Math 104 Homework 3

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February 8, 2024

1 Exercises 1.5

1.5.4 (a) Show that the open interval (a, b) is equinumerous to the set of real numbers $\mathbb{R}$.

Proof: We will show that $(a, b) \sim (-1, 1)$ and $(-1, 1) \sim \mathbb{R}$ and thus, $(a, b) \sim \mathbb{R}$.
There is a 1-1 correspondence

$$f : (a, b) \to (-1, 1)$$

$$f(x) = -1 + \frac{2}{b-a}(x - a)$$

. This is a bijection between these two intervals. It is one-to-one because it is linear and it is onto because $f(a) = -1$ and $f(b) = 1$ and all the values in between are taken on at some point in the interval.

To show that $(-1, 1) \sim \mathbb{R}$, consider the correspondence

$$g : (-1, 1) \to \mathbb{R}$$

$$g(x) = \tan\left(\frac{\pi}{2}x\right)$$

The image on g over this domain is all of $\mathbb{R}$, so it is onto. It is also strictly increasing, so it is one-to-one. Thus, $(a, b) \sim (-1, 1) \sim \mathbb{R}$. □

(b) Show that the unbounded interval $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.

Proof: Consider the function

$$f : (a, \infty) \to \mathbb{R}$$

$$f(x) = \ln(x - a)$$

f is onto because the range of natural log is $\mathbb{R}$. f is one-to-one because it is strictly increasing. Hence, f is a 1-1 correspondence and (a, ∞) has the same cardinality as $\mathbb{R}$. □

(c) Show that the closed interval $[0, 1)$ is equinumerous to the open interval $(0, 1)$ by exhibiting a one-to-one and onto function between the two sets.

Proof: We can think of this map in a way that is analogous to a map that goes from the set of non-negative integers numbers to the set of natural numbers by shifting each number one slot (adding 1 to each number).

$$\{0, 1, 2, 3, 4, ...\} \to \{1, 2, 3, 4, 5, ...\}$$

Of course for the real numbers, we cannot simply add 1 to each number. Instead, we develop a function that utilizes this "bumping things up a slot" on some of the domain and maps the rest of the domain to itself. The function is defined piece-wise.

$$f : [0, 1) \to (0, 1)$$

$$f(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{1}{3} & x = \frac{1}{2} \\ \frac{1}{4} & x = \frac{1}{3} \\ \vdots & \\ x & \text{else } (x \neq 0 \text{ and } x \text{ is not in the form } \frac{1}{n} \text{ for any } n \in \mathbb{N}) \end{cases}$$

This function maps $\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$ to $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\}$ and all other values to themselves. f is a 1-1 correspondence.

First, we show that f is onto. For every $y \in (0, 1)$, y is either in the form $\frac{1}{n}$ for some $n \in \mathbb{N}$ or it is not. If it is, then $y = f(x)$ for $x = \frac{1}{n-1}$ or $x = 0$. If it is not, then $y = f(y)$. Since every element of $(0, 1)$ has a well-defined pre-image, f is onto.

Next we show that f is one-to-one. Suppose $f(x_1) = f(x_2)$. There are 3 cases:

- If $f(x_1) = f(x_2) = \frac{1}{2}$, then $x_1 = x_2 = 0$
- If $f(x_1) = f(x_2) = \frac{1}{n}$ for some $n \geq 3$, then $x_1 = x_2 = \frac{1}{n-1}$.
- If $f(x_1) = f(x_2)$ are not in that form, then $x_1 = x_2 = f(x_1)$

1.5.6 (a) Give an example of a countable collection of disjoint open intervals.

An example is the collection of open intervals $(0, 1), (1, 2), (2, 3), ..., (n - 1, n)$ for $n \in \mathbb{N}$. This collection is countable because there is a 1-1 correspondence with $\mathbb{N}$.

$$(n - 1, n) \mapsto n$$

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

No such collection of intervals exists. We can show that for any infinite collection of intervals A, there is a 1-1 mapping from A to $\mathbb{Q}$. Consider an interval $I \in A$ such that $I = (a, b)$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number $r \in (a, b)$. In fact there exist infinite rationals, so select any of them. Because the intervals are disjoint, no two intervals can have the same r. Therefore, the mapping that sends each interval in A to its' corresponding element $r \in \mathbb{Q}$ is one-to-one and well-defined. Thus, A has 1-1 correspondence with some subset of $\mathbb{Q}$, a countable set. So A must be countable. $\square$

1.5.8 Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show that B must be finite or countable.

Proof: Suppose B is not finite. Take any $b \in B$. There is a finite number of elements in B that are greater than or equal to b. In fact, there are fewer than $\frac{2}{b}$ such elements. To see this, suppose there were $n \geq \frac{2}{b}$ greater elements.

$$\begin{aligned} b + \{\text{elements greater than or equal to } b\} &\geq b + nb \\ &\geq b + \frac{2}{b}b \\ &= b + 2 > 2 \end{aligned}$$

So we can define a 1-1 correspondence $f : \mathbb{N} \to B$ by lining up the elements of B in descending order. Again, this correspondence is possible because for any $b \in B$ there are only finite elements larger than it. Hence, B is finite or countable. $\square$

2 Exercises 2.2

2.2.1 What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Def: A sequence (x_n) verconges to x if there exists an $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, it is true that $n \geq N$ implies $|x_n - x| < \varepsilon$.

Can you provide an example of a vercongent sequence? Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

An example of a vercongent sequence is $\{1, 1, 1, 1, ...\}$, which verconges to 1 (for instance). Certainly, there exists a value for ϵ such that $|1 - 1| < \epsilon$. In fact, any positive value will do.

There are many vercongent sequences that are divergent. Take $\{1, -1, 1, -1, ...\}$, for example. The sequence verconges to 0 (for instance), but clearly does not converge.

In the previous two examples I used the phrasing "for instance" when describing the value they verconged to, because a sequence that verconges to one value actually verconges to infinite values. For example, the sequence $\{1, 1, 1, 1, ...\}$ also verconges to 2 if $\epsilon > 1$, 3 if $\epsilon > 2$, etc.

This definition is basically describing the notion of being bounded. The part of the definition that places a condition on all $N \in \mathbb{N}$ and $n \geq N$ is actually just saying for all $n \in \mathbb{N}$. If a sequence verconges to 0, then there exists an M, such that $|x_n| < M$ for all $n \in \mathbb{N}$. And if a sequence verconges to 0 if and only if it verconges to any value. So vercongence to any value and boundedness are essentially equivalent.

2.2.2 (a) Verify, using the definition of convergence of a sequence, that the following sequence converges to the proposed limit:

$$\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5}.$$

Proof: Let $\epsilon > 0$. We want to find a value for $N \in \mathbb{N}$ such that for all $n \geq N$

$$\left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| < \epsilon$$

By simplifying this inequality, we find a suitable candidate for N.

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| < \epsilon$$
$$\left|\frac{5(2n+1)-(5n+4)2}{5(5n+4)}\right| < \epsilon$$
$$\left|\frac{-3}{25n+20}\right| < \epsilon$$
$$\frac{3}{25n+20} < \epsilon$$
$$\frac{3}{\epsilon} < 25n+20$$
$$\frac{3}{25\epsilon} - \frac{20}{25} < n$$

Select $N > \frac{3}{25\epsilon} - \frac{20}{25}$. Then for all $n \geq N$

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \left|\frac{3}{25n+20}\right| < \left|\frac{3}{25(\frac{3}{25\epsilon} - \frac{20}{25}) + 20}\right| = \frac{3}{\frac{3}{\epsilon} - 20 + 20} = \epsilon$$

So by the definition of convergence of sequence, $\left(\frac{2n+1}{5n+4}\right) \to \frac{2}{5}$. $\square$

(b) Verify, using the definition of convergence of a sequence, that the following sequence converges to the proposed limit:

$$\lim_{n\to\infty} \frac{2n}{n^3+3} = 0.$$

Proof: Let $\epsilon > 0$. We want to find a value for $N \in \mathbb{N}$ such that for all $n \geq N$

$$\left|\frac{2n}{n^3+3} - 0\right| < \epsilon$$

First we employ the fact that $\left|\frac{2n}{n^3+3}\right| < \left|\frac{2n}{n^3}\right|$. So solving the following inequality is enough to find a good value for N.

$$\left|\frac{2n}{n^3}\right| < \epsilon$$
$$\left|\frac{2}{n^2}\right| < \epsilon$$
$$\left|\frac{2}{\epsilon}\right| < n^2$$
$$\sqrt{\frac{2}{\epsilon}} < n$$

Let $N > \sqrt{\frac{2}{\epsilon}}$. Then for all $n \geq N$

$$\left|\frac{2n}{n^3+3} - 0\right| < \left|\frac{2n}{n^3}\right| = \left|\frac{2}{\left(\sqrt{\frac{2}{\epsilon}}\right)^2}\right| = \epsilon$$

So by the definition of convergence of sequence, $\left(\frac{2n}{n^3+3}\right) \to 0$. $\square$

(c) Verify, using the definition of convergence of a sequence, that the following sequence converges to the proposed limit:

$$\lim_{n\to\infty} \frac{\sin(n^2)}{\sqrt[3]{n}} = 0.$$

Proof: Let $\epsilon > 0$. We want to find a value for $N \in \mathbb{N}$ such that for all $n \geq N$

$$\left|\frac{\sin(n^2)}{\sqrt[3]{n}} - 0\right| < \epsilon$$

Since $\sin(x)$ ranges between -1 and 1, we know that $\left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| \leq \left|\frac{1}{\sqrt[3]{n}}\right|$. Solving the below inequality provides a candidate for N.

$$\begin{aligned}\left|\frac{1}{\sqrt[3]{n}}\right| &< \epsilon\\ \frac{1}{\epsilon} &< \sqrt[3]{n}\\ \left(\frac{1}{\epsilon}\right)^3 &< n\end{aligned}$$

Let $N > \left(\frac{1}{\epsilon}\right)^3$. Then for all $n \geq N$

$$\left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| \leq \left|\frac{1}{\sqrt[3]{n}}\right| < \frac{1}{\sqrt[3]{\left(\frac{1}{\epsilon}\right)^3}} = \epsilon$$

So by the definition of convergence of sequence, $\left(\frac{\sin(n^2)}{\sqrt[3]{n}}\right) \to 0$. $\square$

2.2.4 Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case..

(a) A sequence with an infinite number of ones that does not converge to one.

The sequence $\{1, -1, 1, -1, ...\}$ of alternating 1's and -1's has an infinite number of 1's but does not converge to any number.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

This is not possible. For any epsilon neighborhood not containing 1, there would be an infinite number of elements in the sequence that lie outside of the neighborhood.

(c) A divergent sequence such that for every $n \in \mathbb{N}$, it is possible to find n consecutive ones somewhere in the sequence.

The sequence $\{1, 2, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 2, ...\}$ of increasingly long 1-sequences separated by 2's works. For any $n \in \mathbb{N}$ this sequence contains n consecutive 1's. However, any epsilon neighborhood that doesn't include 1 or doesn't include 2 will fail to contain an infinite number of elements. Thus, the sequence diverges.