Math 104 Homework 9

Natalie Brewer

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1 Exercises 5.2

5.2.2 Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, let's assume the functions are defined on all of $\mathbb{R}$.

(a) Functions f and g not differentiable at zero but where fg is differentiable at zero.

Let $f(x) = |x|$ and let $g(x) = |x|$ as well. These functions are not differentiable at zero. However, $(fg)(x) = f(x)g(x) = x^2$ is.

(b) A function f not differentiable at zero and a function g differentiable at zero where fg is differentiable at zero.

Let $f(x) = |x|$ and let $g(x) = 0$. Then $(fg)(x) = 0$ is differentiable at zero.

(c) A function f not differentiable at zero and a function g differentiable at zero where $f + g$ is differentiable at zero.

This is not possible. By the algebraic differentiability function, if g and $f+g$ are both differentiable at zero then,

$$f'(0) = (f + g - g)'(0) = (f + g)'(0) - g'(0)$$

(d) A function f differentiable at zero but not differentiable at any other point.

Let $f(x) = xt(x)$ where $t(x)$ is Thomae's function.

$$t(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ in lowest terms} \\ 1, & \text{if } x \in \mathbb{I} \end{cases}$$

Then,

$$\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} t(x) = 0$$

However, this function is not differentiable at any other point because it is not continuous at any rational.

5.2.9 Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(a) If f' exists on an interval and is not constant, then f' must take on some irrational values.

True. If f' is not constant than there exists an interval $[a, b]$ within the interval it exists on, such that $f'(a) < f'(b)$ (or $f'(a) > f'(b)$). By the density of irrationals there exists some $\alpha \in \mathbb{I}$ such that $f'(a) < \alpha < f'(b)$ or $f'(a) > \alpha > f'(b)$. Then by Darboux's theorem, there exists some $c \in [a, b]$ such that $f'(c) = \alpha$.

(b) If f' exists on an open interval and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$.

False. Consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x) + x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

This function is differentiable at zero and $f'(0) > 0$ because

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(1/x) + x}{x} = \lim_{x \to 0} (x \sin(1/x) + 1) = 1 > 0$$

However this function is oscillating with increasingly small period as you get closer to zero so there is no neighborhood around zero with the stated property.

(c) If f is differentiable on an interval containing zero and if $\lim_{x \to 0} f'(x) = L$, then it must be that $L = f'(0)$.

True. Let $g(x) = f(x) - f(0)$ and let $h(x) = x$. Then we can apply L'Hospital's rule because $g(0) = h(0) = 0$ and $h'(x) \neq 0$ for all x. This gives

$$L = \lim_{x \to 0} \frac{f'(x)}{1} = \lim_{x \to 0} \frac{g'(x)}{h'(x)} = \lim_{x \to 0} \frac{g(x)}{h(x)} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0)$$

2 Exercises 5.3

5.3.2 Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

Suppose for contradiction that f is not one-to-one. Then there exists some $a \neq b$ such that $f(a) = f(b)$. By Rolle's theorem, there exists some point $c \in (a, b)$ such that $f'(c) = 0$, a contradiction.

To show that the converse statement need not be true, take $f(x) = x^3$ which is one-to-one but has $f'(0) = 0$.

5.3.7 A fixed point of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval and $f'(x) \neq 1$, then f can have at most one fixed point.

Suppose for contradiction that f has more than one fixed point, so there exist $a < b$ such that $f(a) = a$ and $f(b) = b$. By the Mean Value theorem, there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1$$

But this contradicts our assumption that $f'(x) \neq 1$ for all x in the interval.

5.3.8 Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x\to 0} f'(x) = L$, show that $f'(0)$ exists and equals L.

Let $g(x) = f(x) - f(0)$ and let $h(x) = x$. Then we can apply L'Hospital's rule because $g(0) = h(0) = 0$ and $h'(x) \neq 0$ for all x. This gives

$$L = \lim_{x\to 0} \frac{f'(x)}{1} = \lim_{x\to 0} \frac{g'(x)}{h'(x)} = \lim_{x\to 0} \frac{g(x)}{h(x)} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0)$$

3 Extra Credit

5.2.7 Let $g_a(x) = \begin{cases} x^a \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

Find a particular (potentially noninteger) value for a so that:

(a) g_a is differentiable on $\mathbb{R}$ but such that g_a' is unbounded on $[0, 1]$.

The derivative of $g_a(x)$ for $x \neq 0$ is:

$$g_a'(x) = x^a \cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) + ax^{a-1} \sin\left(\frac{1}{x}\right)$$

$$= -x^{a-2} \cos\left(\frac{1}{x}\right) + ax^{a-1} \sin\left(\frac{1}{x}\right)$$

For $g_a(x)$ to be differentiable at 0, we need the following limit to exist.

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x^a \sin\left(\frac{1}{x}\right)}{x} =$$

This means that a must be greater than 1, for $g(0)$ to be differentiable on all of $\mathbb{R}$. So, a must be less than 2 for x^{a-2} to have a negative exponent and for $-x^{a-2} \cos\left(\frac{1}{x}\right)$ to be unbounded.

So let $a = 1.5$.

(b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at zero.

We need $a > 1$ for g_a to be differentiable. From the previous part, we also need $a > 2$ for g_a' to be continuous. Now, if we set a to a value less than 3, we can make the cos term not differentiable at 0, but the sin term differentiable. Let $a = 2.5$.

$-x^{.05} \cos\left(\frac{1}{x}\right)$ is not differentiable. But $2.5x^{1.5} \sin\left(\frac{1}{x}\right)$ is.

(c) g_a is differentiable on $\mathbb{R}$ and g_a' is differentiable on $\mathbb{R}$, but such that g_a'' is not continuous at zero.

Following the same technique, let $a > 3$ so that $-x^{a-2} \cos\left(\frac{1}{x}\right)$ is differentiable. Now consider

$$g_a''(x) = -x^{a-4} \sin\left(\frac{1}{x}\right) - (a-2)x^{a-3} \cos\left(\frac{1}{x}\right) + ax^{a-3} \cos\left(\frac{1}{x}\right) + a(a-1)x^{x-2} \sin\left(\frac{1}{x}\right)$$

For this function to not be continuous at 0, we need $a < 4$ so let $a = 3.5$.