Introduction and Basic Definitions

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Introduction

In all areas of human activity, the problem of choosing the best from several possibilities means finding the optimal possibility among them. The word “optimal” originates from the Latin “optima”, which means “the best” To choose the optimal possibility one must solve a problem of finding a maximum or a minimum, i.e., the greatest or the smallest values of some quantities. A maximum or a minimum is called an extremum. Therefore, the problems of finding a maximum or a minimum are called extremal problems. These types of problems were first introduced in antiquity, and they have been the focus of attention and relevance throughout the history of mathematics. There are extremal problems that have no useful content and are only the result of an abstraction. In contrast, extremal problems that represent the mathematical model of any real-life problem are called optimization problems. Mathematical programming studies the methods of solving optimization problems. To apply the optimization process correctly we need several steps

- To specify the problem.
- To formulate mathematical model.
- To identify whether a solution exists.
- To determine the algorithm to solve the problem.
- To solve the problem.
- To interpret the solution.

Depending on the complexity of the problem, one or more of these stages may be combined. The term mathematical programming was introduced before computers were created, and it does not necessarily mean the creation of a software code. At that time, program simply meant an algorithm, a plan. Mathematical programming is divided into several directions: linear, nonlinear, dynamic, integer, geometric programming and several others. At the same time, mathematical programming is part of an even broader discipline - operations research.

Notations

$\boldsymbol{R_n}$ - n-dimensional space of real row vectors.

$\boldsymbol{R^n}$ - n-dimensional space of real column vectors.

$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$-lower case letters indicate scalar values.

$\mathbf{x}, \boldsymbol{y}, \boldsymbol{z}$ -lower case print letters indicate vectors.

$(\boldsymbol{x^1}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n}) \in \boldsymbol{R_n}$ standard notation of row vectors.

$\boldsymbol{x^i}$ - i-th component of vector $\mathbf{x}$.

$\mathbf{x} = (\boldsymbol{x}, \boldsymbol{y})$ and $\mathbf{x} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ -traditional notations of two-dimensional and three-dimensional vectors.

$\mathbf{locmin}(\mathbf{i})$, or simply $\mathbf{locmin}(\mathbf{locmax}, \mathbf{locextr})$ is the set of local points of minimum (maximum, extremum).

$\mathbf{glmin}(\mathbf{i})$ or simply $\mathbf{glmin}(\mathbf{glmax}, \mathbf{glextr})$ is the set of global points of minimum (maximum, extremum).

$\inf\limits_{x \in M} \boldsymbol{f}$ - infimum of function $\boldsymbol{f}: \boldsymbol{M} \rightarrow \boldsymbol{R}$.

$\|\mathbf{x}\|$ - Euclidean norm of vector $\mathbf{x}$. $\|\mathbf{x}\| = (\sum_{i=1}^{n} (\boldsymbol{x^i})^2)^{1/2}$

$\boldsymbol{B_r}(\mathrm{x})$ - open ball: $\boldsymbol{B_r}(\mathrm{x}) = \{\mathbf{y} | \ \|\mathbf{y} - \mathrm{x}\| < \boldsymbol{r}\}$

$\overline{\boldsymbol{B_r}(\mathrm{x})}$ - closed ball: $\overline{\boldsymbol{B_r}(\mathrm{x})} = \{\mathbf{y} | \ \|\boldsymbol{y} - \boldsymbol{x}\| \leq \boldsymbol{r}\}$

$\boldsymbol{R_+}$ - a set of non-negative real numbers.

Basic Definitions

Definition 1. The following notation:

$$\boldsymbol{f}(\mathrm{x}) \rightarrow \boldsymbol{min}, \qquad \mathrm{x} \in \boldsymbol{M}, \tag{1.1}$$

Where $\boldsymbol{f} : \boldsymbol{X} \rightarrow \boldsymbol{R}$, $\boldsymbol{X}$ is the open set in $\boldsymbol{R^n}$ and $\boldsymbol{M} \subset \boldsymbol{X} \subset \boldsymbol{R^n}$, is **called** the finite dimensional minimization problem, or simply **minimization problem**. $\boldsymbol{f}$ is called the **objective function** or criterion, $\boldsymbol{M}$ is called the **feasible set**. Solving the problem (1.1) means that we must find both the **minimum points** or the minimums and the **minimum value**s of the objective function. □

Solving (1.1) means to find out two issues: whether or not the minimum points of the problem (1.1) exist, and how to find them.

There are two types of points of a minimum.

Definition 2. The feasible point $\hat{\mathbf{x}} \in \boldsymbol{M}$ is called the **global minimum** for (1.1) problem, or a point of **global minimum** of the function $\boldsymbol{f}(\cdot)$, over $\boldsymbol{M}$, if for any $\mathbf{x} \in \boldsymbol{M}$:

$$\boldsymbol{f}(\hat{\mathbf{x}}) \leq \boldsymbol{f}(\mathbf{x}),\ \forall \mathbf{x} \in \boldsymbol{M}.$$

Symbolically it can be written as follows: $\hat{\mathbf{x}} \in \boldsymbol{glmin}(\mathbf{1.1})$.

For example:

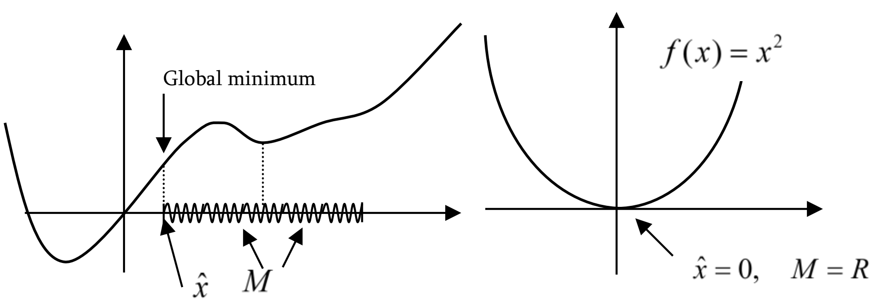

Fig.1.1.

The left function in the Fig.1.1 also has a point of local minimum.

In some problems, the global minimum does not exist. For example, $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}^3,\ \boldsymbol{M} = \boldsymbol{R}$.

Note. The following equations hold: "global" = "absolute", "local" = "relative"

Definition 3. The feasible point $\hat{\mathbf{x}} \in \boldsymbol{M}$ is called the **local minimum** for (1.1) problem, or a point **of local minimum** of the function $\boldsymbol{f}(\cdot)$, if there exists $\boldsymbol{\varepsilon} > \mathbf{0}$ such that for any

$$\mathbf{x} \in \boldsymbol{M}\ with\ \|\mathbf{x} - \hat{\mathbf{x}}\| \leq \boldsymbol{\varepsilon}: \qquad \boldsymbol{f}(\hat{\mathbf{x}}) \leq \boldsymbol{f}(\mathbf{x})$$

Symbolically it can be written as follows: $\hat{\mathbf{x}} \in \boldsymbol{locmin}(\mathbf{1.1})$.

Definition 4. The feasible point $\hat{\mathbf{x}} \in \boldsymbol{M}$ is called the **strict local minimum** for (1.1) problem, or a point **of strict local minimum** of the function $\boldsymbol{f}(\cdot)$, if there exists $\boldsymbol{\varepsilon} > \mathbf{0}$ such that for any

$$\mathbf{x} \in \boldsymbol{M} \setminus \{\hat{\mathbf{x}}\}\ with\ \|\mathbf{x} - \hat{\mathbf{x}}\| \leq \boldsymbol{\varepsilon}: \qquad \boldsymbol{f}(\hat{\mathbf{x}}) < \boldsymbol{f}(\mathbf{x})$$

Obviously, if $\hat{\mathbf{x}} \in \boldsymbol{glmin}(\mathbf{1.1})$, then $\hat{\mathbf{x}} \in \boldsymbol{locmin}(\mathbf{1.1})$. while the converse statement is not true, as shown in the following figure:

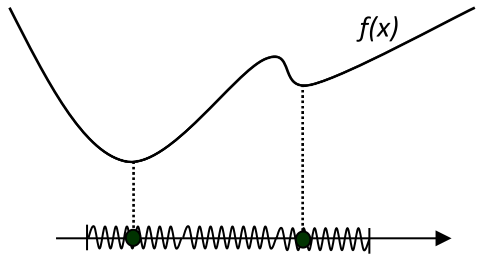

Fig 1.2.

Definition 5. The following notation:

$$\boldsymbol{f}(\mathbf{x}) \rightarrow \boldsymbol{max}, \qquad \mathbf{x} \in \boldsymbol{M}, \tag{1.2}$$

Where $\boldsymbol{f} : \boldsymbol{X} \rightarrow \boldsymbol{R}$, $\boldsymbol{X}$ is the open set in $\boldsymbol{R^n}$ and $\boldsymbol{M} \subset \boldsymbol{X} \subset \boldsymbol{R^n}$, is called the finite dimensional maximization problem, or simply **maximization problem**. $\boldsymbol{f}$ is called the **objective function** or criterion, $\boldsymbol{M}$ is called the **feasible set**. Solving the problem (1.2) means that we must **find** both the **maximum points** or the maximums and the **maximum value**s of the objective function. ▯

Definition 6. The following notation:

$$\boldsymbol{f}(\mathbf{x}) \rightarrow \boldsymbol{extr}, \qquad \mathbf{x} \in \boldsymbol{M}, \tag{1.3}$$

Where $\boldsymbol{f} : \boldsymbol{X} \rightarrow \boldsymbol{R}$, $\boldsymbol{X}$ is the open set in $\boldsymbol{R^n}$ and $M \subset X \subset R^n$and $\boldsymbol{extr} \in \{\boldsymbol{min}, \boldsymbol{max}\}$ is called the finite dimensional extremal problem, or simply **extremal problem**. $\boldsymbol{f}$ is called the **objective function**, $\boldsymbol{M}$ is called the **feasible set**. The minimum and maximum of the objective function are called its extremums, while the points of minimum and maximum of the (1.1) and (1.2) problems are the extremum points of the problem (1.3). Solving the problem (1.3) means that we must find both the **extremum points** and the **extremum value**s of the objective function. ▯

For most applications, $\boldsymbol{M} \neq \boldsymbol{R^n}$. $\mathbf{x} \in \boldsymbol{M}$ is called a condition or constraint, so when $\boldsymbol{M} \neq \boldsymbol{R^n}$, we have a constrained extremal problem. If $\boldsymbol{M} = \boldsymbol{R^n}$, then the condition $\mathbf{x} \in \boldsymbol{M}$ is automatically fulfilled, and the corresponding problem is called unconstraint extremal problem**.**

Types of constraints

Consider an extremal problem:

$$\boldsymbol{f(x)} \rightarrow \boldsymbol{extr}, \qquad \boldsymbol{x \in M}, \qquad (1.5)$$

The simpler the constraint, the easier it is to solve (1.5). In the simplest case, when $M = R^n$, we have a problem without constrains. In the following we will meet the following restrictions:

1. Equality type constraints:

$$\boldsymbol{M} = \{\boldsymbol{x} | \boldsymbol{f_1}(\mathbf{x}) = \boldsymbol{b_1}, \dots, \boldsymbol{f_m}(\mathbf{x}) = \boldsymbol{b_m}\}$$

Where the functions $\boldsymbol{f_i}: \boldsymbol{R_n} \rightarrow \boldsymbol{R}, \boldsymbol{i} = \mathbf{1}, \dots, \boldsymbol{m}$ are usually continuous, while $\boldsymbol{b_1}, \dots, \boldsymbol{b_m}$ are the given numbers.

In the case of equation-type constraints, it is convenient to write an extremal problem (1.5) as follows:

$$\boldsymbol{f(x^1, \dots, x^n)} \rightarrow \boldsymbol{extr}, \qquad \boldsymbol{f_1(x^1, \dots, x^n) = b_1, \dots, f_m(x^1, \dots, x^n) = b_m}$$

2. Inequality type constraint:

$$\boldsymbol{M} = \{\boldsymbol{x} | \boldsymbol{g_1}(\mathbf{x}) \leq \boldsymbol{b_1}, \dots, \boldsymbol{g_m}(\mathbf{x}) \leq \boldsymbol{b_m}\}$$

Where the functions $\boldsymbol{g_i}: \boldsymbol{R_n} \rightarrow \boldsymbol{R}, \boldsymbol{i} = \mathbf{1}, \dots, \boldsymbol{m}$ are continuous

In the case of inequality-type constraints the extremal problem (1.5) takes the following form, which is called the Mathematical Programming Problem:

$$\boldsymbol{f(x^1, \dots, x^n)} \rightarrow \boldsymbol{extr}, \qquad \boldsymbol{g_1(x^1, \dots, x^n \leq) b_1, \dots, g_m(x^1, \dots, x^n) \leq b_m}$$

3. Combined case, where we have simultaneously both types of constraints.