WHOOPS, THERE IS A FIBONACCI REPRESENTATION OF $\mathscr{H}_{k,q}(S_n)$

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Let k be a field, and let $q \neq 0 \in k$ be the parameter of the Hecke algebra $\mathscr{H} := \mathscr{H}_{k,q}(S_n)$. Let F^m be the vector space with basis given by the strings $\{*, p\}^{m+2}$ in which the character $*$ doesn't appear twice in a row. We will act on this by $\mathscr{H}$ by defining the action of T_i "locally", and changing character $i+1$ only dependent on characters $i, i+1, i+2$. We will use "hat" notation e.g. $(1+T_1)(pppp) = (\widehat{ppp}p)$. Then, we may define our action as follows:

$$\begin{aligned}
\widehat{(*pp)} &:= a(*pp) \\
\widehat{(*p*)} &:= b(*p*) \\
\widehat{(p*p)} &:= c(p*p) + d(ppp) \\
\widehat{(pp*)} &:= a(pp*) \\
\widehat{(ppp)} &:= d(p*p) + e(ppp).
\end{aligned}$$

for constants

$$\begin{aligned}
a &= -1 \\
b &= q \\
c &= \tau(q\tau - 1) \\
d &= \tau^{3/2}(q+1) \\
e &= \tau(q - \tau) \\
\tau &= \frac{2}{1+\sqrt{5}}.
\end{aligned}$$

It is easy to verify that these are compatible with the quadratic and braid relations, and hence define a representation F^m of $\mathscr{H}$, henceforth referred to as the Fibonacci representation. This has 4 subrepresentations dependent on the first and last character. We will characterize these subrepresentations fully in the following section.

1. Characterization of F^m via Specht Modules

Let $F^m = F^m_{*p} \oplus F^m_{p*} \oplus F^m_{**} \oplus F^m_{pp}$ be the decomposition of the Fibonacci representation into the 4 subrepresentations depending on the values of the first and last character. We'll supress the superscripts when the dimension is clear.

Note that F^m has dimension the $m+1$st fibonacci number f_{m+2}, we have $\dim F^m_{**} = f_{m+1}$, $F^m_{*p} \simeq F^m_{p*}$ has dimension f_m, and $\dim F^m_{pp} = f_{n-1}$. Further, note that $m = 2n$ gives that $\dim F^{2m}_{pp} = \dim D^{(2,\dots,2)}$; we will prove that these modules are isomorphic via the following propositions:

(1) F^m_{*p} is irreducible, and $\mathrm{Res}\, F^m_{*p} \simeq F^{m-1}_{pp} \simeq F^{m-1}_{*p} \oplus F^{m-1}_{**}$.
(2) $\mathrm{Res}\, F^m_{**} \simeq F^{m-1}_{*p}$.
(3) F^m decomposes into a direct sum of irreducible representations:

$$F^m \simeq 3F^m_{*p} \oplus 2F^m_{**}$$

ıtem Let $D_{m,k} := D^{(m,m-k)'}$ be the nearly-two-column Specht module. Then,

$$\begin{aligned}\text{Res } D_{m,0} &\simeq D_{m-1,1}\\ \text{Res } D_{m,1} &\simeq D_{m-1,0} \oplus D_{m-1,2}\\ \text{Res } D_{m,2} &\simeq D_{m-1,1} \oplus D_{m-1,3}\\ \text{Res } D_{m,3} &\simeq D_{m-1,2}.\end{aligned}$$

(4) The claims are henceforth conjectural:

$$\begin{aligned}F^{2n}_{**} &\simeq D_{n,0}\\ F^{2n-1}_{**} &\simeq D_{n+1,3}\\ F^{2n}_{*p} &\simeq D_{n+1,2}\\ F^{2n-1}_{*p} &\simeq D_{n,1}.\end{aligned}$$

If these are true, then

$$\begin{aligned}F^{2n} &\simeq 3D_{n+1,2} \oplus 2D_{n,0}\\ F^{2n-1} &\simeq 3D_{n,1} \oplus 2D_{n+1,3}.\end{aligned}$$

(5) In the $2n = 8$ case, let K be the intersection of kernels of $(1 + T_i)$ for W; then, we have $W/K \simeq F_{**}$.

We can start by studying low-dimensional cases. First, note that F^2_{*p} is the sign representation $D^{(2)}$ and F^2_{**} is the trivial representation $D^{(1)^2}$.

F^2_{pp}, which is a 2-dimensional representation of a semisimple commutative algebra, and hence decomposes into a direct sum of two subrepresentations. In particular, we can use the basis $\{(p*p), (ppp)\}$ and explicitly write the matrix

$$\rho_{T_1} = \begin{bmatrix} c & d \\ d & e \end{bmatrix}$$

having characteristic polynomial $(c-\lambda)(e-\lambda)-d^2 = \lambda^2-(c+e)\lambda+(ce-d^2)$. The reader can verify that this has roots are -1 and q. The eigenspaces with eigenvalues -1 and q are subrepresentations isomorphic to the sign and trivial representation, hence F_{pp} is isomorphic to a direct sum of the trivial and sign representaitons: $F^2_{pp} \simeq F^2_{*p} \oplus F^2_{**}$.

Now let's prove that F^3_{**} is irreducible; this has basis $\{*p*p\}, \{*ppp\}$, and the following matrices:

$$\rho_{T_1} = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}; \qquad \rho_{T_2} = \begin{bmatrix} c & d \\ d & e \end{bmatrix}.$$

A subrepresentation must be one-dimensional, and hence an eigenspace of each of these matrices; since $b \neq a$, the first has eigenspaces given by the spans of basis elements, and since $d \neq 0$, these are not eigenspaces of the second. Hence F^3_{**} is irreducible. Now we may move on to the general case.

Proposition 1.1. *The representation $F_{*p} := F^m_{*p}$ is irreducible.*

Proof. We will prove this inductively in m. We've already proven it for F^3_{*p} and F^4_{*p}, so suppose that F^{m-2}_{*p} is irreducible.

Let $\{v_i\}$ be the basis for F_{*p}. Then, each v_i is cyclic; indeed, we can transform every basis vector into $(*p\dots p)$ by multiplying by the appropriate $\frac{1}{d-c}(T_i - c)$, and we can transform $(*p\dots p)$ into any basis vector by multiplying be the appropriate $\frac{1}{d-e}(T_i - e)$. Hence it is sufficient to show that each $v \in F_{*p}$ generate some basis element.

Let v' be the basis element $(*p*p\dots p)$, which is many copies of $*p$, followed by an extra p if m is odd. We will show that each $v \in F$ generates v'.

Say that a basis element v_i is *represented in* v if it has nonzero coefficient in v. Suppose that no elements beginning $(*p*p)$ are represented in v_i; then, all such elements are represented in $T_3 v$, so we may assume that at least one is represented in v.

Note that $(T_2 - a)v$ is a nonzero element where only elements beginning $(*p*p)$ are represented; if F' is the subspace of F_{*p} spanned by v_i beginning $(*p*p)$, then $\text{Res}^{\mathscr{H}(S_m)}_{\mathscr{H}(S_{m-2})} F' \simeq F^{m-2}_{*p}$, and v' is mapped to the analogous element in F^{m-2}_{*p}. Hence irreducibility of F^{m-2}_{*p} implies that v' is generated by $(T_2 - a)$, and F^m_{*p} is irreducible. □

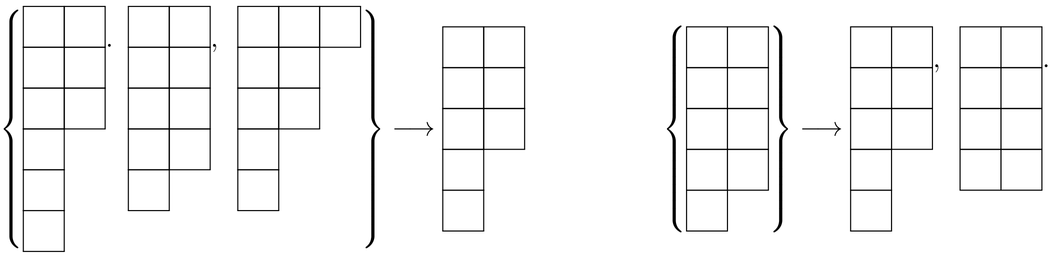

Figure 1. Illustration of the partitions of 9 which can, via row removal, yield $(n, n-2)'$ alone, or both $(n, n-2)'$ and $(n-1, n-1)'$.

There is a bit of bookkeeping to do; we've equivocated by saying asserting that the subalgebras $\mathscr{H}_{k,q}(S_{m-r}) \subset \mathscr{H}_{k,q}(S_m)$ are equivalent, inclding restrictions. This works because they are conjugate, and conjugation gives an isomorphism of the restriction of a representation to separate subalgebras.

Knowing this, the restriction statements are clear; $\mathrm{Res}F^m_{*p} \simeq F^{m-1}_{pp}$ by considering the last $m-2$ transpositions, and $\mathrm{Res}F^{m-1}_{*p} \simeq F^{m-1}_{*p} \oplus F^{m-1}_{**}$ by considering the first $m-2$. Similarly, $\mathrm{Res}F^m_{**} \simeq F^{m-1}_{*p}$ by considering the first $m-2$ transpositions. This gives that $F \simeq 3F_{*p} \oplus 2F_{**}$.

Now we may move on and use Young Tableau to characterize F. Recall that the socle of D^λ is given by $\bigoplus_{\mu \xrightarrow{\text{good}} \lambda} D^\mu$, and that D^λ is semisimple iff every $\mu \xrightarrow{\text{normal}} \lambda$ is good.

Proposition 1.2. *The irreducible components of F are given by the following isomorphisms:*

$$
\begin{aligned}
F^{2n}_{**} &\simeq D^{(n,n)'}\\
F^{2n-1}_{**} &\simeq D^{(n+1,n-2)'}\\
F^{2n}_{*p} &\simeq D^{(n+1,n-1)'}\\
F^{2n-1}_{*p} &\simeq D^{(n,n-1)'}.
\end{aligned}
$$

Proof. We will prove this by induction on n; we have already proven the base case F^2, so suppose that we have proven these isomorphisms for F^{2n-2}. We will prove the isomorphisms for F^{2n-1} and F^{2n}.

By semisimplicity, $F^{2n-1}_{**} \simeq D^{\lambda_{**}}$ and $F^{2n-1}_{*p} \simeq D^{\lambda_{*p}}$ for some diagrams λ_{**} and λ_{*p}. We will show that $\lambda_{**} = (n+1, n-2)'$ and $\lambda_{*p} = (n+1, n-1)'$.

First, note that we have

$$\mathrm{Res}\ D^{\lambda_{**}} \simeq D^{(n,n-2)'} \simeq \mathrm{Res}\ D^{(n+1,n-2)'}$$

and

$$\mathrm{Res}\ D^{\lambda_{*p}} \simeq D^{(n,n-2)} \oplus D^{(n-1,n-1)} \simeq \mathrm{Res}\ D^{(n,n-1)'}.$$

By semisimplicity, every normal cell in λ_{**} and λ_{*p} is good, and every good cell is removed in a summand of the restriction.

In particular, for λ_{**}, the only tableaus which can remove a cell to yield $D^{(n,n-2)'}$ are $(n+1, n-2)'$, $(n, n-1)'$, and $(n, n-2, 1)'$ as illustrated in Figure 1; we have already seen that $D^{(n,n-1)'}$ does not have irreducible restriction, so we are left with $(n+1, n-2)'$ and $(n, n-2, 1)'$. To have irreducible restriction, λ_{**} must have 1 as its only normal number; we may directly check that $(n, n-2, 1)'$ doesn't satisfy this, as we have the following:

$$
\begin{aligned}
\beta_\lambda(1,2) &= 3 - 2 + (n-2) = n-1\\
\beta_\lambda(1,3) &= 3 - 1 + n = n+2\\
\beta_\lambda(2,3) &= 2 - 1 + 3 = 4.
\end{aligned}
$$

At least one of $\beta(1,2)$ and $\beta(1,3)$ is nonzero, and hence at least one of M_2 and M_3 is empty. Hence at least one of 2 or 3 is normal, and $\lambda_{**} = (n+1, n-2)$.

For λ_{*p}, we immediately see from Figure 1 that the only option is $(n, n-1)$.

We can perform a similar argument for the F^{2n} case, finding now that

$$\text{Res } D^{\mu_{**}} \simeq D^{(n,n-1)'} \simeq \text{Res } D^{(n,n)'}$$

and

$$\text{Res } D^{\mu_{*p}} \simeq D^{(n,n-1)'} \oplus D^{(n+1,n-2)'} \simeq \text{Res } D^{(n+1,n-1)'}.$$

Through a similar process, we see that $\mu_{*p} = (n+1, n-1)'$. We narrow down μ_{**} to one of $(n,n)'$ or $(n, n-1, 1)'$, and note that

$$\beta_\mu(1,2) = 3 - 2 + (n-1) = n$$
$$\beta_\mu(1,3) = 3 - 1 + n = n + 2$$
$$\beta_\mu(2,3) = 2 - 1 + 2 = 3$$

and hence at least one of 2 or 3 is normal, $\text{Res}D^{(n,n-1,1)'}$ is not irreducible, and $\mu_{**} = (n,n)'$, finishing our proof. □

Hence F is semisimple, and we have its decomposition into quotients of specht modules. We've proven almost everything that we've set out to; all that's left is explicit transition matrices $W \to F_{**}$.