

Contour Integration Exercises

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Exercises for contours discussed in class: First go through each problem and try to guess the correct contour. Afterwards, show each statement.

1. Show using the residue theorem

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} dx = \frac{\pi}{a}$$

and then compare with using the arctangent function.

2. Show using the residue theorem

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)^2} dx = \frac{\pi}{2a^3}$$

Compare with differentiating the previous problem

- 3.

$$\int_0^{\infty} \frac{x^2}{x^4 + 1} dx = \frac{\pi}{2\sqrt{2}}$$

- 4.

$$\int_0^{2\pi} \frac{\cos(\theta)}{2 + \cos(\theta)} d\theta = 2\pi \left(1 - \frac{2}{\sqrt{3}}\right)$$

- 5.

$$\int_0^{\infty} \frac{\log x}{x^a(x+1)} dx = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)} dx, \quad 0 < a < 1$$

- 6.

$$\int_0^{\infty} \frac{\log x}{x^3 - 1} dx = \frac{4\pi^2}{27}$$

7. (2011 January Problem 1)

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 3x^2 + 4} dx = \frac{\pi}{2\sqrt{7}}$$

- 8.

$$\int_0^{\infty} \frac{1}{x^4 + 1} dx = \frac{\pi}{\sqrt{2}}$$

9.

$$\int_0^{2\pi} \frac{1}{2 + b \sin(\theta)} d\theta = \frac{2\pi}{\sqrt{a^2 - b^2}}, \quad a > b > 0$$

10.

$$\int_0^\infty \frac{\log x}{x^3 + 1} = \frac{-2\pi^2}{27}$$

11.

$$\int_{-\infty}^\infty \frac{\cos x}{(1 + x^2)^2} = \frac{\pi}{e}$$

12.

$$\int_0^\infty \frac{\log(x)}{x^2 - 1} = \frac{\pi^2}{4}$$

Challenge: can you find two different contours one could use for this integral?

13.

$$\int_0^\infty \frac{\sin(x)}{x} dx = \pi$$

14.

$$\int_{-\infty}^\infty \frac{\sin(ax)}{x(x^2 + 1)} dx = \frac{2}{\pi}$$

15.

$$\int_0^\infty \frac{x^3 \sin x}{(x^2 + 1)^2} dx = \frac{\pi}{2e}$$

16.

$$\int_{-\pi}^\pi \frac{1}{1 + \sin^2 \theta} d\theta = 2\sqrt{\pi}$$

17.

$$\int_0^\infty \frac{\log(x)}{x^a(x - 1)} dx = \frac{2\pi^2}{1 - \cos(2\pi a)}, \quad 0 < a < 1$$

18.

$$\frac{1}{2\pi} \int_{-\pi}^\pi \frac{1 - r^2}{1 - 2r \cos \theta + r^2} d\theta = 1$$

(This is the integral of the Poisson kernel)

Filling in details from Class:

1. Prove Rules 1-3 for finding residues on the Contour integration Cheat sheet.

2. A method of showing

$$\int_0^\infty \frac{1}{1+x^b} dx = \frac{\pi}{b \sin(\frac{\pi}{b})}$$

was explained in class. Fill in the details

3. Two methods of showing

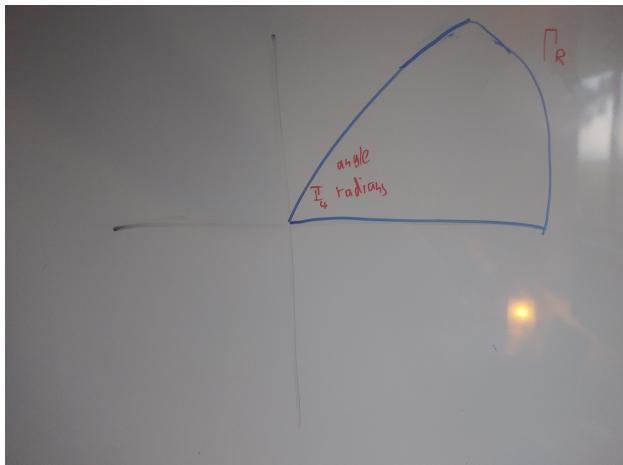
$$\int_0^\infty \frac{x^a}{1+x^2} dx = \pi \frac{\sin(\frac{\pi}{2}a)}{\sin(\pi a)}$$

were explained in class. Fill in the details.

Instructive Examples with contours not discussed in class

1. Use the following contour to show that

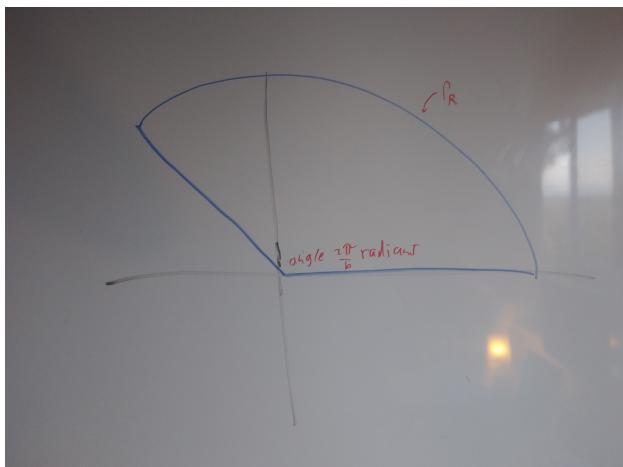
$$\int_0^\infty \sin(x^2) dx = \int_0^\infty \cos(x^2) dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$



2. Show that

$$\int_0^\infty \frac{1}{1+x^b} dx = \frac{\pi}{b \sin(\frac{\pi}{b})}$$

by integrating over the contour



Credit: most problems are from [1]

References

- [1] Theodore W. Gamelin. *Complex Analysis*. UTM. Springer, 2001.