Math 104 Homework 7

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1 Exercises 3.2

3.2.6 Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

(a) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.

False. Consider the set $\mathbb{R}\backslash\{\sqrt{2}\}$. This set clearly contains every rational number but it is not the entirety of $\mathbb{R}$. It is also open because $\mathbb{R}\backslash\{\sqrt{2}\} = (-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$, a union of open sets.

(b)The Nested Interval Property remains true if the term "closed interval" is replaced by "closed set."

False. For example, $\bigcap_{n=1}^{\infty}[n, \infty) = \emptyset$ where each set $[n, \infty)$ is closed.

(c) Every nonempty open set contains a rational number.

True. Let $O \subseteq \mathbb{R}$ be an open set. For any $x \in O$, there exists a neighborhood $V_\epsilon(x) \subseteq O$. By the density of $\mathbb{Q}$, this interval contains a rational.

(d) Every bounded infinite closed set contains a rational number.

False. Consider the set $A = \{\sqrt{2}\} \cup \{\frac{1}{n} + \sqrt{2} : n \in \mathbb{N}\}$. A is bounded within the interval $[\sqrt{2}, 1+\sqrt{2}]$ and A is closed because it only has one limit point, $\sqrt{2}$, which it contains. It is also clearly infinite. However, it does not contain any rationals because all elements of the form $\frac{1}{n} + \sqrt{2}$ must be irrational.

(e) The Cantor set is closed.

True. The Cantor set $C = \bigcap_{n=1}^{\infty}$ is an intersection of closed intervals so by our theorem for intersections of closed sets, it is also closed.

2 Exercises 3.3

3.3.5 Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

(a) The arbitrary intersection of compact sets is compact.

True. Let F_λ be a compact set. Then F_λ is closed and bounded. Then the intersection $\bigcap_{\lambda\in\Lambda} F_\lambda$ is also clearly bounded. It is closed by the theorem for arbitrary intersections of closed sets.

(b) The arbitrary union of compact sets is compact.

False. Let $K_n = [0, n]$ be a compact set (closed and bounded). The union $\bigcap_{n=1}^{\infty} K_n = [0, \infty)$ is not compact since it is not bounded.

(c) Let A be arbitrary, and let K be compact. Then, the intersection $A \cap K$ is compact.

False. Let $A = (1, 2)$ and $K = [0, 3]$. Then the intersection $A \cap K = (1, 2) = A$ is not closed and thus, not compact.

(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$

False. Let $F_n = [n, \infty)$ be a closed set. However, the intersection $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

3.3.12 Using the concept of open covers (and explicitly avoiding the Bolzano–Weierstrass Theorem), prove that every bounded infinite set has a limit point.

Let I be a bounded infinite set. There exists a closed interval such that $I \subseteq [-M, M]$.

Suppose I has no limit points. Every $x \in [-M, M]$ has some epsilon-neighborhood $V_\epsilon(x)$ that does not intersect I at any point other than itself.

These neighborhoods form an open cover for $[-M, M]$ and thus for I as well. By Heine-Borel, there exists a finite subcover for $[-M, M]$, $\{V_{\epsilon_1}(x_1), ..., V_{\epsilon_N}(x_N)\}$. This is also an open cover for I.

However, each neighborhood $V_{\epsilon_n}(x_n)$ contains at most one element of I, so this implies that I has at most N elements. This contradicts our assumption that I is infinite.

$\square$

3.3.13 (Extra Credit). Let's call a set *clompact* if it has the property that every closed cover (i.e., a cover consisting of closed sets) admits a finite subcover. Describe all of the clompact subsets of $\mathbb{R}$.

Clompact sets in $\mathbb{R}$ are merely finite sets. For contradiction, suppose that F is a clompact set with infinite elements. Let $\{\{x\} : x \in A\}$ be a closed cover consisting of singleton sets for each element of A. If there were a finite subcover, then this would imply that A is finite, a contradiction.

$\square$

3 Exercises 3.4

3.4.7 A set E is totally disconnected if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A$, $y \in B$, and $E = A \cup B$.

(a) Show that $\mathbb{Q}$ is totally disconnected.

Let $x, y \in \mathbb{Q}$ and $x < y$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists an irrational number, p, between x and y.

$$\exists r \in \mathbb{Q} : \frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}} \implies x < p < y$$

where $p = \sqrt{2}r$.

Let $A = (-\infty, p) \cap \mathbb{Q}$ and $B = (p, \infty) \cap \mathbb{Q}$. Since $p \notin \mathbb{Q}$, it is clear that $\mathbb{Q} = A \cup B$. $\mathbb{Q}$ is totally disconnected because A and B are separated sets:

$$A \cap \bar{B} = ((-\infty, p) \cap \mathbb{Q}) \cap ([p, \infty) \cap \mathbb{Q}) = \emptyset$$
$$\bar{A} \cap B = ((-\infty, p] \cap \mathbb{Q}) \cap ((p, \infty) \cap \mathbb{Q}) = \emptyset$$

(b) Is the set of irrational numbers totally disconnected?

Yes. For any $x, y \in \mathbb{R}\backslash\mathbb{Q}$ where $x < y$, there exists $r \in \mathbb{Q}$ such that $x < r < y$.

Let $A = (-\infty, r) \cap \mathbb{R}\backslash\mathbb{Q}$ and $B = (r, \infty) \cap \mathbb{R}\backslash\mathbb{Q}$. We can see that $\mathbb{R}\backslash\mathbb{Q} = A \cup B$ and that these sets are separated.

$$A \cap \bar{B} = ((-\infty, r) \cap \mathbb{R}\backslash\mathbb{Q}) \cap ([r, \infty) \cap \mathbb{R}\backslash\mathbb{Q}) = \emptyset$$

$$\bar{A} \cap B = ((-\infty, r] \cap \mathbb{R}\backslash\mathbb{Q}) \cap ((r, \infty) \cap \mathbb{R}\backslash\mathbb{Q}) = \emptyset$$

3.4.8 Follow these steps to show that the Cantor set is totally disconnected in the sense described in Exercise 3.4.7.

Let $C = \bigcup_{n=0}^{\infty} C_n$, as defined in Section 3.1.

(a) Given $x, y \in C$, with $x < y$, set $\varepsilon = y - x$. For each $n = 0, 1, 2, \ldots$, the set C_n consists of a finite number of closed intervals. Explain why there must exist an N large enough so that it is impossible for x and y both to belong to the same closed interval of C_N.

The closed intervals of the set C_n each have a length of $(\frac{1}{3})^n$, which converges to 0. So if there exists $N \in \mathbb{N}$ such that

$$\left|(\frac{1}{3})^n - 0\right| < \epsilon = y - x$$

whenever $n \geq N$. This means that you can find an n large enough so that the length of each closed interval is shorter than the distance between x and y, making it impossible for them to belong to the same interval of C_n.

(b) Show that C is totally disconnected.

Given $x, y \in C$, define A as the closed interval of C containing x and B as the set of closed intervals of C excluding the one that contains x. We can do this because the previous part showed that x and y belong to separate intervals. Now $C = A \cup B$ clearly and $x \in A$ and $y \in B$.

We can also see that A and B are separated because $\bar{A} = A$ and $\bar{B} = B$ and $A \cup B = \emptyset$. Thus, C is totally disconnected.

4 Exercises 4.2

4.2.3 Review the definition of Thomae's function $t(x)$ from Section 4.1.

(a) Construct three different sequences (x_n), (y_n), and (z_n), each of which converges to 1 without using the number 1 as a term in the sequence.

$$(x_n) = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\} = \frac{n}{n+1}$$
$$(y_n) = \left\{\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots\right\} = \frac{n+1}{n}$$
$$(z_n) = \left\{\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \ldots\right\} = \frac{n}{n+2}$$

(b) Now, compute $\lim_{n\to\infty} t(x_n)$, $\lim_{n\to\infty} t(y_n)$, and $\lim_{n\to\infty} t(z_n)$.

$$t(x_n) = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\} = \frac{1}{n+1}$$
$$t(y_n) = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} = \frac{1}{n}$$
$$t(z_n) = \left\{\frac{1}{3}, \frac{1}{2}, \frac{1}{5}, \ldots\right\} = \begin{cases} \frac{1}{n+2} & \text{for odd } n \\ \frac{1}{n/2+1} & \text{for even } n \end{cases}$$

The limit for all three sequences is $\lim_{n\to\infty} t(x_n) = \lim_{n\to\infty} t(y_n) = \lim_{n\to\infty} t(z_n) = 0$.

(c) Make an educated conjecture for $\lim_{x\to 1} t(x)$, and use Definition 4.2.1B to verify the claim. Given $\varepsilon > 0$, consider the set of points $\{x \in \mathbb{R} : t(x) \geq \varepsilon\}$. Argue that all the points in this set are isolated.

Conjecture: $\lim_{x\to 1} t(x) = 0$

Given $\epsilon > 0$, we want to find a delta-neighborhood around 1 such that $x \in V_\delta(1)$ implies $t(x) \in V_\epsilon(0)$.

If $\epsilon > 1$, then $\delta = \epsilon$ will work because the neighborhood will include $x = 0$ and $t(0) = 1 < 0 + \epsilon$, all irrationals which trivially satisfy $t(x) = 0 < \epsilon$, and all rationals because $t(\frac{m}{n}) = \frac{1}{n} < \epsilon$.

If $\epsilon < 1$, then there exists $N \in \mathbb{N}$ such that $\frac{1}{N+1} \leq \epsilon < \frac{1}{N}$. We know that $x = 0$ will not be in the neighborhood and that all irrationals satisfy the equation, so we focus on the rationals.

$$\frac{1}{n} < \epsilon$$
$$\frac{1}{n} < \frac{1}{N}$$
$$n > N + 1$$

So we want the denominator of $\frac{m}{n}$ to be greater than $N+1$. Setting $\delta = \frac{1}{N+1}$ will accomplish this. Suppose $|x - 1| < \frac{1}{N+1}$.

$$1 - \frac{1}{N+1} < x < 1 + \frac{1}{N+1}$$
$$\frac{N}{N+1} < x < \frac{N+2}{N+1}$$

So the denominator of x must be greater than $N+1$, and thus $t(x) < \epsilon$.

4.2.8 Compute each limit or state that it does not exist. Use the tools developed in this section to justify each conclusion.

(a) $\lim_{x\to 2} \frac{|x-2|}{x-2}$

Does not exist. Consider the sequences $(x_n) = 2 - \frac{1}{n}$ and $(y_n) = 2 + \frac{1}{n}$. We have $x_n \neq 2$ and $y_n \neq 2$ for all $n \in \mathbb{N}$ and also $\lim x_n = \lim y_n = 2$. However,

$$\lim f(x_n) = \lim \frac{|2 - \frac{1}{n} - 2|}{2 - \frac{1}{n} - 2} = -1$$

$$\lim f(y_n) = \lim \frac{|2 + \frac{1}{n} - 2|}{2 + \frac{1}{n} - 2} = 1$$

which are different so the limit does not exist.

(b) $\lim_{x\to \frac{7}{4}} \frac{|x-2|}{x-2}$

The limit is -1. Let $\epsilon > 0$. We want δ such that $|x - 7/4| < \delta$ implies $|\frac{|x-2|}{x-2} + 1| < \epsilon$. Since $\frac{|x-2|}{x-2}$ either equals 1 or -1, we just want to be close enough to 7/4 so that it is always negative. Therefore setting $\delta = 1/4$ works.

(c) $\lim_{x\to 0}(-1)^{[1/x]}$

Does not exist. Consider the sequences $(x_n) = \frac{1}{2n}$ and $(y_n) = \frac{1}{2n+1}$. Notice that $x_n \neq 0$ and $y_n \neq 0$ and $\lim x_n = \lim y_n = 0$. However,

$$\lim f(x_n) = \lim(-1)^{1/x_n} = \lim(-1)^{2n} = 1$$

$$\lim f(y_n) = \lim(-1)^{1/y_n} = \lim(-1)^{2n+1} = -1$$

which are different so the limit does not exist.

(d) $\lim_{x\to 0} \sqrt[3]{x}(-1)^{[1/x]}$

The limit is 0. Given $\epsilon > 0$, let $\delta = \epsilon^3$. Then if $|x - 0| < \epsilon^3$, it is also true that

$$|\sqrt[3]{x}(-1)^{[1/x]} - 0| < |\sqrt[3]{\epsilon^3}(-1)^{[1/x]}| = |\epsilon(-1)^{[1/x]}| = \epsilon$$

4.2.10 (Right and Left Limits). Introductory calculus courses typically refer to the right-hand limit of a function as the limit obtained by "letting x approach a from the right-hand side."

(a) Give a proper definition in the style of Definition 4.2.1 for the right-hand and left-hand limit statements:

$$\lim_{x\to a^+} f(x) = L \quad \text{and} \quad \lim_{x\to a^-} f(x) = M.$$

Right-hand limit: Let $f : A \to \mathbb{R}$ and let c be a limit point of A. We say that $\lim_{x\to a^+} f(x) = L$ provided that, for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < x - c < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

Left-hand limit: We say that $\lim_{x\to a^-} f(x) = L$ provided that, for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < c - x < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

(b) Prove that $\lim_{x\to a} f(x) = L$ if and only if both the right and left-hand limits equal L.

($\Longrightarrow$) Suppose for contradiction that the right limit did NOT equal L. Then for some $\epsilon > 0$, it must be that for all $\delta > 0$ there exists $x \in A$ such that $0 < c - x < \delta$ but $|f(x) - L| > \epsilon$. This implies that there is no delta such that whenever $|x - c| < \delta$ it is also true that $|f(x) - L| > \epsilon$, meaning the limit is not L, a contradiction.

A similar proof for the case where the left limit is not L, since $0 < x - c < \delta$ also implies $|x - c| < \delta$.

($\Longleftarrow$) The right and left limits both equal L. This means that for any $\epsilon > 0$, there exists $\delta_1 > 0$ such that whenever $0 < x - c < \delta_1$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$. There also exists $\delta_2 > 0$ such that whenever $0 < c - x < \delta_2$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

Let $\delta = \min \delta_1, \delta_2$. Now $|x - c| < \delta$ implies $|f(x) - L| < \epsilon$. This was for an arbitrary ϵ so the limit must be L.

$\square$

4.2.11 (Squeeze Theorem). Let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ at some limit point c of A, show that $\lim_{x\to c} g(x) = L$ as well.

Let $\epsilon > 0$ be arbitrary.We want to find a δ such that $|x - c| < \delta$ implies $|g(x) - L| < \epsilon$.

Since $\lim_{x\to c} f(x) = L$, there exists δ_f such that whenever $|x - c| < \delta_f$

$$|f(x) - L| < \epsilon$$

Since $\lim_{x\to c} h(x) = L$, there exists δ_h such that whenever $|x - c| < \delta_h$

$$|h(x) - L| < \epsilon$$

Take $\delta = \min\{\delta_f, \delta_h\}$. Now whenever $|x - c| < \delta$, it is also true that

$$-\epsilon < f(x) - L < \epsilon$$

$$-\epsilon < h(x) - L < \epsilon$$

Adding L to both sides and using the fact that $f(x) \leq g(x) \leq h(x)$ gives

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon$$

so $|g(x) - L| < \epsilon$.