SOME GRAPHICAL REALIZATIONS OF TWO-ROW SPECHT MODULES OF IWAHORI–HECKE ALGEBRAS OF THE SYMMETRIC GROUP

MILES JOHNSON AND NATALIE STEWART
MENTOR : ORON PROPP
PROJECT SUGGESTED BY : ROMAN BEZRUKAVNIKOV

JULY 31, 2019

Abstract. We consider the Iwahori–Hecke algebra of the symmetric group on $2n+r$ letters with parameter q. Let e be the smallest positive integer such that the q-number $[e]_q = 0$, or set $e = \infty$ if none exist. We modify Khovanov's crossingless matchings to include $2n$ "nodes" and r "anchors," and prove in the case $e > n+r+1$ that the associated module is isomorphic to the Specht module $S^{(n+r,n)}$ which corresponds to the partition $(n+r,n) \vdash 2n+r$. We then give heuristics in support of the general case, including explicit composition series for $e = n+r+1$ and for $2n+r \leq 7$. Lastly, when $e = 5$, we prove an isomorphism between the irreducible quotient $D^{(n+r,n)}$ with $r \leq 3$ and some subrepresentations of Jordan–Shor's Fibonacci representation. We provide explicit transition matrices between this representation and the crossingless matchings representation for $2n+r \leq 6$.

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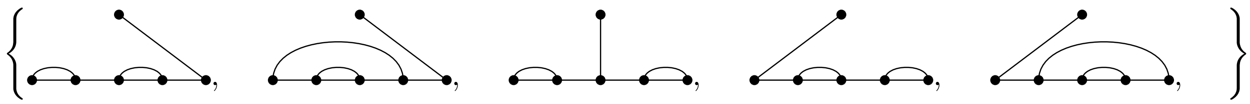


Figure 1. The basis for M_5^1.

1. Introduction

Let S_{2n+r} be the symmetric group on $2n+r$ letters with $2n+r \geq 2$, let $\mathscr{H} := \mathscr{H}_{k,q}(S_{2n+r})$ be the corresponding Hecke algebra over a field k with parameter $q \in k^\times$ having a fixed square root $q^{1/2}$, and let $\{T_i\}$ be the simple reflections generating $\mathscr{H}$.

Let $[m]_q = 1 + q + \cdots + q^{m-1}$ be the q-number of m. Let e be the smallest positive integer such that $[e]_q = 0$, and set $e = \infty$ if no such integer exists. Note that either $q = 1$ and e is the characteristic of k (with 0 replaced by ∞), or $q \neq 1$ and q is a primitive eth root of unity.

When $q = 1$, the Hecke algebra $\mathscr{H}$ is isomorphic to the group algebra $k[S_{2n+r}]$; hence the representation theory of $\mathscr{H}$ generalizes the representation theory of the symmetric group. The Hecke algebra is also well known to be connected to the representation theory of the general linear group over a finite field.[8]

It is a classical result that $\mathscr{H}$ is semisimple precisely when $e > 2n+r$, in which case the irreducible representations of $\mathscr{H}$ are given by *Specht modules* S^λ, which are indexed by the partitions λ of $2n+r$. Further, $\mathscr{H}$ admits a cellular basis with cell modules given by S^λ. In particular, these admit quotients D^λ such that the set $\left\{D^\lambda \mid D^\lambda \neq 0, \lambda \vdash n\right\}$ is a complete set of pairwise-nonisomorphic irreducible $\mathscr{H}$-modules. This set is indexed by the partitions $\lambda \vdash n$ which are e-restricted.[10, 11]

These representations D^λ have explicit constructions, but many of their properties are unknown. For instance, the dimension of these modules is unknown outside of some special cases.[8] However, there does exist an algorithm due to Lascoux–Leclerc–Thibon–Ariki which computes the decomposition matrices of the Iwahori–Hecke Algebra $\mathscr{H}_{\mathbb{C},q}(S_{2n+r})$ for q an eth root of unity.[1, 7]

The cellular basis for S^λ and associated construction for D^λ tend to be complicated and often computationally intractable. We aim to give simple graphical realizations of S^λ and D^λ in some cases that $\lambda = (n+r, n)$. These realizations expose the structure of irreducibility and branching in an intuitive and computationally simple way.

Note that we follow the convention of Murphy and Kleshchev, which is dual to the conventions of Dipper, James, and Mathas; one may translate our results to the latter convention by transposing all partitions.[6, 8, 10]

Throughout this paper, we analyze the *two-row partitions* $(n+r, n) \vdash 2n+r$ and their corresponding modules $S^{(n+r,n)}$ and $D^{(n+r,n)}$.

Crossingless matchings. The following definition modifies the crossingless matchings defined by Khovanov.[5]

Definition 1.1. Define a *crossingless matching on $2n+r$ nodes and r anchors* to be an isotopy class of $n+r$ nonintersecting paths in the slice $\mathbb{R} \times [0,1]$ connecting $2n+r$ disctinct points of $\mathbb{R} \times \{0\}$ and r points of $\mathbb{R} \times \{1\}$ such that none of the latter points are conected. Let M^r_{2n+r} have basis given by these matchings. This is illustrated in Figure 1.

Let the length of an arc (i,j) be $j - i + 1$.

We endow M^r_{2n+r} with an action by specifying $(1+T_i)w_j$ for some basis element w_j of M_{2n+r}. We do so by concatenating in "vertical lines" below each point other than the ith and $i+1$st, concatenating paths between the ith and $i+1$st points as well as points below them, removing any "loops" this forms, and taking isotopy to some matching w_l if possible; if this is not possible, then there are anchors at $i, i+1$ and we set $(1+T_i)w_j := 0$; if this is possible and there is a "loop," set $(1+T_i)w_j := (1+q)w_j$ and otherwise set $(1+T_i)w_j := q^{1/2}w_l$. This action is illustrated in Figure 2, and we verify that this is well-defined in Appendix A.2.

Note that the representations M^r_{0+r} and $S^{(r)}$ are isomorphic to the sign representation. In fact, we will prove that $M^r_{2n+r} \simeq S^{(n+r,n)}$ whenever $e > n+r+1$. Additionally, we will prove that M^r_{2n+r} is irreducible everywhere that $e > n$ and $S^{(n+r,n)}$ is irreducible. We will conjecture that $M^r_{2n+r} \simeq S^{(n+r,n)}$ with e unrestricted as well.

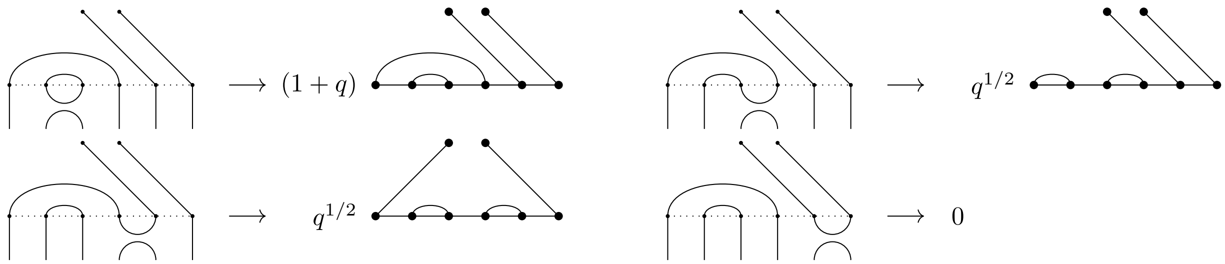


Figure 2. Illustration of the actions $(1+T_i)w_{|M_6^2|}$. In general, we act by deleting loops, isotoping onto a new crossingless matching, and scaling by either $q^{1/2}$, $(q+1)$, or 0.

This proves a graphical characterization of S^λ in many cases, and hence D^λ in the cases where $S^\lambda \simeq D^\lambda$. However, the representation theory of $\mathscr{H}$ is tied to the modules D^λ even in cases where S^λ is not irreducible; we will present a characterization of the module $D^{(n+r,n)}$ when $e = 5$ and $r \leq 3$.

Fibonacci representation. Now suppose that $e = 5$ and k contains the algebraic number $\sqrt{q+q^4}$ for reasons which will be apparent soon. For convenience set $m := 2n+4$. The following is a modification of Shor–Jordan's Fibonacci representation of the braid group.[12]

Definition 1.2. Let V^m be the k-vector space with basis given by the strings $\{*, \mathtt{0}\}^{n+1}$ such that the bit $*$ never appears twice consecutively. We will refer to V^m as the *Fibonacci representation* and suppress the superscript whenever it is clear from context.

We wish to endow this with a $\mathscr{H}$-action which acts on a basis vector only dependent on bits $i, i+1, i+2$, sending each basis vector to a combination of the other basis vectors having the same bits $1, \ldots, i, i+2, \ldots, n+1$ as follows:

$$\begin{aligned}
T_1\ (*\mathtt{00}) &:= \alpha_1\,(*\mathtt{00}) \\
T_1\ (\mathtt{00}*) &:= \alpha_1\,(\mathtt{00}*) \\
T_1\ (*\mathtt{0}*) &:= \alpha_2\,(*\mathtt{0}*) \\
T_1\ (\mathtt{0}*\mathtt{0}) &:= \varepsilon_1\,(\mathtt{0}*\mathtt{0}) + \delta\,(\mathtt{000}) \\
T_1\ (\mathtt{000}) &:= \delta\,(\mathtt{0}*\mathtt{0}) + \varepsilon_2\,(\mathtt{000})
\end{aligned} \tag{1.1}$$

for constants

$$\begin{aligned}
\tau &:= q + q^4 \\
\alpha_1 &:= -1 \\
\alpha_2 &:= q \\
\varepsilon_1 &:= \tau(q\tau - 1) \\
\delta &:= \tau^{3/2}(q+1) \\
\varepsilon_2 &:= \tau(q - \tau)
\end{aligned} \tag{1.2}$$

with T_i acting similarly on the substring $i, i+1, i+2$. We will verify that this is a representation of $\mathscr{H}$ in Appendix A.3

This contains four subrepresentations spanned by strings with beginning and ending with specified bits. Label the subrepresentation spanned by strings $(* \ldots *)$ by V_{**}, $V_{*\mathtt{0}}$, $V_{\mathtt{0}*}$, and $V_{\mathtt{00}}$; these bits are preserved by the action (1.1). We will show that these are isomorphic to particular D^λ.

Overview of paper.

In Section 2 we give corollaries to standard theorems concerning Specht modules. First, James-Mathas provides a sharp characterization of the irreducibility S^λ for $\lambda \vdash 2n+r$ which is e-regular, called the *Carter criterion.*[8] We specialize this to the case that $\lambda = (n+r, n)$ to give a combinatorial condition for irreducibility of $S^{(n+r,n)}$. We note that this irreducibility depends only on e when $e > n$; otherwise it depends on both e and the characteristic of the field. Further, we use Kleshchev–Brundan's modular branching rules to prove our first significant statement: if $S^\lambda \simeq D^\lambda$ and $e > n$, then a particular length-2 composition series uniquely determines λ; further, an irreducible restriction to $D^{(n,n-1)}$ determines λ as well.[2, 6]

In Section 3, we first prove that every basis vector in M is cyclic, then that M contains no sign subrepresentation. Then, we prove the following theorem.

Theorem 3.4. *Suppose $e > n$ and $S^{(n+r,n)}$ is irreducible. Then M^r_{2n+r} is irreducible.*

Following this, we prove the existence of a particular filtration with factors given by other crossingless matchings representations; using irreducibility, this becomes a composition series, and an inductive argument combined with the branching of Section 2 allow us to prove the following:

Theorem 3.7. *Suppose $e > n + r + 1$. Then, $M^r_{2n+r} \simeq S^{(n+r,n)}$.*

Last, we finish the section by proving more general statements concerning sign subrepresentations of M, which contribute to conjectures later in the paper.Fill in with specific information later.

In Section 4, we begin by establishing isomorphisms as in 4.5 for the subrepresentations of V^2, as well as irreducibility of V^3_{*0}. We then use these cases to prove that V^m_{*0} is irreducibile for all m, which implies that that V^m_{**} is irreducible. From this, we inductively prove the following theorem:

Theorem 4.5. *The following isomorphisms characterize the subrepresentations of the Fibonacci representation:*

$$\begin{aligned} V^{2n}_{**} &\simeq D^{(n,n)} \\ V^{2n-1}_{**} &\simeq D^{(n+1,n-2)} \\ V^{2n}_{*0} &\simeq D^{(n+1,n-1)} \\ V^{2n-1}_{*0} &\simeq D^{(n,n-1)}. \end{aligned}$$

Overview of Conjecture and Empirics goes here.

Acknowledgements. The authors thank Prof. Roman Bezrukavnikov for suggesting this project, as well as Dr. Slava Gerovitch for organizing the SPUR+ program. We would also like to thank Profs. David Jerison and Ankur Moitra for their role in SPUR+ as well as general advice. We would also thank Professor Alexander Kleshchev for our helpful conversations concerning branching theorems. Lastly, we would like to express our gratitude to our mentor Oron Propp for his help and advice in both acquiring background knowledge and in executing the mathematics in this paper; this project would not be possible without him.

2. Preliminaries on Specht modules

For this section and the rest of the paper, assume $n > 0$.

Throughout the rest of the paper, it will be useful to have precise notation; identify each partition with a tuple $\lambda = (\lambda_1^{a_1}, \dots, \lambda_l^{a_l})$ having $\lambda_i > \lambda_{i+1}$, $a_i > 0$, and $\sum_i a_i\lambda_i = 2n + r$. Identify each of these with a subset $[\lambda] \subset \mathbb{N}^2$ as in [6], and define $\lambda(i) = (\lambda_1^{a_1}, \dots, \lambda_{i-1}^{a_{i-1}}, \lambda_i^{a_i-1}, \lambda_i - 1, \lambda_{i+1}^{a_{i+1}}, \dots, \lambda_l^{a_l})$ to be the partition with the ith row removed. Say that λ is *e-regular* if $\lambda_i - \lambda_{i+1} < e$ for all i and $\lambda_l < e$.

In the following subsection, we cite a theorem of James–Mathas which precisely characterizes the irreducibility of S^λ in the case that λ is e-regular, and we specialize this result to the case of two-row Specht modules. This falls into two cases: either $e > n$, where $S^{(n+r,n)}$ is irreducible iff $e \nmid r + 2, \dots, n + r + 1$, or $e \le n$, where the irreducibility of $S^{(n+r,n)}$ is more complicated and depends also on the characteristic of k. We will focus primarily on the former case.

Following this, we reproduce the branching theorems of Kleshchev–Brundan, which allow us to fully characterize the socle of $\operatorname{Res} D^\lambda$. This and some combinatorial arguments yield the main result of this section, which allows us to determine certain $D^{(n+r,n)}$ via their composition series. This will be instrumental later for characterizing the crossingless matchings representation M^r_{2n+r} as a Specht module, and it will extend to all cases with $e > n + r + 1$.

$n+r+1$	$n+r$	$\dots$	$r+2$	r	$r-1$	$\dots$	1
n	$n-1$	$\dots$	1				

Figure 3. The young diagram corresponding to the partition $(n+r,n)$. The hook lengths are in the center of the corresponding cells.

2.1. **Irreducibility of S^λ.** Let ℓ be the characteristic of k; then, set

$$p := \begin{cases} \ell & \text{if } \ell > 0, \\ \infty & \text{if } \ell = 0. \end{cases}$$

Note that $p = e$ when $q = 1$. For h a natural number, let $\nu_p(h)$ be the p-adic valuation of h. By convention, set $\nu_\infty(h) = 0$ for all h. Define the function $\nu_{e,p} : \mathbb{N} \to \{-1\} \cup \mathbb{N}$ by

$$\nu_{e,p}(h) := \begin{cases} \nu_p(h) & \text{eif } \mid h \\ -1 & \text{eif } \nmid h \end{cases}.$$

Lastly, let h^λ_{ab} be the hook length of node (a,b) in $[\lambda]$ as defined in [6]. With this language, we may express the following theorem, parts (ii)-(iii) of which are known as the *Carter criterion* in the symmetric group case, due to James–Mathas.[8]

Theorem 2.1 (James–Mathas)**.** *The following are equivalent:*

(i) $S^\lambda \simeq D^\lambda$.
(ii) λ *is* e*-regular and* S^λ *is irreducible.*
(iii) $\nu_{e,p}\left(h^\lambda_{ab}\right) = \nu_{e,p}\left(h^\lambda_{ac}\right)$ *for all nodes* (a,b) *and* (a,c) *in* $[\lambda]$. □

Proof. See [8] theorem 5.42. □

This result gives information solely on e-regular partitions, and the general irreducibility of S^λ away from $p = 2$ is not well understood. We will henceforth specialize slightly to the case that $(n+r,n)$ is e-regular.

Corollary 2.2. *If* $r = 0$*, assume* $e > 2$*.*

(i) Suppose $e > n$*. Then,* $S^{(n+r,n)}$ *is irreducible iff* $e \nmid r+2, r+3, \dots, n+r+1$.
(ii) Suppose $e \leq n$*. If* $S^{(n+r,n)}$ *is irreducible, then* $e \mid r+1$.

Note that the condition $e \nmid r+2, r+3, \dots, n+r+1$ implies that $e > n$.

Proof. Our initial assumption on e implies that λ is e-regular, which we will use below.

(i) Note that $\nu_p(h) \neq -1$ for all naturals h and only hook lengths in the top row may vanish mod e by Figure 3; hence we may equivalently prove that e divides no hook lengths in the leftmost n columns of the second row by Theorem 2.1. These hook lengths are precisely $r+2, \dots, n+r+1$.

(ii) Note that we have $\nu_{e,p}\left(h^\lambda_{2,n-e+1}\right) \neq -1$, and $\nu_{e,p}\left(h^\lambda_{2,n+r}\right) = -1$ so $\nu_{e,p}$ acquires at least two values. Suppose that $e \nmid r+1$. Then,

$$\nu_{e,p}\left(h^\lambda_{1,n-e+1}\right) = \nu_{e,p}\left(h^\lambda_{2,n-e+1} + r + 1\right) = -1,$$

giving $S^{(n+r,n)}$ reducible by Theorem 2.1. □

From part (i) we see that irreducibility at $e > n$ is not dependent on p, and we may cover many modular cases without reference to the characteristic of k. We will finish our discussion of irreducibility of S^λ via sharp characterization of the $e \leq n$ case.

Corollary 2.3. *If* $r = 0$*, assume* $e > 2$*. Suppose* $e \leq n$*, and suppose* $p > n+r+1$*. Then,* $S^{(n+r,n)}$ *is irreducible if and only if* $e \mid r+1$.

Proof. This follows from the proof of Corollary 2.2 part (ii) and the fact that $\nu_p(h) = \nu_p(h')$ for all naturalls h, h'. ☐

2.2. **Branching theorems of Specht modules.** In this section as well as later sections, we will consider the restriction of representations of $\mathscr{H}$ to particular subalgebras isomorphic to $\mathscr{H}_{k,q}(S_{2n+r-1})$. We verify in Appendix B that any two subalgebras of $\mathscr{H}$ generated by $2n+r-2$ simple transpositions are canonically isomorphic, and the corresponding restrictions are canonically isomorphic via this isomorphism of algebras. We will hence abuse notation, pick one such subalgebra $\mathscr{H}'$, and notate $\operatorname{Res}^{\mathscr{H}}_{\mathscr{H}'} W$ by $\operatorname{Res} W$ for any $\mathscr{H}$-module W.

Fixing some partition $\lambda \vdash 2n+r$, for $1 \le i \le j \le l$, let $\beta_\lambda(i,j)$ and γ_λ be the quantities

$$\beta_\lambda(i,j) = \lambda_i - \lambda_j + \sum_{t=i}^{j} a_t$$

$$\gamma_\lambda(i,j) = \lambda_i - \lambda_j + \sum_{t=i+1}^{j} a_t.$$

Note that $\beta_\lambda(i,j)$ is the hook length of cell $(a_1 + \cdots + a_{i-1} + 1, \lambda_j)$.

Results due to Kleshchev and Brundan refer to *normal* and *good* numbers; for these, we will use the facts that 1 is always normal and that j is normal when $\beta_\lambda(i,j) \not\equiv 0 \pmod e$ for all $i \le j$. Further, we will use that j is good if and only if j is normal and $\gamma_\lambda(j,j') \not\equiv 0 \pmod e$ for all $j' \ge j$ normal.[2, 6] When $\lambda(i) = \mu$ for i normal, write $\mu \xrightarrow{\text{normal}} \lambda$, and similar in the good case.

The following statements, collectively known as *modular branching rules* of D^λ, were originally written by Kleshchev for Specht modules of the group algebra $k[S_n]$, then generalized to the Hecke algebra case by Brundan.[2, 6] They entirely characterize the socle of $\operatorname{Res} D^\lambda$, as well as semisimplicity od $\operatorname{Res} D^\lambda$.

Theorem 2.4 (Kleshchev-Brundan)**.** *We have the following isomorphisms of vector spaces*

$$\operatorname{Hom}_{\mathscr{H}'}\left(S^\mu, \operatorname{Res} D^\lambda\right) \simeq \begin{cases} k & \textit{if } \mu \xrightarrow{\textit{normal}} \lambda, \\ 0 & \textit{otherwise.} \end{cases}$$

$$\operatorname{Hom}_{\mathscr{H}'}\left(D^\mu, \operatorname{Res} D^\lambda\right) \simeq \begin{cases} k & \textit{if } \mu \xrightarrow{\textit{good}} \lambda \\ 0 & \textit{otherwise} \end{cases}$$

and $\operatorname{Res} D^\lambda$ *is semisimple if and only if every normal number in* λ *is good.* ☐

Using this, we immediately see that, for any rectangular partition (m^ℓ), we have

$$\operatorname{Res} D^{(m^\ell)} \simeq D^{(m^{\ell-1}, m-1)}.$$

The non-rectangular two-row case is more complicated, but we may still describe it fully as follows.

Corollary 2.5. *Suppose $r > 0$. Then, we may characterize the socle of* $\operatorname{Res} D^\lambda$ *as follows:*

$$\operatorname{soc}\left(\operatorname{Res} D^{(n+r,n)}\right) \simeq \begin{cases} D^{(n+r-1,n)} & \textit{if } e \mid r+2 \\ D^{(n+r,n-1)} & \textit{eif } e \nmid r+2,\ e \mid r \\ D^{(n+r-1,n)} \oplus D^{(n+r,n-1)} & \textit{if } e \nmid r+2, r \end{cases}$$

Further, when $e \nmid r$ or $e \mid r+2$, $\operatorname{Res} D^{(n+r,n)}$ *is semisimple.*

Proof. This amounts to computations of the hook lengths $\beta(1,2)$ and $\gamma(1,2)$:

$$\beta_\lambda(1,2) = r+2$$
$$\gamma_\lambda(1,2) = r$$

Since 2 is the largest removable number, $D^{(n+r,n-1)} \subset D^{(n+r,n)}$ if and only if $e \nmid r+2$. Further, if $e \nmid r+2$, then $D^{(n+r-1,n)} \subset D^{(n+r,n)}$ if and only if 1 is good; this is equivalent to $e \nmid r$. ☐

Now that we've characterized how D^λ restrict, we can describe how strongly these restrictions characterize irreducibles. Namely, we will prove that some D^λ having the same composition series as $D^{(n+r,n)}$ is sufficient to determine that $\lambda = (n+r,n)$ in the case that condition (i) of Corollary 2.2 holds and either $r \neq 0$ or $e \neq 4$.

Proposition 2.6. *Let λ be an e-regular partition of $2n+r$.*

(i) Suppose $r > 0$, suppose $e \nmid r+1, r+2, \dots, n+r+1$, and suppose either $e \mid r$ or $e \nmid r-2$. If D^λ has the composition series

$$0 \subset D^{(n+r-1,n)} \subset \operatorname{Res} D^\lambda \tag{2.1}$$

with factor $\operatorname{Res} D^\lambda / D^{(n+r-1,n)} \simeq D^{(n+r,n)}$, then $\lambda = (n+r,n)$.

(ii) Suppose $r = 0$, suppose $e \nmid 4$, and suppose $D^{(n,n-1)} \simeq \operatorname{Res} D^\lambda$. Then $\lambda = (n,n)$.

Proof. Note that $e > n$. Further, note that the above characterizations are necessary regardless of e-regularity; in the case that μ below fails too be e-regular, this proposition will prove that λ does not satisfy 2.1, a contradiction.

(i) Let $\varpi := (n+r-1,n,1)$, let $\varsigma := (n+r-1,n+1)$, and let $\mu := (n+r,n)$. Since $D^{(n+r-1,n)} \subset \operatorname{Res} D^\lambda$, we have $(n+r-1,n) \longrightarrow \lambda$, implying $\lambda \in \{\varpi, \varsigma, \mu\}$. We will show that ϖ, ς do not have socle compatible with (2.1), allowing us to conclude $\lambda = \mu$.

If ϖ or ς are not e-regular, then $D^\varpi = 0$ or $D^\varsigma = 0$, and we may immediately rule these out; henceforth assume that these are each e-regular.

First suppose that that $\lambda = \varpi$. We will break into cases with r.

- Suppose that $r > 1$. Note that $e \nmid r+1 = \beta_\varpi(1,2)$, so 2 is normal. Further, $\gamma_\varpi(2,3) = n \not\equiv 0 \pmod e$, so 2 is good and $D^{(n+r-1,n-1,1)} \subset D^\varpi$, which is not a composition factor in (2.1). Hence, by the Jordan-Hölder theorem [3], we have $\lambda \neq \varpi$.
- Suppose that $r = 1$. Then, $\varpi = (n,n,1)$ has $\gamma_\varpi(1,2) = n \not\equiv 0 \pmod e$, giving $D^{(n,n-1,1)} \subset D^\varpi$ and hence $\lambda \neq \varpi$ as in the previous case.

Now suppose that $\lambda = \varsigma$ and break into cases with r:

- Suppose $r > 2$. Then, by Corollary 2.5, we require that $e \nmid r$ and $e \mid r-2$; these are not satisfied, so $\lambda \neq \varsigma$.
- Suppose $r = 2$. Then $\operatorname{Res} D^\varsigma \simeq D^{(n+1,n)}$ is irreducible, as $\varsigma = (n+1,n+1)$ has rows of the same length; this contradicts (2.1).
- Suppose $r < 2$. Then ς is not a partition.

This completes the proof.

(ii) Since the socle of D^λ is irreducible, we require that 1 is the only normal number and $\lambda(1) = (n,n-1)$. This reduces to the cases of $\varsigma := (n+1,n-1)$ and $\mu := (n,n)$; if $\lambda = \varsigma$, then we have that $e \mid \beta_\varsigma(1,2) = 4$, a contradiction. Hence $\lambda = \mu$. $\square$

3. Crossingless matchings and Specht modules

In this section, we will analyze the crossingless matchings representation $M := M^r_{2n+r}$ with the goal of proving $M^r_{2n+r} \simeq S^{(n+r,n)}$ under certain conditions in e. We begin by proving irreducibility of M^r_{2n+r} whenever $e \nmid r+2, r+3, \dots, n+r+1$; when $e > n$, this is true if and only if $S^{(n+r,n)}$ is irreducible by Corollary 2.2. This is proven via an inductive process; if $e \nmid n+r+1$, then M contains no sign subrepresentation, and this allows us to "project" down to the case $M^r_{2(n-1)+r}$ and deduce irreducibility of M from irreducibility of this representation.

We will use the following base case to the correspondence throughout:

Lemma 3.1. *Note that $S^{(1^n)}$ is the sign representation and $S^{(n)}$ the trivial representation. We have the following isomorphisms*

(i) $M^0_2 \simeq S^{(2)}$,

(ii) $M^r_r \simeq S^{(1^r)}$.

Each of these are 1-dimensional, so they are irreducible.

Proof. (i) follows from the fact that $(1+T_1)w = (1+q)w$, and hence $T_1 w = qw$ for any nonzero vector $w \in M^0_2$. Similarly, (ii) follows from the fact that $(1+T_i)w = 0$, so $T_i w = -w$ for any i and any $w \in M^r_r$. $\square$

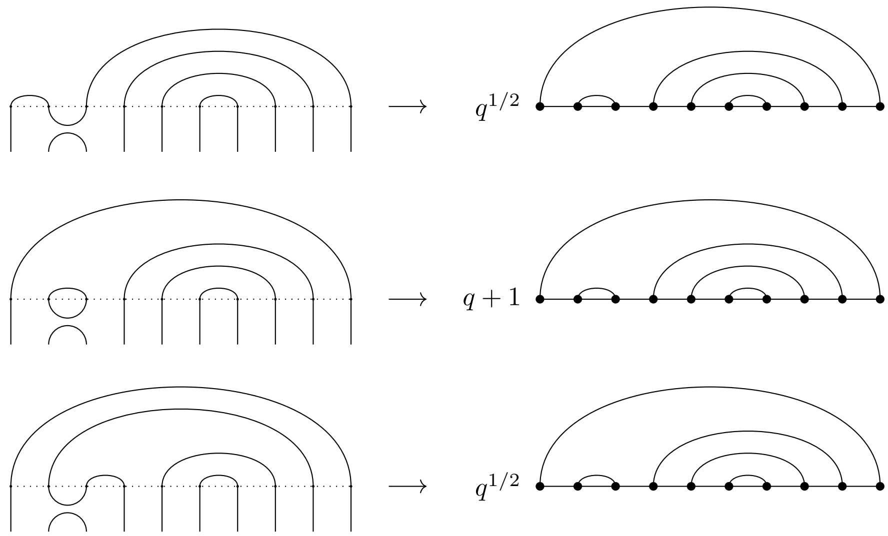


Figure 4. Illustrated is the row constructed for transposition $(1+T_2)$; clearly these are the only basis elements mapping to multiples of the desired element, and they relate to each other. replacing the outermost and/or innermost arc with an anchor typifies the rows constructed with three nonzero coefficients.

3.1. **Irreducibility of M.** We refer to a vector $w \in M$ satisfying $\mathscr{H}w = M$ as *cyclic*. It is a classical result that a representation W is irreducible if and only if every nonzero element of W is cyclic.[3] We will prove irreducibility by showing that every nonzero $w \in M$ is cyclic; the following lemma is integral in showing this.

Lemma 3.2. *Every basis vector in M^r_{2n+r} is cyclic.*

Proof. We have already proven this in the $r = 0$ case, so suppose that $r > 0$.

Note that, between anchors at indices $a < a'$ having no arc at index b with $a < b < a'$, the $M^0_{a'-a}$ case allows us to generate the basis vector with all length-2 arcs between a, a' and identical arcs/anchors outside of this sub-matching. At the ends, we apply the M^0_a case or the M^0_{2n+r-a} case in the same way for the first a or last $2n + r - a$ indices.

Applying this between each arc gives us a vector with anchors and length-2 arcs, and we may use the appropriate $(1 + T_i)$ to move anchors to any positions. Then, we may use the reverse process from above to generate the correct matchings between arcs and generate any other basis vector. □

Let $K^r_{2n+r} := \bigcap_{i=1}^{2n+r-1} \ker(1 + T_i) = \ker \bigoplus_{i=1}^{2n+r-1}(1 + T_i)$. This will be an important technical tool in our proof of irreducibility.

Lemma 3.3. *Suppose $e \nmid n + r + 1$. Then, $K := K^r_{2n+r} = 0$.*

Proof 1. Consider the matrix $A = \bigoplus(1 + T_i)$ having kernel K. It is sufficient by lemma 3.8 to show that A includes a row $[0, \dots, 0, 1, 0, \dots, 0]$ with a nonzero entry only on the row j.

Now, we may characterize the rows of A as follows; if the row corresponding to $(1 + T_i)$ and mapping onto the element $w_l \in W$ is nonzero, then it is of the form $\left[a_1, \dots, a_{|W|}\right]$ where $a_l = 1 + q$, $a_m = q^{1/2}$ whenever $(1 + T_i)w_m = q^{1/2}w_l$, and $a_m = 0$ otherwise.

Seeing this, the row corresponding to $(1 + T_{n+r})$ and w_j has nonzero entries $q^{1/2}$ at w_j and $(1 + q)$ at the vector w agreeing with w_j at all indices except having arcs at $(n+r-1, n+r)$ and $(n+r+1, n+r+2)$. Similar justification leads the row corresponding to $(1 + T_{n+r-1})$ at w to have nonzero entries $q^{1/2}$ at w and $(1 + q)$ at w_j and the vector with anchors $1, \dots, r$, arc $(n + r - 3, n + r - 2)$, and all other arcs maximum length.

We may iterate this process as illustrated in Figure 4, eventually ending at a row with two nonzero entries, either an arc $(1, 2)$ or an arc $(2, 3)$, and all anchors otherwise left-aligned and arcs of maximum length. These rows together form an $(n+r) \times \left|M_{2n+r}^r\right|$ submatrix of A which has a nonzero row in the row corresponding to j, and has (by removing zero rows) the same row space as the following square matrix:

$$B_{n+r} := \begin{bmatrix} q+1 & q^{1/2} & & & & & \\ q^{1/2} & q+1 & q^{1/2} & & & 0 & \\ & q^{1/2} & q+1 & q^{1/2} & & & \\ & & \ddots & \ddots & & & \\ & 0 & & & q^{1/2} & q+1 & q^{1/2} \\ & & & & & q^{1/2} & q+1 \end{bmatrix}. \tag{3.1}$$

We will show that this matrix is invertible; then, a sequence of elementary row operations will yield the identity, and in particular, when applied to A, will yield a row with a nonzero entry only on row j, giving $K = 0$.

We may prove invertibility of this matrix by proving that $\det B_{n+r} = [n+r+1]_q$ inductively on $n+r$. This is satisfied for our base case $n+r=1$, so suppose that it is true for each $m < n+r$. Then,

$$\begin{aligned} \det B_{n+r} &= (q+1)\det B_{n+r-1} - q \det B_{n+r-2} \\ &= (q+1)(1+\cdots+q^{n+r-1}) - (q+\cdots+q^{n+r-1}) \\ &= 1+\cdots+q^{n+r} \\ &= [n+r+1]_q . \end{aligned}$$

Hence $\det B_{n+r} \neq 0$, and $K=0$. $\square$

Theorem 3.4. *Suppose that $e \nmid r+2, r+3, \ldots, n+r+1$. Then the representation M_{2n+r}^r is irreducible.*

Proof. We proceed by induction on n. The base case the base case $n=0, r\neq 0$ and $n=1, r=0$ follow from Lemma 3.1.

Take an arbitrary vector $w \in M$. By Lemma 3.3 there exists some $(1+T_i) \in \mathscr{H}$ such that $(1+T_i)w \neq 0$. Note that

$$\operatorname{im}(1+T_i) = \operatorname{Span}\{w_j \mid w_j \text{ contains the arc } (i,i+1)\}.$$

Hence, as vector spaces, there is an isomorphism $\varphi : \operatorname{im}(1+T_i) \to M_{2(n-1)+r}^r$ "deleting" the arc $(i,i+1)$.

We will show that, for every element $(1+T_j') \in \mathscr{H}(S_{2(n-1)+r})$, there is some element $h_j \in \mathscr{H}(S_{2n+r})$ such that the following commutes:

$$\begin{tikzcd} \operatorname{im}(1+T_i) \arrow[r, "\varphi", "\sim"'] \arrow[d, "h_j"] & M_{2(n-1)+r}^r \arrow[d, "1+T_j'"] \\ \operatorname{im}(1+T_i) \arrow[r, "\varphi", "\sim"'] & M_{2(n-1)+r}^r \end{tikzcd}$$

Indeed, when $i+1 \neq j$ this is given by $h_j = 1+T_j$, and we have $h_{i+1} = q^{-1}(1+T_i)(1+T_{i+1})(1+T_{i-1})$, as given by Figure 5.

Note that, by definition, $e \nmid r+2, \ldots, n+r$ as well, and hence $(n+r-1, n-1)$ satisfies the hypotheses of the proposition as well. Then, by the inductive hypothesis, there is some element $h' \in \mathscr{H}(S_{2(n-1)+r})$ sending $\varphi((1+T_i)w)$ to the image of a basis vector of M_{2n+r}^r via φ; then, the action $\mathscr{H}$ generates the endomorphism $\varphi^{-1}h'\varphi$ of M, which sends $(1+T_i)w$ to a basis vector in M_{2n+r}^r. This implies that w is cyclic, and hence M_{2n+r}^r is irreducible. $\square$

3.2. Correspondence with Specht modules.

The following theorem due to Mathas (theorem 5.5 in [9]) generalizes the classical branching theorem of the symmetric group. It will not be necessary for our present proof of the correspondence, but analogy with M is suggestive.

Theorem 3.5 (Characteristic-free classical branching theorem). *Let λ be a partition of m with ℓ removable nodes. Then, $\operatorname{Res} S^\lambda$ has an $\mathscr{H}_{k,q}(S_{m-1})$-module filtration*

$$0 = S^{0,\lambda} \subset S^{1,\lambda} \subset \cdots \subset S^{\ell,\lambda} = \operatorname{Res} S^\lambda$$

such that $S^{t,\lambda}/S^{t-1,\lambda} \simeq S^{\lambda(t)}$ for all $1 \leq t \leq \ell$. $\square$

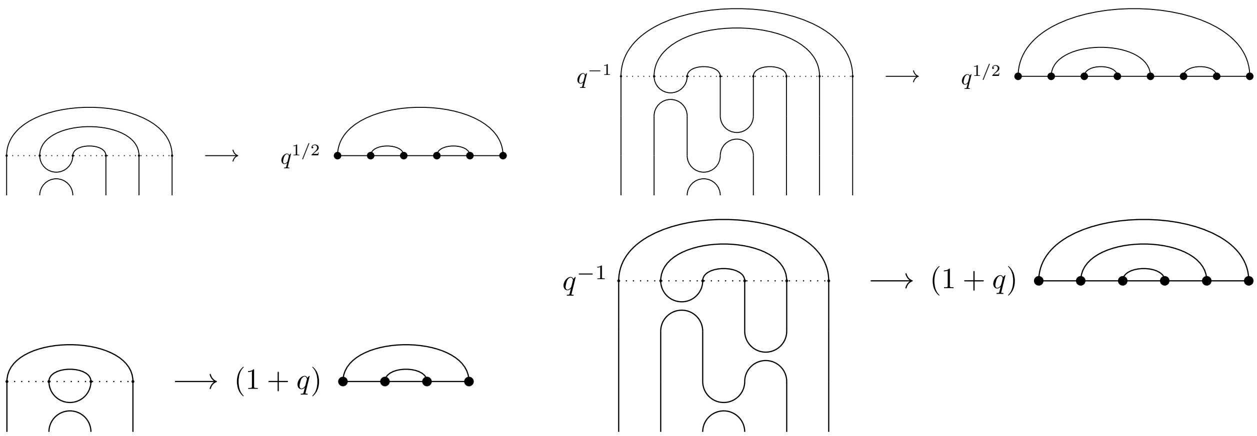


Figure 5. The correspondence between the action of $(1 + T_2)$ on $w'_5 \in M_6^0$ and the action of $q^{-1}(1 + T_3)(1 + T_4)(1 + T_2)$ on the corresponding vector in M_8^0 having arc $(3, 4)$ first, then on $w'_2 \in M_4^0$. This demonstrates that the action works with or without creating a loop.

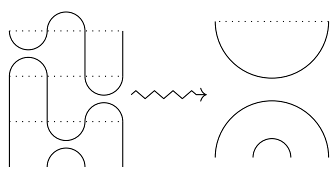


Figure 6. The isotopy demonstrating the correspondence between h_{i+1} and T'_i; the scaling issue with having three simple reflections is handled by the q^{-1} factor in h_{i+1}.

In particular, this holds in cases where S^λ fails to be irreducible. If we replace S^λ with the appropriate M^r_{2n+r} above, we find the statement of the following proposition.

Proposition 3.6. *Suppose that $n > 0$.*

(i) Suppose that $r > 0$. Then, a filtration of $\operatorname{Res} M^r_{2n+r}$ is given by

$$0 \subset M^{r-1}_{2n+r-1} \subset \operatorname{Res} M^r_{2n+r}, \tag{3.2}$$

with $\operatorname{Res} M^r_{2n+r}/M^{r-1}_{2n+r-1} \simeq M^{r+1}_{2n+r-1}$.

(ii) We have the following isomorphism of representations:

$$M^1_{2n-1} \simeq \operatorname{Res} M^0_{2n}. \tag{3.3}$$

In the case that $e \nmid r+1, \ldots, n+r+1$, 3.2 and 3.3 are composition series.

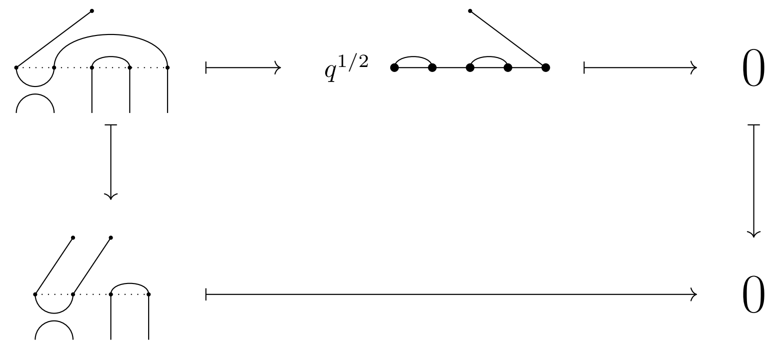
Proof. **(i)** Note that we may identify the subrepresentation of $\operatorname{Res} M^r_{2n+r}$ having anchor at index $2n+r$ with M^{r-1}_{2n+r-1}.

Let $U := \operatorname{Res} M^r_{2n+r}/M^{r-1}_{2n+r-1}$, and let $\pi : \operatorname{Res} M^r_{2n+r} \twoheadrightarrow U$ be the associated projection to U. Let $\phi : U \to M^{r+1}_{2n+r-1}$ be the k-linear map which regards the arc $(i, 2n+r)$ in U as an anchor at i in M^{r+1}_{2n+r-1}. It is not hard to verify that this is a well-defined isomorphism of vector spaces, so we must show that it is $\mathscr{H}$-linear.

Given a basis vector w_j containing arc $(i, 2n+r)$, ϕ is clearly compatible with $(1+T_{i'})$ with $i' \neq i, i-1$. Further, it is easy to verify that ϕ is compatible with T_i by definition of the relevant modules, as well as with $(1 + T_{i-1})$ when w_j does not have anchor $i-1$. The case that w_j has anchor $i-1$ is illustrated in the following commutative diagram:

$$\begin{tikzcd}
M^r_{2n+r} \arrow[r, "1+T_i"] \arrow[d, "\phi\circ\pi"] & M^r_{2n+t} \arrow[r, "\pi"] & U \arrow[d, "\phi"] \\
M^{r+1}_{2(n-1)+r+1} \arrow[rr, "1+T_i"] & & M^{r+1}_{2(n-1)+r+1}
\end{tikzcd}$$

which we may diagram chase as follows



In each of these cases, linearity of $\mathscr{H}$-action and ϕ give

$$\begin{aligned}
\phi(T_j w_i) &= \phi(-w_j + (1+T_j)w) \\
&= -\phi(w_j) + (1+T_j)\phi(w_j) \\
&= T_j\phi(w_j).
\end{aligned}$$

Hence ϕ is an isomorphism of representations, and the statement is proven.

(ii) This follows from an analogous proof: now, $\phi : \operatorname{Res} M^0_{2n} \to M^1_{2(n-1)+1}$ is an isomorphism of representaitons, which is proven to be $\mathscr{H}$-linear by the same proof. $\square$

We've now assembled the basic pieces necessary to prove our correspondence in the case $e > n+r+1$.

Theorem 3.7. *Suppose $e > n+r+1$. Then, $M^r_{2n+r} \simeq S^{(n+r,n)}$.*

Proof. The case $n = 0$ is already proven, by lemma 3.1, so suppose $n > 0$. In order to use Proposition 2.6, suppose for now that either $e \nmid 4$ or $r > 0$.

We will prove this inductively; the base case $2n + r = 2$ is implied by 3.1, so suppose that $M^s_{2n+s} \simeq S^{(m+s,s)}$ whenever $2m + s < 2n + r$ and $m + s \leq n + r$, so that $e > m + s + 1$.

Suppose $r > 0$. By Theorem 3.4, we know that $M^r_{2n+r} \simeq D^\lambda$ for some e-restricted partition λ. By the inductive hypothesis and Corollary 2.2, we have a composition series given by the short exact sequence

$$0 \longrightarrow D^{(n+r-1,n)} \longrightarrow \text{Res } D^\lambda \longrightarrow D^{(n+r,n-1)} \longrightarrow 0 \tag{3.4}$$

Hence the theorem is given by Proposition 2.6 (i).

Now suppose $r = 0$ and $e \neq 4$. Similarly, by Theorem 3.4, we know that $M^r_{2n+r} \simeq D^\lambda$ for some e-restricted partition λ, and by the inductive hypothesis and Corollary 2.2, we have the irreducible restriction $\operatorname{Res} D^\lambda \simeq D^{(n,n-1)}$. Then, the theorem is given by Proposition 2.6 (ii).

Now, suppose $e = 4$ and $r = 0$; then $4 > n + 1$, so $n \leq 2$. We've already proven the $n = 1$ case via the trivial representation, so suppose $n = 2$. Then, from the proof of Proposition 2.6, we know that $M^0_4 \simeq D^\lambda$, where $\lambda \in \{(n,n),(n+1,n-1)\}$. We have already proven that $M^2_4 \simeq D^{(n+1,n-1)}$, and we may verify that $\dim M^0_4 = 2 \neq 3 = \dim M^2_4$, so we have that $\lambda = (n,n)$ and the theorem is proven for $e = 4$. $\square$

3.3. Kernels and further work.

Proposition 3.8. *Let w be an arbitrary vector in W^r_{2n+r}. I claim that if $w \in \cap ker(1+T_i)$, the coordinate c of the rainbow element R in w is nonzero.*

Proof. Let Y be the set of basis elements with nonzero coordinate in w. Let k be the greatest integer such that there exists $y \in Y$ where $y(1) = \ldots = y(k) = 0$ should this be $y(1) - 1 = \cdots = y(k) - k = 0$? Also, we should avoid using k as an integer, as it's used elsewhere as a field., and let $U \subset Y$ be the set of such y. In other words, U is the set of basis elements in Y which have the most anchors to the far left.

Suppose $k < r$. Then for each $y \in U$ there exists a minimal $i_y > k+2$ such that $y(i_y) = 0$. In other words, i_y is the position of the next leftmost anchor in y. Fix $\tilde{y}$ such that $i_{\tilde{y}} \leq i_y$ for all y. Then I claim the basis element $y' := q^{-1/2}(1+T_{i_{\tilde{y}}-1})\tilde{y}$ has nonzero coordinate in $(1+T_{i_{\tilde{y}}-1})w$, implying $w \notin \cap \ker(1+T_i)$. To see this, we can show that $\tilde{y}$ is the only element in Y such that $q^{-1/2}(1+T_{i_{\tilde{y}}-1})\tilde{y} \sim y'$. y' still has k anchors on the left, and $i_{y'} < i_{\tilde{y}}$, so $y' \notin Y$. If $x \in Y, \notin U$, the basis element proportional to $(1+T_{i_{\tilde{y}}-1})x$ will have k anchors at the far left only if the next anchor is at a position $i_{x'} > i_{\tilde{y}}$, so it cannot be y'. If $x \in U$ the basis element proportional to $(1+T_{i_{\tilde{y}}-1})x$ will have anchor at $i_{y'}$ if and only if $i_x = i_{\tilde{y}}$ and $x(i_{\tilde{y}}) = \tilde{y}(i_{\tilde{y}})$. Since this is the only match altered by action $(1+T_{i_{\tilde{y}}-1})$ on x, if $(1+T_{i_{\tilde{y}}-1})x \sim y'$ this implies $x = \tilde{y}$. So if $k < r$ w is not in the desired kernel.

Suppose $k = r$ but $R \notin U$ (so $R \notin Y$). Let us define a sequence of subsets of U in the following way: $U_0 := U$, $U_{i+1} := \{u \in U_i | u(r+i+1) = 2n+2r-i+1\}$. Since $R \notin U$, $\exists t < n-1$ such that $U_{t+1} = \varnothing$. Choose $\tilde{u} \in U_t$ such that $\tilde{u}(r+t+1) \geq u(r+t+1)$ for all $u \in U_t$. Consider the basis element $u' := q^{-1/2}(1+T_{\tilde{u}(r+t+1)})\tilde{u}$. I claim that $\tilde{u}$ is the only element in Y such that $(1+T_{\tilde{u}(r+t+1)})\tilde{u} \sim u'$, again implying that w is not in the desired kernel. u' still has k anchors on the left, $u'(r+i) = 2n+2r-i+2$, $1 \leq i \leq t$, and $u'(r+t+1) > \tilde{u}(r+t+1)$, so $u' \notin Y$. If $x \in Y, \notin U$, the basis element x' proportional to $(1+T_{\tilde{u}(r+t+1)})x$ will have r leftmost anchors only if $x'(r+t+1) < \tilde{u}(r+t+1)$, so $x' \neq u'$. Similarly, if $x \in U, \notin U_t$, the basis element x' will have the property $x'(r+t) = 2n+2r-t+2$ only if $x'(r+t+1) < \tilde{u}(r+t+1)$, so $x' \neq u'$. If $x \in U_t$, $x'(r+t+1) = u'(r+t+1)$ if and only if $x(r+t+1) = \tilde{u}(r+t+1)$ and $x(x(r+t+1)+1) = \tilde{u}(\tilde{u}(r+t+1)+1)$ (since $u' \notin Y$). These are the only matches altered by the action $(1+T_{\tilde{u}(r+t+1)})$, so this implies $x = \tilde{u}$. Thus we have proved that if $R \notin Y$, w is not in the desired kernel. $\square$

Necessary characterization of the Kernel goes here.

Proposition 3.9. *Suppose $e = n+r+1$. Then K is nontrivial.*

4. Fibonacci representations and quotients of Specht modules

We have difficulty characterizing the Specht module above when it is irreducible. We may instead attempt to characterize the irreducible quotient D^λ of the Specht module. In particular, note that, whenever $e \leq r+2$, the restriction $\operatorname{Res} D^{(n+r,n)}$ decomposes into a direct sum of partitions $(m+s, m)$ such that $e \leq s+2$ and $2m+s = 2n+r-1$. This forms a recurrence among these representations, allowing a combinatorial description of their dimension. In this section we henceforth assume $e = 5$ and note that this recurrence resembles the Fibonacci recurrence; we follow this to characterize the restriction $D^{(n+r,n)}$ with $r \leq 3$.

Remark. Let $d_m^{0,3}$ be the dimension $\dim D^{(n,n)}$ with $2n = m$ when m is even, and $\dim D^{(n+3,n)}$ with $2n+3 = m$ when m is odd. Similarly define $d_m^{1,2}$. Then, Corollary 2.5 gives

$$\begin{aligned} d_m^{0,3} &= d_{m-1}^{1,2} \\ d_m^{1,2} &= d_{m-1}^{1,2} + d_{m-1}^{0,3} \\ &= d_{m-1}^{0,3} + d_{m-2}^{0,3}. \end{aligned}$$

Carefully following this and noting the base cases $d_2^{0,3} = d_2^{1,2} = 1$, one may note that this recurrences proves that $d_m^{1,2} = d_{m+1}^{0,3} = f_n$, where f_n is the nth *Fibonacci number*. This matches the dimension our Fibonacci subrepresentations, motivating their definition. $\square$

We can start our study of V by studying low-dimensional cases. First, note that V_{*0}^2 is the sign representation $D^{(1^2)}$ and V_{*0}^2 is the trivial representation $D^{(2)}$.

V_{00}^2 is a 2-dimensional representation of a semisimple commutative algebra, and hence decomposes into a direct sum of two subrepresentations. In particular, we can use the basis $\{(0{*}0), (000)\}$ and explicitly write the matrix

$$\rho_{T_1} = \begin{bmatrix} \varepsilon_1 & \delta \\ \delta & \varepsilon_2 \end{bmatrix}$$

having characteristic polynomial $(\varepsilon_1 - \lambda)(\varepsilon_2 - \lambda) - \delta^2 = \lambda^2 - (\varepsilon_1 + \varepsilon_2)\lambda + (\varepsilon_1\varepsilon_2 - \delta^2)$. We may verify that, for $\lambda = -1$, this evaluates to

$$-((-1+q+q^2)(1+q^3+q^4+q^5+2q^6+q^7))\,[5]_q = 0$$

and for $\lambda = q$ this evaluates to

$$-(q^2(-1+q+q^2)(1+q+q^2+q^3+2q^4+q^5))\,[5]_q = 0$$

hence ρ_{T_1} has eigenvalues -1 and q.

The eigenspaces with eigenvalues -1 and q are subrepresentations isomorphic to the sign and trivial representation, hence $V_{\mathtt{00}}$ is isomorphic to a direct sum of the trivial and sign representations: $V_{\mathtt{00}}^2 \simeq V_{*\mathtt{0}}^2 \oplus V_{**}^2$.

Now we may prove that $V_{*\mathtt{0}}^3$ is irreducible; this has basis $\{(*\mathtt{0}*\mathtt{0}), (*\mathtt{000})\}$, and the following matrices:

$$\rho_{T_1} = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{bmatrix}; \qquad \rho_{T_2} = \begin{bmatrix} \varepsilon_1 & \delta \\ \delta & \varepsilon_2 \end{bmatrix}.$$

A proper nontrivial subrepresentation of $V_{*\mathtt{0}}^3$ must be one-dimensional, and hence an eigenspace of each of these matrices; since $\alpha_2 \neq \alpha_1$, the first has eigenspaces given by the spans of basis elements, and since $\delta \neq 0$, these are not eigenspaces of the second. Hence $V_{*\mathtt{0}}^3$ is irreducible.

These establish the low-dimensional behavior that we will use as base cases below. The rest of this section will proceed first by proving that $V_{*\mathtt{0}}$ and V_{**} are irreducible; then, we will use combinatorial arguments to prove that these are isomorphic to the correct two-row Specht module quotients.

Proposition 4.1. *The representation $V_{*0} := V_{*0}^m$ is irreducible.*

Proof. We will prove this inductively in m. We've already proven it for $V_{*\mathtt{0}}^2$ and $V_{*\mathtt{0}}^3$, so suppose that $V_{*\mathtt{0}}^{m-2}$ is irreducible.

Let $\{v_i\}$ be the basis for $V_{*\mathtt{0}}$. Then, each v_i is cyclic; indeed, we can transform every basis vector into $(*\mathtt{0}\ldots\mathtt{0})$ by multiplying by the appropriate $\frac{1}{\delta-\varepsilon_1}(T_i - \varepsilon_1)$, and we can transform $(*\mathtt{0}\ldots\mathtt{0})$ into any basis vector by multiplying be the appropriate $\frac{1}{\delta-\varepsilon_2}(T_i - \varepsilon_2)$. Hence it is sufficient to show that each $v \in V_{*\mathtt{0}}$ generate some basis element.

Let v' be the basis element $(*\mathtt{0}*\mathtt{0}\ldots\mathtt{0})$, which is many copies of $*\mathtt{0}$, followed by an extra $\mathtt{0}$ if m is odd. We will show that each $v \in F$ generates v'.

Suppose that no elements beginning $(*\mathtt{0}*\mathtt{0})$ are represented in v_i; then, all such elements are represented in $T_3 v$, so we may assume that at least one is represented in v.

Note that $\operatorname{im}(T_2 - \alpha_1) = \operatorname{Span}\{\text{Basis vectors beginning } (*\mathtt{0}*\mathtt{0})\}$ and $(T_2 - \alpha_1)v \neq 0$. Further, note that $\operatorname{Res}^{\mathscr{H}(S_m)}_{\mathscr{H}(S_{m-2})} \operatorname{im}(T_2 - \alpha_1) \simeq V_{*\mathtt{0}}^{m-2}$ as representations. Hence irreducibility of $V_{*\mathtt{0}}^{m-2}$ implies that v' is generated by $(T_2 - \alpha_1)v$, and $V_{*\mathtt{0}}^m$ is irreducible. $\square$

Now, we may begin considering restrictions:

Lemma 4.2. *The following branching rules hold:*

$$V_{00}^{m-1} \simeq \operatorname{Res} V_{*0}^m \simeq V_{**}^{m-1} \oplus V_{*0}^{m-1}$$
$$\operatorname{Res} V_{**}^m \simeq V_{*0}^{m-1}.$$

Proof. The first line follows by considering the last two $m-2$ transpositions for the left isomorphism, then the first two for the right isomorphism. This is well-behaved by Appendix B.

Similarly, the second line follows by considering the last $m-2$ transpositions. $\square$

This immediately gives a rather strong characterization of V.

Corollary 4.3. *The representation V_{**} is irreducible.* $\square$

Corollary 4.4. *The representation V decomposes into a direct sum of irreducible representations as follows:*

$$V \simeq 3V_{*0} \oplus 2V_{**}.$$

$\square$

Now we may use these in order to apply Young Tableau to characterize V.

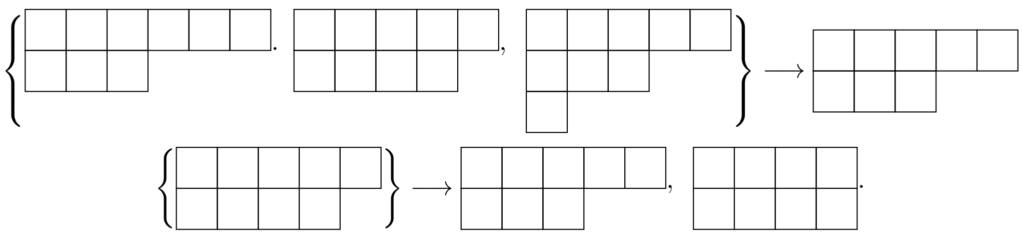


Figure 7. Illustration of the partitions of 9 which can, via row removal, yield $(n, n-2)$ alone, or both $(n, n-2)$ and $(n-1, n-1)$.

Theorem 4.5. *The irreducible components of V are given by the following isomorphisms:*

$$\begin{aligned} V^{2n}_{**} &\simeq D^{(n,n)} \\ V^{2n-1}_{**} &\simeq D^{(n+1,n-2)} \\ V^{2n}_{*0} &\simeq D^{(n+1,n-1)} \\ V^{2n-1}_{*0} &\simeq D^{(n,n-1)}. \end{aligned}$$

Proof. We will prove this by induction on n; we have already proven the base case V^2, so suppose that we have proven these isomorphisms for V^{2n-2}. We will prove the isomorphisms for V^{2n-1} and V^{2n}.

By irreducibility, $V^{2n-1}_{**} \simeq D^{\lambda}$ and $V^{2n-1}_{*0} \simeq D^{\mu}$ for some diagrams λ and μ. We will show that $\lambda = (n+1, n-2)$ and $\mu = (n+1, n-1)$.

First, note that we have

$$\text{Res } D^{\lambda} \simeq D^{(n,n-2)} \simeq \text{Res } D^{(n+1,n-2)}$$

and

$$\text{Res } D^{\mu} \simeq D^{(n,n-2)} \oplus D^{(n-1,n-1)} \simeq \text{Res } D^{(n,n-1)}.$$

By irreducibility of $\operatorname{Res} D^{\lambda}$, the only normal number in λ is 1. Further, the only tableaux which can remove a cell to yield $D^{(n,n-2)}$ are $(n+1, n-2)$, $(n, n-1)$, and $(n, n-2, 1)$ as illustrated in Figure 7; we have already seen that $D^{(n,n-1)}$ does not have irreducible restriction, so we are left with $(n+1, n-2)$ and $\varsigma = (n, n-2, 1)$. We may directly check that ς doesn't satisfy this, as we have the following:

$$\begin{aligned} \beta_{\varsigma}(1,2) &= 3 - 2 + (n-2) = n - 1 \\ \beta_{\varsigma}(1,3) &= 3 - 1 + n = n + 2 \\ \beta_{\varsigma}(2,3) &= 2 - 1 + 3 = 4. \end{aligned}$$

At least one of $\beta_{\varsigma}(1,2)$ and $\beta_{\varsigma}(1,3)$ is nonzero, since $\beta_{\varsigma}(1,3) - \beta_{\varsigma}(1,2) = 3 \not\equiv 0 \pmod{e}$, and hence at least one of M_2 and M_3 is empty. Hence at least one of 2 or 3 is normal in ς, and $\lambda = (n+1, n-2)$.

For μ, we immediately see from Figure 7 that the only option is $(n, n-1)$.

We can perform a similar argument for the V^{2n} case, finding now that

$$\text{Res } D^{\lambda'} \simeq D^{(n,n-1)} \simeq \text{Res } D^{(n,n)}$$

and

$$\text{Res } D^{\mu'} \simeq D^{(n,n-1)} \oplus D^{(n+1,n-2)} \simeq \text{Res } D^{(n+1,n-1)}.$$

Through a similar process, we see that $\mu' = (n+1, n-1)$. We narrow down λ' to one of (n, n) or $\varpi := (n, n-1, 1)$, and note that

$$\begin{aligned} \beta_{\varpi}(1,2) &= 3 - 2 + (n-1) = n \\ \beta_{\varpi}(1,3) &= 3 - 1 + n = n + 2 \\ \beta_{\varpi}(2,3) &= 2 - 1 + 2 = 3 \end{aligned}$$

and hence at least one of 2 or 3 is normal, $\operatorname{Res} D^{\varpi}$ is not irreducible, and $\lambda' = (n, n)$, finishing our proof. □

Corollary 4.6. *We have the following isomorphisms of representations:*

$$V^{2n} \simeq 3D^{(n+1,n-1)} \oplus 2D^{(n,n)}$$
$$V^{2n-1} \simeq 3D^{(n,n-1)} \oplus 2D^{(n+1,n-2)}$$

5. Conjecture

Recall that $K^r_{2n+r} := K$ is the direct sum of all copies of the sign representation in W. Hence the following characterises sign subrepresentations of W completely:

Proposition 5.1. *$K \subset M^r_{2n+r}$ is trivial when $e \neq n+r+1$, and $\dim K = 1$ when $e = n+r+1$.* □

Proposition 5.2. *Suppose $e < n+r+1$, and suppose n' is such that $e = n'+r+1$. Note that $h := (1+T_1)(1+T_3)\dots(1+T_{n-n'})$ maps M^r_{2n+r} onto $M^r_{2n'+r}$. Then, the preimage $h^{-1}(K^r_{2n+r})$ is a subrepresentation of M^r_{2n+r}, and the series*

$$0 \subset h^{-1}(K^r_{2n+r}) \subset M^r_{2n+r}$$

is a composition series of M^r_{2n+r}. □

Proposition 5.3. *Denote the composition factor $M^r_{2n+r}/h^{-1}(K^r_{2n+r})$ by U^r_{2n+r}. Then, there exist some naturals m, s satisfying $2m+s = 2n+r$ and $m+s > n+r$ such that the following is an isomorphism of $\mathscr{H}$-modules*

$$h^{-1}\left(K^r_{2n+r}\right) \simeq U^s_{2m+s}$$

. □

Proposition 5.4. *For the same m, s as above, we have the following composition series of Specht modules:*

$$0 \longrightarrow D^{(m+s,m)} \longrightarrow S^{(n+r,n)} \longrightarrow D^{(n+r,n)} \longrightarrow 0.$$

Proposition 5.5. *$M^r_{2n+r} \simeq S^{(n+r,n)}$ and $U^s_{2m+s} \simeq D^{(m+s,m)}$.*

Appendix A. Compatibility of Representations with the Relations

In general, we defined the representations $V := V^{2n+r}$ and $M := M_{2n+r}$ for the free algebra on generators $\{T_1, \dots, T_{2n+r-1}\}$. Recall that we may give a presentation of $\mathscr{H}$ having generators T_i and relations

$$(T_i - q)(T_i + 1) = 0 \tag{A.1}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{A.2}$$

$$T_i T_j = T_j T_i \qquad |i-j| > 1. \tag{A.3}$$

We call (A.1) the *quadratic relation* and (A.2), (A.3) the *braid relations*. It is easily seen that a representation of $\mathscr{H}$ is equivalent to a representation of the free algebra $k\langle T_i \rangle$ which are compatible with the relations. We will prove in the following sections that V and M are compatible with the Hecke algebra relations.

A.1. Explicit Definition of M.

We will give a more careful definition of the crossingless matchings representation here.

Definition A.1. A *crossingless matching on $2n+r$ indices with r anchors* is a partition of $\{1, \dots, 2n+r\}$ into n parts of size 2 and r of size 1 such that no two parts (a, a') and (b, b') satisfy $a < b < a' < b'$, and no parts $(c), (a, a')$ satisfy $a < c < a'$. We will call these arcs and anchors, respectively. Then, define M^r_{2n+r} to be the k-vector space with basis the set of crossingless matchings on $2n+r$ indices with r anchors.

In order to endow M^r_{2n+r} with an $\mathscr{H}$-action, consider some basis element w_j and some element $(1+T_i)$ of $\mathscr{H}$. The elements $\{1\} \cup \{1 + T_i | 1 \le i < 2n+r\}$ generate $\mathscr{H}$, so it is sufficient to define the action of $1 + T_i$ on w_j.

If w_j has arc $(i, i+1)$, define $(1+T_i)w_j := (1+q)w_j$. If w_j has anchors $W(i) = i$ and $W(i+1) = i+1$, define $(1+T_i)w_j := 0$. If w_j has anchor $W(i) = i$ and arc $W(i+1) = b$, define $(1+T_i)w_j := q^{1/2} w_l$, where $w_l(i) = i+1$, $w_l(b) = b$, and all other arcs agree with w_j. If w_j has arcs $W(i) = a$ and $W(i+1) = b$, then define $(1+T_i)w_j := q^{1/2} w_l$, where $w_l(i) = i+1$,$w(a) = b$, and all other acts agree with w_j. We verify that this is well-defined in Section A.2.

We may alternately sharpen our topological definition;

Definition A.2. Fix $2n+r$ distinct points $a_1, \dots, a_{2n+r}$ points along $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ and r distinct points $b_1, \dots, b_r$ along $\mathbb{R} \times \{1\}$. Then, define M^r_{2n+r} to have basis given by the isotopy classes of $n+r$ paths connecting the points $a_1, \dots, a_{2n+r}, b_1, \dots, b_r$ such that no distinct b_i, b_j are connected by a path.

We will take some basis element $w_j \in M^r_{2n+r}$ and define the action $(1+T_i)w_j$. To do so, map w_j through the natural embedding $\mathbb{R} \times [0,1] \hookrightarrow \mathbb{R} \times \left[\frac{1}{2}, 1\right]$, and form the figure w^i_j by adjoining the lines connecting a_l and $a_l + \left(0, \frac{1}{2}\right)$ for all $l \neq i, i+1$ as well as paths from a_i to a_{i+1} and $a_i + \left(0, \frac{1}{2}\right)$ to $a_{i+1} + \left(0, \frac{1}{2}\right)$. This has either 0 or 1 path components which do not intersect $\mathbb{R} \times \{0, 1\}$; these form "loops."

Take the figure $\tilde{w}^i_j$ without this component. If $\tilde{w}^i_j$ is not isotopic to some w_l, then define $(1+T_i)w_j := 0$. If $\tilde{w}^i_j$ is isotopic to some w_l, define $(1+T_i)w_j := (1+q)w_l$ if w^i_j has a loop and $(1+T_i)w_j := q^{1/2} w_l$ otherwise. This process is illustrated in Figure 2.

Let the length of an arc (i,j) be $l(i,j) := j - i + 1$. Note that the crossingless matchings on $2n$ indices with no anchors can all be identified with a list of n integers describing the lengths of the arcs from left to right; This basis is illustrated for M^1_5 in Figure 1.

Remark. This definition gives a graphical calculus for working with our module. It should be clear that, if w^i_j has a loop then $w_l(i) = i+1$ and $w_l = w_j$. Further, this easily defines an arbitrary composition:

$$(1+T_{i_1}) \cdots (1+T_{i_\ell}) w_j = q^{(\ell - t)\frac{1}{2}} (1+q)^t w_l$$

if the figure we make via $(1+T_{i_1}) \cdots (1+T_{i_\ell})$ is isotopic to w_l after removing t loops.

Note that the crossingless matchings on $2n$ indices with no anchors can all be identified with a list of n integers describing the lengths of the arcs from left to right; call this the *increasing lexicographical order basis*. Further, we have a surjection $M^0_{2n+2r} \twoheadrightarrow M^r_{2n+r}$ which takes basis elements to basis elements; this induces an order on the basis for M^r_{2n+r}, which we will henceforth refer to as the *induced lexicographical order basis.* This basis is illustrated for M^1_5 in Figure 1.

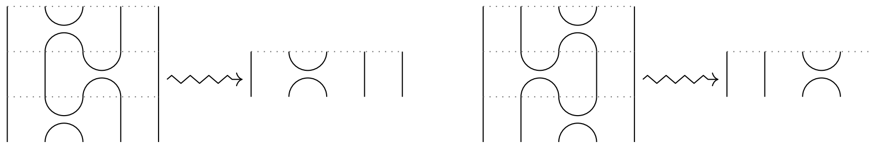


Figure 8. The above give visual intuition for isotopies giving rise to equalities between $(1+T_i)(1+T_{i-1})(1+T_i)w_j$ and $q(1+T_i)$, and between $(1+T_{i+1})(1+T_i)(1+T_{i+1})w_j$ and $q(1+T_i)$.

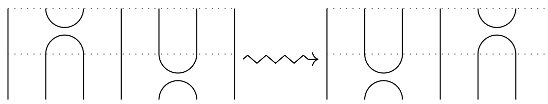


Figure 9. The above give visual intuition for isotopies giving rise to equalities between $(1+T_i)(1+T_j)w_l$ and $(1+T_j)(1+T_i)w_l$.

A.2. **Compatibility for the Crossingless Matchings Representations.** We verify the relations on the crossingless matchings representation M. Take some basis vector $w_i \in M$. We will first check (A.1) by case work:

- Suppose there is an arc $(i, i+1)$. Then,
$$(T_i - q)(T_i + 1)w = (1+q)\left((1+T_i)w - (1+q)w\right) = 0,$$
giving (A.1).
- Suppose there is no arc $(i, i+1)$ and indices $i, i+1$ do not both have anchors; then $(T_i+1)w = q^{1/2}w'$ for some basis vector w' having arc $(i, i+1)$, and the computation follows as above for (A.1).
- Suppose $i, i+1$ are anchors; then $(T_i+1)w = 0$, giving (A.1).

Now we verify (A.2). Let $h := (1+T_i)(1+T_{i+1})(1+T_i)$, and let $g := (1+T_{i+1})(1+T_i)(1+T_{i+1})$. Note the following expansion:

$$\begin{aligned} hw &= 1 + 2T_i + T_i^2 + T_{i+1} + T_iT_{i+1} + T_{i+1}T_i + T_iT_{i+1}T_i \\ &= 1 + (1+q)T_i + T_{i+1} + T_iT_{i+1} + T_{i+1}T_i + T_iT_{i+1}T_i. \end{aligned}$$

This equality, with i and $i+1$ interchanged, holds for g. Hence we have

$$(h-g)w = q(T_i - T_{i+1}) + T_iT_{i+1}T_i - T_{i+1}T_iT_{i+1}.$$

Hence we may equivalently check that $(h-g)w = q(T_i - T_{i+1})$. In fact, $hw = q(1+T_i)$ and $gw = q(1+T_{i+1})$ by Figure 8, giving compatibility.

Lastly, we have the equation

$$(1+T_i)(1+T_j) - (1+T_j)(1+T_i) = T_iT_j - T_jT_i$$

and hence we simply need to verify that $(1+T_i)$ and $(1+T_j)$ commute, as shown in Figure 9

A.3. **Compatibility for the Fibonacci Representations.** We verify the relations on the Fibonacci representation V. Note that (A.3) follows easily from the "local" nature of V, and the others may be verified explicitly on strings of length 3 and 4. By considering the coefficients in order of (1.1), the quadratic relation (A.1) gives the following polynomials in q:

$$\begin{aligned} (\alpha_1 - q)(\alpha_1 + 1) &= 0, \\ (\alpha_2 - q)(\alpha_2 + 1) &= 0, \\ \varepsilon_1\delta + \delta\varepsilon_2 &= (q-1)\delta, \\ \varepsilon_1^2 + \delta^2 &= (q-1)\varepsilon_1 + q, \\ \varepsilon_2^2 + \delta^2 &= (q-1)\varepsilon_2 + q \end{aligned} \tag{A.4}$$

The first two of these are easily verified. Since $\delta \neq 0$, the third is equivalently given by

$$(q-1) = \varepsilon_1 + \varepsilon_2 = \tau(q\tau - 1 + q - \tau) = (\tau^2 + \tau)(q-1)$$

or that $\left(\tau^2 + \tau - 1\right)(q-1) = 0$. One may verify that

$$\tau^2 + \tau - 1 = q^6 + 2q^5 + q^4 + q^3 + q^2 - 1 = (-1 + q + q^2)\,[5]_q = 0.$$

The fourth is given by the quadratic

$$\tau^2\left[(q\tau - 1)^2 - \tau(q+1)\right] = \tau(q-1)(q\tau - 1) + q$$

or equivalently,

$$(\tau^2 + \tau - 1)\left[q\left(qt^2 + 1\right) + t\right] = 0.$$

The fifth is similarly given by

$$(\tau^2 + \tau - 1)\left[q\left(qt + 1\right) + t^2\right] = 0.$$

All of these vanish for $e = 5$, giving compatibility with (A.1).

We now verify (A.2). We may order the basis for V^4 as follows:

$$\{(\mathtt{0000}), (\mathtt{*00*}), (\mathtt{000*}), (\mathtt{*000}), (\mathtt{*0*0}), (\mathtt{0*0*}), (\mathtt{00*0}), (\mathtt{0*00})\}.$$

Then, in verifying the braid relation (A.2) in this order, we encounter the following quadratics (with tautologies and repetitions omitted):

$$\begin{gathered}
\alpha_1\varepsilon_2^2 + \alpha_2\delta^2 = \alpha_1^2\varepsilon_2 \\
\alpha_1\delta\varepsilon_2 + \alpha_2\varepsilon_1\delta = \alpha_1\alpha_2\delta \\
\alpha_2\varepsilon_1^2 + \alpha_1\delta^2 = \alpha_2^2\varepsilon_1 \\
\alpha_1\varepsilon_1^2 + \delta^2\varepsilon_2 = \alpha_1^2\varepsilon_1 \\
\delta\varepsilon_2^2 + \alpha_1\varepsilon_1\delta = \alpha_1\delta\varepsilon_2
\end{gathered}$$

Substituting in τ and dividing by δ whenever possible, these are equivalent to the vanishing of the following polynomials in q:

$$\begin{aligned}
-q(1+q)(1+q^2+q^3)(2+q+3q^2+2q^3)\,[5]_q &= 0 \\
(1+2q+q^3+q^4)\,[5]_q &= 0 \\
(1+q)^2(1+q^2+q^3)(1+3q^3-q^4+q^6)\,[5]_q &= 0 \\
(1+q)^2(1+q^2+q^3)(1+5q+5q^2+3q^3+3q^4+3q^5+q^6)\,[5]_q &= 0 \\
(1+q)(1+q^2+q^3)(-1+2q+q^2+q^3+q^4)\,[5]_q &= 0.
\end{aligned}$$

Notably, each of these vanish when $e = 5$, giving compatibility with (A.2).

Appendix B. Restrictions to Conjugate Subalgebras

Throughout the text, for some representation V, we refer to $\operatorname{Res}^{\mathscr{H}(S_m)}_{\mathscr{H}(S_l)} V$ without specifying exactly which subalgebra $\mathscr{H}(S_l)$. For instance, in section 4, we explicitly state that the subrepresentations $V_{*0} \oplus V_{**}$ and V_{00} are isomorphic because they both may be characterized by such a restriction. We will verify that this is justified, using a more general fact about resctrictions to conjugate subalgebras.

Proposition B.1. *Suppose B, B' are subalgebras of a k-algebra A with $B = uB'u^{-1}$ for some unit $u \in A^\times$, and let V be a left A-module. Let $\phi : V \to V$ be the linear automorphism specified by $v \mapsto uv$. Then, the following commutes for any $b \in B$:*

$$\begin{tikzcd}
V \arrow[r, "\phi"] \arrow[d, "b"] & V \arrow[d, "ubu^{-1}"] \\
V \arrow[r, "\phi"] & V
\end{tikzcd}$$

Hence, through the identification of B and B' via conjugation by u, we have $\operatorname{Res}^A_B V \simeq \operatorname{Res}^A_{B'} V$

Proof. It suffices to note that $(ubu^{-1})uv = ubv$. □

Corollary B.2. *Suppose $\mathscr{H}', \mathscr{H}''$ are two subalgebras of $\mathscr{H}(S_m)$ generated by l simple reflections and V is a representation of $\mathscr{H}$. Then,* $\operatorname{Res}^{\mathscr{H}}_{\mathscr{H}'} V \simeq \operatorname{Res}^{\mathscr{H}}_{\mathscr{H}''} V$.

Proof. Let $\mathscr{H}'$ and $\mathscr{H}''$ be the subalgebras of $\mathscr{H}(S_m)$ generated by the reflections $\{T_{i_1}, \dots, T_{i_l}\}$ and $\left\{T_{i_1}, \dots, T_{i_{j-1}}, T_{i_j+1}, T_{i_{j+1}}, \dots, T_{i_l}\right\}$ for $1 \le i_1 < \cdots < i_{j-1} < i_j + 1 < i_{j+1} < \cdots < i_l \le n$. It is sufficient to prove that $\mathscr{H}'$ and $\mathscr{H}''$ are conjugate; then transitivity gives conjugacy of any $S_l \subset S_m$, and the previous proposition gives isomorphisms of the representations.

We will show that $\mathscr{H}'' = T_{i_j} \mathscr{H}' T_{i_j}^{-1}$. It suffices to show that $T_{i_j} T_w T_{i_j}^{-1} \in \mathscr{H}''$ for w a word generated by simple transpositions $s_{i_1}, \dots, s_{i_l} \in S_m$. First, note that $l(w) < l(s_{i_j} w)$, implying $T_{i_j} T_w = T_{s_{i_j} w}$ by lemma 1.12 in [8]. Further, by the same lemma, we have

$$\begin{aligned} T_{s_{i_j} w} T_{i_j}^{-1} &= q^{-1} \left(T_{s_{i_j} w} T_{i_j} + (1-q) T_{s_{i_j} w} \right) \\ &= q^{-1} \left(T_{q s_{i_j} w s_{i_j}} + (q-1) T_{s_{i_j} w} + (1-q) T_{s_{i_j} w} \right) \\ &= T_{s_{i_j} w s_{i_j}} \end{aligned}$$

which is in $\mathscr{H}''$. ∎

$$\varphi_6^0, \varphi_5^1 = \begin{bmatrix} 0 & 0 & -q^{3/2}+1 & 0 & 0 \\ 0 & -q^{3/2}+1 & [4]_{q^{1/2}} & 0 & 0 \\ 0 & 0 & [4]_{q^{1/2}} & 0 & -q^{3/2}+1 \\ -[4]_{q^{1/2}} & [4]_{q^{1/2}} & q^{1/2}\left(q^{1/2}+1\right) & 0 & [4]_{q^{1/2}} \\ [4]_{q^{1/2}} & 0 & [4]_{q^{1/2}} & -[4]_{q^{1/2}} & 0 \end{bmatrix}$$

$$\varphi_6^2 = \begin{bmatrix} 0 & 0 & -q-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q-1 & -q^{1/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & q+1 & 0 & q^{1/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{1/2} & 0 & 0 & 0 & -q^2-1 & 0 \\ q^{1/2} & -q^{1/2} & -[3]_{q^{1/2}} & 0 & 0 & 0 & -q^{1/2} & 0 \\ -q^{1/2} & q^{1/2} & 0 & [3]_{q^{1/2}} & 0 & 0 & 0 & q^{1/2} \\ -q^{1/2} & 0 & -q^{1/2} & 0 & q^{1/2} & 0 & 0 & 0 \\ q^{1/2} & 0 & 0 & 0 & -q^{1/2} & -[3]_{q^{1/2}} & 0 & -q^{1/2} \end{bmatrix}$$

$$\varphi_5^3 = \begin{bmatrix} -q^{1/2}\left(q^{1/2}+1\right) & [4]_{q^{1/2}} & q^{1/2} \\ q^{1/2} & q^{3/2} & 0 \\ 0 & -q-1 & \end{bmatrix} \qquad \varphi_4^0 = \begin{bmatrix} 0 & -[4]_{q^{1/2}} \\ -1 & 1 \end{bmatrix}$$

$$\varphi_3^1 = \begin{bmatrix} 0 & -[4]_{q^{1/2}} \\ q^{1/2}\left(q^{1/2}+1\right) & -q^{1/2}\left(q^{1/2}+1\right) \end{bmatrix} \qquad \varphi_4^2 = \begin{bmatrix} 0 & [4]_{q^{1/2}} & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Figure 10. We define representations $\varphi^r_{2n+r} : M^r_{2n+r}/K^r_{2n+r} \to V_s^{2n+r}$, where $s = \texttt{**}$ if $r = 0, 3$ and $s = \texttt{*0}$ otherwise. These are given with respect to the basis on M^r_{2n+r} induced by the increasing lexicographic order basis on M^0_{2n+2r} and the quotient $M^0_{2n+2r} \twoheadrightarrow M^r_{2n+r}$. These are also given with respect to the basis on V given by increasing lexicographic order $\texttt{*} < \texttt{0}$.

Appendix C. Data

In this section we aim both to support conjecture with data and to provide transition matrices wherever possible between our two graphical representations.[4]

Remark. These results are restricted by two things; first and with less recourse, this was computed in Magma, a closed source language with bugs that could not be resolved in certain cases. Second, the memory necessary to store the Hecke algebra via basis elements and structure constants grows with $(n!)^3$; this is prohibitively large when $m > 7$. An implementation as a quotient by a free algebra or only specifying multiplication

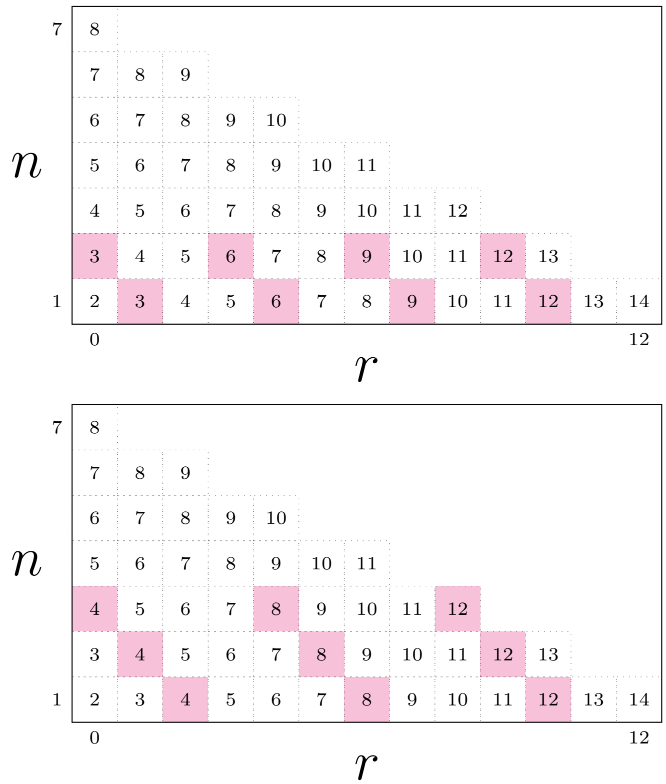


Figure 11. Illustration of the modules M^r_{2n+r} having sign submodules for $p = \infty$ and $e = 3, 4$ respectively. The value $n + r + 1$ is filled in the squares, and modules having sign submodules are colored magenta. For $p = \infty$ and $2n + r \leq 14$, it has been verified, through a combination of theorems here and empirical computations, that K^r_{2n+r} is nontrivial if and only if $e | n + r + 1$ and $e < n$.

by generators may fix this, however, this is not possible with the Magma language, and the closed source renders modification of the language impossible.

We give in Figure 10 some isomorphisms between M/K and V for $e \geq n + r + 1$ with $p = \infty$; all but one of these are cases with $e > n + r + 1$, so $K = 0$ and this is an isomorphism with our crossingless matchings representation. The case $n = 1$, $r = 3$ gives an example of an isomorphism not proven in general, but proven via our computation. All of these computations are done for q a primitive 5th root of unity in the algebraic extension of the Cyclotomic field $\mathbb{Q}(\zeta_{10})$ by a root of the polynomial $x^2 - \tau$.

We give in Figure 11 some data supporting a conjecture concerning sign subrepresentations of M^r_{2n+r}. The computations to support this were done over $\mathbb{C}$ with q a primitive 5th root of unity.

It is known that, for small $2n + r$, each Specht module $S^{(n+r,n)}$ has a composition series of length 2.[8] We give in Figures 12 through 14 the map $\iota^r_{e,2n+r} : U^r_{2n+r} \hookrightarrow M^r_{2n+r} \twoheadrightarrow U^r_{2n+r}$, which conjecturally illustrates the inclusion of the first composition factor of $S^{(n+r,n)}$ into $S^{(n+r,n)}$ for all $2n + r \leq 7$.

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$$
\begin{aligned}
\iota^1_{3,3} = \iota^0_{3,4} &= \begin{bmatrix} 1 & 1 \end{bmatrix}^{\intercal} \\
\iota^1_{3,5} = \iota^0_{3,6} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}^{\intercal} \\
\iota^4_{3,6} &= \begin{bmatrix} -1 & -1 & 0 & 1 & 1 \end{bmatrix}^{\intercal} \\
\iota^3_{3,7} &= \begin{bmatrix} 1 & 0 & -1 & -1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^{\intercal} \\
\iota^1_{3,7} &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}^{\intercal}
\end{aligned}
$$

Figure 12. The maps of representations $\iota^r_{e,2n+r} : U^r_{2n+r} \hookrightarrow M^r_{2n+r}$ for $e = 3$.

$$
\begin{aligned}
\iota^2_{4,4} &= \begin{bmatrix} 1 & \alpha & 1 \end{bmatrix}^{\intercal} \\
\iota^1_{4,4} &= \begin{bmatrix} 1 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & \frac{1}{2}\alpha \end{bmatrix}^{\intercal} \\
\iota^2_{4,6} &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & -1 & -\alpha & 0 \\
0 & 0 & 1 & 0 & 0 & \alpha & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & -\alpha & -1 \\
0 & 0 & 0 & 0 & 1 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & \frac{1}{2}\alpha
\end{bmatrix}^{\intercal} \\
\iota^0_{4,6} &= \begin{bmatrix} \alpha & 1 & 1 & \alpha & 1 \end{bmatrix}^{\intercal} \\
\iota^1_{4,7} &= \begin{bmatrix}
1 & 0 & 0 & 0 & \frac{1}{2}\alpha & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 1 & 1/2 & \alpha & \frac{1}{2} \\
0 & 1 & 0 & 0 & -1 & 0 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & 0 & \alpha & \frac{1}{2}\alpha & -1 & \frac{1}{2}\alpha \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & \alpha & 1 & 0 & -\alpha & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & \alpha & 0 & 0 & -1 & -\alpha & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 0 & 0 & 0 & \frac{1}{2}\alpha & -1 & \frac{1}{2}\alpha \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & \frac{1}{2}\alpha
\end{bmatrix}^{\intercal}
\end{aligned}
$$

Figure 13. The maps of representations $\iota^r_{e,2n+r} : U^r_{2n+r} \hookrightarrow M^r_{2n+r}$ for $e = 4$.

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$$\iota^3_{5,5} = \begin{bmatrix} 1 & \beta & \beta & 1 \end{bmatrix}^\intercal$$
$$\iota^2_{5,6} = \begin{bmatrix} \beta & \beta & 1 & 1 & \beta+1 & \beta & \beta & \beta & 1 \end{bmatrix}^\intercal$$
$$\iota^3_{5,7} = \begin{bmatrix} 1 & \beta & \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \beta & \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & \beta & 1 & 0 & 0 & 0 \\ \beta & \beta & 1 & 0 & 1 & \beta+1 & \beta & 0 & \beta & \beta & 0 & 1 & 0 & 0 \\ -\beta-1 & -\beta & -\beta & 0 & -\beta & -\beta-1 & -\beta-1 & 0 & -\beta-1 & -\beta-1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \beta & 0 & 0 & \beta & 0 & 0 & 0 & 1 \end{bmatrix}^\intercal$$
$$\iota^1_{5,7} = \begin{bmatrix} 1 & \alpha & \alpha & \alpha & -\beta & \alpha & -\beta & \alpha & -\beta & 1 & \alpha & \alpha & \alpha & -\beta \end{bmatrix}^\intercal$$
$$\iota^4_{6,6} = \begin{bmatrix} 1 & \gamma & 2 & \gamma & 1 \end{bmatrix}^\intercal$$
$$\iota^3_{6,7} = \begin{bmatrix} \gamma & 2 & \gamma & 1 & 1 & 2\gamma & 3 & \gamma & 2 & 2\gamma & 2 & 2 & \gamma & 1 \end{bmatrix}^\intercal$$
$$\iota^5_{6,7} = \begin{bmatrix} 1 & \delta & \varepsilon & \varepsilon & \delta & 1 \end{bmatrix}^\intercal$$

Figure 14. The maps of representations $\iota^r_{e,2n+r} : U^r_{2n+r} \hookrightarrow M^r_{2n+r}$ for $e = 5, 6$.

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