

Abstract

Detecting latent signal in high dimensions: estimation in Gaussian mixtures and nonparametric models

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Detecting hidden signal lying in high-dimensional data is a central objective in statistics. In this thesis, we explore algorithms and fundamental limits for estimating signal in target datasets. This work consists of two main parts, and we now summarize each.

In the first part (Chapters 2-5), we provide theoretical analyses of three types of Gaussian mixture models in high dimensions. Throughout this part of the thesis, we observe signal-plus-noise data where the noise is Gaussian in \mathbb{R}^d and the signal is distributed according to a mixing distribution parametrized by a collection of low-dimensional subspaces of \mathbb{R}^d . That is, the mixture signal we wish to learn about has been corrupted by high-dimensional noise. We study clustering, parameter estimation, and density estimation under various assumptions on the mixing distribution. The results in these sections are largely theoretical; we study statistical convergence rates, as well as fast algorithms for estimation.

In the second part, (Chapters 6-7), we study two tasks: latent signal detection when the data are a linearly transformed graphical model, and variational posterior inference when we wish to allow the posterior to have a flexible form. Both tasks require the estimation of a nonparametric density in high dimensions, and for this we turn to an algorithm called score matching that allows for computationally feasible estimation of a flexible density in high dimensions.

Detecting latent signal in high dimensions: estimation in Gaussian mixtures and nonparametric models

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Chapter 1

Introduction

In this introductory section, we provide an overview of each chapter and its main assumptions. We then provide notation that will be used throughout the thesis. We introduce themes that are common to Part 1 of the thesis in Section 1.2 and discuss the algorithm that is common to Part 2 of the thesis in Section 1.3.

2. We explore fundamental limits of estimation in a high-dimensional Gaussian mixture on k means when no lower bound assumptions are placed on the separation or collinearity of the means or weights of the mixing distribution. In this context, clustering is impossible, but mixing distribution and density estimation are possible, and we provide the optimal statistical convergence rates for both tasks, as well as a polynomial time algorithm for mixing distribution estimation. We also provide an extension in the direction of fast algorithms for density estimation
3. We begin the exploration of fundamental limits of estimation in a high-dimensional Gaussian mixture on k subspaces, with no lower bounds on collinearity of the subspaces or on the weights of the mixing distribution. This work is preliminary and is included because it ties closely to the k -means model.

4. We study fundamental limits of clustering in high-dimensional Gaussian mixtures on k manifolds, where the manifolds are assumed to be well separated. We explore lower bounds on the amount of separation that is necessary for optimal error rates to be achievable. In particular, we explore the performance of spectral clustering and its limitations on signal-plus-noise data, and we provide novel algorithms for the refinement of spectral clustering.
5. We study the maximum likelihood estimator (MLE) in high-dimensional signal-plus-noise models when the noise is Gaussian and the distribution on the signal is parametrized by a low-dimensional subspace. This includes the Gaussian mixture models discussed above and extends more broadly to any signal-plus-noise model where the signal follows a low-dimensional parametric distribution and the noise is high-dimensional Gaussian. We show how a naïve use of the classical bracketing entropy technology does not prove the MLE achieves the sharp convergence rate, and we explain how to properly use this technology to prove that the MLE can achieve the correct parametric rate. We demonstrate the technique on the Gaussian k -means model discussed in Chapter 2.
6. We introduce *graphical component analysis*, a technique for estimating latent signal where linearly transformed signal components are dependent with dependence that is modeled by a graphical model. We couple score matching with orthogonal matrix optimization to perform a novel type of high-dimensional density estimation.
7. We propose *nonparametric variational inference via score matching*, a novel type of variational inference that uses a kernel-density based form of score matching to allow for the estimation of a flexible, generative form for the posterior. Our proposed method allows for nonparametric posterior inference while avoiding the optimization of a dual representation of an objective that is so prevalent in

the nonparametric density and posterior estimation literature.

1.1 Notation

Let $[n] \triangleq \{1, \dots, n\}$. Let S^{d-1} and Δ^{d-1} denote the unit sphere and the probability simplex in \mathbb{R}^d , respectively. Let $S^{d \times r}$ be the set of matrices in $\mathbb{R}^{d \times r}$ whose columns form an orthonormal basis of some r -dimensional subspace of \mathbb{R}^d , i.e., Let $S^{d \times r} = \{V \in \mathbb{R}^{d \times r} : V^\top V = I_r\}$. Because we will work with subspaces of \mathbb{R}^d of dimension less than or equal to r , we will sometimes use $S^{d \times r}$ to indicate matrices V which may have some columns equal to zero, i.e., $V^\top V = \text{diag}(I_r, 0, \dots, 0)$.

Let e_j be the vector with a 1 in the j th coordinate and zeros elsewhere. For a matrix A , let $\|A\|_2 = \sup_{x: \|x\|_2=1} \|Ax\|_2$ and $\|A\|_F^2 = \text{tr}(A^\top A)$. For two positive sequences $\{a_n\}, \{b_n\}$, we write $a_n \lesssim b_n$ or $a_n = O(b_n)$ if there exists a constant C such that $a_n \leq Cb_n$ and we write $a_n \lesssim_k b_n$ and $a_n = O_k(b_n)$ to emphasize that C may depend on a parameter k . For functions $\mathcal{L}(\theta), f(\theta)$, we write $\mathcal{L}(\theta) \stackrel{c}{=} f(\theta)$ to indicate that these functions are equal up to terms containing the relevant argument θ .

For $\epsilon > 0$, an ϵ -covering of a set A with respect to a metric ρ is a set \mathcal{N} such that for all $a \in A$, there exists $b \in \mathcal{N}$ such that $\rho(a, b) \leq \epsilon$; denote by $N(\epsilon, A, \rho)$ the minimum cardinality of ϵ -covering sets of A . An ϵ -packing in A with respect to the metric ρ is a set $\mathcal{M} \subset A$ such that $\rho(a, b) > \epsilon$ for any distinct a, b in \mathcal{M} ; denote by $M(\epsilon, A, \rho)$ the largest cardinality of ϵ -packing sets in A .

For distributions P and Q , let p and q denote their relative densities with respect to some dominating measure μ , respectively. The total variation distance is defined as $\text{TV}(P, Q) \triangleq \frac{1}{2} \int |p(x) - q(x)| \mu(dx)$. If $P \ll Q$, the KL divergence and the χ^2 -divergence are defined as $\text{KL}(P||Q) \triangleq \int p(x) \log \frac{p(x)}{q(x)} \mu(dx)$ and $\chi^2(P||Q) \triangleq \int \frac{(p(x)-q(x))^2}{q(x)} \mu(dx)$, respectively. Let $\text{supp}(P)$ denote the support set of a distribution

P . Let $\mathcal{L}(U)$ denote the distribution of a random variable U . For a one-dimensional distribution ν , denote the r th moment of ν by $m_r(\nu) \triangleq \mathbb{E}_{U \sim \nu}[U^r]$. Given a d -dimensional distribution Γ , for each $\theta \in \mathbb{R}^d$, we denote

$$\Gamma_\theta \triangleq \mathcal{L}(\theta^\top U), \quad U \sim \Gamma; \quad (1.1.1)$$

in other words, Γ_θ is the pushforward of Γ by the projection $u \mapsto \theta^\top u$; in particular, the i th marginal of Γ is denoted by $\Gamma_i \triangleq \Gamma_{e_i}$, with e_i being the i th coordinate vector. Similarly, for $V \in \mathbb{R}^{d \times k}$, denote

$$\Gamma_V \triangleq \mathcal{L}(V^\top U), \quad U \sim \Gamma. \quad (1.1.2)$$

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, we write $f'(z)$ to mean the derivative of the function f with respect to the argument z . For a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, let $\nabla f \in \mathbb{R}^d = (\partial f / \partial z_1, \dots, \partial f / \partial z_d)^\top$ be the gradient of f , $\nabla^2 f$ be the Hessian, and $\triangle f = \sum_{i \leq d} \partial^2 f(z) / \partial z_i^2 = \text{tr}(\nabla^2 f)$.

1.2 Theme for Gaussian mixtures: dimension reduction and moments

The signal in a high-dimensional dataset is often generated from different sources. For instance, a genetic dataset may contain information from people with different health conditions, or a medical video dataset may contain videos of people performing different exercises. To model such data, we turn to *mixture models*, which posit that the data are generated from a collection of sub-populations, each governed by a

different distribution. In chapters 2-5 of this thesis, we will observe data of the form:

$$X_i \stackrel{\text{i.i.d.}}{\sim} P_\Gamma, \text{ for } i = 1, \dots, n, \text{ where} \quad (1.2.1)$$

$$P_\Gamma = \Gamma * N(0, \sigma^2 I_d), \text{ and } \Gamma = \sum_{j=1}^k w_j \Gamma_j.$$

In (1.2.1), $*$ denotes the convolution. The latent distribution Γ is the *mixing distribution*. The weights w_1, \dots, w_k satisfy $w_j \geq 0$ and $\sum_{j=1}^k w_j = 1$. For simplicity, we assume $\sigma^2 = 1$. In this work, Γ will be a distribution that is parametrized by k subspaces or manifolds. The subspaces parametrizing Γ may be zero-dimensional affine spaces (as in k -means, or the classical Gaussian location mixture model), linear subspaces (as in subspace clustering), or manifolds. We study these scenarios in Chapters 2, 3, and 4, respectively. In Chapter 5, we explore maximum likelihood estimation in the context of a model of the form (1.2.1); we obtain a result specifically for the Gaussian location mixture model, but the methods therein are applicable to any of the models studied in Chapters 2-4, as well as to related models.

In this work, crucially, the dimension d will be potentially as large as the sample size n . Data generated according to the model (1.2.1) have the following signal-plus-noise structure:

$$X_i = U_i + Z_i, \text{ where} \quad (1.2.2)$$

$$Z_i \sim_{\text{i.i.d.}} N(0, \sigma^2 I_d) \text{ and } U_i \sim_{\text{i.i.d.}} \Gamma.$$

Learning in these finite mixture models takes three forms: clustering, parameter estimation, and density estimation. All three problems fall under the guise of unsupervised learning. In well-separated, standard mixture models such as the Gaussian location mixture model, all three aspects of the problem are well understood. What is less understood are optimal rates of convergence and algorithms when the model may be less ideal but more realistic.

Let ϕ_d denote the standard normal density in d dimensions. Then the density of

P_Γ is given by

$$p_\Gamma(x) = \int \sigma^{-d} \phi_d((x - u) / \sigma) d\Gamma(u). \quad (1.2.3)$$

Any signal-plus-noise model of the form (1.2.1) where Γ lies on some low-dimensional subspace of \mathbb{R}^d really has two components: the a Gaussian distribution on the irrelevant noise variables, and a Gaussian mixture distribution on the relevant variables. Let Γ be a distribution on a subspace of \mathbb{R}^d of rank at r , and let $V = [v_1, \dots, v_r]$ be the matrix whose columns form an orthonormal basis for this subspace. Let $V_C = [v_{r+1}, \dots, v_d]$ be the matrix whose columns are an orthonormal basis of the complement, so $[V, V_C]$ is an orthonormal matrix. We can write:

$$p_\Gamma(x) = \phi_{d-r}(V_C^\top x) p_{\Gamma_V}(V^\top x). \quad (1.2.4)$$

The reduction of densities into relevant and independent irrelevant noise leads to a crucial fact about statistical distances: they can be reduced to statistical distances in a lower dimensional space, as we now state and prove.

Lemma 1.2.1. *Let $\Gamma, \Gamma' \in S$, where S is a subspace of \mathbb{R}^d of dimension at most r . Let $V = [v_1, \dots, v_r]$ be the matrix whose columns form an orthonormal basis for S . Let $D \in \{\chi^2, KL, H^2\}$. Then*

$$D(\Gamma * N(0, I_d), \Gamma' * N(0, I_d)) = D(\Gamma_V * N(0, I_r), \Gamma'_V * N(0, I_r)).$$

Proof. We prove the fact for the chi-square divergence; proofs for the other divergences are exactly similar. Let $V_C = [v_{r+1}, \dots, v_d]$ be the matrix whose columns are

an orthonormal basis of the complement, so $[V, V_C]$ is an orthonormal matrix.

$$\begin{aligned}\chi^2(\Gamma * N(0, I_d) || \Gamma' * N(0, I_d)) &= \int_{x \in \mathbb{R}^d} \phi_{d-r}(V_C^\top x) \frac{(p_{\Gamma_V}(V^\top x) - p_{\Gamma'_V}(V^\top x))^2}{p_{\Gamma'_V}(V^\top x)} dx \\ &= \int_{x \in \mathbb{R}^r} \frac{(p_{\Gamma_V}(x) - p_{\Gamma'_V}(x))^2}{p_{\Gamma'_V}(x)} dx.\end{aligned}$$

□

The other important theme we will see repeatedly is that Gaussian mixtures of the form (1.2.1) can be represented in terms of the moments of Γ ; it turns out that convergence rates in such mixtures depend crucially on the moment space of Γ . These moments appear because densities like (1.2.3) can be orthogonally decomposed using the *Hermite polynomials*. Let H_j be the degree- j Hermite polynomial, defined via

$$H_j(x) \triangleq j! \sum_{i=1}^{\lfloor j/2 \rfloor} \frac{(-1/2)^i}{i!(j-2i)!} x^{j-2i}, \text{ for } x \in \mathbb{R}. \quad (1.2.5)$$

Its multivariate counterpart is defined as follows. For each multi-index $\mathbf{j} = (j_1, \dots, j_d) \in \mathbb{Z}_+^d$, define the \mathbf{j} th Hermite polynomial as

$$H_{\mathbf{j}}(x) = \prod_{i=1}^d H_{j_i}(x_i), \quad x \in \mathbb{R}^d. \quad (1.2.6)$$

which is a degree- $|\mathbf{j}|$ polynomial in x . Recall that for $x, u \in \mathbb{R}$, $\phi(x-u) = \phi(x) \sum_{j \geq 0} H_j(x) \frac{u^j}{j!}$ (see [Abramowitz and Stegun, 1964, 22.9.17]). The multivariate extension of this fact is

$$\phi_d(x-u) = \phi_d(x) \sum_{\mathbf{j} \in \mathbb{Z}_+^d} \frac{H_{\mathbf{j}}(x)}{\mathbf{j}!} \prod_{i=1}^d u_i^{j_i}, \quad x, u \in \mathbb{R}^d.$$

And therefore

$$p_\Gamma(x) = \mathbb{E}_{U \sim \Gamma}[\phi_d(x - U)] = \phi_d(x) \sum_{\mathbf{j} \in \mathbb{Z}_+^d} \frac{H_{\mathbf{j}}(x)}{\mathbf{j}!} \underbrace{\mathbb{E}_{U \sim \Gamma} \left[\prod_{i=1}^d U_i^{j_i} \right]}_{m_{\mathbf{j}}(\Gamma)}. \quad (1.2.7)$$

Let $\mathbf{j} = (j_1, \dots, j_d) \in \mathbb{Z}_+^d$. The \mathbf{j} th multivariate moment of Γ is

$$m_{\mathbf{j}}(\Gamma) = \mathbb{E}_{U \sim \Gamma}[U_{j_1} \cdots U_{j_d}]. \quad (1.2.8)$$

Let $|\mathbf{j}| \triangleq j_1 + \dots + j_d$. Then $m_{\mathbf{j}}(\Gamma)$ is the \mathbf{j} th entry of the moment tensor $M_{|\mathbf{j}|}(\Gamma)$, which we now define. For any d -dimensional random vector U , its order- ℓ *moment tensor* is

$$M_\ell(\Gamma) \triangleq \mathbb{E}_{U \sim \Gamma}[\underbrace{U \otimes \cdots \otimes U}_{\ell \text{ times}}]. \quad (1.2.9)$$

We will also use the notation $M_\ell(U) = M_\ell(\Gamma)$ where $U \sim \Gamma$. Note that $M_1(\Gamma) = \mathbb{E}[U]$ and $M_2(U - \mathbb{E}[U])$ are the mean and the covariance matrix of U , respectively. For a positive integer j , let $S_{j,d} = \{\mathbf{j} \in \mathbb{Z}_+^d : |\mathbf{j}| = j\}$. Let $\Delta m_{\mathbf{j}}(\Gamma, \Gamma') = m_{\mathbf{j}}(\Gamma) - m_{\mathbf{j}}(\Gamma')$.

We now recall some basic facts about tensors that will be useful in this work. The *rank* of an order- ℓ tensor $T \in (\mathbb{R}^d)^{\otimes \ell}$ is defined as the minimum r such that T can be written the sum of r rank-one tensors, namely [Kruskal, 1977]:

$$\text{rank}(T) \triangleq \min \left\{ r : T = \sum_{i=1}^r \alpha_i u_i^{(1)} \otimes \cdots \otimes u_i^{(\ell)}, \quad u_i^{(j)} \in \mathbb{R}^d, \alpha_i \in \mathbb{R} \right\}, \quad (1.2.10)$$

We will also use the *symmetric rank* [Comon et al., 2008]:

$$\text{rank}_s(T) \triangleq \min \left\{ r : T = \sum_{i=1}^r \alpha_i u_i^{\otimes \ell}, \quad u_i \in \mathbb{R}^d, \alpha_i \in \mathbb{R} \right\}. \quad (1.2.11)$$

An order- ℓ tensor T is *symmetric* if $T_{j_1, \dots, j_\ell} = T_{j_{\pi(1)}, \dots, j_{\pi(\ell)}}$ for all $j_1, \dots, j_\ell \in [d]$ and all permutations π on $[\ell]$. By definition, moment tensors are symmetric. The Frobenius

norm of a tensor T is defined as $\|T\|_F \triangleq \sqrt{\langle T, T \rangle}$, where the tensor inner product is defined as $\langle S, T \rangle = \sum_{j_1, \dots, j_\ell \in [d]} S_{j_1, \dots, j_\ell} T_{j_1, \dots, j_\ell}$. The spectral norm (operator norm) of a tensor T is defined as

$$\|T\| \triangleq \max\{\langle T, u_1 \otimes u_2 \otimes \dots \otimes u_\ell \rangle : \|u_i\| = 1, i = 1, \dots, \ell\}. \quad (1.2.12)$$

Denote the set of d -dimensional order- ℓ symmetric tensors by $\mathbb{S}_\ell(\mathbb{R}^d)$. For a symmetric tensor, the following result attributed to Banach ([Banach, 1938, Friedland and Lim, 2018]) is crucial this work:

$$\|T\| = \max\{|\langle T, u^{\otimes \ell} \rangle| : \|u\| = 1\}. \quad (1.2.13)$$

For $T \in \mathbb{S}_\ell(\mathbb{R}^d)$, if $\text{rank}_s(T) \leq r$, then the spectral norm can be bounded by the Frobenius norm as follows [Qi, 2011]:

$$\frac{1}{\sqrt{r^{\ell-1}}} \|T\|_F \leq \|T\| \leq \|T\|_F. \quad (1.2.14)$$

An important observation is that the moment of the projection of a random vector can be expressed in terms of the moment tensor as follows: for any $u \in \mathbb{R}^d$,

$$m_\ell(\langle X, u \rangle) = \mathbb{E}[\langle X, u \rangle^\ell] = \mathbb{E}[\langle X^{\otimes \ell}, u^{\otimes \ell} \rangle] = \langle M_\ell(X), u^{\otimes \ell} \rangle.$$

Consequently, the difference between two moment tensors measured in the spectral norm is equal to the maximal moment difference of their projections. Indeed, thanks to (1.2.13),

$$\|M_\ell(X) - M_\ell(Y)\| = \sup_{\|u\|=1} |m_\ell(\langle X, u \rangle) - m_\ell(\langle Y, u \rangle)|. \quad (1.2.15)$$

Furthermore, if U is a discrete random variable with a few atoms, then its moment tensor has low rank. Specifically, if U is distributed according to some k -atomic

distribution $\Gamma = \sum_{i=1}^k w_i \delta_{\mu_i}$, then

$$M_\ell(\Gamma) = \sum_{i=1}^k w_i \mu_i^{\otimes \ell}, \quad (1.2.16)$$

whose symmetric rank is at most k .

We now present a simple result showing that statistical distances between Gaussian mixture distributions of the form (1.2.1) can be upper bounded by distances between the moment tensors of their corresponding mixing distributions. This result also relies on two simple facts that are stated and proved below in Lemma 1.2.3 and Lemma 1.2.4. We include them in this section because they are used elsewhere in Chapters 2-5.

Lemma 1.2.2 (Moment upper bound on statistical distances). *Let Γ, Γ' be distributions supported on $B(0, R) \cap S$ where S is a subspace of \mathbb{R}^d of rank at most r . Then*

$$\chi^2(P_\Gamma || P_{\Gamma'}) \leq C \sum_{j \geq 1} \frac{\|M_j(\Gamma) - M_j(\Gamma')\|_F^2 r^j}{j!}. \quad (1.2.17)$$

where the constant C may depend on R and r but does not depend on d .

Proof of Lemma 1.2.2. By Lemma 1.2.4, we may assume without loss of generality that Γ' has mean zero. We also note that the following orthogonality property of the multivariate Hermite polynomials is inherited from that of univariate Hermite polynomials: for $Z \sim N(0, I_r)$,

$$\mathbb{E}[H_{\mathbf{j}}(Z) H_{\mathbf{j}'}(Z)] = \mathbf{j}! \mathbf{1}_{\{\mathbf{j}=\mathbf{j}'\}}. \quad (1.2.18)$$

We obtain

$$\begin{aligned}
\chi^2(P_\Gamma \| P_{\Gamma'}) &= \chi^2(P_{\Gamma_V} \| P_{\Gamma'_V}) && \text{by Lemma 1.2.1} \\
&\leq e^{R^2/2} \int_{\mathbb{R}^r} \frac{(p_{\Gamma_V}(x) - p_{\Gamma'_V}(x))^2}{\phi_r(x)} && \text{by Lemma 1.2.3} \\
&= e^{R^2/2} \mathbb{E}_{N(0, I_r)} \left(\sum_{\mathbf{j} \in \mathbb{Z}_+^r} \frac{H_{\mathbf{j}}(x) \Delta m_{\mathbf{j}}(\Gamma_V, \Gamma'_V)}{\mathbf{j}!} \right)^2 && \text{by the expansion (1.2.7)} \\
&= e^{R^2/2} \sum_{\mathbf{j} \in \mathbb{Z}_+^r} \frac{(\Delta m_{\mathbf{j}}(\Gamma, \Gamma'))^2}{\mathbf{j}!} && \text{by (1.2.18)} \\
&\leq e^{R^2/2} \sum_{\ell \geq 1} \frac{\|M_\ell(\Gamma) - M_\ell(\Gamma')\|_{\mathbb{F}}^2}{\ell!} r^\ell,
\end{aligned}$$

where the final step is by the fact that $(|\mathbf{j}|)! \leq \mathbf{j}! r^{|\mathbf{j}|}$ for any $\mathbf{j} \in \mathbb{Z}_+^r$. \square

Lemma 1.2.3. *Let $\Gamma \in \mathbb{R}^d$ be a distribution supported on $B(0, R)$ with mean zero. Let $U \sim \Gamma$. Then*

$$\mathbb{E}_\Gamma \exp(x^\top U - \|U\|_2^2/2) \geq \exp(-R^2/2) \quad (1.2.19)$$

Proof. By Jensen's Inequality and the fact that $\mathbb{E}_\Gamma Y = 0$, $\mathbb{E}_\Gamma (\exp x^\top U - \|U\|_2^2/2) \geq \exp(-\mathbb{E}_\Gamma(\|U\|_2^2/2))$. The result follows because $\|U\|_2 \leq R$. \square

Lemma 1.2.4. *For any random vectors X and Y and any deterministic $\mu \in \mathbb{R}^d$,*

$$\|M_\ell(X - \mu) - M_\ell(Y - \mu)\| \leq \sum_{k=0}^{\ell} \binom{\ell}{k} \|M_k(X) - M_k(Y)\| \|\mu\|^{\ell-k}$$

Proof. Using (1.2.15) and binomial expansion, we have:

$$\begin{aligned}
\|M_\ell(X - \mu) - M_\ell(Y - \mu)\| &= \sup_{\|u\|=1} |m_\ell(\langle X, u \rangle - \langle \mu, u \rangle) - m_\ell(\langle Y, u \rangle - \langle \mu, u \rangle)| \\
&\leq \sup_{\|u\|=1} \sum_{k=0}^{\ell} \binom{\ell}{k} |m_k(\langle X, u \rangle) - m_k(\langle Y, u \rangle)| |\langle \mu, u \rangle|^{\ell-k} \\
&\leq \sum_{k=0}^{\ell} \binom{\ell}{k} \|M_k(X) - M_k(Y)\| \|\mu\|^{\ell-k}
\end{aligned}$$

where in the step we used the Cauchy-Schwarz inequality. \square

1.3 Theme for graphical models: score matching

Latent signal detection often requires the estimation of a high-dimensional density. The study of high-dimensional density estimation has a long history in statistics, and the problem is, in general, intractable since it usually requires the computation of a normalizing constant in high dimensions.

To circumvent this issue, we will rely on a powerful tool known as score matching, which provides a way to estimate a high-dimensional density without computing the normalizing constant. Suppose we have data z generated from a distribution \mathbb{P} with density p . We will estimate p with a density q . In score matching, we optimize the density q by minimizing the Fisher divergence between p and q ; the loss in q is therefore:

$$\mathcal{L}(q) = \mathbb{E}_p \|\nabla \log p - \nabla \log q\|_2^2. \quad (1.3.1)$$

In (7.1.1), the derivatives are taken with respect to the argument z . Clearly, the normalizing constant in q disappears in the objective (1.3.1); however, this objective does not appear to be something we can optimize in q since it explicitly involves the true density p . However, it turns out that (1.3.1) simplifies to an objective of the form $\mathbb{E}_p O_q$ where O_q is an operator depending only on the density q . The simplification is

due simply to integration by parts; we state and prove this here in one dimension so that the intuition is clear, and we refer the reader to [Hyvarinen, 2005] for a general proof.

Lemma 1.3.1 (The score matching objective can be optimized in q). *Let $\hat{q} = \operatorname{argmin}_q \mathcal{L}(q)$ with $\mathcal{L}(q)$ as in (7.1.1). Then*

$$\hat{q} = \operatorname{argmin}_q \int_z p(z) \left(\frac{1}{2} \|\nabla \log q\|_2^2 + \Delta \log q \right) dz.$$

Proof.

$$\mathcal{L}(q) \stackrel{c}{=} \int p(z) \left(\frac{\partial \log q(z)}{\partial z} \right)^2 - 2 \int p(z) \frac{\partial \log p(z)}{\partial z} \frac{\partial \log q(z)}{\partial z}.$$

Since $\frac{\partial \log p(z)}{\partial z} = p'(z)/p(z)$, the second term in the sum is $-2 \int p'(z) \frac{\partial \log q(z)}{\partial z}$. Using integration by parts,

$$\int p'(z) \frac{\partial \log q(z)}{\partial z} \stackrel{c}{=} - \int p(z) \frac{\partial^2 q(z)}{\partial z^2}.$$

□

The objective (1.3.1) simplifies further in the case where q is assumed to be an exponential family. In fact, in this case, the solution for q has a closed form; it requires only the computation of a simple quadratic. We now state and prove this fact.

Lemma 1.3.2 (Score matching for exponential families.). *Let $q(z) = \frac{e^{g(z)}}{\int e^{g(z)} dz}$, and suppose $g : \mathbb{R}^d \rightarrow \mathbb{R}$, with $g(z) = \gamma^\top \phi(z)$ where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^K$ and $\gamma \in \mathbb{R}^K$. Let $\mathcal{L}(q)$*

be as in (7.1.1). Let $\hat{\gamma} = (\mathbb{E}_{p(z)} A(z))^{-1} \mathbb{E}_{p(z)} k(z)$, where

$$k(z) = \sum_{i \leq d} \frac{\partial^2 \phi(z)}{\partial z_i^2}$$

$$A(z) = \sum_{i \leq d} \frac{\partial \phi(z)}{\partial z_i} \frac{\partial \phi(z)}{\partial z_i}^\top.$$

Note $A \in \mathbb{R}^{K \times K}$, $k \in \mathbb{R}^K$. Then

$$\operatorname{argmin}_q \mathcal{L}(q) \propto \exp(\hat{\gamma}^\top \phi(z)). \quad (1.3.2)$$

Proof. This follows directly from Lemma 1.3.1 and plugging in the form of the exponential family. We have

$$\frac{\partial \phi(z)}{\partial z_i} = \left(\frac{\partial \phi_1(z)}{\partial z_i}, \dots, \frac{\partial \phi_K(z)}{\partial z_i} \right)^\top$$

$$\frac{\partial^2 \phi(z)}{\partial z_i^2} = \left(\frac{\partial^2 \phi_1(z)}{\partial z_i^2}, \dots, \frac{\partial^2 \phi_K(z)}{\partial z_i^2} \right)^\top$$

Since $\log q(z) \propto g(z)$,

$$\begin{aligned} \frac{1}{2} \|\nabla \log q\|_2^2 + \Delta \log q &= \sum_{i \leq d} \left(\frac{1}{2} \left(\frac{\partial g(z)}{\partial z_i} \right)^2 + \frac{\partial^2 g(z)}{\partial z_i^2} \right) \\ &= \sum_{i \leq d} \left(\frac{1}{2} \left(\gamma^\top \frac{\partial \phi(z)}{\partial z_i} \right)^2 + \gamma^\top \frac{\partial^2 \phi(z)}{\partial z_i^2} \right) \\ &= \frac{1}{2} \gamma^\top \left(\sum_{i \leq d} \frac{\partial \phi(z)}{\partial z_i} \frac{\partial \phi(z)}{\partial z_i}^\top \right) \gamma + \gamma^\top \sum_{i \leq d} \frac{\partial^2 g(z)}{\partial z_i^2} \\ &= \frac{1}{2} \gamma^\top A(z) \gamma + \gamma^\top k(z). \end{aligned}$$

So $\mathcal{L}(q) \stackrel{c}{=} \frac{1}{2} \gamma^\top (\mathbb{E}_{p(z)} A(z)) \gamma + \gamma^\top (\mathbb{E}_{p(z)} k(z))$, and the conclusion follows from optimizing this quadratic in γ . \square

1.3.1 Score matching via kernel density estimation

Score matching in the form of (1.3.1) is not the only possibility for optimizing a non-parametric density while avoiding computation of the normalizing constant. Notice a weakness of this form of score matching: it requires the computation of a Hessian matrix. The works [Vincent, 2011] and [Saremi et al., 2018] propose a kernel-density estimation based type score matching that avoids the Hessian calculation. We discuss this method in general in Section 1.3.1.

Suppose we observe data x distributed according to a distribution P with density p . Suppose we also observe a noisy form of x , ξ , satisfying $\xi = x + z$, where $z \sim N(0, \sigma^2 I_d)$. Recall that in score matching, we optimize in θ the score matching divergence $\int p(x) \|\nabla_x \log p(x) - \nabla \log_x q_\theta(x)\|_2^2 dx$. [Vincent, 2011] propose that instead of using the integration-by-parts trick to obtain an optimizeable empirical version of this loss, we instead use an explicit representation of $p(x)$; in fact, a kernel density estimate. Specifically, let us observe data points $x_1, \dots, x_n \in \mathbb{R}^d$. Let

$$p(\xi) = \frac{1}{n} \sum_{i=1}^n K(\xi|x_i),$$

where K is a kernel function. [Vincent, 2011] then uses the kernel-density score matching loss:

$$\mathcal{L}_{kde}(\theta) = \mathbb{E}_{p(\xi)} \|\nabla_\xi \log p(\xi) - \nabla_\xi \log q_\theta(\xi)\|_2^2. \quad (1.3.3)$$

It turns out that this can be represented in a simple way that makes it easy to optimize in practice, as we now show.

Lemma 1.3.3. *Let $\mathcal{L}_{kde}(\theta)$ be as defined in (1.3.3). Then optimizing $\mathcal{L}_{kde}(\theta)$ in θ is equivalent to optimizing the loss:*

$$\mathcal{L}_{kde}(\theta) \stackrel{c}{=} \mathbb{E}_{(\xi, x) \sim J} \|\nabla_\xi \log K(\xi|x_i) - \nabla_\xi \log q_\theta(\xi)\|_2^2,$$

where J is the joint distribution with density $\frac{1}{n} \sum_{i=1}^n K(\xi|x_i)$, i.e., it is the empirical expectation (over the empirical density of x) of the conditional density $K(\xi|x)$.

Proof.

$$\mathcal{L}_{kde}(\theta) \stackrel{c}{=} \mathbb{E}_{p(\xi)} \|\nabla_{\xi} \log q_{\theta}(\xi)\|_2^2 - 2\mathbb{E}_{p(\xi)} \langle \nabla_{\xi} \frac{1}{n} \sum_{i=1}^n K(\xi|x_i), \nabla_{\xi} \log q_{\theta}(\xi) \rangle. \quad (1.3.4)$$

Writing out the second term:

$$\begin{aligned} \mathbb{E}_{p(\xi)} \langle \nabla_{\xi} \frac{1}{n} \sum_{i=1}^n K(\xi|x_i), \nabla_{\xi} \log q_{\theta}(\xi) \rangle &= \int_{\xi} \frac{1}{n} \sum_{i=1}^n K(\xi|x_i) \langle \nabla_{\xi} \frac{1}{n} \sum_{i=1}^n K(\xi|x_i), \nabla_{\xi} \log q_{\theta}(\xi) \rangle d\xi \\ &= \int_{\xi} \langle \frac{1}{n} \sum_{i=1}^n \nabla_{\xi} K(\xi|x_i), \nabla_{\xi} \log q_{\theta}(\xi) \rangle d\xi \quad (1.3.5) \\ &= \frac{1}{n} \sum_{i=1}^n \int_{\xi} \langle \nabla_{\xi} K(\xi|x_i), \nabla_{\xi} \log q_{\theta}(\xi) \rangle d\xi. \end{aligned}$$

where (1.3.5) follows because $\nabla_{\xi} \sum_{i=1}^n K(\xi|x_i) = \frac{\sum_{i=1}^n \nabla_{\xi} K(\xi|x_i)}{\sum_{j=1}^n K(\xi|x_j)}$. And so plugging in to (1.3.4),

$$\begin{aligned} \mathcal{L}_{kde}(\theta) &\stackrel{c}{=} \frac{1}{n} \sum_{i=1}^n \int_{\xi} \|\nabla_{\xi} \log q_{\theta}(\xi)\|_2^2 - 2 \frac{1}{n} \sum_{i=1}^n \int_{\xi} \langle \nabla_{\xi} K(\xi|x_i), \nabla_{\xi} \log q_{\theta}(\xi) \rangle d\xi \\ &= \frac{1}{n} \sum_{i=1}^n \int_{\xi} (\|\nabla_{\xi} \log q_{\theta}(\xi)\|_2^2 - 2 \langle \nabla_{\xi} K(\xi|x_i), \nabla_{\xi} \log q_{\theta}(\xi) \rangle) d\xi \\ &\stackrel{c}{=} \mathbb{E}_{(\xi, x) \sim J} \|\nabla_{\xi} \log K(\xi|x_i) - \nabla_{\xi} \log q_{\theta}(\xi)\|_2^2. \end{aligned}$$

□

We now explain how [Saremi et al., 2018] use Lemma 1.3.3 to obtain a simple score matching objective that avoids Hessian computations. Suppose that $K(\xi|x) = \phi_{\sigma}(\xi - x)$, where ϕ_{σ} is the Gaussian kernel with bandwidth σ^2 , i.e., $K(\xi|x) = \exp(-\|\xi - x\|_2^2 / 2\sigma^2)$. Then $\nabla_{\xi} \log K(\xi|x) = \frac{x - \xi}{\sigma^2}$, and using Lemma 1.3.3, the

loss (1.3.3) becomes

$$\mathcal{L}_{kde}(\theta) = \mathbb{E}_{(\xi, x) \sim J} \|x - \xi - \nabla_{\xi} \log q_{\theta}(\xi)\|_2^2.$$

In practice, for each observed data point x_i , we generate Gaussian samples $\xi_{i1}, \dots, \xi_{im}$.

We then use the loss:

$$\mathcal{L}(\theta) = \sum_{i \in [n], j \in [m]} \|x_i - \xi_{i,j} - \sigma^2 \nabla_{\xi} \log q_{\theta}(\xi_{ij})\|_2^2.$$

In [Saremi et al., 2018], they parametrize let q be a multilayer perceptron. Let nl be the number of hidden layers and let ℓ be the number of units per hidden layer; we assume we use the same number per layer for simplicity. Let $M_1 \in \mathbb{R}^{p \times \ell}$, $M_a \in \mathbb{R}^{\ell \times \ell}$ for $a \in [nl]$. Let $M = (M_1, \dots, M_{nl})$ and $b = (b_1, b_{nl})$, and let $\theta = (M, b)$. Let

$$\log p_{M,b}(s) = -M_3 \sigma(M_2 \sigma(M_1 s + b_1) + b_2) + b_3. \quad (1.3.6)$$

Chapter 2

Mixtures of means: parameter and density estimation

2.1 Introduction

We now consider the problems of parameter and density estimation in (1.2.1) when Γ is a distribution on k atoms in \mathbb{R}^d . This is known as the Gaussian location mixture model; we shall also refer to it as the Gaussian mixture on k means. This is one of the most widely studied mixture models because of its simplicity and wide applicability, but rates of convergence for both parameter and density estimation in this model are not well understood when the mixing distribution centers are both high dimensional and allowed to be arbitrarily close to each other. Consider the k -component Gaussian location mixture model (which we shall refer to as k -GM) in d dimensions:

$$\begin{aligned} X_i &\stackrel{\text{i.i.d.}}{\sim} P_\Gamma, \text{ for } i = 1, \dots, n, \text{ where} \\ P_\Gamma &= \Gamma * N(0, \sigma^2 I_d), \text{ and } \Gamma = \sum_{j=1}^k w_j \delta_{\mu_j}. \end{aligned} \tag{2.1.1}$$

In (2.1.1), δ_x denotes the Dirac delta measure at $x \in \mathbb{R}^d$. The unknown parameter is Γ , a discrete distribution on k atoms μ_1, \dots, μ_k , each in \mathbb{R}^d .

In (2.1.1), a $n^{-1/2}$ convergence rate for parameter estimation is obtained when $d = 1$, μ_1, \dots, μ_k are fixed and well separated, and k is known. A more realistic setting is one in which the mixing components may not be well separated. For instance, some of them may be equal or very close together, so that the model is effectively mis-identified. In a pioneering work in this area, [Chen, 1995] considered k -GM under such weak assumptions. In particular, he showed that when $\Gamma = \frac{1}{3}\delta_{2h} + \frac{2}{3}\delta_{-h}$, the rate for estimating h is $n^{-1/4}$ when the truth is allowed to be in a neighborhood of radius $n^{-1/4}$ around zero.

For general k -GM in one dimension, where k is upper bounded but the mixing distribution atoms may be arbitrarily close, the minimax convergence rate is $n^{-1/(4k-2)}$ when the error is measured in the 1-Wasserstein distance (defined later in this section) on the mixing distribution ([Heinrich and Kahn, 2018], [Wu and Yang, 2019]). When the true number of mixing components k_0 is unknown but upper bounded by some k , a convergence rate of $n^{-1/(4(k-k_0)+2)}$ is obtained for general one dimensional mixtures, including Gaussian mixtures; see [Heinrich and Kahn, 2018] and [Wu and Yang, 2019].

We consider rates of convergence for Gaussian location mixtures when the atoms of Γ may arbitrarily overlap, the weights may be arbitrarily small, and crucially, the dimension d may be as large as the sample size n . This extends [Wu and Yang, 2019] to a high-dimensional setting. As discussed above, it is known that in such a scenario, when d is fixed, the minimax rate for parameter estimation is much slower than $n^{-1/2}$, and the rate for density estimation is parametric. We seek to understand how this extends for arbitrary d .

Several recent works have explored multivariate Gaussian mixture estimation, both in high and low dimensions. For statistical rates, [Ho and Nguyen, 2016, Theorem 1.1] obtained convergence rates for mixing distribution estimation in Wasser-

stein distances for low-dimensional multivariate location-scale Gaussian mixtures, both over- and exact-fitted. Their rates for over-fitted mixtures are determined by algebraic dependencies among a set of polynomial equations whose order depends on the level of overfitting; the rates are potentially much slower than $n^{-1/2}$. The estimator analyzed in [Ho and Nguyen, 2016] is the maximum-likelihood estimator (MLE), which involves non-convex optimization and is typically approximated by the Expectation-Maximization (EM) algorithm.

In the computer science literature, a long line of research starting with [Dasgupta, 1999] has developed fast algorithms for individual parameter estimation in multivariate Gaussian mixtures under fairly weak separation conditions, see, e.g., [Arora and Kannan, 2005, Belkin and Sinha, 2009, Kalai et al., 2010, Moitra and Valiant, 2010, Hsu and Kakade, 2013, Hardt and Price, 2015, Hopkins and Li, 2018]. Since these works focus on individual parameter estimation, some separation assumption on the mixing distribution is necessary.

We now give more background on the problem of estimation in (2.1.1). Denote the probability simplex on k elements by $\Delta_k = \{w = (w_1, \dots, w_k) : \sum_{j=1}^k w_j = 1, w_j \geq 0\}$. We will let $\mathcal{G}_{k,d}$ be the class of k -atomic distributions supported on a ball of radius R in d dimensions, where, throughout the paper, R is assumed to be an absolute constant. Let $\|\cdot\|_2$ denote the standard Euclidean norm in \mathbb{R}^d . Formally,

$$\mathcal{G}_{k,d} \triangleq \left\{ \Gamma = \sum_{j=1}^k w_j \delta_{\mu_j} : \mu_j \in \mathbb{R}^d, \|\mu_j\|_2 \leq R, w = (w_1, \dots, w_k) \in \Delta_k \right\}.$$

Let

$$P_\Gamma = \Gamma * N(0, I_d) = \sum_{j=1}^k w_j N(\mu_j, I_d)$$

and $\mathcal{P}_{k,d} = \{P_\Gamma : \Gamma \in \mathcal{G}_{k,d}\}$ denotes the collection of k -Gaussian mixtures (k -GMs) whose centers lie in a ball of radius R . Let ϕ_d denote the standard normal density in

d dimensions. Then the density of P_Γ is given by

$$p_\Gamma(x) = \sum_{j=1}^k w_j \sigma^{-d} \phi_d((x - \mu_j)/\sigma),$$

which is a parametric distribution with the $2k-1$ parameters μ_1, \dots, μ_k and w_1, \dots, w_k . In many scenarios, it is possible to estimate the individual weights and location parameters with small error. Under our assumptions, we may be able to estimate some of the components or weights with large error but still have small error in the estimation of the entire mixing distribution. Therefore, it is natural for us to frame the model as in (2.1.1).

We first discuss the problem of parameter estimation in (2.1.1). We use a Wasserstein distance to measure the error on the mixing distribution. For $q \geq 1$, the Wasserstein- q distance (with respect to the Euclidean distance) is defined as

$$W_q(\Gamma, \Gamma') \triangleq \left(\inf_{\nu \in \mathcal{P}(\Gamma, \Gamma')} \mathbb{E}_\nu \|U - U'\|_2^q \right)^{\frac{1}{q}}, \quad (2.1.2)$$

where the infimum is taken over all couplings of Γ and Γ' , i.e., joint distributions of U and U' with marginals Γ and Γ' respectively. The W_1 distance is a natural and commonly used loss function for mixture models and deconvolution problems. It is invariant under permutation of the distributions it compares, so it sidesteps the unidentifiability of mixing distributions. It allows for the development of a meaningful statistical theory in an assumption-free framework. Note that if Γ and Γ' have the same number of atoms and those atoms are well separated and the weights are bounded away from zero, estimation in W_1 distance translates to recovery of the individual parameters up to permutation; see Lemma 1 of [Wu and Yang, 2019]. In the widely studied case of the symmetric 2-GM in which the samples are drawn from $P_\Gamma = \frac{1}{2}N(\mu, I_d) + \frac{1}{2}N(-\mu, I_d)$ with mixing distribution $\Gamma_\mu = \frac{1}{2}(\delta_{-\mu} + \delta_\mu)$, $W_1(\Gamma_\mu, \Gamma_{\mu'})$

coincides with the commonly used loss function $\min\{\|\mu - \mu'\|_2, \|\mu + \mu'\|_2\}$.¹

Assume $k \geq 2$ in (2.1.1). A natural conjecture is that the d -dimensional analogue of the $n^{-1/(4k-2)}$ -rate of [Wu and Yang, 2019] is $(d/n)^{1/(4k-2)}$. This conjecture is incorrect. The main result of this paper on the minimax rate for estimating Γ in setting (2.1.1) is the following theorem.

Theorem 2.1.1 (Estimating the mixing distribution). *Let P_Γ be the k -GM defined in (2.1.1). Given n i.i.d. samples from P_Γ , the minimax risk of estimating Γ over the class $\mathcal{G}_{k,d}$ satisfies*

$$\inf_{\hat{\Gamma}} \sup_{\Gamma \in \mathcal{G}_{k,d}} \mathbb{E}_\Gamma W_1(\hat{\Gamma}, \Gamma) \asymp_k \left(\frac{d}{n}\right)^{1/4} \wedge 1 + \left(\frac{1}{n}\right)^{1/(4k-2)}, \quad (2.1.3)$$

where the notation \asymp_k means that both sides agree up to constant factors depending only on k . Furthermore, if $n \geq d$, there exists an estimator $\hat{\Gamma}$, computable in $O(nd^2) + O_k(n^{5/4})$ time, and a positive constant C_k , such that for any $\Gamma \in \mathcal{G}_{k,d}$ and any $0 < \delta < \frac{1}{2}$, with probability at least $1 - \delta$,

$$W_1(\hat{\Gamma}, \Gamma) \leq C_k \left(\left(\frac{d}{n}\right)^{1/4} + \left(\frac{1}{n}\right)^{1/(4k-2)} \left(\log \frac{1}{\delta}\right)^{1/2} \right). \quad (2.1.4)$$

We now explain the intuition behind the rate. The location parameters μ_1, \dots, μ_k of Γ span a subspace of \mathbb{R}^d dimension at most k . Let $\{v_1, \dots, v_k\}$ be an orthonormal basis of this subspace, and let $V = [v_1, \dots, v_k]$. For each $j \in [k]$, $\mu_j = V\psi_j$, where $\psi_j = V^\top \mu_j \in \mathbb{R}^k$ encodes the coefficients of μ_j in the basis vectors in V . Therefore, we can identify a k -atomic distribution Γ on \mathbb{R}^d with a pair (V, γ) , where $\gamma = \sum_{j \in [k]} w_j \delta_{\psi_j}$ is a k -atomic distribution on \mathbb{R}^k .

The error in estimating Γ consists of two parts: the high-dimensional component, which results from estimating the subspace on which the centers lie, and the low-

¹This can be seen directly from the definition of W_1 from (2.1.2); simply take the minimum over all couplings and note that any coupling in this case is a distribution on at most four points. Then $W_1(\Gamma_\mu, \Gamma_{\mu'}) = \min_{0 \leq w \leq 1} w\|\mu - \mu'\|_2 + (1 - w)\|\mu + \mu'\|_2$.

dimensional component, which results from estimating the k -atomic distribution in k dimensions. We shall see in Section 2.2 that the error from estimating the subspace is $(d/n)^{1/4}$, while the error from estimating the low-dimensional distribution is $n^{-1/(4k-2)}$. Estimating a high-dimensional Gaussian mixture balances a tradeoff in the error induced by the dimension d and by the number of components k . Indeed, we see that there is a threshold $d^* = n^{(2k-3)/(2k-1)}$ (e.g. $d^* = n^{1/3}$ for $k = 2$), such that for $d > d^*$, the subspace-estimation error, $(d/n)^{1/4}$, dominates the error; otherwise, the low-dimensional error, $(1/n)^{1/(4k-2)}$, does.

As discussed above, the slow rate of $n^{-1/(4k-2)}$ for the low dimensional part of the error can be explained by the weak assumptions placed on the model, making it more difficult to estimate than a standard parametric model. The $(d/n)^{1/4}$ in the subspace estimation component of the error in Theorem 2.1.1 may not be immediately intuitive. If the atoms are bounded away from zero, estimation should be doable at the $(d/n)^{1/2}$ rate, but if not, it is more difficult to estimate the subspace on which the atoms lie. It turns out that this results in the slower rate of $(d/n)^{1/4}$. For a more general statement, see Lemma 2.2.5.

In addition to a minimax rate of convergence, we seek an algorithm providing an estimator that achieves this rate. There are two broad categories of approaches to estimation for mixtures of Gaussians: method-of-moments and maximum likelihood. This paper first provides a polynomial time algorithm based on the former approach. It is derived from the denoised-method-of-moments (DMM) method of [Wu and Yang, 2019], which is an improved variant of the usual method of moments.

The case where d does not grow with n and the case where $d = 1$ are not fundamentally different in terms of the optimal rate of convergence. However, even in the case where d does not grow with n but is greater than one, algorithmic challenges immediately arise. The algorithm presented in this paper solves two distinct

problems presented by the high dimensional mixture: the problem of estimating the subspace (handling a d that may grow with n) and the problem of estimating the low-dimensional mixture which nonetheless is not univariate.

This work also demonstrates that density estimation for high dimensional Gaussian mixtures can be done at the sharp parametric rate. Given distributions P, Q , let p and q denote their relative densities with respect to some dominating measure μ , respectively. The squared Hellinger distance between P and Q is defined as $H^2(P, Q) \triangleq \int \left(\sqrt{p(x)} - \sqrt{q(x)} \right)^2 \mu(dx)$. We now give a brief review of the recent literature on density estimation for Gaussian mixtures, then state our result.

The result in Theorem 2.1.2 below, which follows a long line of research on Gaussian location mixture model density estimation, is the first we know of that avoids logarithmic terms in the numerator. It is known that if $d = 1$ and the mixing distribution support is not finite, i.e., k or R may grow with n , then a $\log n$ term in the numerator of (2.1.5) cannot be avoided, see [Kim, 2014] and [Ibragimov, 2001]. But it is natural to think that in (2.1.1), the sharp parametric rate should be obtainable for density estimation since the mixing distribution support is finite and bounded. Results in this direction when d is fixed are obtained by [Wu and Yang, 2019], who obtain the sharp parametric rate when $d = 1$. In low dimensions, [Ho and Nguyen, 2016, Theorem 2.1] obtained an $O(\sqrt{\log n/n})$ -Hellinger guarantee for the MLE Gaussian location-scale mixtures. For high-dimensional location mixtures, the best result till now was the total variation guarantee of $\tilde{O}(\sqrt{kd/n})$ for location mixtures, where \tilde{O} hides polylogarithmic factors, from [Ashtiani et al., 2018].

The algorithm therein runs in time that is exponential in d . To our knowledge, there is no polynomial-time algorithm that achieves the sharp density estimation guarantee in Theorem 2.1.2 (or the slightly suboptimal rate in [Ashtiani et al., 2018]), even for constant k . The works of [Kalai et al., 2010, Moitra and Valiant, 2010] showed that their polynomial-time parameter estimation algorithms also provide density es-

timators without separation conditions, but the resulting rates of convergence are far from optimal. [Feldman et al., 2006, Acharya et al., 2014, Li and Schmidt, 2017] provided polynomial-time algorithms for density estimation with improved statistical performance. In particular, [Acharya et al., 2014] obtained an algorithm that runs in time $\tilde{O}(n^2d + (d/n)^{k^2})$ and achieves a total variation error of $\tilde{O}((d/n)^{1/4})$. The running time was further improved in [Li and Schmidt, 2017], which achieves the rate $\tilde{O}((d/n)^{1/6})$ for 2-GM.

The above-mentioned works all focus on finite mixtures, which is also the scenario considered in this paper. A related strain of research (e.g., [Genovese and Wasserman, 2000, Ghosal and van der Vaart, 2001, Zhang, 2009, Saha and Guntuboyina, 2017]) studies the so-called *nonparametric mixture model*, in which the mixing distribution Γ may be an arbitrary probability measure.

In this case, the nonparametric maximum likelihood estimator (known as the NPMLE) entails solving a convex (but infinite-dimensional) optimization problem, which, in principle, can be solved by discretization [Koenker and Mizera, 2014]. For statistical rates, it is known that in one dimension, the optimal L_2 -rate for density estimation is $\Theta((\log n)^{1/4}/\sqrt{n})$ and the Hellinger rate is at least $\Omega(\sqrt{\log n/n})$ [Ibragimov, 2001, Kim, 2014], which shows that the parametric rate (2.1.5) is only achievable for finite mixture models. For the NPMLE, [Zhang, 2009] proved the Hellinger rate of $O(\log n/\sqrt{n})$ in one dimension; this was extended to the multivariate case by [Saha and Guntuboyina, 2017]. In particular, [Saha and Guntuboyina, 2017, Theorem 2.3] obtained a Hellinger rate of $C_d\sqrt{k(\log n)^{d+1}/n}$ for the NPMLE when the true model is a k -GM. In high dimensions, this is highly suboptimal compared to the parametric rate in (2.1.5), although the dependency on k is optimal.

Theorem 2.1.2 (Density estimation). *Let P_Γ be as in (2.1.1). Then the minimax*

risk of estimating P_Γ over the class $\mathcal{P}_{k,d}$ satisfies:

$$\inf_{\hat{P}} \sup_{\Gamma \in \mathcal{G}_{k,d}} \mathbb{E}_\Gamma H(\hat{P}, P_\Gamma) \asymp_k \left(\frac{d}{n} \right)^{1/2} \wedge 1. \quad (2.1.5)$$

Furthermore, there exists a proper density estimate $P_{\hat{\Gamma}}$ and a positive constant C_k , such that for any $\Gamma \in \mathcal{G}_{k,d}$ and any $0 < \delta < \frac{1}{2}$, with probability at least $1 - \delta$,

$$H(P_{\hat{\Gamma}}, P_\Gamma) \leq C_k \left(\frac{d \log(1/\delta)}{n} \right)^{1/2}. \quad (2.1.6)$$

The sharp parametric rate for density estimation follows from local cover number calculations, i.e., to achieve (2.1.5), it is necessary to show that a covering number on a radius- δ ball in Hellinger distance on the densities has size no more than $(\delta/\epsilon)^{dC_k}$. To achieve this for a problem parametrized by some θ in a metric space Θ , it is necessary to show that $H(P_\theta, P_{\theta_*}) \asymp \rho(\theta, \theta_*)$, where ρ is the relevant metric on the parameter space.

Previous work, e.g., [Ghosal and van der Vaart, 2001] and [Ho and Nguyen, 2016], considered cover numbers in the atoms and weights space, but this results in global, rather than local, cover numbers because closeness in Hellinger distance between two mixture densities of the form (2.1.1) does not imply closeness between the atoms and weights when the atoms and weights may be arbitrarily small; consider, for example, the distribution on \mathbb{R} : $\epsilon \delta_1 + (1 - \epsilon) \delta_\epsilon$. That is, arguing in the locations and weights space results in upper bounds on the Hellinger distance but not matching lower bounds. For [Ghosal and van der Vaart, 2001], a global entropy could not be avoided since they are in essentially a nonparametric setting, but for our setting, sharper tools are available. In the proof of Theorem 2.1.2, we eschew analyzing cover numbers in the atoms and weights space in favor of covering numbers in the moment space, which turns out to be the most relevant parameter space.

It should be noted that the use of moment tensors in Section 2.3 is theoretic-

cal; we use the moment tensors as a tool to obtain a packing number bound on a small Hellinger ball. Moment tensors have also been used in algorithms for parameter estimation in Gaussian mixtures; in particular, see [Hsu and Kakade, 2013] and [Anandkumar et al., 2014]. Now [Hsu and Kakade, 2013, Theorem 3] translate their tensor-based algorithm into a finite-sample bound for the error in location estimation. Their rates require separation assumptions on the model Γ and are not optimal in general. As shown in [Diakonikolas et al., 2017, Question 1.1], a lower bound on computational time for density estimation is $\text{poly}(n, k)$, and this has not been achieved by an proper estimator that also achieves the correct rate of convergence.

Finally, we note that the rate in Theorem 2.1.2 was achieved for $d = 1$ in [Wu and Yang, 2019]. However, the proof therein relates the Hellinger distance to the moments distance, and the concentration of the moments in that work has a non-optimal dependence on k ; in fact, it is exponential in k . The result in Theorem 2.1.2, when achieved for $d = 1$, and likewise the one-dimensional MLE result in Lemma 5.5.1, achieve optimal dependence on k .

2.2 Mixing distribution estimation

In this section we present the algorithm that achieves the optimal rate for estimating the mixing distribution in Theorem 2.1.1. The procedure is described in Sections 2.2.1 and 2.2.2. The proof of correctness is given in Sections 2.2.3 and 2.2.4.

2.2.1 Dimension reduction via PCA

Recall that $V \in S^{d \times k}$ is the matrix whose columns form an orthonormal basis for the subspace spanned by the atoms of Γ . An optimal procedure is to first project the data onto a subspace that is “close enough” to the space spanned by the v_j , then to

estimate the mixing distribution γ using the low-dimensional data. We would like to project the data onto the subspace spanned by the columns of V , but since we cannot access it, we use the noisy version. For simplicity, consider a sample of $2n$ observations $X_1, \dots, X_{2n} \stackrel{\text{i.i.d.}}{\sim} P_\Gamma$. We construct an estimator $\hat{\Gamma}$ of Γ in the following way:

- (a) Estimate the subspace V using the first half of the sample. Given $\{X_1, \dots, X_n\}$, let

$$\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n X_i X_i^\top - I_d. \quad (2.2.1)$$

Let $\hat{V} = [\hat{v}_1, \dots, \hat{v}_k] \in \mathbb{R}^{d \times k}$ be the matrix whose columns are the top k orthonormal eigenvectors of $\hat{\Sigma}$.

- (b) Project the second half of the sample onto the learned subspace \hat{V} :

$$x_i \triangleq \hat{V}^\top X_{i+n}, \quad i = 1, \dots, n. \quad (2.2.2)$$

Thanks to independence, conditioned on \hat{V} , x_1, \dots, x_n are iid samples from a k -GM in k dimensions, with mixing distribution

$$\gamma \triangleq \Gamma_{\hat{V}} = \sum_{j=1}^k w_j \delta_{\hat{V}^\top \mu_j} \quad (2.2.3)$$

obtained by projecting the original d -dimensional mixing distribution Γ onto \hat{V} .

- (c) To estimate $\hat{\gamma}$, we apply a multivariate version of the denoised method of moments to x_1, \dots, x_n to obtain a k -atomic distribution on \mathbb{R}^k :

$$\hat{\gamma} = \sum_{j=1}^k \hat{w}_j \delta_{\hat{\psi}_j}. \quad (2.2.4)$$

This procedure is explained next and detailed in Algorithm 1.

(d) Lastly, we report

$$\hat{\Gamma} = \hat{\gamma}_{\hat{V}^\top} = \sum_{j=1}^k \hat{w}_j \delta_{\hat{V} \hat{\psi}_j} \quad (2.2.5)$$

as the final estimate of Γ .

Alternatively, we could achieve a slightly better dimension reduction by first centering the data by subtracting the sample mean, then projecting to a subspace of dimension $k - 1$ rather than k . We would then add back the sample mean after obtaining the final estimator. To simplify the presentation, we forgo the centering step.

2.2.2 Estimating a mixing distribution in low dimensions

We now explain how we estimate a k -GM in k dimensions from i.i.d. observations. As mentioned in Section 6.1, the idea is to use many projections to reduce the problem to one dimension. We first present a conceptually simple estimator $\hat{\gamma}^\circ$ with an optimal statistical performance but unfavorable run time $n^{O(k)}$. We then describe an improved estimator $\hat{\gamma}$ that retains the statistical optimality and can be executed in time $n^{O(1)}$.

To make precise the reduction to one dimension, a relevant metric is the *sliced Wasserstein distance* [Rabin et al., 2011], which measures the distance of two d -dimensional distributions by the maximal W_1 -distance of their one-dimensional projections:

$$W_1^{\text{sliced}}(\Gamma, \Gamma') \triangleq \sup_{\theta \in S^{d-1}} W_1(\Gamma_\theta, \Gamma'_\theta). \quad (2.2.6)$$

Here we recall that Γ_θ defined in (1.1.1) denotes the projection, or pushforward, of Γ onto the direction θ . Computing the sliced Wasserstein distance can be difficult and in practice is handled by gradient descent heuristics [Rabin et al., 2011]; we will, however, only rely on its theoretical properties. The following result, which is proved in Section 2.2.4, shows that for low-dimensional distributions with few atoms, the full Wasserstein distance and the sliced one are comparable up to constant factors.

Lemma 2.2.1 (Sliced Wasserstein distance). *For any k -atomic distributions Γ, Γ' on \mathbb{R}^d ,*

$$W_1^{\text{sliced}}(\Gamma, \Gamma') \leq W_1(\Gamma, \Gamma') \leq k^2 \sqrt{d} \cdot W_1^{\text{sliced}}(\Gamma, \Gamma').$$

Having obtained via PCA the reduced samples $x_1, \dots, x_n \sim \gamma * N(0, I_k)$ in (2.2.2), Lemma 2.2.1 suggests the following “meta-procedure”: Suppose we have an algorithm (call it a 1-D algorithm) that estimates the mixing distribution based on n i.i.d. observations drawn from a k -GM in one dimension. Then

1. For each $\theta \in S^{k-1}$, since $\langle \theta, x_i \rangle \stackrel{\text{i.i.d.}}{\sim} \gamma_\theta * N(0, 1)$, we can apply the 1-D algorithm to obtain an estimate $\hat{\gamma}_\theta \in \mathcal{G}_{k,1}$;
2. We obtain an estimate of the multivariate distribution by minimizing a proxy of the sliced Wasserstein distance:

$$\hat{\gamma}^\circ = \operatorname{argmin}_{\gamma' \in \mathcal{G}_{k,k}} \sup_{\theta \in S^{k-1}} W_1(\gamma'_\theta, \hat{\gamma}_\theta). \quad (2.2.7)$$

Then by Lemma 2.2.1 (with $d = k$) and the optimality of $\hat{\gamma}^\circ$, we have

$$\begin{aligned} W_1(\hat{\gamma}^\circ, \gamma) &\lesssim_k \sup_{\theta \in S^{k-1}} W_1(\hat{\gamma}_\theta^\circ, \gamma_\theta) \\ &\leq \sup_{\theta \in S^{k-1}} W_1(\hat{\gamma}_\theta, \gamma_\theta) + \sup_{\theta \in S^{k-1}} W_1(\hat{\gamma}_\theta, \hat{\gamma}_\theta^\circ) \\ &\leq 2 \sup_{\theta \in S^{k-1}} W_1(\hat{\gamma}_\theta, \gamma_\theta). \end{aligned} \quad (2.2.8)$$

Recall that the optimal W_1 -rate for k -atomic one-dimensional mixing distribution is $O(n^{-\frac{1}{4k-2}})$. Suppose there is a 1-D algorithm that achieves the optimal rate *simultaneously* for all projections, in the sense that

$$\mathbb{E} \left[\sup_{\theta \in S^{k-1}} W_1(\hat{\gamma}_\theta, \gamma_\theta) \right] \lesssim_k n^{-\frac{1}{4k-2}}. \quad (2.2.9)$$

This immediately implies the desired

$$\mathbb{E}[W_1(\hat{\gamma}^\circ, \gamma)] \lesssim_k n^{-\frac{1}{4k-2}}. \quad (2.2.10)$$

However, it is unclear how to solve the min-max problem in (2.2.7) where the feasible sets for γ and θ are both non-convex. The remaining tasks are two-fold: (a) provide a 1-D algorithm that achieves (2.2.9); (b) replace $\hat{\gamma}^\circ$ by a computationally feasible version.

Achieving (2.2.9) by denoised method of moments In principle, any estimator for a one-dimensional mixing distribution with exponential concentration can be used as a black box; this achieves (2.2.9) up to logarithmic factors by a standard covering and union bound argument. In order to attain the sharp rate in (2.2.9), we consider the Denoised Method of Moments (DMM) algorithm introduced in [Wu and Yang, 2019], which allows us to use the chaining technique to obtain a tight control of the fluctuation over the sphere.

The DMM method is an optimization-based approach that introduces a denoising step before solving the method of moments equations. For location mixtures, it provides an exact solver to the non-convex optimization problem arising in generalized method of moments [Hansen, 1982]. For Gaussian location mixtures with unit variance, the DMM algorithm proceeds as follows:

- (a) Given $Y_1, \dots, Y_n \stackrel{\text{i.i.d.}}{\sim} \nu * N(0, 1)$ for some k -atomic distribution ν supported on $[-R, R]$, we first estimate the moment vector $\mathbf{m}_{2k-1}(\nu) \triangleq (m_1(\nu), \dots, m_{2k-1}(\nu))$ by their unique unbiased estimator $\tilde{m} = (\tilde{m}_1, \dots, \tilde{m}_{2k-1})$, where $\tilde{m}_r = \frac{1}{n} \sum_{i=1}^n H_r(Y_i)$. Then $\mathbb{E}[\tilde{m}_r] = m_r(\nu)$ for all r . This step is common to all approaches based on the method of moments.
- (b) In general the unbiased estimate \tilde{m} is not a valid moment vector, in which case

the method-of-moment-equation lacks a meaningful solution. The key idea of the DMM method is to denoise \tilde{m} by its projection onto the space of moments:

$$\hat{m} \triangleq \operatorname{argmin}\{\|\tilde{m} - m\| : m \in \mathcal{M}_r\}, \quad (2.2.11)$$

where the moment space

$$\mathcal{M}_r \triangleq \{\mathbf{m}_r(\pi) : \pi \text{ supported on } [-R, R]\} \quad (2.2.12)$$

consists of the first r moments of all probability measures on $[-R, R]$. The moment space is a convex set and characterized by positive semidefinite constraints (of the associated Hankel matrix); we refer the reader to the monograph [Shohat and Tamarkin, 1943] or [Wu and Yang, 2019, Sec. 2.1] for details. This means that the optimization problem (2.2.11) can be solved efficiently as a semidefinite program (SDP); see [Wu and Yang, 2019, Algorithm 1].

- (c) Use Gauss quadrature to find the unique k -atomic distribution $\hat{\nu}$ such that $\mathbf{m}_{2k-1}(\hat{\nu}) = \hat{m}$. We denote the final output $\hat{\nu}$ by $\text{DMM}(Y_1, \dots, Y_n)$.

The following result shows the DMM estimator achieves the optimal rate in (2.2.9) simultaneously for all one-dimensional projections (for a single θ , this is shown in [Wu and Yang, 2019, Theorem 1]):

Lemma 2.2.2. *For each $\theta \in S^{k-1}$, let $\hat{\gamma}_\theta = \text{DMM}(\langle \theta, x_1 \rangle, \dots, \langle \theta, x_n \rangle)$ where $x_1, \dots, x_n \stackrel{i.i.d.}{\sim} \gamma * N(0, I_k)$ as in (2.2.2). There is a positive constant C_k such that, for any $\delta \in (0, \frac{1}{2})$, with probability at least $1 - \delta$,*

$$\max_{\theta \in S^{k-1}} W_1(\hat{\gamma}_\theta, \gamma_\theta) \leq C_k n^{-1/(4k-2)} \sqrt{\log \frac{1}{\delta}}.$$

Solving (2.2.7) efficiently using marginal estimates We first note that in order to achieve the optimal rate in (2.2.10), it is sufficient to consider any approximate minimizer of (2.2.7) up to an additive error of ϵ , as long as $\epsilon = O(n^{-\frac{1}{4k-2}})$. Therefore, to find an ϵ -optimizer, it suffices to maximize over θ in an ϵ -net (in ℓ_2) of the sphere, which has cardinality $(\frac{1}{\epsilon})^k = n^{O(1)}$, and, likewise, minimize γ over an ϵ -net (in W_1) of $\mathcal{G}_{k,k}$. The W_1 -net can be constructed by combining an ϵ -net (in ℓ_2) for each of the k centers and an ϵ -net (in ℓ_1) for the weights, resulting in a set of cardinality $(\frac{1}{\epsilon})^{O(k^2)} = n^{O(k)}$. This naïve discretization scheme leads to an estimator of γ with optimal rate but time complexity $n^{O(k)}$. We next improve it to $n^{O(1)}$.

The key idea is to first estimate the marginals of γ , which narrows down its support set. It is clear that a k -atomic joint distribution is not determined by its marginal distributions, as shown by the example of $\frac{1}{2}\delta_{(-1,-1)} + \frac{1}{2}\delta_{(1,1)}$ and $\frac{1}{2}\delta_{(-1,1)} + \frac{1}{2}\delta_{(1,-1)}$, which have identical marginal distributions. Nevertheless, the support of the joint distribution must be a k -subset of the Cartesian product of the marginal support sets. This suggests that we can select the atoms from this Cartesian product and weights by fitting all one-dimensional projections, as in (2.2.7).

Specifically, for each $j \in [k]$, we estimate the j th marginal distribution of γ by $\hat{\gamma}_j$, obtained by applying the DMM algorithm on the coordinate projections $\langle e_j, x_1 \rangle, \dots, \langle e_j, x_n \rangle$. Consider the Cartesian product of the support of each estimated marginal as the candidate set of atoms:

$$\mathcal{A} \triangleq \text{supp}(\hat{\gamma}_1) \times \dots \times \text{supp}(\hat{\gamma}_k).$$

Throughout this section, let

$$\epsilon_{n,k} \triangleq n^{-\frac{1}{4k-2}},$$

and fix an $(\epsilon_{n,k}, \|\cdot\|_2)$ -covering \mathcal{N} for the unit sphere S^{k-1} and an $(\epsilon_{n,k}, \|\cdot\|_1)$ -covering

\mathcal{W} for the probability simplex Δ^{k-1} , such that²

$$\max\{|\mathcal{N}|, |\mathcal{W}|\} \lesssim \left(\frac{C}{\epsilon_{n,k}}\right)^{k-1}. \quad (2.2.13)$$

Define the following set of candidate k -atomic distributions on \mathbb{R}^k :

$$\mathcal{S} \triangleq \left\{ \sum_{j \in [k]} w_j \delta_{\psi_j} : (w_1, \dots, w_k) \in \mathcal{W}, \psi_j \in \mathcal{A} \right\}. \quad (2.2.14)$$

Note that \mathcal{S} is a random set which depends on the sample; furthermore, each $\psi_j \in \mathcal{A}$ has coordinates lying in $[-R, R]$ by virtue of the DMM algorithm.

The next lemma shows that with high probability there exists a good approximation of γ in the set \mathcal{S} .

Lemma 2.2.3. *Let \mathcal{S} be given in (2.2.14). There is a positive constant C_k such that, for any $\delta \in (0, \frac{1}{2})$, with probability $1 - \delta$,*

$$\min_{\gamma' \in \mathcal{S}} W_1(\gamma', \gamma) \leq C_k n^{-1/(4k-2)} \sqrt{\log \frac{1}{\delta}}. \quad (2.2.15)$$

We conclude this subsection with Algorithm 1, which provides a full description of an estimator for k -atomic mixing distributions in k dimensions. The following result shows its optimality under the W_1 loss:

Lemma 2.2.4. *There is a positive constant C_k such that the following holds. Let $x_1, \dots, x_n \stackrel{i.i.d.}{\sim} \gamma * N(0, I_k)$ for some $\gamma \in G_{k,k}$. Then Algorithm 1 produces an estimator $\hat{\gamma} \in \mathcal{G}_{k,k}$ such that, for any $\delta \in (0, \frac{1}{2})$, with probability $1 - \delta$,*

$$W_1(\gamma, \hat{\gamma}) \leq C_k n^{-1/(4k-2)} \sqrt{\log \frac{1}{\delta}}. \quad (2.2.16)$$

²This is possible by, e.g., [Rudelson and Vershynin, 2009, Prop. 2.1] and [Ghosal and van der Vaart, 2001, Lemma A.4] for the sphere and probability simplex, respectively.

Algorithm 1: Parameter estimation for k -GM in k dimensions

Input: Dataset $\{x_i\}_{i \in [n]}$ with each point in \mathbb{R}^k , order k , radius R .

Output: Estimate $\hat{\gamma}$ of k -atomic distribution in k dimensions.

For $j = 1, \dots, k$:

 Compute the marginal estimate $\hat{\gamma}_j = \text{DMM}(\{e_j^\top x_i\}_{i \in [n]})$;

 Form the set \mathcal{S} of k -atomic candidate distributions on \mathbb{R}^k as in (2.2.14) ;

For each $\theta \in \mathcal{N}$:

 Estimate the projection by $\hat{\gamma}_\theta = \text{DMM}(\{\theta^\top x_i\}_{i \in [n]})$;

For each candidate distribution $\gamma' \in \mathcal{S}$ **and each direction** $\theta \in \mathcal{N}$:

 Compute $W_1(\gamma'_\theta, \hat{\gamma}_\theta)$;

 Report

$$\hat{\gamma} = \arg \min_{\gamma' \in \mathcal{S}} \max_{\theta \in \mathcal{N}} W_1(\gamma'_\theta, \hat{\gamma}_\theta). \quad (2.2.17)$$

Remark 1. The total time complexity to compute the estimator (2.2.5) is $O(nd^2) + O_k(n^{5/4})$. Indeed, the time complexity of computing the sample covariance matrix is $O(nd^2)$, and the time complexity of performing the eigendecomposition is $O(d^3)$, which is dominated by $O(nd^2)$ since $d \leq n$. By (2.2.13), both \mathcal{W} and \mathcal{N} have cardinality at most $(C/\epsilon_{n,k})^{k-1} = O_k(n^{1/4})$. Each one-dimensional DMM estimate takes $O_k(n)$ time to compute [Wu and Yang, 2019, Theorem 1]. Thus computing the one-dimensional estimator $\hat{\gamma}_\theta$ for all $\theta = e_i$ and $\theta \in \mathcal{N}$ takes time $O_k(n^{5/4})$. Since both γ'_θ and $\hat{\gamma}_\theta$ are k -atomic distributions by definition, their W_1 distance can be computed in $O_k(1)$ time. Finally, since $|\mathcal{A}| \leq k^k$ and $|\mathcal{S}| \leq |\mathcal{W}||\mathcal{A}| = O_k(n^{1/4})$, searching over $\mathcal{S} \times \mathcal{N}$ therefore takes time at most $O_k(n^{1/4}) * O_k(n^{1/4}) = O_k(n^{1/2})$. Therefore, the overall time complexity of Algorithm 1 is $O_k(n^{5/4})$.

2.2.3 Proof of the main theorem

The proof is outlined as follows. Recall that the estimate $\hat{\Gamma}$ in (2.2.5) is supported on the subspace spanned by the columns of \hat{V} , whose projection is $\hat{\gamma}$ in (2.2.4). Similarly, the projection of the ground truth Γ on the space \hat{V} is denoted by $\gamma = \Gamma_{\hat{V}}$ in (2.2.3). Note that both γ and $\hat{\gamma}$ are k -atomic distributions in k dimensions. Let $\hat{H} = \hat{V}\hat{V}^\top$ be the projection matrix onto the space spanned by the columns of \hat{V} . By the triangle inequality,

$$\begin{aligned} W_1(\Gamma, \hat{\Gamma}) &\leq W_1(\Gamma, \Gamma_{\hat{H}}) + W_1(\Gamma_{\hat{H}}, \hat{\Gamma}) \\ &\leq W_1(\Gamma, \Gamma_{\hat{H}}) + W_1(\gamma, \hat{\gamma}). \end{aligned} \quad (2.2.18)$$

We will upper bound the first term by $(d/n)^{1/4}$ (using Lemmas 2.2.5 and 2.2.6 below) and the second term by $n^{-1/(4k-2)}$ (using the previous Lemma 2.2.4).

We first control the difference between Γ and its projection onto the estimated subspace \hat{V} . Since we do not impose any lower bound on $\|\mu_j\|_2$, we cannot directly show the accuracy of \hat{V} by means of perturbation bounds such as the Davis-Kahan theorem [Davis and Kahan, 1970]. Instead, the following general lemma bounds the error by the difference of the covariance matrices.

Lemma 2.2.5. *Let $\Gamma = \sum_{j=1}^k w_j \delta_{\mu_j}$ be a k -atomic distribution. Let $\Sigma = \mathbb{E}_{U \sim \Gamma}[UU^\top] = \sum_{j=1}^k w_j \mu_j \mu_j^\top$ with eigenvalues $\lambda_1 \geq \dots \geq \lambda_d$. Let Σ' be a symmetric matrix and H'_r be the projection matrix onto the subspace spanned by the top r eigenvectors of Σ' . Then*

$$W_1^2(\Gamma, \Gamma_{H'_r}) \leq k(\lambda_{r+1} + 2\|\Sigma - \Sigma'\|_2).$$

And if $r = k$, then $\lambda_{r+1}, \dots, \lambda_d = 0$ since Σ has rank at most k , so

$$W_1^2(\Gamma, \Gamma_{H'_r}) \leq 2k\|\Sigma - \Sigma'\|_2.$$

We will apply Lemma 2.2.5 with Σ' being the sample covariance matrix $\hat{\Sigma}$. The following lemma provides the concentration of $\hat{\Sigma}$ we need to prove the upper bound on the high-dimensional component of the error in Theorem 2.1.1.

Lemma 2.2.6. *Let $\Gamma \in \mathcal{G}_{k,d}$ and $\Sigma = \mathbb{E}_{U \sim \Gamma}[UU^\top]$. Let $\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n X_i X_i^\top - I_d$, where $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_\Gamma$. Then there exists a positive constant C such that, with probability at least $1 - \delta$,*

$$\|\hat{\Sigma} - \Sigma\|_2 \leq C \left(\sqrt{\frac{d}{n}} + k \sqrt{\frac{\log(k/\delta)}{n}} + \frac{\log(1/\delta)}{n} \right).$$

Proof of Theorem 2.1.1. We first show that the estimator (2.2.5) achieves the tail bound stated in (2.1.4), which, after integration, implies the average risk bound in (2.1.3). To bound the first term in (2.2.18), note that the rank of $\Sigma = \mathbb{E}_{U \sim \Gamma}[UU^\top]$ is at most k . Applying Lemmas 2.2.5 and 2.2.6 yields that, with probability $1 - \delta$,

$$W_1(\Gamma, \Gamma_{\hat{H}}) \leq \sqrt{2Ck} \left(\left(\frac{d}{n} \right)^{1/4} + \left(\frac{k^2 \log(k/\delta)}{n} \right)^{1/4} + \sqrt{\frac{\log(1/\delta)}{n}} \right), \quad (2.2.19)$$

where we used the fact that $W_1(\Gamma, \Gamma') \leq W_2(\Gamma, \Gamma')$ by the Cauchy-Schwarz inequality. To upper bound the second term in (2.2.18), recall that \hat{V} was obtained from $\{X_1, \dots, X_n\}$ and hence is independent of $\{X_{n+1}, \dots, X_{2n}\}$. Thus conditioned on \hat{V} ,

$$x_i = \hat{V}^\top X_{i+n} \stackrel{i.i.d.}{\sim} \gamma * N(0, I_k), \quad i = 1, \dots, n.$$

Let $\hat{\gamma}$ be obtained from Algorithm 1 with input x_1, \dots, x_n . By Lemma 2.2.4, with probability $1 - \delta$,

$$W_1(\gamma, \hat{\gamma}) \leq C_k n^{-1/(4k-2)} \sqrt{\log \frac{1}{\delta}}. \quad (2.2.20)$$

Note that $(k^2 \log(k/\delta)/n)^{1/4} + (\log(1/\delta)/n)^{1/2}$ in (2.2.19) is dominated by $\epsilon = 2C_k n^{-1/(4k-2)} \sqrt{\log \frac{1}{\delta}}$. The desired (2.1.4) follows from combining (2.2.18), (2.2.19), and (2.2.20).

Finally, the lower bound in (2.1.3) is obtained by combining the $\Omega((d/n)^{1/4} \wedge 1)$ lower bound in [Wu and Zhou, 2019, Theorem 10] for the special case of d -dimensional symmetric 2-GM and the $\Omega(n^{-1/(4k-2)})$ lower bound in [Wu and Yang, 2019, Proposition 7] for 1-dimensional k -GM. \square

2.2.4 Proofs of supporting lemmas

In this subsection we prove Lemmas 2.2.1–2.2.6.

Proof of Lemma 2.2.1. For the lower bound, simply note that for any $\theta \in S^{d-1}$, $\|U - U'\|_2 \geq |\theta^\top U - \theta^\top U'|$ by the Cauchy-Schwartz inequality. Taking expectations on both sides with respect to the optimal W_1 -coupling $\mathcal{L}(U, U')$ of Γ and Γ' yields the lower bound.

For the upper bound, we show that there exists $\theta \in S^{d-1}$ that satisfies the following properties:

1. The projection $y \mapsto \theta^\top y$ is injective on $\text{supp}(\Gamma) \cup \text{supp}(\Gamma')$;
2. For all $y \in \text{supp}(\Gamma)$ and $y' \in \text{supp}(\Gamma')$, we have

$$\|y - y'\|_2 \leq k^2 \sqrt{d} |\theta^\top y - \theta^\top y'|. \quad (2.2.21)$$

This can be done by a simple probabilistic argument. Let θ be drawn from the uniform distribution on S^{d-1} , which fulfills the first property with probability one. Next, for any fixed $x \in \mathbb{R}^d$, we have

$$\mathbb{P}\{|\theta^\top x| < t \|x\|_2\} \leq \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{\Gamma((d-1)/2)}{2\pi^{((d-1)/2)}} \int_{-t}^t (1 - u^2)^{(d-3)/2} du < t\sqrt{d}.$$

Let $\mathcal{X} = \{y - y' : y \in \text{supp}(\Gamma), y' \in \text{supp}(\Gamma')\}$, whose cardinality is at most k^2 . By a union bound,

$$\mathbb{P}\{\exists x \in \mathcal{X} \text{ s.t. } |\theta^\top x| < t \|x\|_2\} < k^2 t \sqrt{d},$$

and thus

$$\mathbb{P}\{|\theta^\top x| \geq t\|x\|_2, \forall x \in \mathcal{X}\} > 1 - k^2 t \sqrt{d}.$$

This probability is strictly positive for $t = 1/(k^2 \sqrt{d})$. Thus, there exists $\theta \in S^{d-1}$ such that (2.2.21) holds. Since $\langle \theta, \cdot \rangle$ is injective on the support of Γ and Γ' , denote its inverse by $g : \mathbb{R} \rightarrow \text{supp}(\Gamma) \cup \text{supp}(\Gamma')$. Then any coupling of the pushforward measures Γ_θ and Γ'_θ gives rise to a coupling of Γ and Γ' in the sense that if $\mathcal{L}(V, V')$ is a coupling of Γ_θ and Γ'_θ then $\mathcal{L}(g(V), g(V'))$ is a coupling of Γ and Γ' . By (2.2.21), we have

$$\|g(V) - g(V')\|_2 \leq k^2 \sqrt{d} |V - V'|.$$

Taking expectations of both sides with respect to $\mathcal{L}(V, V')$ being the optimal W_1 -coupling of Γ_θ and Γ'_θ yields the desired upper bound. \square

Next we prove Lemma 2.2.2. Note that a simple union bound here would lead to a rate of $(\log n/n)^{1/(4k-2)}$. To remove the unnecessary logarithmic factors, we use the chaining technique (see the general result in Lemma 2.5.4), which entails proving the concentration of the increments of a certain empirical process.

Note that the sub-Gaussian type concentration in [Wu and Yang, 2019] was proved using a variance bound and then the so-called “median trick” and Chebychev. The median trick works as follows. We split the data into T samples, $\{x_t\}_{t \in [T]}$. We run the algorithm on each sample and take our estimator to be the median of these estimators; we use facts about the concentration of the binomial distribution to show concentration of the resulting median estimator. For the chaining technique, we need to have sub-Gaussian type concentration of the increments $|f_r(\theta_1, x) - f_r(\theta_2, x)|$. We can indeed apply the median trick to the increments $|f_r(\theta_1, x_t) - f_r(\theta_2, x_t)|$ and obtain sub-Gaussian type concentration of the increments. But $\text{med}_{t \in [T]} |f_r(\theta_1, x_t) - f_r(\theta_2, x_t)| \neq \text{med}(f_r(\theta_1, x_t)) - \text{med}(f_r(\theta_2, x_t))$. That is, obtaining sub-Gaussian type concentration of the median of the increments would not result in a statement about concentration

of the actual process $\text{med}_{t \in [T]} f_r(\theta, x_t)$ that we use in our estimation. We instead rely on some concentration results for polynomials from the literature.

Proof of Lemma 2.2.2. By the continuity of $\theta \mapsto W_1(\hat{\gamma}_\theta, \gamma_\theta)$ and the monotone convergence theorem, it suffices to show that there exists a constant C_k such that, for any finite subset $\Theta \subset S^{k-1}$,

$$\mathbb{P} \left[\max_{\theta \in \Theta} W_1(\hat{\gamma}_\theta, \gamma_\theta) \leq C_k n^{-1/(4k-2)} \sqrt{\log \frac{1}{\delta}} \right] \geq 1 - \delta. \quad (2.2.22)$$

Throughout the proof, C_k stands for a constant depending only on k whose value may vary from line to line.

Recall that $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} P_\gamma$, where $\gamma \in \mathcal{G}_{k,k}$. Define the empirical process

$$\tilde{m}_r(\theta) \triangleq \frac{1}{n} \sum_{i=1}^n H_r(\theta^\top X_i),$$

where H_r is the degree- r Hermite polynomial defined in (1.2.5). Define the centered random process indexed by θ :

$$f_r(\theta) \triangleq \sqrt{n} (\tilde{m}_r(\theta) - \mathbb{E} \tilde{m}_r(\theta)) = \sqrt{n} (\tilde{m}_r(\theta) - m_r(\gamma_\theta)). \quad (2.2.23)$$

Let $r \in [2k-1]$. By Lemma 2.5.3, there are positive constants C, c_k such that $\mathbb{P}\{|f_r(\theta_1) - f_r(\theta_2)| \geq \|\theta_1 - \theta_2\|_2 \lambda\} \leq C \exp(-c_k \lambda^{2/r})$. So we can apply Lemma 2.5.4 (with $\Theta \subseteq S^{k-1}$, $\rho(\theta_1, \theta_2) = \|\theta_1 - \theta_2\|_2$, $\epsilon_0 = 2$, and $\alpha = 2/r$). Note that the maximal ϵ -packing of Θ has size $M(\epsilon, S^{k-1}, \|\cdot\|_2) \leq (4/\epsilon)^k$. Fix $\theta_0 \in \Theta$. Then, by Lemma 2.5.4, with probability $1 - C \exp(-c_k t^{2/r})$,

$$\max_{\theta \in \Theta} |f_r(\theta) - f_r(\theta_0)| \leq C_k \left(t + \int_0^1 (\log(1/u))^{r/2} du \right) = C_k \left(t + \Gamma \left(1 + \frac{r}{2} \right) \right).$$

By (2.5.2) in Lemma 2.5.3, $|f_r(\theta_0)| \leq C_k t$ with probability $1 - C \exp(-c_k t^{2/r})$. There-

fore, with probability $1 - \frac{\delta}{2k-1}$,

$$\max_{\theta \in \Theta} |\tilde{m}_r(\theta) - m_r(\gamma_\theta)| = \frac{1}{\sqrt{n}} \max_{\theta \in \Theta} |f_r(\theta)| \leq C_k \frac{(\log(C_k/\delta))^{\frac{2k-1}{2}}}{\sqrt{n}}.$$

We take a union bound over $r \in [2k-1]$ and obtain that, with probability $1 - \delta$,

$$\max_{\theta \in \Theta, r \in [2k-1]} |\tilde{m}_r(\theta) - m_r(\gamma_\theta)| \leq C_k \frac{(\log(C_k/\delta))^{\frac{2k-1}{2}}}{\sqrt{n}}. \quad (2.2.24)$$

Recall that for each θ , the DMM estimator results in a k -atomic distribution $\hat{\gamma}_\theta$, such that $m_r(\hat{\gamma}_\theta) = \hat{m}_r(\theta)$ for all $r = 1, \dots, 2k-1$, where $(\hat{m}_1(\theta), \dots, \hat{m}_{2k-1}(\theta))$ is the Euclidean projection of $\tilde{m}(\theta) = (\tilde{m}_1(\theta), \dots, \tilde{m}_{2k-1}(\theta))$ onto the moment space \mathcal{M}_{2k-1} (see (2.2.12)). Thus,

$$\max_{r \in [2k-1]} |m_r(\hat{\gamma}_\theta) - m_r(\gamma_\theta)| \leq 2\sqrt{2k-1} \max_{r \in [2k-1]} |\tilde{m}_r(\theta) - m_r(\gamma_\theta)|.$$

By the moment comparison inequality in Lemma 2.5.1, we have

$$W_1(\hat{\gamma}_\theta, \gamma_\theta) \lesssim_k \max_{r \in [2k-1]} |\tilde{m}_r(\theta) - m_r(\gamma_\theta)|^{1/(2k-1)}.$$

Finally, maximizing both sides over $\theta \in \Theta$ and applying (2.2.24) yields the desired (2.2.22). \square

Proof of Lemma 2.2.3. By Lemma 2.2.2, there is a positive constant C_k such that, for any $\delta \in (0, \frac{1}{2})$, with probability $1 - \delta$,

$$W_1(\hat{\gamma}_i, \gamma_i) \leq \epsilon \triangleq C_k n^{-1/(4k-2)} \sqrt{\log \frac{1}{\delta}}, \quad \forall i \in [k].$$

Let $\gamma = \sum_{j=1}^k w_j \delta_{\mu_j}$. Fix $j \in [k]$. For any $i \in [k]$, by definition of W_1 distance in

(2.1.2),

$$w_j \cdot \min_{x \in \text{supp}(\hat{\gamma}_i)} |x - e_i^\top \mu_j| \leq W_1(\hat{\gamma}_i, \gamma_i).$$

Thus there exists $\mu_{ji} \in \text{supp}(\hat{\gamma}_i)$ such that

$$w_j |\mu_{ji} - e_i^\top \mu_j| \leq W_1(\hat{\gamma}_i, \gamma_i) \leq \epsilon.$$

Let $\mu'_j = (\mu_{j1}, \dots, \mu_{jk})^\top \in \mathcal{A}$. Then

$$w_j \|\mu'_j - \mu_j\|_2 \leq \sqrt{k} w_j \|\mu'_j - \mu_j\|_\infty \leq \sqrt{k} \epsilon.$$

Since \mathcal{W} is an $n^{-\frac{1}{4k-2}}$ -covering of the probability simplex with respect to $\|\cdot\|_1$, there exists a weights vector $w' = (w'_1, \dots, w'_k) \in \mathcal{W}$ such that $\|w' - w\|_1 \leq \epsilon$.

Consider the distributions $\gamma' \triangleq \sum_{j=1}^k w'_j \delta_{\mu'_j} \in \mathcal{S}$ and $\gamma'' \triangleq \sum_{j=1}^k w_j \delta_{\mu'_j}$. Note that γ and γ'' have the same weights. Using their natural coupling we have $W_1(\gamma, \gamma'') \leq \sum_{j=1}^k w_j \|\mu_j - \mu'_j\|_2$. Note that γ'' and γ' have the same support. Using the total variation coupling (see [Gibbs and Su, 2002, Theorem 4]) of their weights w and w' (and the fact that total variation equals half of the ℓ_1 -distance), we have $W_1(\gamma'', \gamma') \leq R \|w' - w\|_1$. Thus,

$$W_1(\gamma, \gamma') \leq \sum_{j=1}^k w_j \|\mu_j - \mu'_j\|_2 + R \|w' - w\|_1 \leq C_k \epsilon. \quad \square$$

Proof of Lemma 2.2.4. Throughout this proof we use the abbreviation $\epsilon \equiv \epsilon_{n,k}$. First,

$$W_1(\hat{\gamma}, \gamma) \lesssim_k \sup_{\theta \in S^{k-1}} W_1(\hat{\gamma}_\theta, \gamma_\theta) \lesssim_k 2R\epsilon + \max_{\theta \in \mathcal{N}} W_1(\hat{\gamma}_\theta, \gamma_\theta).$$

The first inequality is by the upper bound in Lemma 2.2.1, with $d = k$. The second inequality follows by definition of \mathcal{N} , for any $\theta \in S^{k-1}$, there is a $u \in \mathcal{N}$ such that $\|\theta - u\| \leq \epsilon$. Now each estimated marginal $\hat{\gamma}_j$ is supported on $[-R, R]$ by nature

of the DMM algorithm, so $\hat{\gamma}$ has atoms on the hypercube $[-R, R]^k$, and similarly for γ . By the Cauchy-Schwarz Inequality and the natural coupling between γ_u, γ_θ , $W_1(\gamma_u, \gamma_\theta) \leq \sqrt{k}R\|\theta - u\|_2$. Now let $\gamma' \in \operatorname{argmin}_{\gamma'' \in \mathcal{S}} W_1(\gamma'', \gamma)$. We have

$$\max_{\theta \in \mathcal{N}} W_1(\hat{\gamma}_\theta, \gamma_\theta) \leq \max_{\theta \in \mathcal{N}} W_1(\hat{\gamma}_\theta, \gamma'_\theta) + \max_{\theta \in \mathcal{N}} W_1(\gamma'_\theta, \gamma_\theta)$$

Now for the first term, we have

$$\begin{aligned} \max_{\theta \in \mathcal{N}} W_1(\hat{\gamma}_\theta, \gamma'_\theta) &\leq \max_{\theta \in \mathcal{N}} W_1(\hat{\gamma}_\theta, \hat{\gamma}_\theta) + \max_{\theta \in \mathcal{N}} W_1(\hat{\gamma}_\theta, \gamma'_\theta) \\ &\leq 2 \max_{\theta \in \mathcal{N}} W_1(\hat{\gamma}_\theta, \gamma'_\theta) && \text{by definition of } \hat{\gamma} \\ &\leq 2 \max_{\theta \in \mathcal{N}} W_1(\hat{\gamma}_\theta, \gamma_\theta) + 2 \max_{\theta \in \mathcal{N}} W_1(\gamma_\theta, \gamma'_\theta). \end{aligned}$$

Putting all this together and applying the lower bound in Lemma 2.2.1, we obtain the deterministic bound

$$W_1(\hat{\gamma}, \gamma) \lesssim_k 2R\epsilon + 2 \max_{\theta \in \mathcal{N}} W_1(\hat{\gamma}_\theta, \gamma_\theta) + 3W_1(\gamma', \gamma).$$

Applying Lemma 2.2.2 and Lemma 2.2.3 completes the proof. \square

It remains to show Lemmas 2.2.5 and 2.2.6.

Proof of Lemma 2.2.5. Let \mathcal{V}_r^\perp be the subspace of \mathbb{R}^d that is orthogonal to the space spanned by the top r eigenvectors of Σ' , and let $y_j = \operatorname{argmax}_{x \in \mathcal{V}_r^\perp \cap S^{d-1}} |\mu_j^\top x|$. Then $\|\mu_j - H'_r \mu_j\|_2^2 = (\mu_j^\top y_j)^2$. Furthermore, for each j , $y_j^\top w_j \mu_j \mu_j^\top y_j \leq y_j^\top (\sum_{\ell=1}^k w_\ell \mu_\ell \mu_\ell^\top) y_j = y_j^\top \Sigma y_j$. It remains to bound the latter. Let $\lambda'_1 \geq \dots \geq \lambda'_d$ be the sorted eigenvalues

of Σ' . Now

$$\begin{aligned}
|y_j^\top \Sigma y_j| &\leq |y_j^\top (\Sigma - \Sigma') y_j| + |y_j^\top \Sigma' y_j| \\
&\leq \|\Sigma - \Sigma'\|_2 + \lambda'_{r+1} && \text{since } y_j \in \mathcal{V}_r^\perp \\
&\leq 2\|\Sigma - \Sigma'\|_2 + \lambda_{r+1},
\end{aligned}$$

where the last step follows from Weyl's inequality [Horn and Johnson, 1991]. By the natural coupling between Γ and $\Gamma_{H'_r}$,

$$W_2^2(\Gamma, \Gamma_{H'_r}) \leq \sum_{j=1}^k w_j \|\mu_j - H'_r \mu_j\|_2^2 \leq k(\lambda_{r+1} + 2\|\Sigma - \Sigma'\|_2). \quad \square$$

The result follows since $W_1(\Gamma, \Gamma_{H'_r}) \leq W_2(\Gamma, \Gamma_{H'_r})$ by the Cauchy-Schwarz Inequality.

Proof of Lemma 2.2.6. Write $X_i = U_i + Z_i$ where $U_i \stackrel{\text{i.i.d.}}{\sim} \Gamma$ and $Z_i \stackrel{\text{i.i.d.}}{\sim} N(0, I_d)$ for $i = 1, \dots, n$. Then

$$\hat{\Sigma} - \Sigma = \left(\frac{1}{n} \sum_{i=1}^n U_i U_i^\top - \Sigma \right) + \left(\frac{1}{n} \sum_{i=1}^n Z_i Z_i^\top - I_d \right) + \left(\frac{1}{n} \sum_{i=1}^n U_i Z_i^\top + Z_i U_i^\top \right).$$

We upper bound the spectral norms of three terms separately. For the first term, let $\Gamma = \sum_{j=1}^k w_j \delta_{\mu_j}$ and $\hat{w}_j = \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{U_i = \mu_j\}$. Then $\frac{1}{n} \sum_{i=1}^n U_i U_i^\top = \sum_{j=1}^k \hat{w}_j \mu_j \mu_j^\top$. Therefore, by Hoeffding's inequality and the union bound, with probability $1 - 2ke^{-2t^2}$,

$$\left\| \frac{1}{n} \sum_{i=1}^n U_i U_i^\top - \Sigma \right\|_2 \leq R^2 \sum_{j=1}^k |\hat{w}_j - w_j| \leq \frac{R^2 k t}{\sqrt{n}}. \quad (2.2.25)$$

For the second term, by standard results in random matrix theory (see, e.g., [Vershynin, 2012, Corollary 5.35]), and since $d < n$, there exists a positive constant C such that, with

probability at least $1 - e^{-t^2}$,

$$\left\| \frac{1}{n} \sum_{i=1}^n Z_i Z_i^\top - I_d \right\|_2 \leq C \left(\sqrt{\frac{d}{n}} + \frac{t}{\sqrt{n}} + \frac{t^2}{n} \right). \quad (2.2.26)$$

To bound the third term, let $A = \frac{1}{n} \sum_{i=1}^n U_i Z_i^\top + Z_i U_i^\top$, and \mathcal{N} be an $\frac{1}{4}$ -covering of S^{d-1} of size 2^{cd} for an absolute constant c . Then

$$\|A\|_2 = \max_{\theta \in S^{d-1}} |\theta^\top A \theta| \leq 2 \max_{\theta \in \mathcal{N}} |\theta^\top A \theta| = 4 \max_{\theta \in \mathcal{N}} \left| \frac{1}{n} \sum_{i=1}^n (\theta^\top U_i)(\theta^\top Z_i) \right|.$$

For fixed $\theta \in S^{d-1}$, conditioning on U_i , we have $\sum_{i=1}^n (\theta^\top U_i)(\theta^\top Z_i) \sim N(0, \sum_{i=1}^n (\theta^\top U_i)^2)$. Since $\sum_{i=1}^n (\theta^\top U_i)^2 \leq nR^2$, we have

$$\mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n (\theta^\top U_i)(\theta^\top Z_i) \right| > \frac{R\tau}{\sqrt{n}} \right\} \leq \mathbb{P} \{ |Z_1| \geq \tau \} \leq 2e^{-\frac{\tau^2}{2}}.$$

Therefore, by a union bound, with probability $1 - 2e^{cd - \frac{\tau^2}{2}}$

$$\max_{\theta \in \mathcal{N}} \left| \frac{1}{n} \sum_{i=1}^n (\theta^\top U_i)(\theta^\top Z_i) \right| \leq \frac{R\tau}{\sqrt{n}}.$$

By taking $\tau = C(\sqrt{d} + t)$ for some absolute constant C , we obtain that with probability $1 - e^{-t^2}$,

$$\left\| \frac{1}{n} \sum_{i=1}^n U_i Z_i^\top + Z_i U_i^\top \right\|_2 \leq CR \left(\sqrt{\frac{d}{n}} + \frac{t}{\sqrt{n}} \right). \quad (2.2.27)$$

□

2.3 Density estimation

In this section we prove the density estimation guarantee of Theorem 2.1.2 for finite Gaussian mixtures. Our strategy is to prove a tight upper bound on the local entropy of Hellinger balls for k -GMs. In Lemma 2.3.4 we show that any ϵ -Hellinger ball in $\mathcal{P}_{k,d}$ can be covered by at most $\left(\frac{C\epsilon}{\delta}\right)^{Cd}$ δ -Hellinger balls, where C only depends on k .

This allows us to invoke the construction of Le Cam and Birgé (see [Le Cam, 1973, Birgé, 1983, Birgé, 1986]) to arrive at an estimator that achieves the parametric rate of $O_k(\frac{d}{n})$ in the squared Hellinger loss. We note that the Le Cam-Birgé estimator is a theoretical construction based on (exponentially many) pairwise tests. Finding a computationally efficient proper density estimate that attains the parametric rate in Theorem 2.1.2, or even within logarithmic factors thereof, is open. This problem is much more challenging than estimating the mixing distribution, for which we have constructed a polynomial-time optimal estimator in Section 2.2. In fact, we show in Section 2.3.3, estimation of the mixing distribution can be reduced to proper density estimation both statistically and computationally.

2.3.1 Moment characterization of k -atomic Gaussian mixtures

The following result gives a characterization of statistical distances (squared Hellinger, KL, or χ^2 -divergence) between k -GMs in terms of the moment tensors up to constant factors that do not depend on d .

Theorem 2.3.1 (Moment characterization of statistical distances). *For any pair of k -atomic distributions Γ, Γ' supported on the ball $B(0, R)$ in \mathbb{R}^d , for any $D \in \{H^2, \text{KL}, \chi^2\}$,*

$$c \max_{\ell \leq 2k-1} \|M_\ell(\Gamma) - M_\ell(\Gamma')\|_{\text{F}}^2 \leq D(P_\Gamma, P_{\Gamma'}) \leq C \max_{\ell \leq 2k-1} \|M_\ell(\Gamma) - M_\ell(\Gamma')\|_{\text{F}}^2. \quad (2.3.1)$$

where the constants c, C may depend on k and R but not d .

To prove Theorem 2.3.1 we need a few auxiliary lemmas. The following lemma bounds the difference of higher-order moment tensors of k -atomic distributions using those of the first $2k - 1$ moment tensors. The one-dimensional version was shown

in [Wu and Yang, 2019, Lemma 10] using polynomial interpolation techniques; however, it is hard to extend this proof to multiple dimensions as multivariate polynomial interpolation (on arbitrary points) is much less well-understood. Fortunately, this difficulty can be sidestepped by exploiting the relationship between moment tensor norms and projections in (1.2.15).

Lemma 2.3.2. *Let U, U' be k -atomic random variables in \mathbb{R}^d . Then for any $j \geq 2k$,*

$$\|M_j(U) - M_j(U')\| \leq 3^j \max_{\ell \in [2k-1]} \|M_\ell(U) - M_\ell(U')\|.$$

Proof.

$$\begin{aligned} \|M_j(U) - M_j(U')\| &\stackrel{(a)}{=} \sup_{\|v\|=1} |m_j(\langle U, v \rangle) - m_j(\langle U', v \rangle)| \\ &\stackrel{(b)}{\leq} 3^j \sup_{\|v\|=1} \max_{\ell \in [2k-1]} |m_\ell(\langle U, v \rangle) - m_\ell(\langle U', v \rangle)| \\ &\stackrel{(c)}{=} 3^j \max_{\ell \in [2k-1]} \|M_\ell(U) - M_\ell(U')\|, \end{aligned}$$

where (a) and (c) follow from (1.2.15), and (b) follows from [Wu and Yang, 2019, Lemma 10]. \square

The lower bound part of Theorem 2.3.1 can be reduced to the one-dimensional case, which is covered by the following lemma. The proof relies on Newton interpolating polynomials and is deferred till Section 2.3.4.

Lemma 2.3.3. *Let γ, γ' be k -atomic distributions supported on $[-R, R]$. Then for any $(2k-1)$ -times differentiable test function h ,*

$$H(\gamma * N(0, 1), \gamma' * N(0, 1)) \geq c \left| \int h d\gamma - \int h d\gamma' \right|, \quad (2.3.2)$$

where c is some constant depending only on k , R , and $\max_{0 \leq i \leq 2k-1} \|h^{(i)}\|_{L_\infty([-R, R])}$.

Proof of Theorem 2.3.1. Since

$$H^2(P, Q) \leq \text{KL}(P\|Q) \leq \chi^2(P\|Q), \quad (2.3.3)$$

(see, e.g., [Tsybakov, 2009, Section 2.4.1]), it suffices to prove the lower bound for H^2 and the upper bound for χ^2 .

Let $U \sim \Gamma$ and $U' \sim \Gamma'$, $X \sim P_\Gamma = \Gamma * N(0, I_d)$ and $X' \sim P_{\Gamma'} = \Gamma' * N(0, I_d)$. Then $\langle \theta, X \rangle \sim P_{\Gamma_\theta}$ and $\langle \theta, X' \rangle \sim P_{\Gamma'_\theta}$. By the data processing inequality,

$$H(P_\Gamma, P_{\Gamma'}) \geq \sup_{\theta \in S^{d-1}} H(P_{\Gamma_\theta}, P_{\Gamma'_\theta}). \quad (2.3.4)$$

Applying Lemma 2.3.3 to all monomials of degree at most $2k - 1$, we obtain

$$H(P_\Gamma, P_{\Gamma'}) \geq c \sup_{\theta \in S^{d-1}} \max_{\ell \leq 2k-1} |m_\ell(\langle \theta, U \rangle) - m_\ell(\langle \theta, U' \rangle)| = c \max_{\ell \leq 2k-1} \|M_\ell(U) - M_\ell(U')\|, \quad (2.3.5)$$

for some constant c depending on k and R , where the last equality is due to (1.2.15).

Thus the desired lower bound for Hellinger follows from the tensor norm comparison in (1.2.14).

To show the upper bound, for notational convenience, let $B = V^\top U \sim \nu$ and $B' = V^\top U' \sim \nu'$. For the upper bound, by Lemma 1.2.2,

$$\begin{aligned} \chi^2(P_\nu\|P_{\nu'}) &\stackrel{(c)}{\leq} e^{\frac{R^2}{2}} e^{2k} \max_{\ell \in [2k-1]} \|M_\ell(B) - M_\ell(B')\|_F^2 + e^{\frac{R^2}{2}} \sum_{\ell \geq 2k} \frac{(4k^2)^\ell}{\ell!} \|M_\ell(B) - M_\ell(B')\|^2 \\ &\stackrel{(d)}{\leq} e^{\frac{R^2}{2}} \left(e^{2k} + \underbrace{\sum_{\ell \geq 2k} \frac{(36k^2)^\ell}{\ell!}}_{\leq e^{36k^2}} \right) \max_{\ell \in [2k-1]} \|M_\ell(B) - M_\ell(B')\|_F^2, \end{aligned}$$

where (c) follows from the tensor norm comparison inequality (1.2.14), since the symmetric rank of $M_\ell(B) - M_\ell(B')$ is at most $2k$ for all ℓ ; (d) follows from Lemma 2.3.2.

Note that if $\mathbb{E}[B'] \neq 0$, by the shift-invariance of χ^2 -divergence, applying Lemma 1.2.4 to $\mu = \mathbb{E}[B']$ (which satisfies $\|\mu\| \leq R$) yields the desired upper bound. \square

2.3.2 Local entropy of Hellinger balls

Recall from Section 1.1 that $N(\epsilon, A, \rho)$ the ϵ -covering number of A with respect to ρ , i.e., the minimum size of a ϵ -covering set A_ϵ such that, for any $v \in A$, there exists $\tilde{v} \in A_\epsilon$ with $\rho(v, \tilde{v}) < \epsilon$. The main result of this section is the following lemma, which bounds the covering number of a Hellinger ball in k -GMs. From this the upper bound in Theorem 2.1.2 immediately follows by invoking the Le Cam-Birgé construction ([Birgé, 1986, Theorem 3.1]; for the high-probability bound (2.1.6) see e.g., [Wu, 2017, Theorem 18.3]).

Lemma 2.3.4 (Local entropy of k -GM). *For any $\Gamma_0 \in \mathcal{G}_{k,d}$, let $\mathcal{P}_\epsilon = \{P_\Gamma : \Gamma \in \mathcal{G}_{k,d}, H(P_\Gamma, P_{\Gamma_0}) \leq \epsilon\}$. Then, for any $\delta \leq \epsilon/2$,*

$$N(\delta, \mathcal{P}_\epsilon, H) \leq (\epsilon/\delta)^{c \cdot d}, \quad (2.3.6)$$

where the constant c only depends on k and R .

Proof. Let $\mathcal{M}_\epsilon = \{M(\Gamma) : P_\Gamma \in \mathcal{P}_\epsilon\}$, where $M(\Gamma) = (M_1(\Gamma), \dots, M_{2k-1}(\Gamma))$ consists of the moment tensors of Γ up to degree $2k-1$. Let $c' = \sqrt{c}$ and $C' = \sqrt{C}$ where c and C are from Theorem 2.3.1. To obtain a δ -covering of \mathcal{P}_ϵ , next we show that it suffices to construct a $\frac{\delta}{2C'}$ -covering of \mathcal{M}_ϵ with respect to $\rho(M, M') \triangleq \max_{\ell \leq 2k-1} \|M_\ell - M'_\ell\|_F$ and thus

$$N(\delta, \mathcal{P}_\epsilon, H) \leq N(\delta/(2C'), \mathcal{M}_\epsilon, \rho). \quad (2.3.7)$$

To this end, let \mathcal{N} be the optimal $\frac{\delta}{2C'}$ -covering of \mathcal{M}_ϵ with respect to ρ , and we show that $\mathcal{N}' = \{P_\Gamma : \Gamma = \operatorname{argmin}_{\Gamma' : P_{\Gamma'} \in \mathcal{P}_\epsilon} \rho(M(\Gamma'), M), M \in \mathcal{N}\}$ is a δ -covering of \mathcal{P}_ϵ . For any $P_\Gamma \in \mathcal{P}_\epsilon$, by the covering property of \mathcal{N} , there exists moment tensors $M \in \mathcal{N}$

such that $\rho(M, M(\Gamma)) < \frac{\delta}{2C'}$. By the definition of \mathcal{N}' , there exists $P_{\tilde{\Gamma}} \in \mathcal{N}'$ such that $\rho(M(\tilde{\Gamma}), M) < \frac{\delta}{2C'}$. Therefore, $\rho(M(\tilde{\Gamma}), M(\Gamma)) < \frac{\delta}{C'}$ and thus $H(P_{\tilde{\Gamma}}, P_{\Gamma}) < \delta$ by Theorem 2.3.1. It also follows from Theorem 2.3.1 and the fact that Γ_0, Γ are both k -atomic that

$$\mathcal{M}_\epsilon \subseteq M(\Gamma_0) + \{\Delta : \|\Delta_\ell\|_F \leq \epsilon/c', \text{rank}_s(\Delta_\ell) \leq 2k, \forall \ell \leq 2k-1\}, \quad (2.3.8)$$

where $\Delta = (\Delta_1, \dots, \Delta_{2k-1})$ and $\Delta_\ell \in \mathbb{S}_\ell(\mathbb{R}^d)$. Let $\mathcal{D}_\ell = \{\Delta_\ell \in \mathbb{S}_\ell(\mathbb{R}^d) : \|\Delta_\ell\|_F \leq \epsilon/c', \text{rank}_s(\Delta_\ell) \leq 2k\}$, and $\mathcal{D} = \mathcal{D}_1 \times \dots \times \mathcal{D}_{2k-1}$ be the Cartesian product. By monotonicity,

$$N(\delta/(2C'), \mathcal{M}_\epsilon, \rho) \leq N(\delta/(2C'), \mathcal{D}, \rho) \leq \prod_{\ell=1}^{2k-1} N(\delta/(2C'), \mathcal{D}_\ell, \|\cdot\|_F). \quad (2.3.9)$$

The conclusion follows from Lemma 2.3.5. \square

Lemma 2.3.5. *Let $\mathcal{T} = \{T \in \mathbb{S}_\ell(\mathbb{R}^d) : \|T\|_F \leq \epsilon, \text{rank}_s(T) \leq r\}$. Then, for any $\delta \leq \epsilon/2$,*

$$N(\delta, \mathcal{T}, \|\cdot\|_F) \leq \left(\frac{C\ell\epsilon}{\delta}\right)^{dr} \left(\frac{C\epsilon}{\delta}\right)^{r^\ell}, \quad (2.3.10)$$

for some absolute constant C .

Proof. For any $T \in \mathcal{T}$, $\text{rank}_s(T) \leq r$. Thus $T = \sum_{i=1}^r a_i v_i^{\otimes \ell}$ for some $a_i \in \mathbb{R}$ and $v_i \in S^{d-1}$. Furthermore, $\|T\|_F \leq \epsilon$. Ideally, if the coefficients satisfied $|a_i| \leq \epsilon$ for all i , then we could cover the r -dimensional ϵ -hypercube with an $\frac{\epsilon}{2}$ -covering, which, combined with a $\frac{1}{2}$ -covering of the unit sphere that covers the unit vectors v_i 's, constitutes a desired covering for the tensor. Unfortunately the coefficients a_i 's need not be small due to the possible cancellation between the rank-one components (consider the counterexample of $0 = v^{\otimes \ell} - v^{\otimes \ell}$). Next, to construct the desired covering we turn to the Tucker decomposition of the tensor T .

Let $u = (u_1, \dots, u_r)$ be an orthonormal basis for the subspace spanned by (v_1, \dots, v_r) .

In particular, let $v_i = \sum_{j=1}^r b_{ij} u_j$. Then

$$T = \sum_{\mathbf{j}=(j_1, \dots, j_\ell) \in [r]^\ell} \alpha_{\mathbf{j}} \underbrace{u_{j_1} \otimes \cdots \otimes u_{j_\ell}}_{\triangleq u_{\mathbf{j}}}, \quad (2.3.11)$$

where $\alpha_{\mathbf{j}} = \sum_{i=1}^r a_i b_{ij_1} \cdots b_{ij_\ell}$. In tensor notation, T admits the following *Tucker decomposition*

$$T = \alpha \times_1 U \cdots \times_\ell U \quad (2.3.12)$$

where the symmetric tensor $\alpha = (\alpha_{\mathbf{j}}) \in \mathbb{S}_\ell(\mathbb{R}^r)$ is called the core tensor and U is a $r \times d$ matrix whose rows are given by u_1, \dots, u_r .

Due to the orthonormality of (u_1, \dots, u_r) , we have for any $\mathbf{j}, \mathbf{j}' \in [r]^\ell$,

$$\langle u_{\mathbf{j}}, u_{\mathbf{j}'} \rangle = \prod_{i=1}^{\ell} \langle u_{j_i}, u_{j'_i} \rangle = \mathbf{1}_{\{\mathbf{j}=\mathbf{j}'\}}. \quad (2.3.13)$$

Hence we conclude from (2.3.11) that

$$\|\alpha\|_F = \|T\|_F. \quad (2.3.14)$$

In particular $\|\alpha\|_F \leq \epsilon$. Therefore,

$$\mathcal{T} \subseteq \mathcal{T}' \triangleq \left\{ T = \sum_{\mathbf{j} \in [r]^\ell} \alpha_{\mathbf{j}} u_{j_1} \otimes \cdots \otimes u_{j_\ell} : \|\alpha\|_F \leq \epsilon, \langle u_i, u_j \rangle = \mathbf{1}_{\{i=j\}} \right\}. \quad (2.3.15)$$

Let \tilde{A} be a $\frac{\delta}{2}$ -covering of $\{\alpha \in \mathbb{S}_\ell(\mathbb{R}^r) : \|\alpha\|_F \leq \epsilon\}$ under $\|\cdot\|_F$ of size $(\frac{C\epsilon}{\delta})^{r^\ell}$ for some absolute constant C ; let \tilde{B} be a $\frac{\delta}{2\ell\epsilon}$ -covering of $\{(u_1, \dots, u_r) : \langle u_i, u_j \rangle = \mathbf{1}_{\{i=j\}}\}$ under the maximum of column norms of size $(\frac{C\ell\epsilon}{\delta})^{dr}$. Let $\tilde{\mathcal{T}}' = \{\sum_{\mathbf{j} \in [r]^\ell} \tilde{\alpha}_{\mathbf{j}} \tilde{u}_{j_1} \otimes \cdots \otimes \tilde{u}_{j_\ell} : \tilde{\alpha} \in \tilde{A}, \tilde{u} \in \tilde{B}\}$. Next we verify the covering property.

For any $T \in \mathcal{T}'$, there exists $\tilde{T} \in \tilde{\mathcal{T}}'$ such that $\|\alpha - \tilde{\alpha}\|_F \leq \frac{\delta}{2}$ and $\max_{i \leq r} \|u_i - \tilde{u}_i\| \leq$

$\frac{\delta}{2\ell\epsilon}$. Then, by the triangle inequality,

$$\left\|T - \tilde{T}\right\|_F \leq \sum_{\mathbf{j}} |\alpha_{\mathbf{j}}| \|u_{j_1} \otimes \cdots \otimes u_{j_\ell} - \tilde{u}_{j_1} \otimes \cdots \otimes \tilde{u}_{j_\ell}\|_F + \sum_{\mathbf{j}} |\alpha_{\mathbf{j}} - \tilde{\alpha}_{\mathbf{j}}| \|\tilde{u}_{j_1} \otimes \cdots \otimes \tilde{u}_{j_\ell}\|_F. \quad (2.3.16)$$

The second term is at most $\|\alpha - \tilde{\alpha}\|_F \leq \delta/2$. For the first term, it follows from the triangle inequality that

$$\|u_{j_1} \otimes \cdots \otimes u_{j_\ell} - \tilde{u}_{j_1} \otimes \cdots \otimes \tilde{u}_{j_\ell}\|_F \leq \sum_{i=1}^{\ell} \|u_{j_1} \otimes \cdots \otimes (u_{j_i} - \tilde{u}_{j_i}) \otimes \cdots \otimes \tilde{u}_{j_\ell}\|_F \leq \frac{\delta}{2\epsilon}. \quad (2.3.17)$$

Therefore, the first term is at most $\frac{\delta}{2\epsilon} \|\alpha\|_F \leq \delta/2$. \square

2.3.3 Proof of main theorem

Proof of Theorem 2.1.2. The upper bound follows from Lemma 2.3.4 and the LeCam-Birgé estimator [Le Cam, 1973, Birgé, 1983].

For the lower bound, Let $\mathcal{M} = \{M(\Gamma) : \Gamma \in \mathcal{G}_{k,d}\}$, where $M(\Gamma) = (M_1(\Gamma), \dots, M_{2k-1}(\Gamma))$. By Theorem 2.3.1, we obtain an ϵ_n -cover in KL -divergence on the space of distributions via an cover on the moment tensor space, i.e.,

$$N(\epsilon_n, \mathcal{P}_{k,d}, KL) \leq N(\epsilon_n, \mathcal{M}, \rho),$$

where $\rho(M, M') = \max_{\ell \in [2k-1]} \|M_\ell - M'_\ell\|_F$. By Lemma 2.3.4, $N(\epsilon, \mathcal{M}, \rho) \leq (C/\epsilon)^{c_k d}$, where C, c_k are positive constants. We moreover have from Lemma 2.3.4, and the relationship between packing and cover numbers³, that $M(\epsilon, \mathcal{P}_{k,d}, H) \leq (C'/\epsilon_{n,H})^{c_k d}$.

Thus we have

$$N(\epsilon, \mathcal{P}_{k,d}, KL) \asymp M(\epsilon, \mathcal{P}_{k,d}, H) \asymp \left(\frac{C}{\epsilon}\right)^{c_k d}.$$

We apply Theorem 1 of [Yang and Barron, 1999] as follows. Select $\epsilon_n = \sqrt{d \log(n/d)/n}$,

³ $N(\epsilon, \mathcal{P}, H) \leq M(\epsilon, \mathcal{P}, H) \leq M(\epsilon/2, \mathcal{P}, H)$; see, e.g., [van der Vaart and Wellner, 1996], page 98, Definition 2.2.3.

so $n\epsilon_n^2 = d \log(n/d)$. And $\log N(\epsilon_n, \mathcal{P}_{k,d}, KL) \asymp c_k d (\log(n/d) - \log \log(n/d))$ so we satisfy:

$$n\epsilon_n^2 \geq N(\epsilon_n, \mathcal{P}_{k,d}, KL).$$

Now set $\epsilon_{n,H} = C\sqrt{d/n}$ for an appropriate positive constant C , so that $\log M(\epsilon_{n,H}, \mathcal{P}_{k,d}, H) = \frac{c_k d}{2} \log(C''n/d)$. Then we have that as needed, $\log M(\epsilon_{n,H}, \mathcal{P}_{k,d}, H) \geq 4n\epsilon_n^2 + 2 \log 2$, and we obtain a lower bound on the rate of $\epsilon_{n,H} = C\sqrt{d/n}$. \square

The next result shows that optimal estimation of the mixing distribution can be reduced to that of the mixture density, both statistically and computationally, provided that the density estimate is proper, i.e., is a valid k -GM. Note that this does not mean an optimal density estimate $P_{\hat{\Gamma}}$ automatically yields an optimal estimator of the mixing distribution $\hat{\Gamma}$ for Theorem 2.1.1. Instead, we rely on an intermediate step that allows us to estimate the appropriate subspace and then perform density estimation in this low-dimensional space.

Theorem 2.3.6. *Suppose that for every $d \in \mathbb{N}$ and $\Gamma \in \mathcal{G}_{k,d}$, there exists a proper density estimator $P_{\hat{\Gamma}'}$ as a function of $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} P_{\Gamma}$, such that*

$$\mathbb{E}H(P_{\hat{\Gamma}'}, P_{\Gamma}) \leq c_k(d/n)^{1/2}, \quad (2.3.18)$$

for some constant c_k . Then there is an estimator $\hat{\Gamma}$ of the mixing distribution Γ and a positive constant C_k such that

$$\mathbb{E}W_1(\hat{\Gamma}, \Gamma) \leq C_k \left(\left(\frac{d}{n} \right)^{1/4} + \left(\frac{1}{n} \right)^{\frac{1}{4k-2}} \right). \quad (2.3.19)$$

Proof of Theorem 2.3.6. We first construct the estimator $\hat{\Gamma}$ using $X_1, \dots, X_{2n} \stackrel{i.i.d.}{\sim} P_{\Gamma}$. Let $\hat{\Gamma}' \in \mathcal{G}_{k,d}$ be the estimator from $\{X_i\}_{i \leq n}$ satisfying

$$\mathbb{E}H(P_{\hat{\Gamma}'}, P_{\Gamma}) \leq c_k \sqrt{d/n}, \quad (2.3.20)$$

for a positive constant c_k , as guaranteed by (2.3.18). Let $\hat{V} \in \mathbb{R}^{d \times k}$ be a matrix whose columns form an orthonormal basis for the space spanned by the atoms of $\hat{\Gamma}'$, $\hat{H} = \hat{V}\hat{V}^\top$, and $\gamma = \Gamma_{\hat{V}}$. Note that conditioned on \hat{V} , $\{\hat{V}^\top X_i\}_{i=n+1, \dots, 2n}$ is an i.i.d. sample drawn from the k -GM P_γ . Invoking (2.3.18) again, there exists an estimator $\hat{\gamma} = \sum_{j=1}^k \hat{w}_j \delta_{\hat{\psi}_j} \in \mathcal{G}_{k,k}$ such that

$$\mathbb{E}H(P_{\hat{\gamma}}, P_\gamma) \leq c_k \sqrt{k/n}. \quad (2.3.21)$$

We will show that $\hat{\Gamma} \triangleq \hat{\gamma}_{\hat{V}^\top} = \sum_{j=1}^k \hat{w}_j \delta_{\hat{\psi}_j}$ achieves the desired rate (2.3.19). Recall from (2.2.18) the risk decomposition:

$$W_1(\Gamma, \hat{\Gamma}) \leq W_1(\Gamma, \Gamma_{\hat{H}}) + W_1(\gamma, \hat{\gamma}). \quad (2.3.22)$$

Let $\Sigma = \mathbb{E}_{U \sim \Gamma}[UU^\top]$ and $\hat{\Sigma} = \mathbb{E}_{U \sim \hat{\Gamma}'}[UU^\top]$ whose ranks are at most k . Then \hat{H} is the projection matrix onto the space spanned by the top k eigenvectors of $\hat{\Sigma}$. It follows from Lemma 2.2.5 that $W_1(\Gamma, \Gamma_{\hat{H}}) \leq \sqrt{2k\|\Sigma - \hat{\Sigma}\|_2}$. By Lemma 2.3.3 and the data processing inequality of the Hellinger distance,

$$\|\Sigma - \hat{\Sigma}\|_2 = \sup_{\theta \in S^{d-1}} |m_2(\Gamma_\theta) - m_2(\hat{\Gamma}'_\theta)| \leq C_k \sup_{\theta \in S^{d-1}} H(P_{\Gamma_\theta}, P_{\hat{\Gamma}'_\theta}) \leq C_k H(P_\Gamma, P_{\hat{\Gamma}'}).$$

Therefore, by (2.3.20), we obtain that

$$\mathbb{E}W_1(\Gamma, \Gamma_{\hat{H}}) \leq C_k \left(\frac{d}{n}\right)^{1/4}. \quad (2.3.23)$$

We condition on \hat{V} to analyze the second term on the right-hand side of (2.3.22). By Lemmas 2.2.1 and 2.5.1,

$$W_1(\gamma, \hat{\gamma}) \leq k^{5/2} \sup_{\theta \in S^{k-1}} W_1(\gamma_\theta, \hat{\gamma}_\theta) \leq C_k \sup_{\theta \in S^{k-1}, r \in [2k-1]} |m_r(\gamma_\theta) - m_r(\hat{\gamma}_\theta)|^{\frac{1}{2k-1}}.$$

Again, by Lemma 2.3.3 and the data processing inequality, for any $\theta \in S^{k-1}$ and $r \in [2k-1]$,

$$|m_r(\gamma_\theta) - m_r(\hat{\gamma}_\theta)| \leq C_k H(P_{\hat{\gamma}_\theta}, P_{\gamma_\theta}) \leq C_k H(P_{\hat{\gamma}}, P_\gamma)$$

Therefore, by (2.3.21), we obtain that

$$\mathbb{E}W_1(\gamma, \hat{\gamma}) \leq C_k \left(\frac{1}{n} \right)^{\frac{1}{4k-2}}. \quad (2.3.24)$$

The conclusion follows by applying (2.3.23) and (2.3.24) in (2.3.22). \square

2.3.4 Proofs of supporting lemmas

Lemma 2.3.3 provides a tighter relationship between mixture density estimation error in Hellinger distance and mixing distribution moment estimation than the one provided by Lemma 11 of [Wu and Yang, 2019], allowing for the sharp connection between Hellinger distance and 1-Wasserstein distance provided in Theorem 2.1.2. Note that the relationship between the Hellinger and L_2 distances used in the proof of Lemma 2.3.3 is not necessarily sharp, since it relies on a simple upper bound on the densities in question. For high dimensions, the upper bound on the densities could be large. However, we will deal with k -dimensional densities, so this upper bound is good enough.

We start by recalling the basics of polynomial interpolation in one dimension. For any function h and any set of $m+1$ distinct points $\{x_0, \dots, x_m\}$, there exists a unique polynomial P of degree at most m , such that $P(x_i) = h(x_i)$ for $i = 0, \dots, m$. For our purpose, it is convenient to express P in the Newton form (as opposed to the more

common Lagrange form):

$$P(x) = \sum_{j=0}^m a_j \prod_{i=0}^{j-1} (x - x_i), \quad (2.3.25)$$

where the coefficients are given by the finite differences of h , namely, $a_j = h[x_0, \dots, x_j]$, which in turn are defined recursively via:

$$\begin{aligned} h[x_i] &= h(x_i) \\ h[x_i, \dots, x_{i+r}] &= \frac{h[x_{i+1}, \dots, x_{i+r}] - h[x_i, \dots, x_{i+r-1}]}{x_{i+r} - x_i}. \end{aligned}$$

Proof of Lemma 2.3.3. Let $U \sim \gamma$ and $U' \sim \gamma'$. Note that $p_\gamma, p_{\gamma'}$ are bounded above by $\frac{1}{\sqrt{2\pi}}$, so we have

$$H^2(p_\gamma, p_{\gamma'}) = \int \left(\frac{p_\gamma - p_{\gamma'}}{\sqrt{p_\gamma} + \sqrt{p_{\gamma'}}} \right)^2 \geq \frac{\sqrt{2\pi}}{4} \|p_\gamma - p_{\gamma'}\|_2^2.$$

Thus to show (2.3.2), it suffices to show there is a positive constant c such that

$$\|p_\gamma - p_{\gamma'}\|_2 \geq c |\mathbb{E}[h(U)] - \mathbb{E}[h(U')]|. \quad (2.3.26)$$

Next, by suitable orthogonal expansion we can express $\|p_\gamma - p_{\gamma'}\|_2$ in terms of the “moments” of the mixing distributions (see [Wu and Verdú, 2010, Sec. VI]). Let $\alpha_j(y) = \sqrt{\frac{\sqrt{2}\phi(\sqrt{2}y)}{j!}} H_j(\sqrt{2}y)$. Then $\{\alpha_j : j \in \mathbb{Z}_+\}$ form an orthonormal basis on $L^2(\mathbb{R}, dy)$ in view of (1.2.18). Since p_γ is square integrable, we have the orthogonal expansion $p_\gamma(y) = \sum_{j \geq 0} a_j(\gamma) \alpha_j(y)$, with coefficient

$$a_j(\gamma) = \langle \alpha_j, p_\gamma \rangle = \mathbb{E}[\alpha_j(U + Z)] = \mathbb{E}[(\alpha_j * \phi)(U)] = \frac{1}{2^{\frac{j+1}{2}} \pi^{\frac{1}{4}} \sqrt{j!}} \mathbb{E}[U^j e^{-\frac{U^2}{4}}]$$

where the last equality follows from the fact that [?, 7.374.6, p. 803]

$$(\phi * \alpha_j)(y) = \frac{1}{2^{\frac{j+1}{2}} \pi^{\frac{1}{4}} \sqrt{j!}} y^j e^{-\frac{y^2}{4}}.$$

Therefore

$$\|p_\gamma - p_{\gamma'}\|_2^2 = \sum_{j \geq 0} \frac{1}{j! 2^{j+1} \sqrt{\pi}} \left(\mathbb{E}[U^j e^{-U^2/4}] - \mathbb{E}[U'^j e^{-U'^2/4}] \right)^2. \quad (2.3.27)$$

In particular, for each $j \geq 0$,

$$|\mathbb{E}[U^j e^{-U^2/4}] - \mathbb{E}[U'^j e^{-U'^2/4}]| \leq \sqrt{j! 2^{j+1} \sqrt{\pi}} \|p_\gamma - p_{\gamma'}\|_2. \quad (2.3.28)$$

In view of (2.3.28), to bound the difference $|\mathbb{E}[h(U)] - \mathbb{E}[h(U')]|$ by means of $\|p_\gamma - p_{\gamma'}\|_2$, our strategy is to interpolate $h(y)$ by linear combinations of $\{y^j e^{-y^2/4} : j = 0, \dots, 2k-1\}$ on all the atoms of U and U' , a total of at most $2k$ points. Clearly, this is equivalent to the standard polynomial interpolation of $\tilde{h}(y) \triangleq h(y) e^{y^2/4}$ by a degree- $(2k-1)$ polynomial. Specifically, let $T \triangleq \{t_1, \dots, t_{2k}\}$ denote the set of atoms of γ and γ' . By assumption, $T \subset [-R, R]$. Denote the interpolating polynomial of \tilde{h} on T by $P(y) = \sum_{j=0}^{2k-1} b_j y^j$. Then

$$\begin{aligned} |\mathbb{E}[h(U)] - \mathbb{E}[h(U')]| &= |\mathbb{E}[P(U) e^{-U^2/4}] - \mathbb{E}[P(U') e^{-U'^2/4}]| \\ &\leq \sum_{j=0}^{2k-1} |b_j| |\mathbb{E}[U^j e^{-U^2/4}] - \mathbb{E}[U'^j e^{-U'^2/4}]| \\ &\leq \|p_\gamma - p_{\gamma'}\|_2 \sum_{j=0}^{2k-1} |b_j| \sqrt{j! 2^{j+1} \sqrt{\pi}}. \end{aligned}$$

It remains to bound the coefficient b_j independently of the set T . This is given by the next lemma. □

Lemma 2.3.7. *Let h be an m -times differentiable function on the interval $[-R, R]$,*

whose derivatives are bounded by $|h^{(i)}(x)| \leq M$ for all $0 \leq i \leq m$ and all $x \in [-R, R]$. Then for any $m \geq 1$ and $R > 0$, there exists a positive constant $C = C(m, R, M)$, such that the following holds. For any set of distinct nodes $T = \{x_0, \dots, x_m\} \subset [-R, R]$, denote by $P(x) = \sum_{i=0}^m b_i x^i$ the unique interpolating polynomial of degree at most m of h on T . Then $\max_{0 \leq j \leq m} |b_j| \leq C$.

Proof. Express P in the Newton form (2.3.25):

$$P(y) = \sum_{i=0}^m h[x_0, \dots, x_i] \prod_{j=0}^{i-1} (y - x_j)$$

By the intermediate value theorem, finite differences can be bounded by derivatives as follows: (c.f. [Stoer and Bulirsch, 2002, (2.1.4.3)])

$$|h[x_0, \dots, x_i]| \leq \frac{1}{i!} \sup_{|\xi| \leq R} |h^{(i)}(\xi)|.$$

Let $\prod_{j=0}^{i-1} (y - x_j) = \sum_{j=0}^i c_{ij} y^j$. Since $|x_j| \leq R$, $|c_{ij}| \leq C_1 = C_1(R, m)$ all i, j . This completes the proof. \square

2.4 Numerical studies

We now present numerical results. We compare the estimator (2.2.5) to the classical EM algorithm. The algorithm that computes (2.2.5) relies on an exhaustive search and is not meant to be practical, but it turns out that it can be competitive with the EM algorithm both statistically and computationally, as our experiments show.

All simulations are run in Python. The DMM algorithm relies on the CVXPY ([Diamond and Boyd, 2016]) and CVXOPT ([Andersen et al., 2013]) packages; see Section 6 of [Wu and Yang, 2019] for more details on the implementation of DMM. We also use the Python Optimal Transport package ([Flamary and Courty, 2017]) to compute the d -dimensional 1-Wasserstein distance.

In all experiments, we let $d = 100$ and $\sigma = 1$. We let n range from 10,000 to 200,000 in increments of 10,000. We initialize EM randomly, and our stopping criterion for the EM algorithm is either after 1000 iterations or once the relative change in log likelihood is below 10^{-6} . For the dimension reduction step in the computation of (2.2.5), we first center our data, then do the projection using the top $k-1$ eigenvectors of $\hat{\Sigma}$ in (2.2.1). Thus when $k = 2$, we project onto a one-dimensional subspace and only run DMM once, so the grid search of Algorithm 1 is never invoked. We note that sample-splitting is used for the estimator (2.2.5) for purposes of analysis only; in the actual experiments, we do not sample split.

When $k = 3$, we project the data to a 2-dimensional subspace after centering. In this case, we need to choose \mathcal{W}, \mathcal{N} , the $\epsilon_{n,k}$ -nets on the simplex Δ^{k-1} and on the unit sphere S^{k-2} , respectively. Here \mathcal{W} is chosen by discretizing the probabilities and \mathcal{N} is formed by gridding the angles $\alpha \in [-\pi, \pi]$ and using the points $(\cos \alpha, \sin \alpha)$. Note that here, $|\mathcal{W}| \leq (C_1/\epsilon_{n,k})^{k-1}$, $|\mathcal{N}| \leq (C_2/\epsilon_{n,k})^{k-2}$. For example, when $n = 10000$, $1/\epsilon_{n,k} \approx 3$. In the experiments, we choose $C_1 = 1, C_2 = 4$. In our experience, an even coarser grid \mathcal{N} can be used and still achieve fairly high accuracy in the well-separated models, while gaining some speed.

In Fig. 2.1, we compare the performance on the symmetric 2-GM, where the samples are drawn from the distribution $\frac{1}{2}N(\mu, I_d) + \frac{1}{2}N(-\mu, I_d)$. For Fig. 2.1(a), $\mu = 0$, i.e., the components completely overlap. And for Fig. 2.1(b) and Fig. 2.1(c), μ is uniformly drawn from the sphere of radius 1 and 2, respectively. In Fig. 2.1(d), the model is $P_\Gamma = \frac{1}{4}N(\mu, I_d) + \frac{3}{4}N(-\mu, I_d)$ where μ is drawn from the sphere of radius 2. That is, the model is the same as the one used in Fig. 2.1(c) except that the weights are uneven. We still run PCA and DMM without any grid search.

Our algorithm and EM perform similarly for the model with overlapping components; our algorithm is more accurate than EM in the model where $\|\mu\|_2 = 1$, but EM improves as the model components become more separated.

There is little difference in the performance of either algorithm in the uneven weights scenario. In terms of running time, computing (2.2.5) takes about the same time as EM for smaller values of n , but EM slows much more as n increases since it accesses all the samples on each iteration. For the largest sample size in the experiments, computing (2.2.5) is about 6 times faster than EM.

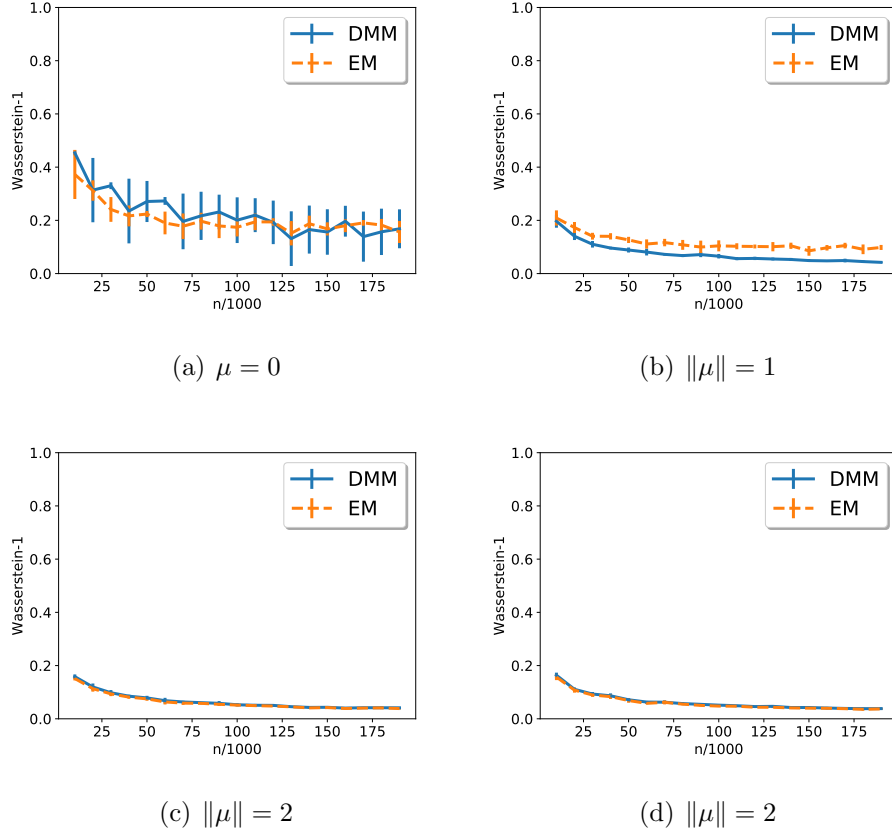


Figure 2.1: In the first three figures, $P_{\Gamma} = \frac{1}{2}N(\mu, I_d) + \frac{1}{2}N(-\mu, I_d)$, for increasing values of $\|\mu\|_2$. In the final figure, $P_{\Gamma} = \frac{1}{4}N(\mu, I_d) + \frac{3}{4}N(-\mu, I_d)$ where $\|\mu\|_2 = 2$.

In Fig. 2.2, we compare the performance on the 3-GM model $\frac{1}{3}N(\mu, I_d) + \frac{1}{3}N(0, I_d) + \frac{1}{3}N(-\mu, I_d)$. We follow the same pattern of increasing the separation of the components in each experiment. For Fig. 2.2(a), $\mu = 0$, i.e., the components completely overlap. And for Fig. 2.2(b) and Fig. 2.2(c), μ is uniformly drawn from the sphere of radius 1 and 2, respectively.

In the experiments where $k = 3$, we see the opposite phenomenon in terms of the relative performance of our algorithm and EM: the former improves more as the centers become more separated. This seems to be because in, for instance, the case where $\mu = 0$, the error in each coordinate for DMM is fairly high, and this is compounded when we select the two-coordinate final distribution. The performance of our algorithm improves rapidly here because as the model becomes more separated, the errors in each coordinate become very small. Note that since we have made the model more difficult to learn by adding a center at 0, the errors are higher than for the $k = 2$ example in every experiment for both algorithms. In terms of running time, for $k = 3$, EM is faster than our algorithm for smaller sample size because of the grid search being invoked for Algorithm 1. But EM slows much more rapidly as the sample size increases and is actually slower than our algorithm for large values of n .

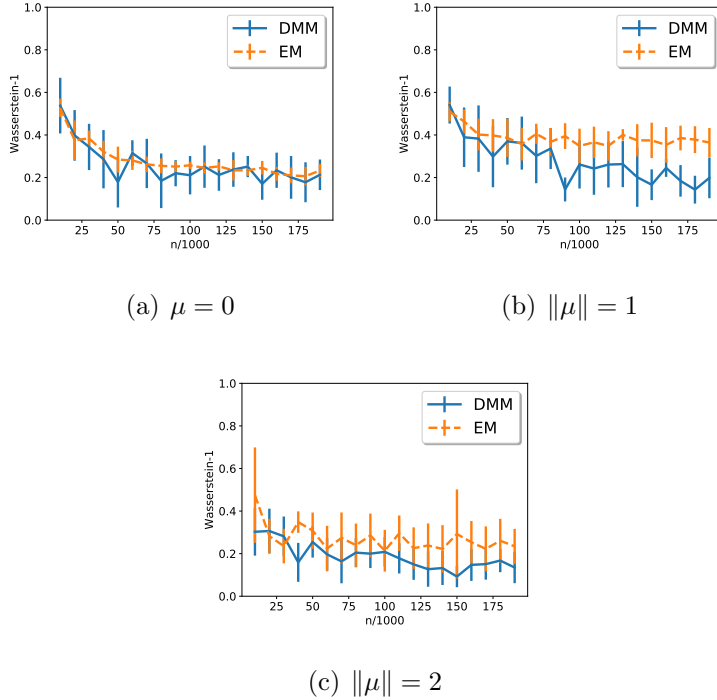


Figure 2.2: $P_{\Gamma} = \frac{1}{3}N(\mu, I_d) + \frac{1}{3}N(0, I_d) + \frac{1}{3}N(-\mu, I_d)$ for increasing values of $\|\mu\|_2$.

2.5 Auxiliary lemmas and proofs

The following moment comparison inequality bound the Wasserstein distance between two univariate k -atomic distributions using their moment differences:

Lemma 2.5.1 ([Wu and Yang, 2019, Proposition 1]). *For any $\gamma, \gamma' \in \mathcal{G}_{k,1}$,*

$$W_1(\gamma, \gamma') \lesssim_k \max_{r \in [2k-1]} |m_r(\gamma) - m_r(\gamma')|^{1/(2k-1)}.$$

Lemma 2.5.2 (Hypercontractivity inequality [Schudy and Sviridenko, , Theorem 1.9]). *Let $Z \sim N(0, I_d)$. Let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ be a polynomial of degree at most q . Then for any $t > 0$,*

$$\mathbb{P}\{|g(Z) - \mathbb{E}g(Z)| \geq t\} \leq e^2 \exp\left(-\left(\frac{t^2}{C\text{Var } g(Z)}\right)^{1/q}\right),$$

where C is a universal constant.

Lemma 2.5.3. *Fix $r \in [2k-1]$. Let $f_r(\theta)$ be the process defined in (2.2.23). Let $\lambda > 0$. There are positive constants C, c_k such that, for any $\theta_1, \theta_2 \in S^{k-1}$,*

$$\mathbb{P}\{|f_r(\theta_1) - f_r(\theta_2)| \geq \|\theta_1 - \theta_2\|_2 \lambda\} \leq C \exp(-c_k \lambda^{2/r}). \quad (2.5.1)$$

$$\mathbb{P}\{|f_r(\theta_1)| \geq \lambda\} \leq C \exp(-c_k \lambda^{2/r}). \quad (2.5.2)$$

Proof. Define $\Delta \triangleq \sqrt{n}(\tilde{m}_r(\theta_1) - \tilde{m}_r(\theta_2))$. Then, $f_r(\theta_1) - f_r(\theta_2) = \Delta - \mathbb{E}\Delta$. Recall that $\tilde{m}_r(\theta) = \frac{1}{n} \sum_{i=1}^n H_r(\theta^\top X_i)$, where $X_i = U_i + Z_i$ and $U_i \stackrel{\text{i.i.d.}}{\sim} \gamma$ and $Z_i \stackrel{\text{i.i.d.}}{\sim} N(0, I_k)$. Conditioning on $U = (U_1, \dots, U_n)$, we have

$$\mathbb{E}(\Delta|U) = \frac{1}{\sqrt{n}} \sum_{i=1}^n ((\theta_1^\top U_i)^r - (\theta_2^\top U_i)^r).$$

Now $|(\theta_1^\top U_i)^r - (\theta_2^\top U_i)^r| \leq rR^r \|\theta_1 - \theta_2\|_2$ since $\|U_i\|_2 \leq R$. By Hoeffding's inequality,

$$\mathbb{P}\{|\mathbb{E}(\Delta|U) - \mathbb{E}\Delta| \geq \|\theta_1 - \theta_2\|_2 \lambda\} \leq 2 \exp\left(-\frac{\lambda^2}{2r^2 R^{2r}}\right). \quad (2.5.3)$$

We now condition on U and analyze $|\Delta - \mathbb{E}(\Delta|U)|$. Since Δ is a polynomial of degree r in Z_1, \dots, Z_n , by Lemma 2.5.2,

$$\mathbb{P}\{|\Delta - \mathbb{E}(\Delta|U)| \geq \|\theta_1 - \theta_2\|_2 \lambda\} \leq e^2 \exp\left(-\left(\frac{\|\theta_1 - \theta_2\|_2^2}{C\text{Var}(\Delta|U)}\right)^{1/r} \lambda^{2/r}\right). \quad (2.5.4)$$

It remains to upper-bound $\text{Var}(\Delta|U)$. We have

$$\text{Var}(\Delta|U) = \frac{1}{n} \sum_{i=1}^n \text{Var}(H_r(\theta_1^\top X_i) - H_r(\theta_2^\top X_i)|U_i).$$

Since the standard deviation of a sum is no more than the sum of the standard deviations,

$$\sqrt{\text{Var}(H_r(\theta_1^\top X_i) - H_r(\theta_2^\top X_i)|U_i)} \leq \sum_{j=0}^{\lfloor r/2 \rfloor} c_{j,r} \sqrt{\mathbb{E}\left(\left((\theta_1^\top X_i)^{r-2j} - (\theta_2^\top X_i)^{r-2j}\right)^2 |U_i\right)},$$

where $c_{j,r} = (-1/2)^r / j!(r-2j)!$. For any $\ell \leq r$, we have $|(\theta_1^\top X)^\ell - (\theta_2^\top X)^\ell| \leq \ell \|X\|_2^\ell \|\theta_1 - \theta_2\|_2$ and thus

$$\mathbb{E}\left(\left((\theta_1^\top X_i)^\ell - (\theta_2^\top X_i)^\ell\right)^2 |U_i\right) \leq \ell^2 \|\theta_1 - \theta_2\|_2^2 \mathbb{E}(\|X_i\|_2^{2\ell} |U_i).$$

Since $\|X_i\|_2^{2\ell} \leq 2^{2\ell-1} (\|Z_i\|_2^{2\ell} + \|U_i\|_2^{2\ell})$ and $\mathbb{E}\|Z_i\|_2^{2\ell} \leq (ck\ell)^\ell$ for a constant c ,

$$\mathbb{E}\left(\left((\theta_1^\top X_i)^\ell - (\theta_2^\top X_i)^\ell\right)^2 |U_i\right) \leq \|\theta_1 - \theta_2\|_2^2 \cdot \ell^2 2^{2\ell-1} (R^{2\ell} + (ck\ell)^\ell).$$

We conclude that $\text{Var}(\Delta|U) \leq C_k \|\theta_1 - \theta_2\|_2^2$. Therefore, (2.5.4) holds with probability $1 - e^2 \exp(-c_k \lambda^{2/r})$. Then (2.5.1) follows from (2.5.3) and (2.5.4).

The second inequality, (2.5.2), can be proved by a similar application of Hoeffding's Inequality and Lemma 2.5.2. \square

The following lemma is adapted from [Pollard, 2016, Section 4.7.1].

Lemma 2.5.4. *Let Θ be a finite subset of a metric space with metric ρ . Let $f(\theta)$ be a random process indexed by $\theta \in \Theta$. Suppose that for $\alpha > 0$, and for $\lambda > 0$, we have*

$$\mathbb{P}\{|f(\theta_1) - f(\theta_2)| \geq \rho(\theta_1, \theta_2)\lambda\} \leq C_\alpha \exp(-c_\alpha \lambda^\alpha), \quad \forall \theta_1, \theta_2 \in \Theta. \quad (2.5.5)$$

Let $\theta_0 \in \Theta$ be a fixed point and $\epsilon_0 = \max_{x,y \in \Theta} \rho(x, y)$. Then there is a constant C'_α such that with probability $1 - C_\alpha \exp(-c_\alpha t^\alpha)$,

$$\max_{\theta \in \Theta} |f(\theta) - f(\theta_0)| \leq C'_\alpha \int_0^{\epsilon_0/2} \left(t + \log^{\frac{1}{\alpha}} \frac{\epsilon_0 |\mathcal{M}(r, \Theta, \rho)|}{r} \right) dr.$$

Proof. We construct an increasing sequence of approximating subsets by maximal packing. Let $\Theta_0 = \{\theta_0\}$. For $i = 0, 1, 2, \dots$, let Θ_{i+1} be a maximal subset of Θ containing Θ_i that constitutes an ϵ_{i+1} -packing, where $\epsilon_{i+1} = \epsilon_i/2$. Since Θ is finite, the procedure stops after a finite number of iterations, resulting in $\Theta_0 \subseteq \Theta_1 \subseteq \dots \subseteq \Theta_m = \Theta$. By definition,

$$N_i \triangleq |\Theta_i| \leq M(\epsilon_i, \Theta, \rho).$$

For $i = 0, \dots, m-1$, define a sequence of mappings $\ell_i : \Theta_{i+1} \rightarrow \Theta_i$ by $\ell_i(t) \triangleq$

$\arg \min_{s \in \Theta_i} \rho(t, s)$ (with ties broken arbitrarily). Now

$$\begin{aligned}
\mathbb{P} \left\{ \max_{\theta \in \Theta} |f(\theta) - f(L_0(\theta))| \geq \sum_{i=0}^{m-1} \epsilon_i \lambda_i \right\} &\leq \mathbb{P} \left\{ \sum_{i=0}^{m-1} \max_{s \in \Theta_{i+1}} |f(s) - f(\ell_i(s))| \geq \sum_{i=0}^{m-1} \epsilon_i \lambda_i \right\} \\
&\leq \mathbb{P} \left\{ \sum_{i=0}^{m-1} \max_{s \in \Theta_{i+1}} |f(s) - f(\ell_i(s))| \geq \sum_{i=0}^{m-1} \max_{s \in \Theta_{i+1}} \rho(s, \ell_i(s)) \lambda_i \right\} \\
&\leq \sum_{i=0}^{m-1} \mathbb{P} \left\{ \max_{s \in \Theta_{i+1}} |f(s) - f(\ell_i(s))| \geq \epsilon_i \lambda_i \right\} \\
&\leq \sum_{i=0}^{m-1} N_{i+1} \exp(-c \lambda_i^\alpha).
\end{aligned}$$

The first inequality is just by the triangle inequality. The second inequality follows from a union bound on the sum. The third inequality is because Θ_i is a maximal ϵ_i -packing, we have $\rho(t, \ell_i(t)) \leq \epsilon_i$ for all $t \in \Theta_{i+1}$. The fourth inequality then follows from the assumption (2.5.5). Thus with probability at least $1 - \sum_{i=0}^{m-1} N_{i+1} C_\alpha \exp(-c_\alpha \lambda_i^\alpha)$,

$$\max_{\theta \in \Theta} |f(\theta) - f(\theta_0)| \leq \sum_{i=0}^{m-1} \lambda_i \epsilon_i. \quad (2.5.6)$$

Set $\lambda_i = (t^\alpha + \frac{1}{c_\alpha} \log(2^{i+1} N_{i+1}))^{1/\alpha}$. Note that $\epsilon_{i+1} = \epsilon_0 2^{-(i+1)}$. Then $\lambda_i \leq C''_\alpha G(\epsilon_{i+1})$ for a constant C''_α , where $G(r) \triangleq t + (\log(\epsilon_0 M(r, \Theta, \rho)/r))^{1/\alpha}$ is a decreasing function for $r \leq \epsilon_0$. By (2.5.6), there is another constant C'_α such that with probability $1 - C_\alpha \exp(-c_\alpha t^\alpha)$,

$$\max_{\theta \in \Theta} |f(\theta) - f(\theta_0)| \leq C'_\alpha \sum_{i=0}^{m-1} G(\epsilon_{i+1}) \epsilon_i \leq C'_\alpha \int_{\epsilon_{m+1}}^{\epsilon_1} 4G(r) dr \leq 4C'_\alpha \int_0^{\epsilon_1} G(r) dr. \quad \square$$

2.6 Alternative methods

2.6.1 Alternative proofs

In this section, we provide alternative proofs for some of the lemmas used in this work. We use the notation $A \gtrsim B$ to mean $x'Ax \geq x'Bx$ for any $x \in \mathbb{R}^d$.

Alternative proof of Lemma 2.2.5. Let the k subscript denote the k -rank approximation. Let $\hat{H} = \hat{V}_k V_k^\top$. Note that $\hat{H}\hat{\Sigma} = \hat{\Sigma}_k$. First,

$$\begin{aligned} \left\| \hat{H}\Sigma - \Sigma \right\|_F &\leq \left\| \hat{H}\Sigma - \hat{\Sigma}_k \right\|_F + \left\| \hat{\Sigma}_k - \Sigma \right\|_F \\ &\lesssim_k \left\| \hat{H}\Sigma - \hat{\Sigma}_k \right\|_2 + \left\| \hat{\Sigma} - \Sigma \right\|_2 \quad \text{by Lemma 4.5.8 and since } \hat{\Sigma}, \hat{\Sigma}_k \text{ have rank } k \\ &\lesssim_k \left\| \hat{\Sigma} - \Sigma \right\|_2. \end{aligned}$$

(We could also do the above step by going directly to spectral norm in step 1.) So by Lemma 2.2.6, with high probability,

$$\left\| \hat{H}\Sigma - \Sigma \right\|_F \lesssim_k (kd/n)^{1/2}.$$

Let $\hat{P} = \hat{H} - I_d$, which is the orthogonal projection operator onto the space orthogonal to that spanned by the vectors in \hat{V}_k . We have shown that $\left\| \hat{P}\Sigma \right\|_F \lesssim_k (d/n)^{1/2}$. And $\left\| \hat{P}\Sigma\hat{P} \right\|_F \leq \left\| \hat{P}\Sigma \right\|_F \left\| \hat{P} \right\|_F \lesssim_k \left\| \hat{P}\Sigma \right\|_F$ since $\left\| \hat{P} \right\|_F = \sqrt{k}$. Now for any $j \in [k]$ and $x \in S^{d-1}$, $x^\top (\hat{P}\Sigma\hat{P}) x \geq x^\top (\hat{P}w_j\mu_j\mu_j^\top \hat{P}) x$. So we can apply Lemma 2.6.1 to say

$$w_j \left\| \hat{P}\mu_j\mu_j^\top \hat{P} \right\|_F \leq \left\| \hat{P}\Sigma\hat{P} \right\|_F,$$

which implies the result. \square

Lemma 2.6.1. *Let $A, B \in \mathbb{R}^{d \times d}$ be symmetric, positive semi-definite matrices with*

$A \succcurlyeq B$. Then

$$\|A\|_F \geq \|B\|_F.$$

Proof. By definition of the Frobenius norm, we must show that $\text{tr}(A^2) \geq \text{tr}(B^2)$.

Since $A - B \succcurlyeq 0$, we have $(A - B)^2 \succcurlyeq 0$, and thus

$$\begin{aligned} \text{tr}(A^2 + B^2) &\geq \text{tr}(AB) + \text{tr}(BA) \\ \text{tr}(A^2 + B^2) &\geq 2\text{tr}(AB), \end{aligned}$$

because $\text{tr}(AB) = \text{tr}(BA)$ since A, B are symmetric. Subtracting $2\text{tr}(B^2)$ from both sides yields:

$$\begin{aligned} \text{tr}(A^2 - B^2) &\geq 2\text{tr}(AB) - 2\text{tr}(B^2) \Rightarrow \\ \text{tr}(A^2 - B^2) &\geq 2\text{tr}((A - B)B) \Rightarrow \\ \text{tr}(A^2 - B^2) &\geq 2\text{tr}((A - B)^{1/2}(A - B)^{1/2}B) \Rightarrow \\ \text{tr}(A^2 - B^2) &\geq 2\text{tr}((A - B)^{1/2}B(A - B)^{1/2}), \end{aligned}$$

where the last step again follows because $(A - B)^{1/2}$ and B are symmetric. Now $A - B$ is symmetric and positive semi-definite, and B is symmetric and positive semi-definite. Thus the right-hand side is nonnegative, so the desired conclusion follows. \square

Lemma 2.6.2. *Let $A \in \mathbb{R}^{n \times p}$ and let A_k be the rank- k approximation of A . Let $P \in \mathbb{R}^{n \times p}$ be rank k . Then*

$$\|A_k - P\|_F^2 \leq 8k \|A - P\|_{OP}^2.$$

Proof. Since A_k and P are rank k , $A_k - P$ has rank at most $2k$. And for any rank $2k$

matrix B , $\|B\|_F^2 \leq 2k \|B\|_2^2$. So,

$$\begin{aligned} \|A_k - P\|_F^2 &\leq 2k \|A_k - P\|_2^2 \\ &\leq 4k \|A_k - A\|_2^2 + 4k \|A - P\|_2^2 \\ &\leq 8k \|A - P\|_2^2 \end{aligned}$$

The third step follows because $\|A - A_k\|_F^2 \leq \|A - P\|_F^2$ for any rank- k matrix P by definition of A_k . \square

Another proof of Lemma 4.5.8. The rank- k approximation of A minimizes $\|A_k - A\|_F$ over rank k matrices, so $\|A_k - A\|_F^2 \leq \|P - A\|_F^2$. Adding and subtract P in $\|A_k - A\|_F^2$ yields

$$\|A_k - P\|_F^2 + \|P - A\|_F^2 + 2\langle A_k - P, P - A \rangle \leq \|A - P\|_F^2$$

which then yields:

$$\|A_k - P\|_F^2 \leq 2\langle A_k - P, A - P \rangle \quad (2.6.1)$$

So

$$\begin{aligned} \|A_k - P\|_F &\leq \sup_{\|U\|_F=1, \text{rank}(U)=k} \langle U, Z \rangle \\ &\leq \sqrt{k} \|Z\|_2 \end{aligned}$$

\square

Lemma 2.6.3 is closely related to Lemma 2.3.2.

Lemma 2.6.3. *Let $\mathbf{j} = (j_1, \dots, j_d) \in \mathbb{Z}_+^d$, and let Γ, Γ' be k -atomic distributions on \mathbb{R}^d with atoms spanning a space of rank r . Then there is a $\theta \in S^{d-1}$ and a constant*

$c_{\mathbf{j}}$ such that

$$\Delta m_{\mathbf{j}}(\Gamma, \Gamma') = \Delta m_{|\mathbf{j}|}(\Gamma_{\theta_{\mathbf{j}}}, \Gamma'_{\theta_{\mathbf{j}}}).$$

And $c_{\mathbf{j}} \leq r^{(|\mathbf{j}|-1)/2}$.

Proof. Let $U \in \mathbb{R}^d$. There is a $\theta_{\mathbf{j}} \in S^{d-1}$ and a constant $c_{\mathbf{j}}$ such that $U_1^{j_1} \dots U_d^{j_d} = c_{\mathbf{j}}(\theta_{\mathbf{j}}^\top U)^{|\mathbf{j}|}$. Thus

$$\Delta m_{\mathbf{j}}(\Gamma, \Gamma') = c_{\mathbf{j}} \Delta m_{|\mathbf{j}|}(\Gamma_{\theta_{\mathbf{j}}}, \Gamma'_{\theta_{\mathbf{j}}}).$$

And the upper bound on $c_{\mathbf{j}}$ holds because of (1.2.14). \square

Alternative proof of Theorem 2.3.1. Let $V \in S^{d \times k}$ have columns that form an orthonormal basis of the space spanned by the atoms of Γ, Γ' . Let r be the dimension of this space, and note that $r \leq 2k$. To simplify notation, let $\gamma = \Gamma_V, \gamma' = \Gamma'_V$. Now

$$\chi^2(P_\Gamma || P_{\Gamma'}) \leq e^{R^2/2} \sum_{\mathbf{j} \in Z_+^d} \frac{(\Delta m_{\mathbf{j}}(\gamma, \gamma'))^2}{\mathbf{j}!} \quad (2.6.2)$$

$$= e^{R^2/2} \sum_{\mathbf{j}} \frac{c_{\mathbf{j}}^2 \left(\Delta m_{|\mathbf{j}|}(\gamma_{\theta_{\mathbf{j}}}, \gamma'_{\theta_{\mathbf{j}}}) \right)^2}{\mathbf{j}!} \quad (2.6.3)$$

$$\leq e^{R^2/2} \sum_{\mathbf{j}} \frac{c_{\mathbf{j}}^2 \max_{\ell \in [2k-1]} \left(3^{|\mathbf{j}|} \Delta m_{\ell}(\gamma_{\theta_{\mathbf{j}}}, \gamma'_{\theta_{\mathbf{j}}}) \right)^2}{\mathbf{j}!} \quad (2.6.4)$$

$$\leq \max_{\ell \in [2k-1]} \sup_{\theta \in S^{d-1}} (\Delta m_{\ell}(\Gamma_{\theta}, \Gamma'_{\theta}))^2 e^{R^2/2} \sum_{\mathbf{j}} \frac{c_{\mathbf{j}}^2 9^{|\mathbf{j}|}}{\mathbf{j}!},$$

where (2.6.2) is by Lemma 1.2.2, (2.6.3) is by Lemma 2.6.3, and (2.6.4) is by Lemma 10 of [Wu and Yang, 2019]. And

$$\sum_{\mathbf{j}} \frac{c_{\mathbf{j}}^2 9^{|\mathbf{j}|}}{\mathbf{j}!} \leq \sum_{\mathbf{j}} \frac{r^{|\mathbf{j}|-1} 9^{|\mathbf{j}|}}{\mathbf{j}!} = \left(\sum_{j \geq 1} \frac{r^{j-1} 9^j}{j!} \right)^r,$$

which is a constant depending on r only. □

2.6.2 Failure of the MLE in weights selection

We now provide more discussion on the intuition behind Algorithm 1. Suppose $k = 2$ and the locations do not lie on a single line in \mathbb{R}^d . Via the DMM algorithm, we estimate the two one-dimensional distributions $\tilde{\gamma}^{(e_1)}, \tilde{\gamma}^{(e_2)}$ with corresponding atoms $(\tilde{\psi}_1^{(e_1)}, \tilde{\psi}_2^{(e_1)})$ and $(\tilde{\psi}_1^{(e_2)}, \tilde{\psi}_2^{(e_2)})$. Now how do we combine these atoms into two-dimensional estimates? And what weights should we match with them?

One approach is to simply select one of the k estimated sets of weights as our estimate. But this will not work for two reasons. One is the identifiability issue mentioned above. The second is that we do not have a guarantee on each individual weight. For instance, suppose that on one coordinate, one of the locations has norm smaller than $\epsilon_{n,k}$. Then the DMM algorithm can easily estimate an associated weight that is very far from the truth.

Another naive approach to the above-mentioned problem is to consider every combination of centers into vectors, pair each combination with a set of weights via maximum likelihood, and select the overall best estimator again via maximum likelihood. Alternatively, we might simply pair sets of locations with potential weights from a discretization of the simplex, then select the best of these via maximum likelihood.

To obtain guarantees on a maximum likelihood procedure, we will need obtain an upper bound of $1/n$ on the KL divergence between the mixture densities. But Example 1 shows that there exists a model where one-dimensional estimates cannot be used in any way that would obtain the correct bound on the KL divergence. Note that this does not necessarily mean that this type of MLE procedure for selecting the weights fails, only that we do not have the tools to prove it works.

Example 1. Let $k = 2$ and suppose we have k -atomic distributions in \mathbb{R}^k . Let

$\epsilon = n^{-1/6}$. Define

$$\gamma = \frac{2}{3}\delta_{(\epsilon, \epsilon)} + \frac{1}{3}\delta_{(-2\epsilon, -2\epsilon)}.$$

The one-dimensional, coordinate-wise marginal distributions are

$$\begin{aligned}\gamma_1 &= \frac{2}{3}\delta_\epsilon + \frac{1}{3}\delta_{-2\epsilon}. \\ \gamma_2 &= \frac{2}{3}\delta_\epsilon + \frac{1}{3}\delta_{-2\epsilon}.\end{aligned}$$

Consider the following estimates of each one-dimensional marginal distribution.

$$\begin{aligned}\hat{\gamma}_1 &= \frac{2}{3}\delta_\epsilon + \frac{1}{3}\delta_{-2\epsilon}. \\ \hat{\gamma}_2 &= \frac{2}{3}\delta_{-\epsilon} + \frac{1}{3}\delta_{2\epsilon}.\end{aligned}$$

Now $m_r(\hat{\gamma}_1) = m_r(\gamma_1)$ for all r . And $\max_{r \in [2k-1]} |m_r(\hat{\gamma}_2) - m_r(\gamma_2)| = m_3(\hat{\gamma}_2) - m_3(\gamma_2) \lesssim \epsilon^3$. By Lemma 2.5.1, for each one-dimensional coordinate $j \in [k]$, $W_1(\hat{\gamma}_j, \gamma_j) \leq C\epsilon$ for a constant C . So $\hat{\gamma}_1, \hat{\gamma}_2$ are estimates that could be produced by the DMM algorithm.

Let \mathcal{S}_c be as defined as in Algorithm 1. In this example,

$$\mathcal{S}_c = \{(\epsilon, -\epsilon)^\top, (\epsilon, 2\epsilon)^\top, (-2\epsilon, -\epsilon)^\top, (-2\epsilon, 2\epsilon)^\top\}.$$

Define

$$\hat{\gamma} = \sum_{j=1}^4 w_j \delta_{\mu_j},$$

where μ_1, \dots, μ_4 are the vectors in \mathcal{S}_c and $w = (w_1, \dots, w_4) \in \Delta^3$. We might pick some of the weights to be zero so we have a 2-atomic distribution. But it turns out that no matter which weights we select (even if we use maximum likelihood to select

them), we can show that in this case, $KL(P_\gamma||P_{\gamma'}) \geq \epsilon^4$. For a formal justification, see Lemma 2.6.4.

Lemma 2.6.4. *Let $\hat{\gamma}, \gamma, \epsilon$ be as defined in Example 1. Then there is a positive constant C such that*

$$KL(P_{\hat{\gamma}}||P_\gamma) \geq C\epsilon^4.$$

Proof. By the data processing inequality, $KL(P_{\hat{\gamma}}||P_\gamma) \geq \sup_{\theta \in S^{k-1}} KL(\hat{\gamma}_\theta * N(0, 1) || \gamma_\theta * N(0, 1))$. Let $\theta = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^T$. Note that $\gamma_\theta = \delta_0$, so $\gamma_\theta * N(0, 1)$ is just the $N(0, 1)$ distribution. Define the support set obtained by taking the inner product of θ with each point in \mathcal{S}_c via:

$$\theta^\top \mathcal{S}_c \triangleq \left\{ \sqrt{2}\epsilon, -\frac{\epsilon}{\sqrt{2}}, -\frac{4\epsilon}{\sqrt{2}}, \right\}.$$

Then $\hat{\gamma}_\theta = w_1\delta_{\theta^\top \mu_1} + (w_2 + w_3)\delta_{\theta^\top \mu_2} + w_4\delta_{\theta^\top \mu_4}$. Choosing $f(x) = tx^2$ for a constant $t < 1/2$, and using the Donsker-Varadhan representation of the KL -divergence,

$$KL(P_{\hat{\gamma}_\theta}||P_{\gamma_\theta}) \geq \mathbb{E}_{X \sim P_{\hat{\gamma}_\theta}} tX^2 - \log \mathbb{E}_{X \sim N(0,1)} e^{tX^2} = \mathbb{E}_{X \sim P_{\hat{\gamma}_\theta}} tX^2 + \frac{1}{2} \log(1 - 2t).$$

Now regardless of which weights w were chosen, $\mathbb{E}_{X \sim P_{\hat{\gamma}_\theta}} tX^2 \propto Ct\epsilon^2 + t$ for a constant C . Let $t = C\epsilon^2/2(C\epsilon^2 + 1)$. This yields a lower bound proportional to ϵ^4 . \square

2.6.3 Symmetric 2-GM

In this section, we explore a simple, classical example of the model (2.1.1): the symmetric, two-component Gaussian mixture model. Since this model is a subset of the more general model (2.1.1), all results from the previous sections hold. However, in many cases, the estimation procedures, theorems, and proofs can be simplified in the symmetric 2-GM. We give these simplifications here because they illuminate much

of the fundamental ideas from the previous sections, and they also tell us something about a model we will consider in the next chapter, the k -spaces model.

Let

$$P_\mu = \Gamma * N(0, I_d) \text{ where } \Gamma = \frac{1}{2}\delta_\mu + \frac{1}{2}\delta_{-\mu}. \quad (2.6.5)$$

Let $\mathcal{G}_d = \{\Gamma = \frac{1}{2}\delta_\mu + \frac{1}{2}\delta_{-\mu} : \mu \in \mathbb{R}^d, \|\mu\|_2 \leq R\}$. For this model, we will use the loss

$$\ell(\mu, \mu') = \min(\|\mu - \mu'\|_2, \|\mu + \mu'\|_2), \quad (2.6.6)$$

which is in fact equal to $W_1(\Gamma_\mu, \Gamma_{\mu'})$, as discussed previously. For the model (2.6.5), spectral algorithms achieve the sharp rate for both parameter estimation and density estimation. Let $\hat{\mu}\hat{\mu}^\top$ be the rank-1 approximation of $\hat{\Sigma}$ in (2.2.1). Then it is easy to show that

$$\|\hat{\mu}\hat{\mu}^\top - \mu\mu^\top\|_2 \leq (d/n)^{1/2}, \quad (2.6.7)$$

with high probability; see for instance, Lemma 4.5.8 and Lemma 2.2.6. We now present a lemma that shows we can directly our error on $\hat{\mu}$ to the covariance gap in (2.6.7).

Lemma 2.6.5. *Let $\mu \in \mathbb{R}^d$ and let $\|\mu\|_2 \leq R$. Let $\hat{\mu} \in \mathbb{R}^d$ satisfy $\|\hat{\mu}\hat{\mu}^\top - \mu\mu^\top\|_2 \leq \epsilon$. Then*

$$\ell(\hat{\mu}, \mu) \leq \sqrt{2\epsilon}. \quad (2.6.8)$$

Lemma 2.6.5 shows that the bound in (2.6.7) translates directly to a bound on $\ell(\hat{\mu}, \mu)$; when μ is arbitrary, a square root factor is lost, but when $\mu \in S^{d-1}$, the rate of estimation is the same for the covariance matrix $\hat{\mu}\hat{\mu}^\top$ and $\hat{\mu}$. This is intuitive, since if we allow $\|\mu\|_2$ to be small, we expect that it is more difficult to estimate its associated direction. (2.6.8) translates to the $(d/n)^{1/4}$ rate on $\ell(\hat{\mu}, \mu)$, i.e., we obtain a parameter

estimator directly from a simple spectral algorithm. Moreover, by Theorem 2.3.1, since both the first and third moment tensors are zero by symmetry, we have

$$H(P_\mu, P_{\mu'})^2 \asymp \|\mu\mu^\top - \mu'\mu'^\top\|_F^2. \quad (2.6.9)$$

Moreover, in this simple model we are able to obtain a maximum likelihood result for both density estimation and parameter estimation.

Lemma 2.6.6. *Let $\hat{\mu} = \operatorname{argmax}_{\|\mu\|_2 \leq R} \sum_{i=1}^n \log p_\mu(x_i)$. Then with probability at least $1 - \delta$,*

$$\ell(\hat{\mu}, \mu) \leq \left(\frac{d \log(1/\delta)}{n} \right)^{1/4}. \quad (2.6.10)$$

Proof. This follows from Lemma 2.6.5 and Theorem 5.1.1. \square

We now prove Lemma 2.6.5. We moreover provide an alternative proof of (2.6.9). This is a somewhat simpler version of Theorem 2.3.1 that applies to the symmetric 2-GM only.

Proof of Lemma 2.6.5. We first prove (2.6.8). Without loss of generality, let $\|\mu\|_2 \geq \|\hat{\mu}\|_2$. Now $\|\hat{\mu}\hat{\mu}^\top - \mu\mu^\top\|_2 \leq \epsilon$ implies

$$\left| \frac{\langle \hat{\mu}, \mu \rangle^2}{\|\mu\|_2^2} - \|\mu\|_2^2 \right| \leq \epsilon. \quad (2.6.11)$$

Now

$$\begin{aligned} \|\mu\|_2^2 - \epsilon &\leq \frac{\langle \hat{\mu}, \mu \rangle^2}{\|\mu\|_2^2} && \text{by (2.6.11) and the triangle inequality} \\ &\leq \frac{|\langle \hat{\mu}, \mu \rangle|}{\|\mu\|_2^2} \|\hat{\mu}\|_2 \|\mu\|_2 && \text{by Cauchy-Schwarz} \\ &\leq |\langle \hat{\mu}, \mu \rangle| && \text{since } \|\hat{\mu}\|_2 \leq \|\mu\|_2. \end{aligned}$$

And since $\|\hat{\mu}\|_2 \leq \|\mu\|_2$, we also obtain $\|\hat{\mu}\|_2^2 - \epsilon \leq |\langle \hat{\mu}, \mu \rangle|$. Thus

$$\|\hat{\mu}\|_2^2 + \|\mu\|_2^2 - 2\epsilon \leq 2|\langle \hat{\mu}, \mu \rangle|,$$

which implies the result. \square

Lemma 2.6.7. *Let $\Gamma, \Gamma_* \in \mathcal{G}_d$. Then for any $D \in \{H^2, \text{KL}, \chi^2\}$,*

$$D(P_\Gamma, P_{\Gamma_*}) \asymp_k \|\mu\mu^\top - \mu_*\mu_*^\top\|_2.$$

Proof. The lower bound is immediate from Lemma 2.3.3. For the upper bound, let

$f(x, \mu) = \frac{1}{2}e^{\langle x, \mu \rangle - \|\mu\|_2^2/2} + \frac{1}{2}e^{\langle x, -\mu \rangle - \|\mu\|_2^2/2}$. By Lemma 1.2.3, it suffices to show that

$$\int_{x \in \mathbb{R}^d} \phi_d(x) (f(x, \mu) - f(x, \mu_*))^2 \leq C \|\mu\mu^\top - \mu_*\mu_*^\top\|_2^2.$$

Let H be the orthogonal projection operator onto the space spanned by μ, μ_* , and let

$V \in S_H$. Let $\tilde{\mu} = V^\top \mu, \tilde{\mu}_* = V^\top \mu_*$. By the rotation invariance of the Gaussian distribution, we reduce ourselves to 2-dimensional space, i.e., $\int_{x \in \mathbb{R}^d} \phi_d(x) (f(x, \mu) - f(x, \mu_*))^2 =$

$\int_{x \in \mathbb{R}^2} \phi_2(x) (f(x, \tilde{\mu}) - f(x, \tilde{\mu}_*))^2 dx$. Now, let $\Gamma = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. Then we can write

$f(x, \tilde{\mu}) = \mathbb{E}_{y \sim \Gamma} e^{\langle x, y\tilde{\mu} \rangle - \|\tilde{\mu}\|_2^2/2}$. Throughout the rest of the proof, $x \in \mathbb{R}^2$. Now

$$f(x, \tilde{\mu}) - f(x, \tilde{\mu}_*) = e^{-\|\tilde{\mu}\|_2^2/2} \mathbb{E}_{y \sim \Gamma} (e^{\langle x, y\tilde{\mu} \rangle} - e^{\langle x, y\tilde{\mu}_* \rangle}) + \mathbb{E}_{y \sim \Gamma} e^{\langle x, y\tilde{\mu}_* \rangle} (e^{-\|\tilde{\mu}\|_2^2/2} - e^{-\|\tilde{\mu}_*\|_2^2/2}).$$

For the first term, since Γ is a symmetric distribution, its odd moments are zero. So,

$$\begin{aligned}
\mathbb{E}_{y \sim \Gamma} (e^{\langle x, y \tilde{\mu} \rangle} - e^{\langle x, y \tilde{\mu}_* \rangle}) &= \sum_{j \geq 1} \frac{\langle x, \tilde{\mu} \rangle^{2j} - \langle x, \tilde{\mu}_* \rangle^{2j}}{j!} \\
&\leq \left| \langle x, \tilde{\mu} \rangle^2 - \langle x, \tilde{\mu}_* \rangle^2 \right| \sum_{j \geq 1} \frac{j (\max(\langle x, \tilde{\mu} \rangle^2, \langle x, \tilde{\mu}_* \rangle^2))^{j-1}}{j!} \\
&\leq \|x\|_2^2 \|\mu \mu^\top - \mu_* \mu_*^\top\|_2 \sum_{j \geq 0} \frac{\|x\|_2^{2j} R^j}{j!} \\
&= \|\mu \mu^\top - \mu_* \mu_*^\top\|_2 \|x\|_2^2 e^{\|x\|_2^2 R},
\end{aligned} \tag{2.6.12}$$

where (2.6.12) follows because $\|\tilde{\mu} \tilde{\mu}^\top - \tilde{\mu}_* \tilde{\mu}_*^\top\|_2 = \|\mu \mu^\top - \mu_* \mu_*^\top\|_2$. Thus

$$\begin{aligned}
\int_{x \in \mathbb{R}^2} \phi_2(x) (f(x, \mu) - f(x, \mu_*))^2 dx &\leq \|\mu \mu^\top - \mu_* \mu_*^\top\|_2^2 \int_{x \in \mathbb{R}^2} \|x\|_2^4 e^{2\|x\|_2^2 R} dx \\
&\leq C \|\mu \mu^\top - \mu_* \mu_*^\top\|_2^2.
\end{aligned}$$

And for the second term, since $e^{\langle x, y \tilde{\mu}_* \rangle} > 0$,

$$\begin{aligned}
\mathbb{E}_{y \sim \Gamma} e^{\langle x, y \tilde{\mu}_* \rangle} \left(e^{-\|\tilde{\mu}\|_2^2/2} - e^{-\|\tilde{\mu}_*\|_2^2/2} \right) &\leq \mathbb{E}_{y \sim \Gamma} e^{\langle x, y \tilde{\mu}_* \rangle} |e^{-\|\tilde{\mu}\|_2^2/2} - e^{-\|\tilde{\mu}_*\|_2^2/2}| \\
&\leq \mathbb{E}_{y \sim \Gamma} e^{\langle x, y \tilde{\mu}_* \rangle} \left| \|\tilde{\mu}_*\|_2^2 - \|\tilde{\mu}\|_2^2 \right| \\
&\leq \mathbb{E}_{y \sim \Gamma} e^{\langle x, y \tilde{\mu}_* \rangle} \|\mu \mu^\top - \mu_* \mu_*^\top\|_2.
\end{aligned}$$

And $\int \phi_2(x) (\mathbb{E}_{y \sim \Gamma} e^{\langle x, y \tilde{\mu}_* \rangle})^2 dx \leq C$. □

2.6.4 Discussion

We have focused on the Gaussian location mixture model (2.1.1) in high dimensions, where the variance parameter σ^2 and the number of components k are known, and the centers lie in a ball of bounded radius. Below we discuss weakening these assumptions and other open problems.

Unbounded centers While the assumption of bounded support is necessary for estimating the mixing distribution (otherwise the worst-case W_1 -loss is infinity), it is not needed for density estimation [Acharya et al., 2014, Li and Schmidt, 2017, Ashtiani et al., 2018]. In fact, [Acharya et al., 2014] first uses a crude clustering procedure to partition the samples into clusters whose means are close to each other, then zooms into each cluster to perform density estimation. For the lower bound, the worst case occurs when each cluster is equally weighted and highly separated, so that the effective sample size for each component is n/k , leading to the lower bound of $\Omega(\frac{kd}{n})$. Finally, the results of NPMLE in [Ghosal and van der Vaart, 2001, Zhang, 2009, Saha and Guntuboyina, 2017] do not impose bounded assumptions, which is partly responsible for the logarithmic factors in the obtained rates.

Location-scale mixtures We have assumed that the covariance of our mixture is known and common across components. There is a large body of work studying general location-scale Gaussian mixtures, see, e.g., [Moitra and Valiant, 2010, Ho and Nguyen, 2016, Ashtiani et al., 2018]. The introduction of the scale parameters makes the problem significantly more difficult. For parameter estimation, if all clusters share the same unknown scale parameter, the optimal rate is shown to be $n^{-1/(4k)}$ in [Wu and Yang, 2019]; otherwise, the optimal rate remains unknown even in one dimension except for $k = 2$ [Hardt and Price, 2015].

Number of components This work assumes that the parameter k is known and fixed. Since the centers are allowed to overlap arbitrarily, k is effectively an upper bound on the number of components. If k is allowed to depend on n , the optimal W_1 -rate is shown in [Wu and Yang, 2019] to be $\Theta(n^{-1/(4k-2)})$ and $\Theta(\frac{\log \log n}{\log n})$ for $k = O(\frac{\log n}{\log \log n})$ and $k = \Omega(\frac{\log n}{\log \log n})$, respectively. Extending this result to the high-dimensional setting of Theorem 2.1.1 is an interesting future direction.

The problem of selecting the mixture order k has been extensively studied. For in-

stance, many authors have considered likelihood-ratio based tests; however, standard asymptotics for such tests may not hold [Hartigan, 1985]. Various workarounds have been considered, including method of moments [Lindsay, 1989, Dacunha-Castelle and Gassiat, 1997] tests inspired by the EM algorithm [Li and Chen, 2010] and quadratic approximation of the log-likelihood ratio [Liu and Shao, 2003].

Connection to spectral clustering Another major strain of the Gaussian mixture literature is on clustering. As previously noted, under our weak assumptions, clustering is not possible. But we here briefly discuss the connection between the parameter estimation technique used to obtain Theorem 2.1.1 and the classical spectral clustering technique studied in, for instance, [Vempala and Wang, 2004, Kannan et al., 2005, Achlioptas and Mcsherry, 2010, Löffler et al.,]. In spectral clustering, data are first projected to a best subspace, and then a clustering algorithm such as k -means is performed on the projected data. The best subspace in these papers is the same as the one estimated via PCA, and the clustering error analysis depends on the error in the subspace estimation and the error of the clustering method performed on the low-dimensional data, analogously to the analysis of mixing distribution estimation in Theorem 2.1.1. Our specific statement on the rate of estimating the subspace, Lemma 2.2.5, is closely connected to Theorem 3 of [Vempala and Wang, 2004].

Adaptivity The rate in Theorem 2.1.1 is optimal in the worst-case scenario where the centers of the Gaussian mixture can overlap. To go beyond this pessimistic result, in one dimension, [Heinrich and Kahn, 2018] showed that when the atoms of Γ form k_0 well-separated (by a constant) clusters (see [Wu and Yang, 2019, Definition 1] for a precise definition), the optimal rate is $n^{-1/(4(k-k_0)+2)}$, interpolating the rate $n^{-1/(4k-2)}$ in the worst case ($k_0 = 1$) and the parametric rate $n^{-1/2}$ in the best case ($k_0 = k$). Furthermore, this can be achieved adaptively by either the minimum distance estimator [Heinrich and Kahn, 2018, Theorem 3.3] or the DMM algorithm

[Wu and Yang, 2019, Theorem 2].

In the high-dimensional case, by Lemma 2.2.5, the optimal projection \hat{H} preserves separation of the atoms of Γ . Letting \hat{H} be the projection to the optimal subspace of Algorithm 1,

$$\left\| \hat{V}^\top \mu_i - \hat{V}^\top \mu_{i'} \right\|_2 = \left\| \hat{H} \mu_i - \hat{H} \mu_{i'} \right\|_2 \geq \left\| \mu_i - \mu_{i'} \right\|_2 - \frac{\left\| \mu_i - \hat{H} \mu_i \right\|_2}{w_i} - \frac{\left\| \mu_{i'} - \hat{H} \mu_{i'} \right\|_2}{w_{i'}}.$$

Let $\lambda = C \max((d/n)^{1/4}, n^{-1/(4k-2)})$ for an appropriate positive constant C . And replace weights lower bound assumption of [Wu and Yang, 2019, Definition 1] with

$$w_i = c_k \text{ for all } i \in S_\ell, S_{\ell'}. \quad (2.6.13)$$

Then assuming the location separation condition of [Wu and Yang, 2019, Definition 1] and (2.6.13) hold, we have

$$\left\| \hat{V}^\top \mu_i - \hat{V}^\top \mu_{i'} \right\|_2 \gtrsim \lambda.$$

Therefore, if Γ has k_0 clusters, projecting the atoms of Γ onto the optimal subspace maintains the k_0 clusters. If the dimension of \hat{V} is one, and we simply perform DMM on the projected data $\{V^\top X_i\}_{i \in [n]}$, we will be adaptive to separation on the model locations; rates like that of Theorem 2 of [Wu and Yang, 2019] will hold for the low-dimensional mixing estimation component of the rate in Theorem 2.1.1.

Two complications occur if we try to generalize this to general k -component mixtures. First, Algorithm 1 is not adaptive in general due to the coarseness of the grid search. This can be easily remedied by using an ϵ_n net with $\epsilon_n = n^{-1/2}$ rather than $n^{-1/(4k-2)}$ in Algorithm 1. This results in an algorithm of time $n^{O(k)}$, as discussed in Section 2.2. Alternatively, by the same reasoning as in the proof of Theorem 2.3.6, the density estimator of Theorem 2.1.2 should be able to provide a k -GM that is

adaptive to stronger assumptions.

However, even if we were to use a finer grid search, the analysis of that algorithm depends on the sliced W_1 upper bound of Lemma 2.2.1. In the bound $W_1(\Gamma, \Gamma') \leq C_k \sup_{\theta \in S^{k-1}} W_1(\Gamma_\theta, \Gamma'_\theta)$, it is not clear that the θ used in the upper bound maintains any separation between the locations of Γ, Γ' . Thus it remains an open problem to demonstrate that the estimator in Theorem 2.3.6 is adaptive, and to find a fast algorithm that is adaptive.

2.7 Extensions

2.7.1 Algorithm for density estimation

In this section, we propose an alternative method for obtaining the density estimation result Theorem 2.1.2. What we hope to prove is that statistical distances between Gaussian mixtures on k means can be represented by the covariance distances and low-dimensional moments distances. This would allow us to reprove Theorem 2.1.2 using a different style of counting argument. But more importantly, it would allow us to obtain a fast algorithm for density estimation based on PCA and low-dimensional moment estimation; this algorithm is described in Algorithm 2. Crucially, our proposed algorithm would run in time $O(n^{ck})$. It is known [Diakonikolas et al., 2017] that a density estimation algorithm whose resulting estimator achieves the correct statistical rate must have time at least $n^{O(k)}$; thus, our algorithm would be computationally optimal, and, if we can show it achieves the correct rate as in Theorem 2.1.2, the estimator resulting from this algorithm would be statistically optimal.

Conjecture 2.7.1. *Let $\Gamma, \Gamma' \in \mathcal{G}_{k,d}$ and let them be symmetric about zero. Let their covariance matrices be Σ, Σ' , respectively. Let $V' \in S^{d \times k}$ be a matrix whose columns form an orthonormal basis of the space spanned by the atoms of Γ' . Let $\gamma = \Gamma_{V'}, \gamma' =$*

$\Gamma'_{V'}$. For any $D \in \{H^2, \text{KL}, \chi^2\}$,

$$D(P_\Gamma, P_{\Gamma'}) \asymp_k \max \left(\|\Sigma - \Sigma'\|_2^2, \max_{\ell \in [2k-1]} \sup_{\theta \in S^{k-1}} (\Delta m_\ell(\gamma_\theta, \gamma'_\theta))^2 \right). \quad (2.7.1)$$

Proof of Lemma 2.7.1. Let H' be the orthogonal projection operator onto the space spanned by the k atoms of Γ' . By Theorem 2.3.1 and the triangle inequality,

$$\chi^2(\mathbb{P}_\Gamma || \mathbb{P}_{\Gamma'}) \lesssim_k \underbrace{\max_{\ell \in [2k-1]} \|M_\ell(\Gamma) - M_\ell(\Gamma_{H'})\|_F^2}_{(1)} + \underbrace{\max_{\ell \in [2k-1]} \|M_\ell(\Gamma_{H'}) - M_\ell(\Gamma')\|_F^2}_{(2)}.$$

By (1.2.14), for $U \sim \Gamma, U' \sim \Gamma'$,

$$(2) \lesssim_k \max_{\ell \in [2k-1]} \sup_{\theta \in S^{d-1}} (\Delta m_\ell(\theta^\top H U, \theta^\top U'))^2 = \max_{\ell \in [2k-1]} \sup_{\theta \in S^{k-1}} (\Delta m_\ell(\theta^\top V'^\top U, \theta^\top V'^\top U'))^2,$$

which gives the second part of the bound. We provide an outline for how to prove the covariance gap component of the upper bound below in 2.7.1.

For the covariance part of the lower bound, we apply the Data Processing Inequality and the second moment in Lemma 2.3.3, as usual:

$$H(P_{\Gamma'}, P_\Gamma) \geq \sup_{\theta \in S^{d-1}} H(P_{\Gamma'_\theta}, P_{\Gamma_\theta}) \geq \sup_{\theta \in S^{d-1}} |m_2(\Gamma'_\theta) - m_2(\Gamma_\theta)| \geq \|\Sigma' - \Sigma\|_2.$$

And the moments part of the lower bound similarly follows from Data Processing and Lemma 2.3.3. \square

We note that we can already show that something slightly weaker than Lemma 2.7.1 is true; namely, that

$$D(P_\Gamma, P_{\Gamma'}) \asymp_k \max \left(\|\Sigma - \Sigma'\|_2^4, \max_{r \in [2k-1]} \sup_{\theta \in S^{k-1}} (\Delta m_r(\gamma_\theta, \gamma'_\theta))^2 \right). \quad (2.7.2)$$

This means that the algorithm in Algorithm 2 yields an estimator that achieves a rate

of $(d/n)^{1/4}$ for density estimation. Though this is not the sharp rate, it is sharper than what has been achieved so far by fast algorithms; the best known so far is $(d/n)^{1/4} * \text{polylog}(n)$ from [Acharya et al., 2014].

Sketch proof for covariance gap upper bound in (2.7.2). We must show that

$$\underbrace{\max_{\ell \in [2k-1]} \|M_\ell(\Gamma) - M_\ell(\Gamma_{H'})\|_F^2}_{(1)} \lesssim \|\Sigma - \Sigma'\|_2^2.$$

First, let's suppose that the distributions Γ, Γ' are symmetric. Then the following steps are clear with the exception of (2.7.3).

$$\begin{aligned} (1) &\lesssim_k \max_{r \in [2k-1]} \|M_{2\ell}(\Gamma) - M_{2\ell}(H'\Gamma)\|_F^2 \\ &\lesssim_k \|\Sigma - H'\Sigma H'\|_F^2 \\ &\lesssim_k \|\Sigma - \Sigma'\|_F^2 + \|\Sigma' - H'\Sigma H'\|_F^2 \\ &\lesssim_k 2 \|\Sigma - \Sigma'\|_F^2 \\ &\lesssim_k 2 \|\Sigma - \Sigma'\|_2^2, \end{aligned} \tag{2.7.3}$$

where the last step follows since the matrices are low rank. We must show the crucial step (2.7.3); I think it is not actually obviously true. Here is an attempt. Let

$$\Gamma = \sum_{j \in [k]} w_j \delta_{\mu_j}.$$

$$\begin{aligned} (1) &\lesssim_k \max_{\ell \in [2k-1]} \sup_{\theta \in S^{d-1}} |\mathbb{E}_{U \sim \Gamma} ((\theta^\top U)^{2\ell} - (\theta^\top H'U)^{2\ell})|^2 \\ &= \max_{\ell \in [2k-1]} \sup_{\theta \in S^{d-1}} \left| \sum_{j=1}^k w_j ((\theta^\top \mu_j)^{2\ell} - (\theta^\top H'\mu_j)^{2\ell}) \right|^2. \end{aligned}$$

Now the θ that achieves the sup lies in the space spanned by the $\mu_j, H'\mu_j$. If we can show that it must contain elements of all the basis vectors in that space, which seems

plausible for it to achieve the sup, then we will have

$$\begin{aligned}
\max_{\ell \in [2k-1]} \sup_{\theta \in S^{d-1}} \left| \sum_{j=1}^k w_j ((\theta^\top \mu_j)^{2\ell} - (\theta^\top H' \mu_j)^{2\ell}) \right|^2 &= \max_{\ell \in [2k-1]} \left| \sum_{j=1}^k w_j \|\mu_j\|_2^{2\ell} - \|H' \mu_j\|_2^{2\ell} \right| \\
&\leq \max_{\ell \in [2k-1]} \ell R^{2(\ell-1)} \sum_{j=1}^k w_j \left| \|\mu_j\|_2^2 - \|H' \mu_j\|_2^2 \right| \\
&\lesssim_{k,R} \sum_{j=1}^k w_j \|\mu_j - H' \mu_j\|_2^2 \\
&\lesssim_{k,R} k \|\Sigma - \Sigma'\|_2 \text{ by Lemma 2.2.5.}
\end{aligned}$$

Thus in this way, we could actually avoid the steps after (2.7.3) and obtain the covariance bound directly. □

Lemma 2.7.2. *Let $\mathbb{P}_\Gamma \in \mathcal{P}_{k,d}$ be the true model. Let \mathbb{P}_0 be the true model, and let $\mathcal{P}(\delta) := \{P_{\Gamma'} : H(P_{\Gamma'}, P_\Gamma) \leq \delta\}$. Then*

$$\mathcal{N}(\epsilon, \mathcal{P}(\delta), H) \leq C \left(\frac{\delta}{\epsilon} \right)^{dk+k^k}.$$

Proof of Lemma 2.7.2. Define $\mathcal{G}(\delta) = \{\Gamma' \in \mathcal{G}_{k,d} : H(P_{\Gamma'}, P_\Gamma) \leq \delta\}$. Let $V' \in S^{d \times k}$ be a matrix whose columns are an orthonormal basis for the k -dimensional subspace spanned by the atoms of Γ' . By Lemma 2.7.1 and Lemma 2.6.3,

$$\mathcal{G}(\delta) \subseteq \tilde{\mathcal{G}}(\delta) \triangleq \{\Gamma' : \|\Sigma' - \Sigma\|_2 \leq \delta \text{ and } \max_{r \in [2k-1], \mathbf{r} \in S_{r,k}} |\Delta m_{\mathbf{r}}(\Gamma_{V'}, \Gamma'_{V'})| \leq \delta\}$$

We proceed to form an ϵ -net in the Hellinger metric on $\tilde{\mathcal{G}}(\delta)$. Let $\Gamma' \in \mathcal{G}(\delta)$. Select a covariance matrix, Σ^* satisfying:

$$\|\Sigma' - \Sigma^*\|_2 \leq \epsilon. \tag{2.7.4}$$

This on the covariance space has size no more than $(\delta/\epsilon)^{dk}$. Note that this net defines the kd direction parameters and the second moment parameter. Let V^* be a matrix whose columns form an orthonormal basis for the subspace spanned by the atoms of Γ^* . Let $\gamma' = \Gamma'_{V^*}, \gamma^* = \Gamma^*_{V^*}$. Let $(r_1, r_2) \in S_{2,k}$.

$$\begin{aligned}
|\Delta m_{(r_1, r_2)}(\gamma', \gamma^*)| &\lesssim_k \sup_{\theta \in S^{k-1}} |\Delta m_2(\gamma'_\theta, \gamma^*_\theta)| && \text{by Lemma 2.6.3} \\
&= \sup_{\theta \in S^{k-1}} |\theta^\top V^{*\top} (\mathbb{E}_{U' \sim \Gamma'} U U'^\top - \mathbb{E}_{U^* \sim \Gamma} U^* U^{*\top}) V^* \theta| \\
&\leq \|\Sigma' - \Sigma^*\|_2.
\end{aligned}$$

So we have guaranteed both a covariance gap and second-order moments gap. It remains to form the rest of the distribution Γ^* by determining the remaining moments. We form a Γ^* in this way to satisfy

$$\max_{r \in [2k-1], \mathbf{r} \in S_{r,k}} |\Delta m_{\mathbf{r}}(\gamma', \gamma^*)| \leq \epsilon. \quad (2.7.5)$$

The resulting net has size no more than $(\delta/\epsilon)^{ck^k}$. The total size of the net is the size of the covariance net times the size of the moments net. \square

The result in Lemma 2.7.1 gives us an alternative way to obtain a covering number bound on the relevant parameter space (in this case, the covariance and low-dimensional moments space), which provides another way to obtain the rate via the LeCam Birge estimator or the MLE.

This result also suggests a fast algorithm for density estimation, which we now detail. The first step of the algorithm proceeds by reducing the dimension of our data from d to k , exactly as described in Section 2.2.1. It then remains to estimate the low-dimensional mixing distribution γ . Let $\epsilon_n = n^{-1/2}$. We form an ϵ_n -net on $\mathcal{G}_{k,k}$ as follows. Let \mathcal{W}, \mathcal{A} be $\|\cdot\|_1, \|\cdot\|_2$ ϵ_n -nets on Δ^{k-1} and $\{(\mu_1, \dots, \mu_k)^\top : \mu_j \in \mathbb{R}^k, \|\mu_j\|_2 \leq R\}$, respectively. Let \mathcal{N} be an ϵ_n -net on S^{k-1} . We can guarantee that there is a

constant C such that

$$\max\{|\mathcal{W}|, |\mathcal{A}|, |\mathcal{N}|\} \leq \left(\frac{C}{\epsilon_n}\right)^k \propto O(n^k). \quad (2.7.6)$$

Define the set of candidate distributions via

$$\mathcal{S} = \mathcal{A} \times \mathcal{W}. \quad (2.7.7)$$

Lemma 2.7.3. *Let \mathcal{S} be as in (2.7.7). With probability at least $1 - \delta$,*

$$\min_{\gamma' \in \mathcal{S}} \max_{r \in [2k-1]} \sup_{\theta \in S^{k-1}} |\Delta m_r(\gamma'_\theta, \gamma_\theta)| \leq \left(\frac{\log(1/\delta)}{n}\right)^{1/2}.$$

Proof. Fix $r \in [2k-1], \theta \in S^{k-1}$. Let $\gamma' = \sum_{j=1}^k w'_j \delta_{\mu'_j}, \gamma = \sum_{j=1}^k \delta_{\mu_j}$.

$$\begin{aligned} |\Delta m_r(\gamma'_\theta, \gamma_\theta)| &\leq \sum_{j=1}^k |w'_j - w_j| |(\theta^\top \mu_j)^r| + \sum_{j=1}^k w_j |(\theta^\top \mu'_j)^r - (\theta^\top \mu_j)^r| \\ &\leq R^r \epsilon_n + \sum_{j=1}^k w_j R^{r-1} \|\mu'_j - \mu_j\|_2 \\ &\lesssim_{R,r} \epsilon_n. \end{aligned}$$

This holds for any r, θ . □

We will rely on vanilla Method-of-Moments (MOM), as described in [Wu and Yang, 2019].

Algorithm 2: Density estimation for k -GM in k dimensions

Input: Dataset $\{x_i\}_{i \in [n]}$ with each point in \mathbb{R}^k , order k , radius R .

Output: Estimate $P_{\hat{\gamma}}$ of k -atomic distribution in k dimensions.

Form the set \mathcal{S} of k -atomic candidate distributions on \mathbb{R}^k as in (2.7.7) ;

For each $\theta \in \mathcal{N}$:

Estimate the projection by $\hat{\gamma}_\theta = \text{MOM}(\{\theta^\top x_i\}_{i \in [n]})$;

For each candidate distribution $\gamma' \in \mathcal{S}$ **and each direction** $\theta \in \mathcal{N}$:

Compute $\max_{r \in [2k-1]} |m_r(\gamma'_\theta) - m_r(\hat{\gamma}_\theta)|$;

Report

$$\hat{\gamma} = \arg \min_{\gamma' \in \mathcal{S}} \max_{\theta \in \mathcal{N}} \max_{r \in [2k-1]} |m_r(\gamma'_\theta) - m_r(\hat{\gamma}_\theta)| \quad (2.7.8)$$

Theorem 2.7.4. *There exists a proper density estimate $P_{\hat{\Gamma}}$, computable in $O(n^k)$ time, and a positive constant C_k , such that for any $\Gamma \in \mathcal{G}_{k,d}$ and any $\delta \in (0, 1/2)$, with probability $1 - \delta$,*

$$H(P_{\hat{\Gamma}}, P_{\Gamma}) \leq C_k \left(\frac{d \log(1/\delta)}{n} \right)^{1/2}.$$

Proof. The bound is achieved by the estimator (2.2.5), this time with $\hat{\gamma}$ computed via Algorithm 2. The covariance matrix gap in the upper bound in Lemma 2.7.1 is upper bounded by $(d/n)^{1/2}$ by Lemma 2.2.6. The moments gap in Lemma 2.7.1 is obtained in the following way. Let $\gamma' \in \arg \min_{\gamma'' \in \mathcal{S}} \sup_{\theta \in S^{k-1}} |\Delta m_r(\gamma'_\theta, \gamma_\theta)|$. Let $r \in [2k-1]$. Let $\theta^* = \arg \max_{\theta \in S^{k-1}} |\Delta m_r(\hat{\gamma}_\theta, \gamma_\theta)|$, which exists since S^{k-1} is compact and the moments gap is continuous. Let $u \in \mathcal{A}$ satisfy $\|\theta^* - u\|_2 \leq \epsilon_n$. Note that since $\hat{\gamma}, \gamma \in \mathcal{G}_{k,d}$ and have atoms bounded by R , $|\Delta m_r(\hat{\gamma}_{\theta^*}, \hat{\gamma}_u)| \leq rR^{r-1} \|\theta^* - u\|_2$

and similarly for γ . By the triangle inequality,

$$\begin{aligned} |\Delta m_r(\hat{\gamma}_{\theta^*}, \gamma_{\theta^*})| &\leq |\Delta m_r(\hat{\gamma}_{\theta^*}, \hat{\gamma}_u)| + |\Delta m_r(\hat{\gamma}_u, \gamma_u)| + |\Delta m_r(\gamma_u, \gamma_{\theta^*})| \\ &\lesssim 2rR^{r-1}\epsilon_n \max_{u \in \mathcal{N}} |\Delta m_r(\hat{\gamma}_u, \gamma_u)|. \end{aligned}$$

Now

$$\begin{aligned} \sup_{\theta \in S^{k-1}} |\Delta m_r(\hat{\gamma}, \gamma)| &\lesssim_{k,r,R} 2\epsilon_n + \max_{\theta \in \mathcal{N}} |\Delta m_r(\hat{\gamma}_\theta, \gamma_\theta)| \\ &\leq 2\epsilon_n + \max_{\theta \in \mathcal{N}} |\Delta m_r(\hat{\gamma}_\theta, \gamma'_\theta)| + \max_{\theta \in \mathcal{N}} |\Delta m_r(\gamma'_\theta, \gamma_\theta)| \\ &\leq 2\epsilon_n + 2 \max_{\theta \in \mathcal{N}} |\Delta m_r(\gamma'_\theta, \hat{\gamma}_\theta)| + \max_{\theta \in \mathcal{N}} |\Delta m_r(\gamma'_\theta, \gamma_\theta)| \text{ by definition of } \hat{\gamma} \\ &\leq 2\epsilon_n + 2 \max_{\theta \in \mathcal{N}} |\Delta m_r(\hat{\gamma}_\theta, \gamma_\theta)| + 3 \max_{\theta \in \mathcal{N}} |\Delta m_r(\gamma'_\theta, \gamma_\theta)|. \end{aligned}$$

The first term is bounded by ϵ_n with probability at least $1 - \delta$ by (2.2.24). The second is bounded ϵ_n with probability at least $1 - \delta$ by Lemma 2.7.3. \square

2.7.2 Mixing distribution maximum likelihood

In this section, we discuss the problem of obtaining a rate of convergence for the parameter mixing distribution MLE. Let us have data X_1, \dots, X_n from the model (2.1.1). Define the mixing distribution MLE:

$$\hat{\Gamma} = \operatorname{argmax}_{\Gamma \in \mathcal{G}_{k,d}} \sum_{i=1}^n \log p_\Gamma(X_i). \quad (2.7.9)$$

By Theorem 5.1.1, the proper (k -GM) density MLE (nearly) achieves the parametric rate $(d \operatorname{polylog}(d, n)/n)^{1/2}$. (Although, currently that proof is only for centered distributions; we must extend it.) It is interesting to consider whether this means that the derived mixing distribution MLE $\hat{\Gamma}$ achieves the optimal convergence rate stated in Theorem 2.1.1.

Consider first for example the symmetric, 2-GM, when there is a single parameter μ . By Theorem 5.1.1 and Lemma 2.3.3, $(d \text{polylog}(d, n)/n)^{1/2} \geq H(P_{\hat{\mu}}, P_{\mu}) \gtrsim \sup_{\theta \in S^{d-1}} |m_2(\theta^\top \hat{\mu}) - m_2(\theta^\top \mu)| = \|\hat{\mu} \hat{\mu}^\top - \mu \mu^\top\|_2$. By Lemma 2.6.5, this means

$$\ell(\hat{\mu}, \mu) \lesssim \left(\frac{d \text{polylog}(d, n)}{n} \right)^{1/4}.$$

Thus in this simple example, it is straightforward to show that the parameter MLE itself achieves the correct rate in Theorem 2.1.1. We now study whether this holds in general for $\hat{\Gamma}$. By Theorem 5.1.1 and Theorem 2.3.6, a sample-split version of the MLE does obtain the rate stated in Theorem 2.1.1. It is now of interest to know whether the true MLE, without sample splitting, can achieve the rate. We believe the following conjecture is true. We state it and provide two methods of proof that seem promising.

Conjecture 2.7.5. *Let $\hat{\Gamma}$ be the MLE as in (2.7.9). Then there is a positive constant C_k such that with probability at least $1 - \delta$,*

$$W_1(\hat{\Gamma}, \Gamma) \leq C_k \left(\left(\frac{d \log(1/\delta)}{n} \right)^{1/4} + \left(\frac{\log(1/\delta)}{n} \right)^{1/(4k-2)} \right). \quad (2.7.10)$$

Proof ideas for 2.7.5. Let $V = [v_1, \dots, v_k] \in S^{d \times k}$ be the matrix whose columns form an orthonormal basis for the space spanned by the atoms of the true mixing distribution Γ . Let $H = VV^\top$. Define $\widehat{\Gamma}_H = \arg\max_{\Gamma \in \mathcal{G}_{k,d}} \sum_{i=1}^n \log p_\Gamma(HX_i)$, i.e., it is the k -atomic MLE obtained when all the data are projected onto the true direction V . Clearly, this estimator is not something we could compute in practice; we will rely on its theoretical properties. The crucial idea is that since V is the true direction, when data are projected onto this direction and the MLE is computed, its error in

Hellinger distance is bounded by the low-dimensional rate, i.e., by Theorem 5.1.1,

$$H^2(P_{\widehat{\Gamma}_H}, P_{\Gamma}) = \tilde{O}\left(\frac{k}{n}\right)^{1/2}. \quad (2.7.11)$$

Now

$$W_1(\hat{\Gamma}, \Gamma) \leq W_1(\hat{\Gamma}, \widehat{\Gamma}_H) + W_1(\widehat{\Gamma}_H, \Gamma). \quad (2.7.12)$$

For the second term,

$$\begin{aligned} W_1(\widehat{\Gamma}_H, \Gamma) &\leq H^{1/(4k-2)}(P_{\widehat{\Gamma}_H}, P_{\Gamma}) && \text{by Lemma 2.5.1 and Lemma 2.3.3} \\ &\leq \tilde{O}(1/n)^{1/(4k-2)}. && \text{by Theorem 5.1.1} \end{aligned}$$

It remains to tackle the first term in (2.7.12). First,

$$\begin{aligned} H(P_{\hat{\Gamma}}, P_{\widehat{\Gamma}_H}) &\lesssim H(P_{\hat{\Gamma}}, P_{\Gamma}) + H(P_{\Gamma}, P_{\widehat{\Gamma}_H}) \\ &= \tilde{O}\left(\frac{d}{n}\right)^{1/2} + \tilde{O}\left(\frac{k}{n}\right)^{1/2} \\ &= \tilde{O}\left(\frac{d}{n}\right)^{1/2}. \end{aligned}$$

It remains to translate this rate from Hellinger to its square root in W_1 . This does not seem straightforward, since typically we related W_1 and Hellinger via Lemma 2.3.3 and Lemma 2.5.1, but doing so here would lead to a rate of $(d/n)^{1/(4k-2)}$. The idea is that is as follows. The MLE $\hat{\Gamma}$ is a k -GM with form $\hat{\Gamma} = \sum_{j=1}^k \hat{w}_j \delta_{\hat{\mu}_j}$. Let $\hat{\Sigma} = \sum_{j=1}^k \hat{w}_j \hat{\mu}_j \hat{\mu}_j^\top$. Let $\hat{V} = [\hat{v}_1, \dots, \hat{v}_k]$ be the space spanned by the k atoms of $\hat{\Gamma}$. We already know by Theorem 5.1.1 and Lemma 2.3.3 that $\|\hat{\Sigma} - \Sigma\|_2 = \tilde{O}(d/n)^{1/2}$, but this does not necessarily mean that \hat{V} and V are close; for the eigenvalues of $\hat{\Sigma}, \Sigma$ could be small. However, if they are close, it seems that computing the MLE with the same data onto two spaces that are close would yield k -GM's with similar weights,

we might be able to employ a natural coupling for the analysis. If the spaces are not close, then $\left\|\hat{\Sigma}\right\|_2, \|\Sigma\|_2$ are small anyway and the rate should hold. \square

Chapter 3

Mixtures of subspaces: parameter and density estimation

3.1 Introduction

In this section, we consider a mixture model when the mixing distribution lies on a finite collection of linear subspaces of \mathbb{R}^d . For simplicity, we will allow each subspace S_1, \dots, S_k to have rank $r \geq 1$. For each $j \in [k]$, let S_j be spanned by the orthonormal basis $\{v_{j1}, \dots, v_{jr}\}$. We observe data as in (1.2.1), i.e., we observe data from the model

$$X_1, \dots, X_n \sim_{i.i.d.} P_\Gamma, \text{ where} \tag{3.1.1}$$

$$P_\Gamma = \Gamma * N(0, I_d) \text{ and } \Gamma = \sum_{j=1}^k w_j \Gamma_j, \tag{3.1.2}$$

where each Γ_j is a certain uniform distribution on S_j ; we now specify this. We have

$$X_i = U_i + Z_i, \text{ where}$$

$$U_i = \sum_{\ell \in [r]} A_{i\ell} V_\ell,$$

where $U_\ell \sim \Psi_\ell = \sum_{j=1}^k w_j \delta_{v_{j\ell}}$, and crucially, U_1, \dots, U_r are coupled so that when $U_1 = v_{j1}, U_2 = v_{j2}$, and so on. For an arbitrary $U \sim \Gamma$, we can also write

$$U = VA \tag{3.1.3}$$

In (3.1.3), $V = [V_1, \dots, V_r] \in \mathbb{R}^{d \times r}$ and each $V_\ell \sim \Psi_\ell$ and the V_ℓ are coupled as above.

We also write $U = VA$ where $V \sim \Psi$, with

$$\Psi = \sum_{j=1}^k w_j \delta_{[v_{j1}, \dots, v_{jr}]},$$

i.e., Ψ is a distribution taking k values, each value being a matrix $W \in S^{d \times r}$, with $V^\top V = I_r$. And $A = (A_1, \dots, A_r)^\top \in \mathbb{R}^r$, where each $A_\ell \sim_{i.i.d.} Uni(-1, 1)$. We denote the distribution of the vector A via $Uni[-1, 1]^r$. The distribution P_Γ in (3.1.1) has density p_Γ :

$$p_\Gamma(x) = \mathbb{E}_{U \sim \Gamma} \phi(x - U) = \mathbb{E}_{A \sim Uni[-1, 1]^r} \mathbb{E}_{V \sim \Psi} \phi(x - VA)$$

Define the parameter and density classes:

$$\mathcal{G}_{k,r,d} = \left\{ \Gamma = \sum_{j=1}^k w_j Uni(S_j) : S_j \text{ spanned by } v_{j1}, \dots, v_{jr} \right\},$$

$$\mathcal{P}_{k,r,d} = \{p_\Gamma : \Gamma \in \mathcal{G}_{k,r,d}\}.$$

3.2 Learning a single subspace

We first a simple variant of the problem (3.1.1). Let $\mathcal{G} = \{\theta \in \mathbb{R}^d : \|\theta\|_2 \leq R\}$. Suppose we wish to estimate θ in the model

$$X_1, \dots, X_n \sim_{i.i.d.} P_\theta, \text{ where} \quad (3.2.1)$$

$$\mathbb{P}_\theta = N(A\theta, \sigma^2 I_d), \quad (3.2.2)$$

where A is a random variable in \mathbb{R} with a symmetric distribution \mathbb{Q} . That is, for each $i \in [n]$,

$$X_i = A_i \theta + Z_i$$

where $A_i \sim \mathbb{Q}$, θ is a fixed vector in \mathbb{R}^d , and $Z_i \sim N(0, \sigma^2 I_d)$. Crucially in this problem, θ is not necessarily a unit vector and may have a small norm. Let p_θ be the density of the distribution \mathbb{P}_θ . Let q be the density of the distribution \mathbb{Q} , i.e. the distribution of A . Suppose that q is defined on $[-c, c]$, where $c \in \mathbb{R} \cup \{\infty\}$. Note that

$$p_\theta(x) = \int_{-c}^c \exp\left(-\frac{\|x - a\theta\|_2^2}{2\sigma^2}\right) q(a) da$$

Note that when $A \sim \text{Rademacher}(1/2)$, this corresponds exactly to the symmetric 2-GM in Section 2.6.3. Our first result is that the convergence rate for estimating θ here is exactly as in the symmetric 2-GM of Section 2.6.3. Since this is in fact an infinite mixture, we cannot directly apply the results from Chapter 2. However, even simpler proofs allow us to obtain the rate here.

Lemma 3.2.1. *Let $X_1, \dots, X_n \sim_{i.i.d.} P_\theta$ as in (3.2.2). The minimax risk of estimating θ over the class \mathcal{G} satisfies*

$$\inf_{\hat{\theta}} \sup_{\theta} \mathbb{E}_\theta \ell(\hat{\theta}, \theta) \asymp \left(\frac{d}{n}\right)^{1/4}. \quad (3.2.3)$$

And there exists an estimator $\hat{\theta}$, computable in $O(n^3)$ time, such that with probability at least $1 - \delta$,

$$\ell(\hat{\theta}, \theta) \asymp \left(\frac{d \log(1/\delta)}{n} \right)^{1/4}. \quad (3.2.4)$$

Proof. We first prove the upper bound in (3.2.4). Let $\hat{\theta} = \sqrt{\hat{\lambda}} \hat{u}$, where \hat{u} is the first unit eigenvector of $X^\top X/n - \sigma^2 I_d$ and $\hat{\lambda}$ is the corresponding eigenvalue. Now $\hat{\theta} \hat{\theta}^\top$ is the rank-1 approximation of $\hat{\Sigma} - \sigma^2 I_d$, so by Lemma 4.5.8, $\left\| \hat{\theta} \hat{\theta}^\top - \theta \theta^\top \right\|_2 \leq \left\| \hat{\Sigma} - \sigma^2 I_d - \theta \theta^\top \right\|_2$. And

$$\hat{\Sigma} - \sigma^2 I_d = \theta \theta^\top \frac{1}{n} \sum A_i^2 + \frac{1}{n} \sum A_i (\theta Z_i^\top + Z_i \theta^\top) + \frac{1}{n} \sum Z_i Z_i^\top - \sigma^2 I_d.$$

The concentration at the rate of $\sqrt{d/n}$ follows by standard random matrix theory as in the proof of Lemma 2.2.6.

For the lower bound, we rely on Fano's Lemma. For shorthand, we write $\mathcal{M} = \inf_{\hat{\theta}} \sup_{\theta} \mathbb{E}_{\theta} \left\| \hat{\theta} - \theta \right\|_2^2$. Let $\mathcal{H} = \{\pm 1\}^d$, the hypercube in d dimensions. By Gilbert-Varshamov, there is a set $\mathcal{V} \subset \mathcal{H}$ such that $|\mathcal{V}| \geq e^{d/8}$ and such that for all $\nu, \nu' \in \mathcal{V}$, $\|\nu - \nu'\|_2 \geq \sqrt{d}/2$. Now we define our set of parameters $\{\theta(\mathbb{P}_{\nu})\}_{\nu \in \mathcal{V}}$. For each $\nu \in \mathcal{V}$, let

$$\theta = \epsilon \nu$$

for ϵ which we will define later. Then for $\theta, \theta' \in \{\theta(\mathbb{P}_{\nu})\}_{\nu \in \mathcal{V}}$,

$$\|\theta - \theta'\|_2 = \epsilon \|\nu - \nu'\|_2 \gtrsim \epsilon \sqrt{d}$$

So let

$$\delta = \epsilon \sqrt{d}$$

Now we'll use the "centered" version of Fano's lemma, with $\nu_0 = 0$. By Lemma 3.2.2,

$D_{KL}(\mathbb{P}_\theta||\mathbb{P}_0) \lesssim \|\theta\|_2^4/\sigma^4$ for any $\theta = \epsilon\nu$. So

$$D_{KL}(P_\nu||P_{\nu_0}) \lesssim \frac{\|\theta\|_2^4}{\sigma^4} \leq \frac{\epsilon^4 \|\nu\|_2^4}{\sigma^4} \leq \frac{\epsilon^4 d^2}{\sigma^4}$$

And by our choice of \mathcal{V} and by Gilbert-Varshamov, $\log |\mathcal{V}| \geq d/8$. So by the center version of Fano's Method,

$$\begin{aligned} \mathcal{M} &\gtrsim \delta^2 \left(1 - \frac{n D_{KL}(P_\nu||P_0)}{\log |\mathcal{V}_0|} \right) \\ &\geq \epsilon^2 d \left(1 - \frac{n \epsilon^4 d^2}{d \sigma^4} \right) \end{aligned}$$

Now choosing ϵ such that $n \epsilon^4 d / \sigma^4 = 1/2$ means choosing

$$\epsilon = \left(\frac{\sigma}{nd} \right)^{1/4}$$

And then $\delta = \epsilon \sqrt{d} = (\sigma d / n)^{1/4}$. Then the lower bound is:

$$\mathcal{M} \gtrsim \delta^2 = \sqrt{\frac{\sigma d}{n}}$$

as needed. □

Lemma 3.2.2. *Let \mathbb{P}_θ be the $N(A\theta, \sigma^2 I_d)$ model. Then*

$$KL(\mathbb{P}_\theta||\mathbb{P}_0) \lesssim \frac{\|\theta\|_2^4}{2\sigma^4}$$

Proof. Let $B \sim \mathbb{Q}$ be independent of A . I will write $\mathbb{E}_A, \mathbb{E}_B$ for shorthand to indicate the expectation over the distribution \mathbb{Q} for random variables A, B . First,

$KL(\mathbb{P}_\theta || \mathbb{P}_0) \leq \log \mathbb{E}_{p_\theta} \left(\frac{p_\theta(x)}{p_0(x)} \right)$. Now $\frac{p_\theta(x)}{p_0(x)} = \mathbb{E}_A \exp \left(\frac{Ax'\theta - A^2\|\theta\|^2}{2\sigma^2} \right)$.

$$\mathbb{E}_{p_\theta} \left(\frac{p_\theta(x)}{p_0(x)} \right) = \mathbb{E}_B \mathbb{E}_A \mathbb{E}_{X \sim N(B\theta, \sigma^2 I_d)} e^{AX'\theta/\sigma^2 - A^2\|\theta\|^2/2\sigma^2} \text{ by Fubini}$$

$$= \mathbb{E}_B \mathbb{E}_A e^{AB\|\theta\|^2/\sigma^2} \text{ using the Gaussian MGF}$$

$$= \sum_{k=0}^{\infty} \mathbb{E}_A \mathbb{E}_B \frac{(AB\|\theta\|^2)^k}{\sigma^{2k} k!} \text{ by Taylor-expanding exp and Fubini}$$

$$= \sum_{k=0}^{\infty} \frac{\|\theta\|^{2k}}{\sigma^{2k} k!} \mathbb{E} B^k \mathbb{E} A^k$$

Now since A, B are symmetric random variables, the odd terms cancel out. Let $m = \max_k \mathbb{E} A^{2k} \mathbb{E} B^{2k}$. Then

$$\begin{aligned} \mathbb{E}_{f_\theta} \left(\frac{p_\theta(x)}{p_0(x)} \right) &= \sum_{k=0}^{\infty} \frac{\|\theta\|^{4k}}{\sigma^{4k} (2k)!} \mathbb{E} B^{2k} \mathbb{E} A^{2k} \\ &\leq \sum_{k=0}^{\infty} \frac{\|\theta\|^{4k}}{\sigma^{4k} 2^k k!} \mathbb{E} B^{2k} \mathbb{E} A^{2k} \\ &= \exp \left(\frac{m \|\theta\|^4}{2\sigma^4} \right) \end{aligned}$$

where the last step follows because $2^k \leq (2k) * (2k-1) * \dots * (2k-k+1)$, so $(2k)! \geq 2^k k!$.

Apply the log gives the result. \square

3.3 Density and parameter estimation in k -spaces

We now discuss how we plan to extend results of Section 3.2. We note that a crucial factor in the slowness of the rate in Lemma 3.2.1 was that θ was arbitrary; it could be very small. For the lower bound proof, this meant that we could apply Fano

with a viable single point in the space being the $N(0, I_d)$ model. But in general, for the model (3.1.1), this will not be the case; we will require that the basis vector θ have unit norm. If we had assumed this in Lemma 3.2.1, we would expect to obtain something like

$$D(P_\theta, P_{\theta'}) \asymp \left\| \theta\theta^\top - \theta'\theta'^\top \right\|_F^2.$$

And given that $\|\theta\|_2 = 1$, this should translate to

$$\ell(\hat{\theta}, \theta) \asymp \left\| \theta\theta^\top - \theta'\theta'^\top \right\|_F^2 \asymp \left(\frac{d}{n} \right)^{1/2}.$$

That is, via spectral estimation we should obtain the parametric rate for θ , and we moreover should obtain the parametric rate for density estimation. This would also be the lower bound, since again by the KL divergence bound and its translation to ℓ , we would not lose a square root factor. This is intuitive, since if $\theta \in S^{d-1}$, and since $A \sim \text{Uni}(-1, 1)$, we obtain enough information about the direction of θ to estimate it at the parametric rate. More generally, we expect that in the k -spaces model of (3.1.1), we should be able to estimate the space at the parametric rate, and this should translate to a rate for density estimation.

We now state some conjectures about density and parameter estimation in the model (3.1.1), as well as some proof sketches. In these conjectures, we see that the uniform random variables A are essentially irrelevant; they are like nuisance parameters, at least assuming we know their distribution and moments.

Conjecture 3.3.1. *Let the model be as in (3.1.1). Then the minimax rate of estimating P_Γ over $\mathcal{P}_{k,r,d}$ satisfies*

$$\inf_{\hat{P}_\Gamma} \sup_{P_\Gamma \in \mathcal{P}_{k,r,d}} \mathbb{E}_\Gamma H(P_\Gamma, P_\Gamma) \asymp \left(\frac{d}{n} \right)^{1/2}.$$

Proof idea for Conjecture 3.3.1. By Lemma 1.2.2, for any $D \in \{\chi^2, KL, H^2\}$,

$$D(P_\Gamma, P_{\Gamma'}) \lesssim_{k,r} \sum_{j \geq 1} \frac{\|M_j(\Gamma) - M_j(\Gamma')\|_F^2}{j!}.$$

Now Γ, Γ' lie on a subspace of rank at most $2kr$. Recall that for $U \sim \Gamma, U = VA$ where $V \sim \Psi, A \sim \text{Uni}[-1, 1]^r$. Therefore,

$$\begin{aligned} \|M_j(\Gamma) - M_j(\Gamma')\|_F^2 &\lesssim_{k,r} \sup_{\theta \in S^{d-1}} |m_j(\Gamma_\theta) - m_j(\Gamma'_\theta)| \\ &= \sup_{\theta \in S^{d-1}} |\mathbb{E}_A \mathbb{E}_{V \sim \Psi} (\theta^\top V A)^j - \mathbb{E}_{A'} \mathbb{E}_{V' \sim \Psi'} (\theta^\top V' A')^j| \\ &= \sup_{\theta \in S^{d-1}} |\mathbb{E}_{A, A'} \left(\mathbb{E}_{V \sim \Psi} (\theta^\top V A)^j - \mathbb{E}_{V' \sim \Psi'} (\theta^\top V' A')^j \right)| \\ &\leq \sup_{\theta \in S^{d-1}} \mathbb{E}_{A, A'} \left| \left(\mathbb{E}_{V \sim \Psi} (\theta^\top V A)^j - \mathbb{E}_{V' \sim \Psi'} (\theta^\top V' A')^j \right) \right| \end{aligned}$$

Conditioning on A, A' , we are taking expectations over k -atomic distributions on orthonormal matrices. We should be able to prove something similar to Lemma 10 of [Wu and Yang, 2019] and obtain an upper bound on $D(P_\Gamma, P_{\Gamma'})$ in terms of a finite number of moment tensors.

It remains to then find a proper density estimator. We might try a moment-based algorithm, or we might try the MLE or Le-Cam Birge estimators. To obtain sharp rates, these would require matching lower bounds. The lower bound technology again requires more work to obtain something akin to Lemma 2.3.3. \square

We now turn to our conjecture about parameter estimation. We conjecture that there will be a two-part rate similar to that in Theorem 2.1.1. This is because we could easily perform a similar procedure to Algorithm 1; we can estimate the (kr) -dimensional subspace, project the data to the estimated subspace, then proceed to do mixing estimation on the low-dimensional data. But in this scenario, as hypothesized earlier in this section, we think the rate of estimating the subspace should be sharper

than in Theorem 2.1.1 because we have full information about the subspaces; we do not have vectors that may be arbitrarily small.

We further conjecture that the rate for the low-dimensional mixing distribution estimation is in fact very similar to that in Theorem 2.1.1. This is because the space complexity of the r -dimensional spaces increases, but the moment concentration should actually not change much.

Conjecture 3.3.2. *Let the model be as in (3.1.1). Then the minimax rate of estimating Γ over $\mathcal{G}_{k,r,d}$ satisfies*

$$\inf_{\hat{\Gamma}} \sup_{\Gamma \in \mathcal{G}_{k,r,d}} E_{\Gamma} W_1(\hat{\Gamma}, \Gamma) \asymp_{k,r} \left(\frac{d}{n}\right)^{1/2} + \left(\frac{kr}{n}\right)^{1/c_r(4k-2)}.$$

Beginning of proof idea for Conjecture 3.3.2. First, we observe that the A (uniform) component of Γ is really a nuisance parameter; the crucial component of the distribution is Ψ . Let $\Gamma, \Gamma' \in \mathcal{G}_{k,r,d}$. We will reduce $W_1(\Gamma, \Gamma')$ to facts about Ψ, Ψ' . Let ν be any coupling of Γ, Γ' . Let ϕ be a coupling of Ψ, Ψ' . A valid coupling is one in which the A, A' are always the same. So

$$\begin{aligned} W_1(\Gamma, \Gamma') &= \inf_{\nu} \mathbb{E}_{\nu} \|U - U'\|_2 = \inf_{\nu} \mathbb{E}_{\nu} \|VA - V'A'\|_2 \\ &\leq \inf_{\phi} \mathbb{E}_{\phi} \mathbb{E}_A \|(V - V')A\|_2 \\ &\leq \mathbb{E}_A \|A\|_2 \inf_{\phi} \mathbb{E}_{\phi} \|V - V'\|_2 \\ &\lesssim W_1(\Psi, \Psi'). \end{aligned}$$

Now, for any $V, V' \in S^{d \times r}$, $\|V - V'\|_2 = \|\sin D\|_2$, where D is the diagonal matrix whose entries are the principal angles between the two subspaces spanned by the columns of V, V' . To estimate $W_1(\Psi, \Psi')$, we will need the moments of the principal angles. It seems these can be obtained by the low-dimensional method-of-moments on the coordinates of $\gamma = \Gamma_V$, but more work is needed to fully explore this. We

suspect that the c_r in Conjecture 3.3.2 is at most r ; it may not even be needed. \square

Chapter 4

Mixtures of manifolds: kernel spectral clustering and refinement

4.1 Introduction

Clustering is an important task in statistics, with applications to a wide range of scientific fields. The goal is to separate data into groups such that points within a group are close to each other but points between groups are well-separated. Data analysts often wish to cluster data that lies in a high-dimensional space, and many of the advances in this area over the past decades have relied on the observation that the data, while high-dimensional, truly lies on a low-dimensional subspace of the ambient space.

Consider facial image data consisting of images of people's faces in different positions and lighting. Each data point is a vector in, e.g., $\mathbb{R}^{32 \times 32}$, with each vector component representing a pixel's brightness. In a typical example of such a dataset, [Tenenbaum et al., 2000] show that the data can be represented by a three-dimensional manifold, with the dimensions capturing the lighting and the vertical and horizontal inclinations of the face. A good clustering algorithm should be able to sep-

arate a dataset consisting of several faces, each shown in multiple different positions or lighting, into the correct groups, as opposed to naively clustering them according to position or lighting.

A classical technique for handling such data is to find the low-dimensional linear subspace that is nearest to the data, a procedure known as Principal Component Analysis (PCA). This method assumes that the data lie on a single low-dimensional subspace. But in practice, the data may lie on multiple subspaces. If the subspace membership of each point were known, it would be easy to simply do PCA separately on each group of data points. But in general, the clustering may be unknown. Moreover, it is more realistic to assume that the data lie, not exactly on a low-dimensional space, but consist of low-dimensional signal plus some high-dimensional noise.

We consider the problem of clustering data into k clusters, where k is known, when the data are generated from k spaces, S_1, \dots, S_k , embedded in \mathbb{R}^p . For instance, each S_j might be a manifold of dimension m_j , where $m_j \ll p$. Let $m = \max_{s \in [k]} m_j$. Suppose we have n data points, $X_1, \dots, X_n \in \mathbb{R}^p$. Let $\tau \in \{1, \dots, k\}^n$ be the true assignment vector. The data are truly grouped into k clusters, C_1, \dots, C_k . That is, $C_s = \{i : \tau(i) = s\}$. We assume X_i has the following structure:

$$X_i = M_i + Z_i \tag{4.1.1}$$

in which M_i is distributed according to some distribution on $S_{\tau(i)}$. And $Z_i \sim N(0, I_p)$. All vectors are assumed to be independent. The assumption on the structure of X_i is standard in the literature, for example, see [Karoui, 2010a] and [Karoui and tieng Wu, 2014]. In matrix form, we let $X, M, Z \in \mathbb{R}^{n \times p}$ be the matrices with rows X_i, M_i, Z_i , respectively.

We moreover assume that the spaces on which the signals lie are separated. This is in contrast to, for instance, the subspace clustering problem, in which it is assumed

that the data lie on k linear subspaces which are allowed to overlap. Let

$$d_{st} = \inf_{\tau(i)=s, \tau(j)=t, s \neq t} \|M_i - M_j\|_2$$

And let $d_{\min} = \min_{s \neq t} d_{st}$, $d_{\max} = \max_{s \neq t} d_{st}$. We assume that M has rank bounded by $r \leq mk$; Lemma 4.5.11 shows why this assumption is reasonable. We also assume that for $i \neq j$ where $\tau(i) = \tau(j)$, $\|M_i - M_j\|_2 \leq b$, where b is some positive constant. And we assume that

$$\max_{\tau(i)=s, \tau(j)=t, s \neq t} \|M_i - M_j\|_2^2 \leq 8b^2 + 4d_{st}^2$$

This is reasonable if we assume that on each manifold there is a point that achieves the minimum distance to each of the other manifolds. For an estimator $\hat{\tau}$ of τ , our loss, or mis-clustering rate, is $\mathcal{L}(\tau, \hat{\tau}) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{\hat{\tau}(i) \neq \tau(i)\}$.

Spectral methods, which make no assumption about the shapes of the clusters, are widely-used for community detection on such data. These methods depend on a kernel matrix, or on its closely-related Laplacian matrix. The kernel random matrix A , based on X_1, \dots, X_n , is a symmetric matrix with entries $A_{ij} = \mathcal{K}(X_i, X_j)$ where \mathcal{K} is a kernel function.

Let $U_k \in \mathbb{R}^{n \times k}$ be the matrix whose columns are the first k orthonormalized eigenvectors of A . The analyst then applies K -means, with $K = k$, or some other clustering algorithm, to each row of U_k , instead of to the p -dimensional original points. The cluster assigned to row i of U_k is then assigned to X_i . Many variants of this method exist. The K -means algorithm as the clustering step could be replaced by other clustering methods, and indeed, in this paper, we rely on a different algorithm.

For an intuitive understanding of why spectral methods work, it is helpful to turn to graph theory. A graph $G = (V, E)$ is a set of nodes V and an edge set E . The edge set contains pairs of nodes that are connected by a potentially weighted edge.

An adjacency matrix A representing a graph is a matrix with entries A_{ij} equal to the weight of the edge (i, j) . Let $L = D - A$, where D is the diagonal matrix with entries D_{ii} equal to the sums of row i of A . L is called the Laplacian matrix, and it can be shown that if G has k connected components, then L has an eigenvalue of 0 with multiplicity k , and that the eigenvectors of the 0 eigenvalue are constant on the indices corresponding to the k connected components of the graph. Thus, performing k -means on the $n \times k$ matrix whose columns are the first k eigenvectors of L - i.e. the eigenvectors corresponding to the 0 eigenvalue - will perfectly separate the connected components of the graph.

We now extend this heuristic to real data. If the data lie, for instance, on a low-dimensional manifold, then the Laplacian of the kernel matrix should capture local information about the manifold. Indeed, [Belkin and Niyogi, 2008] show a connection between the graph Laplacian and the Laplace-Beltrami operator, which is known to provide information about a manifold’s connectivity. And [von Luxburg et al., 2008] show consistency of spectral clustering by demonstrating that the eigenvectors of the Laplacian matrix converge uniformly to the eigenfunctions of the Laplacian operator.

A more realistic extension of the low-dimensional manifold setting is one we consider, in which the data are a combination of low-dimensional signal and high-dimensional noise. In this setting, practical algorithms have been developed; see e.g. [Niu et al., 2001] for an algorithm and a performance comparison among several popular methods. The authors of this paper point out that spectral clustering may be sensitive to noisy, irrelevant dimensions in the data, and their algorithm iterates between spectral clustering and searching for the “best” projection of the data onto a low-dimensional space.

There are many ways to formally justify spectral clustering; one method comes from perturbation theory. The graph adjacency matrix A , or the kernel matrix \mathcal{K} , is considered to be a noisy, or perturbed, version of a matrix P that is ideal for

clustering. The bound on the mis-clustering rate of spectral clustering then depends crucially on the operator norm of $A - P$.

To our knowledge, [Karoui, 2010a] is the first to develop theory about the kernel random matrix in the signal-plus-noise framework. They compare the kernel matrix formed from the signal-plus-noise to the pure signal kernel matrix. In this paper and in [Karoui and tieng Wu, 2014], the bounds suggest that the distance between the manifolds, d , must grow with the data dimension, p , in order for spectral clustering to result in few errors on these sorts of data. Yet one of the reasons for the popularity of spectral clustering is that the method is fairly effective in practice even when p is large relative to d .

In this work, we develop a new kernel spectral method that achieves an error rate of $Ck^2d_{\max}^2/d_{\min}^4$, where $C > 0$ is a constant, thus showing that it does not grow with p . Our bound requires only the weak assumption that $p \leq Cn/(k \log^2(n/k))$, rather than d . Thus we show that spectral clustering can work even if p is large relative to d . To obtain our bound, we propose a new variant of kernel spectral clustering, one that relies on the low-rank approximation of the kernel matrix, along with performing K -lines, instead of K -means, on the resulting low-dimensional matrix. We rely on perturbation theory methods and analyze $\|A - P\|$, but we carefully choose P so that it is both an ideal clustering matrix and is as close as possible to A . This is in contrast to the approach of other authors, who typically compare A to $\mathbb{E}A$ or to the pure signal matrix.

We moreover develop and theoretically justify an algorithm that improves upon spectral clustering by using a dimension-reduction and testing procedure. Our algorithm achieves an error rate that is exponential in d . The algorithm's testing procedure is inspired by the testing procedures of [Zhang and Zhou, 2015]. We provide both theoretical justification for the algorithm, rigorously proving that it achieves the exponential rate. We demonstrate its success on simulated and real data.

This paper is organized as follows. We introduce the model and algorithm in 4.2. We state the major theoretical results in 4.3. Proofs of the main results and key lemmas used are in 4.4.

Notation We now introduce some notation for this section only. We occasionally must refer to points according to the manifold they are generated from. In this case, we write $X_{t,i} = M_{t,i} + Z_{t,i}$ where for $i = 1, \dots, n/k$, $M_{t,i}$ is uniformly distributed on S_t .

Let $Z_{t,(1)} \leq \dots \leq Z_{t,(n/k)}$ be the Gaussian noise random variables, sorted according to ℓ_2 norm, from S_t . Since the manifolds are of the same size, note that $\mathbb{E} \|Z_{s,(i)}\|^2 = \mathbb{E} \|Z_{t,(i)}\|^2$ for all $t \neq s$. So we often write $\mathbb{E} \|Z_{(i)}\|^2$, without the other subscript. Let M_{ti} denote the uniform random variables from S_t associated with the order statistics $Z_{t,(i)}$. We never sort the uniform random variables themselves.

We write $\|v\|$ for the Euclidean norm of a vector v . We write $\|B\|_{OP}$ and $\|B\|_F$ for the operator and Frobenius norms, respectively, of a matrix B . Let $A_{[i]}$ denote the i th row of the matrix A .

For a matrix $B \in \mathbb{R}^{n \times p}$, we often make use of the r -rank approximation of B . By this we mean the following. B can be written $B = \sum_{i \leq \min(n,p)} \lambda_i u_i v_i'$. For $r \leq \min(n, p)$, the r -rank approximation of B is defined as

$$\sum_{i=1}^r \lambda_i u_i v_i'$$

Many of our matrices will consist of k^2 sub-matrices, each in $\mathbb{R}^{(n/k) \times (n/k)}$. For any such matrix B , we write $(B)_{ij}^{st}$ to indicate the (i, j) th entry of the (s, t) sub-matrix of B . Similarly, for any matrix, we write $(B)_{ij}$ to indicate the (i, j) entry of the matrix B .

Define the Euclidean Distance Kernel matrix:

$$A_{ij}^{(EK)} := \|X_i - X_j\|^2$$

Throughout, we will use the Gaussian Kernel random matrix A with entries:

$$A_{ij}^{(GK)} := \exp\left(-\frac{A_{ij}^{(EK)}}{\rho^2}\right).$$

This kernel is widely used by data analysts and in the literature. However, our result is based almost entirely on work for the Euclidean Distance kernel that is then extended to the Gaussian Kernel result via a Taylor expansion, imitating [Karoui, 2010b].

Now let the matrix $P^{EK} \in \mathbb{R}^{n \times n}$ be the matrix consisting of k^2 sub-matrices, each in $\mathbb{R}^{(n/k) \times (n/k)}$, with

$$\begin{aligned} (P^{EK})_{ij}^{st} &= \mathbb{E} \|Z_{s,(i)}\|^2 + \mathbb{E} \|Z_{t,(j)}\|^2 + 4d_{st}^2 \mathbf{1}\{s \neq t\} \\ &= \mathbb{E} \|Z_{(i)}\|^2 + \mathbb{E} \|Z_{(j)}\|^2 + 4d_{st}^2 \mathbf{1}\{s \neq t\} \end{aligned}$$

Since the expectations of the sorted random variables on each manifold are the same. And define,

$$(P^{GK})_{ij}^{st} = \exp\left(-\frac{(P^{EK})_{ij}^{st}}{\rho^2}\right).$$

And let $R^{(GK)} := A^{(GK)} - P^{(GK)}$ and define $R^{(EK)}$ analogously. We will prove our bound on the error rate when we perform spectral clustering on $R^{(GK)}$ using the relation between the Gaussian and Euclidean kernels and a Taylor expansion.

Throughout, denote the χ_p^2 density, distribution, and tail probability functions by f, F , and \bar{F} , respectively. Let the hazard rate be denoted by h , where $h = f/\bar{F}$. And let $\{Y_i\}_{i=1}^n \sim_{iid} \chi^2(p)$. Let $\{Y_{(i)}\}_{i=1}^n$ be the order statistics of this sample, with

$Y_{(1)} \leq Y_{(2)} \leq \dots Y_{(n)}$. Let $S := \sum_{i=1}^n (Y_{(i)} - \mathbb{E}Y_{(i)})^2$.

In the refinement algorithm, we perform estimation of our manifold points. In this context, though the original points M_i are random, we think of them as fixed, and write μ_i . We let these be the entries of M .

We say that an algorithm achieves a “perfect clustering” on data points if the result of the algorithm is an assignment $\hat{\tau}$ such that there exists a permutation $\pi : [k] \rightarrow [k]$ such that $\pi(\hat{\tau}) = \tau$, where τ is the true cluster assignment.

We make use of the following assumptions in our results.

$$d_{\min}^2 \gtrsim \max \left(rk \left(1 + \frac{p}{n} \right), 32r\sigma^2 \right) \text{ (distance assumption)}$$

$$n \gtrsim \max \left(\frac{2k(d^2 - 16\sigma^2)}{128\sigma^2}, 32 \log p + 18 \right) \text{ (sample size assumption)}$$

$$p \geq 3 \text{ (ambient dimension assumption)}$$

$$\rho \geq \max \left(p + 2\sqrt{3p \log n} + 6 \log n, p \log \frac{n}{k}, d_{\min}, \frac{d_{\max}^2}{\sqrt{k}} \right) \text{ (bandwidth assumption)}$$

4.2 Algorithms

The first algorithm is an extension of Lloyd’s algorithm for the K -means problem to linear subspaces. We follow [Vidal, 2009] for our exposition. Suppose we have data points $P_1, \dots, P_n \in \mathbb{R}^n$, and suppose the points truly lie on k different q -dimensional linear spaces. We wish to find τ and the orthogonal projections of the points onto their respective subspaces. The optimization is:

$$\min_{\tau \in [k]^n} \min_{\{H_j\}_{j \leq k}} \sum_{j \leq k} \sum_{\{i: \tau(i)=j\}} \|P_i - H_j P_i\|_2^2$$

For a data matrix P , let $P_{C_j} \in \mathbb{R}^{n_j \times r}$ be the points assigned to cluster j . In our setting, we will apply k -spaces to the rows of k -rank approximation of $A^{(GK)}$. Call this approximation \hat{P} . Notice that \hat{P} has n rows, each of dimension n . But as Lemma

4.4.3 shows, each set of rows associated with one cluster is in fact rank 1. So we apply K -spaces with $q = 1$.

Algorithm 3: K -spaces spectral clustering algorithm for the Gaussian Kernel.

Input: Data points P_1, \dots, P_n arranged in a data matrix $P \in \mathbb{R}^{n \times n}$, number of communities k

Output: clustering result $\hat{\tau}^{init} \in [k]^n$

Initialize $\hat{\tau}^0$;

while *not converged* **do**

$\hat{C}_j^t = \{i : \hat{\tau}^t(i) = j\};$
 $H_j^t = \hat{V}_{\hat{C}_j^t} \hat{V}_{\hat{C}_j^t}^T$ where $P_{\hat{C}_j^t}^q = U_{\hat{C}_j^t} D_{\hat{C}_j^t} V_{\hat{C}_j^t}^T$ is the rank-1 approximation of the nodes in C_j^t ;
 $\hat{\tau}^{t+1}(i) = \operatorname{argmin}_j \left\| \hat{P}_i - H_j^t \hat{P}_i \right\|_2^2 ;$

end

The second algorithm proceeds as follows. We perform an initial clustering, then estimate the manifolds using dimension reduction. Using this manifold estimate, we classify the points of S_1 based on the estimated manifold that they are closest to. Note that we use sample-splitting for the algorithm; the main reason is that it is mathematically tractable; we are able to obtain a sharp bound despite having an initialization that only has $o(1)$ error. In our real data example, we use a single global initializer, then refine on each node. This may work as well or better than the sample-splitting algorithm, but it is harder to analyze. There are several parameters that must be chosen in the algorithm, including ρ , the denominator for the Gaussian kernel, K for the K -nearest neighbors, and more. We propose selecting the parameters

via cross-validation.

Algorithm 4: Refinement Algorithm.

Input: data points $\{X_i\}_{i=1}^n$, number of communities k , effective dimensionality

r , and the minimum distance between manifolds d_{\min}

Output: clustering result $\hat{\tau} \in [k]^n$

1 Data splitting ;

Denote $X^{(1)} = \{X_1, X_2, \dots, X_{n/2}\}$ and $X^{(2)} = \{X_{n/2+1}, X_{n/2+2}, \dots, X_n\}$;

2 Manifold Estimation: ;

For $j = 1, 2$, let W_j be the matrix containing the first r right-singular vectors of $X^{(j)}$;

For $j = 1, 2$, let $\mathcal{P}_{W_j} = W_j W_j'$;

Let $\hat{M}^{(1)} = X^{(1)} \mathcal{P}_{W_2}$ and $\hat{M}^{(2)} = X^{(2)} \mathcal{P}_{W_1}$;

Let $\hat{M} = [\hat{M}^{(1)}, \hat{M}^{(2)}]^T$. Let $\hat{\mu}_1, \dots, \hat{\mu}_n$ be the rows of \hat{M} ;

3 Initialization ;

Apply Algorithm Algorithm 3 on $X^{(2)}$ to obtain $\hat{\tau}^{init}(i)$ on $i = n/2 + 1, \dots, n$;

4 Refinement: for $i = 1, 2, \dots, n/2$, let

$$\hat{\tau}(i) = \operatorname{argmax}_{s \in [k]} \sum_{\{j: \hat{\tau}^{init}(j)=s\}} 1\{\|\hat{\mu}_i - \hat{\mu}_j\| \leq \delta\}, \text{ where } \delta = d_{\min}/4$$

5 Consensus: repeat Steps 3-4 but exchange the role of $X^{(1)}$ and $X^{(2)}$; obtain

$\hat{\tau}(i)$ for $i = n/2 + 1, \dots, n$. Combine them together to obtain $\hat{\tau}$.

4.3 Major Results

The error rate of spectral clustering depends on the spectral norm of $A - P$, where typically P is chosen to be a matrix with ideal clustering properties. Typically, $P = EA$ or $P_{ij} = K(\mu_i, \mu_j)$, the signal kernel matrix. If our kernel is the Euclidean distance kernel, then it might be reasonable to imitate [Karoui, 2010a] and use for P

the matrix with entries

$$P_{ij} = \mathbb{E} \|Z_i - Z_j\|^2 + \|M_i - M_j\|_2^2 = 2p + \|M_i - M_j\|_2^2$$

Notice that

$$(A - P)_{ij} \leq \|Z_i\|^2 - p + \|Z_j\|^2 - p + Z_i' Z_j$$

Let B be the $n \times n$ matrix with entries $B_{ij} = \|Z_i\|^2 - p$. Then it is rank 1, so has spectral equal to its Frobenius norm, which is approximately $n^2 p$, since $\|Z_i\|^2 \sim \chi_p^2$. This would result in an error rate of about $\frac{k^2}{d^4} p \log n$ for spectral clustering, which requires that p be smaller than d^4 . Our method improves upon this by choosing a different P and a different method of clustering.

Theorem 4.3.1. *Suppose the ambient dimension and sample size assumptions hold. Then there is a $c > 0$ such that with probability at least $1 - \frac{1}{n} - \frac{1}{\log^2(n/k)}$,*

$$\|R^{(EK)}\|_{OP}^2 \leq c(n^2 + p^2 + n^2 b^4 + n^2 d_{\max}^2 + knp \log^2(n/k))$$

Theorem 4.3.2. *Suppose the ambient dimension, sample size, and bandwidth assumptions hold. Let $\hat{\tau}^{init}$ be the clustering obtained from Algorithm Algorithm 3 to a dataset of n data points of the form given in (4.1.1). Then there is a constant $c > 0$ such that with probability at least $1 - \frac{2}{n} - \frac{1}{k \log^2(n/k)}$,*

$$\frac{1}{n} \mathcal{L}(\tau, \hat{\tau}^{init}) \leq c \frac{k^2}{d_{\min}^4} \left(1 + \frac{p^2}{n^2} + d_{\max}^2 + b^4 + \frac{kp \log^2(n/k)}{n} \right)$$

Theorem 4.3.3. *Suppose the sample size, ambient dimension, and distance assumptions hold. Let $\hat{\tau}$ be the estimator of τ obtained from Algorithm Algorithm 4. Then*

with probability at least $1 - 1/n - 1/k \log^2(n/k) - \exp(-(d^2 - 16\sigma^2)/256\sigma^2)$,

$$\frac{1}{n} \mathcal{L}(\tau, \hat{\tau}) \leq 4k \exp\left(-\frac{d^2 - 16\sigma^2 r}{256\sigma^2}\right)$$

4.4 Proofs of the main results

4.4.1 Key Lemmas

Lemma 4.4.1. *Suppose $P \in \mathbb{R}^{n \times n}$ has rank k . Suppose applying Algorithm Algorithm 3 to P yields a perfect cluster assignment. Let the minimum Euclidean distance between two rows of P corresponding to different clusters be bounded below by α^2 . Let A_k be the k -rank approximation of a matrix $A \in \mathbb{R}^{n \times n}$. Then Algorithm 3 on the rows of A_k results in a $\hat{\tau}$ that satisfies*

$$\mathcal{L}(\tau, \hat{\tau}) \lesssim \frac{k \|A - P\|^2}{\alpha^2}$$

Proof. Define:

$$T := \left\{ i : \|(A_k)_{[i, \cdot]} - P_{[i, \cdot]}\|_2^2 < \frac{\alpha^2}{2} \right\}$$

For nodes outside of T , Algorithm Algorithm 3 is not guaranteed to assign the nodes to the correct cluster. Thus

$$\mathcal{L}(\tau, \hat{\tau}) \leq |T^C| = \#\left\{ i : \|(A_k)_{[i, \cdot]} - P_{[i, \cdot]}\|_2^2 \geq \frac{\alpha^2}{2} \right\}$$

And $\|A_k - P\|_F^2 = \sum_{i \leq n} \|(A_k)_{[i,]} - P_{[i,]}\|_2^2$, so

$$\begin{aligned} |T^C| &\leq \frac{2 \|A_k - P\|_F^2}{\alpha^2} \\ &\leq \frac{2 * 8k \|A - P\|^2}{\alpha^2} \end{aligned} \quad \text{by Lemma 4.5.8 .}$$

□

Lemma 4.4.2. *Suppose ρ satisfies the bandwidth assumptions. Let α be the minimum Euclidean distance between two rows of $P^{(GK)}$. Then there is a constant $c > 0$ such that*

$$\alpha^2 \geq c \frac{n d_{\min}^4}{k \rho^4}$$

Proof. Suppose without loss of generality that $d_{\min} = d_{12}$. Then

$$\begin{aligned} \alpha^2 &= 2 \sum_{j=1}^{n/k} \left(e^{-\frac{\mathbb{E}\|Z_{(i)}\|^2 + \mathbb{E}\|Z_{(j)}\|^2}{\rho^2}} \left(e^{-\frac{d^2}{\rho^2}} - 1 \right) \right)^2 + \sum_{s,t} \sum_{j=1}^{n/k} \left(e^{-\frac{\mathbb{E}\|Z_{(i)}\|^2 + \mathbb{E}\|Z_{(j)}\|^2}{\rho^2}} \left(e^{-\frac{d_{1s}^2}{\rho^2}} - e^{-\frac{d_{2t}^2}{\rho^2}} \right) \right)^2 \\ &\geq \frac{2n}{k} \left(e^{-\frac{2\mathbb{E}\|Z_{(n/k)}\|^2}{\rho^2}} \right)^2 \left(e^{-\frac{d^2}{\rho^2}} - 1 \right)^2 \\ &\geq \frac{2n}{k} e^{-\frac{4\mathbb{E}\|Z_{(n/k)}\|^2}{\rho^2}} \left(\frac{d^2}{\rho^2} \right)^2 \text{ since } e^{-x} \geq 1 - x \end{aligned}$$

By Lemma 4.5.14, $\mathbb{E}\|Z_{(n/k)}\|^2 \leq p \log(n/k)$. Thus $e^{-4\mathbb{E}\|Z_{(n/k)}\|^2/\rho^2}$ is bounded below by a constant. □

Lemma 4.4.3. *$P^{(GK)}$ is rank k , and Algorithm 3 yields a perfect cluster assignment when applied to $P^{(GK)}$.*

Proof. Recall that $P^{(GK)}$ is composed of k^2 sub-matrices in $\mathbb{R}^{(n/k) \times (n/k)}$. Divide $P^{(GK)}$ into the first $\frac{n}{k}$ rows, the second $\frac{n}{k}$ rows, and so on. Any one of these blocks of rows

is rank 1, since:

$$P_{[j,]}^{(GK)} = \exp \left(\frac{\mathbb{E} \|Z_{(i)}\|^2 - \mathbb{E} \|Z_{(j)}\|^2}{\rho^2} \right) * P_{[i,]}^{(GK)}$$

The rows between these blocks are linearly independent, so $P^{(GK)}$ has rank k . And each such cluster of points lies on a 1-dimensional subspace of \mathbb{R}^n . Now a sufficient condition for Algorithm 3 to achieve a perfect clustering is that there is a positive distance between the subspaces the points belong to. By Lemma 4.4.2, the distance between the subspaces in $P^{(GK)}$ is at least α^2 , a positive constant. \square

Lemma 4.4.4. *Consider any $i \in \{1, \dots, n/2\}$. Let $X_i = \mu_i + Z_i$, and let \mathcal{P}_{W_2} be as in Algorithm Algorithm 4. Then there is a positive constant C such that with probability at least $1 - 1/n$,*

$$\|\mu_i - \mathcal{P}_{W_2}\mu_i\|_2^2 \leq Ckmr \left(1 + \frac{p}{n} + \beta^2 \right)$$

A similar bound holds for $\|\mu_i - \mathcal{P}_{W_1}\mu_i\|_2^2$ when $i \in \{n/2, \dots, n\}$.

Proof. Assume without loss of generality that $i \in \{1, \dots, n/2\}$. Let W_2 be as in Algorithm Algorithm 4, and for simplicity, write $\mathcal{P}_W = W_2W_2'$, the projection operator from the second sample. Recall that $V \in \mathbb{R}^{n \times p}$ is the matrix of the km unique centers that approximate the manifolds. Now,

$$\begin{aligned} \|V - \mathcal{P}_W V\|_F^2 &\lesssim \|V - M^{(2)}\|_F^2 + \|M^{(2)} - \mathcal{P}_W M^{(2)}\|_F^2 + \|\mathcal{P}_W M^{(2)} - \mathcal{P}_W V\|_F^2 \\ &\leq 2 \|V - M^{(2)}\|_F^2 + \|M^{(2)} - \mathcal{P}_W M^{(2)}\|_F^2 \end{aligned}$$

And,

$$\|M^{(2)} - \mathcal{P}_W M^{(2)}\|_F^2 \lesssim \|M^{(2)} - \mathcal{P}_W X^{(2)}\|_F^2 + \|\mathcal{P}_W X^{(2)} - \mathcal{P}_W M^{(2)}\|_F^2 \quad (4.4.1)$$

$$\lesssim r \|M^{(2)} - X^{(2)}\|_2^2 + r \|\mathcal{P}_W X^{(2)} - \mathcal{P}_W M^{(2)}\|_2 \quad (4.4.2)$$

$$\lesssim 2r \|M^{(2)} - X^{(2)}\|_2^2 \quad (4.4.3)$$

$$\lesssim 2r(p+n) \quad (4.4.4)$$

Step (4.4.2) follows from Lemma 4.5.8 and the fact that $\mathcal{P}_W X^{(2)} - \mathcal{P}_W M^{(2)}$ is rank r , so its Frobenius norm is bounded by r times its operator norm. And Step (4.4.3) follows since $\|\mathcal{P}_W X^{(2)} - \mathcal{P}_W M^{(2)}\|_2^2 \leq \|\mathcal{P}_W\|_2^2 \|X^{(2)} - M^{(2)}\|_2^2$, and the operator norm of an orthogonal projection matrix is 1. The final step is with probability at least $1 - 1/n$, and follows from Lemma 4.5.15. Now we have

$$\begin{aligned} \|V - \mathcal{P}_W V\|_F^2 &\lesssim \|V - M^{(2)}\|_F^2 + r(p+n) \\ &\leq n\beta^2 + r(p+n) \end{aligned}$$

And V has only km unique rows, v_1, \dots, v_{km} , each repeated $n/(km)$ times. Thus $\|V - \mathcal{P}_W V\|_F^2 = \sum_{j=1}^{km} \frac{n}{km} \|v_j - \mathcal{P}_W v_j\|_2^2$, which implies that

$$\sum_{j=1}^{km} \|v_j - \mathcal{P}_W v_j\|_2^2 \lesssim \frac{kmn\beta^2}{n} + \frac{kmr(p+n)}{n}$$

And,

$$\begin{aligned} \|\mu_1 - \mathcal{P}_W \mu_1\|_2^2 &\lesssim \|\mu_1 - v_1\|_2^2 + \|v_1 - \mathcal{P}_W v_1\|_2^2 + \|\mathcal{P}_W v_1 - \mathcal{P}_W \mu_1\|_2^2 \\ &\lesssim 2 \|\mu_1 - v_1\|_2^2 + \|v_1 - \mathcal{P}_W v_1\|_2^2 \\ &\lesssim 2\beta^2 + km\beta^2 + (kmr) \left(1 + \frac{p}{n}\right) \end{aligned}$$

And the proof for an $i \in \{n/2+1, \dots, n\}$ would be exactly similar, using $W_1 W_1'$ instead of $W_2 W_2'$. \square

4.4.2 Proofs of the main theorems

Proof of Theorem 4.3.1. Define the following matrices:

$$\begin{aligned} R^{(1)} &= \left(\|Z_{s,(i)}\|^2 - \mathbb{E} \|Z_{(i)}\|^2 \right)_{ij}^{st} \\ R^{(2)} &= \left(\|Z_{t,(j)}\|^2 - \mathbb{E} \|Z_{(j)}\|^2 \right)_{ij}^{st} \\ R^{(3)} &= (Z_i' Z_j)_{ij} \\ R^{(4)} &= (\langle Z_i - Z_j, M_i - M_j \rangle)_{ij} \\ R^{(5)} &= (\|M_{si} - M_{tj}\|_2^2 - d_{st}^2 \mathbf{1}\{s \neq t\})_{ij}^{st} \end{aligned}$$

We now obtain high-probability bounds on each of the operator norms of these matrices. First,

$$\|A^{(EK)} - P^{(EK)}\|_2^2 \leq 4 \left(2 \|R^{(1)}\|_2^2 + \|R^{(3)}\|_2^2 + \|R^{(4)}\|_2^2 + \|R^{(5)}\|_2^2 \right)$$

since $R^{(1)}$ and $R^{(2)}$ have the same norms. Notice that for $R^{(3)}$ and $R^{(4)}$, we have dropped the manifold and order statistic notation since the matrix with entries $(Z_i' Z_j)_{ij}$ has the same operator norm as the matrix with entries $(Z_{s,(i)}' Z_{t,(j)})_{ij}^{st}$, and similarly for the inner product matrix. Now, $R^{(1)}$ has rank 1, so its operator norm is equal to its Frobenius norm. We have:

$$\begin{aligned} \|R^{(1)}\|_F^2 &= \sum_{t=1}^k \sum_{s=1}^k \frac{n}{k} * \left(\sum_{j=1}^{n/k} \left(\|Z_{s,(j)}\|^2 - \mathbb{E} \|Z_{s,(j)}\|^2 \right)^2 \right) \\ &= k^2 \frac{n}{k} \left(\sum_{j=1}^{n/k} \left(\|Z_{s,(j)}\|^2 - \mathbb{E} \|Z_{s,(j)}\|^2 \right)^2 \right) \end{aligned}$$

since the noise distribution is the same on all the manifolds. Thus

$$\mathbb{E} \|R^{(1)}\|_F^2 = kn \mathbb{E} \sum_{j=1}^{n/k} \left(\|Z_{s,(j)}\|^2 - \mathbb{E} \|Z_{s,(j)}\|^2 \right)^2 \lesssim knp \log(n/k) \text{ by Lemma ??}$$

And,

$$\begin{aligned} \text{Var} \|R^{(1)}\|_F^2 &= k^2 \sum_{s,t=1}^k \text{Var} \left(\|R_1^{st}\|_F^2 \right) \text{ since the sub-matrices contain independent entries} \\ &= k^4 \text{Var} \|R_1^{st}\|_F^2 \\ &= k^4 \frac{n^2}{k^2} \text{Var} \left(\sum_{j=1}^{n/k} \left(\|Z_{s,(j)}\|^2 - \mathbb{E} \|Z_{s,(j)}\|^2 \right)^2 \right) \\ &\lesssim k^2 n^2 p^2 \log^2(n/k) \end{aligned}$$

by Lem

Using these bounds, along with the Markov Inequality,

$$\begin{aligned} \mathbb{P}\{ \|R_1\|_F^2 - \mathbb{E} \|R_1\|_F^2 > knp \log^2(n/k) \} &\leq \frac{\text{Var} \|R_1\|_F^2}{k^2 n^2 p^2 \log^4(n/k)} \\ &\leq \frac{k^2 n^2 p^2 \log^2(n/k)}{k^2 n^2 p^2 \log^4(n/k)} \\ &= \frac{1}{\log^2(n/k)} \end{aligned}$$

Now $R^{(3)} = ZZ'$. And by a slight adjustment of the analysis in Theorem 5.39 of [Vershynin, 2012],

$$\|Z\| \lesssim \sqrt{n} + \sqrt{p} + \log n, \quad \text{with probability at least } 1 - \frac{1}{n}.$$

Now $\|ZZ'\| \leq \|Z\| \|Z'\| = \|Z\|^2$, so $\|ZZ'\| \lesssim n + p + \log^2 n$. So

$$\|ZZ'\|^2 \lesssim n^2 + p^2 + \log^2 n, \quad \text{with probability at least } 1 - \frac{1}{n}.$$

For $R^{(4)}$, notice that if we condition on $M_i - M_j$, $\langle Z_i - Z_j, M_i - M_j \rangle \sim N(0, \|M_i - M_j\|_2^2)$.

So

$$\begin{aligned} \mathbb{P}\{\|R^{(4)}\|_2^2 > t\} &\leq \mathbb{P}\{\|R^{(4)}\|_F^2 > t\} \leq n^2 \mathbb{P}\{N(0, 1) > t/2n^2(b^2 + d_{\max}^2)\} \\ &\leq n^2 \exp\left(-\frac{t^2}{4n^4(b^4 + d_{\max}^4)}\right) \end{aligned}$$

which leaves us with a bound on the squared operator norm of $n^2(b^2 + d^2) \log(n^2/\delta)$

with probability at least $1 - \delta$. To analyze $R^{(5)}$, we recall that

$$\|M_{si} - M_{tj}\|_2^2 \leq 4(b_s^2 + d_{st}^2 + b_t^2) \leq 8b^2 + 4d_{st}^2$$

So

$$(\|M_{si} - M_{tj}\|_2^2 - 4d_{st}^2 \mathbf{1}\{s \neq t\})^2 \leq 64b^4$$

□

Proof of Theorem 4.3.2.

$$\begin{aligned} \frac{1}{n} L(\tau, \tilde{\tau}) &\lesssim \frac{k \|R^{GK}\|^2}{nr^2} \text{ by Lemma 4.4.1 and Lemma 4.5.8.} \\ &\leq \frac{k^2 \rho^4}{n^2 d_{\min}^4} \|R^{GK}\|^2 \text{ by Lemma 4.4.2.} \\ &\lesssim \frac{k^2 \rho^4}{n^2 d_{\min}^4} \left(\frac{\|R^{EK}\|^2}{\rho^4} + \frac{n^2}{\rho^4} + \frac{n^2 b^8}{\tau^8} \right) \text{ by Lemma 4.5.9} \\ &\lesssim \frac{k^2 \rho^4}{n^2 d_{\min}^4} \left(\frac{n^2 + p^2 + n^2(b^4 + d_{\max}^2 + b^2) + nkp \log(n/k)}{\rho^4} + \frac{n^2}{\rho^4} + \frac{n^2(b^8 + d_{\max}^8)}{\rho^8} \right) \text{ by Theorem 4.3.1} \\ &= \frac{k^2}{d_{\min}^4} \left(c + \frac{p^2}{n^2} + b^4 + d_{\max}^2 + b^2 + \frac{pk \log(n/k)}{n} \right) \end{aligned}$$

for a constant $c > 0$.

□

Proof of Theorem 4.3.3. Let γ^{init} be the error rate from Theorem 4.3.2. Let $\alpha =$

$Ckmr(1 + p/n + \beta^2)$. Define the following sets:

$$\begin{aligned}
B &= \left\{ \forall i \in \{1, \dots, n/2\}, \|\mu_i - \mathcal{P}_{W_2}\mu_i\|_2^2 \leq \alpha \text{ and } \forall i \in \{n/2 + 1, \dots, n\}, \|\mu_i - \mathcal{P}_{W_1}\mu_i\|_2^2 \leq \alpha \right\} \\
D^{init} &= \left\{ i \in \{n/2 + 1, \dots, n\} : \hat{\tau}^{init}(i) = \tau(i) \right\} \\
E^{init} &= \left\{ |(D^{init})^C| \leq n\gamma^{init} \right\}
\end{aligned}$$

Let $E := B \cap E^{init}$. By Theorem 4.3.2 and Lemma 4.4.4

$$\mathbb{P}\{E^C\} \leq \mathbb{P}\{B^C\} + \mathbb{P}\{(E^{init})^C\} \leq \frac{1}{n} + \frac{1}{k \log^2(n/k)} \quad (4.4.5)$$

By Lemma ??, we have only to analyze $(k-1) \max_{j \neq \tau(i)} \mathbb{P}\{\hat{\tau}(i) = j \text{ and } E\}$ for a single node i . The error rate will be the same for all $i \in [n]$, so for simplicity, we let $i = 1$. Assume without loss of generality that $\tau(1) = 1$ and that the probability is the same for all j ; we will let $j = 2$ below. Let $B_\delta(x)$ be the ball of radius δ around a point x . Define

$$Y_i = \mathbf{1}\{\hat{\mu}_i \in B_\delta(\hat{\mu}_1)\}$$

Now we can bound $\mathbb{P}\{\hat{\tau}(i) = 2 \text{ and } E\}$ above by

$$\mathbb{P}\left\{ \sum_{i \in \hat{C}_2} Y_i - \sum_{i \in \hat{C}_1} Y_i > 0 \text{ and } E \text{ and } \|\mathcal{P}_{W_2}Z_1\|_2 + \|\mathcal{P}_{W_1}Z_i\|_2 \leq d/4 \right\} + 2\mathbb{P}\{\chi_r^2 > d^2/64\sigma^2\}$$

The bound of $2\mathbb{P}\{\chi_r^2 > d^2/64\sigma^2\}$ follows because both $\|\mathcal{P}_W Z_1\|_2^2$ and $\|\mathcal{P}_W Z_i\|_2^2$ are distributed as $\sigma^2 \chi_r^2$. And by Lemma 4.5.12.

$$\mathbb{P}\{\chi_r^2 \geq d^2/64\sigma^2\} \leq \exp\left(-\frac{d^2 - 64\sigma^2 r}{512\sigma^2}\right)$$

We use the sub-exponential part of the chi-square tail since $d^2/64\sigma^2 \geq 2r$. Let $i \in \hat{C}_2^{init} \cap D^{init}$. To upper bound $\mathbf{1}\{\|\hat{\mu}_i - \hat{\mu}_1\|_2 \leq \delta\}$, we lower bound $\|\hat{\mu}_i - \hat{\mu}_1\|_2$. Now

$$\begin{aligned} \|\hat{\mu}_i - \hat{\mu}_1\|_2 &= \|\mathcal{P}_{W_1}\mu_i + \mathcal{P}_{W_1}Z_i - \mu_i + \mu_i - \mu_1 + \mu_1 - \mathcal{P}_{W_2}\mu_1 - \mathcal{P}_{W_2}Z_1\|_2 \\ &\geq \|\mu_i - \mu_1\|_2 - \|\mu_i - \mathcal{P}_{W_1}\mu_i\|_2 - \|\mu_1 - \mathcal{P}_{W_2}\mu_1\|_2 - \|\mathcal{P}_{W_1}Z_i\|_2 - \|\mathcal{P}_{W_2}Z_1\|_2 \\ &\geq \|\mu_i - \mu_1\| - 2\sqrt{Ckmr(1 + p/n + \beta^2)} - d/4 \end{aligned} \quad \text{by Lemma 4}$$

when $\|\mathcal{P}_{W_2}Z_1\|_2, \|\mathcal{P}_{W_1}Z_i\|_2$ are both bounded above by $d/8$. Now let $i \in \hat{C}_1^{init} \cap D^{init}$. We must lower bound $\mathbf{1}\{\|\hat{\mu}_i - \hat{\mu}_1\| \leq \delta\}$. Similarly to the above, we obtain:

$$\|\hat{\mu}_i - \hat{\mu}_1\| \leq \|\mu_i - \mu_1\| + 2\sqrt{Ckmr(1 + p/n + \beta^2)} + d/4$$

when $\|\mathcal{P}_{W_2}Z_1\|_2, \|\mathcal{P}_{W_1}Z_i\|_2$ are both bounded above by $d/8$. Now in either case above, if i is not in this set, the above bounds do not hold. But on the set E , there are at most $n\gamma^{init}$ such nodes. Let

$$\begin{aligned} W_{2i} &= \mathbf{1}\{\|\mu_i - \mu_1\| \leq \delta + 2Ckmr(1 + p/n + \beta^2) + d/4\} \\ W_{1i} &= \mathbf{1}\{\|\mu_i - \mu_1\| \leq \delta - 2Ckmr(1 + p/n + \beta^2) - d/4\} \end{aligned}$$

Then we have:

$$\begin{aligned} \mathbb{P}\left\{\sum_{i \in \hat{C}_2} Y_i - \sum_{i \in \hat{C}_1} Y_i > 0 \text{ and } E \text{ and } \|\mathcal{P}_{W_2}Z_1\|_2 + \|\mathcal{P}_{W_1}Z_i\|_2 \leq d/4\right\} &\leq \mathbb{P}\left\{\sum_{i \in B_2} W_{2i} - \sum_{i \in B_1} W_{1i} > -n\gamma\right\} \\ &\leq \mathbb{P}\left\{\sum_{i \in C_2} W_{2i} - \sum_{i \in C_1} W_{1i} > -2n\gamma\right\} \\ &\leq \mathbb{P}\left\{\sum_{i \in C_1} W_{1i} < 2n\gamma^{init}\right\} \end{aligned}$$

The second step follows because $\sum_{i \in B_2} W_{2i} \leq \sum_{i \in C_2} W_{2i}$, since $B_2 \in C_2$, and

$$|B_1| \geq |C_1| - n\gamma^{init}$$

The final step follows because $\mathbf{1}\{\|\mu_i - \mu_1\| \leq \delta + \sqrt{2Ckmr(1 + p/n + \beta^2)} + d/4\} = 0$ for $i \in C_2$ when $\delta \leq d/4$ and $d \geq 2\sqrt{2Ckmr(1 + p/n + \beta^2)}$.

Since we have shifted to the non-random sum of the true nodes, $\sum_{i \in C_1} W_{1i} \sim \text{Binomial}(n/k, p_1)$, where $p_1 = \max(1, \frac{\delta}{b})$. For a random variable Y distributed as $\text{Binomial}(n/k, p_1)$, we have the following Chernoff bound for $\lambda \in [0, 1]$:

$$\mathbb{P}\{Y < (1 - \lambda)\frac{np_1}{k}\} \leq \exp\left(-\frac{\lambda^2 np_1}{k}\right)$$

Recall that we have assumed that $\gamma^{init} = c/k$ for a positive constant c .

$$\begin{aligned} \mathbb{P}\left\{Y < n\gamma^{init}\right\} &= \mathbb{P}\left\{Y < \frac{4cn}{k}\right\} \\ &= \mathbb{P}\left\{Y \leq \frac{np_1}{k} \frac{4c}{p_1}\right\} \\ &\leq \exp\left(-\frac{np_1}{k} \left(\frac{p_1 - 4c}{p_1}\right)^2\right) \end{aligned}$$

where this Chernoff bound holds because $4c/p_1 \in (0, 1)$. Now we can write $p_1 = 1 - h$, where h is tiny. Let $c' = (1-h) * \left(\frac{1-h-4c}{1-h}\right)^2$, which is very close to 1. Then our exponent is nc'/k . By our assumption that

$$n \geq \frac{2k(d^2 - 16\sigma^2)}{128\sigma^2}$$

the desired bound holds. □

4.5 Auxiliary lemmas and proofs

4.5.1 Results for order statistics

Lemma 4.5.1. *Assume m is an even integer and $m \leq 6$. Assume $n > \max(2m + 2, 16m \log p + 4m)$.*

There is a constant $c > 0$ such that for each i and for any even integer $m \leq 6$,

$$\mathbb{E} \frac{1}{(h(Y_{(i)}))^m} \leq cp^{m/2}$$

Proof of Lemma 4.5.1. We split the domain of Y into the right tail (III), the middle (II), and the left tail (I). On the right tail, we use Lemma ?? to bound h . In the middle region, we have a lower bound on h via Lemma 4.5.3. On the left tail, we rely on concentration inequalities for order statistics of the χ_p^2 distribution. Let $i \geq n/2$. Let $c_1 > 0$ be as in Lemma 4.5.3 and denote $Y_{(i)}$ by Y to simplify the notation. Now,

$$\begin{aligned} \mathbb{E} \frac{1}{h(Y)^m} &= \mathbb{E} \frac{1}{h(Y)^m} \left\{ Y \in [0, p - c_1 \sqrt{p}] \right\} + \mathbb{E} \frac{1}{h(Y)^m} \left\{ \left| \frac{Y - p}{\sqrt{p}} \right| \leq c_1 \right\} + \mathbb{E} \frac{1}{h(Y)^m} \left\{ Y > p + c_1 \sqrt{p} \right\} \\ &\triangleq (I) + (II) + (III) \end{aligned}$$

Consider (III). Corollary 4.5.6 says that for some $c_3 > 0$:

$$\begin{aligned} (III) &\leq \mathbb{E} \left(\frac{\bar{F}(Y)}{f(Y)} \right)^m \left\{ Y > p + c_1 \sqrt{p} \right\} \\ &\leq \mathbb{E} \frac{1}{\left(\frac{1}{2} - \frac{p/2-1}{Y} \right)^m} \left\{ Y > p + c_1 \sqrt{p} \right\} \\ &\leq \mathbb{E} \frac{1}{\left(\frac{1}{2} - \frac{p/2-1}{p+c_1\sqrt{p}} \right)^m} \left\{ Y > p + c_1 \sqrt{p} \right\} \\ &\leq c_3 p^{m/2} \end{aligned}$$

For (II), we bound \bar{F} above by 1 and use Lemma 4.5.3 to obtain, for a constant c_2 ,

$$(II) \leq \mathbb{E} \left(\frac{1}{f(Y)} \right)^m \left\{ \left| \frac{Y - p}{\sqrt{p}} \right| \leq c_1 \right\} \leq c_2 p^{m/2}. \quad (4.5.1)$$

For (I), we further split this section into three parts:

$$\begin{aligned} (I) &= \mathbb{E} \frac{1}{h(Y)^m} \left\{ Y \in (0, 1] \right\} + \mathbb{E} \frac{1}{h(Y)^m} \left\{ Y \in \left(1, \frac{p}{2} \right) \right\} + \mathbb{E} \frac{1}{h(Y)^m} \left\{ Y \in \left(\frac{p}{2}, p - c_1 \sqrt{p} \right) \right\} \\ &\triangleq (I_1) + (I_2) + (I_3) \end{aligned}$$

In each, again bound \bar{F} above by 1. For $(I)_1$, we use the density of the i th order statistic and the fact that $F(x) \leq xf(x)$ for $x \in (0, 1]$, since the chi-square density is increasing for $x \leq 1$ when $p \geq 3$.

$$\begin{aligned} (I)_1 &\leq \int_0^1 \frac{1}{f^m(x)} f(x) (F(x))^{n-i-1} (\bar{F}(x))^i dx \\ &\leq \int_0^1 \frac{1}{(f(x))^m} f(x) (xf(x))^{n-i-1} dx \\ &\leq \int_0^1 x (f(x))^{n-i-m} dx \\ &\leq \int_0^1 x (f(x))^{n/2-m} dx \\ &\leq 1 \end{aligned}$$

For $(I)_2$, recall that the $f^m(x) = \left(\frac{x^{p-2}e^{-x}}{c_p^2} \right)^{m/2}$ where $c_p = \Gamma(p/2)2^{p/2}$. For $x \in (1, \frac{p}{2}]$,

this is bounded below by $\frac{e^{-pm/4}}{c_p^m}$. So:

$$\begin{aligned}
(I)_2 &\leq c_p^m e^{pm/4} \mathbb{P} \left\{ Y \in \left(1, \frac{p}{2} \right) \right\} \\
&\leq c_p^m e^{pm/4} \left(\mathbb{P} \left\{ Y_i \leq \frac{p}{2} \right\} \right)^{i-1} && \text{by Lemma 4.5.13} \\
&\leq c_p^m e^{pm/4} e^{-(i-1)p/16} && \text{since } \mathbb{P}\{Y_i \leq \frac{p}{2}\} \leq e^{-p/16} \text{ by Lemma 4.5.12} \\
&\leq c_p^m e^{pm/4} e^{-p(n/2-1)/16} && \text{since } i \leq n/2 \\
&\leq c_p^m e^{pm/4} e^{-pn/32}
\end{aligned}$$

To bound $c_p^m e^{pm/4}$, recall that $\Gamma(x) \leq x! \leq x^x$ for $x > 2$. So:

$$c_p^m = (\Gamma(p/2))^m 2^{pm/2} \leq \left(\frac{p}{2}\right)^{pm/2} 2^{pm/2} = p^{pm/2}$$

Solving for a bound on n :

$$p^{pm/2} e^{pm/4} e^{-pn/32} \leq p^{m/2} \Leftrightarrow n \geq \frac{16m}{p} (p-1) \log p + 4m$$

Thus as long as $n \geq 16m \log p + 4m$, we have the desired conclusion. For $(I)_3$, we rely

on Lemma 4.5.3, which tells us that $\frac{1}{(f(x))^m} \lesssim p^{m/2} e^{\frac{m(x-p)^2}{2p}}$ if $x \in (\frac{p}{2}, p - c_1\sqrt{p})$. So:

$$\begin{aligned}
(I)_3 &\leq p^{m/2} \mathbb{E} e^{\left(\frac{Y-p}{\sqrt{2p}}\right)^2} \left\{ Y \in \left(\frac{p}{2}, p - c_1\sqrt{p}\right) \right\} \\
&\leq p^{m/2} \sum_{j=1}^{\sqrt{p}/c_1} e^{\frac{(-c_1 j)^2}{2}} \mathbb{P} \left\{ Y \in \left(p - (j+1)c_1\sqrt{p}, p - jc_1\sqrt{p}\right] \right\} \\
&\leq p^{m/2} \sum_{j=1}^{\sqrt{p}/c_1} e^{\frac{c_1^2 j^2}{2}} (F(p - jc_1\sqrt{p}))^{n/2} \\
&\leq p^{m/2} \sum_{j=1}^{\sqrt{p}/c_1} e^{c_1^2 j^2/2} e^{-jn/2} \\
&\leq p^{m/2}
\end{aligned}$$

For $i < n/2$, we divide up the expectation into the same three parts as we did previously. The methods here are essentially the same as those used in the previous part, except in reverse order. From Equation 4.5.1 we have $(II) \leq c_2 p^{m/2}$. Recall from Lemma ?? that \tilde{h} is increasing and that $\tilde{h}(x) = f(-x)/F(-x)$. This implies that F/f is also increasing. Thus,

$$(I) \leq \mathbb{E} \frac{F^m(p - c_1\sqrt{p})}{f^m(p - c_1\sqrt{p})} 1 \left\{ \frac{Y_{(i)} - p}{\sqrt{p}} < -c_1 \right\} \leq f^{-m}(p - c_1\sqrt{p}) \leq c_4 p^{m/2},$$

where the last inequality is by Lemma 4.5.3. And,

$$(III) \leq \mathbb{E} \left(\frac{1}{f^m(Y_{(i)})} \right) 1 \left\{ \frac{Y_{(i)} - p}{\sqrt{p}} > c_1 \right\} \quad (4.5.2)$$

$$\leq p^{m/2} e^{c_1^2/2} e^{-c_1^2(n-i+1)/8} \quad (4.5.3)$$

$$\leq p^{m/2} \quad (4.5.4)$$

where step 4.5.3 follows from Lemma 4.5.3 and Lemma 4.5.13 in Appendix B. \square

Lemma 4.5.2. *Let $S := \sum_{i=1}^n \left(Y_{(i)} - \mathbb{E}Y_{(i)}\right)^2$. Then there are constants $c_1, c_2 > 0$*

such that, with probability at least $1 - c_1/\log^2 n$,

$$S \leq c_2 p \log^2 n,$$

Proof of Lemma 4.5.2. For $i \geq \frac{n}{2}$, by Theorem 2.5 of [Boucheron and Thomas, 2012],

$$\text{Var}(Y_{(i)}) \leq \frac{2}{n-i+1} \mathbb{E} \frac{1}{(h(Y_{(i-1)}))^2}.$$

Note that [Boucheron and Thomas, 2012] uses the reverse ordering as this, so their result is stated slightly differently. We have reversed their ordering of the order statistics to be more consistent with traditional notation. For $i < \frac{n}{2}$, we use the following transformation. Let $\tilde{Y}_i = -Y_i$ for each $i \in [n]$. Then \tilde{Y}_i has the density \tilde{f} and distribution function \tilde{F} which satisfy $\tilde{f}(x) = f(-x)$ and $\tilde{F}(x) = 1 - F(-x)$. The hazard rate of Y_i have $\tilde{h}(x) = \tilde{f}(x)/(1 - \tilde{F}(x)) = f(-x)/F(-x)$, for $x < 0$. \tilde{h} is an increasing function, since:

$$\begin{aligned} \tilde{h}'(x) &= \frac{f(-x)^2 - f'(-x)F(-x)}{F^2(-x)} \\ &= \frac{f(-x) \left(f(-x) + \left(\frac{p/2-1}{x} + \frac{1}{2} \right) F(-x) \right)}{F^2(-x)} \\ &> 0, \end{aligned}$$

where we use the fact that $f'(x) = (\frac{p/2-1}{x} - \frac{1}{2})f(x)$. Again by Theorem 2.9 of [Boucheron and Thomas

$$\text{Var}(Y_{(i)}) = \text{Var}(\tilde{Y}_{(n-i+1)}) \leq \frac{2}{i} \mathbb{E} \frac{1}{(\tilde{h}(\tilde{Y}_{(n-i)}))^2}$$

Thus for $i < n/2$,

$$\text{Var}(Y_{(i)}) \leq \frac{2}{i} \mathbb{E} \left(\frac{F^2(-\tilde{Y}_{(n-i)})}{f^2(-\tilde{Y}_{(n-i)})} \right) = \frac{2}{i} \mathbb{E} \left(\frac{F^2(Y_{(i+1)})}{f^2(Y_{(i+1)})} \right).$$

In summary, we have:

$$\text{Var}(Y_{(i)}) \leq \begin{cases} \frac{2}{n-i+1} \frac{1}{h^2(Y_{(i-1)})}, & i \geq \frac{n}{2}; \\ \frac{2}{i} \frac{1}{h^2(Y_{(i+1)})}, & i < \frac{n}{2}. \end{cases}$$

Combining this with Lemma 4.5.1, we obtain:

$$\text{Var}(Y_{(i)}) \leq \begin{cases} \frac{2(c_2+c_3+1)p}{n-i+1}, & i \geq \frac{n}{2}; \\ \frac{2(c_2+c_4+1)p}{i}, & i < \frac{n}{2}. \end{cases}$$

Then for some constant $c_5 > 0$,

$$\mathbb{E}S = \sum_{i=1}^n \text{Var}(Y_{(i)}) \leq 4(c_2 + c_3 \vee c_4 + 1)p \sum_{i=1}^{n/2} \frac{1}{i} \leq 4(c_2 + c_3 \vee c_4 + 1)p \int_1^{n/2} \frac{1}{x} dx = c_5 p \log(n/2).$$

Note that for any two random variables U and V ,

$$\mathbb{E}UV = \text{Cov}(U, V) + \mathbb{E}U\mathbb{E}V \leq \sqrt{\text{Var}U\text{Var}V} + \mathbb{E}U\mathbb{E}V \quad (4.5.5)$$

So,

$$\begin{aligned} \text{Var}S &= \sum_{i=1}^n \text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)})^2 + \sum_{i \neq j} \text{Cov}((Y_{(i)} - \mathbb{E}Y_{(i)})^2, (Y_{(j)} - \mathbb{E}Y_{(j)})^2) \\ &\leq \sum_{i=1}^n \text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)})^2 + \sum_{i \neq j} \sqrt{\text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)})^2 \text{Var}(Y_{(j)} - \mathbb{E}Y_{(j)})^2} \\ &= \left(\sum_{i=1}^n \sqrt{\text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)})^2} \right)^2 \end{aligned} \quad (4.5.6)$$

It remains to find an upper bound for $\text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)})^2$. Let $Z := (Y_{(i)} - \mathbb{E}Y_{(i)})^2$, $Z_j = (Y_{(i)}^j - \mathbb{E}Y_{(i)})^2$, where $Y_{(i)}^j$ is the i th order statistic of the set $\{Y_1, \dots, Y_{j-1}, Y_{j+1}, \dots, Y_n\}$.

Note that $\mathbb{E}Y_{(i)}$ is just a constant that depends on i, n, p .

$$Z_j = \begin{cases} (Y_{(i+1)} - \mathbb{E}Y_{(i)})^2, & \text{if } Y_j \leq Y_{(i)}; \\ Z, & \text{otherwise} \end{cases}$$

That is, if we remove Y_j from the sample and it is smaller or equal to $Y_{(i)}$, then the i th order statistic of the new sample is now $Y_{(i+1)}$. By Theorem 2.1 of [Boucheron and Thomas, 2012],

$$\text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)})^2 \leq \sum_{j=1}^n \mathbb{E}(Z - Z_j)^2 = i\mathbb{E}\left((Y_{(i)} - \mathbb{E}Y_{(i)})^2 - (Y_{(i+1)} - \mathbb{E}Y_{(i)})^2\right)^2.$$

Thus:

$$\begin{aligned} \text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)})^2 &\leq i\mathbb{E}\left((Y_{(i)} - \mathbb{E}Y_{(i)})^2 - (Y_{(i+1)} - Y_{(i)} + Y_{(i)} - \mathbb{E}Y_{(i)})^2\right)^2 \\ &\leq i\mathbb{E}\left((Y_{(i+1)} - Y_{(i)})^2 + 2(Y_{(i+1)} - Y_{(i)})(Y_{(i)} - \mathbb{E}Y_{(i)})\right)^2 \end{aligned}$$

Let $D_i = Y_{(i+1)} - Y_{(i)}$. Then:

$$\begin{aligned} \text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)})^2 &\leq i\mathbb{E}\left(D_i^2 + 2D_i(Y_{(i)} - \mathbb{E}Y_{(i)})\right)^2 \\ &= i\left(\mathbb{E}D_i^4 + 4\mathbb{E}D_i^3(Y_{(i)} - \mathbb{E}Y_{(i)}) + 4\mathbb{E}D_i^2(Y_{(i)} - \mathbb{E}Y_{(i)})^2\right) \\ &\triangleq i(A + 4B + 4C). \end{aligned}$$

By (4.5.5),

$$B \leq \sqrt{\text{Var}D_i^3\text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)}) + \mathbb{E}D_i^3\mathbb{E}(Y_{(i)} - \mathbb{E}Y_{(i)})} \leq \sqrt{\mathbb{E}D_i^6\text{Var}Y_{(i)}} + 0.$$

and

$$C \leq \sqrt{\text{Var}D_i^2\text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)})^2 + \mathbb{E}D_i^2\text{Var}Y_{(i)}} \leq \sqrt{\mathbb{E}D_i^4\text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)})^2 + \mathbb{E}D_i^2\text{Var}Y_{(i)}},$$

Combining Lemma 4.5.4 and Lemma 4.5.1, $\mathbb{E}D_i^2 \leq c'_1 i^{-2}p$, $\mathbb{E}D_i^4 \lesssim c'_1 i^{-4}p^2$, $\mathbb{E}D_i^6 \leq c'_1 i^{-6}p^3$ for some constant $c'_1 > 0$. And for $i < n/2$, $\text{Var}Y_{(i)} \leq c'_1 i^{-1}p$, while for $i \geq n/2$, $\text{Var}Y_{(i)} \leq c'_1 (n - i + 1)^{-1}p$. So for $i < n/2$,

$$\text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)})^2 \leq i \left(c'_1 i^{-4}p^2 + 4\sqrt{c'_1 i^{-4}p^2 \text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)})^2} + 4c'_1 i^{-3}p^2 + 4\sqrt{c'_1 i^{-7}p^4} \right).$$

By solving this quadratic equation we have

$$\text{Var}(Y_{(i)} - \mathbb{E}Y_{(i)})^2 \leq \frac{c'_2 p^2}{i^2},$$

for some constant $c'_2 > 0$. Thus continuing from (4.5.6), we have:

$$\begin{aligned} \text{Var}S &\leq 4(c'_2 \vee c'_3)^2 \left(\sum_{i=1}^{n/2} \frac{p}{i} \right)^2 \\ &\leq c'_4 p^2 \log^2 n, \end{aligned}$$

for some constant $c'_4 > 0$. Thus we have shown that

$$\mathbb{E}S \leq cp \log n$$

and

$$\text{Var}S \leq cp^2 \log^2 n$$

And by the Markov Inequality:

$$\mathbb{P}(|S - \mathbb{E}S| \geq cp \log^2 n) \leq \frac{\text{Var}S}{c^2 p^2 \log^4 n} \lesssim \frac{1}{\log^2 n}.$$

Thus with probability $1 - 1/\log^2 n$, $S - \mathbb{E}S \leq |S - \mathbb{E}S| \leq cp \log^2 n$. So $S \leq \mathbb{E}S + cp \log^2 n \leq c_2 p \log^2 n$ by the bound on $\mathbb{E}S$. \square

Lemma 4.5.3. *Let $f(x)$ be the density function of $\chi^2(p)$. Then*

1. *There exists constants $c_1, c > 0$ such that for any $x \in [p - c_1\sqrt{p}, p + c_1\sqrt{p}]$ we have $f(x) \geq c/\sqrt{p}$;*
2. *For $t \in (-\sqrt{p}/2, +\infty)$, $f(p + t\sqrt{p}) \geq c \exp(-t^2/2)/\sqrt{p}$. That is, $f(x) \geq ce^{-\frac{(x-p)^2}{2p}}$.*
3. *For $t \in [\sqrt{p}, +\infty)$, $f(x) \geq \exp(-x/2)$.*

Proof. First, we show that there is a constant $c > 0$ such that $f(p) \geq c/\sqrt{p}$. Recall that

$$f(p) = \frac{p^{p/2-1}e^{-p/2}}{2^{p/2}\Gamma(p/2)} = \frac{(p/2)^{p/2}e^{-p/2}}{p\Gamma(p/2)}$$

Now $p \geq 3$ is an integer. First assume that p is even, so $\Gamma(p/2) = (\frac{p}{2} - 1)!$. By Stirling's bounds,

$$\Gamma(p/2) \leq \left(\frac{p}{2} - 1\right)! \leq e \left(\frac{p}{2} - 1\right)^{p/2-1/2} e^{-p/2-1}$$

Thus, if p is even,

$$f(p) \geq \frac{(p/2)^{p/2}e^{-p/2}}{pe(p/2-1)^{p/2-1/2}e^{-p/2-1}} = \frac{(p/2)^{p/2}}{p(p/2-1)^{p/2-1/2}} \geq \frac{c}{\sqrt{p}}$$

since $\frac{p/2}{p/2-1}$ is bounded below by a constant and

$$p \left(\frac{p}{2} - 1\right)^{-1/2} = p \left(\frac{2}{p-2}\right)^{1/2} = \sqrt{2} \left(\frac{p^2}{p-2}\right)^{1/2} \leq \sqrt{2}c_2\sqrt{p}$$

where c_2 is some positive constant. Now suppose p is odd. Let f_p denote the density of a χ_p^2 random variable. Note that

$$\frac{c}{\sqrt{p}} \leq f_p(p) \leq f_{p-1}(p-1)$$

Therefore, there is a positive constant c_1 such that

$$\frac{c_1}{\sqrt{p-1}} \leq \frac{\sqrt{p-1}}{\sqrt{p}} \frac{c}{\sqrt{p-1}} \leq f_{p-1}(p-1)$$

Now we prove (1).

$$f(p + c_1\sqrt{p})/f(p) \geq \left(\frac{c_1}{\sqrt{p}} + 1\right)^{\frac{p}{2}-1} \exp\left(-\frac{c_1\sqrt{p}}{2}\right) \quad (4.5.7)$$

$$= \exp\left(\left(\frac{p}{2} - 1\right) \log\left(\frac{c_1}{\sqrt{p}} + 1\right) - \frac{c_1\sqrt{p}}{2}\right) \quad (4.5.8)$$

$$\geq \exp\left(\left(\frac{p}{2} - 1\right) \left(\frac{c_1}{\sqrt{p}} - \frac{c_1^2}{p}\right) - \frac{c_1\sqrt{p}}{2}\right) \quad \text{since } \log(1+x) \geq x - x^2 \quad (4.5.9)$$

$$= \exp\left(-\frac{c_1}{\sqrt{p}} - \frac{c_1^2}{2} + \frac{c_1^2}{p}\right) \quad (4.5.10)$$

$$\geq \exp\left(-\frac{c_1^2}{2} - \frac{1}{4}\right) \quad (4.5.11)$$

$$\triangleq c_3, \quad (4.5.12)$$

In (4.5.11), we simply minimized the function in p and found that the minimum is achieved at $p = 4c_1^2$, with the minimum as given. We also checked, via the second derivative, that this is indeed a minimum. By a similar argument, we have that $f(p - c_1\sqrt{p})/f(p) \geq c_3$. Thus for any $x \in [p - c_1\sqrt{p}, p + c_1\sqrt{p}]$, $f(x) \geq c_2c_3/\sqrt{p}$. For

2), we do the same Taylor expansion as above:

$$f(p + t\sqrt{p})/f(p) \geq \exp\left(-\frac{t}{\sqrt{p}} - \frac{t^2}{2} + \frac{t^2}{p}\right) \geq \exp(-t^2).$$

For (3), notice that $(2p)^{\frac{p}{2}-1} \geq 2^{p/2}\Gamma(p/2)$ where $\Gamma(p/2) = (p/2)!$ if $p/2$ is an integer and $\Gamma(p/2) \leq \sqrt{2}((p+1)/2)!$ if otherwise. Thus $f(x) \geq \exp(-x/2)$ immediately from its density function. \square

Lemma 4.5.4. *Let $Y_{(1)} \leq \dots \leq Y_{(n)}$ be the order statistics of a sample of n points from a distribution F with a non-decreasing hazard rate h . Let E_i be a standard exponential random variable. Let $X_{(1)} \leq \dots \leq X_{(n)}$ be the order statistics of a sample of n points from the standard exponential distribution.*

$$\mathbb{E}(Y_{(i+1)} - Y_{(i)})^m \leq \frac{m!}{i^m} \mathbb{E} \frac{1}{h^m(Y_{(i)})}$$

Proof. Let $U(t) = F^-(1 - 1/t)$, as in [Boucheron and Thomas, 2012]. Note that $\frac{d}{dt}U(e^t) = 1/h(U(e^t))$. So $U \circ \exp$ is concave since h is non-decreasing. Thus:

$$\begin{aligned} U(\exp(X_{(i+1)})) - U(\exp(X_{(i)})) &= U(\exp(X_{(i)} + X_{(i+1)} - X_{(i)})) - U(\exp(X_{(i)})) \\ &\leq \frac{d}{dt}(U \circ \exp)(t) \Big|_{X_{(i)}} (X_{(i+1)} - X_{(i)}) \\ &= \frac{1}{h(U(e^{X_{(i)}}))} (X_{(i+1)} - X_{(i)}) \end{aligned}$$

Now by Theorem 2.5 of [Boucheron and Thomas, 2012], $Y_{(i)} =_d U(\exp(X_{(i)}))$. Thus:

$$Y_{(i+1)} - Y_{(i)} =_d U(\exp(X_{(i+1)})) - U(\exp(X_{(i)})) \leq \frac{X_{(i+1)} - X_{(i)}}{h(U(\exp(X_{(i)})))} =_d \frac{X_{(i+1)} - X_{(i)}}{h(Y_{(i)})}$$

And again by Theorem 2.5 of [Boucheron and Thomas, 2012], $X_{(i+1)} - X_{(i)} \sim E_i/i$.

The m th moment of a standard exponential is $m!$. Thus,

$$\mathbb{E}(Y_{(i+1)} - Y_{(i)})^m \leq \mathbb{E} \frac{(E_i/i)^m}{h(Y_{(i)})^m} = \frac{m!}{i^m} \mathbb{E} \frac{1}{h(Y_{(i)})^m}$$

□

Lemma 4.5.5. *The hazard function h of a χ_p^2 random variable is non-decreasing as long as $p \geq 2$.*

Proof of Lemma 4.5.5. Apply Lemma 4.5.7 to f and \bar{F} , which are both defined on $[0, \infty)$ and which satisfy $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \bar{F}(x) = 0$. The derivative of $\bar{F}(x)$ is $-f(x)$, which is always negative on $(0, \infty)$. The derivative of f is proportional to $(\frac{p/2-1}{x} - \frac{1}{2})f(x)$. The ratio is thus $\frac{1}{2} - \frac{p/2-1}{x}$, which is increasing in x as long as $p \geq 3$. □

Corollary 4.5.6. *For $\chi^2(p)$ with $p \geq 2$, we have $f \geq (\frac{1}{2} - \frac{p/2-1}{x})\bar{F}$ for $x > p$.*

Proof of Corollary 4.5.6. This is a direct consequence of Lemma 4.5.5 by taking the derivative of h . Note that we have $f' = (\frac{p/2-1}{x} - \frac{1}{2})f$. We prove the lemma by noticing

$$h'(x) = \frac{f(x)(f(x) - (\frac{1}{2} - \frac{p/2-1}{x})\bar{F}(x))}{(1 - F(x))^2} \geq 0.$$

□

The following lemma is a slight adaptation of Proposition 1.1 in [Pinelis, 2006].

Lemma 4.5.7. *Suppose f and g are two continuously differentiable functions defined on the same interval, (a, b) , and that $\lim_{x \rightarrow b^+} f(x) = \lim_{x \rightarrow b^+} g(x) = 0$. And suppose g is either always positive or always negative on its domain (it cannot change sign and cannot be 0). Then $\frac{f'}{g}$ increasing implies that $(\frac{f}{g})' > 0$.*

Proof. Note that $(\frac{f}{g})' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$. The denominator is positive. It remains to show that the numerator is also positive for all $x \in (a, b)$. For $y \in (a, b)$, define

$m_x(y) := g(y)f'(x) - f(y)g'(x)$. Now fix an $x \in (a, b)$ and let $y \in (x, b)$.

$$\frac{\partial}{\partial y} m_x(y) = g'(y)f'(x) - f'(y)g'(x) = g'(y)g'(x) \left(\frac{f'(x)}{g'(x)} - \frac{f'(y)}{g'(y)} \right)$$

Now $g'(y)g'(x) > 0$ since g does not change sign on its domain. And since $y > x$ and $\frac{f'}{g'}$ is increasing, $\left(\frac{f'(x)}{g'(x)} - \frac{f'(y)}{g'(y)} \right) < 0$. Thus $\frac{\partial}{\partial y} m_x(y) < 0$, so $m_x(y)$ is decreasing for $y \in (x, b)$. Moreover, $m_x(y)$ is continuous since g, g', f, f' are. So in fact, $m_x(y)$ is decreasing on $[x, b)$. And:

$$\lim_{y \rightarrow b^+} m_x(y) = \lim_{x \rightarrow b^+} g(y)f'(x) - \lim_{x \rightarrow b^+} f(y)g'(x) = 0$$

That is, $m_x(y)$ is decreasing to 0 on $[x, b)$, which implies $m_x(x) > 0$, as needed. \square

4.5.2 Inequalities related to the upper bound

Lemma 4.5.8. *Let $A \in \mathbb{R}^{n \times p}$ and let A_k be the rank k approximation of A . Let P have rank k . Then*

$$\|A_k - P\|_F^2 \leq 8k \|A - P\|^2$$

Proof of Lemma 4.5.8. Since A_k and P are rank k , $A_k - P$ has rank at most $2k$. And for any rank $2k$ matrix A , $\|A\|_F^2 \leq 2k \|A\|_2^2$. So,

$$\begin{aligned} \|A_k - P\|_F^2 &\leq 2k \|A_k - P\|_2^2 \\ &\leq 4k \|A_k - A\|_2^2 + 4k \|A - P\|_2^2 \quad \text{by the triangle inequality} \\ &\leq 8k \|A - P\|_2^2 \end{aligned}$$

The third step follows because $\|A - A_k\|_F^2 \leq \|A - P\|_F^2$ for any rank- k matrix P . \square

Lemma 4.5.9. *Suppose ρ satisfies the bandwidth assumptions. Then:*

$$\|R^{(GK)}\|^2 \lesssim \left\| \frac{R^{(EK)}}{\rho^2} \right\|^2 + \frac{3n^2}{\rho^4} + \frac{b^8 n^2}{\rho^8}$$

Proof of Lemma 4.5.9. Fix $s, t \in [k]$. Let $i, j \in [n/k]$. By Taylor's Theorem, we know there exists

$$\xi_{ij}^{st} \in \left(0, \frac{\|X_{s,(i)} - X_{t,(j)}\|^2}{\rho^2} \right)$$

and

$$c_{ij}^{st} \in \left(0, \frac{\mathbb{E} \|Z_{s,(i)}\|^2 + \mathbb{E} \|Z_{t,(j)}\|^2 + d_{st}^2 \{s \neq t\}}{\rho^2} \right)$$

such that:

$$(R^{(GK)})_{ij}^{st} = -\frac{(R^{(EK)})_{ij}^{st}}{\rho^2} + \underbrace{\frac{e^{-\xi_{ij}^{st}}}{2} \left(\frac{\|X_{s,(i)} - X_{t,(j)}\|^2}{\rho^2} \right)^2}_{(I)} - \underbrace{\frac{e^{-c_{ij}^{st}}}{2} \left(\frac{\mathbb{E} \|Z_{s,(i)}\|^2 + \mathbb{E} \|Z_{t,(j)}\|^2 + \mathbb{E} \|M_{si} - M_{tj}\|^2}{\rho^2} \right)}_{(II)}$$

Since ξ_{ij}^{st} and c_{ij}^{st} are positive, $e^{-\xi_{ij}^{st}}, e^{-c_{ij}^{st}} \in (0, 1)$. The terms (I) and (II) are real and nonnegative, and the matrices with these entries are symmetric. So Lemma 4.5.10 implies that we can drop the exponential coefficients in analyzing the operator norm.

Using the bound $(a + b)^2 \leq a^2 + b^2$ repeatedly,

$$\|R^{(GK)}\|_2^2 \leq c \left(\left\| \frac{R^{(EK)}}{\rho^2} \right\|^2 + \|N^{(1)}\|^2 + \|N^{(2)}\|^2 + \|N^{(3)}\|^2 + \|N^{(4)}\|^2 \right)$$

for some constant $c > 0$, where

$$\begin{aligned}(N^{(1)})_{ij} &= \left(\frac{\|Z_j - Z_j\|^2}{\rho^2} \right)^2 \\(N^{(2)})_{ij} &= \frac{\|M_{s,i} - M_{t,j}\|_2^4}{\rho^4} \\(N^{(3)})_{ij}^{st} &= \left(\frac{\mathbb{E} \|Z_{s,(i)}\|^2 + \mathbb{E} \|Z_{t,(j)}\|^2}{\rho^2} \right)^2 \\(N^{(4)})_{ij}^{st} &= \left(\frac{d_{st}^2}{\rho^4} \mathbf{1}\{s \neq t\} \right)\end{aligned}$$

We proceed to bound each operator norm separately. For $N^{(4)}$, we use the operator norm bound; for the rest, we use the Frobenius norm. First,

$$\begin{aligned}\mathbb{P} \left\{ \|N^{(1)}\|_F^2 > \frac{n^2}{\rho^4} \right\} &\leq n^2 \mathbb{P} \left\{ \frac{\|Z_i - Z_j\|^8}{\rho^8} > \frac{1}{\rho^4} \right\} && \text{by a union bound} \\&= n^2 \mathbb{P} \{ \|Z_i - Z_j\|^2 > \rho \} \\&= n^2 \mathbb{P} \{ \chi_p^2 \geq p + 2\sqrt{3p \log n} + 6 \log n \} \\&\leq n^2 \exp(-3 \log n)\end{aligned}$$

where the third step is since $\|Z_i - Z_j\|^2 \sim 2\chi_p^2$ and since $\rho \geq 2(p + 2\sqrt{3p \log n} + 6 \log n)$.

The final step is by Lemma 4.5.12. So with probability $1 - \frac{1}{n}$,

$$\|N^{(1)}\|^2 \lesssim \frac{n^2}{\rho^4}.$$

And for any i, j , $\|M_i - M_j\|^2 \leq 6b^2 + 3d_{\max}^2$. So there is a constant $c > 0$ such that

$$\|N^{(2)}\|_F^2 \leq \frac{cn^2(b^8 + d_{\max}^8)}{\rho^8} \leq \frac{cn^2b^8}{\rho^8} + \frac{cn^2}{\rho^4}$$

where the last step follows since $\rho \geq d_{\max}^2$. And $\mathbb{E} \|Z_{(n/k)}\|^2 \leq p \log \frac{n}{k}$ by Lemma 4.5.14, so $\rho \geq \mathbb{E} \|Z_{s,(i)}\|^2 + \mathbb{E} \|Z_{t,(j)}\|^2$ for all s, t, i, j , by our assumption on ρ . Thus

all squared entries of $N^{(3)}$ are bounded by $\frac{1}{\rho^4}$, so

$$\|N^{(3)}\|_F^2 \leq \frac{n^2}{\rho^4}.$$

Finally,

$$\|N^{(4)}\|^2 \leq \frac{n^2 d_{\max}^8}{k^2 \rho^8} \leq \frac{n^2}{\rho^4}.$$

since $\rho \geq d_{\max}^2 / \sqrt{k}$. □

Lemma 4.5.10. *If E is a matrix with bounded entries and M is a real symmetric matrix with positive entries, then $\|E \circ M\|_{OP} \leq \max_{i,j} |E_{ij}| \|M\|_{OP}$.*

Proof. See Lemma A.5 of [Karoui, 2010b]. □

Lemma 4.5.11. *Let $U' \in \mathbb{R}^{n \times p}$ be a matrix whose rows μ'_i lie on a smooth submanifold M of \mathbb{R}^p . Then there exists an m such that U can be “well-approximated” an m -rank matrix U , in that:*

$$\|U - U'\|_F^2 \lesssim n * o(1/n)$$

Proof. [Sketch:] Since M is smooth, we can find m centers such that each μ'_i is “close” to a μ_i that is a linear combination of these m centers. (E.g., μ_i could just be cv_i for one v_i .) Put these μ_i into the matrix U ; then U is rank m . And the result holds. □

4.5.3 Standard inequalities in probability

Lemma 4.5.12. *Let $X \sim \chi^2(p)$. Then*

$$\mathbb{P}\left(|X - p| \geq t\sqrt{p}\right) \leq \begin{cases} 2 \exp(-t^2/8), & \text{if } 0 \leq t \leq \sqrt{p} \\ 2 \exp(-\sqrt{p}t/8), & \text{if } t > \sqrt{p}. \end{cases}$$

Proof of Lemma 4.5.12. This is a standard Chernoff bound; see for instance the comments for Lemma 1 in Section 4 of [Laurent and Massart, 2000]. Note that we could also write

$$\begin{aligned}\mathbb{P}\{X \geq p + 2\sqrt{pt} + 2t\} &\leq e^{-t} \\ \mathbb{P}\{X \leq p - 2\sqrt{pt}\} &\leq e^{-t}\end{aligned}$$

□

Lemma 4.5.13. *Let $Y_1, \dots, Y_n \sim \chi_p^2$, and let $Y_{(1)} \leq \dots \leq Y_{(n)}$ be the order statistics. Then:*

$$\begin{aligned}\mathbb{P}\left\{\frac{Y_{(i)} - p}{\sqrt{p}} > t\right\} &\leq \exp(-(n-i-1)t^2/8) && \text{and} \\ \mathbb{P}\left\{\frac{Y_{(i)} - p}{\sqrt{p}} \leq t\right\} &\leq \exp(-(i-1)t^2/8)\end{aligned}$$

Proof of Lemma 4.5.13. For the first part, if $Y_{(i)}$ is larger than something, this implies that $n-i$ of the other random variables must be larger than that thing. The argument for the second part is reversed. Then we use Lemma 4.5.12. □

Lemma 4.5.14. *Let $Y_1, \dots, Y_n \sim \chi_p^2$ where $p \geq 3$. Then $\mathbb{E}Y_{(n)} \leq p \log n$.*

Proof. Let $M := Y_{(n)}$, the maximum. We use the Chernoff method and let $\theta < 1/2$ so that the MGF of a χ_p^2 random variable is defined. Using Jensen's Inequality:

$$e^{\mathbb{E}\theta M} \leq \mathbb{E}e^{\theta M} = \mathbb{E}\max_{i \leq n} e^{\theta Y_i} \leq n \mathbb{E}e^{\theta Y_i} = n(1-2\theta)^{-p/2}$$

where in the final step, we just inserted the χ_p^2 moment-generating function. Thus for $\theta < 1/2$:

$$\mathbb{E}M \leq \frac{\log n}{\theta} - \frac{p \log(1-2\theta)}{2\theta}$$

Now, it is well-known that for all $x > 0$, $\frac{-x}{1-x} \leq \log(1-x)$. Thus:

$$\mathbb{E}M \leq \frac{\log n}{\theta} + \frac{p}{1-2\theta}$$

for all $\theta \in (0, 1/2)$. Pick a θ in this range, e.g. $\theta = 1/4$. Then we have a bound of $4 \log n + 2p$, which is bounded above by $8 \max(\log n, p)$. \square

Lemma 4.5.15. *Let Z be a $n \times p$ matrix with independent entries, each distributed $N(0, \sigma^2)$. Then with probability at least $1 - 1/n$, there is a $c > 0$ such that*

$$\|Z\|_2^2 \leq c\sigma^2(n+p)$$

Proof. See e.g. the proof of Theorem 5.39 of [Vershynin, 2012]. \square

4.5.4 Refinement

Lemma ?? provides an error rate when sample-splitting was used in the algorithm. Sample splitting is used the error rate depends on a set E where $\mathbb{P}\{E^C\}$ is large enough so that we don't want it to play a role in the bound. Consider the proof of Lemma ?. If we instead examined had a separate E_i for each node, and if e.g. $\delta = 1/\log n$, then in the proof, we would have

$$\frac{1}{n}\mathcal{L}(\hat{\tau}, \tau) \lesssim \frac{\sum_{i \leq n} \mathbb{P}\{\hat{\tau}(i) \neq \tau(i) \text{ and } E_i\} + \mathbb{P}\{E_i^C\}}{nt}$$

We would then need to choose t to offset the $\delta = 1/\log n$; i.e. the error rate would have an exponential and a logarithmic term, which would be terrible.

Lemma 4.5.16. *[Error rate: ordinary version] Let $\hat{\tau}$ be the result of Algorithm Al-*

gorithm 4. Then with probability at least $1 - \sqrt{\sum_{u=1}^n \mathbb{P}\{\hat{\tau}(u) \neq \tau(u)\}/n}$,

$$\frac{1}{n} \mathcal{L}(\hat{\tau}, \tau) \leq \left(\frac{\sum_{u=1}^n \mathbb{P}\{\hat{\tau}(u) \neq \tau(u)\}}{n} \right)^{1/2}$$

Note that if $\mathbb{P}\{\hat{\tau}(u) \neq \tau(u)\}$ is the same for all $u \in [n]$, then with probability at least $1 - \sqrt{\mathbb{P}\{\hat{\tau}(u) \neq \tau(u)\}/n}$,

$$\frac{1}{n} \mathcal{L}(\hat{\tau}, \tau) \leq (\mathbb{P}\{\hat{\tau}(u) \neq \tau(u)\})^{1/2}$$

Proof. Let $t > 0$. By the Markov Inequality,

$$\mathbb{P}\{\mathcal{L}(\hat{\tau}, \tau) > nt\} \leq \frac{\sum_{u=1}^n \mathbb{P}\{\hat{\tau}(u) \neq \tau(u)\}}{nt}$$

Now choose

$$t = \sqrt{\sum_{u=1}^n \mathbb{P}\{\hat{\tau}(u) \neq \tau(u)\}/n}$$

to obtain the result. □

Lemma 4.5.17. [Error rate: sample-splitting version] Let $\hat{\tau}$ be the result of Algorithm Algorithm 4. Let $s(i)$ be the sample of node i , i.e.,

$$s(i) = \begin{cases} 1 & \text{if } i \in \{1, \dots, n/2\} \\ 2 & \text{if } i \in \{n/2 + 1, \dots, n\} \end{cases}$$

Let $\tau_1 = \{\tau(1), \dots, \tau(n/2)\}$ and $\tau_2 = \{\tau(n/2 + 1), \dots, \tau(n)\}$. Define $\hat{\tau}_1, \hat{\tau}_2$ similarly.

Let E_1, E_2 be sets with $\max_{j \in [2]} \mathbb{P}\{E_j^C\} \leq \delta$.

Then with probability at least $1 - 2\delta - \sqrt{\sum_{u=1}^n \mathbb{P}\{\hat{\tau}(u) \neq \tau(u) \text{ and } E_{s(u)}\}/n}$,

$$\frac{1}{n} \mathcal{L}(\hat{\tau}, \tau) \leq 2 \left(\frac{\sum_{u=1}^n \mathbb{P}\{\hat{\tau}(u) \neq \tau(u) \text{ and } E_{s(u)}\}}{n} \right)^{1/2}$$

Note that if $\mathbb{P}\{\hat{\tau}(u) \neq \tau(u) \text{ and } E_{s(u)}\}$ is the same for all $u \in [n]$, this simplifies to

$$\frac{1}{n} \mathcal{L}(\hat{\tau}, \tau) \leq 2 \left(\mathbb{P}\{\hat{\tau}(u) \neq \tau(u) \text{ and } E_{s(u)}\} \right)^{1/2}$$

Proof. Let $t > 0$. Then

$$\begin{aligned} \mathbb{P}\{\mathcal{L}(\hat{\tau}, \tau) > nt\} &\leq \mathbb{P}\{\mathcal{L}(\hat{\tau}_1, \tau_1) > nt/2 \text{ and } E_1\} + \mathbb{P}\{\mathcal{L}(\hat{\tau}_2, \tau_2) > nt/2 \text{ and } E_2\} + \mathbb{P}\{E_1^C\} + \mathbb{P}\{E_2^C\} \\ &\leq \frac{2 \sum_{u=1}^n \mathbb{P}\{\hat{\tau}(u) \neq \tau(u) \text{ and } E_{s(u)}\}}{nt} + 2\delta \end{aligned}$$

by the Markov Inequality and by the assumption on E_1 and E_2 . Choose

$$t = \sqrt{\sum_{u=1}^n \mathbb{P}\{\hat{\tau}(u) \neq \tau(u) \text{ and } E_{s(u)}\} / n}$$

to obtain the result. \square

Lemma 4.5.18. *[Error rate: leave-one-out version] Let $\hat{\tau}$ be the result of Algorithm Algorithm 4. For each $u \in [n]$, let E_u be a set with $\mathbb{P}\{E_u^C\} \leq \delta_u \leq \delta$. Then with probability at least $1 - \sqrt{\sum_{u=1}^n \mathbb{P}\{\hat{\tau}(u) \neq \tau(u) \text{ and } E_u\} / n + \delta}$,*

$$\frac{1}{n} \mathcal{L}(\hat{\tau}, \tau) \leq \left(\frac{\sum_{u=1}^n \mathbb{P}\{\hat{\tau}(u) \neq \tau(u) \text{ and } E_u\}}{n} + \delta \right)^{1/2}$$

If $\mathbb{P}\{\hat{\tau}(u) \neq \tau(u) \text{ and } E_u\}$ is the same for all $u \in [n]$, this simplifies to

$$\frac{1}{n} \mathcal{L}(\hat{\tau}, \tau) \leq (\mathbb{P}\{\hat{\tau}(u) \neq \tau(u) \text{ and } E_u\} + \delta)^{1/2}$$

Proof. Let $t > 0$. Then

$$\begin{aligned} \mathbb{P}\{\mathcal{L}(\hat{\tau}, \tau) > nt\} &\leq \frac{\sum_{u=1}^n (\mathbb{P}\{\hat{\tau}(u) \neq \tau(u) \text{ and } E_u\} + \delta_u)}{nt} \\ &\leq \frac{\sum_{u=1}^n \mathbb{P}\{\hat{\tau}(u) \neq \tau(u) \text{ and } E_u\} + n\delta}{nt} \end{aligned}$$

by the Markov Inequality. Choose

$$t = \sqrt{\frac{\sum_{u=1}^n \mathbb{P}\{\hat{\tau}(u) \neq \tau(u) \text{ and } E_u\}}{n}} + \delta$$

to obtain the result. □

Lemma 4.5.19. *For any $u \in [n]$,*

$$\mathbb{P}\{\hat{\tau}(u) \neq \tau(u)\} \leq \sum_{j \in ([k] \setminus \tau(u))} \mathbb{P}\{\hat{\tau}(u) = j\} \leq (k-1) \max_{j \in ([k] \setminus \tau(u))} \mathbb{P}\{\hat{\tau}(u) = j\}$$

Lemma 4.5.20. *Let z, v, u be vectors in \mathbb{R}^p . Then*

$$\langle z, u - v \rangle \geq \|u\|_2^2 - \|v\|_2^2 \text{ implies that } \langle z, \frac{u - v}{\|u - v\|_2} \rangle \geq \|u\|_2 - \|v\|_2$$

Proof. Note that

$$\|u\|_2^2 - \|v\|_2^2 (\|u\|_2 - \|v\|_2) (\|u\|_2 + \|v\|_2) \geq (\|u\|_2 - \|v\|_2) \|u - v\|_2$$

Dividing both sides of the first inequality in the lemma statement by $\|u - v\|_2$ yields the result. □

Chapter 5

Maximum likelihood in high-dimensional parametric models

5.1 Introduction

A natural approach to any estimation problem is the maximum likelihood estimator (MLE). In mixture models of the form (1.2.1) that we have heretofore considered, the likelihood is non-convex, and typically, the Expectation Maximization (EM) algorithm is used to approximate the MLE. It is nonetheless of interest to understand the performance of the MLE itself, even though it may not be easily computable.

The main result of this section is Theorem 5.1.1, which states that the MLE in the k -means Gaussian mixture model of (2.1.1) achieves the (nearly) parametric rate. We currently can obtain the correct rate only up to some log factors and only for centered high-dimensional mixtures. These weaknesses seem improveable, and we plan to extend this to the sharp rate for general Gaussian location mixtures in future work.

Theorem 5.1.1. Let X_1, \dots, X_n be i.i.d. from the model P_Γ as in (2.1.1). Let $\Gamma \in \mathcal{G}_{d,k}$ where $\mathcal{G}_{d,k}$ has the added condition that the distributions therein are centered at zero. Let $\hat{P}_\Gamma = \operatorname{argmax}_{P_\Gamma} \sum_{i=1}^n \log p_\Gamma(X_i)$. With probability at least $1 - \delta$,

$$H(\hat{P}_\Gamma, P_\Gamma) \leq \operatorname{polylog}(d, n) \left(\frac{d \log(1/\delta)}{n} \right)^{1/2}. \quad (5.1.1)$$

Before proceeding to the proof of Theorem 5.1.1, we will discuss the analysis of the MLE in a general, high-dimensional parametric setting. We will show why a naive approach to analyzing the MLE in such settings leads to a non-sharp analysis. We will then show a refined approach that leads to the correct rate, and we will demonstrate how this approach can be applied to achieve Theorem 5.1.1.

Let us first place ourselves in an abstract parametric setting. Let us have a true parametric model P_Γ , with density p_Γ , where $\Gamma \in \mathcal{G}$, some high-dimensional parameter space. Let $\mathcal{P} = \{p_\Gamma : \Gamma \in \mathcal{G}\}$ be our class of densities. For an alternative model $p_{\Gamma'}$, define $\bar{p}_{\Gamma'} = (p_{\Gamma'} + p_\Gamma)/2$. For $\delta \in (0, 1]$, let $\bar{\mathcal{P}}^{1/2}(\delta) \triangleq \{\sqrt{\bar{p}_{\Gamma'}}(x) : \bar{p}_{\Gamma'} \in \bar{\mathcal{P}} \text{ and } H(\bar{p}_{\Gamma'}, p_\Gamma) \leq \delta\}$.

For $X_1, \dots, X_n \sim_{i.i.d.} P_\Gamma$, define the MLE:

$$\hat{p}_\Gamma \in \operatorname{argmax}_{p_\Gamma : P_\Gamma \in \mathcal{P}_{d,k}} \sum_{i=1}^n \log p_\Gamma(X_i). \quad (5.1.2)$$

A rate of convergence for the MLE usually can be found by evaluating the *bracketing entropy integral* [Wong and Shen, 1995, van der Vaart and Wellner, 1996, van de Geer, 2000]. We now introduce the notion of a bracketing number.

Definition 1 (Bracketing Number). Let \mathcal{F} be a class of real-valued functions on a set \mathcal{X} . Let $\|f\|_2 = (\int f(x)^2 dx)^{1/2}$. A bracket is a pair of functions $[l, u]$ such that $l(x) \leq f(x) \leq u(x)$ for all $x \in \mathcal{X}$. An ϵ -bracket is a bracket $[l, u]$ with $\|u - l\|_2 \leq \epsilon$. The ϵ -bracketing number of \mathcal{F} with respect to the $\|\cdot\|_2$ -norm, written $N_{[]}(\epsilon, \mathcal{F}, \|\cdot\|_2)$, is the minimum number of ϵ -brackets needed to cover \mathcal{F} . The *bracketing entropy* is

the log of the bracketing number.

The following lemma shows how the bracketing number of a local Hellinger ball on the densities can be converted into a rate of convergence for the MLE.

Lemma 5.1.2. *Suppose there is a positive constant C such that*

$$N_{[]} (s, \mathcal{P}^{1/2}(\delta), \|\cdot\|_2) = \left(\frac{C\delta}{s} \right)^d.$$

Then there is a constant c such that with probability at least $1 - \delta$,

$$H(\hat{P}_\Gamma, P_\Gamma) \leq c \left(\frac{d \log(1/\delta)}{n} \right)^{1/2}.$$

For parametric problems, we expect that the local bracketing number should resemble the local covering number on the relevant parameter space. A standard approach to computing the bracketing number is to relate it to a cover number in the relevant parameter space via, e.g., Lemma 2.7.11 of [van der Vaart and Wellner, 1996], which is provided here.

Lemma 5.1.3 (Lemma 2.7.11 of [van der Vaart and Wellner, 1996]). *Let Θ be a metric space equipped with metric ρ . Let $\theta \in \Theta$. Let \mathcal{X} be a metric space. Let $f_\theta : \mathcal{X} \rightarrow \mathbb{R}$ be a function parametrized by θ . Let $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$. Let $N(\epsilon, \Theta, \rho)$ be the ϵ covering number in the ρ metric on Θ . Suppose for every $\theta, \theta_* \in \Theta$, and for all $x \in \mathcal{X}$,*

$$|f_\theta(x) - f_{\theta_*}(x)| \leq \rho(\theta, \theta_*) F(x), \tag{5.1.3}$$

where $F(x)$ is square-integrable under Lebesgue measure. Then

$$N_{[]} (\epsilon \|F\|_2, \mathcal{F}, \|\cdot\|_2) \leq N(\epsilon, \Theta, \rho).$$

The $F(x)$ in (5.1.3) will be $F(x) = \sup_{\theta \in \Theta} |\frac{\partial}{\partial \theta} f_\theta(x)|$.

5.2 Naïve bracketing

5.2.1 Failure of naïve bracketing

We show how a naïve application of the bracketing technology in a standard but high-dimensional Gaussian location fails to achieve the correct rate. Let $\Theta = \{\theta \in \mathbb{R}^d : \|\theta\|_2 = 1\}$. Consider a Gaussian location model in d dimensions,

$$P_{\theta_0} = N(\theta_0, I_d), \quad (5.2.1)$$

where $\theta_0 \in \Theta$. Now $p_\theta(x) = (2\pi)^{-d/2} \exp(-\|x - \theta\|_2^2/2)$. We know that in this example, we can compute the MLE in closed form, and it achieves $H(\hat{P}, P_0) \asymp \sqrt{d/n}$. Now let us show how applying the bracketing technology fails to achieve this if we apply it in the wrong way. We have

$$|\sqrt{p_\theta(x)} - \sqrt{p_{\theta_*}(x)}| \leq F(x) \|\theta - \theta_*\|_2, \quad (5.2.2)$$

where $F(x) = \sup_{\theta \in S^{d-1}} \frac{1}{2} \|x - \theta\|_2 e^{-\|x - \theta\|_2^2/4}$. But this means $\|F\|_2 \propto e^{\sqrt{d}}$, which means

$$N_{[]}(\epsilon, \mathcal{P}^{1/2}(\delta), \|\cdot\|_2) = \left(\frac{\|F\|_2}{\epsilon} \right)^d = \left(\frac{e^d \delta}{\epsilon} \right)^d.$$

Plugging this bracketing number bound into Lemma 5.1.2 yields a rate of $\sqrt{d^{3/2}/n}$, which is far from the sharp rate of $\sqrt{d/n}$ that we know is achievable by the MLE in this model. Thus, this naïve application of bracketing fails to yield the correct rate for the MLE.

Another commonly-used argument relies truncation rather than Lemma 5.1.3, but it too results in non-optimal dependence on d . By the same arguments as in [Ho and Nguyen, 2016], we can relate $N_{[]}(\epsilon, \mathcal{P}, \|\cdot\|_2)$ to $N(\epsilon, \mathcal{P}, \|\cdot\|_\infty)$. Let $\eta > 0$ to

be chosen later. Let the bracketing function pairs be:

$$\begin{aligned} p_i^L(x) &= \max\{p_{\Gamma_i}(x) - \eta, 0\} \\ p_i^U(x) &= \min\{p_{\Gamma_i}(x) + \eta, F(x)\}, \end{aligned}$$

where $F(x) \propto e^{-\|x\|_2^2/8}$. Then $\mathcal{P} \subseteq \cup_{i \in N} \{p_i^L, p_i^U\}$, and for any $B \geq 2R$,

$$\begin{aligned} \int (p_i^U(x) - p_i^L(x)) \, dx &\leq \int_{\|x\| < B} 2\eta \, dx + \int_{\|x\|_2 \geq B} F(x) \, dx \\ &\leq \eta B^d + B^{d-1} e^{-B^2/8} \\ &\lesssim \eta (\log(1/\eta))^{d/2} \quad \text{choosing } B = (\log(1/\eta))^{1/2}. \end{aligned}$$

So

$$N_{[]}(\eta(\log(1/\eta))^{d/2}, \mathcal{P}, \|\cdot\|_1) \lesssim \log\left(\frac{1}{\eta}\right)^d.$$

Let $\epsilon = \eta(\log(1/\eta))^{d/2}$. Now [Ho and Nguyen, 2016] state that the following is true, but this seems erroneous.

$$\log\left(\frac{1}{\eta}\right)^d \propto \log\left(\frac{1}{\epsilon}\right)^d. \quad (5.2.3)$$

But this is really not the case. Try just solving for η :

$$\eta \approx \frac{\epsilon}{(\log(1/\epsilon))^{d/2}}$$

So

$$\left(\frac{1}{\eta}\right)^d = \left(\frac{(\log(1/\epsilon))^{d/2}}{\epsilon}\right)^d$$

Our bracketing number will be exponential in d^2 , leading to a rate of $\sqrt{d^2/n}$, which

is even worse than before. Another way to see this is as follows. By definition of ϵ , and letting $\gamma = 2$,

$$\log \frac{1}{\epsilon} = \log \frac{1}{\eta} - \frac{d}{2} \log \log \frac{1}{\eta}.$$

So for (5.2.3) to hold, we need:

$$\frac{d}{2} \log \log(1/\eta) \propto \log(1/\eta). \quad (5.2.4)$$

In the integral we finally evaluate in Theorem 7.4 of [van de Geer, 2000], $\epsilon \in (\delta^2, \delta)$; here η is near ϵ . If $\delta = \sqrt{d/n}$, the condition (5.2.4) becomes

$$\frac{n}{d} > \left(\log \frac{n}{d} \right)^d.$$

If n/d is some small constant, then $\log(n/d)$ will also be very small and potentially this holds, but it doesn't necessarily hold if $n/d > \exp 1$ and d (and n) are large. So it's possible that [Ho and Nguyen, 2016] holds for their case, when d is small, but it does not hold in general. We need better technology.

5.2.2 A suggestive analysis

The reliance on $\|F\|_2$ in Section 5.2.1 is surprising. We now provide a simple analysis of an MLE variant that yields a rate with the correct dependence on d . This provides us with the intuition that the MLE in high-dimensional parametric models should perform at the parametric rate.

Let the model be P_{θ_0} with $\theta_0 \in \Theta$ as in Section 5.2.1, and suppose that instead of computing the actual MLE, we approximate using the P_{θ} that maximizes the likelihood over a net on Θ . The rate for such an estimator is provided in Lemma 5.2.1. Let $\ell_n(\theta) = \sum_{i=1}^n \ell(x_i, \theta)$.

Lemma 5.2.1. *Let \mathcal{N} be a net of minimal size on Θ . Let $\hat{p}_\theta \in \operatorname{argmax}_{\theta \in \mathcal{N}} \sum_{i=1}^n \log p_\theta(x_i)$. There are positive constants c, C such that with probability at least $1 - e^{-cd \log n}$,*

$$H(\hat{P}_\theta, P_0) \leq C \left(\frac{d \log n}{n} \right)^{1/2}. \quad (5.2.5)$$

Proof. First, $N(\epsilon, \Theta, \|\cdot\|_2) \leq (C/\epsilon)^d$. And it can be shown (using a dimension reduction argument), that $\chi^2(P_\theta \| P_{\theta_*}) \leq c \|\theta - \theta_*\|_2^2$, where c does not depend on d . By Lemma 5.2.2,

$$\mathbb{P}\{H(\hat{P}, P_0) \geq 2\epsilon\} \leq \left(\frac{C}{\epsilon} \right)^d e^{-n\epsilon^2} + e^{-n\epsilon^2}.$$

□

Lemma 5.2.2. *Let $\mathcal{N} = \mathcal{N}(\epsilon, \Theta, \rho)$ be an ϵ -net on Θ in the ρ metric of minimal size. Let $\hat{\theta} \in \operatorname{argmax}_{\theta \in \mathcal{N}} \ell_n(\theta)$. Suppose $\chi^2(\mathbb{P}_\theta \| \mathbb{P}_{\theta_0}) \leq \rho^2(\theta, \theta_0)$. Then*

$$\mathbb{P}_0\{H(P_{\hat{\theta}}, P_0) \geq 2\epsilon\} \leq |\mathcal{N}| \exp(-n\epsilon^2) + \exp(-n\epsilon^2).$$

Proof. There is a $\theta_* \in \mathcal{N}$ such that $\rho(\theta_*, \theta_0) \leq \epsilon$. By the triangle inequality, definition of \mathcal{N} , and fact that $H(P_\theta, P_{\theta_0}) \leq \rho(\theta, \theta_0)$, $\mathbb{P}_0\{H(P_{\hat{\theta}}, P_{\theta_0}) \geq 2\epsilon\} \leq \mathbb{P}_0\{H(P_{\hat{\theta}}, P_{\theta_*}) \geq \epsilon\}$. Let $B = \{\theta \in \mathcal{N} : H(P_\theta, P_{\theta_0}) \geq 2\epsilon\}$. So for $\theta \in B$, $H^2(P_\theta, P_{\theta_0}) \geq 4\epsilon^2$. Now

$$\begin{aligned} \mathbb{P}_0\{H(P_{\hat{\theta}}, P_{\theta_*}) \geq \epsilon\} &\leq \mathbb{P}_0\{\max_{\theta \in B} \ell_n(\theta) - \ell_n(\theta_*) \geq 0\} \\ &\leq |B| \max_{\theta \in B} \mathbb{P}_0\{\ell_n(\theta) - \ell_n(\theta_0) + 2\epsilon^2 \geq 0\} + \mathbb{P}_0\{\ell_n(\theta_0) - \ell_n(\theta_*) - 2\epsilon^2 \geq 0\} \\ &\leq |B| e^{n\epsilon^2 - \log(1 - H^2(P_\theta, P_{\theta_0})/2)} + e^{-n(2\epsilon^2 - \log(1 + \chi^2(\mathbb{P}_0 \| \mathbb{P}_{\theta_*}))} \\ &= |B| \exp(-n\epsilon^2) + \exp(-n\epsilon^2). \end{aligned}$$

The third inequality follows since $H^2(P_\theta, P_{\theta_0}) \geq 4\epsilon^2$ on B . The final line follows because $\chi^2(\mathbb{P}_0 \| \mathbb{P}_{\theta_*}) \leq \rho^2(\theta_0, \theta_*) \leq \epsilon^2$ by definition of \mathcal{N} . And $|B| \leq |\mathcal{N}|$, which

completes the proof. □

5.3 Bracketing entropy in high-dimensional parametric models

The analysis in Section 5.2.2 suggests that the rate of convergence for the MLE itself depends on the cover number of the relevant parameter space. Any finite, high-dimensional parametric model, including Gaussian mixtures of the form (1.2.1), should achieve the correct rate in high-dimensional parametric models. We made this formal in the k -means Gaussian location mixture of (2.1.1).

We now explain the process for obtaining Theorem 5.1.1. This process of course applies to a Gaussian location model as in Section 5.2.1, since this is a model in $\mathcal{P}_{1,d}$, and it moreover applies to other mixture models of the form in (1.2.1).

In evaluating the bracketing number for high-dimensional mixtures, we will rely on the crucial recognition, as in [Maugis and Michel, 2011], that we can represent a distribution of the form (1.2.1) as the product of two distributions, one being a low-dimensional Gaussian mixture distribution on the relevant variables, the other being a spherical Gaussian on the irrelevant variables. That is, let $\Gamma = \sum_{j=1}^k w_j \delta_{\mu_j} \in \mathcal{G}_{k,d}$. Let V be the space spanned by the k locations of Γ , and let V_C be the space in \mathbb{R}^d orthogonal to it. We in fact have

$$p_\Gamma(x) = \phi_{d-k}(V_C^\top x) \sum_{j=1}^k w_j \phi_k(V^\top(x - \mu_j)). \quad (5.3.1)$$

Let $S^{d \times (d-k)}$ be the set of matrices whose columns form orthonormal bases of $(d-k)$ -dimensional subsets of \mathbb{R}^d . Let $\mathcal{P} = \{g_{V_C}(x) = \phi_{d-k}(V_C^\top x) : V_C \in S^{d \times (d-k)}\}$. For each $p_\Gamma \in \mathcal{P}_{k,d}$, $p_\Gamma(x) = g_{V_C}(x)p_\gamma(V^\top x)$, where $g \in \mathcal{P}$ and $p_\gamma \in \mathcal{G}_{k,k}$, with $\gamma = \Gamma_V$. That is,

we have

$$\mathcal{P}_{k,d} = \{g_{V_C}(x)p_\gamma(V^\top x) : g \in \mathcal{P}, p_\gamma \in \mathcal{P}_{k,k}\}. \quad (5.3.2)$$

We next invoke Theorem B.1 and Proposition B.2 in [Maugis and Michel, 2011]. These allow us to bound the high-dimensional bracketing entropy and low-dimensional one separately, avoiding unfortunate dependence on d .

Lemma 5.3.1.

$$N_{[]}(\epsilon, \mathcal{P}_{k,d}^{1/2}(\delta), \|\cdot\|_2) \leq N_{[]}(\epsilon, \mathcal{P}^{1/2}(\delta), \|\cdot\|_2) * N_{[]}(\epsilon, \mathcal{P}_{k,k}^{1/2}(\delta), \|\cdot\|_2). \quad (5.3.3)$$

Proof. See Inequality B.2, page 55 of [Maugis and Michel, 2011]. \square

Lemma 5.3.2. *There is a positive constant C depending on k, R but not on d such that*

$$N_{[]}(\epsilon, \mathcal{P}^{1/2}(\delta), \|\cdot\|_2) \leq \left(C \frac{d}{\epsilon}\right)^{dc_k}. \quad (5.3.4)$$

$$N_{[]}(\epsilon, \mathcal{P}_{k,k}^{1/2}(\delta), \|\cdot\|_2) \leq \left(\frac{C\delta}{\epsilon}\right)^k. \quad (5.3.5)$$

The proof of Theorem 5.1.1 is now straightforward.

Proof of Theorem 5.1.1. By Lemma 5.3.1 and Lemma 5.3.2, we obtain that $N_{[]}(\epsilon, \mathcal{P}_{k,d}^{1/2}(\delta), \|\cdot\|_2) \leq (\delta d/\epsilon)^{c_k d}$. The result follows from an application of Lemma 5.1.2. \square

5.4 Supporting lemmas and proofs

Proof of Lemma 5.1.2. Note that $\left\|\bar{p}_1^{1/2} - \bar{p}_2^{1/2}\right\|_2 \leq \left\|p_1^{1/2} - p_2^{1/2}\right\|_2 / \sqrt{2}$ by Lemma 4.2 of [van de Geer, 2000]. Therefore we may reduce ourselves to bounding $N_{[]}(\sqrt{2}\epsilon, \mathcal{P}^{1/2}(\delta), \|\cdot\|_2)$. Apply Theorem 7.4 of [van de Geer, 2000] in the following way. Let $\Psi(\delta) := C_1 \sqrt{d}\delta$

for an appropriate positive constant C_1 . We have

$$\begin{aligned}
\int_{\delta^2/2^{13}}^{\delta} \log \sqrt{N_{\square}(s, \mathcal{P}^{1/2}(\delta), \|\cdot\|_2)} ds &\leq \int_{\delta^2/2^{13}}^{\delta} \sqrt{\log \left(\frac{C\delta}{u} \right)^d} du \\
&\leq \sqrt{d} C \delta \int_{\delta/(C2^{13})}^{1/C} \sqrt{\log \frac{1}{t}} dt \quad \text{using the substitution } t = u/C\delta. \\
&\leq \sqrt{d} C \delta \int_0^{1/C} \sqrt{\log \frac{1}{t}} dt \\
&\leq \Psi(\delta).
\end{aligned}$$

And $\Psi(\delta)/\delta^2$ is nonincreasing in δ . Setting $\delta_{d,n} = (d/n)^{1/2}$, there is a positive constant c such that $c\Psi(\delta_{d,n}) \leq \sqrt{n}\delta_{d,n}^2$. By Theorem 7.4 of [van de Geer, 2000], for any $\delta \geq \delta_{d,n}$, $\mathbb{P}\{H(\hat{P}_\Gamma, P_\Gamma) \geq \delta\} \leq ce^{-n\delta^2/c^2}$. \square

Proof of (5.3.4) in Lemma 5.3.2. Recall that $g(x) = e^{-\|V_C^\top x\|_2^2/2}/(2\pi)^{(d-k)/2}$. Now let $V_C, V'_C \in S^{d \times (d-k)}$.

$$\left| \sqrt{g_{V_C}(x)} - \sqrt{g_{V'_C}(x)} \right| \leq \|V_C - V'_C\|_2 F(x),$$

where $F(x) = \|x\|_2 \sup_{V_C \in S^{d \times (d-k)}} e^{-\|V_C^\top x\|_2^2/4}/(2\pi)^{(d-k)/4} = \|x\|_2 \sup_{V_C} \sqrt{\phi_{d-k}(V_C^\top x)}$.

Now

$$\int_{x \in \mathbb{R}^d} F^2(x) dx = \int \|x\|_2^2 \sup_{V_C} \phi_{d-k}(V_C^\top x) dx = \mathbb{E}_{N(0, I_d)} \|x\|_2^2 = d.$$

A net on $S^{d \times (d-k)}$ has size no more than $(1/\epsilon)^{dk}$. Thus

$$N_{\square}(\epsilon, \mathcal{P}^{1/2}(\delta), \|\cdot\|_2) \leq N_{\square}(\epsilon, \mathcal{P}^{1/2}, \|\cdot\|_2) \leq \left(\frac{1}{\epsilon} \right)^{dk}.$$

To sharpen this to the local bracketing number, we would need to show that $p_{\Gamma'} \in \mathcal{P}_{k,d}(\delta)$ implies that the associated orthogonal direction matrix V'_C is within δ of the

true V_C , i.e., that $\|V_C - V'_C\|_2 \lesssim \delta$. Our plan is to attempt to show that

$$\|V'_C - V_C\|_2 \lesssim H(g_{V_C} p_\gamma, g_{V'_C} p_\gamma). \quad (5.4.1)$$

If (5.4.1) holds, then the rest is straightforward. Specifically, we have

$$\begin{aligned} H(g_{V_C} p_\gamma, g_{V'_C} p_\gamma) &\leq H(g_{V_C} p_\gamma, g_{V'_C} p_{\gamma'}) + H(g_{V'_C} p_{\gamma'}, g_{V'_C} p_\gamma) \\ &= H(p_\Gamma, p_{\Gamma'}) + H(p_{\gamma'}, p_\gamma) \\ &\leq 2H(p_\Gamma, p_{\Gamma'}) \text{ by the Data Processing Inequality} \\ &\leq 2\delta. \end{aligned}$$

□

Proof of (5.3.5) in Lemma 5.3.2. We form brackets for densities $\sqrt{p_\gamma} \in \mathcal{P}_{k,k}^{1/2}(\delta)$. We will write $x \in \mathbb{R}^k$ below, instead of $V^\top x$ for $x \in \mathbb{R}^d$, since the x 's were projected to the same direction. Then

$$\left| \sqrt{p_\gamma(x)} - \sqrt{p_{\gamma'}(x)} \right| = \frac{|p_\gamma(x) - p_{\gamma'}(x)|}{\sqrt{p_\gamma(x)} + \sqrt{p_{\gamma'}(x)}} \leq 2e^{R^2/4} \phi_k(x)^{-1/2} |p_\gamma(x) - p_{\gamma'}(x)|,$$

by Lemma 1.2.3 and the fact that our distributions are centered. And using the representation (1.2.7),

$$\begin{aligned} |p_\gamma(x) - p_{\gamma'}(x)| &= \phi_k(x) \sum_{j \geq 1} \sum_{\mathbf{j} \in S_{j,k}} \frac{|H_{\mathbf{j}}(x)|}{\mathbf{j}!} |\Delta m_{\mathbf{j}}(\gamma, \gamma')| \\ &= \phi_k(x) \sum_{j \geq 1} \sum_{\mathbf{j} \in S_{j,k}} \frac{|H_{\mathbf{j}}(x)| c^j}{\mathbf{j}!} \sup_{\theta \in S^{k-1}} |\Delta m_j(\gamma_\theta, \gamma'_\theta)| \quad \text{by Lemma 2.6.3} \\ &\leq \phi_k(x) \sum_{j \geq 1} \sum_{\mathbf{j} \in S_{j,k}} \frac{|H_{\mathbf{j}}(x)| (3c)^j}{\mathbf{j}!} \max_{\ell \in [2k-1]} \sup_{\theta \in S^{k-1}} |\Delta m_\ell(\gamma_\theta, \gamma'_\theta)|, \end{aligned}$$

where the final step is by Lemma 10 of [Wu and Yang, 2019]. That is, we have shown

that

$$\left| \sqrt{p_\gamma(x)} - \sqrt{p_{\gamma'}(x)} \right| \leq F(x) \max_{\ell \in [2k-1]} \|M_\ell(\gamma) - M_\ell(\gamma')\|_2,$$

where

$$F(x) = 2e^{R^2/4} \phi_k^{1/2}(x) \sum_{j \geq 1} \sum_{\mathbf{j} \in S_{j,k}} \frac{|H_{\mathbf{j}}(x)|(3c)^j}{\mathbf{j}!}.$$

It remains to find the size of a local cover on the moment tensors; k -dimensional moment tensors; this was done in Theorem 2.3.1. And recall that we have $\sqrt{p_\gamma}, \sqrt{p_{\gamma'}} \in \mathcal{P}^{1/2}(\delta)$, so $p_\gamma, p_{\gamma'}$ are within δ of the truth in Hellinger distance; this implies that their moment tensors are within δ of the true moment tensor by Lemma 2.3.3. Thus the size of our moments net on $\mathcal{G}_{k,k}(\delta)$ is no more than $(\delta/\epsilon)^k$. We can thus apply Lemma 5.1.3 to obtain that $N_{[]}(\epsilon, \mathcal{P}^{1/2}(\delta), \|\cdot\|_2) \leq (C \|F\|_2 \delta/\epsilon)^k$; it remains to compute $\|F\|_2$.

Note that

$$\sum_{j \geq 1} \sum_{\mathbf{j} \in S_{j,k}} \frac{|H_{\mathbf{j}}(x)|(3c)^j}{\mathbf{j}!} = \prod_{l=1}^k \sum_{j \geq 1} \frac{|H_j(x_l)|(3c)^j}{j!}.$$

So

$$\begin{aligned} \|F\|_2^2 &= 2e^{R^2/2} \int_{x \in \mathbb{R}^k} \phi_k(x) \left(\prod_{l=1}^k \sum_{j \geq 1} \frac{|H_j(x_l)|(3c)^j}{j!} \right)^2 \\ &= \prod_{l=1}^k \mathbb{E}_{x \sim N(0,1)} \left(\sum_{j \geq 1} \frac{|H_j(x)|(3c)^j}{j!} \right)^2. \end{aligned}$$

We have absolute Hermites here so we cannot use their orthogonality; however,

$$\mathbb{E}_{x \sim N(0,1)} \left(\sum_{j \geq 1} \frac{3^j |H_j(x)|}{j!} \right)^2 = \sum_{r \geq 1} \mathbb{E} \frac{9^r H_r^2(x)}{(r!)^2} + \underbrace{\sum_{j \neq l} \mathbb{E} \frac{3^{j+l} |H_j(x) H_l(x)|}{j! l!}}_{(2)}$$

The first term is equal to $\sum_{j \geq 1} \frac{9^j}{j!} \leq e^9$ by evaluating the expected value of the

squared Hermites. For the second term, using Cauchy Schwarz,

$$(2) \leq \sum_{j \neq l} \frac{3^{j+l} \sqrt{\mathbb{E}H_j^2(x) \mathbb{E}H_l^2(x)}}{j!l!} = \sum_{j \neq l} \frac{3^{j+l}}{\sqrt{j!} \sqrt{l!}} \leq \left(\sum_{j \geq 1} \frac{3^j}{\sqrt{j!}} \right)^2 \leq \left(\sum_{j \geq 1} \frac{3^j e^j}{\sqrt{2\pi} j^{j/2+1/4}} \right)^2 = C,$$

for a constant C ; the latter inequality used Stirling's Formula. Thus $\|F\|_2^2 \propto 2e^{R^2/2} (e^9 + C)^k \leq C_1^k$ for a constant C_1 . \square

5.5 Special case: MLE for k -GM in one dimension

Now we show a simple example: the MLE for a one-dimensional mixture is evaluated via arguments about the moment space, just as in Theorem 2.1.2. In this case, we are able to achieve the sharp rate for the MLE. This improves on what was previously known about the MLE for Gaussian mixtures in one dimension with weak separation conditions. However, it currently assumes the distributions Γ are centered; we should be able to weaken this assumption but haven't yet.

Lemma 5.5.1 (MLE in one dimension). *Let $\Gamma \in \mathcal{G}_{k,1}$, and let \hat{P}_Γ be the MLE for P_Γ . Then with probability at least $1 - \delta$,*

$$H(\hat{P}_\Gamma, P_\Gamma) \leq \left(\frac{k \log(1/\delta)}{n} \right)^{1/2}.$$

Proof. Apply Theorem 7.4 of [van de Geer, 2000] and Lemma 5.5.2. \square

Lemma 5.5.2 (Bracketing for k -GM in one dimension). *Let $\Gamma \in \mathcal{G}_{k,1}$. There is a positive constant C such that*

$$N_{[]}(\epsilon, \mathcal{P}_{k,1}^{1/2}(\delta), \|\cdot\|_2) \leq \left(\frac{C\delta}{\epsilon} \right)^{2k-1}.$$

Proof of Lemma 5.5.2. Let \mathcal{G} be the set of all k -atomic distributions in one dimensions

with centers μ_1, \dots, μ_k satisfying $|\mu_j| \leq 1$ for each $j \in [k]$. Let

$$\mathcal{G}(\delta, \Gamma) = \{\Gamma \in \mathcal{G} : H(P_\Gamma, P) \leq \delta\}.$$

By Lemma 2.3.3, for $\Gamma \in \mathcal{G}_\delta(\Gamma)$,

$$\max_{r \in [2k-1]} |m_r(\Gamma) - m_r(\Gamma')| \leq \delta. \quad (5.5.1)$$

A one-dimensional k -atomic distribution is uniquely determined by its first $2k - 1$ moments. To create an ϵ -covering on the distributions, we create a net on the moment space. Let $\mathbb{M}(\delta)$ be the set of moment vectors in \mathbb{R}^{2k-1} moments satisfying (5.5.1), i.e., every vector $m \in \mathbb{M}(\delta)$ satisfies $\|m - m_0\|_\infty \leq \delta$. Now $\mathbb{M}(\delta)$ is a subset of the rectangle $[-\delta, \delta]^{2k-1}$, so an ϵ -covering on $\mathbb{M}(\delta)$ has size upper bounded by $(2\delta/\epsilon)^{2k-1}$. Thus $N(\epsilon, \mathcal{G}_\delta(\Gamma), \|\cdot\|_2) \leq (2\delta/\epsilon)^{2k-1}$. Let $\Gamma \in \mathbb{M}(\delta)$, and let Γ' be the nearest k -atomic distribution in \mathcal{N} . By Lemma 10 of [Wu and Yang, 2019], (5.5.1) implies that we obtain a moments gap for all moments, i.e., for any positive integer r ,

$$|m_r(\Gamma) - m_r(\Gamma')| \leq 3^r \epsilon. \quad (5.5.2)$$

So

$$|p_\Gamma(x) - p_{\Gamma'}(x)| \leq \phi(x) \sum_{r \geq 1} \frac{3^r |H_r(x)| |m_r(\Gamma) - m_r(\Gamma')|}{r!} \leq \epsilon \phi(x) \sum_{r \geq 1} \frac{3^r |H_r(x)|}{r!}. \quad (5.5.3)$$

Now let m_r, m'_r be the r th moments of Γ, Γ' , respectively. Now

$$|\sqrt{p_\Gamma(x)} - \sqrt{p_{\Gamma_*}(x)}| = \frac{|p_\Gamma(x) - p_{\Gamma_*}(x)|}{\sqrt{p_\Gamma(x)} + \sqrt{p_{\Gamma_*}(x)}} \leq \phi^{-1/2}(x) \frac{|p_\Gamma(x) - p_{\Gamma_*}(x)|}{e^{xm_1/2 - m_2/4} + e^{xm'_1/2 - m'_2/4}},$$

by Jensen's inequality. We apply Lemma 5.1.3 with

$$F(x) = \phi^{1/2}(x) \frac{\sum_{r=1}^{\infty} 3^r |H_r(x)|/r!}{e^{xm_1/2-m_2/4} + e^{xm'_1/2-m'_2/4}}.$$

Since the distributions Γ are centered, by Lemma 1.2.3, $F(x) \leq 2e^{R^2/4}\phi_1^{1/2}(x) \sum_{r \geq 1} 3^r |H_r(x)|/r!$. So $\|F\|_2^2 \leq 4e^{R^2/2} \mathbb{E}_{N(0,1)} \left(\sum_{r \geq 1} |H_r(x)|/r! \right)^2$. We have absolute Hermites here so we cannot use their orthogonality; however,

$$\mathbb{E}_{x \sim N(0,1)} \left(\sum_{r \geq 1} \frac{3^r |H_r(x)|}{r!} \right)^2 = \sum_{r \geq 1} \mathbb{E} \frac{9^r H_r^2(x)}{(r!)^2} + \underbrace{\sum_{r \neq l} \mathbb{E} \frac{3^{r+l} |H_r(x) H_l(x)|}{r! l!}}_{(2)}$$

The first term is equal to $\sum_{r \geq 1} \frac{9^r}{r!}$ by evaluating the expected value of the squared Hermites. For the second term, using Cauchy Schwarz,

$$(2) \leq \sum_{r \neq l} \frac{3^{r+l} \sqrt{\mathbb{E} H_r^2(x) \mathbb{E} H_l^2(x)}}{r! l!} = \sum_{r \neq l} \frac{3^{r+l}}{\sqrt{r!} \sqrt{l!}} \leq \left(\sum_{r \geq 1} \frac{3^r}{\sqrt{r!}} \right)^2 \leq \left(\sum_{r \geq 1} \frac{3^r e^r}{\sqrt{2\pi} r^{r/2+1/4}} \right)^2 < \infty.$$

We could sharpen this slightly by not using 3^r on first $2k-1$ moments. \square

Chapter 6

Graphical component analysis for latent signal detection

6.1 Introduction

We now study latent signal detection when the observed data are a linear transformation of an underlying signal that we wish to detect. Independent component analysis (ICA) addresses this setting when the signal is generated with independent components, but this independence assumption is often unrealistic. We propose *graphical component analysis* (GCA), which allows for the latent source components to be dependent with the dependence modeled by a sparse graphical model.

Let $A \in \mathbb{R}^{d \times d}$ be invertible, and define $W = A^{-1}$. Let $G = (V, E)$ be a graph with $|V| = d$. Let us observe data according to the model

$$\begin{aligned} s_i &\sim_{i.i.d.} \mathbb{P}_\theta, \text{ for } i = 1, \dots, n, \text{ and} \\ x_i &= As_i. \end{aligned} \tag{6.1.1}$$

Assume that \mathbb{P}_θ has density

$$p_\theta(s) = \exp(\theta^T \phi(s) - \Psi), \quad (6.1.2)$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^k$, $\theta \in \mathbb{R}^k$, and $\Psi = \int_{s \in \mathbb{R}^d} \exp(\theta^T \phi(s)) ds$. We will also write $g_\theta(s) = \theta^T \phi(s)$. We will place a pairwise graphical model structure on θ , and the aim is to estimate θ , i.e., the joint distribution of the signal; as a byproduct, we will also estimate A .

Estimating a graphical model requires the computation of a normalizing constant, which is intractable if the data dimension is high. To avoid this problem, we rely on the technique of *score matching*, [Hyvarinen, 2005], which allows for the estimation of a graphical model without computation of the normalizing constant. We present an algorithm that jointly estimates the graphical model and mixing transformation, where the graphical model is represented in terms of an orthogonal polynomial basis, and the mixing matrix is estimated using a coordinate descent procedure based on the use of Givens rotations [Shalit and Chechik, 2014]. GCA generalizes both ICA, where the components are independent, and tree component analysis [Bach and Jordan, 2003] (TCA), where they are governed by a graph with no cycles. Our experiments demonstrate how the presence of cycles in the latent graph can result in failures by both ICA and TCA to recover accurate models of the marginal components, while GCA succeeds.

6.2 Approach

The model (6.1.1) is equivalent to $x = DA\tilde{s}$ where D is a diagonal matrix with non-zero entries and \tilde{s} is the correspondingly-scaled version of s . That is, the model is only identifiable up to scaling. We assume $W^T W = W W^T = I_d$ to handle this problem. Note also for any permutation matrix P , $x = AP^{-1}Ps$, i.e., the model is identifiable

up to permutation. We estimate the graphical model up to the equivalence class under permutation of the nodes.

Let the true density of the observed data x be p . Suppose we have a model with density $p_\theta(x)$ and we wish to estimate θ . Recall that the score matching objective is

$$\mathcal{L}(\theta) = \frac{1}{2} \mathbb{E} \|\nabla_x \log p(x) - \nabla_x \log p_\theta(x)\|_2^2.$$

Recall that due to integration by parts, the score-matching objective simplifies to something we can optimize in θ [Hyvarinen, 2005]:

$$\mathcal{L}(\theta) \stackrel{c}{=} \frac{1}{2} \mathbb{E} \|\nabla_x \log p_\theta(x)\|_2^2 + \mathbb{E} \operatorname{tr} (\nabla_x^2 \log p_\theta(x)). \quad (6.2.1)$$

In our case, we will use this objective on $p_{W,\theta}(x)$, the observed data density. By the change of variable density formula, the distribution for x has density:

$$p_{W,\theta}(x) = |W| p_\theta(Wx). \quad (6.2.2)$$

In GCA, we will optimize the objective (6.2.1) in W and θ . Using the transformation (6.2.2) and the chain rule,

$$\nabla_x \log p_{\theta,W}(x) = \nabla_x (\log p_\theta(Wx) + \log |W|) = W^T (\nabla \log p_\theta(Wx)).$$

And

$$\nabla_x^2 \log p_{\theta,W}(x) = W^T (\nabla^2 \log p_\theta(Wx)) W.$$

Now $\log p_\theta(x) = g_\theta(x) - \Psi$, and as in the usual score matching, the normalizer disappears when we take the derivative. Plugging into (6.2.1) and using the assumption

that $WW^T = I_d$, our objective becomes

$$\mathcal{L}(\theta, W) = \frac{1}{2} \mathbb{E} (\nabla g_\theta(Wx)^T \nabla g_\theta(Wx)) + \mathbb{E} \text{tr} (\nabla^2 g_\theta(Wx)). \quad (6.2.3)$$

For a function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ and for $u, v \in V$, let $\partial_v g = \partial g / \partial s_v$, and $\partial_{uv} g = \partial^2 g / \partial s_u \partial s_v$. And for a vector-valued function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^k$, let $\partial_v \psi = (\partial_v \psi_1, \dots, \partial_v \psi_m)^T$, and so on. Define $\Delta g(s) = \sum_{v \in V} \partial_{vv}^2 g(s)$. Now $\nabla g = (\theta^T \partial_1 \phi, \dots, \theta^T \partial_p \phi)^T$ and $(\nabla^2 g)_{vv} = \langle \theta, \partial_{vv}^2 \phi \rangle$. Define

$$\begin{aligned} C(Wx) &= \sum_{v \in V} \partial_v \phi(Wx) \partial_v \phi(Wx)^T. \\ \xi(Wx) &= \sum_{v \in V} \partial_{vv}^2 \phi(Wx). \end{aligned} \quad (6.2.4)$$

Then we see that our population score matching objective becomes just similar to the usual one, except that now it contains W :

$$\mathcal{L}(\theta, W) = \frac{1}{2} \theta^T \mathbb{E} C(Wx) \theta + \theta^T \mathbb{E} \xi(Wx).$$

To compute this in practice, we use the empirical expectation. Let $\hat{\mathbb{E}} f(x) = \sum_{i \in [n]} f(x_i) / n$. Let $\hat{C}_W = \hat{\mathbb{E}} C(Wx)$ and $\hat{\xi}_W = \hat{\mathbb{E}} \xi(Wx)$. We add in the penalty $\|\theta\|_2^2$ to discourage too many edges. Now define

$$\hat{\mathcal{L}}(\theta, W) = \frac{1}{2} \theta^T \hat{C}_W \theta + \theta^T \hat{\xi}_W + \frac{\lambda}{2} \|\theta\|_2^2$$

Since we have the added constraint that W must be orthogonal, the score-matching objective becomes:

$$\min_{\theta, W} \hat{\mathcal{L}}(\theta, W) \text{ s.t. } WW^T = I_d$$

Fix W . The objective is quadratic in θ , and there is a closed-form solution:

$$\hat{\theta} = \left(\hat{C}_W + \lambda I_k \right)^{-1} (-\hat{\xi}_W). \quad (6.2.5)$$

Alternatively, as in [Janofsky, 2015], we may impose $\theta_{vu} = \theta_{uv}$, as well as the group sparsity penalty:

$$\hat{\mathcal{L}}(\theta, W) = \frac{1}{2} \theta^T \hat{C}_W \theta + \theta^T \hat{\xi}_W + \sum_{(u,v) \in E} \|\theta_{u,v}\|_2$$

In the simulations, we use the latter. When we have a graphical model on the components of s , g_θ can be expressed as an undirected pairwise graphical model and that we expand the potential functions with a truncated orthogonal basis. Let us have m_1 basis elements for the individual node potential functions and m_2 basis elements for the pairwise potential functions. Then

$$g_\theta(s) = \sum_{v \in V} \sum_{j \in [m_1]} \theta_v^{(j)} \phi_v^{(j)}(s_v) + \sum_{(u,v) \in E} \sum_{j \in [m_2]} \theta_{u,v}^{(j)} \phi_{uv}^{(j)}(s_u, s_v).$$

Thus, for us, $k = m_1 d + m_2 d(d-1)$.

6.3 Algorithm

We iterate over optimizing $\hat{\mathcal{L}}(\theta, W)$ in θ and in W . The optimization in θ has the simple solution given in (6.2.5). To optimize the orthogonal matrix W , we use the Givens rotation method of [Shalit and Chechik, 2014]. This method reduces the problem of optimizing the matrix to $O(d^2)$ sub-problems of optimizing rotations of the matrix. In each rotation, a pair of nodes is selected and the angle of rotation is optimized.

This is a one-dimensional optimization. For $u, v \in V$, define

$$G_{u,v}(\eta) = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & \cos \eta & \dots & -\sin \eta & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & \dots & \sin \eta & \dots & \cos \eta & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \quad (6.3.1)$$

That is, it has 1's down the diagonal except for columns and zeros everywhere else except for the (u, u) , (v, v) , (u, v) , and (v, u) entries, where it has $\cos \eta$, $\cos \eta$, $-\sin \eta$, $\sin \eta$, respectively. Fix a pair (u, v) . Write G_η for simplicity.

Algorithm 5: GCA. Define $\hat{C}_W, \hat{\xi}_W, \hat{\mathcal{L}}(\theta, W)$ as in (6.2.4). Let $G_{u,v}(\eta)$ be as defined in (6.6.2). See Section 6.6.2 for how to find $\operatorname{argmin}_\eta \hat{\mathcal{L}}(\theta, W_t G_{u,v}(\eta))$.

Input: Dataset $\{x_i\}_{i \in [n]}$, each point in \mathbb{R}^d , penalty coefficient $\lambda \in \mathbb{R}$, cutoff

$\epsilon \in \mathbb{R}^+$, number of steps T

Output: $\theta \in \mathbb{R}^k$, $W \in \mathbb{R}^{d \times d}$.

Initialize at random a W_0 satisfying $WW^T = I_d$;

While $\hat{\mathcal{L}}(\theta, W) > \epsilon$;

$\theta = \left(\hat{C}_W + \lambda I_k \right)^{-1} \left(-\hat{\xi}_W \right)$;

While not converged and $t \in [T]$, **do** ;

Select pair of indices $(u(t), v(t))$ **satisfying** $1 \leq u(t) < v(t) \leq p$;

$\eta_{t+1} = \operatorname{argmin}_\eta \hat{\mathcal{L}}(\theta, W_t G_{u,v}(\eta))$;

$W_{t+1} = W_t G_{u,v}(\eta_{t+1})$;

$t = t + 1$;

6.4 Experiments

We now demonstrate empirically the effectiveness of the GCA algorithm. We compare to ICA and TCA on synthetic data generated in the following way. We fix a positive semidefinite matrix $\Omega \in S_{d \times d}^+$ representing the graphical model structure. We generate $Z_i \sim N(0, \Omega^{-1})$ for $i = 1, \dots, n$. Then for $i = 1, \dots, n$, we let our source data be

$$S_i = f_\alpha(Z_i), \tag{6.4.1}$$

where $f_\alpha(s) \triangleq \text{sgn}(s)|s|^\alpha$, for varying α . Our synthetic source data S_i follow a nonparanormal distribution, as described in [Liu et al., 2009, Liu et al., 2012]. We then transform the S_i using a random orthonormal matrix $A \in \mathbb{R}^{d \times d}$. We generate source data with three types of graphical model structures: independence, tree, and a graph containing cycles. We first present, in Fig. 6.1, an example picture showing the performance of ICA, TCA, GCA in a simple case when $d = 3$. Here, we have $\alpha = 1, 2, 3$ in (6.4.1). We compare the recovered sources, i.e., $\hat{W}x$, from each algorithm to the truth in a case where the underlying source graph contains a single cycle. We see that even in this simple case, GCA is able to accurately recover the marginal distributions, while ICA and TCA do not perform as well.

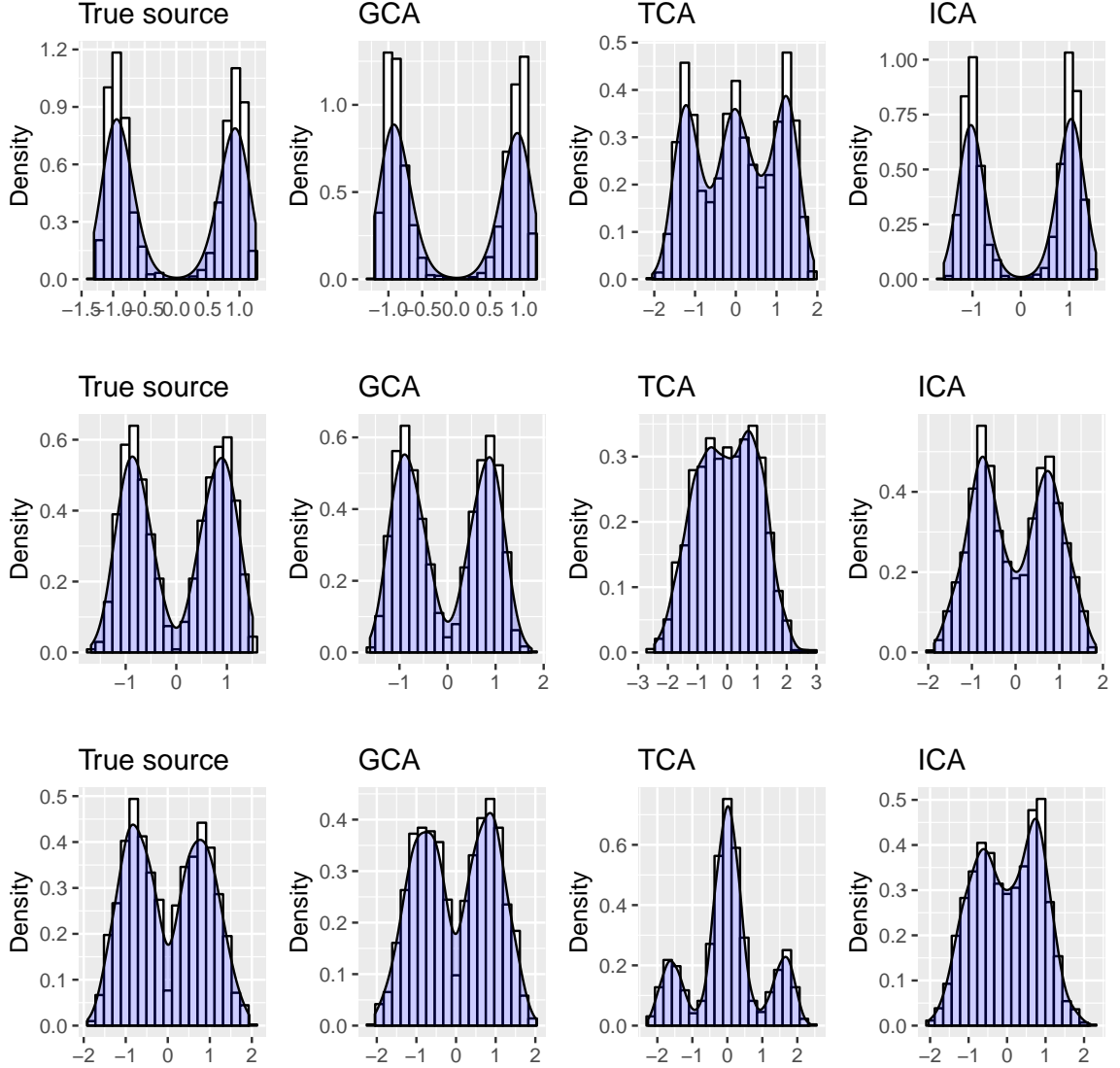


Figure 6.1: Marginal source recovery for GCA, TCA, ICA

We now discuss how to compare the three algorithms in a simulation study. Recall that the three algorithms ICA, TCA, and GCA each produce both an estimate of the de-mixing matrix W and an estimate of the graphical model G on the source components. In the case of ICA, the graphical model is always just the independence model. In the cases of TCA, it is always a tree, and in the case of GCA, any graph structure can be estimated. To compare these algorithms in our preliminary experiments, we use an estimated heldout log likelihood, computed as follows. Given an estimate \hat{W}

of W and an estimated graphical model structure $\hat{\theta}$, as well as a set of heldout (test) data x , we compute

$$|\hat{W}|p_{\hat{\theta}}(\hat{W}x_{test}). \quad (6.4.2)$$

If the graphical model governing $\hat{\theta}$ has no connections (independence) or has a tree structure, computing (6.4.2) can be done via kernel density estimation, with a single kernel density estimate for each node in the independence case or pairwise kernel density estimates in the tree case. However, if $\hat{\theta}$ is arbitrary, it is not possible to compute (6.4.2) (indeed, this is the motivation for using score matching). Now ICA and TCA *always* estimate independence and tree graphs, respectively, so for these algorithms, we can always estimate (6.4.2). But even if the true θ has a simple structure, GCA can in theory estimate an arbitrary graph, so that (6.4.2) is not computable.

In our preliminary experiments in this section, we circumvent this issue in the following way. For the graph with cycles, we use a graph that contain cycles with no more than three nodes; thus, a three-node kernel density estimate can be used. We found that in general, GCA estimated the graph accurately, so that (6.4.2) could be computed for GCA using the graph it estimated. However, to handle any exceptions, i.e., if GCA estimated a more complex graph, we imposed the following restrictions. From the $\hat{\theta}$ estimated by GCA, we select the first three closest neighbors with edge weights of at least 1.0 and form a cycle, remove them from the list of potential cycles, then continue in this way till all nodes are accounted for. This somewhat hacky procedure should be improved, but in our experiments with cycles of exactly three nodes, we found that GCA almost always selected the correct graph; thus, at least our heldout log likelihood experiments on data with cycles (see Fig. 6.2(c)) demonstrate that GCA can be more effective than ICA or TCA on such data.

In Fig. 6.2, we compare the heldout log likelihoods for synthetic source data s with three types of graphical model structures: independent, tree, and cycles with no more than three nodes, respectively. In these experiments, we let $d = 6$, and the training data sample sizes range from 500 to 5000 in increments of 500; the test set size is always 500. For each sample size value, we run 10 experiments. In all experiments, the matrix W is the same; it was chosen at random in the beginning. We can see that GCA performs slightly worse than ICA on independent data, and surpasses it on tree and cycles data.

The implementation of GCA used in this section is in R. A major weakness is its slowness; it runs in time $O(d^3)$ due to the Givens rotation. We ran our 300 experiments simultaneously using the Yale High Performance Computing resources; nonetheless, the maximum dimension we can run an experiment on in 24 hours is $d = 15$.

The experiments we present below use $d = 6$. We can run this algorithm on higher-dimensional data, but we have some concerns about the results so far. In the lower d case, we have seen that GCA estimates the graph fairly accurately, so the heldout log likelihood calculation in Fig. 6.2(c) is an accurate representation of GCA's performance. We are able to observe the performance of the graph estimation in the following way.

Recall that W is only identifiable up to permutation; our estimated sources $\hat{W}x$ might correspond to a permutation of the original sources s . This means we wish to estimate the graph up to permutation. Since in our synthetic experiments we can observe the true de-mixing matrix W , we compare $\hat{W}x$ with Wx and select the best node permutation in the following way. We compare the first node of $\hat{W}x$ to each available one in Wx and find the one its distribution is closest to in ℓ_2 norm. We then remove that one from the nodes in Wx and proceed with the next node in $\hat{W}x$. We continue until all nodes are accounted for. This allows us to avoid selecting the best

permutation by computing all $d!$ possible ones. Note that it relies on our knowledge of the true test source, something we would not have in real-world data.

When $d = 6$, we generate source data where each marginal distribution is unique, and thus the above method of finding the optimal permutation of the graph is viable. For larger d , we have not yet done this. Therefore, for larger d , it is difficult to see if the graph estimation is accurate. Therefore the process of estimating the graph and computing the heldout log likelihood as described above is not necessarily an accurate representation of GCA’s performance. We found in our observations that GCA seemed to perform well on higher-dimensional data, but we refrain from showing higher-dimensional experiments till we have found a better way to evaluate the methods. We also note that the current way of approximating the graph estimated by GCA is probably not ideal for the tree scenario.

Finally, we note that we experimented on a variety of different data structures (different values of α in (6.4.1)), and found that the performance of TCA varies greatly with different data structures. This is worth further exploration. In the experiments here, we used a different data structure for each node of the signal, with α ranging from 0.5 to 3 in increments of 0.5.

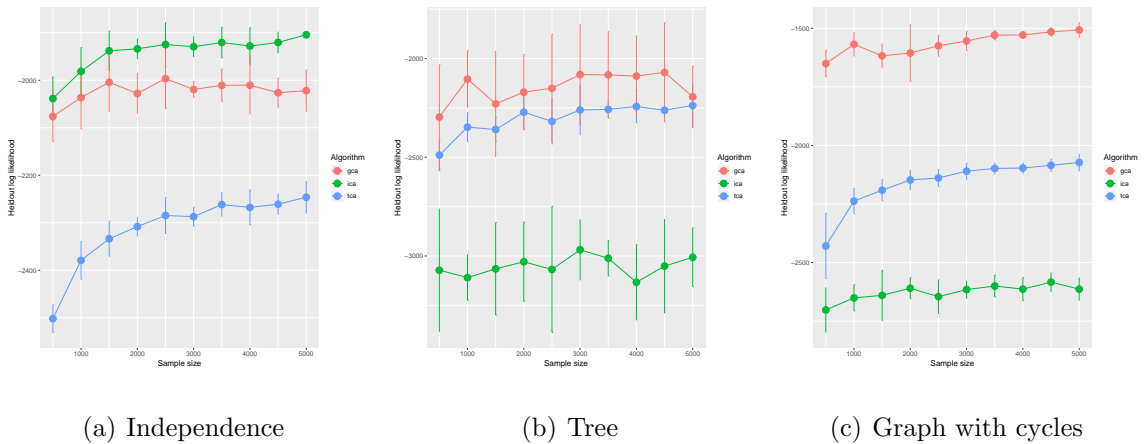


Figure 6.2: Heldout log likelihood on test data for three algorithms

In Fig. 6.3, we demonstrate the graph recovery performance of TCA and GCA

(there is no need to compare these to ICA, since we know it always estimates an independence graph). Using our permuted graph, we compare the estimated graph to the true one from Ω . We display the ROC plots in the figures below. In the case of a tree, GCA does not consistently outperform TCA, but in the case of a graph with cycles, it does. Now d is very small here, so GCA in fact performs nearly perfectly in recovering the two cycles in this graph.

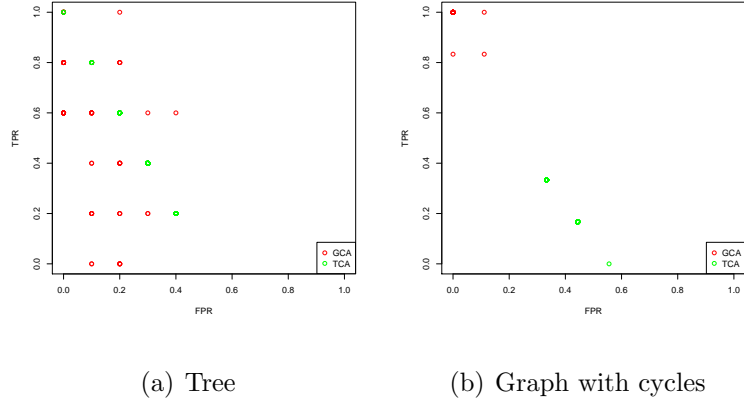


Figure 6.3: ROC plots for graph recovery for TCA and ICA

Though the preliminary experiments above are promising, more work is necessary to obtain a better comparison of the performance of these algorithms, especially on higher-dimensional data. In future work, we plan to compare the heldout Fisher information. The Fisher divergence does not require a normalizing constant. For each algorithm, we will compute the Fisher divergence on a holdout dataset x_{test} ,

$$\hat{\mathbb{E}}_{x_{test}} \left(\frac{1}{2} \hat{\theta}^\top C(\hat{W}x_{test}) \hat{\theta} + \hat{\theta}^\top \xi(\hat{W}x_{test}) \right).$$

This method will circumvent the need to compare only on simple graphs, thus allowing us to compare the performance of GCA and the others on more general graphical model structures.

6.5 Extension: kernel density approach

In this section, we propose a computationally more attractive algorithm than the one presented in Section 6.3. As previously noted, the algorithm presented in Section 6.3 runs in time $O(d^3)$. The run time is in fact prohibitively slow in the currently implementation in the R programming language. In this section, we show how the kernel density method of score matching can be used in the context of GCA in Section ??.

Let $m \in \mathbb{N}$. For each $i \in [n]$, $j \in [m]$,

$$s_i \sim_{i.i.d.} P_S \text{ with density } p_S.$$

$$\xi_{s,i,j} = s_i + z_j, \text{ where } z_j \sim_{i.i.d.} N(0, I_p).$$

Then we have an orthonormal matrix A , and we observe the data:

$$x_i = As_i.$$

$$\xi_{x,i,j} = A\xi_{s,i,j} = As_i + Az_j = As_i + N(0, I_p).$$

We optimize

$$\begin{aligned} \mathcal{L}(W, M, b) &= \sum_{i \in [n], j \in [m]} \|x_i - \xi_{x,i,j} - \sigma^2 \nabla_{\xi} \log p_{M,b}(\xi_{x,i,j})\|_2^2 \\ &= \sum_{i \in [n], j \in [m]} \|x_i - \xi_{x,i,j} - \sigma^2 W^\top \nabla_{\xi_s} \log p_{M,b}(W\xi_{x,i,j})\|_2^2. \end{aligned}$$

This is optimized using stochastic gradient descent. To optimize W , we adapt the method and code of [Jing et al., 2016] to represent W as a product of orthonormal matrices and to optimize via the Givens rotation method as before. Our current implementation and preliminary results are available on in the Jupyter notebook in the Github repository [Graphical component analysis in TensorFlow](#).

6.6 Appendix

In this section, we expand on the approach and algorithm used in Section 6.2, providing some details on how the optimization looks in the specific case of a pairwise graphical model.

6.6.1 Pairwise graphical model

Suppose now that we have an undirected pairwise graphical model and that we expand the potential functions with a truncated orthogonal basis. Let us have m_1 basis elements for the individual node potential functions and m_2 basis elements for the pairwise potential functions. Then

$$\begin{aligned} g(s) &= \sum_{v \in V} \psi_v(s_v) + \sum_{(u,v) \in E} \psi_{u,v}(s_u, s_v) \\ &= \sum_{v \in V} \sum_{k \in [m_1]} \theta_v^k \phi_v^{(k)}(s_v) + \sum_{(u,v) \in E} \sum_{k \in [m_2]} \theta_{u,v}^k \phi_{uv}^{(k)}(s_u, s_v) \end{aligned}$$

There are a few simple ways to write J, C, ξ, θ, ϕ in such a model. Here is my preferred way. As in [Janofsky, 2015], we define the vectors

$$\begin{aligned} \theta_v &= (\theta_v^{(1)}, \dots, \theta_v^{(m_1)})^T \in \mathbb{R}^{m_1} \\ \theta_{vu} &= (\theta_{vu}^{(1)}, \dots, \theta_{vu}^{(m_2)})^T \in \mathbb{R}^{m_2} \text{ for } u \neq v \\ \theta_{v\cdot} &= (-\theta_{v1}^T, \dots, -\theta_{vv}^T, \dots, -\theta_{vp}^T)^T \in \mathbb{R}^{m_1+m_2(p-1)} \\ \theta &= (-\theta_{1\cdot}^T, \dots, -\theta_{p\cdot}^T)^T \in \mathbb{R}^{m_1p+m_2p(p-1)} \end{aligned} \tag{6.6.1}$$

Here, we let $\theta_{uv} = \theta_{vu}$. We define the vectors $\phi_v, \phi_{vu}, \phi_{v\cdot}$, and ϕ analogously. Now $\partial_v \phi = (0, \dots, 0, \partial_v \phi_v, 0, \dots, 0)$. Given these definitions, we can write the terms from (6.2.4) via the following. Everything here is a function of Wx , which I drop

for simplicity. Now,

$$J(Wx) = \begin{pmatrix} | & & & \\ \partial_1 \phi_{1.} & 0 & & 0 \\ | & & & \\ & | & & \\ 0 & \partial_2 \phi_{2.} & 0 & 0 \\ & | & & \\ & & \ddots & \\ & & & | \\ 0 & 0 & 0 & \partial_p \phi_{p.} \\ & & & | \end{pmatrix}$$

and

$$C(Wx) = \begin{pmatrix} \partial_1 \phi_{1.} \partial_1 \phi_{1.}^T & \dots & \dots \\ \dots & \partial_2 \phi_{2.} \partial_2 \phi_{2.}^T & \dots \\ \ddots & & \\ \dots & \dots & \partial_p \phi_{p.} \partial_p \phi_{p.}^T \end{pmatrix}$$

and

$$\xi(Wx) = (-\partial_{11} \phi_{1.}^T, \dots, -\partial_{pp} \phi_{p.}^T)^T$$

Note that $\theta_{vu} = \theta_{uv}$, and $\theta \in \mathbb{R}^{m_1 p + m_2 p(p-1)}$. Note that we can write the objective as

$$\mathcal{L}(\theta, W, x) = \sum_{v \in V} \theta_v^T \partial_v \phi_{v.}(Wx) \partial_v \phi_{v.}(Wx)^T \theta_v + \theta_v^T \partial_{vv}^2 \phi_{v.}(Wx)$$

6.6.2 W optimization: Givens rotation

The Givens orthogonal rotation algorithm is in Algorithm Algorithm 5. The main thing to figure out is how to find $\operatorname{argmin}_\eta \hat{\mathcal{L}}(WG_{u,v}(\eta))$; for this, we compute the first derivative of the function with respect to η . For $u, v \in V$, define

$$G_{u,v}(\eta) = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & \cos \eta & \dots & -\sin \eta & \dots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \dots & \sin \eta & \dots & \cos \eta & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \quad (6.6.2)$$

That is, it has 1's down the diagonal except for columns u and v and zeros everywhere else except for the (u, u) , (v, v) , (u, v) , and (v, u) entries, where it has $\cos \eta$, $\cos \eta$, $-\sin \eta$, $\sin \eta$, respectively. Fix a pair (u, v) . Write G_η for simplicity. To simplify notation, let $f_v = \partial_v \phi$ and $g_v = \partial_{vv}^2 \phi$. And let $y_\eta := WG_\eta x$. Our objective in η is $\mathbb{E}\mathcal{L}(\theta, WG_\eta x)$, where

$$\mathcal{L}(\eta, x) = \mathcal{L}(\theta, WG_\eta x) = \frac{1}{2}\theta^T \left(\sum_{v \in V} f_v(y_\eta) f_v(y_\eta)^T \right) \theta + \theta^T \left(\sum_{v \in V} g_v(y_\eta) \right)$$

Now let $h : \mathbb{R}^p \rightarrow \mathbb{R}$, let η be a scalar, and let $y_\eta := y(\eta) = (y_1(\eta), \dots, y_p(\eta))^T \in \mathbb{R}^p$ be a vector depending on η . Then $\partial_\eta h(y_\eta) = \langle \nabla h(y_\eta), \partial_\eta y_\eta \rangle$. For us, $\partial_\eta y_\eta = W \partial_\eta G_\eta x$,

where

$$\partial_\eta G_{u,v}(\eta) = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & -\sin \eta & \dots & -\cos \eta & \dots & 0 \\ \vdots & & & & & \vdots & \\ 0 & \dots & \cos \eta & \dots & -\sin \eta & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{pmatrix}.$$

This is the matrix whose entries are the derivatives of the entries of G_η with respect to η . Now f_v, g_v are functions from $\mathbb{R}^p \rightarrow \mathbb{R}^K$ (so they're vectors in \mathbb{R}^K). And

$$\begin{aligned} \partial_\eta f_v(y_\eta) &\triangleq (\partial_\eta f_{v1}(y_\eta), \dots, \partial_\eta f_{vK}(y_\eta))^T \\ &= (\langle \nabla f_{v1}(y_\eta), \partial_\eta y_\eta \rangle, \dots, \langle \nabla f_{vK}(y_\eta), \partial_\eta y_\eta \rangle)^T \\ &= J_{f_v}(y_\eta) \partial_\eta y_\eta \end{aligned}$$

Note that by the product rule applied to each element of the matrix $f_v(y_\eta) f_v(y_\eta)^T$, $\partial_\eta (\theta^T (f_v f_v)^T) \theta = \theta^T \left((\partial_\eta f_v) f_v^T + f_v (\partial_\eta f_v)^T \right) \theta$. Thus

$$\partial_\eta \mathcal{L}(\theta, WG_\eta, x) = \frac{1}{2} \theta^T \left(\sum_{v \in V} J_{f_v}(y_\eta) \partial_\eta y_\eta f_v(y_\eta)^T + f_v(y_\eta) \partial_\eta y_\eta^T J_{f_v}(y_\eta)^T \right) \theta + \theta^T \left(\sum_{v \in V} J_{g_v}(y_\eta) \partial_\eta y_\eta \right)$$

So now assuming we can bring the derivative inside the expectation, $\partial_\eta \mathcal{L}(\theta, WG_\eta) = \mathbb{E} \partial_\eta \mathcal{L}(\theta, WG_\eta, x)$.

Graphical model form for Givens rotation

Let $f_v = \partial_v \phi_v$. and $g_v = \partial_{vv}^2 \phi_v$. We can now write our objective in η in the form

$$\mathcal{L}(\theta, WG_\eta, x) = \frac{1}{2} \sum_{v \in V} \theta_v^T f_v(y_\eta) f_v(y_\eta)^T \theta_v + \sum_{v \in V} \theta_v^T g_v(y_\eta)$$

And

$$\partial_\eta \mathcal{L}(\theta, WG_\eta, x) = \frac{1}{2} \sum_{v \in V} \theta_v^T (J_{f_v}(y_\eta) \partial_\eta y_\eta f_v(y_\eta)^T + f_v(y_\eta) \partial_\eta y_\eta^T J_{f_v}(y_\eta)^T) \theta_v + \sum_{v \in V} \theta_v^T J_{g_v}(y_\eta) \partial_\eta y_\eta$$

Chapter 7

Nonparametric variational inference via score matching

7.1 Introduction

Consider the posterior inference problem: we observe data $x_1, \dots, x_n \sim_{i.i.d.} \mathbb{P}_z$, a distribution with density $p(x|z)$. We place a prior $p(z)$ on $z \in \mathbb{R}^d$, and we wish to estimate the posterior density $p(z|x)$:

$$p(z|x) = \frac{p(x|z)p(z)}{p(x)}.$$

Typically $p(x)$ is intractable, so the posterior cannot be directly computed. Markov Chain Monte Carlo (MCMC) methods are a classical tool for this problem, but these methods can be infeasible for large datasets. *Variational inference* has emerged in recent decades as a powerful way to estimate a posterior in high-dimensional settings.

Variational inference posits a family \mathcal{Q} for the posterior density and minimizes a measure of divergence between $q \in \mathcal{Q}$ and the true $p(z|x)$. It is common to choose q to maximize the evidence lower bound, $\mathcal{L}(q) := \mathbb{E}_q \log p(x, z) + H(q)$, which is equivalent to minimizing the KL divergence between the true posterior and q . A

mean-field family, in which $q(z) = \prod_{i \leq d} q_i(z_i)$, is often posited for \mathcal{Q} . When the likelihood and prior are conjugate, this results in a simple coordinate ascent algorithm in which the updates are computable in closed form. In non-conjugate models, the Laplace method, which assumes a Gaussian family for \mathcal{Q} and expands $\log p(x, z)$ to a quadratic term, is a popular approach; see [Wang and Blei, 2013] for more details.

In these and other standard methods, restrictive families are chosen for \mathcal{Q} because of their computational convenience. But such methods have serious weaknesses. For instance, restricting \mathcal{Q} to be a mean-field family means the posterior estimate will fail to capture dependence among the posterior coordinates and will underestimate the posterior variance; see [Blei et al., 2017] for more discussion. The Gaussian form imposed on q in the Laplace method means the estimate will fail to capture multimodality in the posterior.

We propose *nonparametric variational inference*, in which we assume a nonparametric form for q and optimize it using the score matching objective. Classical score matching seeks a density q to minimize a Fisher divergence, defined via

$$\mathcal{L}(q) = \mathbb{E}_p \|\nabla_z \log p(z) - \nabla_z \log q(z)\|_2^2. \quad (7.1.1)$$

This method is appealing for nonparametric density estimation because it does not require the calculation of a normalizing constant; this computation can be prohibitive for nonparametric densities in high dimensions. And it turns out, see Lemma 1.3.1, that (7.1.1) can be expressed as

$$\mathcal{L}(q) = \mathbb{E}_p O(q), \text{ where } O(q) = \|\nabla \log q\|_2^2 + \Delta q. \quad (7.1.2)$$

When we have samples from the true density p , we can optimize an empirical approximation of (7.1.2) in q . For exponential families, (7.1.2) further simplifies into a nice quadratic; see Lemma 1.3.2. It has been used in high dimensional den-

sity estimation for exponential families of the form $\log q \propto f(x)$ where f lies in a reproducing kernel Hilbert space [Sriperumbudur et al., 2017] or $f(x) = \gamma^\top \phi(x)$ where $\phi(x)$ is represented in terms of an orthogonal polynomial basis, e.g., in a pairwise, nonparametric graphical model [Janofsky, 2015, Lin et al., 2016], and these papers also provide rates of convergence for density estimation via score matching; in [Sriperumbudur et al., 2017], score matching is shown to achieve the optimal rate for density estimation based on the smoothness of the function class over which the optimization is done.

The problem of an intractable normalizing constant that arises in high-dimensional density estimation also appears in posterior inference. Score matching seems like a promising technique to apply in this context, but issues immediately arise. The crucial difference between density estimation and posterior inference is that in the former, we have access to samples from the truth p but not to the form of p , while in posterior inference we have access to the nonnormalized form of the posterior, i.e., to $p(x, z)$, but not to samples from the posterior. In an exponential family form for q , the cancellation of the normalizing constant $p(x)$ means the objective is still optimizeable in q , but it nonetheless requires estimation of an integral over $p(x, z)$; see Section 7.2 for more discussion. Moreover, we often desire in posterior inference, as in density estimation, to obtain samples from the posterior, but it is not possible to directly sample from a nonparametric exponential family.

To circumvent the issues classical score matching poses in the posterior inference context, we turn to a related objective:

$$\mathcal{L}(q) = \mathbb{E}_q \|\nabla \log p - \nabla \log q\|_2^2. \quad (7.1.3)$$

We show in Lemma 7.3.4 that (7.1.3) has a close relationship to the objective (7.1.1). Instead of positing an exponential family form for q , we use a novel score matching

variant of [Saremi et al., 2018] to optimize a generative form for q that allows us to sample from the posterior and perform a variational EM algorithm to estimate a model.

Review of recent literature Recent literature has developed methods to allow \mathcal{Q} to be a richer family. For instance, the authors of [Ranganath et al., 2016] use a dual representation of the divergence (7.1.3) and a generative form of q to avoid the use of the density itself in the variational optimization. Their algorithm requires optimization over both the parameters of q and the function in the dual representation. [Li and Turner, 2016] propose a similar method using a generalized dual representation of divergences. These methods perform an optimization of the form:

$$\min_q \max_f \mathbb{E}_q O(f), \quad (7.1.4)$$

where $O(f)$ is an operator on the function f .

The technique of representing a divergence in its dual form has been widely used in the statistics literature for nonparametric density estimation. The classical generative adversarial network (GAN) of [Goodfellow et al., 2014], along with developments such as the Wasserstein-GAN [Arjovsky et al., 2017] and others, have relied on dual representations of the symmetric KL divergence and Wasserstein distance, respectively. Indeed, any f -divergence can be represented using the convex conjugate, or Fenchel dual form (e.g., the Donsker-Varadhan lower bound on the KL divergence). For exponential families, optimizing a representation of the density using a variational lower bound on the normalizing constant was explored in [Dai et al., 2018].

The disadvantage of relying on a dual representation for a divergence is that it requires a procedure that iterates over optimizing a divergence in a function f and minimizing over the density q . This non-convex procedure does not have algorithmic or statistical convergence guarantees. In this work, we propose to optimize the

objective (7.1.3) directly, without resorting to a dual representation.

The authors of [Gershman et al., 2012] propose “nonparametric variational inference,” which uses a Gaussian-mixture form of q and the Delta method to approximate the model. The mixture of Gaussians for q resembles a kernel density estimator from nonparametric estimation. Moreover, this family for q , along with the Delta method, makes the optimization process fairly simple. The Gaussian-mixture family for q is also related to the “mixture of mean-field” family proposed by [Bishop et al., 1998].

7.2 Classical score matching for posterior inference

Suppose we posit an exponential family form for the posterior, i.e., $p(z|x) \propto e^{f(z|x)}$. Suppose further that $f(z|x) = f_x(z) = \gamma^\top \phi(z)$, where $\gamma \in \mathbb{R}^K$ and $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^K$. Letting $A(z), k(z)$ be as defined in Lemma 1.3.2, the objective (7.1.1) simplifies to:

$$\int_z p(z|x) \left(\frac{1}{2} \gamma^\top A(z) \gamma + \gamma^\top k(z) \right) dz,$$

This has optimal solution

$$\begin{aligned} \hat{\gamma} &= \left(\int p(z|x) A(z) dz \right)^{-1} \int p(z|x) k(z) dz \\ &= \left(\int \frac{p(x, z)}{p(x)} A(z) dz \right)^{-1} \int \frac{p(z, x)}{p(x)} k(z) dz \\ &= \bar{A}^{-1} \bar{k}, \end{aligned} \tag{7.2.1}$$

where

$$\begin{aligned}\bar{A} &= \int p(x, z) A(z) \, dz \\ \bar{k} &= \int p(x, z) k(z) \, dz.\end{aligned}$$

Crucially, the solution (7.2.1) requires only integrals over $p(z, x)$, not over the uncomputable $p(z|x)$, since in the quadratic form, the $p(x)$ cancels out. Now (7.2.1) is something we can calculate. There is still an important difficulty: computing the integrals \bar{A}, \bar{k} over the joint density. We propose importance sampling for this, and our proposed algorithm is described in Algorithm 6.

Algorithm 6: Classical score matching for nonparametric variational inference with an exponential family

Input: Dataset $\{x_i\}_{i \in [n]}$ with each point in \mathbb{R}^d , likelihood model $p(x|z)$, prior $p(z)$, orthogonal basis model $\phi(z)$ for posterior, functions $A(z), k(z)$ as defined in Lemma 1.3.2, importance-sampling density $h(z)$

Output: Estimate $p(z|x)$.

Sample $z_1, \dots, z_m \sim_{i.i.d.} P$ where P is a distribution on \mathbb{R}^d with density h ;

Compute $\bar{A} = \sum_{j=1}^m \frac{p(x, z_j)}{h(z_j)} A(z_j)$;

Compute $\bar{k} = \sum_{j=1}^m \frac{p(x, z_j)}{h(z_j)} k(z_j)$;

Report

$$\hat{\gamma} = \bar{A}^{-1} \bar{k} \tag{7.2.2}$$

$$\log p(z|x) \propto \hat{\gamma}^\top \phi(z). \tag{7.2.3}$$

Several major problems arise in Algorithm 6. First, often $p(x|z)$ is extremely

small if n is large, making importance sampling difficult, especially in high dimensional examples. Second, and more importantly, suppose that $\phi(z)$ is a polynomial; such an assumption is common in nonparametric density estimation. For instance, suppose $\phi(z) = 1 + z + z^2 + \dots$. Then the first and second derivative functions, $A(z), k(z)$, will contain constant terms. This means that for example, \bar{k} contains terms like $\int p(z, x) 1 dz$, which is precisely $p(x)$. In such an example, we would simply be estimating $p(x)$ via importance sampling. If we are going to do this, we might as well compute $p(z|x)$ directly. Such a form for ϕ is common, e.g., if the family is Gaussian; see Example 3.

We have implemented Algorithm 6 in a simple, low-dimensional case; see [Nonparametric variational inference via score matching](#).

7.3 Score matching for generative nonparametric variational inference

In this section, we propose an optimization that uses score matching but circumvents some of the issues noted in Section 7.2. We propose to directly optimize (7.1.3) via kernel density estimation and the reparametrization trick, relying on the form of score matching discussed in Section 1.3.1.

First, we posit a generative form for the posterior q . We generate Gaussian noise, then feed it through a multilayer perceptron f_ϕ to generate samples from q . That is, we draw

$$\begin{aligned} z_{0i} &\sim N(0, I_{d_0}) \\ z_i &= f_\phi(z_{0i}). \end{aligned} \tag{7.3.1}$$

We wish to estimate the parameters ϕ . Here $z_i \in \mathbb{R}^d$, and we will let $d_0 < d$. We will let f_ϕ be a multilayer perceptron with some nonlinear activation function. Since the transformation f_ϕ is not invertible, we cannot use the change of variable

density formula to obtain an exact formula for $q(z)$. Thus, it seems impossible to optimize (7.1.3) directly since it requires the form of q in order to compute $\nabla_z \log q(z)$. We will circumvent this issue by representing the posterior q_ϕ via a kernel density representation, exactly as in Section 1.3.1.

7.3.1 Algorithm

To run initial tests of our proposal, we generate synthetic data using a variational-EM algorithm to estimate the generative model parameters. We imitate the variational autoencoder of [Kingma and Welling, 2014], but in our case, the variational inference step is done as described above rather than using traditional variational inference. We posit a model

$$p_\theta(x, z) = \prod_{i=1}^n p_\theta(x_i, z_i) = \prod_{i=1}^n p_\theta(x_i|z_i)p(z_i),$$

and we posit an *amortized mean-field* form for the posterior:

$$q_\phi(z|x) = \prod_{i=1}^n q_\phi(z_i|x_i),$$

where the parameter ϕ is common to all data points. We estimate ϕ and θ as described below. To reproduce data samples, we generate from $q_\phi(z_i|x_i)$, then generate a new sample x according to $p_\theta(x_i|z_i)$. We optimize the parameters θ using EM. The complete algorithm is provided in Algorithm 7.

Algorithm 7: Nonparametric generative variational inference via score matching

Input: Dataset $\{x_i\}_{i \in [n]}$ with each point in \mathbb{R}^d , likelihood model $p(x|z)$, prior $p(z)$, stopping criteria T

Output: Estimate $q_\phi(z|x)$ for the posterior.

Initialize ϕ, θ ;

Set $t = 0$;

While $t < T$

Generate $z_{0i} \sim N(0, I_d)$ for $i \in [n]$ and $\xi_{ij} \sim N(0, I_d)$ for $j \in [m]$;

Optimize the posterior via (stochastic) gradient descent:

$$\min_{\phi} \sum_{i \in [n], j \in [m]} \|f_{\phi}(z_{0i}) - \xi_{ij} - \nabla_{\xi} \log p_{\theta}(x_i, f_{\phi}(\xi_{ij}))\|_2^2 ;$$

Optimize the the likelihood via (stochastic) gradient descent:

$$\max_{\theta} \sum_{i=1}^n \log p_{\theta}(x_i, f(z_{0i})) ;$$

Report estimates

$$\hat{\phi}, \hat{\theta}. \tag{7.3.2}$$

To reproduce samples:

Select x from dataset ;

Draw $z_0 \sim N(0, I_{d_0})$;

Draw $z(x) = f_{\hat{\phi}(x)}(z_0)$;

Draw $\hat{x} = p_{\hat{\theta}}(x|z(x))$.

To produce new samples:

Draw $x = p_{\hat{\theta}}(x|z)$ where $z \sim N(0, I_d)$.

We run preliminary tests of Algorithm 7 on the MNIST handwritten digit dataset using TensorFlow to perform the optimizations. Each data point x_i in the MNIST dataset lies in $d = 784$ dimensions, with each coordinate representing a pixel, which takes a value in $[0, 1]$. We posit the model, for $i \in [n]$,

$$z_i \sim_{i.i.d.} N(0, I_d)$$

$$x_i = \frac{\exp(\eta^\top \beta(z_i))}{1 + \exp(\eta^\top \beta(z_i))}.$$

Here, the model parameter is $\theta = (\eta, \beta)$, where $\eta \in \mathbb{R}^K$ and $\beta : \mathbb{R}^d \rightarrow \mathbb{R}^K$ is a multilayer perceptron. These experiments are available in the Jupyter notebooks in the Github repository [Variational autoencoder via score matching and ratio matching](#).

Ratio matching We propose one other alternative. First, notice the following. For small h ,

$$\nabla_z \log q(z) \approx \frac{\log q(z+h) - \log q(z)}{h} = \frac{1}{h} \log \frac{q(z+h)}{q(z)}$$

This is the basis of ratio-matching, which approximates score matching using the above representation of the derivative. Draw samples z_1, z_2 from a sampling form q , as in (7.3.1), and optimize

$$\mathbb{E}_q \left\| \log \frac{q(z_1)}{q(z_2)} - \log \frac{p(x, z_1)}{p(x, z_2)} \right\|_2^2. \quad (7.3.3)$$

This imitates the method of [Hyvarinen, 2007]. To begin, we use a Gaussian form for q . Since we can sample from this, we use the reparametrization trick to estimate the integral in (7.3.3).

7.3.2 Decomposition of the objective

In this section, we show that the score-matching objective (7.1.3) decomposes into two components: a component matching q to p , and a component restricting q to be smooth. This resembles the evidence lower bound of $\mathbb{E}_q \log p(x, z) + H(q)$ used in classical variational inference; the first term matches the posterior estimate to the log likelihood, the second constrains the variance of q .

Lemma 7.3.1. *Suppose $q(z) = \exp(f(z) - \Psi)$ where $\Psi = \int_z \exp(f(z)) dx$. Let $\mathcal{L}(q) = \mathbb{E}_{q(z|x)} \|\nabla_z \log p(z|x) - \nabla_z \log q(z|x)\|_2^2$. Then*

$$\mathcal{L}(q) =^c \frac{1}{2} \mathbb{E}_q \|\nabla_z \log p(x, z)\|_2^2 + \mathbb{E}_q \Delta \log p(x, z) + \mathbb{E}_q \|\nabla_z f(z)\|_2^2 \quad (7.3.4)$$

The decomposition in (7.3.4) deepens our understanding of (7.1.3) and suggests alternatives to the ideas proposed in Section 7.3.1. In theory, we could directly optimize (7.3.4), since in posterior inference we have the form of $\log p(x, z)$ and therefore we have its gradients. Moreover, $\nabla_z \log q(z)$ does not require the normalizing constant of q , so we could optimize this in q if we had, e.g., an exponential family form for q . The difficulty would be estimating the integral over q when q is from a nonparametric family; there is no general way to do this. In general, to have a viable sampling form for q and an explicit form of $\nabla \log q$, we would need to assume a Gaussian form for q , which we of course wish to avoid. Therefore, we might simply assume a generative form for q and approximate $\nabla \log q$ via a kernel density estimate as previously; optimizing (7.3.4) would thus be an alternative to Algorithm 7.

7.3.3 Connections to the dual representation

The objective (7.1.3) is related to a form of score matching used in [Ranganath et al., 2016] and we now discuss connections between our proposal and that paper and other recent literature. Let $\psi_p(z) = \frac{\partial \log p(z)}{\partial z}$. The authors of [Ranganath et al., 2016] propose

to optimize following variational objective in q :

$$\sup_{f \in \mathcal{F}} |\mathbb{E}_{q(z)}(O_{LS}^p f)(z)| \quad (7.3.5)$$

where

$$O_{LS}^p f = \psi_p f + \nabla f. \quad (7.3.6)$$

As noted in [Ranganath et al., 2016], the objective (7.3.5) is reasonable since it is zero when $q = p$, as we show in Lemma 7.3.2. [Ranganath et al., 2016] optimize (7.3.5) by iterating between maximizing over f and minimizing over q ; a generative multilayer perceptron form similar to ours is used for q , while f is another neural network. This optimization is nonconvex and is not guaranteed to reach some global optimum.

We can relate the objective (7.3.5) to a Stein discrepancy and then to a score matching objective (7.1.3). We state this in Lemma 7.3.3. It turns out that this objective has a close relationship to that in (7.1.1) and (7.3.5), as we state in Lemma 7.3.4. Proofs for these lemmas are provided in Section 7.4.1.

Lemma 7.3.2. *Suppose $\lim_{z \rightarrow \infty} q(z) = \lim_{z \rightarrow -\infty} q(z) = 0$. Then*

$$\mathbb{E}_{q(z)} (\psi_q f(z) + f'(z)) = 0$$

Lemma 7.3.3 (Equivalence of score-matching, Stein discrepancy, and OVI). *Let \mathcal{F} be a function class, and let \mathcal{G} be a class of functions of the form $g(z) = \psi_p(z)f(z) + f'(z)$, for $f \in \mathcal{F}$. Then*

$$\sup_{g \in \mathcal{G}} |\mathbb{E}_q g(z)| = \sup_{g \in \mathcal{G}} |\mathbb{E}_q g(z) - \mathbb{E}_p g(z)| \quad \leftarrow \text{Stein discrepancy} \quad (7.3.7)$$

$$= \mathbb{E}_{q(z)} (\psi_q(z) - \psi_p(z))^2 \quad \leftarrow \text{Score matching} \quad (7.3.8)$$

Lemma 7.3.3 motivates the use of the alternative score matching objective (7.1.3).

We now show how it relates to the objective (7.1.1).

Lemma 7.3.4. *Let p, q be densities that are positive everywhere. Let \mathcal{H} be a class of functions, and let \mathcal{F} be a class of functions of the form $f(z) = \frac{p(z)}{q(z)}h(z)$ for $h \in \mathcal{H}$. Then*

$$\sup_{f \in \mathcal{F}} |\mathbb{E}_q O_{LS}^p f| = \sup_{h \in \mathcal{H}} |\mathbb{E}_p O_{LS}^q h|$$

7.4 Appendix

7.4.1 Proofs

Proof of Lemma 7.3.1. We provide the proof in one dimension only.

$$\mathcal{L}(q) = \int q(z) \left(\frac{\partial}{\partial z} \log p \right)^2 - 2 \int q(z) \frac{q'(z)}{q(z)} \frac{p'(z)}{p(z)} + \int q(z) \left(\frac{\partial}{\partial z} \log q(z) \right)^2$$

The third term is $\mathbb{E}_q f'(z)^2$. And for the middle term, using integration by parts,

$$\int q(z) \frac{q'(z)}{q(z)} \frac{p'(z)}{p(z)} = \int q'(z) \frac{\partial}{\partial z} \log p(z) =^c - \int q(z) \frac{\partial^2}{\partial z^2} \log p(z)$$

□

Proof of Lemma 7.3.2. First, by the product rule,

$$\mathbb{E}_{p(z)} (\psi_q(z) f(z) + f'(z)) = \int_z \frac{p(z)}{q(z)} \frac{\partial (q(z) f(z))}{\partial z} dz$$

Now

$$\int \frac{q(z)}{q(z)} \frac{\partial (q(z) f(z))}{\partial z} dz = \int \frac{\partial}{\partial z} (q(z) f(z)) dz = q(\infty) f(\infty) - q(-\infty) f(-\infty) = 0.$$

□

Proof of Lemma 7.3.3. By Lemma 7.3.2, $\mathbb{E}_{p(z)}\psi_p(z)f(z) + f'(z) = 0$. So

$$\begin{aligned} \sup_{g \in \mathcal{G}} |\mathbb{E}_{p(z)} g(z)| &= \sup_{f \in \mathcal{F}} |\mathbb{E}_{p(z)} (\psi_q(z)f(z) + f'(z) - \psi_p(z)f(z) - f'(z))| \\ &= \sup_{f \in \mathcal{F}} |\mathbb{E}_p(\psi_q - \psi_p)f| \end{aligned} \quad (7.4.1)$$

And this is maximized when $f(z) = \psi_q(z) - \psi_p(z)$. □

Proof of Lemma 7.3.4. This is clear if we use the representation in (7.4.1). That is,

$$\sup_{f \in \mathcal{F}} |\mathbb{E}_q O_{LS}^p f| = \sup_{f \in \mathcal{F}} \mathbb{E}_q(\psi_q - \psi_p)f = \sup_{h \in \mathcal{H}} \mathbb{E}_q \frac{p}{q}(\psi_q - \psi_p)h = \sup_{h \in \mathcal{H}} \mathbb{E}_p(\psi_q - \psi_p)h = \sup_{h \in \mathcal{H}} |\mathbb{E}_p O_{LS}^q h|$$

Alternatively (this is a sanity check):

$$\begin{aligned} \sup_{f \in \mathcal{F}} |\mathbb{E}_q O_{LS}^p f| &= \sup_{f \in \mathcal{F}} |\mathbb{E}_q (\psi_p(z)f(z) + f'(z))| \\ &= \sup_{h \in \mathcal{H}} \left| \int q(z) \frac{p'(z)}{p(z)} \frac{p(z)}{q(z)} h(z) + \int q(z) \left(\frac{q(z)p'(z) - p(z)q'(z)}{q(z)^2} h(z) + \frac{p(z)}{q(z)} h'(z) \right) \right| \\ &= \sup_{h \in \mathcal{H}} \left| \int p'(z)h(z) + \int p'(z)h(z) - \int \frac{q'(z)}{q(z)} p(z)h(z) + \int p(z)h'(z) \right| \\ &= \sup_{h \in \mathcal{H}} \left| \int 2(p'(z)h(z) + p(z)h'(z)) - \int p(z)h'(z) - \int \frac{q'(z)}{q(z)} p(z)h(z) \right| \\ &= \sup_{h \in \mathcal{H}} |\mathbb{E}_p (\psi_q(z)h(z) + h'(z))| \end{aligned}$$

since $p'(z)h(z) + p'(z)h(z) = \frac{\partial(p(z)h(z))}{\partial z}$, which has integral zero by Lemma 7.3.2. □

7.4.2 Examples

The following lemma shows that when we assume $q(z)$ is a Gaussian density, the score matching objective as in (7.1.3) reduces to the Laplace method as in [Wang and Blei, 2013].

In retrospect, this is obvious by Lemma 7.3.2; since we assume a Gaussian form for

the model, as long as we are optimizing over \mathcal{Q} that includes Gaussian densities, we should obtain “the right thing.” But this is nice as a sanity check.

Lemma 7.4.1. *Let $z \in \mathbb{R}^d$. Let $f(z) := \log p(z)$ and suppose for any z_0 ,*

$$f(z) \approx f(z_0) + \nabla f(z_0)^T(z - z_0) + \frac{1}{2}(z - z_0)^T \nabla^2 f(z_0)(z - z_0)$$

. Let $z^ \in \operatorname{argmax} f(z)$. Let \mathcal{Q} be $N(\lambda, V)$ densities. Let $\hat{q} = \operatorname{argmin}_{q \in \mathcal{Q}} \mathbb{E}_q \|\nabla_z \log p - \nabla_z \log q\|_2^2$. Then \hat{q} is the $N(\hat{\lambda}, \hat{V})$ density with*

$$\hat{\lambda} = z^*$$

$$\hat{V} = -\nabla^2 f(z^*)$$

Proof. Let z^* be a mode of f , i.e., $\nabla_z f(z^*) = 0$. Then

$$f(z) \approx f(z_0) + \frac{1}{2}(z - z^*)^T \nabla^2 f(z^*)(z - z^*)$$

So

$$\nabla_z f(z) \approx \nabla^2 f(z^*)(z - z^*)$$

Let q be the $N(\lambda, V)$ density where $\lambda \in \mathbb{R}^d$ and $V \in \mathbb{R}^{d \times d}$; λ and V are the parameters to be optimized. Now $\nabla_z \log q(z) = -V^{-1}(z - \lambda)$. Now $z = V^{1/2}z_0 + \lambda$, where

$z_0 \sim N(0, I_p)$. So

$$\begin{aligned}
\mathcal{L}(q) &= \mathbb{E}_q \|\nabla_z \log q(z) - \nabla_z \log p(z)\|_2^2 \\
&= \mathbb{E}_{z_0 \sim N(0, I_p)} \left\| -V^{-1/2} z_0 - \nabla^2 f(z^*) (V^{1/2} z_0 + \lambda - z^*) \right\|_2^2 \\
&= \mathbb{E}_{z_0} \left\| (-V^{-1/2} - \nabla^2 f(z^*) V^{1/2}) z_0 - \nabla^2 f(z^*) (\lambda - z^*) \right\|_2^2 \\
&= \mathbb{E}_{z_0} \left\| (-V^{-1/2} - \nabla^2 f(z^*) V^{1/2}) z_0 \right\|_2^2 + \left\| \nabla^2 f(z^*) (\lambda - z^*) \right\|_2^2 \\
&= \left\| V^{-1/2} + \nabla^2 f(z^*) V^{1/2} \right\|_F^2 + \left\| \nabla^2 f(z^*) (\lambda - z^*) \right\|_2^2
\end{aligned}$$

Optimizing in λ yields:

$$\hat{\lambda} = z^*$$

Let $M = \nabla^2 f(z^*)$. Let's do one dimension to check:

$$v^{-1/2} + mv^{1/2} = \frac{1 + mv}{\sqrt{v}}$$

And

$$\left(\frac{1 + mv}{\sqrt{v}} \right)^2 = \frac{1 + 2mv + m^2 v^2}{v} = \frac{1}{v} + 2m + m^2 v$$

Now

$$\frac{\partial}{\partial v} \left(\frac{1}{v} + 2m + m^2 v \right) = -\frac{1}{v^2} + m^2$$

Setting this equal to zero yields:

$$V^2 = M^{-2} \Rightarrow V = \pm M^{-1}$$

which is exactly the Laplace method solution. □

Example 2 (Lemma 7.3.1 in the Gaussian case). Consider the objective (7.1.3). In the case where q is Gaussian, as in, Lemma 7.4.1, we can see that only optimizing the first two terms of the score-matching objective would lead to a terribly sharp q (with zero variance). The final term (the penalty on the smoothness of q) forces more smoothness. Let the setting be as in Lemma 7.4.1: q is the $N(\lambda, V)$ and the Taylor expansion for $\log p(x, z)$ holds. Now,

$$\begin{aligned} \frac{1}{2} \mathbb{E}_q \|\nabla_z \log p(z, x)\|_2^2 + \mathbb{E}_q \Delta \log p(z, x) &= \frac{1}{2} \mathbb{E}_q (z - z^*)^T (\nabla^2 f(z^*))^2 (z - z^*) - \mathbb{E}_q \nabla^2 f(z^*) \\ &=^c \frac{1}{2} \text{tr} \left(V^T (\nabla^2 f(z^*))^2 V \right) + (\lambda - z^*)^T (\nabla^2 f(z^*))^2 (\lambda - z^*) \end{aligned}$$

which would be optimized by setting $\hat{\lambda} = z^*$ and $\hat{V} = 0$. But we have the penalty:

$$\mathbb{E}_q \|\nabla \log q\|_2^2 = \text{tr}(V^{-1})$$

Due to this penalty, we end up with a reasonable q .

We now provide further discussion of the potential problems with using score matching as in (7.1.1) for posterior inference. We first show how the optimization would look if q is restricted to be Gaussian.

Example 3 (Score-matching with \mathbb{E}_p for posterior inference when q has Gaussian form). Suppose q has form $N(\mu_x, \Sigma_x)$ where $\mu_x \in \mathbb{R}^d$, $\Sigma_x \in \mathbb{R}^{d \times d}$. Let our prior $p(z)$ be the $N(0, I_d)$ density. Note that we could have $d \gg n$. We have $q(z) \propto e^{g(z)}$ where

$$g(z) = \gamma' \phi(z)$$

where $K = 2d$ and

$$\gamma = (\Sigma^{-1}, \Sigma^{-1}\mu)^T$$

$$\phi(z) = \left(-\frac{1}{2}zz', z\right)^T$$

So (for $d = 1$, to keep it simple):

$$\frac{\partial\phi(z)}{\partial z} = (-z, 1)^T$$

$$\frac{\partial^2\phi(z)}{\partial z^2} = (1, 0)^T$$

We have $A(z) \in \mathbb{R}^{2d \times 2d}$. Let e.g. z^2 indicate (z_1^2, \dots, z_d^2) . We have

$$A(z) = \begin{pmatrix} \text{diag}(z^2) & \text{diag}(-z) \\ \text{diag}(-z) & \text{diag}(1) \end{pmatrix} \text{ and}$$

$$k(z) = (\text{rep}(-1, d), \text{rep}(0, d))^T$$

Each submatrix is in $\mathbb{R}^{d \times d}$. We parameterize γ via B ; it is some multi-layer non-linear function with many parameters B . Our objective, written as a sum, is

$$\mathbb{E}_{p(z|x)} \left(\sum_{j \leq d} \gamma_{1j}^2 z_j^2 - \sum_{j \leq d} \gamma_{2j} \gamma_{1j} z_j + \sum_{j \leq d} \gamma_{2j}^2 - \sum_{j \leq d} \gamma_{1j} \right)$$

Note that as noted in [Janofsky, 2015] and [Lin et al., 2016], there are closed-form solutions of μ, Σ . To see it, note that if $q(z)$ is $N(\mu, \Sigma)$,

$$\log q(z) \propto \frac{-(z - \mu)' \Sigma^{-1} (z - \mu)}{2}$$

So the score-matching objective is

$$\frac{1}{2} \|\nabla_z \log q(z)\|_2^2 + \Delta_z \log q(z) = \frac{1}{2} \|\Sigma^{-1}(z - \mu)\|_2^2 + \text{tr}(\Sigma^{-1})$$

We can directly obtain:

$$\begin{aligned}\hat{\mu} &= \mathbb{E}_{p(z|x)} z \\ \hat{\Sigma}_i &= \mathbb{E}_{p(z|x)} (z - \hat{\mu})(z - \hat{\mu})'\end{aligned}$$

While this is estimable for density estimation, it requires the posterior for us. Why does this happen when we can still not require $p(x)$ if we compute the exponential family parameter? Notice that e.g. in the simple $d = 1$ case, the quadratic we'd obtain for score-matching is:

$$\gamma' \mathbb{E}_{p(z|x)} \begin{pmatrix} z^2 & z \\ z & 1 \end{pmatrix} \gamma - \gamma' \mathbb{E}_{p(z|x)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Optimizing this allows us to remove the $p(x)$, but notice that then, we optimize in $\gamma' \bar{A} \gamma - \gamma' \bar{k}$, the usual thing, where

$$\bar{k} = \begin{pmatrix} p(x) \\ 0 \end{pmatrix}$$

Now again we can approximate $p(x)$ via importance sampling, but we do have to approximate it either way.

7.4.3 Score matching for a bounded density

For a bounded density, score matching has a slightly different objective. See [Janofsky, 2015] for more details. We report the basis of bounded-density score matching here because

it is in fact what we used for the classical score matching for variational inference implementation in [Nonparametric variational inference via score matching](#).

$$\begin{aligned}
k_1(z) &= \sum_{i \leq d} 2(2z_i - 1)z_i(1 - z_i) \frac{\partial \phi(z)}{\partial z_i} \\
k_2(z) &= \sum_{i \leq d} z_i^2(1 - z_i)^2 \frac{\partial^2 \phi(z)}{\partial z_i^2} \\
k(z) &= k_1(z) - k_2(z) \\
A(z) &= \sum_{i \leq d} \frac{\partial \phi(z)}{\partial z_i} \frac{\partial \phi(z)}{\partial z_i}' z_i^2(1 - z_i)^2
\end{aligned}$$

For a bounded density (the second line is for an exponential family), the objective is:

$$\begin{aligned}
h_\gamma(z) &= \sum_{i \leq d} \frac{1}{2} \left(\frac{\partial g(z)}{\partial z_i} z_i(1 - z_i) \right)^2 - 2(2z_i - 1)z_i(1 - z_i) \frac{\partial g(z)}{\partial z_i} + z_i^2(1 - z_i)^2 \frac{\partial^2 g(z)}{\partial z_i^2} \\
&= \sum_{i \leq d} \left(\gamma' \frac{\partial \phi(z)}{\partial z_i} z_i(1 - z_i) \right)^2 - 2(2z_i - 1)z_i(1 - z_i) \gamma' \frac{\partial \phi(z)}{\partial z_i} + z_i^2(1 - z_i)^2 \gamma' \frac{\partial^2 \phi(z)}{\partial z_i^2}
\end{aligned}$$

In one dimension, and for $z \in [0, 1]$, this simplifies to the following. Let $\phi : [0, 1] \rightarrow \mathbb{R}^K$.

$$\begin{aligned}
h_\gamma(z) &= \frac{1}{2} \left(\frac{\partial g(z)}{\partial z} z(1 - z) \right)^2 - 2(2z - 1)z(1 - z) \frac{\partial g(z)}{\partial z} + z(1 - z) \frac{\partial^2 g(z)}{\partial z^2} \\
&= \frac{1}{2} z^2(1 - z)^2 \gamma' A(z) \gamma - 2(2z - 1)z(1 - z) \gamma' k_1(z) + z^2(1 - z)^2 \gamma' k_2(z)
\end{aligned}$$

where

$$k_1(z) = \left(\frac{\partial \phi_1(z)}{\partial z}, \dots, \frac{\partial \phi_K(z)}{\partial z} \right)'$$
$$k_2(z) = \left(\frac{\partial^2 \phi_1(z)}{\partial z^2}, \dots, \frac{\partial^2 \phi_K(z)}{\partial z^2} \right)'$$

$$k(z) = k_1(z) - k_2(z)$$

$$A(z) = k_1(z)k_1(z)'$$

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