4A

a) Independent observations so $lik(\theta) = \prod_{i=1}^{10} f(X_i | \theta)$

$$lik(\theta) = P(X=0|\theta)^2 \cdot P(X=1|\theta)^3 \cdot P(X=2|\theta)^3 \cdot P(X=3|\theta)^2$$

$$= \left(\frac{2}{3}\theta\right)^2 \left(\frac{1}{3}\theta\right)^3 \left(\frac{2}{3}(1-\theta)\right)^3 \left(\frac{1}{3}(1-\theta)\right)^2$$

$$= \frac{2^5}{3^{10}} \theta^5 (1-\theta)^5$$

$$L(\theta) = \log(lik(\theta)) = \log\left(\frac{2^5}{3^{10}}\right) + 5\log(\theta) + 5\log(1-\theta)$$

b) $L'(\theta) = \frac{5}{\theta} - \frac{5}{1-\theta} = 0 \Rightarrow$

$$\frac{5}{\theta} = \frac{5}{1-\theta}$$

$$5 - 5\theta = 5\theta$$

$$5 = 10\theta$$

$$\Downarrow$$

$$\hat{\theta}_{ML} = \frac{1}{2}$$

c) $lik(\theta) = \left(\frac{2}{3}\theta\right)^{n_0} \left(\frac{1}{3}\theta\right)^{n_1} \left(\frac{2}{3}(1-\theta)\right)^{n_2} \left(\frac{1}{3}(1-\theta)\right)^{n_3} = \frac{2^{n_0+n_2}}{3^n} \theta^{n_0+n_1} (1-\theta)^{n_2+n_3}$

$$L(\theta) = \log\left(\frac{2^{n_0+n_2}}{3^n}\right) + (n_0+n_1)\log(\theta) + (n_2+n_3)\log(1-\theta)$$

$$L'(\theta) = \frac{n_0+n_1}{\theta} - \frac{n_2+n_3}{1-\theta} = 0$$

$$\Downarrow$$

$$(n_0+n_1)(1-\theta) = (n_2+n_3)\theta$$

$$n_0+n_1 - n_0\theta - n_1\theta = n_2\theta + n_3\theta$$

$$(n_0+n_1+n_2+n_3)\theta = n_0+n_1$$

$$\hat{\theta}_{ML} = \frac{n_0+n_1}{n}$$

(4B) $Y \sim Bin(n, P(X=0)+P(X=1))$

a) $P(Y=k) = P(N_0+N_1=k) = \binom{n}{p}\left(\frac{2}{3}\theta + \frac{1}{3}\theta\right)^k \left(\frac{2}{3}(1-\theta) + \frac{1}{3}(1-\theta)\right)^{n-k}$

$$= \binom{n}{p}\theta^k (1-\theta)^{n-k}$$

$$E(\hat{\theta}_{ML}) = E\left(\frac{Y}{n}\right) = \frac{1}{n}E(Y)$$

expectation of binomial is np

$$= \frac{1}{n}(n\theta)$$

$$= \theta$$

b) $Var(\hat{\theta}_{ML}) = Var\left(\frac{Y}{n}\right) = \frac{1}{n^2} Var(Y)$

variance of binomial is $np(1-p)$

$$= \frac{1}{n^2} \cdot n\theta(1-\theta)$$

$$= \frac{\theta}{n}(1-\theta)$$

4C

a) First moment: $\mu_1 = E(X) = 0\left(\frac{2}{3}\theta\right) + 1\left(\frac{1}{3}\theta\right) + 2\left(\frac{2}{3}(1-\theta)\right) + 3\left(\frac{1}{3}(1-\theta)\right)$

$$= \frac{1}{3}\theta + \frac{4}{3} - \frac{4}{3}\theta + \frac{3}{3} - \frac{3}{3}\theta$$

$$= \frac{7}{3} - 2\theta$$

Invert: $2\theta = \frac{7}{3} - \mu_1$

$$\theta = \frac{7}{6} - \frac{\mu_1}{2}$$

Add hats: $\hat{\theta}_{MM} = \frac{7 - 3\bar{X}}{6}$

b) $Var(\hat{\theta}_{MM}) = Var\left(\frac{7}{6} - \frac{\bar{X}}{2}\right) = Var\left(\frac{7}{6}\right) + \frac{1}{4}Var(\bar{X}) = \frac{1}{4}Var(\bar{X})$

$$= \frac{1}{n}\left(\frac{1}{18} + \theta - \theta^2\right)$$

$Var(\bar{X}) = \frac{\sigma^2}{n}$

square root

$$\sigma_{\hat{\theta}_{MM}} = \sqrt{\frac{\frac{1}{18} + \theta - \theta^2}{n}}$$

$\sigma^2 = E(X^2) - E(X)^2$

$$= \left(1\left(\frac{1}{3}\theta\right) + 4\left(\frac{2}{3}(1-\theta)\right) + 9\left(\frac{1}{3}(1-\theta)\right)\right) - \left(\frac{7}{3} - 2\theta\right)^2$$

$$= \left(\frac{\theta}{3} + \frac{8}{3} - \frac{8}{3}\theta + 3 - 3\theta\right) - \left(\frac{7}{3} - 2\theta\right)^2$$

$$= \left(-\frac{7}{3}\theta + \frac{17}{3} - \frac{9}{3}\theta\right) - \left(\frac{7}{3} - 2\theta\right)^2$$

$$= \left(\frac{17}{3} - \frac{16}{3}\theta\right) - \left(\frac{7}{3} - 2\theta\right)^2$$

$$= \left(\frac{17}{3} - \frac{16}{3}\theta\right) - \left(\frac{49}{9} + 4\theta^2 - \frac{28\theta}{3}\right)$$

$$= \frac{2}{9} + 4\theta - 4\theta^2$$

c) $Var(\hat{\theta}_{MM}) = \frac{\frac{1}{18} + \theta - \theta^2}{n}$

$Var(\hat{\theta}_{ML}) = \frac{\theta}{n}(1-\theta) = \frac{\theta}{n} - \frac{\theta^2}{n}$ $\rightarrow$

$Var(\hat{\theta}_{MM}) > Var(\hat{\theta}_{ML})$

$$\frac{1}{18n} + \frac{\theta}{n} - \frac{\theta^2}{n} > \frac{\theta}{n} - \frac{\theta^2}{n}$$

For all values of θ, the standard error of $\hat{\theta}_{MM}$ is slightly greater than that of $\hat{\theta}_{ML}$

d) $\hat{\theta}_{ML} = \frac{n_0 + n_1}{n} = \frac{2+3}{10} = \frac{5}{10} = \frac{1}{2}$

$\sigma_{\hat{\theta}_{ML}} = \frac{\theta}{n} - \frac{\theta^2}{n} \approx \frac{\hat{\theta}_{ML}}{n} - \frac{\hat{\theta}_{ML}^2}{n} = \frac{1}{10}\left(\frac{1}{2}\right) - \frac{1}{10}\left(\frac{1}{4}\right) = \frac{1}{40}$

$\hat{\theta}_{MM} = \frac{7}{6} - \frac{\overline{x}}{2} = \frac{7}{6} - \frac{1.5}{2} = \frac{7-4.5}{6} = \frac{5}{12}$

$\overline{x} = \frac{3+2+1+3+2+1+2+1}{10} = \frac{15}{10} = 1.5$

$\sigma_{\hat{\theta}_{MM}} = \frac{1}{18n} + \frac{\theta}{n} - \frac{\theta^2}{n} \approx \frac{1}{18n} + \frac{\hat{\theta}_{MM}}{n} - \frac{\hat{\theta}_{MM}^2}{n} = \frac{1}{180} + \frac{5}{120} - \frac{25}{1440}$

$= 0.04375$

4D 6.10.23

Method of Moments:

$\mu_1 = E(X) = \frac{1}{N}\sum_{i=1}^{N} i = \frac{1}{N} \cdot \frac{N(N+1)}{2} = \frac{N+1}{2}$

invert: $2\mu_1 = N+1$

$N = 2\mu_1 - 1$

add hats: $\hat{N}_{MM} = 2\overline{X} - 1$

For our sample of 1: $\overline{X} = 888$

$\hat{N}_{MM} = 2(888) - 1 = 1775$

MLE: $L(N) = \begin{cases} \frac{1}{N} & \text{if } x \leq N \\ 0 & \text{else} \end{cases}$

where x is the observation.

L(N) is maximized when N is as small as possible ($\frac{1}{N}$ is strictly decreasing) AND $x \leq N$ still.

$\Rightarrow \hat{N}_{ML} = x$

$\hat{N}_{ML} = 888$