

# Contour Integration Cheat Sheet

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**Definition 1 (Residue)** Assume that  $f$  has an isolated singularity at  $z_0$ . The residue of  $f$  at  $z_0$ , denoted  $\text{Res}[f, z_0]$  is the coefficient  $a_{-1}$  of  $(z - z_0)^{-1}$  in the Laurent expansion of  $f$  at  $z_0$ :

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, 0 < |z - z_0| < \rho$$

**Theorem 1 (The Residue Theorem)** Let  $D$  be a bounded set with a piecewise smooth boundary oriented in the counterclockwise direction. Assume that  $f$  is analytic on  $\text{int } D$  except at singularities  $z_1, z_2, \dots, z_m \in \text{int } D$ . Then

$$\int_{\partial D} f(z) dz = 2\pi i \sum_{j=1}^m \text{Res}[f(z), z_j]$$

**Strategies for finding residues** 2-4 on this list straight from in 7.1 of *Complex Analysis* by Gamelin.

1. Partial Fractions decomposition
2. If  $f(z)$  has a simple pole at  $z_0$ , then

$$\text{Res}[f(z), z_0] = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

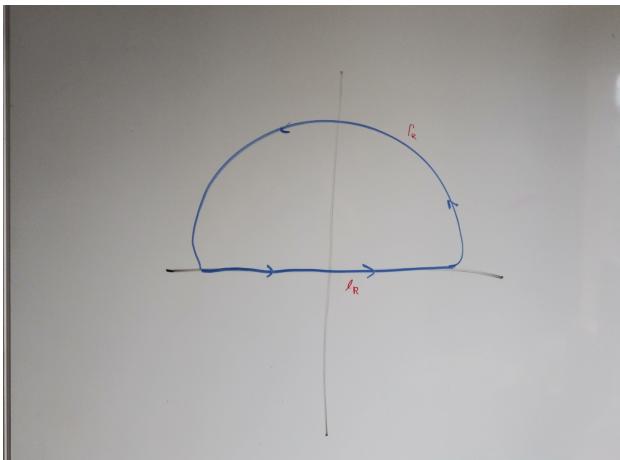
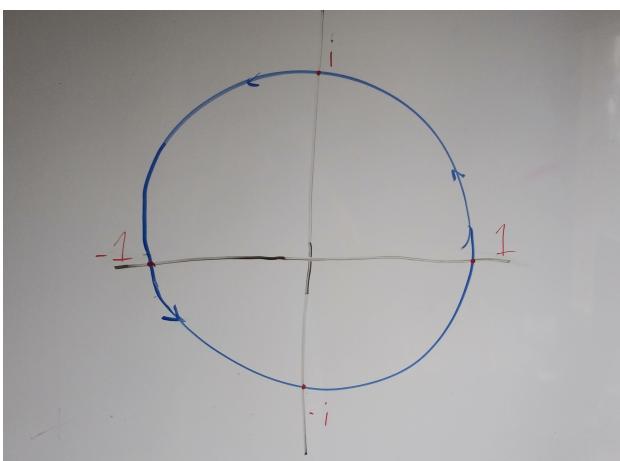
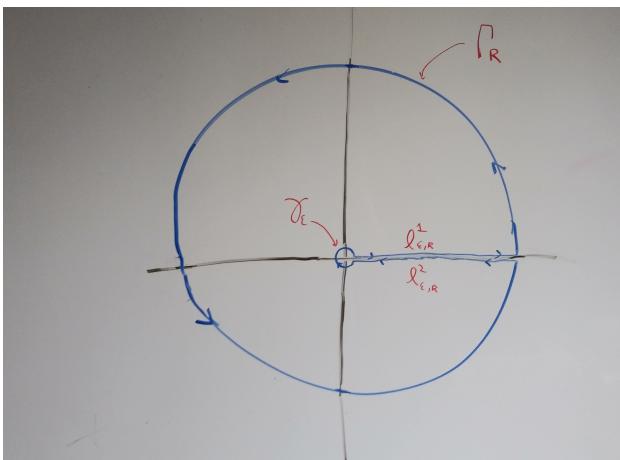
3. If  $f(z)$  has a double pole at  $z_0$ , then

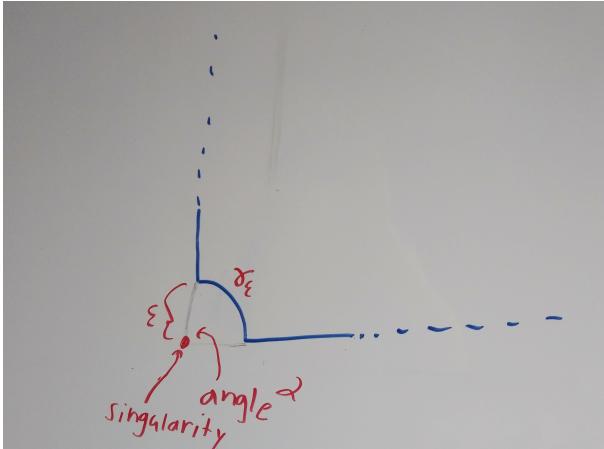
$$\text{Res}[f(z), z_0] = \lim_{z \rightarrow z_0} \frac{d}{dz} ((z - z_0)^2 f(z))$$

4. If  $f, g$  are analytic and  $g$  has a simple zero at  $z_0$ , then

$$\text{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}$$

**Exercise 1** Prove 2-4 above.

Contour	Uses
 <p><b>Semicircle Contour</b> limits: <math>R \rightarrow \infty</math></p>	<ul style="list-style-type: none"> <li>integral from <math>-\infty</math> to <math>\infty</math></li> <li><math>P(x)/Q(x)</math> for <math>Q, P</math> polynomials with <math>\deg Q \geq \deg P + 2</math></li> <li><math>P(x)/Q(x)R(\sin(x), \cos(x))</math> for <math>P, Q, R</math> polynomials with <math>\deg Q \geq \deg P + 1</math> <ul style="list-style-type: none"> <li>If in the previous bullet <math>\deg Q = \deg P + 1</math>, use Jordan's Lemma:</li> </ul> </li> </ul> <p><b>Lemma 1 (Jordan's Lemma)</b> Let <math>\Gamma_R</math> be the semicircle of radius <math>R</math> in the upper half plane. Then</p> $\int_{\Gamma_R}  e^{iz}   dz  < \pi$
 <p><b>Unit Circle Contour</b></p>	<ul style="list-style-type: none"> <li>integral from 0 to <math>2\pi</math></li> <li>for rational functions of <math>\sin \theta, \cos \theta</math></li> <li>Goal: turn into an integral of rational functions around unit circle. Use the substitutions <math>\sin(z) = (e^{iz} - e^{-iz})/2i, \cos(z) = (e^{iz} + e^{-iz})/2</math></li> </ul>
 <p><b>Keyhole Contour</b> limits: <math>\epsilon \rightarrow 0, R \rightarrow \infty</math>, integrating along <math>\ell_{\epsilon,R}^1, \ell_{\epsilon,R}^2</math> is implicitly a limit as well</p>	<ul style="list-style-type: none"> <li>integral from 0 to infinity, denominator a polynomial in <math>P(x)</math></li> <li>for integrand with a branch cut, typically <math>z^a</math> or <math>\log(z)</math></li> <li>the integrand will assume different values on either side of the branch</li> <li>In order for the integrand to have the right form on either side of the branch, irreducible factors of <math>P</math> should be <math>(x-a)</math> for real <math>a</math>. One may first need to perform a change of variable if this is not the case</li> </ul>

Contour	Uses
 <p>small arc angle <math>\alpha</math>, also called a “contour indented at <math>z_0</math>” limits: <math>\epsilon \rightarrow 0</math></p>	<ul style="list-style-type: none"> <li>When integrating through a singularity on the real line and that singularity is a simple pole</li> </ul> <p><b>Theorem 2 (Fractional Residue Theorem)</b>  <i>Let <math>z_0</math> be a simple pole of <math>f</math> and let <math>C_\epsilon</math> be the arc of angle <math>\alpha</math> (in radians) of radius <math>\epsilon</math>. Then</i></p> $\lim_{\epsilon \rightarrow 0} \int_{C_\epsilon} f(z) dz = \alpha i \operatorname{Res}[f(z), z_0]$

References: This cheat-sheet summarizes 7.1-7.5 and 7.7 of *Complex Analysis* by Gamelin. All theorem statements are from this text as well. Look in the textbook for proofs, worked examples, and more information.