Weierstrass Sufficient condition of the existence of global extreme points

While solving any extremal problem the first question one asks is whether its solution exists, because not all problems are solvable. the absence of a solution, to search for it means to waste of time and energy.

There are some effective sufficient methods to determine whether a solution exists. The existence of local minimums is usually proved using advance calculus. The classical Weierstrass theorem and its results guarantees the existence of points of global extremum in a tremendous number of cases.

Thus, before we start looking for extremal points, we must find out if they exist.

The discontinuity of the objective function and the non-compactness of the feasible set are the two main factors that prevent both the existence of a global minimum and the fact of its existence.

Theorem / Weierstrass /. Let $\boldsymbol{M} \subset \boldsymbol{R_n}$ be a closed and bounded set. (i.e., compact). $\boldsymbol{f}: \boldsymbol{M} \to \boldsymbol{R}$ is a continuous function. Then there exist a global minimum and a global maximum in the extremal problem:

$$\boldsymbol{f}(\mathbf{x}) \to \boldsymbol{extr}, \qquad \mathbf{x} \in \boldsymbol{M}, \tag{2.1}$$

Proof. Let $\boldsymbol{k} = \inf_{x \in M}\{\boldsymbol{f}(\mathbf{x})\}$. By definition of infimum, there exists a sequence $\{\mathbf{x}_i\}_{i=1}^{\infty} \subset \boldsymbol{M}$, that $\lim_{i\to\infty} \boldsymbol{f}(\mathbf{x}_i) = \boldsymbol{k}$. Because $\boldsymbol{M}$ is a compact, $\{\mathbf{x}_i\}_{i=1}^{\infty}$ has a subsequence $\left\{\mathbf{x}_{i_j}\right\}_{j=1}^{\infty} \subset \boldsymbol{M}$ which converges to some point $\mathbf{x_0}$ of the set $\boldsymbol{M}$. Since $\boldsymbol{f}(\cdot)$ is the continuous function, we get:

$$\lim_{j\to\infty} \boldsymbol{f}(\mathbf{x}_{i_j}) = \boldsymbol{f}(\mathbf{x_0}). \tag{2.2}$$

At the same time, the sequence $\left\{\boldsymbol{f}(\boldsymbol{x}_{i_j})\right\}_{j=1}^{\infty}$ converges to the same limit, as $\{\boldsymbol{f}(\boldsymbol{x}_i)\}_{i=1}^{\infty}$, i.e.

$$\lim_{j\to\infty} \boldsymbol{f}(\mathbf{x}_{i_j}) = \boldsymbol{k}. \tag{2.3}$$

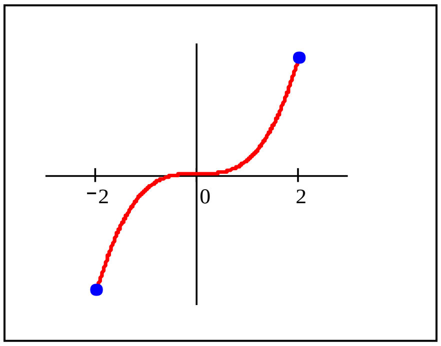
Due to the uniqueness of the limit, from (2.2) and (2.3) we get, that, $\boldsymbol{k} = \boldsymbol{f}(\mathbf{x_0})$, which, according to the definition of $\boldsymbol{k}$ means:

$$\boldsymbol{f}(\mathbf{x_0}) = \boldsymbol{k} \le \boldsymbol{f}(\mathbf{x}), \forall \mathbf{x} \in \boldsymbol{M} \quad \Leftrightarrow \quad \mathbf{x_0} \in \boldsymbol{glmin}(\mathbf{2.1})$$

Weierstrass's theorem is sometimes formulated as follows: a continuous function on a compact set reaches its global minimum and maximum.

Consider some examples.

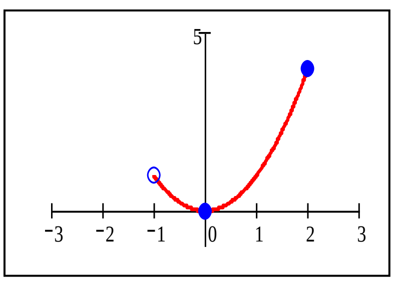
Example 1. $M = [-2, 2]$, $f(x) = x^3$.



In this example M is a compact set, the function $f(x)$ is a continuous function, so the global maximum and minimum are attained. ▯

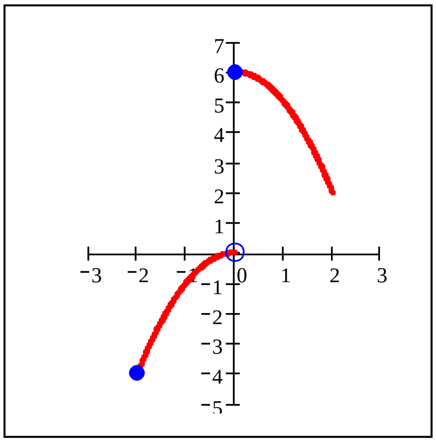
The Weierstrass theorem gives us sufficient conditions for the existence of a global minimum and a global maximum, which means that if the given conditions hold, the global minimum and maximum exist. But the fulfilment of those conditions is not necessary for the existence of a global minimum and maximum.

Example 2. $M = (-1, 2]$, $f(x) = x^2$.



In this example, the function $f(x)$ is continuous while M is not a compact set, but the global minimum and maximum are attained. ▯

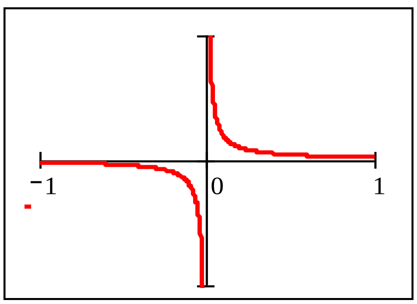
Example 3. $M = [-2, 2]$, $f(x) = \begin{cases} -x^2, & x \in [-2, 0), \\ -x^2 + 6, & x \in [0, 2]. \end{cases}$



In this example $\boldsymbol{M}$ is a compact set, but $\boldsymbol{f}(\boldsymbol{x})$ is not a continuous function. Nevertheless, the global minimum and maximum are still attained. □

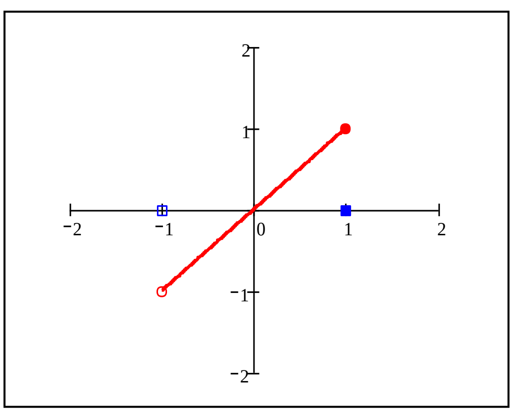
As we have seen, one of the conditions of the Weierstrass theorem is not fulfilled in the second and third examples, but the global minimum and maximum are reached. In the following two examples (4 and 5) a violation of one of the conditions of the Weierstrass theorem leads to the fact that the global minimum, or the global maximum, or both, are not reached.

Example 4. $\boldsymbol{M} = [-\mathbf{1}, \mathbf{1}]$, $\boldsymbol{f}(\boldsymbol{x}) = \begin{cases} 1/\boldsymbol{x}, & \boldsymbol{x} \neq \mathbf{0}, \\ \mathbf{0} & \boldsymbol{x} = \mathbf{0}\,. \end{cases}$



In this example, $\boldsymbol{M}$ is a compact set, the function $\boldsymbol{f}(\boldsymbol{x})$ is not continuous. The global maximum and minimum are not reached. □

Example 5. $\boldsymbol{M} = [-\mathbf{1}, \mathbf{1}]$, $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$.



In this example, $\boldsymbol{M}$ is not a compact set, the $\boldsymbol{f}(\boldsymbol{x})$ function is continuous. The global maximum is reached, the global minimum is not. □

Let's formulate the corollary of the Weierstrass theorem, where the condition on the feasible set is weakened, but the condition on the objective function is added. We will formulate it for both types of the extrema, however, the basic version is formulated for the minimum, while the maximum case is given in parentheses.

Corollary. Let $\boldsymbol{M} \subset \boldsymbol{R_n}$ be a closed set. $\boldsymbol{f}: \boldsymbol{M} \to \boldsymbol{R}$ is a continuous function and the following $\lim\limits_{x \in M, \|x\| \to \infty} \boldsymbol{f}(\mathbf{x}) = +\infty \quad (\lim\limits_{x \in M, \|x\| \to \infty} \boldsymbol{f}(\mathbf{x}) = -\infty)$

Then there exists the global minimum (maximum) in the extremal problem (2.1).

Proof. Consider the case of the minimum, the case of the maximum is similar.

Let's note, that $\lim\limits_{x \in M, \|x\| \to \infty} \boldsymbol{f}(\mathbf{x}) = +\infty$ means the following: for every number $\alpha > 0$ there exist a number $\mathbf{r} > \mathbf{0}$ such that

$$(\boldsymbol{x} \in \boldsymbol{M}, \ \|\mathbf{x}\| > \boldsymbol{r}) \ \Rightarrow \ \boldsymbol{f}(\mathbf{x}) > \boldsymbol{\alpha}. \qquad (2.4)$$

Let $\mathbf{x_0} \in \boldsymbol{M}$ and $\boldsymbol{\alpha} > \mathbf{f}(\mathbf{x_0})$. For some $\mathbf{r} > \mathbf{0}$ (2.4) holds. We can take $\mathbf{r} > \mathbf{0}$ so big, that $\|\mathbf{x_0}\| \le \boldsymbol{r}$. Because the $\boldsymbol{M} \cap \overline{\boldsymbol{B_r}(\mathbf{0})}$ is a compact set, then by virtue of the Weierstrass theorem, there exist $\hat{\boldsymbol{x}} \in \boldsymbol{M} \cap \overline{\boldsymbol{B_r}(\mathbf{0})}$, such that

$$\boldsymbol{f}(\hat{\mathbf{x}}) \le \boldsymbol{f}(\mathbf{x}), \qquad \forall \mathbf{x} \in \boldsymbol{M} \cap \overline{\boldsymbol{B_r}(\mathbf{0})} \qquad (2.5)$$

particularly, $\boldsymbol{f}(\hat{\mathbf{x}}) \le \boldsymbol{f}(\mathbf{x_0}) < \boldsymbol{\alpha}$, In addition, by virtue of (2.4)

$$\boldsymbol{\alpha} < \boldsymbol{f}(\mathbf{x}), \text{ if } \mathbf{x} \in \boldsymbol{M}, \ \|\mathbf{x}\| > \boldsymbol{r}. \qquad (2.6)$$

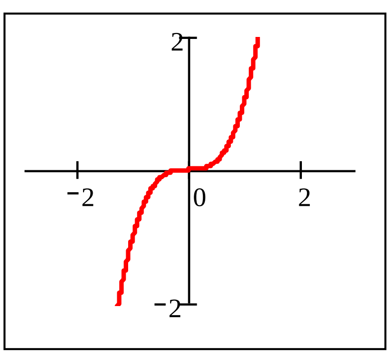
(2.5) and (2.6) together give us:

$$\boldsymbol{f}(\hat{\mathbf{x}}) \le \boldsymbol{f}(\mathbf{x}), \qquad \forall \mathbf{x} \in \boldsymbol{M} \qquad \square$$

In the case of one variable, when n=1, the condition $\lim\limits_{\|x\| \to \infty} \boldsymbol{f}(\mathbf{x}) = +\infty \ (-\infty)$ means that $\lim\limits_{x \to \infty} \boldsymbol{f}(\mathbf{x}) = +\infty \ (-\infty)$ and $\lim\limits_{x \to -\infty} \boldsymbol{f}(\mathbf{x}) = +\infty \ (-\infty)$

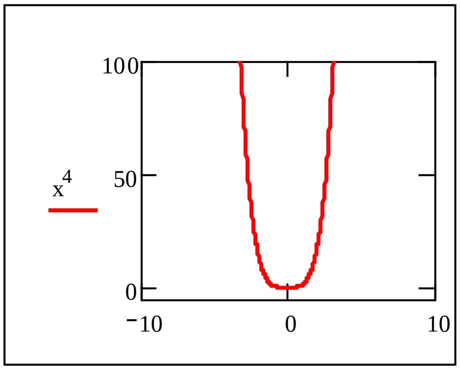
Let us consider the examples.

Example 6. $\boldsymbol{M} = \boldsymbol{R}, \ \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}^3$.



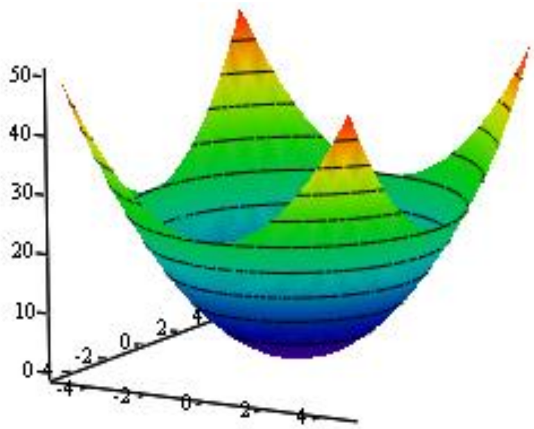
In this example $\boldsymbol{M}$ is a closed set, $\boldsymbol{f}(\boldsymbol{x})$ is a continuous function. In this case, neither $\lim\limits_{x \in M, \|x\| \to \infty} \boldsymbol{f}(\boldsymbol{x}) = +\infty$, nor $\lim\limits_{x \in M, \|x\| \to \infty} \boldsymbol{f}(\boldsymbol{x}) = -\infty$ holds. Both, the global minimum, and the global maximum do not exist.

Example 7.



In this example $\boldsymbol{M}$ is a closed set, the function $\boldsymbol{f}(\boldsymbol{x})$ is continuous. The condition $\lim\limits_{x\in M,\|x\|\to\infty} \boldsymbol{f}(\boldsymbol{x}) = +\infty$ is fulfilled and therefore the global minimum exists. ▯

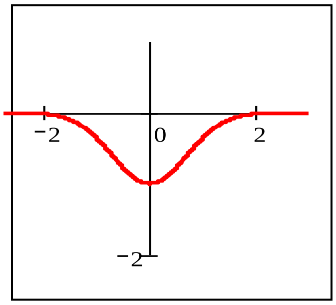
Example 8. $\boldsymbol{M} = \boldsymbol{R}^2$, $\boldsymbol{f}(\mathrm{x}) = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}^2 + \boldsymbol{y}^2$.



In this example $\boldsymbol{M}$ is a whole plane, i.e., a closed set, the function $\boldsymbol{f}(\mathrm{x})$ is continuous, the condition $\lim\limits_{x\in M,\|x\|\to\infty} \boldsymbol{f}(\boldsymbol{x}) = +\infty$ is fulfilled and therefore there is a global minimum. ▯

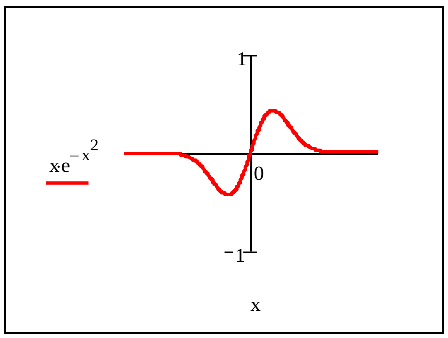
The question naturally arises as to what happens when $\lim\limits_{x\to\infty} \boldsymbol{f}(\mathbf{x})$ exists but is a finite number. It is a relatively exotic case, so we can very briefly say that when the analogous conditions of the corollary of the Weierstrass theorem are fulfilled, that is, when $\boldsymbol{M}$ is a closed set in R_n, $\boldsymbol{f}: \boldsymbol{M} \to \boldsymbol{R}$ is a continuous function and $\lim\limits_{x\in \boldsymbol{M},\|x\|\to\infty} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{k}$ for some $\boldsymbol{k} \in \boldsymbol{R}$ and it is also known that $\mathbf{f}(\boldsymbol{x}_0) < \boldsymbol{k}$ ($\boldsymbol{f}(\mathbf{x}_0) > \boldsymbol{k}$) for some $\mathbf{x}_0$, then there exists a global minimum (maximum) in problem (2.1).

Example 9. $\boldsymbol{M} = \boldsymbol{R}$, $\boldsymbol{f}(\boldsymbol{x}) = -\boldsymbol{e}^{-\boldsymbol{x}^2}$



In this example $\boldsymbol{M}$ is a closed set, the $\boldsymbol{f}(\mathrm{x})$ is a continuous function, $\lim\limits_{|x|\to\infty} \boldsymbol{f}(\mathbf{x}) = \mathbf{0}$ and $\boldsymbol{f}(\mathbf{0}) = -\mathbf{1} < \mathbf{0} = \lim\limits_{|x|\to\infty} \boldsymbol{f}(\mathbf{x})$, therefore the global minimum exists, but the global maximum - does not. □

Example 10. $\boldsymbol{M} = \boldsymbol{R}$, $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{e}^{-\boldsymbol{x}^2}$



In this example $\boldsymbol{M}$ is a closed set, the $\boldsymbol{f}(\mathrm{x})$ is a continuous function, $\lim\limits_{|x|\to\infty} \boldsymbol{f}(\mathbf{x}) = \mathbf{0}$ and $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$, however, there exists $\mathbf{x}_0' \in \boldsymbol{M}$, such that $\boldsymbol{f}(\mathbf{x}_0') < \mathbf{0}$ and there exists $\mathbf{x}_0'' \in \boldsymbol{M}$ such that $f(\mathbf{x}_0'') > \mathbf{0}$, so there exist both global minimum and global maximum.