Contour Integration Exercises

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Exercises for contours discussed in class: First go through each problem and try to guess the correct contour. Afterwards, show each statement.

1. Show using the residue theorem

$$\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}dx = \frac{\pi}{a}$$

and then compare with using the artangent function.

2. Show using the residue theorem

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx = \frac{\pi}{2a^3}$$

Compare with differentiating the previous problem

3.

$$\int_{0}^{\infty} \frac{x^2}{x^4+1}dx = \frac{\pi}{2\sqrt{2}}$$

4.

$$\int_{0}^{2\pi} \frac{\cos(\theta)}{2+\cos(\theta)} = 2\pi(1-\frac{2}{\sqrt{3}})$$

5.

$$\int_{0}^{\infty} \frac{\log x}{x^a(x+1)}dx = \frac{\pi^2\cos(\pi a)}{\sin^2(\pi a)}dx, \quad 0 < a < 1$$

6.

$$\int_{0}^{\infty} \frac{\log x}{x^3-1}dx = \frac{4\pi^2}{27}$$

7. (2011 January Problem 1)

$$\int_{-\infty}^{\infty} \frac{1}{x^4+3x^2+4}dx = \frac{\pi}{2\sqrt{7}}$$

8.

$$\int_{0}^{\infty} \frac{1}{x^4+1}dx = \frac{\pi}{\sqrt{2}}$$

9.

$$\int_0^{2\pi} \frac{1}{2 + b\sin(\theta)} d\theta = \frac{2\pi}{\sqrt{a^2 - b^2}}, \quad a > b > 0$$

10.

$$\int_0^{\infty} \frac{\log x}{x^3 + 1} = \frac{-2\pi^2}{27}$$

11.

$$\int_{-\infty}^{\infty} \frac{\cos x}{(1 + x^2)^2} = \frac{\pi}{e}$$

12.

$$\int_0^{\infty} \frac{\log(x)}{x^2 - 1} = \frac{\pi^2}{4}$$

Challenge: can you find two different contours one could use for this integral?

13.

$$\int_0^{\infty} \frac{\sin(x)}{x} dx = \pi$$

14.

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2 + 1)} dx = \frac{2}{\pi}$$

15.

$$\int_0^{\infty} \frac{x^3 \sin x}{(x^2 + 1)^2} dx = \frac{\pi}{2e}$$

16.

$$\int_{-\pi}^{\pi} \frac{1}{1 + \sin^2\theta} d\theta = 2\sqrt{\pi}$$

17.

$$\int_0^{\infty} \frac{\log(x)}{x^a(x - 1)} dx = \frac{2\pi^2}{1 - \cos(2\pi a)}, \quad 0 < a < 1$$

18.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos\theta + r^2} d\theta = 1$$

(This is the integral of the Poisson kernel)

Filling in details from Class:

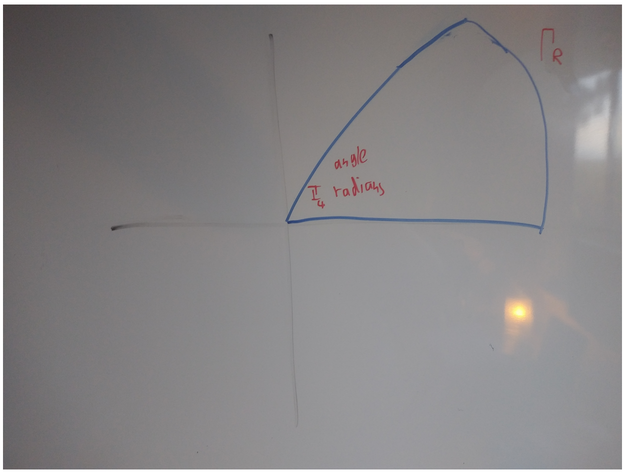
1. Prove Rules 1-3 for finding residues on the Contour integration Cheat sheet.

2. A method of showing
$$\int_0^\infty \frac{1}{1+x^b}dx = \frac{\pi}{b\sin(\frac{\pi}{b})}$$
for $b>1$ was explained in class. Fill in the details

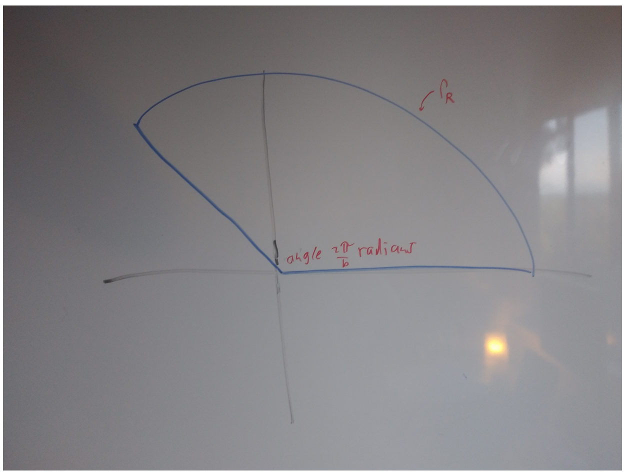
3. Two methods of showing
$$\int_0^\infty \frac{x^a}{1+x^2}dx = \pi\frac{\sin(\frac{\pi}{2}a)}{\sin(\pi a)}$$
were explained in class. Fill in the details.

Instructive Examples with contours not discussed in class

1. Use the following contour to show that
$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$



2. Show that
$$\int_0^\infty \frac{1}{1+x^b}dx = \frac{\pi}{b\sin(\frac{\pi}{b})}$$
for $b>1$ by integrating over the contour



3. (Jan 2008, Problem 4)Show that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
by integrating
$$f(z) = \frac{1}{z^2} \frac{1}{e^{2\pi i z} - 1}$$
about a suitable contour.

Credit: most problems are from [1]

References

[1] Theodore W. Gamelin. *Complex Analysis*. UTM. Springer, 2001.