Math 104 Homework 5

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1 Exercises 2.5

2.5.1 Give an example of each of the following, or argue that such a request is impossible.
(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

This is impossible by Bolzano Weierstrauss, which states that every bounded sequence contains at least one subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

The sequence given by $\{\frac{1}{2}, 1-\frac{1}{2}, \frac{1}{3}, 1-\frac{1}{3}, \frac{1}{4}, 1-\frac{1}{4}, ...\}$. The subsequence of all odd terms converges to 0, while the subsequence of even terms converges to 1.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{\frac{1}{n}\}_{n=1}^{\infty}$.

The example of such a sequence is the one defined by including terms of the form $\frac{1}{n}$ as well as all previous terms repeated:

$$\{\frac{1}{1}, \frac{1}{2}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}...\}$$

(d) A sequence that contains subsequences converging to every point in the infinite set $\{\frac{1}{n}\}_{n=1}^{\infty}$, and no subsequences converging to points outside of this set.

This is impossible because you will always be able to find a subsequence that converges to zero, which is not in that set.

2.5.5 Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Suppose that (a_n) did not converge to a. Then, for every $\epsilon > 0$, for every $N \in \mathbb{N}$, there exists some $n \geq N$ such that

$$|a_n - a| > \epsilon$$

Let (b_n) be the subsequence of all such (a_n). Infinitely many terms of this sort exist. Since (a_n) is bounded, so must be (b_n). By Bolzano Weierstrauss (b_n) contains a convergent subsequence (c_n). We have reached a contradiction becausw (c_n) is also a subsequence of (a_n), however it does not converge to a. $\square$

2.5.9 Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano–Weierstrass Theorem using the Axiom of Completeness.)

By the definition of $s = \sup S$, for any choice of $\epsilon > 0$, there must exist some $x \in S$ such that

$$s - \epsilon < x < a_n \text{ (for infinitely many terms } a_n\text{)}$$

In particular, this holds for one term (a_n). Rearranging, we get

$$s - a_n < \epsilon$$

$$\implies |a_n - s| < \epsilon$$

So we define our sequence by letting a_{n_k} be the term satisfying the above inequality for $\epsilon = \frac{1}{k}$. For this to be a well defined subsequence, it must hold that $n_k < n_{k+1}$. To ensure that this holds, we select each term to be "further" in the sequence than the previous term. We know that a "further along" term exists because there are infinite choices of a_n by the properties of the elements in S.

2 Exercises 2.6

2.6.2 Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

The sequence $(x_n) = \frac{(-1)^n}{n}$ converges to 0 and is thus Cauchy, however it is not monotone.

(b) A Cauchy sequence with an unbounded subsequence.

This is not possible. If a sequence is Cauchy, it is convergent. This means that every subsequence of it converges to the same limit. Therefore, every subsequence is bounded.

(c) A divergent monotone sequence with a Cauchy subsequence.

This is not possible. The Cauchy/convergent subsequence must be bounded, which implies that the monotone sequence is also bounded. By the MCT, the sequence must also be convergent.

(d) An unbounded sequence containing a subsequence that is Cauchy.

The sequence $\{1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, ...\}$ is unbounded but has a convergent and thus Cauchy subsequence:

$$\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$$

2.6.5 Consider the following (invented) definition: A sequence (s_n) is pseudo-Cauchy if, for all $\varepsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \varepsilon$.

Decide which one of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Pseudo-Cauchy sequences are bounded.

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

The true statement is (ii) and the false statement is (i).
Proof of (ii): For a given $\epsilon > 0$, choose N_1 and N_2 such that for all $n \geq \max\{N_1, N_2\}$

$$|x_{n+1} - x_n| < \frac{\epsilon}{2}$$

$$|y_{n+1} - y_n| < \frac{\epsilon}{2}$$

Then,

$$\begin{aligned} |(x_{n+1} + y_{n+1}) - (x_n + y_n)| &= |(x_{n+1} - x_n) + (+y_{n+1} - y_n)| \\ &\leq |(x_{n+1} - x_n)| + |(+y_{n+1} - y_n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Counterexample for (i): We define the sequence recursively:

$$x_1 = 1$$

$$x_{n+1} = x_n + \frac{1}{n}$$

The "steps" between each term shrink infinitely, so it is definitely pseudo-Cauchy. However, the sequence is not bounded.

2.6.7. Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness and the Monotone Convergence Theorem. They also show the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

(b) Use the Cauchy Criterion to prove the Bolzano–Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of completeness discussed at the end of Section 2.6

Proof: Suppose the sequence (a_n) is bounded by $M \in \mathbb{N}$. We can divide the interval [-M,M] into two halves [-M,0] and [0,M]. At least one of these halves contains an infinite number of elements in the sequence. Name this half I_1 and select an element $a_{n_1} \in I_1$.

Divide I_1 in half and select $a_{n_2} \in I_2$, the half containing infinite elements. Make sure that $n_1 < n_2$. Continue this process infinitely to define a subsequence (a_{n_k}).

Consider the length of each interval:

$$\text{length of } I_k = \frac{M}{2^{k-1}}$$

If $n_k, n_j \geq N$, then they both lie within I_N, so the distance between them is less than the interval length:

$$|a_{n_k} - a_{n_j}| < \frac{M}{2^{N-1}}$$

The right hand side shrinks as N grows so for any $\epsilon > 0$, we can set N such that $\frac{M}{2^{N-1}} < \epsilon$ and thus

$$|a_{n_k} - a_{n_j}| < \epsilon$$

for $n_k, n_j \geq N$. $\square$

We implicitly use the Archimedian Property in the step where we argue that there must exist an $N \in \mathbb{N}$ such that the rational number $\frac{M}{2^{N-1}}$ is less than the positive real number ϵ.