Math 104 Homework 1

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1 Exercises 1.2

1.2.1 (a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?

Proof: Suppose for contradiction that $\sqrt{3}$ is rational. Let p and q be integers such that $\frac{p}{q} = \sqrt{3}$ is in least terms. Thus, p and q have no common factors.

$$\left(\frac{p}{q}\right)^2 = 3 \tag{1}$$

$$p^2 = 3q^2 \tag{2}$$

So 3 divides p^2 and hence divides p because 3 is prime. So $p = 3r$ for some $r \in \mathbb{Z}$.

$$p^2 = (3r)^2 = 9r^2 = 3q^2 \tag{3}$$

$$3r^2 = q^2 \tag{4}$$

Now 3 divides q^2 and hence q. So p and q share a common factor 3, contradicting our initial assumption. $\square$

A similar argument works to show that $\sqrt{6}$ is irrational. However, instead of arguing that $6 \mid p^2$ implies that $6 \mid p$ because it is prime, we use the fact that 2 and 3 must divide p because they are the prime factors of 6. This implies that $6 \mid p$.

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

The proof breaks down in the step where we would say $4 \mid p^2$ implies $4 \mid p$. Because 4 is a perfect square, this statement is not necessarily true. Consider the counterexample where $p = 2$. Here, 4 divides $p^2 = 4$, but not p.

1.2.5 Let A and B be subsets of $\mathbb{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

If x belongs to the complement of the intersection $A \cap B$, then x does not belong to the intersection. So x does not belong to both A and B. This means that x must belong to at least one of A^c or B^c.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

Proof: Let $x \in (A^c \cup B^c)$. Then $x \in A^c$ or $x \in B^c$ or both. This means that $x \notin A$ or $x \notin B$ or x is in neither. So x cannot be in both A and B, the intersection. Written using operators, $x \notin (A \cap B)$. This is equivalent to saying $x \in (A \cap B)^c$. □

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Proof: First we show that $x \in (A \cup B)^c \implies x \in A^c \cap B^c$.

Suppose $x \in (A \cup B)^c$. This means that x is not in $A \cup B$. Therefore, x is not in A and x is not in B. Hence, $x \in A^c$ and $x \in B^c$. This implies $x \in A^c \cap B^c$.

Next, we show that $x \in A^c \cap B^c \implies x \in (A \cup B)^c$.

Suppose $x \in A^c \cap B^c$. This means that x is not in A and x is not in B. Therefore, x is not in $A \cup B$. Hence, $x \in (A \cup B)^c$.

We have shown that each set is a subset of the other and therefore,

$$(A \cup B)^c = A^c \cap B^c$$

□

1.2.7 Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

$$f(A) = [0, 4]$$
$$f(B) = [1, 16]$$
$$f(A \cap B) = f([1, 2]) = [1, 4]$$
$$f(A) \cap f(B) = f([1, 2]) = [0, 4] \cap [1, 16] = [1, 4]$$

$$f(A \cup B) = f([0, 4]) = [0, 16]$$
$$f(A) \cup f(B) = [0, 4] \cup [1, 16] = [0, 16]$$

In this case $f(A \cap B) = f(A) \cap f(B)$ and $f(A \cup B) = f(A) \cup f(B)$.

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

Let $A = [0, 1]$ and $B = \{-1\}$. Then

$$f(A) = [0, 1]$$

$$f(B) = \{1\}$$
$$f(A \cap B) = f(\emptyset) = \emptyset$$
$$f(A) \cap f(B) = [0,1] \cap \{1\} = \{1\}$$

(c) Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

Proof: Let $y \in \mathbb{R}$ such that $y \in g(A \cap B)$. Then, $y = g(x)$ for some $x \in (A \cap B)$. Since x is in the intersection, x belongs to both A and B. Thus, y belongs to both $g(A)$ and $g(B)$, i.e. $y \in g(A) \cap g(B)$. $\square$

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Conjecture: For an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A) \cup g(B) = g(A \cup B)$.

Proof: First we show that $g(A) \cup g(B) \subseteq g(A \cup B)$Let y be such that $y \in g(A) \cup g(B)$. Then, $y = g(x)$ for some $x \in A$ or some $x \in B$. So $x \in A \cup B$ and therefore, $y \in g(A \cup B)$.

Now, to show that $g(A \cup B) \subseteq g(A) \cup g(B)$, consider $y \in g(A \cup B)$. Then, $y = g(x)$ for some $x \in A \cup B$. So x is in A or B. Therefore $y = g(x) \in g(A) \cup g(B)$. $\square$

1.2.12 Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = \frac{2y_n - 6}{3}$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.

Proof: Base case: $n = 1$. The statement is satisfied:

$$y_1 = 6 > -6$$

Inductive hypothesis: Suppose that the statement holds for n. We show that it also holds for $n + 1$,

$$\begin{aligned} y_{n+1} = \frac{2y_n - 6}{3} &> \frac{2(-6) - 6}{3} \\ &= \frac{-18}{3} \\ &= -6 \end{aligned}$$

So, $y_{n+1} > -6$ and we are done. $\square$

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Proof: We want to show that for all $n \in \mathbb{N}$, $y_n > y_{n+1}$.

Base case: $n = 1$. We have $y_1 = 6$

$$y_2 = \frac{2y_1 - 6}{3} = \frac{2(6) - 6}{3} = 2 < 6 = y_1$$

So the statement holds for $n = 1$.

Inductive hypothesis: Suppose that the statement holds for n and $y_n > y_{n+1}$. We show that it also holds for $n+1$,

$$y_{n+1} = \frac{2y_n - 6}{3} > \frac{2y_{n+1} - 6}{3} = y_{n+2}$$

Where the middle inequality arises from the inductive hypothesis. $\square$

1.2.13 (a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

Proof: Base case: $n = 1$. Clearly,

$$(A_1)^c = A_1^c$$

Inductive hypothesis: Suppose that the statement holds for n and use this to show that it holds for $n+1$. Grouping the union of the first n sets into one set and applying the usual De Morgan's Law,

$$((A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1})^c = (A_1 \cup A_2 \cup \cdots \cup A_n)^c \cap A_{n+1}^c$$

By the I.H.,

$$\begin{aligned}(A_1 \cup A_2 \cup \cdots \cup A_n)^c \cap A_{n+1}^c &= (A_1^c \cap A_2^c \cap \cdots \cap A_n^c) \cap A_{n+1}^c \\ &= A_1^c \cap A_2^c \cap \cdots \cap A_n^c \cap A_{n+1}^c\end{aligned}$$

(b) It is tempting to appeal to induction to conclude

$$(\bigcup_{i=1}^{\infty} A_i)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbb{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

An example of such a collection of sets is

$$B_1 = \{1, 2, ...\}$$

$$B_2 = \{2, 3, ...\}$$

$$\vdots$$

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Proof: Suppose that $x \in (\bigcup_{i=1}^{\infty} A_i)^c$. Then $x \notin \bigcup_{i=1}^{\infty} A_i$, which means that for all A_i, $x \notin A_i$. Thus, for all A_i, $x \in A_i^c$, so $x \in \bigcap_{i=1}^{\infty} A_i^c$.

We have shown that $(\bigcup_{i=1}^{\infty} A_i)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c$. To show the other direction, let $x \in \bigcap_{i=1}^{\infty} A_i^c$. Then for all A_i, $x \in A_i^c$, which means that $x \notin A_i$ for all A_i. This implies that $x \notin \bigcup_{i=1}^{\infty} A_i$ or rather $x \in (\bigcup_{i=1}^{\infty} A_i)^c$.

This proves that $\bigcap_{i=1}^{\infty} A_i^c \subseteq (\bigcup_{i=1}^{\infty} A_i)^c$ and completes the proof. $\square$