Math 104 Homework 2

Natalie Brewer

Feb. 1, 2024

1 Exercises 1.3

1.3.2 Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

Solution: This request is impossible. Suppose this statement were true. By the definition of supremum, all elements of B would be less than $\sup B$ and therefore less than $\inf B$, a contradiction to $\inf B$ being a lower bound.

(b) A finite set that contains its infimum but not its supremum.

Solution: This is not possible because the supremum of a finite set is always the maximum of the set, which is contained in it.

(c) A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

Solution: The set $\{x \in \mathbb{Q} : 0 < x \leq 1\}$ contains its supremum, 1, but not its infimum, 0.

1.3.6 Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

Solution: Let x be an arbitrary element of $A+B$. x can be written as a sum $a+b$ for some $a \in A$ and $b \in B$. Because s and t are upper bounds, we know that $a \leq s$ and $b \leq t$. It follows that $a+b \leq s+t$.

(b) Now let u be an arbitrary upper bound for $A+B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

Solution: Since u is an upper bound for $A + B$ and a is fixed, then for all b,

$$\begin{aligned} a + b &\leq u \\ b &\leq u - a \end{aligned}$$

So $u - a$ is an upper bound for B. By the definition of supremum, t must be the smallest upper bound and thus, $t \leq u - a$.

(c) Finally, show $\sup(A+B) = s+t$.

Solution: $t \leq u - a$ implies that $a \leq u - t$. So $u - t$ is an upper bound for A (because a was selected arbitrarily).

And since s is the supremum of A, we know that $s \leq u - t$. Lastly, we rearrange to show that $s + t \leq u$ for any upper bound u of $A + B$. Hence, $s + t$ is the least upper bound.

(d) Construct another proof of this same fact using Lemma 1.3.8.

Lemma 1.3.8. *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\varepsilon > 0$, there exists an element $a \in A$ satisfying $s - \varepsilon < a$.*

Solution: By Lemma 1.3.8, $s = \sup A$ implies that $\forall \epsilon > 0$, there exists an $a \in A$ such that $s - \epsilon < a$. By the fact that $t = \sup B$, we can also find a $b \in B$ such that $t - \epsilon < b$. For a given $\epsilon > 0$ we know that $\frac{\epsilon}{2} > 0$ so for some a and b,

$$s - \frac{\epsilon}{2} > a$$

$$t - \frac{\epsilon}{2} > b$$

Adding these inequalities together gives $s + t - \epsilon > a + b$. $\square$

1.3.11. Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

True. Because $A \subseteq B$ and all elements of A are in B, the supremum for B is an upper bound for A. Then by definition, $\sup A$ can't be greater than $\sup B$ because it wouldn't be the least upper bound.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

True. We can take c to be

$$c = \frac{\sup A + \inf B}{2}$$

Because $\sup A \neq \inf B$, this gives a value for c that is strictly greater than $\sup A$ and strictly less than $\inf B$. Therefore for all $a \in A$ and $b \in B$, we have $a < c < b$.

(c) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

False. A counterexample would be two open intervals $A = (0, 1)$ and $B = (1, 2)$. In this case, $a < 1 < b$ for all $a \in A$ and $b \in B$. However, $\sup A = \inf B$.

2 Exercises 1.4

1.4.5 Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Corollary 1.4.4. Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$.

Solution: We start by considering the real numbers $a-\sqrt{2}$ and $b-\sqrt{2}$. These two numbers are in $\mathbb{R}$ so by Theorem 1.4.3 (Density of $\mathbb{Q}$ in $\mathbb{R}$), there exists a rational number r such that $a-\sqrt{2} < r < b-\sqrt{2}$.

We can add $\sqrt{2}$ on all sides to reach the inequality

$$a < r + \sqrt{2} < b$$

The sum $r + \sqrt{2}$ is irrational by part (b) of Exercise 1.4.1, which states that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$. So, we have found an irrational number $t = r + \sqrt{2}$ satisfying $a < t < b$. $\square$

1.4.8 Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \varnothing$, $\sup A = \sup B$, $\sup A \notin A$, and $\sup B \notin B$.

An example is

$$A = \mathbb{Q} \cap (0, 1)$$

$$B = \mathbb{I} \cap (0, 1)$$

These two sets are disjoint, but share the same supremum, 1. Also, $\sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

An example is the sequence of intervals defined by

$$J_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$$

As n gets bigger, the interval gets smaller and smaller, centered on 0. The only value in the intersection $\bigcap_{n=1}^{\infty} J_n$ in 0. Clearly, $0 \in \bigcap_{n=1}^{\infty} J_n$ because it the center of these symmetric intervals. It is the only element in the intersection because any nonzero element would be excluded from *some* interval.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \varnothing$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$.)

An example is

$$L_n = [n, \infty)$$

The infinite intersection is empty because any potential element $x \in \mathbb{R}$ is excluded from the interval defined by the next largest natural number n.

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \varnothing$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

We can rule out all nested intervals since by the Nested Interval Property, $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$.
As for non-nested intervals,

$$\bigcap_{n=1}^{N} I_n \neq \varnothing \implies \text{for all } i,\ \bigcap_{n=1}^{i+1} I_n \subseteq \bigcap_{n=1}^{i} I_n$$

So the intersections themselves are nested. They are also closed and bounded because they are intersections of closed, bounded sets. By the Nested Interval Property, the intersection of these intersections is non empty.

Now since the intersection of the intersections is equal to the that of the intervals, we can conclude that $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$. Thus, the request is impossible.