Math 104 Homework 8

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1 Exercises 4.3

4.3.9 Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Proof: Let x be a limit point of K. There exists a subsequence $(x_n) \subseteq K$ such that $(x_n) \to x$. By the continuity of h, $h(x_n) \to h(x)$.

For all $n \in \mathbb{N}$, $h(x_n) = 0$ since each x_n is in K. So their limit must be 0, implying that $h(x) = 0$ and thus x is in K.

4.3.11 (Contraction Mapping Theorem) Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

(a) Show that f is continuous on $\mathbb{R}$.

Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{c}$. Then if $|x - y| < \delta$, it is also true that

$$|f(x) - f(y)| \leq c|x - y| < c \cdot \frac{\epsilon}{c} = \epsilon$$

(b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots)$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence, we may let $y = \lim_{n\to\infty} y_n$.

Given $\epsilon > 0$, we want to find a bound for $|y_n - y_m|$.

$$\begin{aligned}
|y_n - y_m| &= |f(y_{n-1} - f(y_{m-1})| \\
&\leq c|y_{n-1} - y_{m-1}| = |f(y_{n-2} - f(y_{m-2})| \\
&\leq c^2|y_{n-2} - y_{m-3}| = c^2|f(y_{n-3} - f(y_{m-3})| \\
&\vdots \\
&\leq c^{n-1}|y_1 - y_{m-n+1}|
\end{aligned}$$

This can be smaller than ϵ if $|y_1 - y_{m-n+1}|$ can be bounded by M and we let n be such that $c^{n-1} < \frac{\epsilon}{M}$. Indeed,

$$\begin{aligned}
|y_1 - y_{m-n+1}| &= |y_1 - y_2 + y_2 - y_3 + ... + y_{m-n} - y_{m_n+1}| \\
&\leq |y_1 - y_2| + |y_2 - y_3| + ... + |y_{m-n} - y_{m_n+1}| \\
&= |y_1 - y_2| + |f(y_1) - f(y_2)| + ... + |f(y_{m-n-1}) - f(y_{m_n})| \\
&\leq |y_1 - y_2| + c|y_1 - y_2| + ... + |f(y_{m-n-1}) - f(y_{m_n})| \\
&\vdots \\
&\leq |y_1 - y_2| + c|y_1 - y_2| + ... + c^{m-n-1}|y_1 - y_2| \\
&= |y_1 - y_2| \sum_{i=1}^{m-n-1} c^i
\end{aligned}$$

This series is geometric with $0 < c < 1$ so it is convergent and thus bounded.

(c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.

Since f is continuous, $(y_n) \to y$ implies that $f(y_n) \to f(y)$. The second sequence is the same as the first but without the first term, so it converges to the same limit. Thus $f(y) = y$.

To show uniqueness, suppose that another element x has this property. $f(x) = x$

By part (d), the sequence $\{x, f(x), f(f(x)), ...\}$ has the same limit as (y_n). This sequence also equals $\{x, x, x, ...\}$ so we have $(x_n) \to x = y$. So y is the only element with the given property.

(d) Finally, prove that if x is any arbitrary point in $\mathbb{R}$, then the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y defined in (b).

Consider the sequence $(x_n - y_n) = \{x - y, f(x) - f(y), f(f(x)) - f(f(y)), ...\}$. By the Algebraic Limit Theorem,

$$\begin{aligned}
\lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n &= \lim_{n\to\infty} x_n - y_n \\
&\leq \lim_{n\to\infty} |x_n - y_n| \\
&= \lim_{n\to\infty} |f(x_{n-1}) - f(y_{n-1})| \\
&\leq \lim_{n\to\infty} c^{n-1}|x_1 - y_1| = 0
\end{aligned}$$

Thus, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = y$.

2 Exercises 4.4

4.4.6 Give an example of each of the following, or state that such a request is impossible. For any that are impossible, supply a short explanation for why this is the case.

(a) A continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Let $f(x) = \frac{1}{x}$ and let $x_n = \frac{1}{n}$. This sequence is convergent and therefore Cauchy. However, the sequence $f(x_n) = n$ is not.

(b) A uniformly continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

This is impossible. Let $\epsilon > 0$. Since f is uniformly continuous, there also exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x_n) - f(y)| < \epsilon$. Since$(x_n)$ is Cauchy, there exists $N \in \mathbb{N}$ such that $n, m \geq N$ implies that $|x_n - x_m| < \delta$. So if $n, m \geq N$, then

$$|f(x_n) - f(y)| < \epsilon$$

So $(f(x_n))$ must be Cauchy.

(c) A continuous function $f : [0, \infty) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

This is impossible. The set $[0, \infty)$ is closed so it contains its limit points. So the limit of (x_n) (which exists since it is Cauchy) must be in the domain. By the characterization of continuity, $(x_n) \to c$ implies that $f(x_n) \to f(c)$. Since $f(x_n)$ is convergent, it must be Cauchy.

4.4.11 (Topological Characterization of Continuity) Let g be defined on all of $\mathbb{R}$. If B is a subset of $\mathbb{R}$, define the set $g^{-1}(B)$ by

$$g^{-1}(B) = \{x \in \mathbb{R} : g(x) \in B\}$$

Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is an open set.

($\implies$) Let g be continuous and $O \subseteq \mathbb{R}$ be an open set. Consider the set

$$g^{-1}(O) = \{x \in \mathbb{R} : g(x) \in O\}$$

.

Suppose for contradiction that this set is not open. Then for some $x \in g^{-1}(O)$, every neighborhood $V_\epsilon(x)$ is not contained in $g^{-1}(O)$. In particular, every $\epsilon = 1/n$ neighborhood contains some point outside of $g^{-1}(O)$, call this point x_n. The sequence (x_n) converges to x.

By the continuity of g, if (x_n) converges to x, then it should also be true that $g(x_n) \to g(x)$. However, $g(x_n) \notin O$ for all $n \in \mathbb{N}$ by construction, so we can find an element outside of O within any neighborhood of $g(x)$, which contradicts the openness of O.

($\impliedby$) Let $g^{-1}(O)$ be open whenever $O \subseteq \mathbb{R}$ is open.

Consider $V_\epsilon(g(c))$ which is open. The set $g^{-1}(V_\epsilon(g(c))) = \{x \in \mathbb{R} : g(x) \in V_\epsilon(g(c))\}$ is also open. It contains c, so there exists a neighborhood $V_\delta(c)$ such that $V_\delta(c) \subseteq g^{-1}(V_\epsilon(g(c))$. In other words, there exists a δ such that $x \in V_\delta(c)$ implies $g(x) \in V_\epsilon(c)$. This is the definition of continuity. $\square$

3 Exercises 4.5

4.5.2 Provide an example of each of the following, or explain why the request is impossible.

(a) A continuous function defined on an open interval with range equal to a closed interval.

Consider the function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ where $f(x) = \tan(x)$ this function is defined on this open interval and has a range that equals $(-\infty, \infty)$, which is closed.

(b) A continuous function defined on a closed interval with range equal to an open interval.

Consider $f : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ where $f(x) = \tan^{-1}(x)$. This function is continuous on the closed interval (all of $\mathbb{R}$) and has a range which is an open interval.

(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbb{R}$.

Consider $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (0, \infty)$ where

$$f(x) = \begin{cases} -\tan(x) & -\frac{\pi}{2} < x \leq 0 \\ \tan(x) & 0 < x < \frac{\pi}{2} \end{cases}$$

(d) A continuous function defined on all of $\mathbb{R}$ with range equal to $\mathbb{Q}$.

This is not possible. By theorem 4.5.2, if the $f : E \to \mathbb{R}$ is a function with a connected domain E, then $f(E)$ must be connected as well. In this case, since $E = \mathbb{R}$ is connected, the range of f cannot be $\mathbb{Q}$ because it is disconnected.

4.5.8 (Inverse functions) If a function $f : A \to \mathbb{R}$ is one-to-one, then we can define the inverse function f^{-1} on the range of f in the natural way: $f^{-1}(y) = x$ where $y = f(x)$. Show that if f is continuous on an interval $[a, b]$ and one-to-one, then f^{-1} is also continuous.

Suppose for contradiction that $f^{-1} : f([a, b]) \to [a, b]$ is not continuous on $f([a, b])$. Then there exists some $f(x) \in f([a, b])$ such that we can define a sequence $(y_n) \subseteq f([a, b])$ where $(y_n) \to f(x)$ but $f^{-1}(y_n) \nrightarrow f^{-1}(f(x)) = x$.

Uniquely define a sequence in $[a, b]$ by $(x_n) = f^{-1}(y_n)$, so x_n is such that $y_n = f(x_n)$. For every y_n, there is only one such x_n because f is one-to-one. Now we have that

$$(x_n) \nrightarrow x$$

This means that for some $\epsilon > 0$, there are only finite elements of (x_n) the ϵ-neighborhood $(x-\epsilon, x+\epsilon)$ around x. Therefore there are infinite elements in the complement of this neighborhood:

$$[a, x - \epsilon] \cup [x + \epsilon, b]$$

which is closed and bounded. Thus, there exists a subsequence (x_{n_k}) that converges to a limit x' in $[a, x - \epsilon] \cup [x + \epsilon, b]$. Note that $x' \neq x$. However, by the continuity of f,

$$(x_{n_k}) \to x' \implies f(x_{n_k}) \to f(x')$$

But,

$$f(x_{n_k}) = f(f^{-1}(y_{n_k})) = (y_{n_k}) \to f(x)$$

This implies that $x = x'$ because f is one-to-one, which is a contradiction. $\square$