6A

9.3

Under H_0:

$$X \approx N(50, 25)$$

so $\frac{X-50}{5} \approx N(0,1)$

a) $|X-50| > 10.5$

$$\alpha = P_0(|X-50|>10.5) = 1 - P_0(|X-50| \leq 10.5)$$
$$= 1 - P_0\left(\frac{|X-50|}{5} \leq 2.1\right)$$
$$\approx 1 - \left(\Phi(2.1) - \Phi(-2.1)\right)$$
$$= 1 - (0.964)$$
$$= 0.0357$$

b) Under H_1:

$$X \approx N(100p, 100p(1-p))$$

So $\frac{X-100p}{10\sqrt{p-p^2}} \approx N(0,1)$

$$\text{Power} = 1-\beta = 1 - P_1(|X-50| < 10.5)$$
$$= 1 - P_1(39.5 < X < 60.5)$$
$$= 1 - P\left(\frac{39.5-100p}{10\sqrt{p-p^2}} < \frac{X-100p}{10\sqrt{p-p^2}} < \frac{60.5-100p}{10\sqrt{p-p^2}}\right)$$
$$= 1 - \left(\Phi\left(\frac{6.05-10p}{\sqrt{p-p^2}}\right) - \Phi\left(\frac{3.95-10p}{\sqrt{p-p^2}}\right)\right)$$

6B

9.11

Rejection Region: $(-\infty, z(\frac{\alpha}{2})] \cup [z(\frac{\alpha}{2}), \infty)$

Under H_1

$$\frac{\bar{X}-\mu}{\sigma_{\bar{X}}} \sim N(0,1)$$

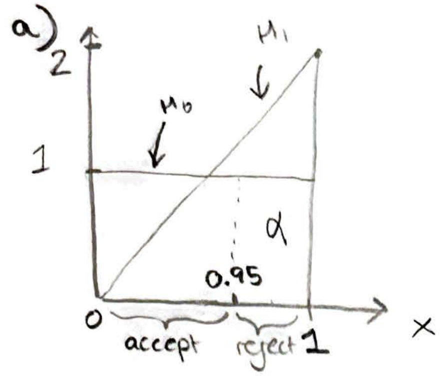
$$\sigma_{\bar{X}} = \frac{\sqrt{100}}{\sqrt{25}} = \frac{10}{5} = 2$$

$$\text{Power} = 1-\beta = 1 - P_1\left(\left|\frac{\bar{X}-\mu_0}{\sigma_{\bar{X}}}\right| < z\left(\frac{\alpha}{2}\right)\right)$$
$$= 1 - P_1\left(z\left(\frac{-\alpha}{2}\right) < \frac{\bar{X}-\mu_0}{2} < z\left(\frac{\alpha}{2}\right)\right)$$
$$= 1 - P_1\left(z\left(\frac{-\alpha}{2}\right) < \frac{\bar{X}-\mu}{2} + \frac{\mu-\mu_0}{2} < z\left(\frac{\alpha}{2}\right)\right)$$
$$= 1 - P_1\left(z\left(\frac{-\alpha}{2}\right) - \frac{\mu-\mu_0}{2} < \frac{\bar{X}-\mu}{2} < z\left(\frac{\alpha}{2}\right) - \frac{\mu-\mu_0}{2}\right)$$
$$= 1 - \left(\Phi\left(z\left(\frac{\alpha}{2}\right) - \frac{\mu-\mu_0}{2}\right) - \Phi\left(z\left(\frac{-\alpha}{2}\right) - \frac{\mu-\mu_0}{2}\right)\right)$$
$$= 1 - \left(\Phi\left(z\left(\frac{\alpha}{2}\right) - \frac{\mu}{2}\right) - \Phi\left(z\left(\frac{-\alpha}{2}\right) - \frac{\mu}{2}\right)\right)$$
$$= 2\Phi\left(z\left(\frac{-\alpha}{2}\right) - \frac{\mu}{2}\right)$$

6C

$H_0: \theta = 1 \rightarrow f(x|H_0) = 1x^{1-1} = 1$
$H_1: \theta = 2 \rightarrow f(x|H_1) = 2x^{2-1} = 2x$

a)



This is simple hypothesis so Neymann-Pearson Lemma applies. MPT is LRT.

Reject H_0 if:

$$\frac{f(x|H_0)}{f(x|H_1)} = \frac{1}{2x} < c \Rightarrow \underbrace{\frac{1}{2c}}_{=K} < x$$

Reject if $x > .95$

$$\alpha = P_0(\text{reject } H_0) = P_0\left(\tfrac{1}{2c} < x\right) = \int_k^1 1\, dx = 0.05$$

$$x \Big|_k^1 = 0.05$$

$$1 - k = 0.05$$

$$k = 0.95$$

b) $\text{Power} = 1 - \beta = P_1(\text{reject } H_0) = P_1(x > .95) = \int_{0.95}^1 2x\, dx$

$$= x^2 \Big|_{0.95}^1$$

$$= 1 - .9025 = .0975$$

c) $\text{p-value} = P_0(X \geq 0.8) = \int_{.8}^1 1\, dx = 0.2$

d) $\overset{\text{Power}}{1-\beta} = P_1(\text{reject } H_0) = P_1(x > .95) = \int_{.095}^1 \theta x^{\theta-1}\, dx = x^\theta \Big|_{.095}^1$

$$= 1 - .095^\theta$$

The power depends on the value of θ so it is not UMP test for $H_1: \theta > 1$.

6D 9.12

$$\text{GLRT}: \quad \Lambda = \frac{\max_{\theta \in \omega_0} \text{lik}(\theta)}{\max_{\theta \in \Omega} \text{lik}(\theta)} \qquad \omega_0 = \{\theta_0\} \quad \Omega = \mathbb{R}$$

$$\frac{\max_{\theta \in \omega_0} \text{lik}(\theta)}{\text{lik}(\hat{\theta}_{ML})} = \frac{\prod_{i=1}^{n} \theta_0 e^{-\theta_0 x_i}}{\prod_{i=1}^{n} \frac{1}{\bar{x}} \cdot e^{-\frac{x_i}{\bar{x}}}} = \frac{\theta_0^n}{\frac{1}{\bar{x}^n}} \prod_{i=1}^{n} e^{-\theta_0 x_i + \frac{x_i}{\bar{x}}}$$

$$= (\theta_0 \bar{x})^n e^{\sum_{i=1}^{n} (-\theta_0 + \frac{1}{\bar{x}}) x_i}$$

$$= (\theta_0 \bar{x})^n e^{(-\theta_0 + \frac{1}{\bar{x}}) \sum_{i=1}^{n} x_i}$$

$$= (\theta_0 \bar{x})^n e^{(-\theta_0 + \frac{1}{\bar{x}})(n\bar{x})}$$

$$= \boxed{(\theta_0 \bar{x})^n e^{-\theta_0 n \bar{x} + n}}$$

$$\text{lik}(\theta) = \prod_{i=1}^{n} \theta \exp(-\theta x_i)$$

$$= \theta^n e^{\sum_i -\theta x_i}$$

$l(\theta) =$

$$\log(\text{lik}\,\theta) = \log \theta^n + \sum_{i=1}^{n} -\theta x_i$$

$$l'(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \theta = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$$

$$\Lambda = (\theta_0 \bar{x})^n e^{(-\theta_0 \bar{x} + 1)n} \leq K$$

$$\Rightarrow \theta_0 \bar{x} \, e^{-\theta_0 \bar{x} + 1} \leq (K)^{1/n}$$

$$\Rightarrow \bar{x} e^{-\theta_0 \bar{x}} \leq \underbrace{\frac{(K)^{1/n}}{\theta_0 e}}_{c}$$

$$\Rightarrow \bar{x} e^{-\theta_0 \bar{x}} \leq c$$

6E 9.20 LRT: Reject if $\frac{1}{2x} < c \Rightarrow \underbrace{\frac{1}{2c}}_{k} < x \Rightarrow k < x$

Power $= P_1(\text{reject } H_0)$

$\alpha = P_0(\text{reject } H_0) = \int_k^1 1\,dx = 0.10$

$1 - k = 0.10$

$k = 0.9$

Power $= P_1(0.9 < x) = \int_{.9}^1 2x\,dx$

$x^2 \Big|_{.9}^1 = 1 - .9^2 = 0.19$

0.19 is the max power, by the Neyman, Pearson Lemma.

6F continued...

c) null distribution: $\binom{n}{x} .5^x .5^{n-x} = \binom{n}{x} .5^n$

$\alpha = P_0(\text{reject } H_0) = P_0\left(\left|X - \frac{n}{2}\right| > k\right)$

$= \sum_{x:\left|x-\frac{n}{2}\right|>k} \binom{n}{x} .5^n$ } This is how we can determine α.

d) $\alpha = \sum_{x:|x-5|>2} \binom{n}{x} .5^n = 1 - \sum_{x=3}^{7} \binom{10}{x} 0.5^{10}$

$= 0.109$ (Done in R)

e) $\alpha = P_0(\text{reject } H_0) = P_0(|X-50| > 10) = 1 - P(|X-50| \leq 10)$

$= 1 - P(40 \leq X \leq 60)$

Under H_0:

$X \sim N(50, 25)$

$\frac{X-50}{5} \sim N(0,1)$

$= 0.046$ (Done in R)

6F

a) $$\Lambda = \frac{\max_{p \in \{.5\}} lik(p)}{\max_{p \in [0,1]} lik(p)} = \frac{\binom{n}{x} 0.5^x \, 0.5^{n-x}}{\binom{n}{x} \left(\frac{x}{n}\right)^x \left(1-\frac{x}{n}\right)^{n-x}}$$

$\hat{p}_{ML}$ which is $\frac{x}{n}$ (Chapter 8)

$$= \frac{n^x \, 0.5^{x+n-x}}{x^x \left(\frac{n-x}{n}\right)^{n-x}}$$

$$= \frac{n^x \, n^{n-x} \, 0.5^n}{x^x \, (n-x)^{n-x}} = \boxed{\frac{(0.5n)^n}{x^x (n-x)^{n-x}}}$$

b) $$\Lambda = \left(\frac{n}{2}\right)^n \cdot \frac{1}{x^x (n-x)^{n-x}} < c$$

$$\Downarrow$$

$$l(\Lambda) = \log(\Lambda) = \log\left(\left(\tfrac{n}{2}\right)^n\right) + \log 1 - \log\left(x^x (n-x)^{n-x}\right) < K$$

$$= \log\left(\tfrac{n}{2}\right)^n - x\log(x) - (n-x)\log(n-x)$$

$$l'(\Lambda) = -\frac{x}{x} - \log(x) + \frac{(n-x)}{n-x} + \log(n-x) = 0$$

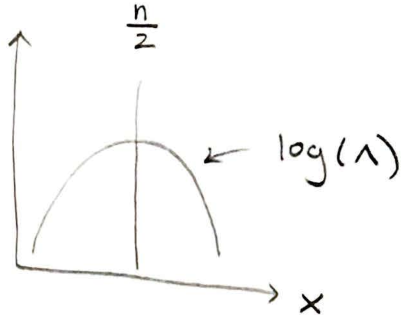
$$-\log(x) + \log(n-x) = 0$$

$$\log\left(\frac{n-x}{x}\right) = 0$$

$$\frac{n-x}{x} = 1$$

$$n-x = x$$

$$x = \frac{n}{2}$$



$$l''(\Lambda) = \frac{-1}{x} - \frac{1}{n-x} < 0$$

The statistic is maximized at $\frac{n}{2}$ and concave down for all x. So as $|X - \frac{n}{2}|$ increases (we move away from $\frac{n}{2}$), we are more likely to reject.

66

a) In R

b) $H_0: \mu = 100$

$H_1: \mu < 100$

test statistic $t = \dfrac{\bar{x} - 100}{\frac{s}{\sqrt{20}}} = \dfrac{89.85 - 100}{\frac{14.904}{\sqrt{20}}} = -3.044$

$t_{19}(0.01) = -2.54 > -3.044$ so we reject the null and say that the smokers' DL is lower than nonsmokers'.

we reject when $P\left(\dfrac{\bar{x} - 100}{\frac{s}{\sqrt{20}}} < c\right) = \alpha = 0.01$

6H

a) $\bar{X} \sim N(\mu_x, \frac{\sigma_x^2}{m})$, $\bar{Y} \sim N(\mu_y, \frac{\sigma_y^2}{n})$

$E(\bar{X}-\bar{Y}) = E(\bar{x}) - E(\bar{Y}) = \mu_x - \mu_y$

$Var(\bar{X}-\bar{Y}) = Var(\bar{X}) + Var(\bar{Y}) = \frac{\sigma_x^2}{m} + \frac{\sigma_y^2}{n}$

So $W \sim N(\mu_x - \mu_y, \frac{\sigma_x^2}{m} + \frac{\sigma_y^2}{n})$

and the pdf is: $f(w) = \dfrac{1}{\left(\frac{\sigma_x^2}{m}+\frac{\sigma_y^2}{n}\right)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{w-(\mu_x-\mu_y)}{\frac{\sigma_x^2}{m}+\frac{\sigma_y^2}{n}}\right)^2}$

If $|W|<K$, accept H_0

$\alpha = P_0(\text{reject } H_0) = 1 - P_0(\text{accept null}) = 1 - P_0(|W|<K)$

$= 1 - P_0(-K < W < K)$

$= 1 - P_0\left(\dfrac{-K}{\sqrt{\frac{\sigma_x^2}{m}+\frac{\sigma_x^2}{n}}} < \dfrac{W}{\sqrt{\frac{\sigma_x^2}{m}+\frac{\sigma_y^2}{n}}} < \dfrac{K}{\sqrt{\frac{\sigma_x^2}{m}+\frac{\sigma_y^2}{n}}}\right)$

$= 2\Phi\left(\dfrac{K}{\sqrt{\frac{\sigma_x^2}{m}+\frac{\sigma_y^2}{n}}}\right) = \alpha$

$\Phi\left(\dfrac{K}{\sqrt{\frac{\sigma_x^2}{m}+\frac{\sigma_y^2}{n}}}\right) = \dfrac{\alpha}{2}$

*Under H_0

$N \sim (0, \frac{\sigma_x^2}{m} + \frac{\sigma_y^2}{n})$

$\Downarrow$

$\dfrac{W-0}{\sqrt{\frac{\sigma_x^2}{m}+\frac{\sigma_y^2}{n}}} \sim N(0,1)$

$\Downarrow$

$z(\frac{\alpha}{2}) = \dfrac{K}{\sqrt{\frac{\sigma_x^2}{m}+\frac{\sigma_y^2}{n}}}$

$K = \dfrac{z(\frac{\alpha}{2})}{\sqrt{\frac{\sigma_x^2}{m}+\frac{\sigma_y^2}{n}}}$