4}$$

Bayes' theorem

Suppose that a sample space S contains events A and B with nonzero probabilities. Then, we have

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

More generally, suppose that a sample space S is a union of mutually disjoint events $B_1, B_2, \ldots, B_n$, and suppose A and all the B_i have nonzero probabilities. If k is an integer with $1 \leq k \leq n$, then

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \ldots + P(A|B_n)P(B_n)} \tag{5}$$

Independent events

If A, B, and C are events in a sample space S, then A and B are **independent** if and only if

$$P(A \cap B) = P(A)P(B) \iff P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

Given event B, event A is said to be **conditionally independent** of event C if and only if

$$P(A|B, C) = P(A|B) \tag{6}$$

Questions

1. In a game of chance, you are required to draw one coin from a bag containing 4 red coins, 7 green coins, and 9 blue coins. You win $20 if you draw a red coin, $15 if you draw a green coin, or nothing if you draw a blue coin. You pay $10 for each game. After each game, the coin drawn is returned to the bag.

   If you play this game 100 times, how much would you expect to win/lose?

2. A bowl contains three coins. Two of them are normal coins and one of them is a two-headed coin.

   (a) You pick one coin at random and toss it. What is the probability that you get a head?

   (b) You pick one coin at random, toss it, and get a head. What is the probability that the coin is the two-headed coin?

3. There are two bags; one bag contains 10 red apples and 25 green apples, and a second bag contains 22 red apples and 15 green apples. An apple is chosen as follows:

   - A bag is first selected by tossing a loaded coin with probability 0.4 of landing heads up.
   - If the coin lands heads up, the first bag is chosen. Otherwise, the second bag is chosen.
   - Then, an apple is picked at random from the chosen bag.

   (a) What is the probability that the chosen apple is green?

   (b) If the chosen apple is green, what is the probability that it was picked from the first bag?

4. After your yearly checkup, the doctor has some good news and some bad news. The bad news is that you tested positively for a serious disease and that the test is 99% accurate (i.e. the probability of testing positive given that you have the disease is 0.99, as is the probability of testing negative given that you don't have the disease). The good news is that this is a rare disease, striking only 1 in 10,000 people. Why is it good news that the disease is rare? What are the chances that you actually have the disease?

5. Consider the Bayes Net of figure 1. We have that $P(E, S, B, W, G) = P(E)P(B)P(S|E, B)P(W|S)P(G|S)$.

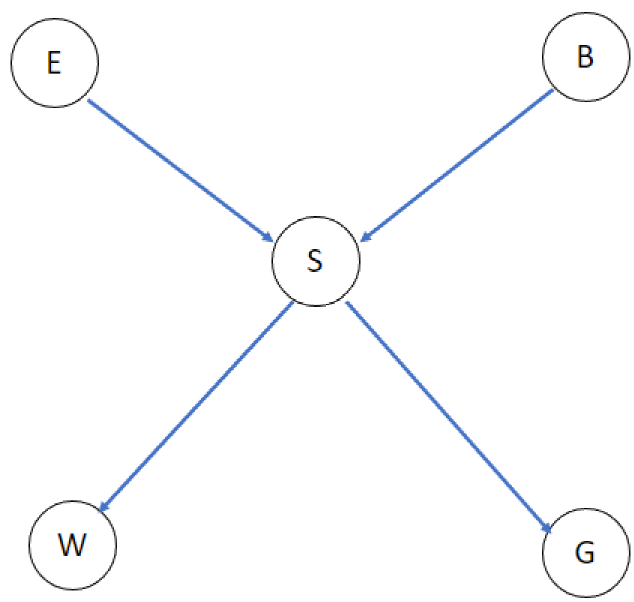

Figure 1: Bayes Net for question 5

Events:

- B: Burglary
- E: Earthquake
- S: Alarm went off
- G: Mrs. Gibbons calls
- W: Dr. Watson calls

P(E)	e	-e
	1/10	9/10

P(B)	b	-b
	1/10	9/10

P(S|E,B)	s	-s
e ∧ b	9/10	1/10
e ∧ -b	2/10	8/10
-e ∧ b	8/10	2/10
-e ∧ -b	0	1

P(W|S)	w	-w
s	8/10	2/10
-s	2/10	8/10

P(G|S)	g	-g
s	1/2	1/2
-s	0	1

Figure 2: CPT of figure 1

(a) Given the alarm went off (S) what is the probability that Mrs. Gibbons phones you (G)?

(b) Given that Mrs. Gibbons phones you (G) what is the probability the alarm went off (S)?

(c) Say that there was a burglary (B) but no earthquake ($\neg E$), what is the expression specifying the probability of Dr. Watson phoning you (W) given the evidence. (You do not need to calculate a numeric answer, just give the probability expression).

a) $P(G|S) = \frac{1}{2}$

b) $P(S|G) = 1$

c) $P(W|B, \neg E, S) =$ $P(S|\neg E, B) \cdot P(\neg E) P(B)$