Unfolding orthogonal polyhedra

A survey of previous works

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Review of orthogonal polyhedra

- A polyhedron is *orthogonal* if its faces are:
  1. *orthogonal to each other* and
  2. *each perpendicular to the x, y, or z axis.*

  Abuse of notation: a polyhedron "is" its bounding surface.
- The *genus* of an connected orientable closed surface is one of the following equivalent data:
  - The number of "handles" it has.
  - The maximum number of cuttings along non-intersecting simple closed curves without disconnecting the surface.
  - $1 - \frac{\chi}{2}$, where χ is the *Euler characteristic*.

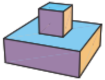

Figure: a genus 0 orthogonal polyhedron.[1]

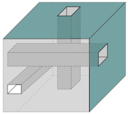

Figure: a genus 3 orthogonal polyhedron.[2]

[1] Erik D. Demaine and Joseph O'Rourke. *Geometric folding algorithms*. Linkages, origami, polyhedra. Cambridge University Press, Cambridge, 2007, pp. xiv+472. ISBN: 978-0-521-85757-4; 0-521-85757-0; 978-0-521-71522-5. DOI: 10.1017/CBO9780511735172. URL: https://doi.org/10.1017/CBO9780511735172, Fig. 22.4

[2] Mirela Damian, Erik Demaine, Robin Flatland, and Joseph O'Rourke. "Unfolding genus-2 orthogonal polyhedra with linear refinement". In: *Graphs Combin.* 33.5 (2017), pp. 1357–1379. ISSN: 0911-0119. DOI: 10.1007/s00373-017-1849-5. URL: https://doi.org/10.1007/s00373-017-1849-5, Fig. 7

Review of unfolding

- An *unfolding* of a surface is an isometry between it and a simple polygon.
- An *edge unfolding* of a polyhedron is an unfolding of a cut of a polyhedron along edges.
- Some orthogonal polyhedra have no edge unfoldings! To find more unfoldings, we need to relax the constraints. . .

Review of unfolding 2

- A *grid with $k \times k'$ refinement* on an orthogonal polyhedron is the graph formed by adding edges given by the intersection of the surface with planes along each face, then replacing each non-split rectangle with a $k \times k'$ grid of rectangles.
- A *grid unfolding of an orthogonal polyhedron with $k \times k'$ refinement* is an edge unfolding of the polyhedron with edges corresponding with a grid with $k \times k'$ refinement.

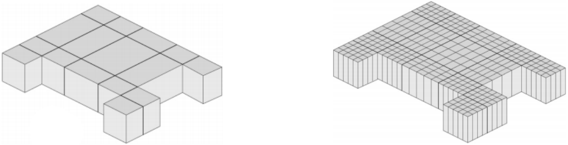

Figure: on the left is a grid. On the right is a grid with 5×4 refinement.[3]

[3] Mirela Damian, Robin Flatland, and Joseph O'Rourke. "Epsilon-unfolding orthogonal polyhedra". In: *Graphs Combin.* 23.suppl. 1 (2007), pp. 179–194. ISSN: 0911-0119. DOI: 10.1007/s00373-007-0701-8. URL: https://doi.org/10.1007/s00373-007-0701-8, Fig. 4.

The central problem

Question

What is the smallest k, k' such that every orthogonal polyhedron has a grid unfolding with $k \times k'$ refinement?

Here is a summary of many special cases of k, k' we know to work; don't sweat terminology you don't know.

Class	Refinement	Reference
Orthogonal polyhedra	Open	
Genus $g \leq 2$ orthogonal polyhedra	$O(n) \times O(n)$	[CY17; Dam+17]
Genus g one-layer polyhedra	2×1 on only $2(g-1)$ faces	[CY17]
Well-separated orthographs	2×1	[HCY17]
Orthotrees	4×4	[DF18a; DF18b]
Orthotubes	1×1	[Bie+98]
Orthostacks	2×1	[Bie+98]
Manhattan towers	5×4	[DFO08]
Regular orthogonal polyhedra w/ $x-$ and $z-$holes	2×1	[HCY17]

We'll focus mostly on the genus $g = 0$ case.

References for "the chart"

[Bie+98] Therese Biedl, Erik Demaine, Martin Demaine, Anna Lubiw, Mark Overmars, Joseph O'Rourke, Steven Robbins, and Sue Whitesides. "Unfolding Some Classes of Orthogonal Polyhedra". In: *Proc. 10th Canad. Conf. Comput. Geom.* Jan. 1998. URL: https://www.researchgate.net/publication/220991543_Unfolding_Some_Classes_of_Orthogonal_Polyhedra.

[DFO07] Mirela Damian, Robin Flatland, and Joseph O'Rourke. "Epsilon-unfolding orthogonal polyhedra". In: *Graphs Combin.* 23.suppl. 1 (2007), pp. 179–194. ISSN: 0911-0119. DOI: 10.1007/s00373-007-0701-8. URL: https://doi.org/10.1007/s00373-007-0701-8.

[DO07] Erik D. Demaine and Joseph O'Rourke. *Geometric folding algorithms*. Linkages, origami, polyhedra. Cambridge University Press, Cambridge, 2007, pp. xiv+472. ISBN: 978-0-521-85757-4; 0-521-85757-0; 978-0-521-71522-5. DOI: 10.1017/CBO9780511735172. URL: https://doi.org/10.1017/CBO9780511735172.

[DFO08] Mirela Damian, Robin Flatland, and Joseph O'Rourke. "Unfolding Manhattan towers". In: *Comput. Geom.* 40.2 (2008), pp. 102–114. ISSN: 0925-7721. DOI: 10.1016/j.comgeo.2007.07.003. URL: https://doi.org/10.1016/j.comgeo.2007.07.003.

[DDF14] Mirela Damian, Erik D. Demaine, and Robin Flatland. "Unfolding orthogonal polyhedra with quadratic refinement: the delta-unfolding algorithm". In: *Graphs Combin.* 30.1 (2014), pp. 125–140. ISSN: 0911-0119. DOI: 10.1007/s00373-012-1257-9. URL: https://doi.org/10.1007/s00373-012-1257-9.

[CY17] Yi-Jun Chang and Hsu-Chun Yen. "Improved algorithms for grid-unfolding orthogonal polyhedra". In: *Internat. J. Comput. Geom. Appl.* 27.1-2 (2017), pp. 33–56. ISSN: 0218-1959. DOI: 10.1142/S0218195917600032. URL: https://doi.org/10.1142/S0218195917600032.

[Dam+17] Mirela Damian, Erik Demaine, Robin Flatland, and Joseph O'Rourke. "Unfolding genus-2 orthogonal polyhedra with linear refinement". In: *Graphs Combin.* 33.5 (2017), pp. 1357–1379. ISSN: 0911-0119. DOI: 10.1007/s00373-017-1849-5. URL: https://doi.org/10.1007/s00373-017-1849-5.

[HCY17] Kuan-Yi Ho, Yi-Jun Chang, and Hsu-Chun Yen. "Unfolding some classes of orthogonal polyhedra of arbitrary genus". In: *Computing and combinatorics.* Vol. 10392. Lecture Notes in Comput. Sci. Springer, Cham, 2017, pp. 275–286. DOI: 10.1007/978-3-319-62389-4_23. URL: https://doi.org/10.1007/978-3-319-62389-4_23.

[DF18a] Mirela Damian and Robin Flatland. "Unfolding low-degree orthotrees with constant refinement". In: (2018). URL: http://www.cs.umanitoba.ca/~cccg2018/papers/session4B-p1.pdf.

Structure of the talk

We'll follow the following structure

1. Review terminology around unfoldings,
2. Review the epsilon-unfolding algorithm for grid-unfolding genus 0 orthogonal polyhedra with exponential refinement.
3. Introduce delta-unfolding algorithm for the genus 0 case with quadratic refinement.
4. Introduce the linear unfolding algorithm for the genus 0 case with linear refinement.
5. Sketch the extension of the linear unfolding algorithm to genus $g \leq 2$ polyhedra.

Review: layers, bands, unfolding trees, rims

We'll review terminology from the *epsilon unfolding strategy* due to Damian, Flatland, and O'Rourke[4]:

- xz faces extend to planes slicing the polyhedron into *layers*.
- Each layer is made of xz faces connecting cylindrical *bands*.
- Suppose we have genus $g = 0$. Then, bands are the vertices in an *unfolding tree*, where adjacencies are determined by z-*beams*.
- Arbitrary choice of *root node* having one rim circling a simply connected face yields a directed unfolding tree.
- Bands are bounded by circular *rims*; the *front rim* connects to a parent (if one exists) and the other is the *back rim*.

[4] Mirela Damian, Robin Flatland, and Joseph O'Rourke. "Epsilon-unfolding orthogonal polyhedra". In: *Graphs Combin.* 23.suppl. 1 (2007), pp. 179–194. ISSN: 0911-0119. DOI: 10.1007/s00373-007-0701-8. URL: https://doi.org/10.1007/s00373-007-0701-8.

Illustration of layers and bands

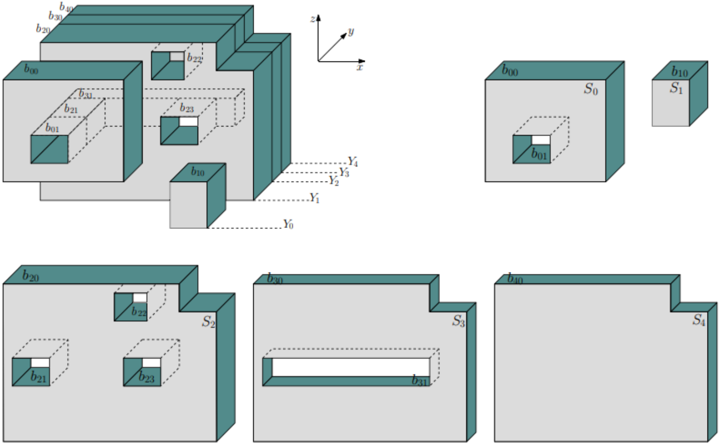

Figure: a decomposition of an orthogonal polyhedron of genus 1 into layers.[5]

[5] Mirela Damian, Erik Demaine, Robin Flatland, and Joseph O'Rourke. "Unfolding genus-2 orthogonal polyhedra with linear refinement". In: *Graphs Combin.* 33.5 (2017), pp. 1357–1379. ISSN: 0911-0119. DOI: 10.1007/s00373-017-1849-5. URL: https://doi.org/10.1007/s00373-017-1849-5, Fig. 1.

Illustration of an unfolding tree

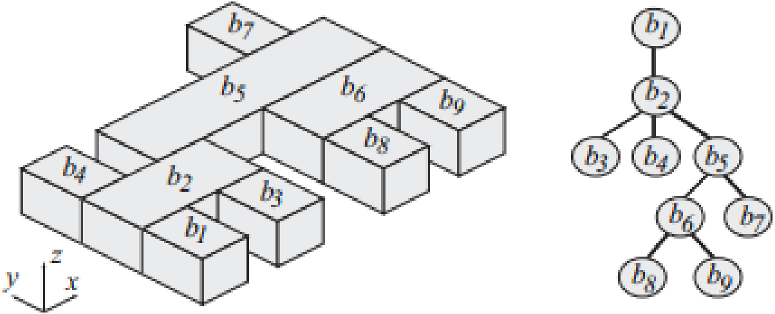

Figure: a particularly simple orthogonal polyhedron and an unfolding tree.[6]

[6] Mirela Damian, Robin Flatland, and Joseph O'Rourke. "Epsilon-unfolding orthogonal polyhedra". In: *Graphs Combin.* 23.suppl. 1 (2007), pp. 179–194. ISSN: 0911-0119. DOI: 10.1007/s00373-007-0701-8. URL: https://doi.org/10.1007/s00373-007-0701-8, Fig. 2.

Meta-outline of the strategy

We recursively construct a *spiral path* which:

- begins and ends at the front rim of the root node,
- spirals around each band at least once, and
- traverses each face at least once.

The thickened path will cover the polyhedron (modulo issues with xz faces), giving an unfolding. We construct this recursively as follows:

- With base case given by a single box, we employ a special gadget.
- Suppose we're at band b_i. We employ a gadget which spirals around the entire band and visits each child node at least once.

Single box spiral path

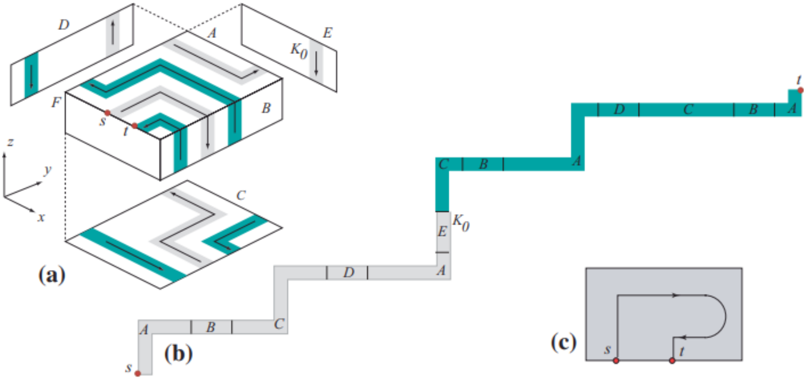

Figure: **(a)** is a spiral path and **(b)** is its unfolding. We will use diagrams such as **(c)** as shorthand[7]

[7] Mirela Damian, Robin Flatland, and Joseph O'Rourke. "Epsilon-unfolding orthogonal polyhedra". In: *Graphs Combin.* 23.suppl. 1 (2007), pp. 179–194. ISSN: 0911-0119. DOI: 10.1007/s00373-007-0701-8. URL: https://doi.org/10.1007/s00373-007-0701-8, Fig. 3.

Recursive step spiral path when all ends are trivial

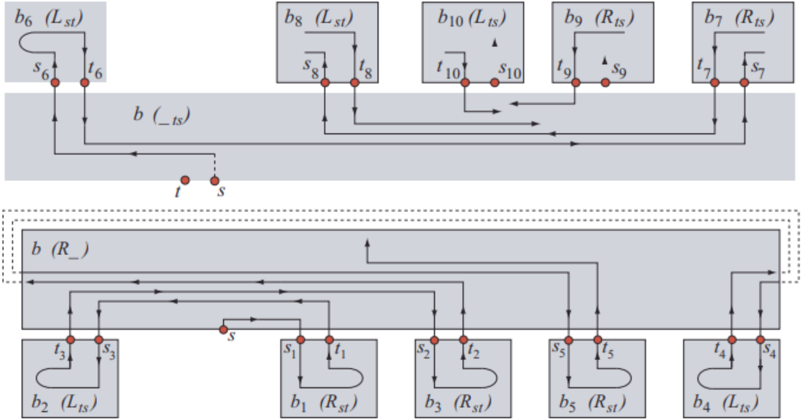

Figure: Given an initial "direction" (left or right), we travel this direction until we see a front child, which we traverse to, then exit and move the opposite direction; we do this until we finish the children, circle around the band, and execute a similar process on the back children. After the last one, we double back. We employ a familiar gadget for the back face if there are no back children.[8]

[8] Mirela Damian, Robin Flatland, and Joseph O'Rourke. "Epsilon-unfolding orthogonal polyhedra". In: *Graphs Combin.* 23.suppl. 1 (2007), pp. 179–194. ISSN: 0911-0119. DOI: 10.1007/s00373-007-0701-8. URL: https://doi.org/10.1007/s00373-007-0701-8, Fig. 4, 5.

Recursive step spiral path when all ends are trivial

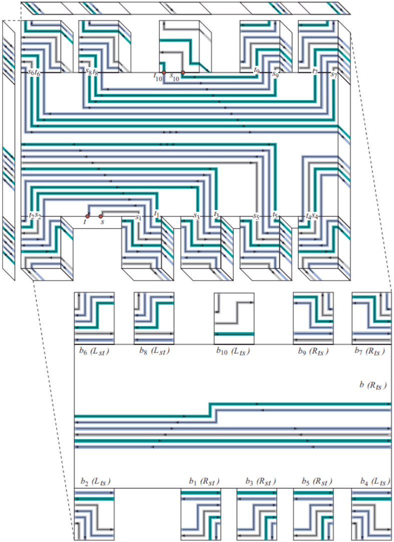

Figure: a full picture of the spiral path in the previous example. [9] We henceforth refrain from showing full examples.

Note that "doubling back" creates many paths on the middle band. This has a "path density" of $4n-2$, where n is the number of children.

We might have to "double back" many times for deep unfolding trees, which will double the path density each time. This yields an upper bound of $2^{O(n)} \times 2^{O(n)}$ refinement in general.

[9] Mirela Damian, Robin Flatland, and Joseph O'Rourke. "Epsilon-unfolding orthogonal polyhedra". In: *Graphs Combin.* 23.suppl. 1 (2007), pp. 179–194. ISSN: 0911-0119. DOI: 10.1007/s00373-007-0701-8. URL: https://doi.org/10.1007/s00373-007-0701-8, Fig. 6

More complicated recursive example

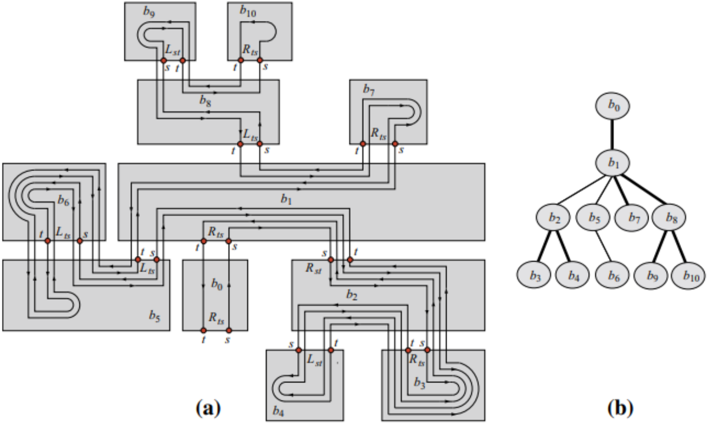

Figure: another example, with **(a)** the unfolding and **(b)** the unfolding tree. Note that the path is "quadrupled" on certain children of high depth.[10]

[10] Mirela Damian, Robin Flatland, and Joseph O'Rourke. "Epsilon-unfolding orthogonal polyhedra". In: *Graphs Combin.* 23.suppl. 1 (2007), pp. 179–194. ISSN: 0911-0119. DOI: 10.1007/s00373-007-0701-8. URL: https://doi.org/10.1007/s00373-007-0701-8, Fig. 7.

Path density of the epsilon-unfolding strategy

We can construct high depth trees which lead to high path density. The following example shows that we get $2^{\Omega(n)}$ path density, and hence $2^{\Omega(n)} \times 2^{\Omega(n)}$ refinement.

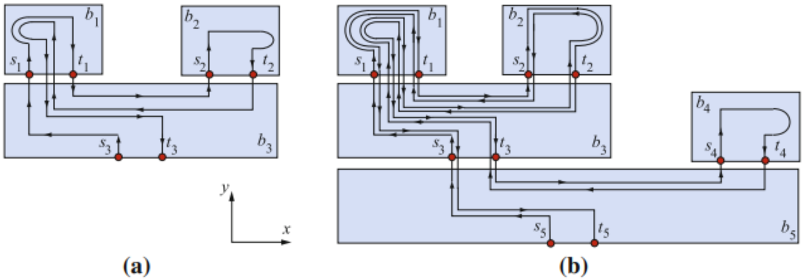

Figure: **(a)** has path density 4, **(b)** has 8, and in general this pattern yields an n box polyhedron with path density $2^{\lfloor n/2 \rfloor}$.[11]

[11] Mirela Damian, Erik D. Demaine, and Robin Flatland. "Unfolding orthogonal polyhedra with quadratic refinement: the delta-unfolding algorithm". In: *Graphs Combin.* 30.1 (2014), pp. 125–140. ISSN: 0911-0119. DOI: 10.1007/s00373-012-1257-9. URL: https://doi.org/10.1007/s00373-012-1257-9, Fig. 7.

Improvements: the delta-unfolding strategy

There have been multiple improvements on epsilon-unfolding which use the same basic strategy. The first is the *delta-unfolding strategy*, due to[12]: define a node in the unfolding tree to be *heavy* if it spans a subtree containing at least half of the children of the parent. Traverse heavy children last.

Path density is related to the number of times the spiral path "visits a child of a band." We guarantee that heavy children are visited once,

[12] Mirela Damian, Erik D. Demaine, and Robin Flatland. "Unfolding orthogonal polyhedra with quadratic refinement: the delta-unfolding algorithm". In: *Graphs Combin.* 30.1 (2014), pp. 125–140. ISSN: 0911-0119. DOI: 10.1007/s00373-012-1257-9. URL: https://doi.org/10.1007/s00373-012-1257-9.

Illustration of the delta-unfolding: heavy back child

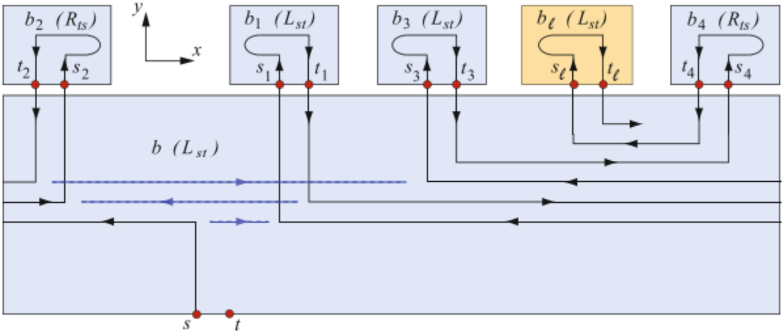

Figure: It's easy to visit heavy back children last: simply choose the correct first back child to visit, which one may choose arbitrarily.[13]

[13] Mirela Damian, Erik D. Demaine, and Robin Flatland. "Unfolding orthogonal polyhedra with quadratic refinement: the delta-unfolding algorithm". In: *Graphs Combin.* 30.1 (2014), pp. 125–140. ISSN: 0911-0119. DOI: 10.1007/s00373-012-1257-9. URL: https://doi.org/10.1007/s00373-012-1257-9, Fig. 10.

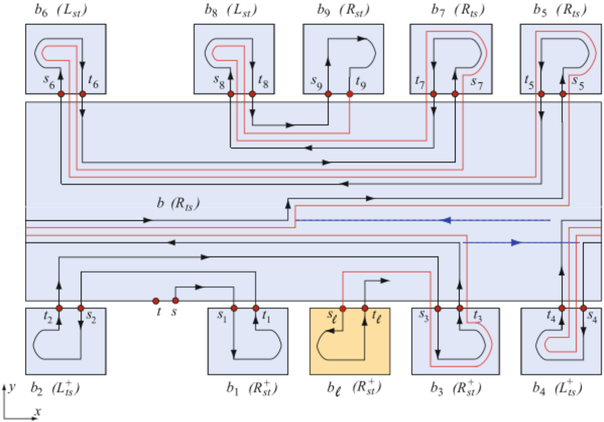

Figure: in the case of a back child, we'll actually have to visit non-heavy children four times via the above strategy.[14]

[14] Mirela Damian, Erik D. Demaine, and Robin Flatland. "Unfolding orthogonal polyhedra with quadratic refinement: the delta-unfolding algorithm". In: *Graphs Combin.* 30.1 (2014), pp. 125–140. ISSN: 0911-0119. DOI: 10.1007/s00373-012-1257-9. URL: https://doi.org/10.1007/s00373-012-1257-9, Fig. 9.

Sketch of asymptotics

Let b be a band, and let $b_1, \ldots, b_{n(b)}$ be the children of b. Suppose b_{heavy} is the heavy child of b if one exists. Write $R(n(b))$ for the asymptotic upper bound for the path density at b. Then, there are at most $O(n(b))$ segments on b's top face, so we have

$$
\begin{aligned}
R(n(b)) &= \max\left\{O(n(b)), 4 \max_{i \text{ non-heavy}} R(n(b_i)), R(n(b_{\text{heavy}}))\right\} \\
&\leq \max\left\{O(n(b)), 4 \max_{i \text{ non-heavy}} R\left(\frac{n(b)}{2}\right), R(n(b)-1)\right\} \\
&\leq \max\left\{O(n(b)), 4R\left(\frac{n(b)}{2}\right), R(n(b)-1))\right\}
\end{aligned}
$$

This leads to $R(n(b)) = O(n^2)$. This can be shown to be sharp using polyhedron forming a full binary tree.

Linear unfolding

We can do better: in fact, we can get linear refinement[15].

Instead of *top down* construction of *one spiral path*, we do the following, where $T(b)$ denotes the subtree spanned by band b.

Inducting on increasing height of $T(b)$, we construct one spiral path with endpoints on the front rim of b per leaf node in $T(b)$, such that:

- the endpoints of each spiral path are adjacent,
- no two paths cross, and
- all bands in the subtree spanned by b have been spiraled around at least once.

We then use a gadget on the root node to stitch these into one path.

[15] Mirela Damian, Erik Demaine, Robin Flatland, and Joseph O'Rourke. "Unfolding genus-2 orthogonal polyhedra with linear refinement". In: *Graphs Combin.* 33.5 (2017), pp. 1357–1379. ISSN: 0911-0119. DOI: 10.1007/s00373-017-1849-5. URL: https://doi.org/10.1007/s00373-017-1849-5.

Linear unfolding example

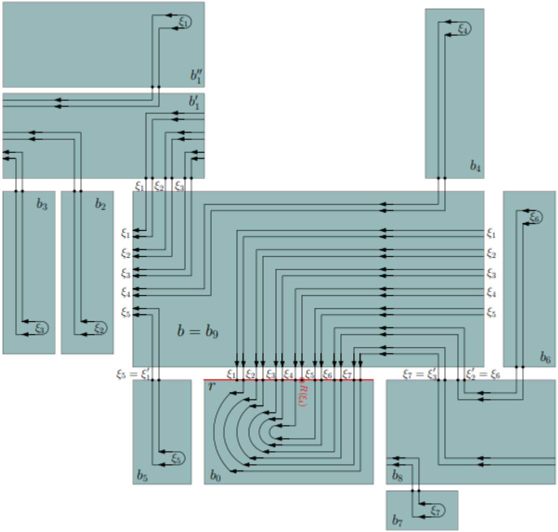

Figure: an example of the linear unfolding strategy. [16]

The directions indicated on paths correspond with the direction of the unfolding tree.

Note that layers don't "double" the path density; instead, they add $O(1)$ to the path density!

[16] Mirela Damian, Erik Demaine, Robin Flatland, and Joseph O'Rourke. "Unfolding genus-2 orthogonal polyhedra with linear refinement". In: *Graphs Combin.* 33.5 (2017), pp. 1357–1379. ISSN: 0911-0119. DOI: 10.1007/s00373-017-1849-5. URL: https://doi.org/10.1007/s00373-017-1849-5, Fig. 4

Linear unfolding 2

The above strategy explains how to work with root node having one child, and internal nodes having both front and back children; we can guarantee this by only adding $O(1)$ cleverly placed cuts.

From the above handwave, we require $O(n) \times O(n)$ refinement.

This is sharp: take an orthogonal polyhedron with $O(n)$ root nodes.

Sketch of genus $g \leq 2$ generalization

We were secretly using three (perhaps nonobvious) facts of genus 0 orthogonal polyhedra P:

1. The unfolding tree of P is acyclic (i.e. it *is* a tree).
2. There is a band with a rim enclosing a face of P which can be chosen to be a root node.
3. The back rim of each leaf node encloses a face of P.

To fix the first, we replace the first tree with a *rim unfolding graph*, and work with a spanning tree of that graph, called a *rim spanning tree.*

Fixes to our problems with genus $g \leq 2$

Suppose the genus of P is $g \leq 2$.

Lemma

There is an orientation of P such that it has a face rim which may be chosen as a root node.

Lemma

There are at most g leaves with back rims not enclosing a face.

We just need a gadget that deals with at most 2 nonface leaf nodes! They provide one in[17].

This gives grid unfolding with linear refinement for genus $g \leq 2$.

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[17] Mirela Damian, Erik Demaine, Robin Flatland, and Joseph O'Rourke. "Unfolding genus-2 orthogonal polyhedra with linear refinement". In: *Graphs Combin.* 33.5 (2017), pp. 1357–1379. ISSN: 0911-0119. DOI: 10.1007/s00373-017-1849-5. URL: https://doi.org/10.1007/s00373-017-1849-5.