Math 104 Homework 6

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1 Exercises 2.7

2.7.2 Decide whether each of the following series converges or diverges:

(a)$\sum_{n=1}^{\infty} \frac{1}{2^n+n}$

Notice that $\frac{1}{2^n+n} < \frac{1}{2^n}$ for all $n \in \mathbb{N}$. We know that the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges because it is a geometric series with $0 < r < 1$. By a comparison test, $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ converges as well.

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

The sin function is bounded by 1 and -1 so $|\frac{\sin(n)}{n^2}| < |\frac{1}{n^2}|$. The series $\sum_{n=1}^{\infty} |\frac{1}{n^2}|$ converges, so by a comparison test, so does $\sum_{n=1}^{\infty} |\frac{\sin(n)}{n^2}|$. Since this series converges absolutely, it must converge regularly.

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$

This series can be written as $\sum_{n=1}^{\infty} \frac{n+1}{2n}$. Notice that $\frac{n+1}{2n} > \frac{n}{2n} = \frac{1}{2}$. By a comparison test, $\sum_{n=1}^{\infty} \frac{1}{2}$ diverges implies that $\sum_{n=1}^{\infty} \frac{n+1}{2n}$ diverges.

(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$

Consider the partial sums of every 3 terms:

$$s_{3m} = \left(1 + \frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{3m-2} + \frac{1}{3m-1} - \frac{1}{3m}\right) = \sum_{n=1}^{3m} \left(\frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n}\right)$$

Notice that $\left(\frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n}\right) > \frac{1}{3n-2} > \frac{1}{3n}$ which diverges. By a comparison test, (s_{3m}) diverges and therefore (s_m) diverges. Then our series diverges as well.

(e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$

This series can be split into an expression involving two series: $\sum_{n=1}^{\infty} \frac{1}{2n-1} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

The first series diverges by a comparison test to $\frac{1}{n}$ and the second series converges since it equals $\frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$. By the Algebraic Limit Theorem for series, the original series must diverge.

2 Exercises 3.2

3.2.5 Prove Theorem 3.2.8.

Thm 2.3.8: A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

Proof:
($\Longrightarrow$) Suppose F is closed and hence contains its limit points.

Let (a_n) be a Cauchy sequence contained in F. Being Cauchy is equivalent to convergence, so (a_n) converges to some limit a. By Theorem 3.2.5, a is a limit point.

By assumption, all limit points of F are contained in F so $a \in F$.

($\Longleftarrow$) Suppose that every Cauchy sequence contained in F has a limit in F. Consider an arbitrary limit point x of F. By Theorem 3.2.5, x must be the limit of some convergent sequence contained in F. All convergent sequences are Cauchy and thus x is the limit of a Cauchy sequence in F. By assumption, $x \in F$ and since x was arbitrary, F is closed. $\square$

3.2.14 A dual notion to the closure of a set is the interior of a set. The interior of E is denoted E° and is defined as

$$E^\circ = \{x \in E : \text{there exists } V_\epsilon(x) \subseteq E\}.$$

Results about closures and interiors possess a useful symmetry.

(a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^\circ = E$.

Proof: E is closed if and only if $\overline{E} = E$
($\Longrightarrow$) Suppose E is closed. Then it contains the set of its limit points, L. Since $L \subseteq E$, then $\overline{E} = L \cup E = E$.

($\Longleftarrow$) Suppose $\overline{E} = E$. Then $L \cup E = E$, which implies that $L \subseteq E$. So E contains all of its limit points and is therefore closed. $\square$

Proof: E is open if and only if $E^\circ = E$
($\Longrightarrow$) Suppose E is open. Then for every $x \in E$, there exists an epsilon neighborhood $V_\epsilon(x) \subseteq E$, which means that every x belongs to E°. Thus, $E^\circ = E$.

($\Longleftarrow$) Suppose that $E^\circ = E$. Then for every $x \in E$. there exists an epsilon neighborhood $V_\epsilon(x) \subseteq E$ because every x belongs to E°. This is the definition of open, so E is open. $\square$

(b) Show that $\overline{E}^c = (E^c)^\circ$, and similarly that $(E^\circ)^c = \overline{E^c}$.

Proof: $\overline{E}^c = (E^c)^\circ$
This can be shown through a series of if and only if statements:

$$\begin{aligned}
x \in \overline{E}^c &\iff x \notin \overline{E} \\
&\iff x \notin E \text{ and } x \text{ is not a limit point of } E \\
&\iff \text{every epsilon neighborhood doesn't interect E, i.e } V_\epsilon(x) \subseteq E^c \\
&\iff x \in (E^c)^\circ
\end{aligned}$$

Proof: $(E^\circ)^c = \overline{E^c}$ Again, we form our proof using several if and only if statements:

$$\begin{aligned}
x \in (E^\circ)^c &\iff x \notin E^\circ \\
&\iff \text{every epsilon nieghborhood is not a subset of } E \\
&\iff \text{for every } V_\epsilon(x) \text{ intersects } E^c \\
&\iff x \text{ is a limit point of } E^c \\
&\iff x \in \overline{E^c}
\end{aligned}$$

$\square$

3 Exercises 3.3

3.3.1 . Show that if K is compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.

Proof: By Theorem 3.3.4 (Characterization of Compactness), K is compact implies that it is also closed and bounded. By the Axiom of Completeness, K being a bounded set of real numbers means that $\sup K$ must exist. By definition of supremum, for every $\epsilon > 0$, there exists some element $x \in K$ such that

$$\sup K - \epsilon < x$$

In particular, we can define a sequence contained in K by letting (x_n) be such an x for when $\epsilon = \frac{1}{n}$. Clearly $(x_n) \to \sup K$.

By the compactness of K, every sequence contains a subsequence that converges to an element in K, but every subsequence of (x_n) must converge to $\sup K$ as well. Thus, $\sup K$ must be in K.

A nearly identical proof can be used to show that $\inf K$ exists and is an element of K. □

3.3.9 Follow these steps to prove the final implication in Theorem 3.3.8.

Assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K.

(a) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim_{n\to\infty} |I_n| = 0$.

Bisect I_0 into two halves. At least one of these halves must intersect K in such a way that it can't be finitely covered. (If a finite subcover existed for both intersections, than we would contradict our assumption that no finite subcover exists.) Call this half I_1

Bisect I_1 and repeat this process to define I_2 and so on. The result is a nested sequence of closed intervals with the desired property and infinitely decreasing length.

(b) Argue that there exists an $x \in K$ such that $x \in I_n$ for all n.

The sequence of intersections given by each interval, $I_0 \cap K, I_1 \cap K, I_2 \cap K, ...$ is itself nested. Further, the intersections are compact sets. This is because K is compact, so every sequence in the intersection contains a subsequence with the correct properties, inherited from K.

Now by the Nested Compact Set Property, the intersection of the intersections, $\bigcap_{n=1}^{\infty}(I_n \cap K)$, is nonempty. This element x belongs to every I_n as well as K.

(c) Because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiction.

Using the fact that $\lim |I_n| = 0$, we can find a very large n such that $I_n \subseteq O_{\lambda_0}$. So I_n is finitely covered by the single open set O_{λ_0}. Therefore, the intersection $I_n \cap K$ is also be finitely covered. This contradicts our assumption that for each intersection, no finite subcover exists. □