Math 104 Homework 4

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1 Exercises from Abbott 2nd Edition

2.2.8 For some additional practice with nested quantifiers, consider the following invented definition:
Def: Let's call a sequence (x_n) zero-heavy if there exists $M \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.
(a) Is the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ zero-heavy?

Yes, the sequence is zero-heavy. Let $M = 1$. For any value of N, the given statement holds. The set $\{x_n : N \leq n \leq N + M\} = \{x_N, x_{N+1}\}$ contains both 0 and 1. This is because it contains the values of two consecutive elements in the alternating sequence. So there exists such an n and the sequence is zero-heavy.

(b) If a sequence is zero-heavy, does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

Yes, suppose for contradiction that a zero-heavy sequence (x_n) contains a finite number of zeroes with the last zero occurring at $n = N_0$. Then for any value of M, by the definition of zero-heavy, there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$. In particular, there exists n satisfying $N_0 + 1 \leq n$ when $x_n = 0$. That means there is a zero in the sequence after the last zero, a contradiction.

(c) If a sequence contains an infinite number of zeros, is it necessarily zero-heavy? If not, provide a counterexample.

No, consider a sequence where the number of non-zero values grows between each zero

$$\{0, 1, 0, 1, 1, 0, 1, 1, 1, 0, ...\}$$

This sequence has an infinite number of zeros but is not zero-heavy. For any value of M, there exists a stretch of 1's longer than M. Setting N to the index of the first 1 in this stretch, means that there does not exist an n where $N \leq n \leq N + M$ and $x_n = 0$.

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is not zero-heavy if ....

A sequence is not zero-heavy if for all $M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $x_n \neq 0$ for all n satisfying $N \leq n \leq N + M$.

2.3.1 Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Proof: Let $\epsilon > 0$ be arbitrary. We want to find an $N \in \mathbb{N}$ such that for all $n \geq N$, it is true that $|\sqrt{x_n} - \sqrt{x}| \leq \epsilon$. There are two cases.

(Case 1: $x = 0$) Since $(x_n) \to x$, there exists N such that whenever $n \geq N$

$$|x_n - 0| = x_n < \epsilon^2$$

$$\implies \sqrt{x_n} < \epsilon$$

(Case 2: $x \neq 0$) In this case, we can use the fact that $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x}$

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}$$

Since $(x_n) \to x$, there exists N such that whenever $n \geq N$, $|x_n - x| < \epsilon\sqrt{x}$. So

$$|\sqrt{x_n} - \sqrt{x}| \leq \frac{|x_n - x|}{\sqrt{x}} < \frac{\epsilon\sqrt{x}}{\sqrt{x}} = \epsilon$$

2.3.3 (Squeeze Theorem) Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Proof: Suppose for contradiction that $\lim y_n \neq l$. Then there are two cases: $\lim y_n > l$ or $\lim y_n < l$. By the Order Limit Theorem, $x_n \leq y_n \leq z_n$ for all n implies that

$$\lim x_n \leq \lim y_n \leq \lim z_n$$

- First case: $\lim y_n > l$. This is a contradiction to $\lim y_n \leq \lim z_n = l$.
- Second case: $\lim y_n < l$. This is a contradiction to $l = \lim x_n \leq \lim y_n$

2.3.7 Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):
(a) Sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges.

Let $(x_n) = (-1)^n$ and let $(y_n) = (-1)^{n+1}$. Both of these sequences diverge, however their sum $(x_n + y_n) = \{0, 0, 0, ...\}$ converges.

(b) Sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges.

This is not possible, by the Algebraic Limit Theorem. Suppose this were true. Then both (x_n) and $(x_n + y_n)$ converge. So by the ALT,

$$\lim(x_n + y_n - x_n) = \lim(x_n + y_n) - \lim(x_n)$$

But this limit is equal to $\lim y_n$, which diverges. This is a contradiction.

(c) A convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges.

The sequence $(b_n) = \frac{1}{n}$ has $b_n \neq 0$ for all n. However, since it converges to 0, the ALT does not apply and $(1/b_n) = n$ diverges.

(d) An unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded.

This is not possible. If (b_n) is convergent, then it is bounded. There exists M such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Similarly, there exists N such that $|a_n - b_n| \leq N$. Using the triangle inequality we get

$$|a_n| \leq |a_n - b_n + b_n| \leq |a_n - b_n| + |b_n| \leq M + N$$

So (a_n) must be bounded by $M + N$.

(e) Two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Let $(a_n) = 0$ and let $(b_n) = (-1)^n$. In this example, $(a_n) \to 0$ and $(a_n b_n) \to 0$, but (b_n) does not converge.

2.3.11 (Cesaro Means)
(a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

Suppose that (x_n) converges to x. We want to show that given an arbitrary $\epsilon > 0$, the exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|y_n - x| < \epsilon$$

First note that

$$\begin{aligned} |y_n - x| &= \left| \frac{x_1 + x_2 + ... + x_n}{n} - x \right| \\ &\leq \frac{|x_1 - x| + |x_2 - x| + ... + |x_n - x|}{n} \end{aligned}$$

Because (x_n) converges, we know that there exists some N_1 such that whenever $n \geq N_1$,

$$|x_n - x| < \frac{\epsilon}{2}$$

Our equation becomes,

$$\begin{aligned} &= \frac{|x_1 - x|}{n} + ... + \frac{|x_{N_1 - 1} - x|}{n} + \frac{|x_{N_1} - x|}{n} + ... + \frac{|x_n - x|}{n} \\ &< \frac{|x_1 - x|}{n} + ... + \frac{|x_{N_1 - 1} - x|}{n} + \frac{n - (N_1 - 1)}{n} \frac{\epsilon}{2} \end{aligned}$$

Further since (x_n) is bounded, we know that for all n,

$$|x_n|, |x| \leq M$$

And thus, $|x_n - x| \leq |x_n| + |x| \leq 2M$. Now we have

$$\leq \frac{N_1 - 1}{n}2M + \frac{n-(N_1-1)}{n}\frac{\epsilon}{2}$$
$$\leq \frac{N_1 - 1}{n}2M + \frac{\epsilon}{2}$$

The numerator of the first term does not vary with n, so we can choose N_2 so that

$$\frac{(N_1-1)2M}{\frac{\epsilon}{2}} < N_2$$

Now, whenever $n \geq N_2$

$$\frac{(N_1-1)2M}{n} < \frac{\epsilon}{2}$$

Substituting into our inequality, we find

$$|y_n - x| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Consider the sequence given by $x_n = (-1)^n = \{-1, 1, -1, 1, ...\}$. (x_n) clearly does not converge. However, the sequence of averages

$$y_n = \frac{-1 + ... + (-1)^n}{n} = \{-1, 0, -\frac{1}{3}, 0, -\frac{1}{5}, ...\}$$

converges to 0.

2.4.3
(a) Show that the sequence

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.

Proof: This sequence is monotonically increasing and bounded above by 2. We will show this by induction, using the recursive definition for the sequence: $a_n = \sqrt{2 + a_{n-1}}$

First off, $\sqrt{2} < \sqrt{2+\sqrt{2}} < 2$ so we have a base case. Now suppose that $a_{k-1} < a_k < 2$.

$$a_{k+1} = \sqrt{2 + a_k} < \sqrt{2+2} = 2$$

and

$$a_{k-1} < a_k$$
$$\implies 2 + a_{k-1} < 2 + a_k$$
$$\implies \sqrt{2 + a_{k-1}} < \sqrt{2 + a_k}$$
$$\implies a_k < a_{k+1}$$

So for all n, it holds that $a_{n-1} < a_n < 2$. Hence by the Monotone Convergence Theorem, the sequence converges to the supremum of the terms, 2. To confirm that this is indeed the limit, use the fact that $x = \lim a_n = \lim a_{n+1} = \lim \sqrt{2 + a_n}$. By the ALT,

$$\implies x = \sqrt{2+x}$$

$$\implies x^2 - x - 2 = 0$$

$$\implies x = 2, -1$$

The sequence is positive, so $x = 2$ is the only possible solution.

2.4.5 (Calculating Square Roots)
(a) Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

Proof by induction.
Base case: $x_1 = 2$. Clearly $x_1^2 = 4 > 2$. Suppose the statement holds for x_n, ie. $x_n^2 > 2$. Then

$$\begin{aligned}
x_{n+1}^2 &= \frac{1}{2}^2 \left(x_n + \frac{2}{x_n}\right)^2 \\
&= \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right)^2 \\
&> \frac{1}{4}\left(2^2 + 4 + \frac{4}{2^2}\right)^2 \\
&= \frac{9^2}{4} > 2
\end{aligned}$$

So for all x_n, it is true that $x_n^2 > 2$. Now this means that

$$\begin{aligned}
x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\
&= x_n - \frac{x_n}{2} - \frac{1}{x_n} \\
&= \frac{x_n}{2} - \frac{1}{x_n} \\
&= \frac{x_n^2 - 2}{2x_n} \geq 0
\end{aligned}$$

Because this sequence is monotonically decreasing with an infimum of $\sqrt{2}$, we conclude that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

The modified version that converges to $\sqrt{c}$ is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

2.4.7 (Limit Superior) Let (a_n) be a bounded sequence.
(a) Let (a_n) be a bounded sequence. Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

Proof: We will show that the sequence (y_n) is monotonically decreasing and bounded. Consider the two sets $\{a_k : k \geq n\}$ and $\{a_k : k \geq n+1\}$. The latter set is a subset of the former and so its supremum must be less than or equal to the supremum of the containing set. Thus, $y_{n+1} \leq y_n$ and the sequence (y_n) is monotonically decreasing.

Since (a_n) is bounded, we know that the supremums of any subsets are also bounded. So by the Monotone Convergence Theorem, the sequence (y_n) converges.

(b) The limit superior of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

Let $z_n = \inf\{a_k : k \geq n\}$. Then we define $\liminf a_n$ by

$$\liminf a_n = \lim z_n$$

It always exists for any bounded sequence because the sequence (z_n) is monotonically increasing (by similar reasoning to part a) and bounded because (a_n) is bounded.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

Proof: For any set A, it is necessarily true that $\inf A \leq \sup A$. So for all $n \in \mathbb{N}$, $\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$. Thus, by the Order Limit Theorem $\liminf a_n \leq \limsup a_n$.

An example of a sequence for which the inequality is strict is the sequence $\{1, 1, 1, ...\}$.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof:

$(\implies)$ For all $n \in \mathbb{N}$, $z_n \leq a_n \leq y_n$. So by the Squeeze Theorem, if $\lim z_n = \lim y_n = a$, then $\lim a_n = a$.

$(\impliedby)$ For the other direction, suppose that $\lim a_n$ exists and call it a. Let $\epsilon > 0$ be arbitrary. Because the limit exists, there must exist an $N \in \mathbb{N}$ such that for all $n \geq N$

$$|a_n - a| < \epsilon$$

$$a - \epsilon < a_n < a + \epsilon$$

So $a + \epsilon$ is an upper bound for $\{a_k : k \geq N\}$. The supremum of this set, y_N, must be less than or equal to any upper bound by definition. This also must hold for the supremum of any subset, or any y_n where $n \geq N$. So for all such n we have

$$a - \epsilon \leq y_n \leq a + \epsilon$$

$$|y_n - a| \leq \epsilon$$

We have shown that $\lim y_n = \limsup a_n = a$. A nearly identical proof can be used to show that $\liminf a_n = a$ and thus $\limsup a_n = \liminf a_n$ □

2 Other Exercises

Show that $\left(\frac{n^2+1}{2n+1}\right)$ diverges to ∞.

Proof: Let M be an arbitrary positive real number. Then we aim to find $N \in \mathbb{N}$ such that $|a_N| \geq M$. This is how we start our search.

$$\begin{aligned}\left|\frac{n^2+1}{2n+1}\right| &\geq M \\ \frac{n^2}{2n+1} &\geq M \\ n^2 &\geq M(2n+1) \\ n^2 - 2Mn - M &\geq 0\end{aligned}$$

Using the quadratic formula, we get

$$n \geq \frac{2M \pm \sqrt{4M^2+4M}}{2} = M \pm \sqrt{M(M+1)}$$

So let $N \geq M + \sqrt{M(M+1)}$. This gives $|a_N| \geq M$ and we are done. $\square$