# Analog Implementation of the Rössler Chaotic System Using Operational Amplifiers

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We construct an analog implementation of the Rössler system—a continuous-time chaotic oscillator—using operational amplifiers, resistors, and capacitors. By emulating each equation through op-amp-based integrators and summing stages, the circuit produces real-time voltages corresponding to the evolving state variables $x(t)$, $y(t)$, and $z(t)$. Nonlinear feedback introduces the $z(x-c)$ term, with the system's parameters $a$, $b$, and $c$ precisely set by resistor ratios and fixed voltage sources. Adjusting these component values reveals qualitative features of chaotic behavior, including aperiodic oscillations and sensitivity to initial conditions, demonstrating the power of analog electronics in modeling complex systems.

## BACKGROUND

The Rössler system, introduced by Otto Rössler in 1976, is a set of three coupled first-order differential equations that exhibit chaotic behavior. As one of the simplest continuous-time chaotic systems, it serves as a foundational example in the study of nonlinear dynamics. The equations are given by:

$$\dot{x} = -y - z \tag{1}$$

$$\dot{y} = x + ay \tag{2}$$

$$\dot{z} = b + z(x - c) \tag{3}$$

where $a$, $b$, and $c$ are tunable parameters that determine the system's behavior. Under certain values (originally $a = 0.2$, $b = 0.2$, $c = 5.7$ as studied by Rössler[4]), the system exhibits a strange attractor with sensitive dependence on initial conditions, a hallmark of chaos.

The Rössler system can possess two fixed points, one typically near the center of the attractor and another farther away. Under the standard chaotic parameter set (e.g., $a = 0.2, b = 0.2, c = 5.7$), one of these fixed points lies close to the origin and serves as a reference point around which the trajectory spirals in phase space. The second fixed point is located farther out, often less relevant for the visible attractor loops [4].

This system can be implemented in hardware using analog electronics, where voltages represent the state variables $x$, $y$, and $z$, and their time derivatives are computed via op-amp-based integrators. In this configuration, operational amplifiers, resistors, and capacitors are used to perform real-time analog integration. We implement three inverting integrators using op-amps, summing input voltages and producing an output proportional to the integral of the input current.

We used a Falstad Circuit Simulator (see Figure 1) throughout the design process to visualize node voltages and verify that the analog implementation correctly reproduced the coupled nonlinear dynamics of the Rössler system.

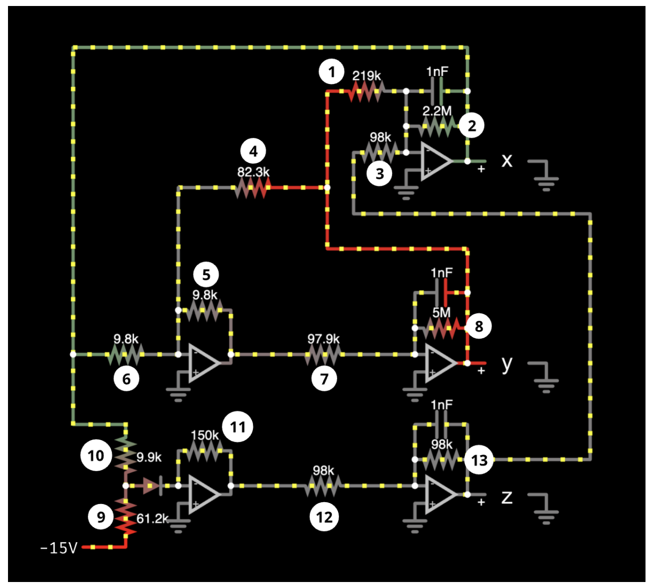


FIG. 1. Labeled Falstad simulation of the analog Rössler circuit. Voltages corresponding to the state variables $x$, $y$, and $z$ evolve in real time through a network of op-amp-based integrators, resistors, and feedback loops. Numbered resistors correspond to components referenced in the analysis.

### Op-Amps and Analog Computation

Operational amplifiers (op-amps) are fundamental components used to implement analog mathematical operations such as amplification, summation, and integration. In the ideal case, an op-amp has infinite gain, infinite input impedance, and zero output impedance. The voltage difference between its inverting and non-inverting inputs is virtually zero when negative feedback is present.

We use the LF356 op-amp, a JFET-input operational amplifier with high input impedance ($10^{12}\Omega$), low input bias current (30 pA), and wide gain bandwidth (5MHz), making it well-suited for fast, low-noise analog computation in integrator and amplifier configurations [3].

### Inverting Amplifier

One standard op-amp configuration is the inverting amplifier. In this setup, the input voltage $V_{\text{in}}$ passes through a resistor $R_{\text{in}}$ to the inverting input, with a feedback resistor $R_f$ connecting the output to the same node. The resulting output voltage is:

$$V_{\text{out}} = -\frac{R_f}{R_{\text{in}}} V_{\text{in}} \tag{4}$$

### Inverting Integrator

An inverting integrator is formed by replacing the feedback resistor with a capacitor. The output voltage is proportional to the negative integral of the input voltage:

$$V_{\text{out}}(t) = -\frac{1}{RC} \int V_{\text{in}}(t) dt \tag{5}$$

Here, $R$ is the input resistor and $C$ is the feedback capacitor. This circuit forms the basis for modeling each differential equation in the Rössler system.

### Implementation Details

In our breadboard implementation of the Rössler system (Figure 2), we use three op-amp integrators to generate $x(t)$, $y(t)$, and $z(t)$ by integrating their respective derivatives $\dot{x}$, $\dot{y}$, and $\dot{z}$. Before these signals reach the integrators, they pass through inverting amplifiers whose resistor ratios set the amplitude of each term, thus determining the coefficients in the differential equations. At the input of each integrator, a $98\,\text{k}\Omega$ resistor provides current-limiting and isolates the summing node from unwanted interactions.

To create the constant voltage offset for $\dot{z}$, we employ a voltage divider ($9.9\,\text{k}\Omega$ and $61.2\,\text{k}\Omega$) from the $-15\,\text{V}$ rail, followed by a 1N4004 diode that ensures negative current flow. This offset, once inverted or attenuated, feeds into the $z$-integrator to realize parameters $b$ and $c$ in $z(x-c)$. Large resistors (e.g., $5\,\text{M}\Omega$) in parallel with each feedback capacitor implement a slow discharge path, preventing op-amp saturation and creating a leaky integrator that stabilizes the circuit over time.

#### *Note on Time Constant and System Scaling*

Each integrator in the circuit is built using a 98 k$\Omega$ resistor and a 1 nF capacitor, giving a consistent time constant of $RC = 98 \times 10^3 \cdot 10^{-9} = 9.8 \times 10^{-5}$ s. This corresponds to a characteristic frequency of $\frac{1}{RC} \approx 10{,}200\,\text{Hz}$, meaning the circuit evolves roughly 10,000 times faster than the unitless time of the original model.

As a result, the circuit does not implement the original Rössler equations directly, but rather a rescaled system:

$$\frac{dV_x}{dt} = \frac{1}{RC} f(V_x, V_y, V_z) \tag{6}$$

This introduces an overall factor of $1/RC$ in front of every derivative, which speeds up the time evolution of the system. However, since this factor multiplies every term equally, it does not affect the relative values of the parameters $a$, $b$, and $c$. The resulting voltage trajectories preserve the shape and dynamics of the Rössler attractor, but evolve more quickly in physical time.

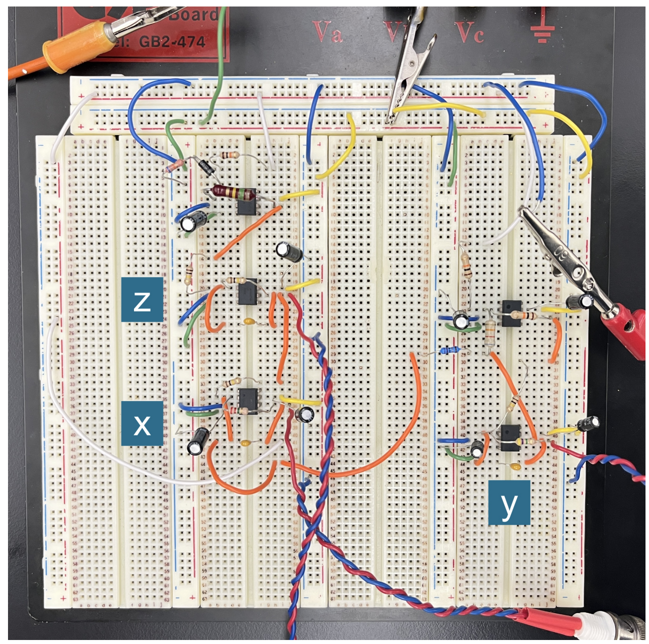


FIG. 2. Breadboard implementation of the Rössler circuit using LF356 op-amps, resistors, capacitors, and a dual power supply. Three integrator stages are visible in the center and right sections of the board, with nonlinear feedback implemented using diode networks and signal mixing. Twisted pairs carry the $x$, $y$, and $z$ outputs to the oscilloscope for visualization.

Crucially, the component values in the circuit directly determine the parameters $a$, $b$, and $c$:

- The coefficient $a$ is defined by the gain of the inverting amplifier feeding $y$ back into the $y$-integrator. This gain is set by the ratio of the resistor carrying the $y$ signal to the feedback resistor. In our circuit, the $x$ and $y$ signals are summed through 9.8 k$\Omega$ and 82.3 k$\Omega$ resistors, respectively, into the inverting input of the $y$-integrator. This gives $a = 9.8/82.3 \approx 0.12$.

- The constant term $b$ is created by a fixed voltage offset derived from a voltage divider made of 9.9 k$\Omega$ and 61.2 k$\Omega$ resistors connected to a $-15$ V supply.

This produces an intermediate voltage of approximately $-12.91$ V, which is then sent through a diode and into an inverting amplifier (with 9.9 k$\Omega$ input and 150 k$\Omega$ feedback resistors), resulting in a theoretical output of approximately $+195.6$ V. However, due to the diode limiting current flow, the actual contribution to $z$ is on the order of 18.4 mV. This voltage sets the constant driving term $b$ in $\dot{z}$, meaning $b \approx 0.02$.

- The parameter $c$ appears in the nonlinear term $z(x - c)$ and is implemented by feeding both the $x$ signal and a fixed voltage into the $z$-integrator through equal 98 k$\Omega$ resistors. The fixed voltage is generated in the same way as $b$, using the voltage divider, diode, and inverting amplifier described above. As with $b$, diode-limited current flow reduces the actual contribution to around 18.4 mV. Because both $x$ and this fixed voltage enter through equal resistors, the op-amp computes $x - (-V)$, effectively realizing $x + V$. This implements the $z(x - c)$ term in hardware, with $c$ corresponding to the magnitude of the fixed voltage, meaning $c \approx 13$ V in circuit units.

This lab explores the use of analog electronics to implement the Rössler system in real time. By carefully designing and tuning each component, the circuit models the system's chaotic behavior, offering insight into how nonlinear dynamics can be realized with simple electronic building blocks.

## DEBUGGING AND TESTING PROCEDURE

Throughout the construction of the Rössler circuit, we encountered multiple issues that required systematic debugging. Initially, the circuit failed to power correctly: although the DC supply was set to $\pm 15$ V, the measured voltages at the power rails were significantly lower. The issue was ultimately resolved after checking wiring continuity and replacing potentially damaged components, though we suspect broken op-amps or incorrect breadboard connections may have been responsible.

Once the power rails were restored, we observed that the circuit still did not produce the expected voltage behavior predicted by the Falstad simulation. Suspecting a faulty op-amp, we isolated and tested each LF356 by building a simple integrator circuit on an unused section of the breadboard. A square wave was fed into each op-amp, and we monitored the output for the expected ramp waveform. This process allowed us to identify a defective op-amp, which we replaced. Following this fix, the full chaotic Rössler attractor appeared on the oscilloscope.

### Safety Considerations

All voltages in this circuit remain below $\pm 15$ V, well under the 45 V guideline. The current draw is minimal, and no components showed significant heating during operation. Thus, no additional safety risks were identified.

## RESULTS

Before analyzing how variations in circuit parameters affect behavior, we first verified that the implemented system qualitatively replicates the expected dynamics of the Rössler attractor. Using an oscilloscope, we plotted $x$ vs $y$ and $x$ vs $z$ in XY mode, revealing clear signatures of chaotic motion.

We observed that the attractor looped around a point near $(x, y) \approx (0, 0)$. This coincides with the expected interior fixed point for these parameter values, while the trajectory never settles there due to the system's chaotic dynamics.

To understand how the component values map onto the system's behavior, we conducted controlled modifications of three key resistors—$R_4$, $R_9$, and $R_{12}$—each corresponding to tunable parameter in the Rössler equations. Their effects are detailed below.

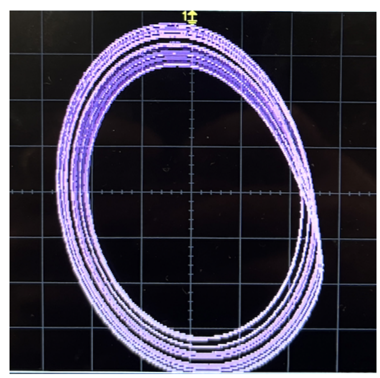

FIG. 3. Oscilloscope plot of $x$ vs $y$ showing the characteristic spiral structure of the Rössler attractor.

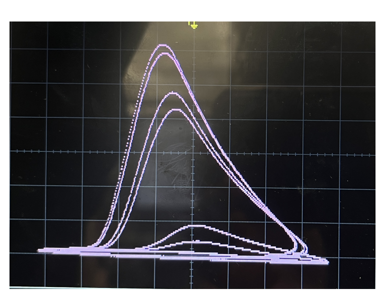

FIG. 4. Oscilloscope plot of $x$ vs $z$ showing the nonlinear relationship between state variables.

### Effect of Varying $R_4$ (Coefficient $a$)

The parameter $a$ in the Rössler equation appears in the linear term $ay$ in $\dot{y} = x + ay$. In our circuit, this is implemented using an inverting summing amplifier that adds scaled versions of $x$ and $y$ to the $y$-integrator input. The $x$ signal is passed through a $9.8\,\mathrm{k}\Omega$ resistor (meaning it's gain is simply $\frac{9.8\mathrm{k}}{9.8\mathrm{k}} = 1$), and the $y$ signal is passed through a resistor $R_4$. Since the $y$ signal passes through the same inverting amplifier but with $R_{in} = R_4$, its gain has magnitude:

$$a = \frac{9.8\,\mathrm{k}\Omega}{R_4}. \tag{7}$$

By varying $R_4$ and keeping the $9.8\,\mathrm{k}\Omega$ resistor fixed, we can tune the value of $a$ and observe the corresponding changes in system dynamics. Table I summarizes representative measurements of $R_4$ and the resulting values of $a$.

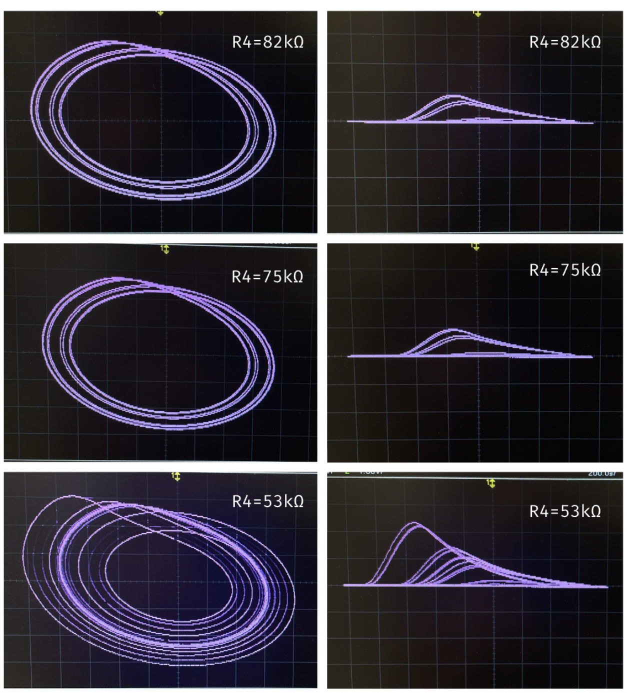


FIG. 5. Oscilloscope traces showing changes in the Rössler attractor as $R_4$ is decreased (i.e., as $a$ increases). Left column shows the $x$-$y$ phase plane; right column shows the $x$-$z$ projection. The attractor grows increasingly stretched and distorted with decreasing $R_4$, illustrating the system's sensitivity to $a$.

TABLE I. Measured values of $R_4$ and resulting parameter $a$, calculated as $a = 9.8/R_4$. Uncertainty assumes a 1% resistor tolerance.

| $R_4$ (kΩ) | $\sigma_R$ (±kΩ) | $a = 9.8/R_4$ |
|---|---|---|
| 27.4 | 0.27 | 0.358 |
| 47.1 | 0.47 | 0.208 |
| 82.3 | 0.82 | 0.119 |
| 100.0 | 1.00 | 0.098 |
| 150.0 | 1.50 | 0.065 |

Qualitatively, we find that decreasing $R_4$ (thus increasing $a$) makes the attractor more elongated and layered in both the $x$–$y$ and $x$–$z$ planes, as seen in Figure 5. This aligns with the expected role of $a$ in $\dot{y} = x + a\,y$: increasing $a$ strengthens the feedback of $y$ into itself, which amplifies fluctuations and stretches the trajectories in phase space. Conversely, larger $R_4$ values (smaller $a$) lead to more compact, less chaotic loops. These observations are consistent with typical Rössler system behavior, where tuning the linear feedback term can shift the attractor between highly distorted, chaotic regimes and simpler periodic or quasi-periodic motion.

### Effect of Varying $R_9$ (Coefficients $b$ and $c$)

We explored the role of resistor $R_9$ by changing its value while holding $R_{10} = 9.9\,\mathrm{k}\Omega$ constant. Resistor $R_9$ appears in the voltage divider and inverting amplifier that generate the fixed voltage input to the $z$-integrator. This voltage sets both the constant term $b$ and the offset $c$ in the nonlinear term $z(x - c)$ of the Rössler equations. Increasing $R_9$ raises the output voltage of the divider, thereby increasing $c$, while also increasing $b$—but only in theory. In practice, the diode that follows the voltage divider suppresses the actual current delivered to the $b$ path, with measured current dropping from approximately $200\,\mu\mathrm{A}$ before the diode to $20\,\mathrm{nA}$ after. We therefore apply a correction factor of 0.0001 to $b$ to reflect this diode-limited behavior.

The parameters are computed as follows:

$$b \approx \left(-15\,\mathrm{V}\right) \times \frac{R_9}{R_9 + R_{10}} \times \left(-\frac{150\,\mathrm{k}\Omega}{9.9\,\mathrm{k}\Omega}\right) \times 0.0001 \tag{8}$$

$$c \approx \left|-15\,\mathrm{V}\right| \times \frac{R_9}{R_9 + R_{10}} \tag{9}$$

where $R_9$ is the variable resistor and $R_{10} = 9.9\,\mathrm{k}\Omega$ is fixed.

Physically, $b$ serves as a small DC offset (the "push" in $\dot{z} = b + z(x - c)$), while $c$ is subtracted from $x$ before the result is multiplied by $z$. The two parameters shape the dynamics of $\dot{z}$ in complementary ways: $b$ injects energy, and $c$ shifts the effective operating point. However, it is the magnitude of $b$ that plays a dominant role in sustaining chaos. If $b$ becomes too small, $z$ no longer grows and the nonlinear term $z(x-c)$ vanishes, reducing the system to a periodic orbit.

TABLE II. Measured values of $R_9$ (with $R_{10} = 9.9\,\text{k}\Omega$ fixed) and resulting coefficients $b$ and $c$ computed using Eqs. (8), (9).

| $R_9$ (kΩ) | $c$ (V) | $b$ (V) | $R_9/(R_9+9.9)$ | Observed Behavior |
|---|---|---|---|---|
| 38 | 11.89 | 0.0177 | 0.793 | Chaotic, large orbit |
| 81 | 13.37 | 0.0199 | 0.891 | Chaotic |
| 100 | 13.65 | 0.0204 | 0.910 | Mildly chaotic |
| 120 | 13.86 | 0.0209 | 0.924 | Quasi-periodic |
| 480 | 14.70 | 0.0222 | 0.980 | Period-1 orbit |

Qualitatively, the oscilloscope traces reveal a clear trend: as $R_9$ increases, the attractor contracts and becomes increasingly regular. At low $R_9$ (38 kΩ), we see a large, dense, and chaotic orbit. Around 81–100 kΩ, the system is still chaotic but begins to show signs of structure. At 120 kΩ, the motion appears nearly periodic, and by 480 kΩ, the trajectory has collapsed into a clean period-1 orbit.

These results are consistent with theoretical expectations. As $R_9$ increases, both $b$ and $c$ increase, but the diode heavily attenuates $b$. With a smaller effective $b$, the $z$ variable receives less drive, leading to reduced nonlinear coupling via the $z(x-c)$ term. The system thus transitions from complex chaotic behavior to regular, low-energy oscillations—an elegant illustration of how even small constant offsets can dramatically affect the long-term dynamics of a chaotic system [4].

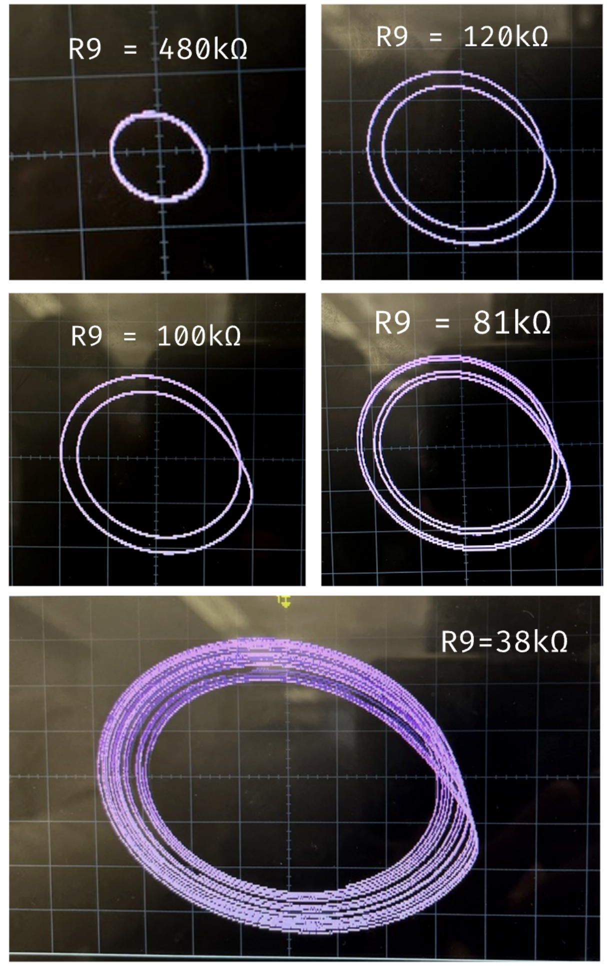


FIG. 6. Oscilloscope traces showing the effect of varying resistor $R_9$ on the system's behavior. As $R_9$ increases from 38 kΩ to 480 kΩ, the chaotic attractor gradually transitions into a stable limit cycle. This is expected since increasing $R_9$ suppresses the effective value of $b$, limiting the energy available to sustain chaotic motion.

### Effect of Varying $R_{12}$ on System Behavior

While the coefficients $b$ and $c$ are primarily set by the voltage divider and inverting amplifier (components $R_9$, $R_{10}$, and $R_{11}$), the resistor $R_{12}$ plays a key role in controlling how much of that voltage actually reaches the $z$-integrator. It appears directly between the output of the inverting amplifier and the summing node of the $z$-stage.

In this configuration, $R_{12}$ acts as a voltage attenuator: the larger its value, the less of the amplified signal makes it into the $z$ equation. Thus, increasing $R_{12}$ decreases both $b$ and $c$ proportionally, while reducing $R_{12}$ increases their contribution to the dynamics.

Figure 7 shows that as $R_{12}$ increases, the attractor contracts and becomes more circular, eventually stabilizing into a periodic orbit. In contrast, smaller values of $R_{12}$ sustain broader, more complex orbits characteristic of chaotic motion. This visual trend supports the interpretation that increasing $R_{12}$ reduces the effective contributions of $b$ and $c$ in the $\dot{z}$ equation, thereby weakening the nonlinear drive that sustains chaos.

This outcome is consistent with expectations: the parameter $b$ acts as the only constant source in the equation $\dot{z} = b + z(x-c)$, sustaining oscillations in $z$ even when $x-c$ is small. If $R_{12}$ is too large, $b$ becomes negligible, $z$ decays, and the nonlinear term vanishes—effectively freezing the chaotic feedback loop and pushing the system toward steady periodic motion.

## EXTENSIONS AND GENERALIZATIONS

Although our analog circuit was designed to implement the Rössler system, it can be adapted to simulate a variety of other chaotic oscillators. A natural extension would be to reconfigure the integrators and feedback net-

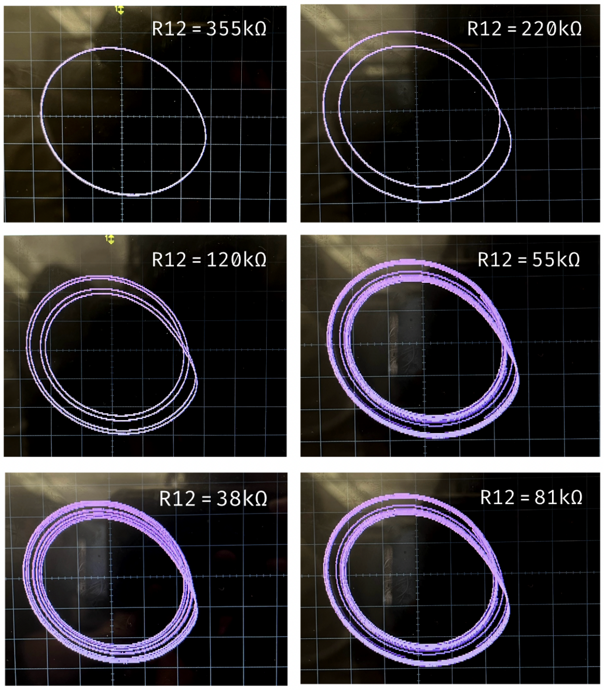


FIG. 7. Oscilloscope traces showing the effect of varying $R_{12}$. Lower resistance values (e.g., $38\,\mathrm{k\Omega}$) allow more of the $b$ and $c$ signal into the $z$-integrator, sustaining chaotic dynamics. As $R_{12}$ increases to $355\,\mathrm{k\Omega}$, the attractor shrinks and the system stabilizes into a simple limit cycle.

works to realize the Lorenz system:

$$\dot{x} = \sigma(y - x), \tag{10}$$

$$\dot{y} = x(\rho - z) - y, \tag{11}$$

$$\dot{z} = xy - \beta z, \tag{12}$$

where $\sigma$, $\rho$, and $\beta$ are constants that give rise to the famous Lorenz attractor. While the same three inverting integrators can be retained, each stage would require updated resistor ratios to implement the new linear terms. Unlike the Rössler case, the Lorenz equations contain a product of two independent signals ($xy$) in $\dot{z}$, so a hardware multiplier or a dedicated four-quadrant multiplier IC (e.g., AD633) must be introduced. This additional component would replace the simplified feedback path used in the Rössler circuit, enabling $\dot{z}$ to depend directly on the product of $x$ and $y$.

Such a modification would be a strong proof-of-concept that simple op-amp integrators can be generalized to a broad range of nonlinear dynamical systems, offering real-time chaos generation and further insights into complex behavior without relying on digital computation.

## CONCLUSION

Through the analog implementation of the Rössler system, we successfully recreated chaotic dynamics in hardware using operational amplifiers, resistors, and capacitors. The design closely followed the structure of the Rössler equations, with each parameter mapped to a physical component. By adjusting key resistors, we observed transitions between chaotic and periodic behavior, consistent with theoretical predictions. The debugging and testing process deepened our understanding of both nonlinear dynamics and analog electronics. This project demonstrates how simple circuit elements can simulate complex mathematical systems in real time.

### Acknowledgements

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