CSC384 Assignment 4

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Question 1

(a) A counter–example model for Φ

Let $\mathcal{M}$ be a structure whose domain contains exactly three elements

$$D = \{a, b, c\}.$$

Interpret the predicate *Above* as the transitive closure of a single stack:

$$Above^{\mathcal{M}} = \{(a, b), (b, c), (a, c)\}.$$

In words, a is above b, b is above c, and by transitivity a is also above c. The first three formulas in Φ only speak about the *Above* predicate:

- $\forall x\, \neg Above(x, x)$ requires *Above* to be irreflexive;
- $\forall x\, \forall y\, \forall z\, ((Above(x, y) \wedge Above(x, z) \wedge y \neq z) \rightarrow (Above(z, y) \vee Above(y, z)))$ says that if x is above two distinct elements then those two elements are comparable;
- $\forall x\, \forall y\, \forall z\, ((Above(x, y) \wedge Above(y, z)) \rightarrow Above(x, z))$ asserts transitivity.

The relation $Above^{\mathcal{M}}$ defined above is irreflexive, linearly orders the elements below each block and is transitive, so $\mathcal{M} \models \Phi$.

The remaining predicates *Under*, *Clear* and *OnTable* do not occur in Φ, so they can be interpreted arbitrarily. To violate the English definition of *Under*, pick

$$Under^{\mathcal{M}} = \{(c, a)\}.$$

According to the informal description, $Under(y, x)$ should hold exactly when x is the unique block immediately above y, or x sits on the table (i.e., has no blocks underneath it) and $y{=}x$. In the present model c is under b and b is under a, so there is no block immediately above c other than b. However $Under^{\mathcal{M}}$ falsely declares c to be under a even though b sits between them. This mis–interpretation of *Under* shows that although $\mathcal{M}$ satisfies Φ it does not reflect the intended English meaning.

(b) Enforcing the intended meanings of *Under*, *Clear*, and *OnTable*

To ensure that all models of the theory match the natural-language definitions of the predicates, augment Φ with sentences capturing those definitions using only the given vocabulary.

The predicate $\mathit{Under}(y, x)$ is intended to hold exactly when either x is the unique block immediately above y, or x has nothing underneath it and $y = x$. This can be enforced with the following two formulas:

$$\forall x \, \forall y \, \big(\mathit{Under}(y, x) \leftrightarrow \mathit{Above}(x, y) \ \wedge \ \neg\exists z \, \big((z \neq x) \wedge \mathit{Above}(z, y) \wedge \neg \mathit{Above}(z, x)\big)\big),$$
$$\forall x \, \big(\mathit{Under}(x, x) \leftrightarrow \mathit{OnTable}(x)\big).$$

The predicate $\mathit{Clear}(x)$ is intended to hold when no block sits above x, and $\mathit{OnTable}(x)$ when x sits on no blocks. These meanings are captured by the bi-implications:

$$\forall x \, \big(\mathit{Clear}(x) \leftrightarrow \neg\exists y \, \mathit{Above}(y, x)\big), \qquad \forall x \, \big(\mathit{OnTable}(x) \leftrightarrow \neg\exists y \, \mathit{Above}(x, y)\big).$$

Adding these sentences to Φ yields a new theory Φ' whose models are precisely those in which *Above* is a finite stacking relation, and *Under*, *Clear*, and *OnTable* behave according to their intended meanings.

Question 2

Let Ψ denote the set of sentences (1)–(5) below, where *between* is a ternary relation. We will argue that Ψ is inconsistent.

$$\forall x \, \forall y \, \forall z \big(\mathit{between}(x, y, z) \to \mathit{between}(z, y, x)\big), \tag{1}$$
$$\forall x \, \forall y \, \forall z \big((\mathit{between}(x, y, z) \wedge \mathit{between}(y, x, z)) \to x = y\big), \tag{2}$$
$$\forall x \, \forall y \, \forall z \, \forall w \big(\mathit{between}(y, x, z) \to (\mathit{between}(y, x, w) \vee \mathit{between}(z, x, w))\big), \tag{3}$$
$$\forall x \, \forall y \, \forall z \big(\mathit{between}(y, x, z) \vee \mathit{between}(z, y, x) \vee \mathit{between}(x, z, y)\big), \tag{4}$$
$$\forall x \, \forall y \, \forall z \big(\mathit{between}(x, y, z) \to \neg \mathit{between}(y, x, z)\big). \tag{5}$$

Unsatisfiability of Ψ

Consider any structure $\mathcal{N}$ with domain D interpreting *between*. Take an arbitrary element $d \in D$ and instantiate $x = y = z = d$ in formula (4). Because all three disjuncts in (4) collapse to the single atom $\mathit{between}(d, d, d)$, the disjunction forces $\mathit{between}(d, d, d)$ to hold. But then formula (5) applied to the same triple says

$$\mathit{between}(d, d, d) \to \neg \mathit{between}(d, d, d),$$

a contradiction. Since d was arbitrary, no interpretation can satisfy all five sentences simultaneously. Therefore Ψ is unsatisfiable.

Two satisfiable modifications

Although Ψ is inconsistent, small changes yield theories that admit natural models.

First modification: drop formula (4)**.** Let Ψ_1 be obtained by omitting (4). Take as the domain a finite totally ordered set, for example $D_1 = \{0, 1, 2\}$ with the usual order $0 < 1 < 2$. Define $between^{\mathcal{N}_1}(x, y, z)$ to mean that y lies strictly between x and z in this order:

$$between^{\mathcal{N}_1}(x, y, z) \quad \text{iff} \quad (x < y < z) \;\vee\; (z < y < x).$$

For triples with repeated elements we take *between* to be false. We now verify that axioms (1), (2), (3) and (5) hold for $\mathcal{N}_1$:

- **Symmetry** (1)**.** If y lies strictly between x and z in the total order, then the same configuration holds when x and z are interchanged, so $between(x, y, z)$ implies $between(z, y, x)$.

- **Anti–symmetry** (5)**.** In our interpretation there is no triple of distinct elements such that one element lies strictly between the other two in both directions. Consequently, if $between(x, y, z)$ holds then $between(y, x, z)$ is false.

- **Uniqueness** (2)**.** This formula asserts that two distinct elements cannot both lie between each other and a common third element. In a linear order, a point cannot simultaneously be between y and z and also have y between it and z, so if both $between(x, y, z)$ and $between(y, x, z)$ held we would have $x = y$.

- **Ray property** (3)**.** If x lies strictly between y and z in the order, then for any element w either w lies to the same side of x as y or to the same side as z. In the first case $between(y, x, w)$ holds; in the second case $between(z, x, w)$ holds.

Since each axiom holds under this interpretation, the modified theory Ψ_1 is satisfiable.

Second modification: restrict formula (5) **to distinct variables.** Define Ψ_2 to be the theory obtained by replacing (5) with

$$\forall x\, \forall y\, \forall z \big((x \neq y \wedge y \neq z) \rightarrow (between(x, y, z) \rightarrow \neg between(y, x, z)) \big). \tag{6}$$

Using the same ordered set $D_2 = \{0, 1, 2\}$, interpret *between* to be true either when y is strictly between x and z or when two of the arguments coincide:

$$between^{\mathcal{N}_2}(x, y, z) \quad \text{iff} \quad (x < y < z) \vee (z < y < x) \vee (y = x) \vee (y = z).$$

With this definition, every ordered triple of distinct elements has exactly one element between the other two, while triples with repeated elements satisfy the disjunction in (4) because $y = x$ or $y = z$.

We now verify that axioms (1)–(4) and (6) hold for $\mathcal{N}_2$:

- **Symmetry** (1)**.** If $between(x, y, z)$ holds, then either y is strictly between x and z in the order or y coincides with x or z. In all of these cases interchanging x and z leaves the relation true, so $between(z, y, x)$ also holds.

- **Anti–symmetry** (6). When $x = y$ or $y = z$ the implication in (6) holds vacuously. Otherwise, suppose $x \neq y$ and $y \neq z$ and $between(x, y, z)$ holds. Then y lies strictly between x and z in the total order, and it is impossible for x to lie strictly between y and z; hence $between(y, x, z)$ is false and the implication holds.

- **Uniqueness** (2). If both $between(x, y, z)$ and $between(y, x, z)$ hold, either $y = x$ or x lies strictly between y and z while also y lies strictly between x and z, which cannot happen in a total order. Thus $x = y$.

- **Ray property** (3). Suppose $between(y, x, z)$ holds; then x lies strictly between y and z. For any element w, either w lies to one side of x with y or lies to the other side with z, so x will be between y and w or between z and w.

- **Coverage** (4). For any x, y, z we must have either $x = y$, $y = z$, or the three elements are distinct. If two are equal, the corresponding disjunct holds because our interpretation makes $between(x, y, z)$ true when $y = x$ or when $y = z$. If all three are distinct, there is exactly one element strictly between the other two, so one of the disjuncts of (4) is satisfied.

Since each axiom holds under this interpretation, the modified theory Ψ_2 is satisfiable.

Question 3

For each sentence below we construct a structure that falsifies that sentence while making the other two sentences true. In all interpretations P is a binary relation and a, b are constants in the domain.

(a) $\forall x \, \forall y \, \forall z \, [P(x, y) \wedge P(y, z) \;\rightarrow\; P(x, z)]$ is false but the other two sentences are true

Let $\mathcal{M}_1$ have domain $D_1 = \{u, v, w\}$, interpret the constants as $a := w$ and $b := w$, and take

$$P^{\mathcal{M}_1} = \{(u, v),\ (v, w)\}.$$

The relation P contains $P(u, v)$ and $P(v, w)$ but not $P(u, w)$, so transitivity fails, making the first sentence false. The second sentence $\forall x \, \forall y[(P(x, y) \wedge P(y, x)) \rightarrow x = y]$ requires antisymmetry. There are no pairs (x, y) and (y, x) with $x \neq y$ in $P^{\mathcal{M}_1}$, hence it holds vacuously. The third sentence

$$\forall x \, \forall y \big(P(a, y) \rightarrow P(x, b) \big)$$

is also satisfied: by choosing a and b both to be w, the antecedent $P(a, y)$ is always false because there are no tuples of the form (w, y) in $P^{\mathcal{M}_1}$, so the implication is true for all x, y.

(b) $\forall x\,\forall y\,[(P(x,y) \wedge P(y,x)) \to x = y]$ is false but the other two sentences are true

Let $D_2 = \{u, v\}$ and interpret the constants as $a := u$ and $b := u$. Set

$$P^{\mathcal{M}_2} = \{(u,u), (u,v), (v,u), (v,v)\},$$

the universal relation on D_2. Because both $P(u,v)$ and $P(v,u)$ hold with $u \neq v$, the antisymmetry sentence is false. To verify that the transitivity sentence holds, note that for every triple $(x,y,z) \in D_2^3$ we have $P(x,y)$ and $P(y,z)$, and we need to check that $P(x,z)$ also holds. We enumerate the eight possible triples and see that the conclusion $P(x,z)$ is indeed in $P^{\mathcal{M}_2}$:

- $(x,y,z) = (u,u,u)$: $P(u,u)$ and $P(u,u)$ hold and $P(u,u)$ holds.
- (u,u,v): $P(u,u)$ and $P(u,v)$ hold and $P(u,v)$ holds.
- (u,v,u): $P(u,v)$ and $P(v,u)$ hold and $P(u,u)$ holds.
- (u,v,v): $P(u,v)$ and $P(v,v)$ hold and $P(u,v)$ holds.
- (v,u,u): $P(v,u)$ and $P(u,u)$ hold and $P(v,u)$ holds.
- (v,u,v): $P(v,u)$ and $P(u,v)$ hold and $P(v,v)$ holds.
- (v,v,u): $P(v,v)$ and $P(v,u)$ hold and $P(v,u)$ holds.
- (v,v,v): $P(v,v)$ and $P(v,v)$ hold and $P(v,v)$ holds.

Thus the implication $P(x,y) \wedge P(y,z) \to P(x,z)$ is satisfied in every case. For the third sentence recall that $a = b = u$. Since P is universal, $P(a,y)$ holds for both $y = u$ and $y = v$, and for every $x \in \{u,v\}$ the relation $P(x,b)$ also holds. For example, when $y = u$ we have $P(a,u)$ and $P(u,b)$ for $x = u$ and $P(v,b)$ for $x = v$; when $y = v$ we have $P(a,v)$ and $P(u,b)$ or $P(v,b)$. In all cases the implication $P(a,y) \to P(x,b)$ holds. Hence $\mathcal{M}_2$ makes the transitivity and third sentences true and the antisymmetry sentence false.

(c) $\forall x\,\forall y\,[P(a,y) \to P(x,b)]$ is false but the other two sentences are true

Take $D_3 = \{0,1,2\}$, interpret $a := 0$ and $b := 2$, and set

$$P^{\mathcal{M}_3} = \{(0,1), (1,2), (0,2)\}.$$

This relation is transitive and antisymmetric. To verify transitivity, notice that the only non-vacuous chain in $P^{\mathcal{M}_3}$ is $P(0,1)$ and $P(1,2)$. In this case $(x,y,z) = (0,1,2)$ satisfies $P(0,1)$ and $P(1,2)$, and the composite $P(0,2)$ also belongs to $P^{\mathcal{M}_3}$. For all other triples $(x,y,z) \in D_3^3$ the antecedent $P(x,y) \wedge P(y,z)$ fails because at least one of (x,y) or (y,z) is not in $P^{\mathcal{M}_3}$, so the implication holds vacuously. It is also antisymmetric because there are no pairs (x,y) and (y,x) with $x \neq y$ in $P^{\mathcal{M}_3}$. However the implication $\forall x\,\forall y\,(P(a,y) \to P(x,b))$ is false. Indeed, $P(a,1)$ holds since $(0,1) \in P^{\mathcal{M}_3}$, but $(2,b)$ is $(2,2)$, which does not belong to $P^{\mathcal{M}_3}$, so the conditional fails when $x = 2$ and $y = 1$. Hence $\mathcal{M}_3$ makes the transitivity and antisymmetry sentences true and the third sentence false.

References

For completeness we recall the standard definitions of the relational properties used above. A binary relation P on a set X is called *antisymmetric* if for all $u, v \in X$, whenever $P(u, v)$ and $P(v, u)$ hold, then $u = v$; equivalently, if $P(u, v)$ with $u \neq v$, then $P(v, u)$ cannot hold. A relation is *transitive* when for all $u, v, w \in X$, $P(u, v)$ and $P(v, w)$ imply $P(u, w)$. These definitions justify our use of transitivity and antisymmetry in the arguments above.