Chapter 1 PLANE CURVES

planecurves version 64

##change coordinates $t, u = t^{-1}$ for $\mathbb{P}^1$ to $u, v = u^{-1}$ throughout##
affine blowing up is only in this chapter redefined in Chap 3 for projection, not used, i think

Plane curves were the first algebraic varieties to be studied. They provide examples that are helpful for understanding varieties of higher dimension, so we begin with them.

affineplane 1.1 The Affine Plane

The n-dimensional *affine space* $\mathbb{A}^n$ is the space of n-tuples of complex numbers. The *affine plane* $\mathbb{A}^2$ is the two-dimensional affine space.

If $f(x_1, x_2)$ is an irreducible polynomial in two variables with complex coefficients, the set of points X of the affine plane at which f vanishes, the *locus of zeros* of f, is called a *plane curve*, or an *affine plane curve*. Using vector notation $x = (x_1, x_2)$,

affcurve (1.1.1)
$$X \;=\; \{x \,|\, f(x) = 0\}$$

The *degree* of the curve X is the degree of its irreducible defining polynomial.

$$figure$$

polyirred **1.1.2. Note.** In contrast with polynomials in one variable, most complex polynomials in two or more variables are irreducible – they cannot be factored. This can be shown by a method called "counting constants". For instance, quadratic polynomials in x_1, x_2 depend on the coefficients of the six monomials in x_1, x_2 of degree at most two. Linear polynomials $ax_1 + bx_2 + c$ depend on three coefficients, but the product of two linear polynomials depends on only five parameters, because a scalar factor can be moved from one of the linear polynomials to the other. So the quadratic polynomials cannot all be written as products of linear polynomials. This reasoning is fairly clear. It can be justified formally in terms of *dimension*, which will be discussed in Chapter . □

As this chapter progresses, we will get some understanding of the geometry of a plane curve. Here we mention just one important point. A plane curve is called a curve because it is defined by one equation in two variables. Its *algebraic* dimension is one. But because our scalars are complex numbers, it will be a surface, geometrically. This is analogous to the fact that the *affine line* $\mathbb{A}^1$ is the plane of complex numbers.

One can see that a plane curve is a surface by inspecting its projection to the affine line. To do this, one writes its defining polynomial f as a polynomial in x_2 whose coefficients are polynomials in x_1:

$$f(x_1, x_2) = c_0(x_1)x_2^d + c_1(x_1)x_2^{d-1} + \cdots + c_d(x_1)$$

Let's suppose that f isn't a polynomial in x alone, so that d is positive. The *fibre* of a map $X \to Z$ over a point q of Z is defined to be the inverse image of q, the set of points of X that map to q. The fibre of the projection $X \to \mathbb{A}^1$ over a point $x_1 = a$ is the set of points (a, b) such that b is a root of the one-variable polynomial

$$f(a, x_2) = c_0(a)x_2^d + c_1(a)x_2^{d-1} + \cdots + c_d(a)$$

There will be finitely many points in the fibre, and the fibre won't be empty unless $f(a, x_2)$ is a constant. So X covers most of the x_1-line, a complex plane, finitely often.

(1.1.3) changing coordinates planecoords

When classifying plane curves, one allows linear changes of variable and translations. If we write x as the column vector $(x_1, x_2)^t$, the coordinates $x' = (x_1', x_2')^t$ after such a change of variable will have the form

(1.1.4) $$Qx' + a = x$$ chgcoord

where Q is an invertible 2×2 matrix with complex coefficients and $a = (a_1, a_2)^t$ is a complex translation vector. This changes a polynomial equation $f(x) = 0$, to $f(Qx'+a) = 0$. One may also multiply a polynomial f by a (nonzero) complex scalar without changing the locus $\{f = 0\}$. Using these operations, all *lines*, plane curves of degree 1, become equivalent.

An *affine conic* is a plane curve of degree two. Every equation $q(x_1, x_2) = 0$ in which q is an irreducible quadratic polynomial is equivalent by a change of coordiantes to one of the two equations

(1.1.5) $$x_1^2 - x_2^2 - 1 = 0 \quad \text{or} \quad x_1^2 - x_2 = 0$$

The proof of this is similar to the one used to classify real conics. The loci of solutions of the two equations might be called a complex 'hyperbola' and 'parabola', respectively. The complex 'ellipse' $x_1^2 + x_2^2 - 1 = 0$ becomes the hyperbola when one multiplies x_2 by i.

On the other hand, there are infinitely many inequivalent cubic curves. Cubic polynomials in two variables depend on the coefficients of the ten monomials in x_1, x_2 of degree at most 3. Linear operators, translations, and scalar multiplication give us only seven parameters to work with, leaving three essential parameters.

1.2 The Projective Plane projplane

The n-dimensional *projective space* $\mathbb{P}^n$ is the set of equivalence classes of *nonzero* vectors $x = (x_0, x_1, ..., x_n)$, the equivalence relation being

(1.2.1) $$(x_0', ..., x_n') \sim (x_0, ..., x_n) \quad \text{if} \quad (x_0', ..., x_n') = (\lambda x_0, ..., \lambda x_n)$$ equivrel

for some nonzero complex number λ. The equivalence classes are the *points* of $\mathbb{P}^n$, and one often refers to a point by a particular vector in its class. Points of $\mathbb{P}^n$ correspond bijectively to one-dimensional subspaces of $\mathbb{C}^{n+1}$. If x is a nonzero vector, the vectors λx, together with the zero vector, form the one-dimensional subspace of the complex vector space $\mathbb{C}^{n+1}$ spanned by x.

The *projective plane* is the two-dimensional projective space.

projline **(1.2.2) the projective line**

Points of the projective line $\mathbb{P}^1$ are equivalence classes of nonzero vectors (x_0, x_1). If x_0 isn't zero, we may multiply by $\lambda = x_0^{-1}$ to normalize the first entry of a point (x_0, x_1) to 1, and write the point it represents in a unique way as $(1, u)$, with $u = x_1/x_0$. There is one remaining point, the one represented by the vector $(0, 1)$. The projective line $\mathbb{P}^1$ can be obtained by adding this point, called the *point at infinity*, to the affine u-line (which is a complex plane). Topologically, the projective line is a two-dimensional sphere.

projpl **(1.2.3) lines in projective space**

A *line* L in projective space $\mathbb{P}^n$ can be described in terms of a pair of distinct points p and q, as the set of points $\{rp + sq\}$, with r, s in $\mathbb{C}$ not both zero. The points of this line correspond bijectively to points of the projective line $\mathbb{P}^1$, by

pline (1.2.4)
$$rp + sq \quad \longleftrightarrow \quad (r, s)$$

The definition of a line in a projective space of any dimension is the same.

Points of the projective plane $\mathbb{P}^2$ are equivalence classes of nonzero vectors (x_0, x_1, x_2). A line in $\mathbb{P}^2$ can also be described as the locus of solutions of a homogeneous linear equation

eqline (1.2.5)
$$s_0x_0 + s_1x_1 + s_2x_2 = 0$$

linesmeet **1.2.6. Lemma.** *Two distinct lines in the projective plane have exactly one point in common, and a pair of distinct points is contained in exactly one line.* □

standcov **(1.2.7) the standard affine cover of $\mathbb{P}^2$**

If the first entry x_0 of a point $p = (x_0, x_1, x_2)$ of $\mathbb{P}^2$ isn't zero, we may normalize it to 1 without changing the point: $(x_0, x_1, x_2) \sim (1, u_1, u_2)$, where $u_i = x_i/x_0$. We did the analogous thing for $\mathbb{P}^1$ above. The representative vector $(1, u_1, u_2)$ is uniquely determined by p, so points with $x_0 \neq 0$ correspond bijectively to points of an affine plane $\mathbb{A}^2$ with coordinates u:

$$(x_0, x_1, x_2) \sim (1, u_1, u_2) \quad \longleftrightarrow \quad (u_1, u_2)$$

We regard the affine plane as a subset of $\mathbb{P}^2$ by this correspondence, and we denote that subset by $\mathbb{U}^0$. The points of $\mathbb{U}^0$, those with $x_0 \neq 0$, are the *points at finite distance*. The *points at infinity* of $\mathbb{P}^2$, those of the form $(0, x_1, x_2)$, are on the *line at infinity* L^0, the locus $\{x_0 = 0\}$. The projective plane is the union of the two sets $\mathbb{U}^0$ and L^0, so when a point is given, we can assume that its first coordinate is either 1 or 0.

When looking at a point of $\mathbb{U}^0$, we may simply set $x_0 = 1$, and write the point as $(1, x_1, x_2)$. To write $u_i = x_i/x_0$ makes sense only when a particular vector (x_0, x_1, x_2) has been given.

There is an analogous correspondence between points $(x_0, 1, x_2)$ and points of an affine plane $\mathbb{A}^2$, and between points $(x_0, x_1, 1)$ and points of $\mathbb{A}^2$. We denote the subsets $\{x_1 \neq 0\}$ and $\{x_2 \neq 0\}$ by $\mathbb{U}^1$ and $\mathbb{U}^2$, respectively. The three sets $\mathbb{U}^0, \mathbb{U}^1, \mathbb{U}^2$ form the *standard covering* of $\mathbb{P}^2$ by three *standard affine open sets*. Since the vector $(0, 0, 0)$ has been ruled out, every point of $\mathbb{P}^2$ lies in at least one of these sets. Points whose three coordinates aren't zero lie in all three.

figure

pointatinfinity **1.2.8. Note.** Which points of $\mathbb{P}^2$ are at infinity depends on which of the standard affine open sets is taken to be the one at finite distance. When the coordinates are (x_0, x_1, x_2), I like to normalize x_0 to 1, as above. Then the points at infinity are those of the form $(0, x_1, x_2)$. But when coordinates are (x, y, z), I may normalize z to 1. Then the points at infinity are the points $(x, y, 0)$. I hope this won't cause too much confusion. □

(1.2.9) aside: the real projective plane realprojplane

The *real projective plane* $\mathbb{RP}^2$ is the set of equivalence classes of nonzero real vectors (x_0, x_1, x_2), the equivalence relation being $(x') \sim (x)$ if $(x') = \lambda(x)$ for some real number λ. It can be thought of as the space of one-dimensional subspaces of the real vector space of dimension three.

Let V denote the real vector space $\mathbb{R}^3$, and let U be the plane $\{x_0 = 1\}$ in V. This plane is analogous to the open subset $\mathbb{U}^0$ of the complex projective plane $\mathbb{P}^2$.

We can project V from the origin o to U, sending a point (x_0, x_1, x_2) of V to the point $(1, u_1, u_2)$, with $u_i = x_i/x_0$. Looking from the origin, U becomes a "picture plane".

figure

Dürer drawing of perspective

The projection to U is undefined at the points $(0, x_1, x_2)$. Lines through the origin that are orthogonal to the x_0-axis don't meet U. They correspond to the points at infinity of $\mathbb{RP}^2$.

The history of the real projective plane is complicated. The projection from 3-space to a picture plane goes back to the the early 16th century, the time of Desargues and Dürer. Bot projective coordinates were introduced 200 years later, by Möbius.

(1.2.10) changing coordinates in the projective plane chgcoordssec

An invertible 3×3 matrix P determines a linear change of coordinates in $\mathbb{P}^2$. With $x = (x_0, x_1, x_2)^t$ and $x' = (x'_0, x'_1, x'_2)^t$ represented as column vectors, the coordinate change is given by

(1.2.11) $$Px' = x$$ chg

As the next proposition shows, four special points, the three points $e_0 = (1,0,0)^t, e_1 = (0,1,0)^t, e_2 = (0,0,1)^t$ and the point $\epsilon = (1,1,1)^t$ determine the coordinates.

1.2.12. Proposition. *Let p_0, p_1, p_2, q be four points of $\mathbb{P}^2$, no three of which lie on a line. There is, up to scalar factor, a unique linear coordinate change $Px' = x$ such that $Pp_i = e_i$ and $Pq = \epsilon$.* fourpoints

proof. We represent the points by specific vectors. The statement that p_0, p_1, p_2 don't lie on a line means that those three vectors are independent. They span $\mathbb{C}^3$. So q will be a combination $c_0p_0 + c_1p_1 + c_2p_2$, and because no three points lie on a line, the coefficients c_i will be nonzero. We can *scale* the vectors p_i (multiply them by nonzero scalars) to make $q = p_0 + p_1 + p_2$. Next, the columns of P can be an arbitrary set of independent vectors. We let them be p_0, p_1, p_2. Then $Pe_i = p_i$, and $P\epsilon = q$. The matrix P is unique up to scalar factor, as is verified by looking the reasoning over. □

(1.2.13) conics projconics

A polynomial $f(x_0, x_1, x_2)$ is *homogeneous, of degree d*, if all monomials that appear with nonzero coefficient have degree d. For example, $x_0^3 + x_1^3 - x_0x_1x_2$ is a homogeneous cubic polynomial.

A *conic* is the locus of zeros of an irreducible homogeneous quadratic polynomial, a combination of the six monomials

$$x_0^2,\ x_1^2,\ x_2^2,\ x_0x_1,\ x_1x_2,\ x_0x_2$$

1.2.14. Proposition. *For any conic C, there is a choice of coordinates so that C becomes the locus* classifyconic

$$x_0x_1 + x_0x_2 + x_1x_2 = 0$$

proof. Since the conic C isn't a line, it will contain three points that aren't colinear. Let's leave the verification of this fact as an exercise. We choose three non-colinear points, and adjust coordinates so that they become the points e_0, e_1, e_2. Let f be the quadratic polynomial in those coordinates whose zero locus is C. Then $f(1,0,0) = 0$, and therefore the coefficient of x_0^2 in f is zero. Similarly, the coefficients of x_1^2 and x_2^2 are zero. So f has the form

$$f = ax_0x_1 + bx_0x_2 + cx_1x_2$$

Since f is irreducible, a, b, c aren't zero. By scaling the variables appropriately, we can make $a = b = c = 1$. We will be left with the polynomial $x_0x_1 + x_0x_2 + x_1x_2$. □

1.3 Plane Projective Curves

projcurve

Algebraic geometry studies the loci in projective space that are defined by systems of *homogeneous* polynomial equations. Homogeneity is required because the vectors $(a_0, ..., a_n)$ and $(\lambda a_0, ..., \lambda a_n)$ represent the same point of $\mathbb{P}^n$. One wants to know that if $f(x) = 0$ is a polynomial equation, and if $f(a) = 0$, then $f(\lambda a) = 0$ for every $\lambda \neq 0$. As we verify now, this will be true if and only if f is homogeneous.

A polynomial f can be written as a sum of its *homogeneous parts*:

homparts (1.3.1)
$$f = f_0 + f_1 + \cdots + f_d$$

where f_0 is the constant term, f_1 is the linear part, etc., and d is the degree of f.

hompartszero **1.3.2. Lemma.** *Let f be a polynomial of degree d, and let $x = (x_0, ..., x_n)$. Then $f(\lambda x) = 0$ for every nonzero complex number λ if and only if $f_i(x)$ is zero for all $i = 0, ..., d$.*

proof. $f(\lambda x_0, ..., \lambda x_n) = f_0 + \lambda f_1(x) + \lambda^2 f_2(x) + \cdot + \lambda^d f_d(x)$. When we evaluate at a given vector x, the right side of this equation becomes a polynomial of degree at most d in λ. Since a nonzero polynomial of degree at most d can have at most d roots, $f(\lambda x)$ will not be zero for *every* λ unless that polynomial is zero – unless $f_i(x) = 0$ for every i. □

locipone **(1.3.3) loci in the projective line**

Before going to plane curves, we describe the zeros in $\mathbb{P}^1$ of a homogeneous polynomial in two variables.

factorhom-poly **1.3.4. Lemma.** *Every nonzero homogeneous polynomial $f(x, y) = a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d$ with complex coefficients is a product of homogeneous linear polynomials that are unique up to scalar factor.*

To prove this, one factors the one-variable complex polynomial $f(x, 1)$ into linear factors, substitutes x/y for x, and multiplies the result by y^d. When one adjusts scalar factors, one will obtain the expected factorization of $f(x, y)$. For instance, to factor $f(x, y) = 2x^2 - 3xy + y^2$, substitute $y = 1$: $2x^2 - 3x + 1 = 2(x-1)(x-\frac{1}{2})$. Substitute $x = x/y$ and multiply by y^2: $f(x, y) = 2(x-y)(x-\frac{1}{2}y)$. The scalar 2 can be distributed arbitrarily among the factors. For instance, $f(x, y) = (x - y)(2x - y)$. □

Adjusting scalar factors, we may write a homogeneous polynomial as a product

factorpoly-two (1.3.5)
$$f(x, y) = (v_1x - u_1y)^{r_1} \cdots (v_kx - u_ky)^{r_k}$$

where no factor $v_ix - u_iy$ is a constant multiple of another, and where $r_1 + \cdots + r_k$ is the degree d of f. The exponent r_i is the *multiplicity* of the linear factor $v_ix - u_iy$.

A linear polynomial $vx - uy$ corresponds to the point (u, v) in the projective line $\mathbb{P}^1$, the unique *zero* of that polynomial, and changing the polynomial by a scalar factor doesn't change its zero. Thus the linear factors of the homogeneous polynomial (1.3.5) determine points of $\mathbb{P}^1$, the *zeros* of f. As with the roots of a one-variable polynomial, we can assign multiplicities to those zeros. The points (u_i, v_i) are zeros of *multiplicity* r_i.

The zero (u_i, v_i) of f corresponds to a root $x = u_i/v_i$ of multiplicity r_i of the one-variable polynomial $f(x, 1)$, except when the zero is the point $(1, 0)$. This happens when the coefficient a_0 of f is zero, and y is a factor of f. One might say that $f(x, 1)$ has a root at infinity in that case.

This sums up the information contained in an algebraic locus in the projective line. It will be a finite set of points with multiplicities.

(1.3.6) intersections with a line intersectline

Let Z be the zero locus in $\mathbb{P}^n$ of a homogeneous polynomial $f(x_0,...,x_n)$ of degree d, and let L be a line in $\mathbb{P}^n$ (1.2.4). Say that L is the set of points $rp+sq$, where $p=(a_0,...,a_n)$ and $q=(b_0,...,b_n)$, so that L corresponds to the projective line $\mathbb{P}^1$ by $rp+sq \leftrightarrow (r,s)$. Let's also assume that L isn't a subset of Z. Then the intersection $Z\cap L$ corresponds to a subset of $\mathbb{P}^1$ that is obtained as follows: We substitute $rp+sq$ into f, obtaining a homogeneous polynomial $\widetilde{f}(r,s)$ of degree d in r,s. For example, if $n=2$ and if $f=x_0x_1+x_0x_2+x_1x_2$, then $\widetilde{f}$ is the following quadratic polynomial in r,s:

$$\begin{aligned}\widetilde{f}(r,s)=f(rp+sq)&=(ra_0+sb_0)(ra_1+sb_1)+(ra_0+sb_0)(ra_2+sb_2)+(ra_1+sb_1)(ra_2+sb_2)\\&=(a_0a_1+a_0a_2+a_1a_2)r^2+\big(\textstyle\sum_{i\neq j}a_ib_j\big)rs+(b_0b_1+b_0b_2+b_1b_2)s^2\end{aligned}$$

The zeros of $\widetilde{f}$ in $\mathbb{P}^1$ correspond to the points of $Z\cap L$, and with multiplicities as described above, there will be d of them. □

1.3.7. Definition With notation as above, the *intersection multiplicity* of Z and L at a point p is the multiplicity of zero of the polynomial $\widetilde{f}$. intersect-linetwo

1.3.8. Corollary. *Let Z be the zero locus in $\mathbb{P}^n$ of a homogeneous polynomial f, and let L be a line in $\mathbb{P}^n$ that isn't contained in Z. When counted with multiplicity, the number of intersections of Z and L is equal to the degree of f.* □ XcapL

(1.3.9) loci in the projective plane lociptwo

If a homogeneous polynomial f is a product, say $f=f_1f_2$, then the locus $\{f=0\}$ is the union of the two loci $\{f_1=0\}$ and $\{f_2=0\}$. This is rather obvious. What isn't obvious is that homogeneous polynomials f and g with no common factor have finitely many common zeros. This is proved below, in Proposition 1.3.13.

1.3.10. Corollary. *Any locus in $\mathbb{P}^2$ defined by a system of homogeneous polynomial equations is a finite union of points and curves.* □ pointscurves

The most interesting loci in the projective plane are the zero sets of single irreducible homogeneous polynomial equations. When f is an irreducible homogeneous polynomial, the locus $\{f=0\}$ is called a *(projective) plane curve*. The *degree* of a plane curve is the degree of its irreducible defining polynomial.

As is true for curves in the affine plane, a plane projective curve will have geometric dimension two. The case of a line, which is homeomorphic to the two-dimensional sphere $\mathbb{P}^1$ illustrates this.

The zero locus of a reducible homogeneous polynomial may be called a *reducible curve*. To keep track of multiple factors of a reducible polynomial f, one can associate an integer combination of curves, called a *divisor*, to it. One writes f as a product of irreducible polynomials, say

(1.3.11) $$f=g_1^{r_1}\cdots g_k^{r_k},$$ factorf

where g_i are irreducible polynomials and where g_j isn't a scalar multiple of g_i if $i\neq j$. If C_i is the plane curve $\{g_i=0\}$, the associated *divisor* is defined to be the integer combination

(1.3.12) $$Z=r_1C_1+\cdots+r_kC_k$$ divisoroff

1.3.13. Proposition. *Homogeneous polynomials $f_1,...,f_r$ in x,y,z whose greatest common divisor is* 1 *have finitely many common zeros.* fgzerofinite

We make a small digression before proving the proposition. The ring $\mathbb{C}[x,y]$ embeds into its field of fractions $F=\mathbb{C}(x,y)$, the field of rational functions in x,y, and one may study the polynomial ring $\mathbb{C}[x,y,z]$ as a subring of the one-variable polynomial ring $F[z]$. This is a useful method because $F[z]$ is a principal ideal domain. Its algebra is simpler.

relprime **1.3.14. Lemma.** *Let $f_1, ..., f_k$ be homogeneous polynomials in x, y, z with no common factor. Their greatest common divisor in $F[z]$ is 1, and therefore there is an equation of the form $\sum g_i' f_i = 1$ with g_i' in $F[z]$.*

proof. Let $\widetilde{h}$ be an element of $F[z]$ that divides $f_1, ..., f_k$, say $f_i = \widetilde{u}_i \widetilde{h}$, and suppose that $\widetilde{h}$ isn't a unit (an element of F). The elements $\widetilde{u}_i$ and $\widetilde{h}$ are polynomials in z whose coefficients are in F. When we clear denominators from the coefficients, we will obtain equations of the form $d_i f_i = u_i h$, where d_i are polynomials in x, y and u_i, h are polynomials in x, y, z. Since $\widetilde{h}$ isn't in F, $\widetilde{h}$ and h have positive degree in z.

Let s be an irreducible factor of h of positive degree in z. Then s divides $d_i f_i$ but doesn't divide d_i which has degree zero in z, so s divides f_i for all i. This contradicts the hypothesis that $f_1, ..., f_k$ have no common factor. $\square$

proof of the proposition. The lemma tells us that we may write $\sum g_i' f_i = 1$, with g_i' in $F[z]$. Clearing denominators from g_i' gives us an equation of the form

$$\sum g_i f_i = d$$

where d is a polynomial in x, y and g_i are polynomials in x, y, z. Taking suitable homogeneous parts of g_i and d produces an equation $\sum g_i f_i = d$ in which all terms are homogeneous.

Lemma 1.3.4 tells us that d is a product of linear polynomials, say $d = \ell_1 \cdots \ell_k$. A common zero of f_i is also a zero of d, and therefore it is a zero of ℓ_j for some j. It suffices to prove that for each j, the polynomials $f_1, ..., f_k, \ell_j$ have finitely many common zeros. Since $f_1, ..., f_k$ have no common factor, there must be at least one f_i that isn't divisible by ℓ_j. Corollary 1.3.8 shows that f_1 and f_i have finitely many common zeros. $\square$

The next corollary is a special case of the Strong Nullstellensatz that will be proved in the next chapter.

idealprincipal **1.3.15. Corollary.** *Let S be an infinite set of points of $\mathbb{P}^2$, and let f be an irreducible homogeneous polynomial that vanishes on S. If another homogeneous polynomial g vanishes on S, then f divides g. Therefore, if an irreducible polynomial vanishes on an infinite set S, that polynomial is unique up to scalar factor.*

proof. If the irreducible polynomial f doesn't divide g, then f and g have no common factor, and therefore they have finitely many common zeros. $\square$

classical-topology **(1.3.16) the classical topology**

The usual topology on the affine plane $\mathbb{A}^2$ will be called the *classical topology*. A subset U of $\mathbb{A}^2$ is open in the classical topology if, whenever U contains a point p, it contains all points sufficiently near to p. We call this the classical topology to distinguish it from another topology, the *Zariski topology*, that will be described in the next chapter.

The projective plane also has a classical topology. A subset U of $\mathbb{P}^2$ is open if, whenever a point p of U is represented by a vector (x_0, x_1, x_2), all vectors (x_0', x_1', x_2') with x_i' sufficiently near to x_i represent points of U.

isopts **(1.3.17) isolated points**

A point p of a topological space X is *isolated* if both $\{p\}$ and its complement $X - \{p\}$ are closed sets. If X is a subset of $\mathbb{A}^n$ or $\mathbb{P}^n$, a point p of X is isolated (in the classical topology) if X doesn't contain points p' distinct from, but arbitrarily close to, p.

noisolated-point **1.3.18. Proposition** *Let n be an integer > 1. The zero locus of a polynomial in $\mathbb{A}^n$ or $\mathbb{P}^n$ contains no isolated points.*

The proof is below.

polyfunction **1.3.19. Lemma.** *A formal polynomial, an element $f(x_1, ..., x_n)$ of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$, is determined by the function that it defines on $\mathbb{C}^n$.*

This lemma shows that we needn't be careful to distinguish formal polynomials from polynomial functions.

proof. To show that formal polynomials f and g which define the same function are equal, we replace f by $f-g$. Then what we must show is that if the function defined by a formal polynomial f is identially zero, then f is the zero polynomial. If f defines the zero function, its partial derivatives are zero too. The partial derivatives are the functions defined by the formal partial derivatives of f. Since the partials have degree $d-1$, we may use induction on the degree to conclude that the formal partial derivatives are zero. This implies that f is a constant, and if f defines the zero function, that constant is zero. □

1.3.20. Lemma. *Let f be a polynomial of degree d in the variables $x_1, ..., x_n$. There is a linear change of variable $Px' = x$, where P is an invertible $n \times n$matrix, such that $f(Px')$ is a monic polynomial of degree d in x'_n.* polymonic

proof. We write $f = f_0 + f_1 + \cdots + f_d$, where f_i is the homogeneous part of f of degree i, and we choose a point p of $\mathbb{A}^n$ at which f_d isn't zero. We change variables so that p becomes the point $(0, ..., 0, 1)$, and we call the new variables $x_1, .., .x_n$. With this change of variable, $f_d(0, ..., 0, x_n) = cx_n^d$ for some nonzero constant c. We can adjust x_n by a scalar factor to make $c = 1$. Then f will be monic. □

proof of Proposition 1.3.18. The proposition is true both for loci in affine space and for loci in projective space. We look at the affine case. Let $f(x_1, ..., x_n)$ be a polynomial with zero locus Z, and let p be a point of Z. We adjust coordinates so that p is the origin $(0, ..., 0)$ and f is monic in x_n. We relabel x_n as y, so that the variables are $x_1, ..., x_{n-1}, y$ and f has the form

$$f = y^d + a_{d-1}(x)y^{d-1} + \cdots + a_0(x)$$

where $x = x_1, ..., x_{n-1}$. When we fix x, $a_0(x)$ is the product of the roots of $f(x, y)$, considered as a polynomial in the variable y. Since p is the origin and $f(p) = 0$, $a_0(0) = 0$. So $a_0(x)$ will tend to zero with x. Then at least one root y of $f(x, y)$ will tend to zero. This gives us points (x, y) of Z that are arbitrarily close to p. □

1.3.21. Corollary. *Let U be the complement of a finite set of points in a plane curve C. A continuous function g on C that is zero at every point of U is identically zero.* □ functioniszero

1.4 Tangent Lines tanlines

(1.4.1) homogenizing and dehomogenizing homdehomone

We will often want to inspect a projective curve $C : \{f(x) = 0\}$ in a neighborhood of a particular point p. To do this we may adjust coordinates so that p is the point $(1, 0, 0)$ and look in the standard affine open set $\mathbb{U}^0$. The intersection C^0 of C with $\mathbb{U}^0$ will be the zero locus of the non-homogeneous polynomial $f(1, x_1, x_2)$, and p will be the origin in the affine x_1, x_2-plane. We call $f(1, x_1, x_2)$ the *dehomogenization* of f. Proposition 1.4.2 below asserts that the dehomogenization of an irreducible polynomial is irreducible.

A simple procedure called *homogenization* inverts dehomogenization. Its description will be clear when we describe dehomogenization again, in a way that is obviously invertible.

Let f be a homogeneous polynomial of degree d, and let $u_i = x_i/x_0$, so that $u_0 = 1$. To dehomogenize f, we divide f by x_0^d, we put one copy of x_0 under each x_i as it appears in the polynomial, and we replace x_i/x_0 by u_i. For example, suppose that $f = 2x_0^3 + x_0^2 x_1 + x_2^3$. We divide by x_0^3:

$$(2x_0^3 + x_0^2 x_1 + x_2^3)/x_0^3 = 2u_0^3 + u_0^2 u_1 + u_2^3 = 2 + u_1 + u_2^3 = f(1, u_1, u_2)$$

The result is the dehomogenized polynomial F, except that the variables x_i have been replaced by u_i.

Now suppose given a non-homogeneous polynomial $F(x_1, x_2)$ of degree d. To *homogenize* F, we invert this process. We replace the variables x_i by $u_i = x_i/x_0$, and then multiply by x_0^d. Since u_i has degree zero in x, so does $F(u_1, u_2)$. The result will be a homogeneous polynomial of degree d, and it won't be divisible by x_0,

Let $\mathcal{N}$ denote the space of non-homogeneous polynomials in u_1, u_2, and let $\mathcal{H}$ denote the space of homogeneous polynomials in x_0, x_1, x_2 that aren't divisible by x_0. Homogenization and dehomogenization are inverse functions

$$\mathcal{N} \longleftrightarrow \mathcal{H}$$

Homogenization and dehomogenization will be discussed again in Chapter foneirred **1.4.2. Proposition.** *A homogeneous polynomial $f(x_0,x_1,x_2)$ that isn't divisible by x_0 is irreducible if and only if its dehomogenization $f(1,x_1,x_2)$ is irreducible.*

proof. Let's denote $f(1,x_1,x_2)$ by $F(x_1,x_2)$. When f isn't divisible by x_0, the degrees of F and f will be equal. Suppose that f is a product gh of (nonconstant) homogeneous polynomials. Then $F=GH$, where $G=g(1,x_1,x_2)$ and $H=h(1,x_1,x_2)$. Since f isn't divisible by x_0, neither are g or h, so G and H have the same degrees as g and h. Therefore F isn't irreducible. Conversely, if $F=GH$, then when we homogenize, we obtain an equation $f=gh$, so f isn't irreducible. □

smsingpts **(1.4.3)** **smooth points and singular points**

Let C be the plane curve defined by an irreducible homogeneous polynomial $f(x_0,x_1,x_2)$, and let f_i denote the partial derivative $\frac{\partial f}{\partial x_i}$. A point of C at which the partial derivatives f_i aren't all zero is called a *smooth point* of C, and a point at which all partial derivatives are zero is a *singular point*.

A curve is *smooth*, or *nonsingular*, if it contains no singular point; otherwise it is a *singular* curve. The fermatcurve *Fermat curve*

$$x_0^d+x_1^d+x_2^d=0 \tag{1.4.4}$$

is smooth because the only common zeros of the partial derivatives $dx_0^{d-1},dx_1^{d-1},dx_2^{d-1}$, $(0,0,0)$, doesn't represent a point of $\mathbb{P}^2$.

The cubic curve $x_0^3+x_1^3-x_0x_1x_2=0$ is singular at the point $(0,0,1)$.

The meaning of smoothness is explained by the Implicit Function Theorem. Let p be a point of C in the standard affine open set $\mathbb{U}^0 : x_0\neq 0$. We set $x_0=1$ and inspect the locus $f(1,x_1,x_2)=0$ in $\mathbb{U}^0$. If $\frac{\partial f}{\partial x_2}$ isn't zero at p, the Implicit Function Theorem tells us that we can solve the equation $f(1,x_1,x_2)=0$ for x_2 locally as an analytic function φ of x_1. Sending x_1 to $(1,x_1,\varphi)$ inverts the projection from C to the affine x_1-line X. So at a smooth point, C is locally homeomorphic to X.

figure: node, cusp, tacnode

somesingpts

A Note about figures. In algebraic geometry, dimensions are too big to allow realistic figures. Even with a plane curve, one is dealing with a locus in the space $\mathbb{C}^2$, which is four-dimensional. In some cases, such as in the figures shown above, depicting the real locus can be helpful., but In most cases one must make do with a schematic figure. The one below is an example. My students tell me that all of my figures look more or less like this:

$$figure$$

□

eulerformula **1.4.5. Euler's Formula.** Let f be a homogeneous polynomial of degree d in the variables $x_0,...,x_n$. Then

$$\sum_i x_i\frac{\partial f}{\partial x_i}=d\,f.$$

It is enough to check this formula for monomials, which is easy. For instance, when $f=x^2y^3z$,

$$xf_x+yf_y+zf_z=x(2xy^3z)+y(3x^2y^2z)+z(x^2y^3)=6x^2y^3z=6\,f$$

□

singpointon-curve **1.4.6. Corollary.**
(i) *If all partial derivatives of a homogeneous polynomial f are zero at a point p of $\mathbb{P}^2$, then f is zero at p, and therefore p is a singular point of the curve or divisor it defines.*
(ii) *The partial derivatives of an irreducible polynomial have no common (nonconstant) factor.*
(iii) *A curve has finitely many singular points.*

proof. **(ii,iii)** Euler's Formula shows that a common factor of the partial derivatives divides f. If f is an irreducible polynomial of degree d, it can have no factor in common with its partial derivatives, which have degree $d-1$. Therefore the partial derivatives have no common factor. Proposition 1.3.13 shows that the partials have finitely many common zeros. $\square$

One can use the same definition for the divisor Z defined by a reducible polynomial $f = g_1^{r_1} \cdots g_k^{r_k}$, so that $Z = r_1C_1 + \cdots + r_kC_k$ (see (1.3.12)). A *singular point* of Z is one at which all of the partial derivatives f_i are zero. But if some r_j is greater than 1, g_j will divide all of the partial derivatives of f. Then every point of C_j will be a singular point of Z.

(1.4.7) tangent lines and flex points tangent

Let C be the plane curve defined by an irreducible homogeneous polynomial f of degree at least 2, A line L is *tangent* to C at a smooth point p if the intersection multiplicity of C and L at p is at least 2, and a smoth point p is a *flex point* of C if the intersection multiplicity at p of C with its tangent line is at least 3 (see (1.3.7)). We will see that there is a unique tangent line at a smooth point.

Let L be a line through a point p and let q be a point of L distinct from p. We represent p and q by specific vectors, and write a variable point of L as $p + tq$. We expand the restriction of f to L in a Taylor's Series, Let $f_i = \frac{\partial f}{\partial x_i}$ and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Then

(1.4.8) $$f(p+tq) \;=\; f(p) \;+\; \Big(\sum_i f_i(p)\, q_i\Big)t \;+\; \tfrac{1}{2}\Big(\sum_{i,j} q_i\, f_{ij}(p)\, q_j\Big)t^2 \;+\; O(3)$$ taylor

The point q is missing from this parametrization of L, but this isn't important.

1.4.9. Notation. The symbol $O(3)$ stands for a polynomial or a power series in which all terms have total degree at least 3 in the variables that are are involved. The only variable in Formula 1.4.8 is t. $\square$ Onotation

The intersection multiplicity of C and L at p is the lowest power of t that has nonzero coefficient in $f(p+tq)$. The term $f(p)$ in (1.4.8) is zero if p lies on C. Then the intersection multiplicity is at least 1. If p is as smooth point of C, L is tangent to C at p if and only if p lies on C and $\sum_i f_i(p)\, q_i$ is zero – if and only if the intersection multiplicity is at least 2.

Let $s_i = f_i(p)$. The equation of the tangent line is

(1.4.10) $$s_0x_0 + s_1x_1 + s_2x_2 = 0$$ eqtanline

Note. Taylor's formula shows that the restriction of f to every line through a singular point p has a multiple zero, but we call L a line a tangent only when it is tangent to C at a smooth point.

Let ∇ be the gradient vector (f_0, f_1, f_2), and let H be the *Hessian matrix* of second partial derivatives:

(1.4.11) $$H \;=\; \begin{pmatrix} f_{00} & f_{01} & f_{02} \\ f_{10} & f_{11} & f_{12} \\ f_{20} & f_{21} & f_{22} \end{pmatrix}$$ hessianmatrix

Let ∇_p and H_p be the evaluations of ∇ and H, respectively, at p. Then, regarding p and q as column vectors, Equation 1.4.8 can be written as

(1.4.12) $$f(p+tq) = \; f(p) \;+\; \big(\nabla_p\, q\big)t \;+\; \tfrac{1}{2}(q^tH_p\, q)t^2 \;+\; O(3)$$ texp

In it, $\langle p, q\rangle$ denotes the symmetric form $v^t H_p\, w$ on $\mathbb{C}^3 \times \mathbb{C}^3$. It makes sense to say that this form vanishes on a pair of points. because the formula $\langle p, q\rangle = 0$ doesn't depend on the vectors that represent those points.

1.4.13. Proposition. *Let p be a point of $\mathbb{P}^2$, let q be a point distinct from p, and let L be the line defined by 1.4.10.* bilinform

(i) *p is a point of C if and only if $\langle p, p\rangle = 0$.*

(ii) *L is a tangent at a smooth point p of C if and only if $\langle p, p\rangle = \langle p, q\rangle = 0$, and*

(iii) *A smooth point p of C is a flex point with tangent line L if and only if $\langle p, p\rangle = \langle p, q\rangle = \langle q, q\rangle = 0$.* $\square$

The proposition follows from the next lemma, which is proved by applying Euler's Formula to the entries of ∇ and H. $\square$

applyeuler **1.4.14. Lemma.** *Let d be the degree of f. Then*
(a) $p^t H_p = (d-1)\nabla_p$*, and* $\nabla_p p = d\, f(p)$.
(b) $\langle p, q\rangle = (d-1)\nabla_p q$*, and* $\langle p, p\rangle = d(d-1)f(p)$.

tangentline **1.4.15. Theorem.** *A smooth point p of a curve $C : \{f = 0\}$ is a flex point if and only if the determinant of the Hessian matrix H_p at p is zero.*

proof. If $\det H_p = 0$, the form is degenerate, and there is a nonzero null vector q. Let p be a smooth point of C, so that $\langle p, p\rangle = 0$. Then $\nabla_p \neq 0$, and therefore $\langle p, v\rangle = (d-1)\nabla_p v$ isn't identically zero (1.4.14). So the null vector q is distinct from p. Then $\langle p, p\rangle = \langle p, q\rangle = \langle q, q\rangle = 0$, so p is a flex point.

Conversely, suppose that p is a flex point and that q is on the tangent line at p, so that $\langle p, p\rangle = \langle p, q\rangle = \langle q, q\rangle = 0$. The restriction of the form to the two-dimensional space W spanned by p and q will be zero. A form on a space V of dimension 3 that restricts to zero on a two-dimensional subspace W is degenerate. If (p, q, v) is a basis with p, q in W, the matrix of the form will look like this:

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix}$$

$\square$

hessnotzero **1.4.16. Proposition.**
(i) *Let $f(x, y, z)$ be an irreducible homogeneous polynomial of degree 2 or more. The Hessian determinant isn't divisible by f. In particular, it isn't identically zero.*
(ii) *A curve has finitely many flex points.*

proof. **(i)** Let C be the curve defined by an irreducible polynomial f. If f divides the Hessian determinant, then every smooth point p of C will be a flex point. We set $z = 1$ and look on the standard affine $\mathbb{U}^2$. We may assume that p is the origin and that $\frac{\partial f}{\partial y} \neq 0$ at p. The Implicit Function Theorem tells us that we can solve the equation $f(x, y, 1) = 0$ for y locally, say $y = \varphi(x)$. The graph $\Gamma : \{y = \varphi(x)\}$ will be equal to C in a neighborhood of p (see (1.4.17) below.) A point q of Γ is a flex point if and only if $\frac{d^2\varphi}{dx^2}$ is zero at q. If this is true for all points near to p, then $\frac{d^2\varphi}{dx^2}$ will be identically zero, which implies that φ is linear: $y = ax$. Then $y = ax$ solves $f = 0$, and therefore $y - ax$ divides $f(x, y, 1)$ (see 1.4.17 below). But since $f(x, y, z)$ is irreducible, so is $f(x, y, 1)$ (1.4.2). Therefore $f(x, y, 1)$ and $f(x, y, z)$ are linear.

(ii) This follows from **(i)** and Proposition 1.3.13. $\square$

ifthm **1.4.17. Review.** (*about the Implicit Function Theorem*)

Let $f(x, y)$ be a polynomial, and suppose that $\frac{df}{dy}$ isn't zero at the origin $(0, 0)$. The Implicit Function Theorem tells us that there is a unique analytic function $\varphi(x)$, defined for small x, such that $\varphi(0) = 0$ and $f(x, \varphi(x))$ is identically zero.

We'll make two further remarks. First, let $\mathcal{R}$ be the ring of analytic functions of x. In the ring $\mathcal{R}[y]$ of polynomials in y with coefficients in $\mathcal{R}$, $y - \varphi(x)$ divides $f(x, y)$. To see this, we divide f by the monic polynomial $y - \varphi(x)$:

divrem (1.4.18)
$$f(x, y) = (y - \varphi(x))q(x, y) + r(x)$$

The quotient $f(x, y)$ is in $\mathcal{R}[y]$, and the remainder $r(x)$ is in $\mathcal{R}$ because it has degree zero in y. Setting $y = \varphi(x)$ in the equation shows that the remainder is zero.

Next, let Γ be the graph of $\varphi(x)$ in a suitable neighborhood of the origin in x, y-space. Since $f(x, y) = (y - \varphi(x))q(x, y)$, the locus $f(x, y) = 0$ has the form $\Gamma \cup \Delta$, where Δ is the zero locus of the quotient $q(x, y)$. We differentiate, finding that $q(0, 0) = \frac{\partial f}{\partial y}(0, 0)$. So $q(0, 0) \neq 0$. Then Δ doesn't contain the origin while Γ does. This implies that Δ is disjoint from Γ, locally (in a neighborhood of p). Therefore, in a sufficently small neighborhood of p, the locus of zeros of f is equal to Γ. $\square$

1.5 Nodes and Cusps

nodes

We take a look here at nodes and cusps, the simplest singularities of curves.

Let C be the projective curve defined by a homogeneous polynomial $f(x,y,z)$ of degree d. We choose coordinates so that the point we wish to inspect is $p=(0,0,1)$, and we set $z=1$. This gives us an affine curve C_0 in $\mathbb{A}^2_{x,y}$, the zero set of the polynomial $\widetilde{f}(x,y)=f(x,y,1)$, and p becomes the origin $(0,0)$. We write

(1.5.1) $$\widetilde{f}(x,y)=f_0(x,y)+f_1(x,y)+f_2(x,y)+\cdots,$$ seriesf

where f_i is the homogeneous part of $\widetilde{f}$ of degree i. Then f_i,unless it is zero, is also the coefficient of z^{d-i} in the polynomial $f(x,y,z)$. If the origin p is a point of C_0, the constant term f_0 will be zero. Then the linear term f_1 will define the tangent direction to C_0 at p, If f_0 and f_1 are both zero, p will be a singular point of C.

(1.5.2) the multiplicity of a singular point singmult

To describe the singularity of C at the origin p, one looks first at the part of $\widetilde{f}$ of lowest degree. The smallest integer r such that $f_r(x,y)$ isn't zero is the *multiplicity* of p. If p is a point of multiplicity r, the polynomial that defines C_0 will have the form

(1.5.3) $$\widetilde{f}(x,y)=f_r(x,y)+f_{r+1}(x,y)+\cdots$$ multr

Let L be a line $\{vx=uy\}$ through p. The interesection multiplicity of C and L at p will be r unless $f_r(u,v)$ is zero. If $f_r(u,v)=0$, the intersection multiplicity will be greater than r. In that case, we call L a *special line*. The special lines at p correspond to the zeros of f_r in $\mathbb{P}^1$. There will be at most r special lines, because f_r has degree r.

(1.5.4) double points dpt

Suppose that p is a *double point*, a point of multiplicity 2, and let the quadratic part of f be

(1.5.5) $$f_2=ax^2+bxy+cy^2$$ quadrat-icterm

The double point p is called a *node* if f_2 has distinct zeros in $\mathbb{P}^1$. There will be two special lines at a node. A node is the simplest singularity that a curve can have.

If the discriminant b^2-4ac of f_2 is zero, f_2 will be a square and there will be just one special line at p. Suppose that $b^2-4ac=0$. We may arrange coordinates so that $c\neq 0$. Then

(1.5.6) $$(bx+2cy)^2=4c(ax^2+bxy+cy^2)$$ discrzero

The unique special line at p is the line $\{bx+2cy=0\}$. The point p is called a *cusp* if the multiplicity of intersection of C and L at p is 3. This will be true if and only if $bx+2cy$ doesn't divide $f_3(x,y)$.

When the discriminant is zero, one can adjust coordinates to make $f_2=y^2$. Then the special line is the line $\{y=0\}$, and p is a a cusp if the coefficient of the monomial x^3 in f_3 isn't zero. The *standard cusp* is the locus $y^2=x^3$.

The simplest example of a double point that isn't a node or cusp is a *tacnode*, a point at which two smooth branches of a curve intersect with the same tangent direction (see Figure 1.4).

(1.5.7) blowing up nodes and cusps blowupnode-cusp

One way to analyze a singular point of a curve is to see what happens when one blows up the plane. We use an affine blowing up here. Let U be the x,y-plane, and let V be a second affine plane, with coordinates x,t. The map $V\xrightarrow{\pi}U$ defined by $x=x$, and $y=xt$:

$$\pi(x,t)=(x,xt)$$

is the *affine blowing up* of the origin $p = (0,0)$ in U.

To describe the geometry of the blowing up, we inspect the fibres of π. When $x \neq 0$, the fibre of π over the point (x,y) consists of a single point (x,t) with $t = x^{-1}y$. When $(x,y) = (0,y)$ and $y \neq 0$, the fibre is empty. The most interesting fibre is the one over the origin $(0,0)$. It is the line

$$T : \{x = 0\}$$

in V, whose points correspond to tangent directions at p in a way that is explained below. The map π is called a blowup because the origin in U, a point, is 'blown up' into a line in V.

figure

Suppose that the origin p is a point of multiplicity r of an affine curve C_0, so that the polynomial that defines C_0 has the form (1.5.3). We choose coordinates so that the part f_r of degree r isn't divisible by x, we substitute $y = xt$ into $\widetilde{f}$, and pull out powers of x:

$$\widetilde{f}(x, xt) = x^r f_r(1,t) + x^{r+1} f_{r+1}(1,t) + \cdots$$

Let g be quotient $\widetilde{f}(x,xt)/x^r$. So

blowup-cusptwo

$$g(x,t) = f_r(1,t) + x\,h(x,t) \tag{1.5.8}$$

where

$$h = f_{r+1}()1,t) + x f_{r+2}(1,t) + \cdots$$

. The curve $C_1 : \{g = 0\}$ in V is the *blowup* of C_0.

The points of C_1 that lie over p are its intersections with the fibre T. We set $x = 0$ to compute those intersections: $g(0,t) = f_r(1,t)$. The intersections are the points $t = \alpha$, where α is a root of this polynomial. They correspond to the zeros of $f_r(x,y)$ in $\mathbb{P}^1_{x,y}$.

figure

A smooth point of C_0 has multiplicity one: $f_1 \neq 0$. Suppose that this is so, and that $f_1 = ax - by$. The tangent line L at p isthe line $ax - by = 0$. Then $f_1(1,t) = a - bt$, and sincef_1 isn't divisible by x, $b \neq 0$. The only intersection of C_1 with T is the point $t = a/b$, and a/b is the slope of the tangent line L at p. In this way, points of T correspond to tangent directions.

If p is a double point of C_0. Then $\widetilde{f}(x,y) = f_2(x,y) + f_3(x,y) + \cdots$. The intersections of C_1 with T are determined by the two roots of $f_2(1,t)$, corresponding to the two zeros of $f_2(x,y)$ in $\mathbb{P}^1_{x,y}$.

blowupcusp **1.5.9. Proposition.** *With assumptions and notation as above:*

(i) *The multiplicity of the point p is equal to the number of intersections, counted with multiplicity, of the blowup curve C_1 with the line T.*

(ii) *A double point p is a node or a cusp if and only if the blowup C_1 is smooth at the points that lie over p. If C_1 is smooth at those points, then*

- **(a)** *p is a node if $f_2(x,y)$ has distinct zeros in $\mathbb{P}^1$, and*
- **(b)** *p is a cusp if $f_2(x,y)$ has a double zero.*

proof. **(ii)** Suppose that p is a double point of C. Recall that coordinates have been chosen so that the quadratic polynomial f_2 isn't divisible by x. So $C_1 \cap T$ consists of two points of multiplicity one or one point of multiplicity two. If $f_2(x,y)$ has two zeros in $\mathbb{P}^1$, p is a node (1.5.5). Then $f_2(1,t)$ has two roots, say α_1, α_2, and is a scalar multiple of $(t - \alpha_1)(t - \alpha_2)$. The partial derivative $\frac{\partial g}{\partial t}$ isn't zero at the points $(0, \alpha_1)$ and $(0, \alpha_2)$, so they are smooth points of C_1. Conversely, if $C_1 \cap T$ consists of two smooth points of C_1, those points correspond to zeros of multiplicity 1 of f_2.

Suppose that $C_1 \cap T$ consists of a single point of multiplicity two. We may choose coordinates so that $f_2 = y^2$, and then $g(x,t) = t^2 + xh(x,t)$. The curve C_1 will be smooth at p_1 if and only if the linear term of g, which is $xh(0,0)$, isn't zero. The constant coefficient $h(0,0)$ of h is $f_3(1,0)$, which is the coefficient of x^3

in $f_3(x,y)$. So C_1 is smooth over p if and only if the coefficient of x^3 is nonzero $-$ if and only if p is a cusp. □

A singularity more complicated than a node or cusp can be "resolved" (made smooth) by repeating the blowing up process a finite number of times. For example, the curve C_0: $y^2 = x^4 + x^5$ has a tacnode at the origin. Blowing up, we obtain the curve C_1: $t^2 = x^2 + x^3$, a nodal curve. A second blowing up resolves the singularity.

1.6 Transcendence degree

transcdeg

##$[K : F] < \infty$ implies same tr deg##

Let $F \subset K$ be a field extension. A set $\alpha = \{\alpha_1, ..., \alpha_n\}$ of elements of K is *algebraically dependent* over F if there is a nonzero polynomial $f(x_1, ..., x_n)$ with coefficients in F, such that $f(\alpha) = 0$. If $f(\alpha) \neq 0$ for every nonzero polynomial f with coefficients in F, then α is *algebraically independent* over F. An infinite set is called algebraically independent if every finite subset is algebraically independent – if there is no polynomial relation among any finite set of elements of α.

The set consisting of a single element α_1 of K is algebraically dependent if and only if α_1 is algebraic over F, and is *transcendental* over F if it isn't algebraic over F. So α_1 is transcendental when the set $\{\alpha_1\}$ is algebraically independent.

A set $\alpha = \{\alpha_1, ..., \alpha_n\}$ is a *transcendence basis* for K over F if it a maximal algebraically independent set – if it isn't contained in a larger algebraically independent set. If K has a finite transcendence basis, the number of elements in a transcendence basis is the *transcendence degree* of the field extension K of F. Lemma 1.6.2 below shows that this number doesn't depend on the transcendence basis.

For example, let $K = F(x_1, ..., x_n)$ be the field of rational functions in n variables. The variables form a transcendence basis of K over F, and the transcendence degree of K over F is n.

We use the customary notation $F[\alpha_1, ..., \alpha_n]$ or $F[\alpha]$ for the algebra generated by a set α, and if $F[\alpha]$ is a domain, a nonzero (commutative) ring with no zero divisors, we may denote its field of fractions by $F(\alpha_1, ..., \alpha_n)$ or $F(\alpha)$.

By the way, in these notes the word *ring* means *commutative ring*, i.e., multiplication is required to be a commutative law of composition.

1.6.1. Lemma. *Let K/F be a field extension, let $\alpha = \{\alpha_1, ..., \alpha_n\}$ be an algebraically independent subset of K over F.* algindtrivialities

(i) *Every element of the field $F(\alpha)$ that is not in F is transcendental over F.*

(ii) *If β is another element of K, the set $\alpha \cup \{\beta\}$ is algebraically dependent if and only if β is algebraic over the field $F(\alpha)$.*

(iii) *The set α is a transcendence basis if and only if every element of K is algebraic over $F(\alpha)$.*

*proof.***(i)** Suppose that a rational function z is written as a fraction $p(\alpha)/q(\alpha)$, where p and q are relatively prime polynomials, and that z satisfies a nontrivial polynomial relation $c_0z^n + c_1z^{n-1} + \cdots + c_n = 0$, with c_i in F. We may assume that $c_0 \neq 0$, and we normalize c_0 to 1. Multiplying the relation by q^n gives us an equation

$$p^n = -q(c_1p^{n-1} + \cdots + c_nq^{n-1})$$

This equation shows that q divides p^n, which contradicts the hypothesis that p and q are relatively prime. □

1.6.2. Lemma. *Let K/F be a field extension. If K has a finite transcendence basis, then all algebraically independent subsets of K are finite, and all transcendence bases have the same number of elements.* trdeg

proof. We show that if K is algebraic over a subfield $F(\alpha_1, ..., \alpha_s)$ and if $\beta = \{\beta_1, ..., \beta_r\}$ is an algebraically independent subset, then $r \leq s$. The fact that all transcendence bases have the same order will follow: If both α and β are transcendence bases, then $r \leq s$ and, since we can interchange α and β, $s \leq r$.

The proof proceeds by reducing to the trivial case that β is a subset of α. Suppose that some element of β, say β_r, isn't in the set α. The set $\beta' = \{\beta_1, ..., \beta_{r-1}\}$ isn't a transcendence basis, so K isn't algebraic over $F(\beta')$. Since K is algebraic over $F(\alpha)$, there is at least one element of α, say α_s, that isn't algebraic over $F(\beta')$. Then $\gamma = \beta' \cup \{\alpha_s\}$ will be an algebraically independent set of order r, and it will contain more elements of the set α than β does. Induction shows that $r \leq s$. □

dualcurve

1.7 The Dual Curve

dualplanesect

(1.7.1) the dual plane

Let $\mathbb{P}$ denote the projective plane with coordinates x_0, x_1, x_2, and let L be the line with an equation

lineequation

$$s_0x_0 + s_1x_1 + s_2x_2 = 0. \tag{1.7.2}$$

The solutions of this equation determine the coefficients s_i only up to a nonzero scalar factor, so L determines a point (s_0, s_1, s_2) that we denote by L^* in another projective plane $\mathbb{P}^*$, the *dual plane*. Moreover, a point $p = (x_0, x_1, x_2)$ in $\mathbb{P}$ determines a line p^* in the dual plane, the line with the equation (1.7.2), when s_i are regarded as the variables and x_i as the scalar coefficients. The equation exhibits a duality between $\mathbb{P}$ and $\mathbb{P}^*$. A point p of $\mathbb{P}$ lies on a line L if and only if the equation is satisfied, and this means that, in $\mathbb{P}^*$, the point L^* contains the line p^*.

dualcurvetwo

(1.7.3) the dual curve

Let C be a plane projective curve defined by an irreducible homogeneous polynomial of degree $d \geq 2$, and let U be the set of its smooth points. Since C has finitely many singular points, U is the complement of a finite set. We define a map

$$U \xrightarrow{t} \mathbb{P}^*$$

as follows: Let p be a point of U, a smooth point of C, and let L be the tangent line to C at p. We define $t(p) = L^*$, where L^* is the point of the dual plane $\mathbb{P}^*$ that corresponds to L. We don't try to define this map at the singular points of C. The degree of C is assumed to be at least two because, if C were a line, U^* would be a point.

The tangent line L at a smooth point $p = (x_0, x_1, x_2)$ of C has the equation $s_0x_0 + s_1x_1 + s_2x_2 = 0$, where s_i is the partial derivative $f_i(x) = \frac{\partial f}{\partial x_i}$. Therefore L^* is the point

ellstarequation

$$(s_0, s_1, s_2) = \big(f_0(x), f_1(x), f_2(x)\big) \tag{1.7.4}$$

Let U^* denote the image of U in $\mathbb{P}^*$. Points of U^* of U correspond to tangent lines at smooth points of C.

$$??figure??$$

phizerogzero

1.7.5. Lemma. *A polynomial $\varphi(s_0, s_1, s_2)$ is identically zero on U^* if and only if $g(x) = \varphi(f_0(x), f_1(x), f_2(x))$ is identically zero on U, and this is true if and only if f divides g.*

See Corollary 1.3.21. □

dualcurvethm

1.7.6. Theorem. *Let C be the plane curve defined by an irreducible polynomial f of degree $d \geq 2$. With notation as above, the closure C^* of U^* in $\mathbb{P}^*$ is a curve in the dual space, the* **dual curve***.*

proof. If an irreducible homogeneous polynomial $\varphi(s_0, s_1, s_2)$ vanishes on U^*, it will be unique up to scalar factor (Corollary 1.3.15). Its zero locus will be the dual curve.

We show first that there is a nonzero polynomial φ, not necessarily irreducible or homogeneous, that vanishes on U^*. The field $\mathbb{C}(x_0, x_1, x_2)$ has transcendence degree 3 over $\mathbb{C}$. Therefore the four polynomials f_0, f_1, f_2, and f are algebraically dependent. There is a nonzero polynomial $\psi(s_0, s_1, s_2, t)$ such that $\psi(f_0, f_1, f_2, f) = 0$. We can cancel factors of t, so we may assume that ψ isn't divisible by t. Let $\varphi(s_0, s_1, s_2) = \psi(s_0, s_1, s_2, 0)$. This polynomial isn't zero when t doesn't divide ψ. If x is a point of U, then $f(x) = 0$, and therefore $\varphi(f_0, f_1, f_2) = \psi(f_0, f_1, f_2, f) = 0$. Lemma 1.7.5 tells us that $\varphi(s)$ vanishes on U^*.

The homogeneous parts of φ vanish on U^* (1.3.2), so we may assume that φ is homogeneous, say of degree r. Then the polynomial $g(x) = \varphi(f_0(x), f_1(x), f_2(x))$ is also homogeneous, of degree $r(d-1)$. It will vanish on U, and therefore on C (1.3.21). So f will divide g. Finally, if $\varphi(s)$ factors, then $g(x)$ factors accordingly, and because f is irreducible, it will divide one of the factors of g. The corresponding factor of φ

will vanish on U^* (1.7.5). So we may replace the polynomial φ, now homogeneous, by one of its irreducible factors. □

It can be painful to determine the defining polynomial of a dual curve explicitly. Several points of the dual curve C^* may correspond to a singular point of C and vice versa, and the degrees of C and C^* are often different. However, the computation is simple for a conic.

1.7.7. Example. (*the dual of a conic*) Let C be the conic $f = 0$, with $f = x_0x_1 + x_0x_2 + x_1x_2$ (see Proposition 1.2.14), and let $(s_0, s_1, s_2) = (f_0, f_1, f_2) = (x_1{+}x_2, x_0{+}x_2, x_0{+}x_1)$. Then exampledualone

(1.7.8) $$s_0^2 + s_1^2 + s_2^2 - 2(x_0^2 + x_1^2 + x_2^2) = 2f \quad \text{and} \quad s_0s_1 + s_1s_2 + s_0s_2 - (x_0^2 + x_1^2 + x_2^2) = 3f$$ exampledualtwo

Setting $f = 0$ gives us the equation of the dual curve:

(1.7.9) $$(s_0^2 + s_1^2 + s_2^2) - 2(s_0s_1 + s_1s_2 + s_0s_2) = 0$$ equationdualthree

It is another conic. □

(1.7.10) a local equation for the dual curve equationofcstar

We label the coordinates in $\mathbb{P}$ and $\mathbb{P}^*$ as x, y, z and u, v, w, respectively, and we work in a neighborhood of a smooth point p of the curve C defined by a polynomial $f(x, y, z)$. We choose coordinates so that $p = (0, 0, 1)$, and that the tangent line L at p is the line $\{y = 0\}$.

Let $\widetilde{f}(x, y) = f(x, y, 1)$. In the affine x, y-plane, the point p becomes $\widetilde{p} = (0, 0)$. So $\widetilde{f}(0, 0) = 0$, and since the tangent line is horizontal, $\frac{\partial \widetilde{f}}{\partial y}(0, 0) \neq 0$. This allows us to solve the equation $\widetilde{f} = 0$ for y as an analytic function $y(x)$ for small x, with $y(0) = 0$. Let $y'(x)$ denote the derivative $\frac{dy}{dx}$. Then $y(0) = y'(0) = 0$.

Let $\widetilde{p}_1 = (x_1, y_1)$ be a point of C_0 near to $\widetilde{p}$, and let y_1 and y_1' denote $y(x_1)$ and $y'(x_1)$, respectively. The tangent line L_1 at $\widetilde{p}_1$ has the equation

(1.7.11) $$y - y_1 = y_1'(x - x_1)$$ localtangent

So the point L_1^* of the dual plane that corresponds to L_1, in projective coordinates, is $(-y_1', 1, y_1'x_1 - y_1)$. Let's drop the subscript 1. Then as x varies,

(1.7.12) $$(u, v, w) = (-y', 1, y'x - y),$$ projlocaltangent

traces out C^* near the point L^* (see (1.4.17)). □

(1.7.13) the bidual bidualone

The *bidual* C^{**} of a curve C is the dual of the curve C^*.

1.7.14. Theorem. *A plane curve of degree at least 2 is equal to its bidual C^{**}.* bidualC

1.7.15. Lemma. *The set V of points p such that C is smooth at p and C^* is smooth at $t(p)$ is the complement of a finite subset of C.* vopen

proof. The points of V are the points of U that aren't in the inverse images of the finitely many singular points of C^*. When we show that the fibres of the map $U \to U^*$ are finite sets it will follow that V is the complement of a finite set.

Let $s = (s_0.s_1, s_2)$ be a point of C^*. We may suppose that one of the coordinates, say s_0, is equal to 1. If s is the image of a smooth point x of C, then $(f_0(x), f_1(x), f_2(x)) = \lambda(1, s_1, s_2)$, which means that $\lambda = f_0(x)$, $f_1(x) = s_1f_0(x)$ and $f_2(x) = s_2f_0(x)$. Then at the point x, $f = 0$ and $g_i = f_i - s_if_0 = 0$ for $i = 1, 2$. The polynomials g_1, g_2 can't both be identically zero (see (1.4.6)**(ii)**). Say $g_1 \neq 0$. Then there are finitely many points x at which f and g_1 are zero, because the irreducible polynomial f and the lower degree polynomials g_1 have no common factor (1.3.13). So the inverse image of s is finite. $\square$

proof of Theorem 1.7.14. Let V be as in Lemma 1.7.15, and let V^* be its image $t(V)$. If L is the tangent line to C at a point p of V, then $t(p) = L^*$ is a smooth point of V^*. We will show that

tangenttangent (1.7.16) *The line p^* is tangent to C^* at the point L^*.*

Assume that this is shown. Let U^* denote the set of smooth points of C^*, and let $U^* \xrightarrow{t^*} \mathbb{P}^{**}$ be the map analogous to the map t. Then $t^*(L^*)$ is the point $(p^*)^*$ of $\mathbb{P}$, and therefore $p^{**} = p$. So for all points p of V, $t^*t(p) = t^*(L^*) = p$. Therefore the restriction of t to V is injective. It defines a bijective map from V to its image V^*, whose inverse function is t^*. So $V \subset C^{**}$. Since V is dense in C and C^{**} is closed, $C \subset C^{**}$. Since C and C^* are curves, $C = C^{**}$. $\square$

To prove (1.7.16), we choose a second point p_1 of V, and we let it approach p. Let L_1 be the tangent line to C at p_1. Because the image of p_1 is $L_1^* = (f_0(p_1), f_1(p_1), f_2(p_1))$, and because the partial derivatives f_i are continuous, $\lim_{p_1 \to p} L_1^* = (f_0(p), f_1(p), f_2(p)) = L^*$.

qapproachesp **1.7.17. Lemma.** *With notation as above, let q be intersection $L_1 \cap L$. Then $\lim_{p_1 \to p} q = p$.*

$$figure$$

Let's assume the lemma for a moment. In the dual space $\mathbb{P}^*$, q^* is the line through the two points L_1^* and L^*. Since $\lim_{p_1 \to p} L_1^* = L^*$, q^* approaches the tangent line to C^* at L^*. On the other hand, the lemma tells us that q^* approaches p^*. Therefore the tangent line at L^* is p^*. $\square$

proof of Lemma 1.7.17. We work analytically in a neighborhood of p. We choose coordinates so that $p = (0, 0, 1)$ and that L is the line $\{y = 0\}$. Let $(x_q, y_q, 1)$ be the coordinates of $q = L \cap L_1$. Since q is a point of L, $y_q = 0$, and x_q can be obtained by substituting $x = x_q$ and $y = 0$ into the equation (1.7.11) for L_1:

$$x_q \;=\; x_1 - y_1/y_1'.$$

Now: When a function has an nth order zero at a point, i.e, when it has the form $y = x^n h(x)$, where $n > 0$ and $h(0) \neq 0$, the order of zero of its derivative at that point is n–1. This is verified by differentiating $x^n h(x)$. Since the function $y(x)$ is zero at p, $\lim_{p_1 \to p} y_1/y_1' = 0$. It follows that $\lim_{p_1 \to p} x_q = 0$ and that $\lim_{p_1 \to p} q = (0, 0, 1) = p$. $\square$

cstarisimage **1.7.18. Corollary.** *If C is a smooth curve, the map $C \xrightarrow{t} C^*$, which is defined at all points of C, is a surjective map.*

proof. Let W denote the image of C in C^*. The map $C^* \xrightarrow{t^*} C$ is defined at the smooth points of C^*, and it inverts t at those points. Therefore W contains the smooth points of C^*. So $C^* = W \cup S$ where S is a finite set. Because C is compact, its image W is compact, and therefore closed in C^*. Then S is open. And since it is a finite set, it is closed. Therefore S consists of isolated points of C^*. Since a curve has no isolated points (1.3.18), S is empty. $\square$

singdual **(1.7.19) singularities of the dual curve**

Let C be a plane curve. A tangent line L at a smooth point p of C is an *ordinary tangent* if p isn't a flex point, and a flex point p is *ordinary* if the the intersection multiplicity of the curve and its tangent line at p is precisely 3. A *bitangent* to a curve C is a line L that is tangent to C at distinct smooth points p and q. A bitangent is *ordinary* if neither p nor q is a flex point, L isn't tangent to C at a third point, and L doesn't contain a singular point of C.

1.7.20. Proposition. *Let p be a smooth point of a curve C, and let L be the tangent line at p. Suppose that L doesn't contain a singular point of C.* dualcusp
(i) *If L is an ordinary tangent at p and not a bitangent, L^* is a smooth point of C^*.*
(ii) *If p is an ordinary flex point and not a bitangent, L^* is a cusp of C^*.*
(iii) *If L is an ordinary bitangent, L^* is a node of C^*.*

proof. We set $z = 1$ and choose affine coordinates so that p is the origin and the tangent line L at p is the line $\{y = 0\}$. Let $\widetilde{f}(x, y) = f(x, y, 1)$. We solve $\widetilde{f} = 0$ for $y = y(x)$ as analytic function of x near zero, as before. The tangent line L_1 to C at nearby points $p_1 = (x, y)$ has the equation (1.7.11), and L_1^* is the point $(u, v, w) = (-y', 1, y'x - y)$ of $\mathbb{P}^*$ (1.7.12).

If L is an ordinary tangent line, $y(x)$ will have a zero of order 2 at $x = 0$. Then $u = -y'$ will have a simple zero there. The Implicit Function Theorem solves for x as a function of u. Then w becomes a function of u, and since y' has a simple zero, C^* is smooth at the origin.

If L is an ordinary bitangent, tangent to C at two points p and q, the reasoning given for an ordinary tangent shows that the images in C^* of small neighborhoods of p and q in C will be smooth at L^*. Their tangent lines p^* and q^* will be distinct. Therefore the blowup of C^* at L^* consists of two smooth points, and L^* is a node (see Proposition 1.5.9).

Suppose p is an ordinary flex. Then y has a triple zero at $x = 0$, say $y = cx^3 + O(4)$ with $c \neq 0$. Then in (1.7.12), $u = -3cx^2 + O(3)$ has a zero of order 2, and $w = 2cx^3 + O(4)$ has a zero of order 3. We blow up C^*, substituting $u = u$ and $w = ut$. Then t has a zero of order 1, so u is a function of t near zero. Therefore the blowup C_1^* is smooth at the point $(u, t) = (0, 0)$. Proposition 1.5.9 shows that L^* is a cusp. □

1.7.21. Corollary. *A plane curve has finitely many bitangents.* finitebitangents

This is true whether or not the bitangents are ordinary, and it This follows from the fact that the dual curve C^* has finitely many singular points (1.4.6). If L is any bitangent, L^* will be a singular point of C^*. □

1.7.22. Example. (*the dual of a cuspidal cubic*) It is painful to compute the equation of the dual of a smooth cubic. We will see in a while that a smooth cubic curve has has 9 flex points (1.11.10), so its dual has 9 cusps. We will also see that the degree of the dual curve is 6 (1.12.1 **(ii)**). cuspcubic

We compute the dual of a cubic with a cusp instead. The curve C defined by the irreducible polynomial $f = y^2z + x^3$ has a cusp at $(0, 0, 1)$ as its only singularity. The Hessian matrix of f is

$$H = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 2z & 2y \\ 0 & 2y & 0 \end{pmatrix}$$

and the Hessian determinant h is $-24xy^2$. The common zeros of f and h are the cusp point $(0, 0, 1)$ and a single flex point $(0, 1, 0)$. We can guess the result. The dual curve C^* will have a flex and a cusp that correspond to the cusp and the flex of C, respectively. It is probably another cuspidal cubic.

The proof of Theorem 1.7.6 gives us a method for finding a polynomial that vanishes on C^*. That is to find a relation among f_x, f_y, f_z, f, and then set $f = 0$.

We scale the partial derivatives of f to simplify notation. Let $u = x^2 = f_x/3$, $v = yz = f_y/2$, and $w = y^2 = f_z$. Then

$$f^2 = y^4z^2 + 2x^3y^2z + x^6 = v^2w + 2uv(xy) + u^3$$

Working modulo f, $v^2w + u^3 = -2uv(xy)$. Squaring both sides,

$$v^4w^2 + 2u^3v^2w + u^6 = 4u^2v^2(x^2y^2) = 4u^2v^2(uw)$$

or $(v^2w - u^3)^2 = 0$. The zero locus of the irreducible polynomial $v^2w - u^3$ is the dual curve. □

1.8 Resultants

resultant

Let F and G be monic polynomials with variable coefficients,

$$(1.8.1) \qquad F(x) = x^m + a_1x^{m-1} + \cdots + a_m \quad \text{and} \quad G(x) = x^n + b_1x^{n-1} + \cdots + b_n$$ polys

The resultant $\mathrm{Res}(F,G)$ of F and G is a polynomial in the coefficients a_i, b_j. Its important property is that, when the coefficients of are in a field, the resultant is zero if and only if F and G have a common factor.

The formula for the resultant is nicest when one allows leading coefficients different from 1. We work with homogeneous polynomials in two variables to prevent the degree from dropping when a leading coefficient happens to be zero.

Let f and g be homogeneous polynomials with complex coefficients, say

hompolys (1.8.2) $$f(x,y) = a_0x^m + a_1x^{m-1}y + \cdots + a_my^m, \quad g(x,y) = b_0x^n + b_1x^{n-1}y + \cdots + b_ny^n$$

If these polynomials have a common zero (u,v) in $\mathbb{P}^1$, then $vx - uy$ will divide both f and g (see **1.3.3**). So the polynomial $h = fg/(vx-uy)$ of degree $m+n-1$ will be divisible by f and g, say $h = pf = qg$, where p and q are homogeneous polynomials of degrees $n-1$ and $m-1$, respectively. Then pf will be a linear combination of the polynomials x^iy^jf, with $i+j = n-1$, and it will also be a linear combination qg of the polynomials $x^ky^\ell g$, with $k+\ell = m-1$. This implies that the $m+n$ polynomials of degree $m+n-1$,

mplusnpolys (1.8.3) $$x^{n-1}f,\ x^{n-2}yf, ..., y^{n-1}f\ ;\ x^{m-1}g,\ x^{m-2}yg, ..., y^{m-1}g$$

will be dependent. For example, when $m = 3$ and $n = 2$, the polynomials

$$\begin{array}{ll}
xf = & a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 \\
yf = & \qquad a_0x^3y + a_1x^2y^2 + a_2xy^3 + a_3y^4 \\
x^2g = & b_0x^4 + b_1x^3y + b_2x^2y^2 \\
xyg = & \qquad b_0x^3y + b_1x^2y^2 + b_2xy^3 \\
y^2g = & \qquad\qquad bx^2y^2 + b_1xy^3 + b_2y^4
\end{array}$$

will be dependent. Conversely, if the polynomials (1.8.3) are dependent, there will be an equation of the form $pf = qg$, with p of degree $n-1$ and q of degree $m-1$. Then at least one zero of g must be a zero of f.

Let $r = m+n-1$. We form a square matrix $\mathcal{R}$, the *resultant matrix*, whose columns are indexed by the $m+n$ monomials $x^r, x^{r-1}y, ..., y^r$ of degree $r = m+n-1$, and whose rows list the coefficients of those monomials in the polynomials (1.8.3). The matrix is illustrated below for the cases $m,n = 3,2$ and $m,n = 1,2$, with dots representing entries equal to zero:

resmatrix (1.8.4) $$\mathcal{R} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdot \\ \cdot & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & \cdot & \cdot \\ \cdot & b_0 & b_1 & b_2 & \cdot \\ \cdot & \cdot & b_0 & b_1 & b_2 \end{pmatrix} \quad \text{or} \quad \mathcal{R} = \begin{pmatrix} a_0 & a_1 & \cdot \\ \cdot & a_0 & a_1 \\ b_0 & b_1 & b_2 \end{pmatrix}$$

By definition, the determinant of $\mathcal{R}$ is the the *resultant* of f and g:

resequalsdet (1.8.5) $$\mathrm{Res}(f,g) = \det \mathcal{R}$$

The coefficients of the polynomials can be in any ring.

The resultant $\mathrm{Res}(F,G)$ of the monic, one-variable polynomials $F(x) = x^m + a_1x^{m-1} + \cdots + a_m$ and $G(x) = x^n + b_1x^{n-1} + \cdots + b_n$ is the determinant of the matrix $\mathcal{R}$, with $a_0 = b_0 = 1$.

homogresult **1.8.6. Corollary.** *Let f and g be homogeneous polynomials in two variables, or monic polynomials in one variable, with coefficients in a field. The resultant* $\mathrm{Res}(f,g)$ *is zero if and only if f and g have a common factor.* □

weights (**1.8.7**) **weighted degree**

When defining the degree of a polynomial, one may assign an integer called a *weight* to each variable. If one assigns weight w_i to the variable x_i, the monomial $x_1^{e_1} \cdots x_n^{e_n}$ gets a *weighted degree*, which is

$$e_1w_1 + \cdots + e_nw_n$$

For example, it is natural to assign weight k to the coefficient a_k of the polynomial $f(x) = x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots \pm a_n$. The reason is that, if f factors into linear factors, $f(x) = (x-\alpha_1)\cdots(x-\alpha_n)$, then a_k will be the kth elementary symmetric function in $\alpha_1, ..., \alpha_n$. When written as a polynomial in α, the degree of a_k will be k.

We leave the proof of the next lemma as an exercise.

1.8.8. Lemma. *Let $f(x,y)$ and $g(x,y)$ be homogeneous polynomials of degrees m and n respectively, with variable coefficients a_i and b_i, as in (1.8.2). Assigning weight i to a_i and b_i, the resultant* $\mathrm{Res}(f,g)$ *is a weighted homogeneous polynomial of degree mn in the variables $\{a_i, b_j\}$.* $\square$ degresult

1.8.9. Proposition. *Let F and G be products of monic linear polynomials, say $F = \prod_i(x-\alpha_i)$ and $G = \prod_j(x-\beta_j)$. Then* resroots

$$\mathrm{Res}(F,G) \quad = \quad \prod_{i,j}(\alpha_i - \beta_j) \quad = \quad \prod_i G(\alpha_i)$$

proof. The equality of the second and third terms is obtained by substituting α_i for x into the formula $G = \prod(x-\beta_j)$. We prove that the first and second terms are equal.

Let the elements α_i and β_j be variables, and let R be the resultant $\mathrm{Res}(F,G)$ and let Π be the product $\prod_{i.j}(\alpha_i - \beta_j)$. When we write the coefficients of F and G as symmetric functions in the roots α_i and β_j, R will be homogeneous. Its (unweighted) degree in α_i, β_j will be mn, the same as the degree of Π (Lemma 1.8.8). To show that $R = \Pi$, we choose i, j and divide R by the polynomial $\alpha_i - \beta_j$, considered as a monic polynomial in α_i:

$$R = (\alpha_i - \beta_j)q + r,$$

where r has degree zero in α_i. The resultant R vanishes when we substitute $\alpha_i = \beta_j$. Looking at the displayed equation, we see that the remainder r also vanishes when $\alpha_i = \beta_j$. On the other hand, the remainder is independent of α_i. It doesn't change when we set $\alpha_i = \beta_j$. Therefore the remainder is zero, and this is true for all i and j. So R is divisible by Π, and since these two polynomials have the same degree, $R = c\Pi$ for some scalar c. To show that $c = 1$, one computes R and Π for some particular polynomials. We suggest using $F = x^m$ and $G = x^n - 1$. $\square$

1.8.10. Corollary. *Let F, G, H be monic polynomials and let c be a scalar. Then* restrivialities

(i) $\mathrm{Res}(F, GH) = \mathrm{Res}(F,G)\,\mathrm{Res}(F,H)$, *and*

(ii) $\mathrm{Res}(F(x-c), G(x-c)) = \mathrm{Res}(F(x), G(x))$. $\square$

1.8.11. Proposition. *Let f and g be homogeneous polynomials (1.8.2) in x, y, with complex coefficients. Then* $\mathrm{Res}(f,g) = 0$ *if and only if f and g have a common zero.* $\square$ leadcoeffnotzero

When both a_0 and b_0 are zero, the point $(1,0)$ of $\mathbb{P}^1$ will be a zero of f and of g. In this case, f and g have a common zero at infinity.

(1.8.12) the discriminant discrimsect

The *discriminant* $\mathrm{Discr}(F)$ of a polynomial $F = a_0x^m + a_1x^{n-1} + \cdots a_m$ is the resultant of F and its derivative F':

$$\mathrm{Discr}(F) = \mathrm{Res}(F, F') \qquad (1.8.13)$$ discrdef

The computation of the discriminant is made using the formula for the resultant of a polynomial F of degree m. The definition makes sense when the leading coefficient a_0 is zero, though the discriminant is zero in that case.

When the coefficients of F are complex numbers, the discriminant is zero if and only if either F has a double root, which happens when F and F' have a common factor, or else F has degree less than m (see (1.8.11)).

The discriminant of the quadratic polynomial $F(x) = ax^2 + bx + c$ is

discrquadr (1.8.14)
$$\det\begin{pmatrix} a & b & c \\ 2a & b & \cdot \\ \cdot & 2a & b \end{pmatrix} = -a(b^2 - 4ac).$$

discrnotzero **1.8.15. Proposition.** *Let K be a field of characteristic zero. The discriminant $\mathrm{Discr}(F)$ of an irreducible polynomial F with coefficients in K isn't zero. Therefore an irreducible polynomial $F(x)$ with coefficients in K has no multiple root.*

proof. When F is irreducible, it cannot have a factor in common with the lower degree polynomial F'. □

This proposition is false when the characteristic of K isn't zero. In characteristic p, the derivative F' might be the zero polynomial.

discrformulas **1.8.16. Proposition.** *Let $F = \prod(x - \alpha_i)$ be a polynomial that is a product of monic linear factors. Then*

$$\mathrm{Discr}(F) = \prod_i F'(\alpha_i) = \pm \prod_{i<j} (\alpha_i - \alpha_j)^2$$

proof. The fact that $\mathrm{Discr}(F) = \prod F'(\alpha_i)$ follows from Proposition 1.8.9. So it suffices to show that $F'(\alpha_i) = \prod_{j, j\neq i}(\alpha_i - \alpha_j)$. By the product rule for differentiation,

$$F'(x) = \sum_j (x-\alpha_1)\cdots\widehat{(x-\alpha_j)}\cdots(x-\alpha_n)$$

where the hat $\widehat{\ }$ indicates that the term is deleted. Substituting $x = \alpha_i$, all terms in the sum except the one with $i = j$ become zero. □

discrprop **1.8.17. Proposition.** *If $F(x)$ and $G(x)$ are monic polynomials,*

$$\mathrm{Discr}(FG) = \pm\,\mathrm{Discr}(F)\,\mathrm{Discr}(G)\mathrm{Res}(F,G)^2$$

proof. This proposition follows from Propositions 1.8.9 and 1.8.16 for polynomials with complex coefficients. It is true for polynomials with coefficients in any ring because it is an identity. □

hensel **(1.8.18) Hensel's Lemma**

The resultant matrix (1.8.4) arises in a second context that we explain here. Suppose given a product $P = FG$ of two polynomials, say

multiply-polys (1.8.19)
$$(c_0x^{m+n} + c_1x^{m+n-1} + \cdots + c_{m+n}) = \big(a_0x^m + a_1x^{m-1} + \cdots + a_m\big)\big(b_0x^n + b_1x^{n-1} + \cdots + b_n\big)$$

We call the equations among the coefficients that are implied by this polynomial equation the *product equations*. The product equations are

$$c_i = a_ib_0 + a_{i-1}b_1 + \cdots + a_0b_i$$

for $i = 0, ..., m{+}n$. For instance, when $m = 3$ and $n = 2$, they are

prodeqns **1.8.20.**

$$\begin{aligned}
c_0 &= a_0b_0 \\
c_1 &= a_1b_0 + a_0b_1 \\
c_2 &= a_2b_0 + a_1b_1 + a_0b_2 \\
c_3 &= a_3b_0 + a_2b_1 + a_1b_2 \\
c_4 &= \qquad\quad\; a_3b_1 + a_2b_2 \\
c_5 &= \qquad\qquad\qquad\;\; a_3b_2
\end{aligned}$$

Let J denote the Jacobian matrix of partial derivatives of $c_1, ..., c_{m+n}$ with respect to the variables $b_1, ..., b_n$ and $a_1, ..., a_m$, holding a_0, b_0 and c_0 constant. When $m, n = 3, 2$,

(1.8.21) $$J = \frac{\partial(c_i)}{\partial(b_j, a_k)} = \begin{pmatrix} a_0 & . & b_0 & . & . \\ a_1 & a_0 & b_1 & b_0 & . \\ a_2 & a_1 & b_2 & b_1 & b_0 \\ a_3 & a_2 & . & b_2 & b_1 \\ . & a_3 & . & . & b_2 \end{pmatrix}$$ prodjacob

1.8.22. Lemma. *The Jacobian matrix J is the transpose of the resultant matrix $\mathcal{R}$ (1.8.4).* □ jacres

1.8.23. Corollary. *Let F and G be polynomials with complex coefficients. The Jacobian matrix is singular if and only F and G have a common root, or $a_0 = b_0 = 0$.* jacobian-notzero

This follows from Corollary 1.8.11. □

Corollary 1.8.23 has an application to polynomials with analytic coefficients. Let

(1.8.24) $$P(t, x) = c_0(t)x^d + c_1(t)x^{d-1} + \cdots + c_d(t)$$ polyforhensel

be a polynomial in x whose coefficients c_i are analytic functions, defined for small values of t, and let $\overline{P} = P(0, x) = \overline{c}_0 x^d + \overline{c}_1 x^{d-1} + \cdots + \overline{c}_d$ be the evaluation of P at $t = 0$, so that $\overline{c}_i = c_i(0)$. Suppose given a factorization $\overline{P} = \overline{F}\,\overline{G}$, where $\overline{F} = x^m + \overline{a}_1 x^{m-1} + \cdots + \overline{a}_m$ is a monic polynomial and $\overline{G} = \overline{b}_0 x^n + \overline{b}_1 x^{n-1} + \cdots + \overline{b}_n$ is another polynomial, both with complex coefficients. We ask whether this factorization of $\overline{P}$ is induced by a factorization of P. Are there polynomials $F(t, x) = x^m + a_1 x^{m-1} + \cdots + a_m$ and $G(t, x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_n$, with F monic, whose coefficients a_i and b_i are analytic functions defined for small t, such that $P = FG$, $F(0, x) = \overline{F}$, and $G(0, x) = \overline{G}$?

1.8.25. Hensel's Lemma. *With notation as above, suppose that $\overline{F}$ and $\overline{G}$ have no common root. Then P factors, as above.* hensellemma

proof. Since F is supposed to be monic, we set $a_0 = 1$. The first product equation tells us that $b_0(t) = c_0(t)$. Corollary 1.8.23 tells us that the Jacobian matrix for the remaining product equations is nonsingular at $t = 0$, so according to the Implicit Function Theorem, the product equations have a unique solution in analytic functions $a_i(t), b_j(t)$ for small t. □

Note that P isn't assumed to be monic. If $\overline{c}_0 = 0$, the degree of $\overline{P}$ will be less than the degree of P. In that case, $\overline{G}$ will have lower degree than G.

figure

1.8.26. Example. Let $P = c(t)x^2 + c_1(t)x + c_2(t)$, and suppose that $\overline{P} = \overline{c}x^2 + \overline{c}_1 x + \overline{c}_2$ has a root $\overline{a}$. Then $\overline{P}$ is a product $(x - \overline{a})(\overline{c}x + \overline{b})$, with $\overline{c}_1 = \overline{b} - \overline{ca}$ and $\overline{c}_2 = -\overline{a}\overline{b}$. The Jacobian matrix (1.8.21) at $t = 0$ is henselex

$$\begin{pmatrix} 1 & \overline{c} \\ -\overline{a} & \overline{b} \end{pmatrix}$$

Its determinant $\overline{d} = \overline{b} + \overline{ca}$, is nonzero if and only if the two factors of $\overline{P}$ are relatively prime.

The single variable Jacobian criterion allows us to solve the equation $P(t, x) = 0$ for x as function of t, provided that $\frac{\partial P}{\partial x}$ isn't zero at $(t, x) = (0, \overline{a})$. It won't be surprising that $\frac{\partial P}{\partial x}(0, \overline{a}) = 2\overline{ca} + \overline{c}_1$ is equal to the Jacobian determinant $\overline{b} + \overline{ca}$. □

1.9 Plane Curves as Coverings of the Projective Line coverline

When f and g are polynomials in several variables, including z, we denote by $\mathrm{Res}_z(f, g)$ and $\mathrm{Discr}_z(f)$ the resultant and the discriminant, computed with respect to the variable z. They will be polynomials in the remaining variables.

1.9.1. Lemma. *Let $f(x, y, z)$ be an irreducible polynomial in $\mathbb{C}[x, y, z]$ that isn't a linear polynomial in x, y. Let K be the rational function field $\mathbb{C}(x, y)$.* firroverK

(i) *f is an irreducible element of $K[z]$.*

(ii) *The discriminant $\mathrm{Discr}_z(f)$ of f with respect to the variable z is a nonzero polynomial in x, y.*

proof. **(i)** First, if f were an irreducible polynomial in x, y, it would have to be linear (1.3.4). Since that case has been ruled out, f has positive degree in z. Say that $f(x,y,z)$ factors in $K[z]$, $f = g'h'$, where g' and h' are polynomials of positive degree in z, with coefficients in K. The coefficients of g' and h' have denominators that are polynomials in x, y. When we clear those denominators, we obtain an equation in $\mathbb{C}[x,y,z]$ of the form $df = gh$, where g and h are polynomials in x, y, z of positive degree in z and d, a common denominator of the coefficients of g' and h', is a polynomial in x, y. Since g and h have positive degree in z, neither of them divides d. Then f must be reducible.

(ii) This follows from **(i)**, together with Proposition 1.8.15. $\square$

Let π denote the *projection* $\mathbb{P}^2 \longrightarrow \mathbb{P}^1$ that drops the last coordinate, sending a point (x,y,z) to (x,y). This projection is defined at all points of $\mathbb{P}^2$ except at the point $q = (0,0,1)$, which is called the *center of projection*. The fibre of π over a point $\widetilde{p} = (x_0, y_0)$ of $\mathbb{P}^1$ is the line L_{pq} through the points $p = (x_0, y_0, 0)$ and $q = (0,0,1)$, with the point q omitted – the set of points (x_0, y_0, z_0).

$figure$

Let C be a plane curve defined by an irreducible homogeneous polynomial $f(x,y,z)$ of degree d. We write f as a polynomial in z:

polyinztwo (1.9.2)
$$f = a_0 z^d + a_1 z^{d-1} + \cdots + a_d$$

with a_i homogeneous, of degree i in x, y. The fibre of C over a point $\widetilde{p} = (x_0, y_0)$ of $\mathbb{P}^1$ is the intersection of C with the line L_{pq}. It consists of the points (x_0, y_0, α) such that α is a root of the one-variable polynomial

effp (1.9.3)
$$f_{\widetilde{p}}(z) = f(x_0, y_0, z)\,.$$

Suppose that q is a point of multiplicity r of C, and let $k = d - r$. The coefficients $a_0, ..., a_{r-1}$ will be zero, and f will have the form
$$f = a_r z^k + a_{r+1} z^{k-1} + \cdots + a_d$$
with $k = d - r$ (see (**1.5.2**)). The discriminant $\mathrm{Discr}_z(f_{\widetilde{p}})$ of $f_{\widetilde{p}}$ can be obtained by evaluating the discriminant of f at $\widetilde{p}$, and because $\mathrm{Discr}_z(f)$ isn't zero, $\mathrm{Discr}_z(f_{\widetilde{p}})$ will be nonzero for all but finitely many points $\widetilde{p}$ of $\mathbb{P}^1$. If $\mathrm{Discr}_z(f_{\widetilde{p}})$ isn't zero, $f_{\widetilde{p}}$ will have k roots, and if $\mathrm{Discr}_z(f_{\widetilde{p}})$ is zero, $f_{\widetilde{p}}$ will have a multiple root. A multiple root occurs when L_{pq} is tangent to C at a smooth point distinct from q, or L_{pq} passes through a singular point distinct from q, or L_{pq} is a special line at q (**1.5.4**). Because all but finitely may fibres consist of k points, C is called a *k-sheeted branched covering* of $\mathbb{P}^1$. The *branch points* are the points $\widetilde{p}$ of $\mathbb{P}^1$ at which the discriminant $\mathrm{Discr}_z(f)$ is zero – those such that the fibre over $\widetilde{p}$ has fewer than k points.

The most important case is that the plane curve $C : \{f = 0\}$ of degree d doesn't pass through q. Then the projection π will be defined everywhere on C, and it will present C as a d-sheeted branched covering of $\mathbb{P}^1$. In this case, we will have $f(0,0,1) = a_0$, and since C doesn't contain q, the coefficient a_0 will be a nonzero constant that we can normalize to 1, so that f becomes a monic polynomial of degree d in z,

Let's suppose that coordinates are chosen so that $q = (0,0,1)$ is in general position.

generic **1.9.4. Note.** In algebraic geometry, the phrases *general position* and *generic* indicate an object (the point q here) that has no special 'bad' properties. Typically, the object will be parametrized somehow, and the word *generic* indicates that the parameter representing the particular object avoids a proper subset that may or may not be described explicitly. For the Proposition 1.9.7 below, we require that q shall not lie on any of the
genericcond following lines:

(1.9.5)

bitangent lines and flex tangent lines,
tangent lines that pass through a singular point of C,
lines that contain more than one singular point, and
special lines through singular points (see (**1.5.2**)).

finlines **1.9.6. Lemma.** *This is a list of finitely many lines that q must avoid.*

proof. Corollary 1.7.21 and Proposition 1.4.16 tell us that there are finitely many bitangents and finitely many flex points. To show that there are finitely many tangent lines through singular points, we project C from a singular point p and apply Lemma 1.9.1. The discriminant isn't identically zero, so it vanishes finitely often. Finally, since there are finitely many singular points, there are finitely many lines through pairs of singular points and finitely many special lines (1.5.3). $\square$

We consider the projection $\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^1$ with center q. Let $\overline{p} = (x_0, y_0)$ denote the image in $\mathbb{P}^1$ of the point $p = (x_0, y_0, z_0)$ of $\mathbb{P}^2$.

1.9.7. Proposition. *Let $f(x,y,z)$ be a homogeneous polynomial with no multiple factors, let C be the (possibly reducible) plane projective curve $\{f = 0\}$, and suppose that q is in general position. If p is a smooth point of C with tangent line L_{pq}, the discriminant $\operatorname{Discr}_z(f)$ has a simple zero at $\overline{p}$. If p is a node of C, $\operatorname{Discr}_z(f)$ has a double zero at $\overline{p}$, and if p is a cusp, $\operatorname{Discr}_z(f)$ has a triple zero at $\overline{p}$.* discrimvanishing

proof. We set $x = 1$, to work in the standard affine open set $\mathbb{U}$ with coordinates y, z. In affine coordinates, the projection is the map $(y, z) \to y$. We may suppose that p is the origin in $\mathbb{U}$. Its image $\overline{p}$ will be the point $y = 0$ of the affine y-line, and the line L_{pq} in $\mathbb{U}$ will be the line $\{y = 0\}$.

Let's denote the defining polynomial of the curve C, restricted to $\mathbb{U}$ by $f(y,z)$ instead of $f(1,y,z)$. In each of the three cases under consideration, the polynomial $\overline{f}(z) = f(0,z)$ will have a double zero at $z = 0$. We will have $\overline{f}(z) = z^2\overline{h}(z)$, with $\overline{h}(0) \neq 0$. Since z^2 and $\overline{h}(z)$ have no common root, we may apply Hensel's Lemma to write $f(y,z) = g(y,z)h(y,z)$, where g and h are polynomials in z whose coefficients are analytic functions of y, defined for small y, g is monic, $g(0,z) = z^2$, and $h(0,z) = \overline{h}$. According to Proposition 1.8.17,

$$\operatorname{Discr}_z(f) = \pm \operatorname{Discr}_z(g) \operatorname{Discr}_z(h) \operatorname{Res}_z(g,h)^2$$

Since q is in general position, $\overline{h}$ will have simple zeros. Then $\operatorname{Discr}_z(h)$ doesn't vanish at $y = 0$. Neither does $\operatorname{Res}_z(g,h)$. So the orders of vanishing of $\operatorname{Discr}_z(f)$ and $\operatorname{Discr}_z(g)$ are equal. We replace f by g, and doing so reduces us to the case that f is a monic quadratic polynomial in z with $f(0,z) = z^2$, say

$$f(y,z) = z^2 + b(y)z + c(y)$$

whose coefficients b and c are analytic functions of y. We write these coefficients as series in y:

$$b(y) = b_0 + b_1 y + b_2 y^2 + \cdots \quad \text{and} \quad c(y) = c_0 + c_1 y + c_2 y^2 + c_3 y^3 + \cdots$$

Since $f(0,z) = z^2$, the constant terms b_0 and c_0 are zero, and

$$\operatorname{Discr}_z(f) = -(b(y)^2 - 4c(y)) = 4c_1 y + (4c_2 - b_1^2)y^2 + (4c_3 - 2b_1 b_2)y^3 + O(4)$$

If C is smooth at p, then $c_1 \neq 0$. If so, then $\operatorname{Discr}_z(f) = 4c_1 y + O(2)$ has a zero of order one. If p is a node, then $c_1 = 0$ and $4c_2 - b_1^2 \neq 0$, so the discriminant has a zero or order two. If p is a cusp, the discriminant of f is $(4c_3 - 2b_1 b_2)y^3 + O(4)$, and $(2z + b_1 y)^2 = 4(z^2 + b_1 yz + c_2 y^2)$ (see 1.5.6). We substitute $z = -\frac{b_1}{2}y$ into the cubic term $f_3 = b_2 y^2 z + c_3 y^3$ of f, obtaining $\frac{1}{2}(-b_1 b_2 + c_3)y^3$. Since p is a cusp, $2z + b_1 y$ doesn't divide f_3, so the result isn't zero. The discriminant has a zero of order 3. $\square$

1.9.8. Corollary. *With notation as in Proposition 1.9.7, let $C : \{f = 0\}$ and $D : \{g = 0\}$ be plane curves that intersect transversally at a point $p = (x_0, y_0, z_0)$. Then $\operatorname{Res}_z(f,g)$ has a simple zero at (x_0, y_0).* transres

Two curves are said to intersect *transversally* at a point p if they are smooth at p and if their tangent lines there are distinct.

proof. Proposition 1.9.7 applies to the product fg, whose divisor is $C \cup D$. It shows that the discriminant $\operatorname{Discr}_z(fg)$ has a double zero at $\widetilde{p}$. We also know that $\operatorname{Discr}_z(fg) = \pm \operatorname{Discr}_z(f) \operatorname{Discr}_z(g) \operatorname{Res}_z(f,g)^2$ (1.8.17). Because coordinates are in general position, $\operatorname{Discr}_z(f)$ and $\operatorname{Discr}_z(g)$ will not be zero at $\widetilde{p}$. Therefore $\operatorname{Res}_z(f,g)$ has a simple zero there. $\square$

1.10 Genus

genus

In this section, we describe the topological structure of smooth plane curves. We defer the proof of one statement.

curveshome-omorphic **1.10.1. Theorem.** *The smooth plane curves of degree d in $\mathbb{P}^2$ are homeomorphic manifolds of dimension two. They are orientable, and connected.*

Unfortunately, the connectedness of a plane curve is a subtle fact whose proof mixes topology and algebra. I don't know a proof that fits into our discussion here. It will be proved later (see Theorem

If one wants to have a proof now, one can begin by showing that the Fermat curve $x^d + y^d + z^d = 0$ is connected, by studying the projection to $\mathbb{P}^1$ from the point $(0, 0, 1)$. I propose this as an exercise. Then one can show that every plane curve is connected by proving a plausible fact: If a family C_t of smooth plane projective curves of degree d is parametrized by t in an interval of the real line, the curves in the family are homeomorphic. One can prove this using a *gradient flow*. This approach has two drawbacks: It leads us far afield, and it applies only to plane curves. If you are interested in following it up, read about gradient flows.

orientability: A two-dimensional manifold is orientable if one can choose one of its two sides in a continuous, consistent way. A smooth curve C is orientable because its tangent space at a point is a one-dimensional complex vector space, the affine line with the equation (1.4.12). Multiplication by i orients the tangent space by defining the counterclockwise rotation. Then the right-hand rule tells us which side of C is "up".

compactness: A plane projective curve is compact because it is a closed subset of the compact space $\mathbb{P}^2$.

The *Euler characteristic* e of a compact, connected, orientable two-dimensional manifold M is the alternating sum $b^0 - b^1 + b^2$ of its Betti numbers. It depends only on the topological structure of M, and it can be computed in terms of a *triangulation*, a subdivision of M into topological triangles, called *faces*, by the formula

vef (1.10.2)
$$e = |\text{vertices}| - |\text{edges}| + |\text{faces}|$$

For example, a tetrahedron is homeomorphic to a sphere. It has four vertices, six edges, and four faces, so its Euler characteristic is $4-6+4 = 2$. Any other topological triangulation of a sphere, such as the one given by the icosahedron, yields the same Euler characteristic.

Every compact, connected, orientable two-dimensional manifold is homeomorphic to a sphere with a finite number of "handles", and its *genus* is the number of handles. A torus has one handle. Its genus is one. The projective line $\mathbb{P}^1$, a sphere, has genus zero.

$$Figure$$

The Euler characteristic and the genus are related by the formula

genuseuler (1.10.3)
$$e = 2 - 2g$$

The Euler characteristic of a torus is zero, and the Euler characteristic of $\mathbb{P}^1$ is two.

To compute the the Euler characteristic of a smooth curve C of degree d, we analyze a generic projection $C \xrightarrow{\pi} \mathbb{P}^1$.

figure

We choose generic coordinates x, y, z in $\mathbb{P}^2$ and project form the point $q = (0, 0, 1)$. When the defining equation of C is written as a monic polynomial in z,

$$f = z^d + a_1 z^{d-1} + \cdots + a_d$$

where a_i is a homogeneous polynomial of degree i in the variables x, y, the discriminant $\text{Discr}_z(f)$ with respect to z will be a homogeneous polynomial of degree $d^2 - d$ in x, y.

The covering $C \xrightarrow{\pi} \mathbb{P}^1$ will be branched at a point p of C if the tangent line at p is the line L_{pq} through p and q (1.9). If so, C and L_{pq} will have $d-1$ intersections (1.9.4). Proposition 1.9.7 tells us that the discriminant $\text{Discr}_z(f)$ has a simple zero at the image of a tangent line. So there will be $d^2 - d$ points $\widetilde{p}$ in $\mathbb{P}^1$ over which the fibre of the map has order $d-1$. They are the branch points of the covering. All other fibres consist of d points.

We triangulate $\mathbb{P}^1$ in such a way that the branch points are among the vertices, and we use the inverse images of the vertices, edges, and faces to triangulate C. Then C will have d faces and d edges lying over each face and edge of $\mathbb{P}^1$, respectively. And there will be d vertices of C lying over a vertex $\widetilde{p}$ of $\mathbb{P}^1$ except when $\widetilde{p}$ is one of the d^2-d branch points. In that case the the fibre will contain only $d-1$ vertices. Therefore the Euler characteristic of C is

$$e(C) = d\,e(\mathbb{P}^1) - (d^2-d) = 2d - (d^2-d) = 3d - d^2.$$

This is the Euler characteristic of any smooth curve of degree d, so we denote it by e_d:

(1.10.4) $$e_d = 3d - d^2.$$ equatione

Formula (1.10.3) shows that the genus g_d of a smooth curve of degree d is

(1.10.5) $$g_d \;=\; \tfrac{1}{2}(d-1)(d-2) = \tbinom{d-1}{2}.$$ equationg

Thus smooth curves of degrees $1,2,3,4,5,6,...$ have genus $0,0,1,3,6,10,...$, respectively. A smooth plane curve cannot have genus 2.

1.11 Bézout's Theorem bezoutthm

Bézout's Theorem counts intersections of plane curves. We state it here in a form that is ambiguous because it contains a term "multiplicity" that hasn't yet been defined.

1.11.1. Bézout's Theorem. Let C and D be distinct curves of degrees m and n, respectively. When intersections are counted with the appropriate multiplicity, the number of intersections is eual to mn. Moreover, the multiplicity at a point is 1 if C and D intersect transversally at that point. bezoutone

As before, C and D intersect transversally at p if they are smooth at p and their tangent lines there are distinct.

1.11.2. Corollary. *Bézout's Theorem is true when one of the curves is a line.* bezoutline

See Corollary 1.3.8. The multiplicity of intersection of a curve and a line is the one that was defined there. □

The proof in the general case requires some algebra that we would rather defer. It will be given later (Theorem It is possible to determine the intersections by counting the zeros of the resultant with respect to one of the variables. To do this, one chooses coordinates x, y, z so that neither C nor D contains the point $(0,0,1)$, and one writes their defining polynomials f and g as polynomials in z with coefficients in $\mathbb{C}[x,y]$. The resultant with respect to z will be a homogeneous polynomial R in x, y, of degree mn. It will have mn zeros in $\mathbb{P}^1_{x,y}$, when counted with multiplicity. If $\widetilde{p} = (x_0, y_0)$ is a zero of R, the one-variable polynomials $f(x_0,y_0,z)$ and $g(x_0,y_0,z)$ have a common root $z = z_0$, and then $p = (x_0,y_0,z_0)$ will be a point of $C \cap D$. It is a fact that the multiplicity of the zero of the resultant R at the image $\widetilde{p}$ is the (as yet undefined) intersection multiplicity of C and D at p. Unfortunately, this won't be obvious. However, one can prove the next proposition using this approach.

1.11.3. Proposition. *Let C and D be distinct plane curves of degrees m and n, respectively.* nocommon-factor
(i) *The curves have at least one point of intersection, and the number of intersections is at most mn.*
(ii) *If all intersections are transversal, the number of intersections is precisely mn.*

It isn't obvious that two curves in the projective plane must intersect. If two curves in the affine plane, such as parallel lines, have no intersection, their closures in the projective plane meet on the line at infinity.

1.11.4. Lemma. *Let f and g be homogeneous polynomials in x, y, z of degrees m and n, repsectively, and suppose that the point $(0,0,1)$ isn't a zero of f or g. If the resultant $\operatorname{Res}_z(f,g)$ with respect to z is identically zero, then f and g have a common factor.* resnotzero *proof.* Let the degrees of f and g be m and n, respectively, and let K denote the field of rational functions $\mathbb{C}(x,y)$. If the resultant is zero, f and g have a common factor in $K[z]$ (Corollary 1.8.6). There will be polynomials p and q in $K[z]$, of degrees at most $n-1$ and $m-1$ in z, respectively, such that $pf = qg$ (see (1.8.2)). We may clear denominators, so we may assume that the coefficients of p and q are in $\mathbb{C}[x,y]$. Then $pf = qg$ is an equation in $\mathbb{C}[x,y,z]$. Since p has degree at most $n-1$ in z, it isn't divisible by g, which has degree n (1.9.2). Since $\mathbb{C}[x,y,z]$ is a unique factorization domain, f and g have a common factor. □

proof of Proposition 1.11.3. **(i)** Let f and g be irreducible polynomials whose zero sets are C and D, respectively. Proposition 1.3.13 shows that there are finitely many intersections. We project to $\mathbb{P}^1$ from a point q that doesn't lie on any of the finitely many lines through pairs of intersection points. Then a line through q passes through at most one intersection, and the zeros of the resultant $\mathrm{Res}_z(f,g)$ that correspond to the intersection points will be distinct. Since the resultant has degree mn (Lemma 1.8.8), there are at most mn zeros and there is at least one of them. Therefore there are at most mn intersections and there is at least one.

(ii) Every zero of the resultant will be the image of an intersection of C and D. To show that there are mn intersections if all intersections are transversal, it suffices to show that the resultant has simple zeros. This is Corollary 1.9.8. □

smoothirred **1.11.5. Corollary.**

(i) *If the divisor X defined by a homogeneous polynomial $f(x,y,z)$ is smooth, then f is irreducible, and therefore X is a smooth curve.*

(ii) *There exist irreducible homogeneous polynomials in three variables, of arbitrary degree.*

proof. **(i)** Suppose that $f = gh$, and let p be a point of intersection of the loci $\{g = 0\}$ and $\{h = 0\}$. The previous proposition shows that such a point exists. The partial derivatives f_i vanish at p, so p is a singular point of X.

(ii) The Fermat polynomial $x^d + y^d + z^d$ is irreducible because its locus of zeros is smooth. □

numberof-flexes **1.11.6. Corollary.** **(i)** *Let d be an integer ≥ 3. A smooth plane curve of degree d has at least one flex point, and the number of flex points is at most $3d(d-2)$.*

(ii) *If all flex points are ordinary, the number of flex points is equal to $3d(d-2)$.*

Thus smooth curves of degrees $2,3,4,5,...$ have at most $0,9,24,45,...$ flex points, respectively.

proof. **(i)** Let $C : \{f(x_0,x_1,x_2) = 0\}$ be a smooth curve of degree d. The entries of the 3×3 Hessian matrix H are the second partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$. They are homogeneous polynomials of degree $d-2$, so the Hessian determinant is homogeneous, of degree $3(d-2)$. The flex points are intersections of a curve C with its *Hessian divisor* $D : \{\det H = 0\}$. Propositions 1.4.16 and 1.11.3 tell us that there are at most $3d(d-2)$ intersections.

(ii) Recall that a flex point is ordinary if the multiplicity of intersection of the curve and its tangent line is 3. Bézout's Theorem asserts that the number of flex points is equal to $3d(d-2)$ if the intersections of C with its Hessian divisor D are transversal, and therefore have multiplicity 1. So the next lemma completes the proof.

transversalH **1.11.7. Lemma.** *A curve $C : \{f = 0\}$ intersects its Hessian divisor D transversally at a point p if and only p is an ordinary flex point of C.*

proof. We prove this by a computation. Let L be the tangent line to C at the flex point p. The Hessian divisor D will be transversal to C at p if and only if it is transversal to L, and this will be true if and only if the order of vanishing of the Hessian determinant, restricted to L, is 1. Let's denote the restriction of the Hessian determinant to L by h.

We adjust coordinates x,y,z so that the flex point is $p = (0,0,1)$ and the tangent line L at p is the line $\{y = 0\}$. We write the polynomial f of degree d as

fwithcoeffs (1.11.8)
$$f(x,y,z) = \sum_{i+j+k=d} a_{ij} x^i y^j z^k,$$

The restriction of f to L is the polynomial

$$f(x,0,z) = \sum_i a_{i0} x^i z^k$$

Since p is a flex point with tangent line L, the coefficients a_{00}, a_{10}, and a_{20} are zero, and p is an ordinary flex point if and only if the coefficient a_{30} is nonzero. The first few terms of f are:

$$f = a_{01}yz^{d-1} + a_{11}xyz^{d-2} + a_{02}y^2z^{d-2} + a_{30}x^3z^{d-3} + \cdots$$

The restriction to L of the Hessian determinant, which we denote by h, is obtained by substituting $y = 0$ and $z = 1$ into $\det H$. We are interested in the linear term of $h(x)$, for which the relevant entries of the Hessian matrix H have degree zero in y and one in x. Then

$$\begin{aligned}
&f_{xx}(x,0,1) = 6a_{30}x + \cdots \\
&f_{xz}(x,0,1) = 0 + \cdots \\
&f_{yz}(x,0,1) = (d{-}1)a_{01} + (d{-}2)a_{11}x + \cdots \\
&f_{zz}(x,0,1) = 0 + \cdots
\end{aligned}$$

We won't need to compute f_{xy} or f_{yy}. Setting $v = 6a_{30}x$ and $w = (d{-}1)a_{10} + (d{-}2)a_{11}x$, the resultant matrix has the form

$$H_p(x,0,1) \;=\; \begin{pmatrix} v & * & 0 \\ * & * & w \\ 0 & w & 0 \end{pmatrix} \;+\; O(2) \qquad (1.11.9)$$

where $*$ are entries that don't affect terms of degree ≤ 1 in the determinant. The determinant is

$$h = -vw^2 + O(2) = -6(d-1)^2a_{30}a_{01}^2x + O(2)$$

This determinant has a zero of order 1 at $x = 0$ if and only if a_{30} and a_{01} aren't zero. Since C is smooth at p and $a_{10} = 0$, the coefficient a_{01} can't be zero. Thus C and D intersect transversally, and C and L intersect with multiplicity 3, if and only if a_{30} is nonzero – if and only if p is an ordinary flex. □

1.11.10. Corollary. *A smooth cubic curve contains exactly* 9 *flex points.* nineflexes

proof. Let f be the irreducible cubic polynomial whose zero locus is a smooth cubic C. The degree of the Hessian divisor D is also 3, so Bézout predicts at most 9 intersections of D with C. To derive the corollary, we show that C intersects D transversally. According to Proposition 1.11.7, a nontransversal intersection would correspond to a point at which the curve and its tangent line intersect with multiplicity greater than 3. This is impossible when the curve is a cubic. □

1.12 The Plücker Formulas

Let C be a smooth plane curve. As before (**1.7.19**), a bitangent L is *ordinary* if both of its tangencies are ordinary and if L isn't tangent to C at a third point. A flex point p is *ordinary* if C and its tangent line L have a contact of order precisely 3 at p. A plane curve C is *ordinary* if it is smooth, and if all of its bitangents and flex points are ordinary. The *Plücker formulas* compute the number of flexes and bitangents of an ordinary plane curve. plucker

For the next proposition, we refer back to the notation of Section 1.9. Let $\pi : C \to X$ be the projection of a plane curve C to the projective line X from a generic point q. The covering C will be branched at the points $\widetilde{p} = (x_0, y_0)$ of X such that, with $p = (x_0, y_0, 0)$, the line L_{pq} is a tangent line to C. It will also be branched at the images of singular points of C.

1.12.1. Proposition. *Let C be a plane curve, projected to* $\mathbb{P}^1$ *from a generic point q of the plane. With notation as above:* classofcurve

(i) *The number* β *of points* $\widetilde{p}$ *such that line* L_{pq} *is tangent to C at a smooth point is equal to the degree of the dual curve* C^*.

(ii) *If C is a smooth curve of degree d, the degree* d^* *of the dual curve* C^* *is* $d^2 - d$.

proof. **(i)** The degree of the dual curve C^* is the number of its intersections with a line in the dual plane. We count intersections with the line q^*.

Let L be a line in $\mathbb{P}^2$ such that L^* is a point of $q^* \cap C^*$. Then L contains q, but since q is generic, Lemma 1.9.6 shows that L doesn't pass through a singular point, and it isn't a bitangent or a flex tangent (1.9.6). So all points L^* of $q^* \cap C^*$ are dual to ordinary tangent lines L at smooth points of C. Proposition 1.7.20 tells us C^* is smooth at the point L^*, and that the tangent line to C^* at L^* is p^*. Since $q^* \neq p^*$, C^* intersects q^* transversally at L^*. So the degree of C^* is equal to the number of intersections, which is β.

(ii) When we project a smooth curve C from q, all branch points are images of tangent lines. The discriminant of the defining polynomial f with respect to the chosen variable z will have degree $d^2 - d$ (see Section 1.9). There will be $d^2 - d$ ordinary tangent lines through q, so $d^* = d^2 - d$. □

Plücker Formulas. Let C be an ordinary curve of degree $d \geq 2$, and let C^* be its dual curve. Let f and b denote the numbers of flex points and bitangents of C, and let δ^* and κ^* denote the numbers of nodes and cusps of C^*, respectively.

(i) The dual curve C^* has no flexes or bitangents. Its singularities are nodes and cusps.

(ii) $f = \kappa^* = 3d(d-2)$, and $b = \delta^* = \frac{1}{2}d(d-2)(d^2-9)$.

proof. **(i)** A bitangent or a flex on C^* would produce a singularity on the bidual C^{**}, which is the smooth curve C.

(ii) Bézout's Theorem counts the flex points (see (1.11.6)). The facts that $\kappa^* = f$ and $\delta^* = b$ are dealt with in Proposition 1.7.20. Thus $\kappa^* = f = 3d(d-2)$.

We project C^* to $\mathbb{P}^1$ from a generic point Q. Let β^* be the number of branch points that correspond to tangent lines through Q at smooth points of C^*. Since $C^{**} = C$, Proposition 1.12.1 tells us that $\beta^* = d$ and $d^* = d^2 - d$.

Next, let F be the defining polynomial for C^*. The discriminant $\mathrm{Discr}_z(F)$ has degree $d^{*2} - d^*$. Proposition 1.9.7 describes the order of vanishing of the discriminant at the images of the tangent lines, the nodes, and the cusps of C^*. It tells us that

$$d^{*2} - d^* = \beta^* + 2\delta^* + 3\kappa^*$$

We substitute the known values of d^*, β^*, κ^* into this formula:

$$(d^2 - d)(d^2 - d - 1) = d + 2\delta^* + 9d(d-2)$$

This gives the formula for δ^*:

$$2\delta^* = d^4 - 2d^3 - 9d^2 + 18d = (d^2 - 2d)(d^2 - 9)$$

□

Note. It isn't easy to count the number of bitangents directly.

someplucker-formulas

1.12.2. Examples.

(i) All curves of degree 2 and all smooth curves of degree 3 are ordinary.

(ii) A curve of degree 2 has no flexes and no bitangents. Its dual curve has degree 2.

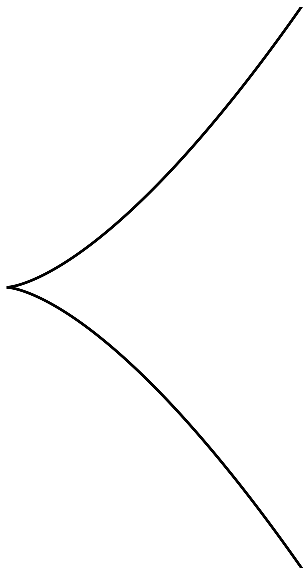
(iii) A smooth curve of degree 3 has 9 flexes and no bitangents. Its dual curve has degree 6.

(iv) An ordinary curve C of degree 4 has 24 flexes and 28 bitangents. Its dual curve has degree 12. □

It is a famous fact that a quartic curve as 28 bitangents. We will make use of this in Chapter Aside from the case of a quartic, the complicated formula for the number of bitangents is rarely used, though it could be useful to know that a formula exists.

figure

goober0



1.12.3. This is a nice caption

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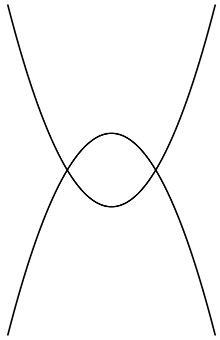

1.12.4. This is a nice caption

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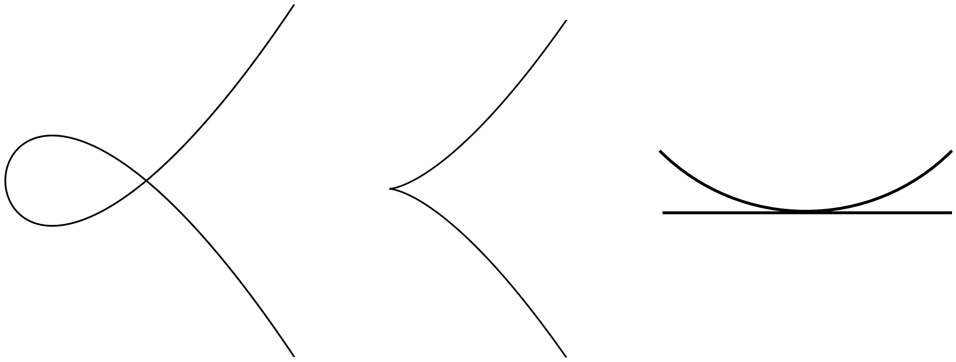
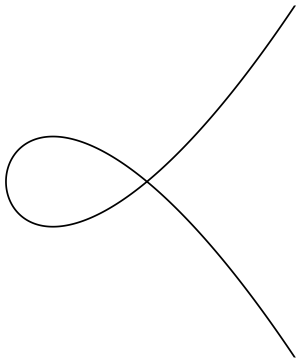


1.12.5. A node with equation $y^2 = x^3 + x$, a cusp with equation $y^2 = x^3$, and a circle $x^2 + y^2 = 1$ intersecting the lines $y = 0, 1$.

goober3



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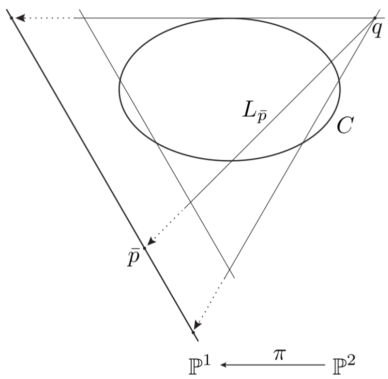

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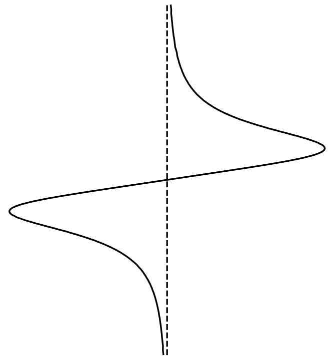

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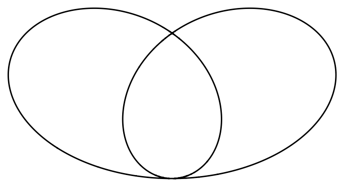

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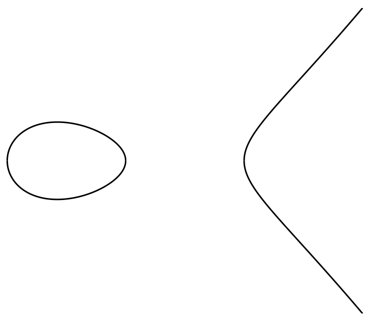

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1.12.11. This is a nice caption

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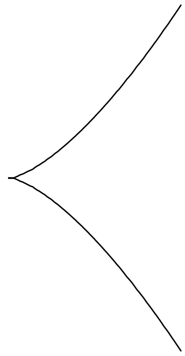

1.12.12. This is a nice caption

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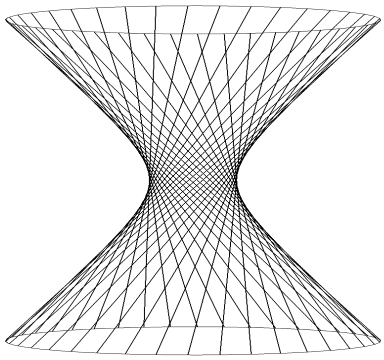

1.12.13. This is a nice caption

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1.12.14. This is a nice caption

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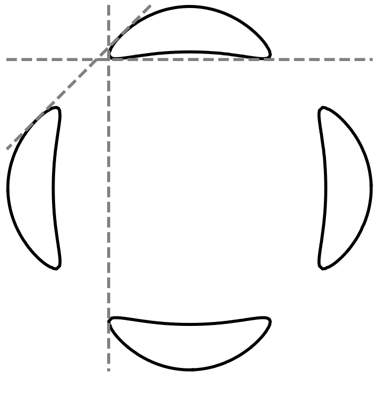

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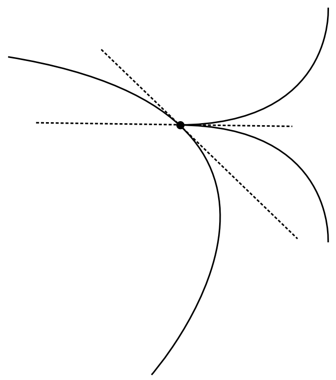

1.12.16. This is a nice caption

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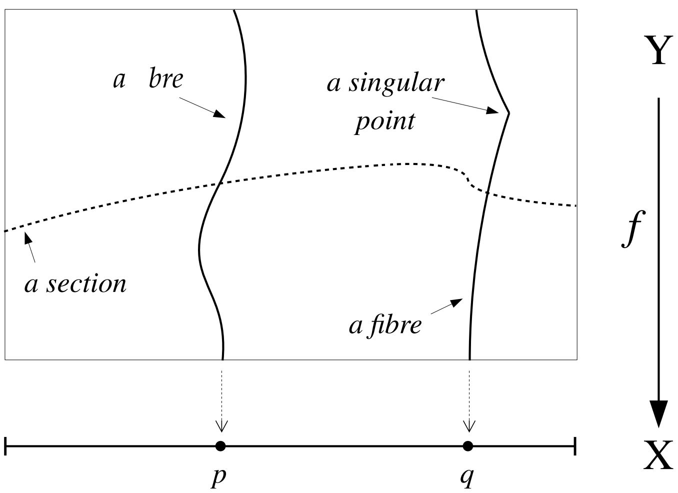

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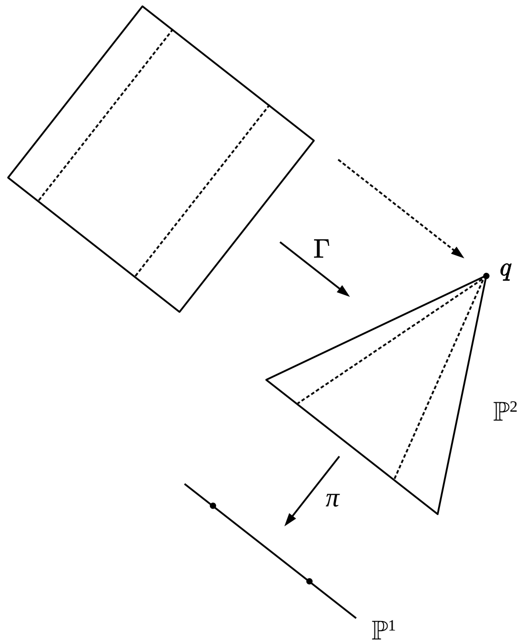

1.12.18. This is a nice caption

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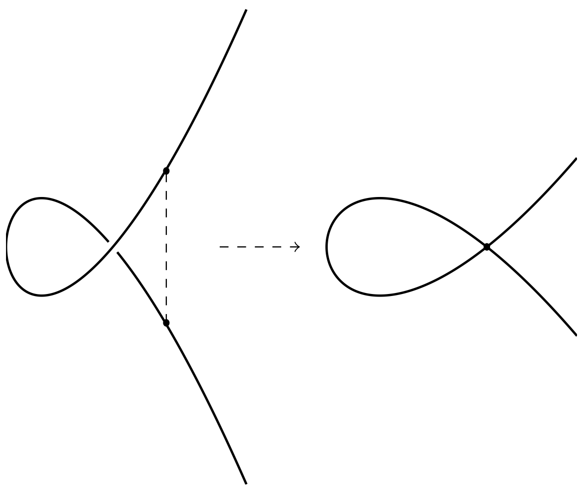

1.12.19. This is a nice caption

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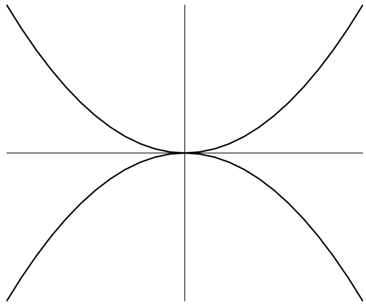

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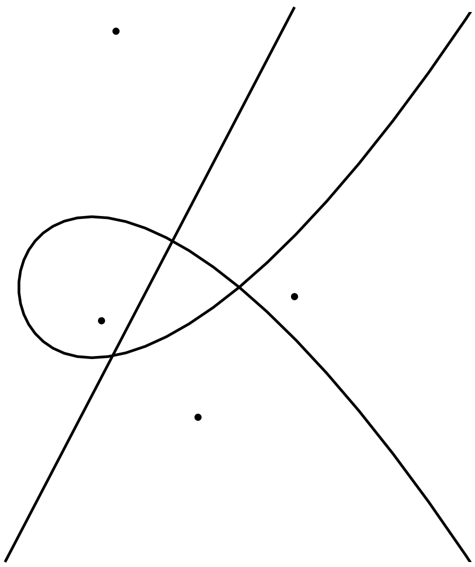

1.12.21. This is a nice caption

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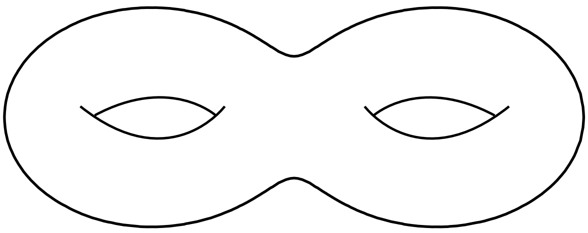

1.12.22. This is a nice caption

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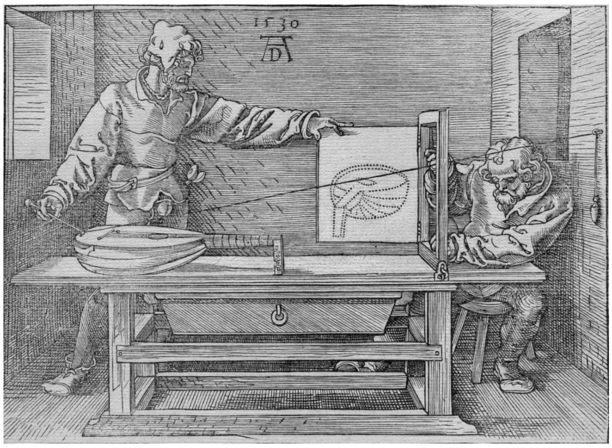

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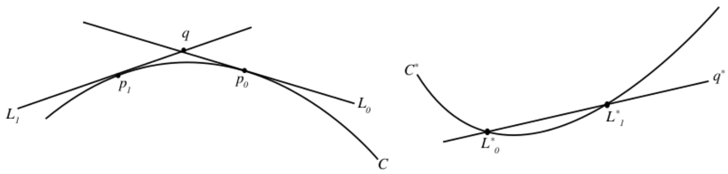

1.12.24. This is a nice caption