ROUGH CUT OF PROVEN WORK ON $\mathscr{H}$.

MILES JOHNSON & NATALIE STEWART

1. Introduction

Let S_{2n+r} be the symmetric group on $2n+r$ indices with $2n+r \geq 2$, let $\mathscr{H} = \mathscr{H}_{k,q}(S_{2n+r})$ be the corresponding Hecke algebra over field k with parameter $q \in k^\times$ having square root $q^{1/2}$, and let $\{T_i\}$ be the reflections generating $\mathscr{H}$. Let $[m]_q = 1+q+\cdots+q^{m-1}$ be the q-number of m. Let e be the smallest positive integer such that $[e]_q = 0$, and set $e = \infty$ if no such integer exists. Either $q = 1$ and e is the characteristic of k (with 0 replaced by ∞), or $q \neq 1$ and q is a primitive eth root of unity.

Throughout the text, we will refer to partitions of $2n+r$; identify each partition with a tuple $\lambda = (\lambda_1^{a_1}, \ldots, \lambda_l^{a_l})$ having $\lambda_i > \lambda_{i+1}$, $a_i > 0$, and $\sum_i a_i \lambda_i = 2n+r$. Identify each of these with a subset $[\lambda] \subset \mathbb{N}^2$ as defined in Kleshev, and define $\lambda(i) = (\lambda_1^{a_1}, \ldots, \lambda_{i-1}^{a_{i-1}}, \lambda_i^{a_i - 1}, \lambda_i - 1, \lambda_{i+1}^{a_{i+1}}, \ldots, \lambda_l^{a_l})$ to be the partition with the ith row removed.

Fixing some partition λ, for $1 \leq i \leq j \leq l$, let $\beta(i,j)$ be the hook length

$$\beta(i,j) = \lambda_i - \lambda_j + \sum_{t=i}^{j} a_t.$$

Then, adopting Kleshev's terminology, j is normal in λ if $\beta(i,j) \not\equiv 0 \pmod{e}$ for all $i < j$, and j is good if it is the largest normal number (these are stronger conditions than generally necessary).

Let $S^{(n+r,n)'}$ be the Specht module corresponding to the young diagram with two columns with height difference r, and let $D^{(n+r,n)'}$ be the corresponding irreducible quotient. The purpose of this writing is to characterize these representation via an isomorphism with two graphical representations of $\mathscr{H}$.

1.1. Crossingless Matchings.

Definition 1.1. A *crossingless matching on $2n+r$ indices with r anchors* is a partition of $\{1, \ldots, 2n+r\}$ into n parts of size 2 and r of size 1 such that no two parts of size two"cross", i.e. there are no parts (a,a') and (b,b') such that $a < b < a' < b'$, and no parts of size one are "inside" of a part of size two, i.e. there are no $c, (a,a')$ such that $a < c < a'$. We will call these arcs and anchors, respectively. Then, define W_{2n+r}^r to be the k-vector space with basis the set of generalized crossingless matchings on $2n+r$ indices with r anchors.

In order for this to be a $\mathscr{H}$-module, endow this with the action given by Figure 1; if a "loop" is created, scale by $q+1$, if a loop is not created and the action involves fewer than 2 anchors, deform into a new crossingless maching and scale by $q^{1/2}$, and if it involves two anchors, scale by 0. We verify that this is well-defined in appendix A.1.

Let the length of an arc (i,j) be $l(i,j) := j - i + 1$. Note that the crossingless matchings on $2n$ indices with no anchros can all be identified with a list of n integers describing the lengths of the arcs from left to right; using this, we may order the crossingless matchings with no anchors in increasing lexicographical order in order to obtain an order on the subbasis containing a particular set of anchors; let the basis be ordered first by the position of the anchors in decreasing lexicographical order, then increasing for the matchings between each anchor. Let this basis be $\{w_i\}$. This basis is illustrated for W_5^1 in Figure 2.

We will prove that $W := W_{2n+r}^r$ and $S := S^{(n+r,n)'}$ are isomorphic as representations in the case that $e > n+r+1$. Note that, when $r = 0$, these have the same dimension given by the nth catalan number C_n.

1.2. Fibonacci Representation.

Now suppose that $k = \mathbb{C}$ and $q = \exp(2\pi i \ell/5)$ is a primitive 5th root of unity. Let V^m be a k-vector space with basis given by the strings $\{*, p\}^{n+1}$ such that the character $*$ never appears twice in a row. We will surpress the superscript whenever it is clear from context.

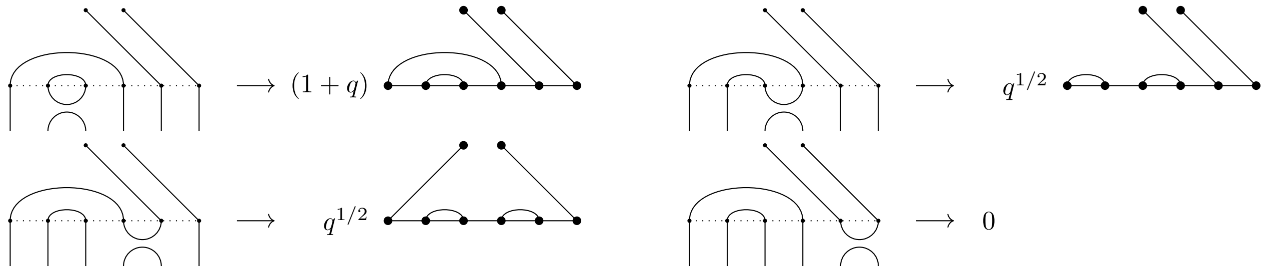

Figure 1. Illustration of the actions $(1+T_i)w_{|W_6^2|}$. In general, we act by deleting loops, deforming into a new crossingless matching, and scaling by either $q^{1/2}$, $(q+1)$, or 0.

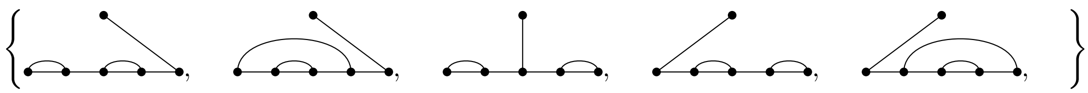

Figure 2. The basis for W_5^1.

We wish to endow this with a $\mathscr{H}$-action which acts on a basis vector only dependent on characters $i, i+1, i+2$, sending each basis vector to a combination of the other basis vectors having the same characters $1, \dots, i, i+2, \dots, n+1$ as follows:

$$
\begin{aligned}
T_1\,(*pp) &:= \alpha(*pp)\\
T_1\,(pp*) &:= \alpha(pp*)\\
T_1\,(*p*) &:= \beta(*p*)\\
T_1\,(p*p) &:= \gamma(p*p) + \delta(ppp)\\
T_1\,(ppp) &:= \delta(p*p) + \varepsilon(ppp)
\end{aligned}
\tag{1}
$$

for constants

$$
\begin{aligned}
\alpha &= -1\\
\beta &= q\\
\gamma &= \tau(q\tau - 1)\\
\delta &= \tau^{3/2}(q+1)\\
\varepsilon &= \tau(q-\tau)\\
\tau &= -1 - q^2 - q^3
\end{aligned}
\tag{2}
$$

with T_i acting similarly on the substring $i, i+1, i+2$. We will verify that this is a representation of $\mathscr{H}$ in Appendix A.2

This contains 4 subrepresentations based on the first and last character of the string, which are not modified by $\mathscr{H}$. Label the subrepresentation of strings $(*\cdots*)$ by V_{**}, and similar for the other 3. It is easy to see that $V_{*p} \simeq V_{p*}$, so that

$$V \simeq 2V_{*p} \oplus V_{**} \oplus V_{pp}.$$

We will show that $V_{pp} \simeq V_{*p} \oplus V_{**}$, and give the following isomorphisms with irreducible quotients of specht modules depending on the parity of the number of indices in $\mathscr{H}$:

$$
\begin{aligned}
V_{**}^{2n} &\simeq D^{(n,n)'}\\
V_{**}^{2n-1} &\simeq D^{(n+1,n-2)'}\\
V_{*p}^{2n} &\simeq D^{(n+1,n-1)'}\\
V_{*p}^{2n-1} &\simeq D^{(n,n-1)'}.
\end{aligned}
\tag{3}
$$

2. Crossingless Matchings and Specht Modules

Our goal is to prove that $W^r_{2n+r} \simeq S^{(n+r,n)'}$ when $e > n+r+1$.

Proposition 2.1. *(i) Suppose that $n, r > 0$. Then, a filtration of $ResW^r_{2n+r}$ is given by*

$$0 \subset W^{r-1}_{2n+r-1} \subset Res\, W^r_{2n+r} \tag{4}$$

with $Res\, W^r_{2n+r}/W^{r-1}_{2n+r-1} \simeq W^{r+1}_{2n+r-1}$.

(ii) We have the following isomorphism of representations:

$$W^1_{2n-1} \simeq Res\, W^0_{2n} \tag{5}$$

Proof. **(i)** Note that we may identify the subrepresentation of Res W^r_{2n+r} having anchor n with W^{r-1}_{2n+r-1}.

Let $U :=$ Res $W^r_{2n+r}/W^{r-1}_{2n+r-1}$. Let $\phi : U \to W^{r+1}_{2n+r-1}$ be the k-linear map which regards the arc $(i, 2n+r)$ in U as an anchor at i in W^{r+1}_{2n+r-1}. It is not hard to verify that this is a well-defined isomorphism of vector spaces, so we must show that it is $\mathscr{H}$-linear.

Given a basis vector w_j with arc $(i, 2n+r)$, ϕ is clearly compatible with $T_{i'}$ with $i' \neq i, i-1$. Further, it's easy to verify that ϕ is compatible with T_i and T_{i-1}, as actions on one anchor were designed for this deformation. When there are anchors $(i, i+1)$, then $\phi(T_i w_j) = T_i \phi(w_j) = 0$, and similar for T_{i-1}. Hence ϕ is an isomorphism of representations, and the statement is proven.

(ii) This follows with the above proof, defining $W^{-1}_{2n-1} := 0$ ☐

Lemma 2.2. *Every basis vector in W^r_{2n+r} is cyclic.*

Proof. We have already proven this in the $r = 0$ case, so suppose that $r > 0$.

Note that, between anchors $a < a'$ having no arc b with $a < b < a'$, the $W^0_{a'-a}$ case allows us to generate the vector with all length-2 arcs between a, a' and identical arcs/anchors outside of this sub-matching.[1]

Applying this between each arc gives us a vector with length-2 arcs and anchors, and we may use the appropriate $(1 + T_i)$ to move anchors to any positions, and the reverse process from above to generate the correct matchings between arcs and generate any other basis vector. ☐

Let $K := \bigcap_{i=1}^{2n+r-1} \ker(1 + T_i) = \ker \bigoplus_{i=1}^{2n+r-1}(1 + T_i)$. This will be a large technical tool in our proof of irreducibility.

Lemma 2.3. *Let w_j be the basis vector with anchors $1, \dots, r$ and all arcs of maximal length. Suppose $w \in K \setminus \{0\}$. Then, w_j is represented in w.*

Proof. We will show this in steps; first, we show that, given that a vector is represented with anchors $1, \dots, s$, there must be a vector represented in w with $(s+1)$st anchor, including when $s = 0$; this implies that a vector is represented with anchors $1, \dots, r$. Then, we will show that, given a vector is represented with anchors $1, \dots, r$ and first s arc-lengths $n, n-2, \dots, n-2s$, there is a vector represented with these and the $(s+1)$st arc-length $n - 2s - 2$. This implies that w_j is represented.

Step 1. Suppose that $s < r$ is the maximal number such that a vector with anchors $1, \dots, s$ is represented. Take the vector w_i which, among vectors represented in w with anchors $1, \dots, s$, has $(s+1)$st anchor at minimal index $t > s+1$, Then, $q^{-1/2}(1 + T_{t-1})w_i$ has anchors $1, \dots, s$ and a earlier index than t, so it was not represented before; further, for any other basis vector $w_l \neq w_i$ to map onto $q^{-1/2}(1 + T_{t-1})w_i$, we would require that w_l has anchors $1, \dots, s$ and some other anchor at index $t' < t$, so it is not represented. Hence w_i is unique among the vectors represented mapping onto $q^{-1/2}(1+T_{t-1})w_i$, and $(1+T_{t-1})w$ represents this vector, giving $w \notin \ker(1 + T_{t-1})$.

When $s = 0$, this is similar, and we simply perform this logic on the 1st anchor. Each lead to contradiction, so we must have $s = r$.

Step 2. This step is similar; suppose that $s < n$ is the maximal number such that a vector with anchors $1, \dots, r$ and first s arc-lengths $n, \dots, n-2s$ is represented. Take the vector w_i which, among vectors represented in w with anchors $1, \dots, r$ and first s arc-lenths $n, \dots, n-2s$, has maximal length t of the arc beginning at index $r+s+1$. Then, $q^{1/2}(1 + T_{r+s+t})w_i$ is mapped to only by w_i and vectors having anchors $1, \dots, r$ and first $s+1$ arc-lengths $n, \dots, n-2s, t'$ with $t' > t$, which are not represented in w; hence

[1] At the ends, we apply the W^0_a case or the W^0_{2n+r-a} case in the same way for the first a or last $2n+r-a$ indices.

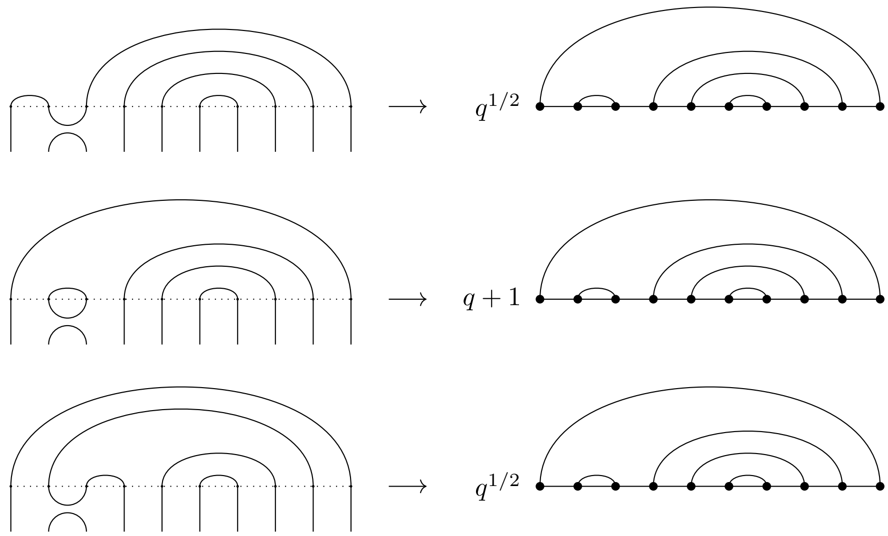

Figure 3. Illustrated is the row constructed for transposition $(1+T_2)$; clearly these are the only basis elements mapping to multiples of the desired element, and they relate to each other. replacing the outermost and/or innermost arc with an anchor typifies the rows constructed with three nonzero coefficients.

$q^{-1/2}(1+T_{r+s+t})w_i$ is represented in $(1+T_{r+s+t})w$, giving $w \notin \ker(1+T_{r+s+t})$. The $s=0$ case is similar, and implies that $s=n$. $\square$

Lemma 2.4. *Suppose $e \nmid n+r+1$. Then, $K=0$.*

Proof. Consider the matrix $A = \bigoplus(1+T_i)$ having kernel K. It is sufficient by lemma 2.3 to show that A includes a row $[0,\dots,0,1,0,\dots,0]$ with a nonzero entry only on the column j.

Now, we may characterize the rows of A as follows; if the row corresponding to $(1+T_i)$ and mapping onto the element $w_l \in W$ is nonzero, then it is of the form $\left[a_1,\dots,a_{|W|}\right]$ where $a_l = 1+q$, $a_m = q^{1/2}$ whenever $(1+T_i)w_m = q^{1/2}w_l$, and $a_m = 0$ otherwise.

Seeing this, the row corresponding to $(1+T_{n+r})$ and w_j has nonzero entries $q^{1/2}$ at w_j and $(1+q)$ at the vector w agreeing with w_j at all indices except having arcs at $(n+r-1,n+r)$ and $(n+r+1,n+r+2)$. Similar justification leads the row corresponding to $(1+T_{n+r-1})$ at w to have nonzero entries $q^{1/2}$ at w and $(1+q)$ at w_j and the vector with anchors $1,\dots,r$, arc $(n+r-3,n+r-2)$, and all other arcs maximum length.

We may iterate this process as illustrated in Figure 3, eventually ending at a row with two nonzero entries, either an arc $(1,2)$ or an arc $(2,3)$, and all anchors otherwise left-aligned and arcs of maximum length. These rows together form an $(n+r)\times\left|W^r_{2n+r}\right|$ sumbatrix of A which has a nonzero column in the row corresponding to j, and has (by removing zero columns) the same column space as the following square matrix:

$$B_{n+r} := \begin{bmatrix} q+1 & q^{1/2} & & & & & \\ q^{1/2} & q+1 & q^{1/2} & & & \text{\Large 0} & \\ & q^{1/2} & q+1 & q^{1/2} & & & \\ & & \ddots & \ddots & & & \\ & \text{\Large 0} & & & q^{1/2} & q+1 & q^{1/2} \\ & & & & & q^{1/2} & q+1 \end{bmatrix}.$$

We will show that this matrix is invertible; then, a sequence of elementary row operations will yield the identity, and in particular, when applied to A, will yield a row with a nonzero entry only on column j, giving $K=0$.

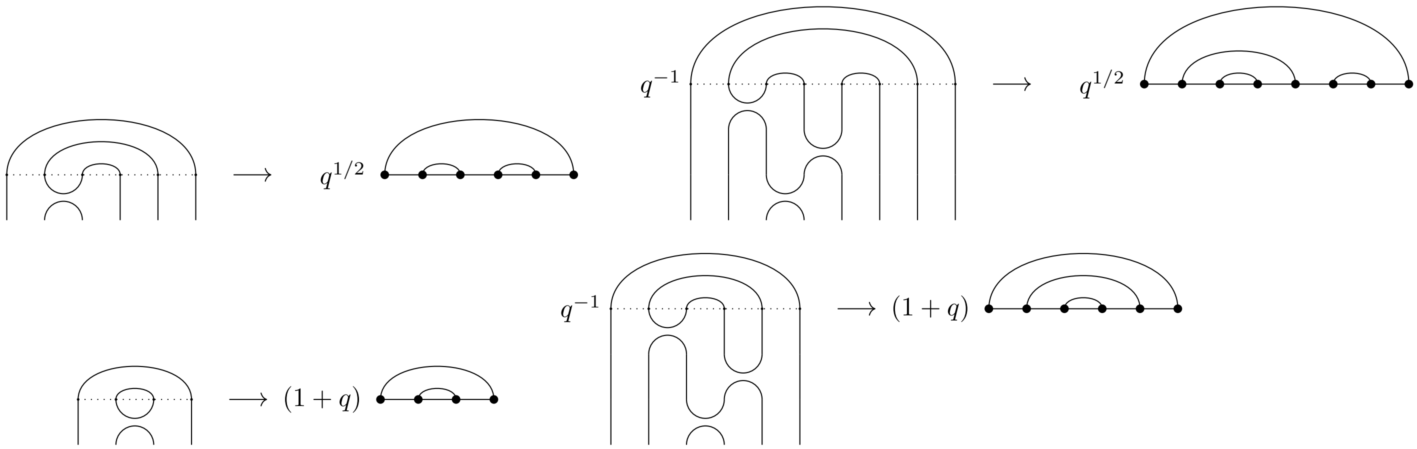

Figure 4. The correspondence between the action of $(1+T_2)$ on $w_5' \in W_6^0$ and the action of $q^{-1}(1+T_3)(1+T_4)(1+T_2)$ on the corresponding vector in W_8^0 having arc $(3,4)$ first, then on $w_2' \in W_4^0$. This demonstrates that the action works with and without creating a loop.

We may prove invertibility of this matrix by proving that $\det B_{n+r} = [n+r+1]_q$ inductively on $n+r$. This is satisfied for our base case $n+r=1$, so supose that it is true for each $m < n+r$. Then,

$$\begin{aligned}\det B_{n+r} &= (q+1)\det B_{n+r-1} - q\det B_{n+r-2}\\ &= (q+1)(1+\cdots+q^{n+r-1}) - (q+\cdots+q^{n+r-1})\\ &= 1+\cdots+q^{n+r}\\ &= [n+r+1]_q\,.\end{aligned}$$

Hence $\det B_{n+r} \neq 0$, and $K=0$. □

Proposition 2.5. *The representation W_{2n+r}^r is irreducible when $e > n+r+1$.*

Proof. We proceed by induction on $2n+r$. Note that, by identification with the trivial and sign representations, the base case $2n+r=2$ is already prove, so suppose we have proven this for each $2m+s < 2n+r$.

Take some $w \in W$ and some $(1+T_i)$ not annhialating w. Note that

$$\operatorname{im}(1+T_i) = \operatorname{Span}\{w_j \mid w_j \text{ contains arc } (i,i+1)\}\,.$$

Hence, as vector spaces, there is an isomorphism $\varphi : \operatorname{im}(1+T_i) \to W_{2(n-1)+r}^r$ "deleting" the arc $(i,i+1)$. This sends every basis vector to another basis vector.

We will show that, for every action $(1+T_j') \in \mathscr{H}(S_{2(n-1)+r})$, there is some action $h_j \in W_{2n+r}^r$ such that the following commutes:

$$\begin{tikzcd}\operatorname{im}(1+T_i) \arrow[r, "\varphi", "\sim"'] \arrow[d, "h_j"] & W_{2(n-1)+r}^r \arrow[d, "1+T_j"]\\ \operatorname{im}(1+T_i) \arrow[r, "\varphi", "\sim"'] & W_{2(n-1)+r}^r\end{tikzcd}$$

Indeed, when $i \neq j$ this is given by $h_j = 1+T_j$, and we have $h_i = q^{-1}(1+T_i)(1+T_{i+1})(1+T_{i-1})$, as given by Figure 4.

Due to the inductive hypothesis, there is some action $h' \in \mathscr{H}(S_{2(n-1)+r})$ sending $\varphi((1+T_i)w)$ to a basis vector; then, the action $\mathscr{H}$ generates the endomorphism $\varphi^{-1}h'\varphi$ sending $(1+T_i)w$ to a basis vector, giving w cyclic and hence W_{2n+r}^r irreducible. □

Corollary 2.6. *Suppose $n,r > 0$ and $e > n+1$. Then, the sequence (4) is a composition series of $Res\, W_{2n+r}^r$.* □

Theorem 2.7. *Suppose $e > n+r+1$, $n > 0$. Then, $W_{2n+r}^r \simeq S^{(n+r,n)'}$.*

Proof. By irreducibility, we know that $W^r_{2n+r} \simeq D^\lambda$ for some e-restricted partition λ. We will proceed in two steps; first we prove that $\lambda = (n+r,n)'$, then we prove that $S^{(n+r,n)'}$ is irreducible.

This will be done inductively; by identification with the trivial and sign representations, the $2n+r=2$ caseholds, so suppose this is true for W^s_{2m+s} whenever $2m+s < 2n+r$ and $m+s \le n+r$ (i.e. $e > m+s+1$).

Step 1. By the inductive hypothesis and irreducibility, we have a composition series given by

$$0 \longrightarrow D^{(n+r-1,n)'} \longrightarrow \operatorname{Res} D^\lambda \longrightarrow D^{(n+r,n-1)'} \longrightarrow 0 \tag{6}$$

In particular, by the Jordan-Hölder theorem, if we have some module $D^\mu \subset \operatorname{Res} D^\lambda$, then $\mu = (n+r-1,n)'$ or $\mu = (n+r,n-1)'$.

By Kleschev, we know that $\operatorname{soc}(D^\lambda) = \bigoplus_\mu D^\mu$, where μ ranges over $\lambda(i)$ for every good number i of λ. Immediately this narrows down λ to three options; the only λ with some $\lambda(i)$ giving $(n+r-1,n)'$ are $\lambda_1 ;= (n+r-1,n,1)'$, $\lambda_2 = (n+r-1,n+1)'$, and $\lambda_3 = (n+r,n)'$.

Note numbers $\beta_l(i,j)$ correspond to hook-lengths, and each $\lambda_\cdot$ has maximum hook length $n+r+1$; hence $\beta_l(i,j) \not\equiv 0 \pmod e$ for all i,j,l, and 3 is a good number in λ_1 and 2 in λ_2. This implies $D^{(n+r-2,n,1)'} \subset D^{\lambda_1}$ and $D^{(n+r-2,n+1)'} \subset D^{\lambda_2}$, giving that $\lambda = \lambda_3$ as desired.

Step 2. Let the hook length of node (a,b) in $[\lambda]$ be written h^λ_{ab}. Let $\nu_p(h^\lambda_{ab})$ be the p-adic valuation of h^λ_{ab}, and let $\nu_{e,p}(h^\lambda_{ab})$ be $\nu_p(h^J)$. Mathas Theorem 5.42 says, for λ e-restricted, S^λ is irreducible if $\nu_{e,p}(h^\lambda_{ab}) = \nu_{e,p}(h^\lambda_{ac})$ for a suitable prime p. In particular, since $h^\lambda_{ab} < e$ for all a,b, we have $\nu_{e,p}(h^\lambda_{ab}) = -1$ for all a,b, and S^λ is irreducible. □

3. The Fibonacci Representation and Specht Modules

We can start our study of V by studying low-dimensional cases. First, note that V^2_{*p} is the sign representation $D^{(2)}$ and V^2_{**} is the trivial representation $D^{(1)^2}$.

V^2_{pp} is a 2-dimensional representation of a semisimple commutative algebra, and hence decomposes into a direct sum of two subrepresentations. In particular, we can use the basis $\{(p*p),(ppp)\}$ and explicitly write the matrix

$$\rho_{T_1} = \begin{bmatrix} \gamma & \delta \\ \delta & \varepsilon \end{bmatrix}$$

having characteristic polynomial $(c-\lambda)(e-\lambda) - d^2 = \lambda^2 - (c+e)\lambda + (ce-d^2)$. The reader can verify that this has roots -1 and q. The eigenspaces with eigenvalues -1 and q are subrepresentations isomorphic to the sign and trivial representation, hence F_{pp} is isomorphic to a direct sum of the trivial and sign representations: $V^2_{pp} \simeq V^2_{*p} \oplus V^2_{**}$.

Now let's prove that V^3_{**} is irreducible; this has basis $\{*p*p\}, \{*ppp\}$, and the following matrices:

$$\rho_{T_1} = \begin{bmatrix} \beta & 0 \\ 0 & \alpha \end{bmatrix}; \qquad \rho_{T_2} = \begin{bmatrix} \gamma & \delta \\ \delta & \varepsilon \end{bmatrix}.$$

A subrepresentation must be one-dimensional, and hence an eigenspace of each of these matrices; since $\beta \neq \alpha$, the first has eigenspaces given by the spans of basis elements, and since $\delta \neq 0$, these are not eigenspaces of the second. Hence V^3_{**} is irreducible. Now we may move on to the general case.

Proposition 3.1. *The representation $V_{*p} := V^m_{*p}$ is irreducible.*

Proof. We will prove this inductively in m. We've already proven it for V^2_{*p} and V^3_{*p}, so suppose that V^{m-2}_{*p} is irreducible.

Let $\{v_i\}$ be the basis for V_{*p}. Then, each v_i is cyclic; indeed, we can transform every basis vector into $(*p\dots p)$ by multiplying by the appropriate $\frac{1}{\delta-\gamma}(T_i - \gamma)$, and we can transform $(*p\dots p)$ into any basis vector by multiplying be the appropriate $\frac{1}{\delta-\varepsilon}(T_i-\varepsilon)$. Hence it is sufficient to show that each $v \in V_{*p}$ generate some basis element.

Let v' be the basis element $(*p*p\dots p)$, which is many copies of $*p$, followed by an extra p if m is odd. We will show that each $v \in F$ generates v'.

Suppose that no elements beginning $(*p*p)$ are represented in v_i; then, all such elements are represented in $T_3 v$, so we may assume that at least one is represented in v.

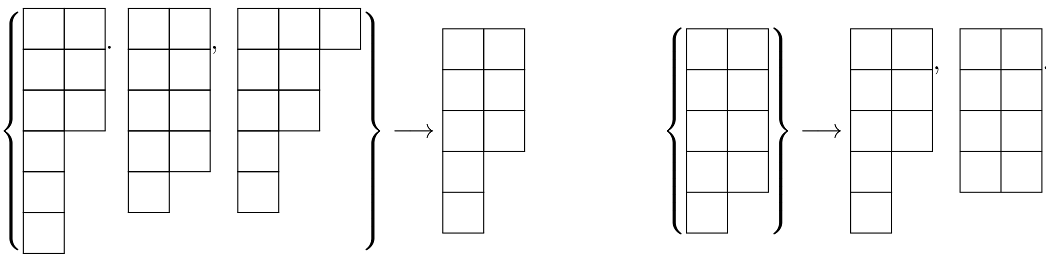

Figure 5. Illustration of the partitions of 9 which can, via row removal, yield $(n,n-2)'$ alone, or both $(n,n-2)'$ and $(n-1,n-1)'$.

Note that $\operatorname{im}(T_2-\alpha) = \operatorname{Span}\{\text{Basis vectors beginning } (*p*p)\}$ and $(T_2-\alpha)v \neq 0$. Further, note that $\operatorname{Res}^{\mathscr{H}(S_m)}_{\mathscr{H}(S_{m-2})} \operatorname{im}(T_2-\alpha) \simeq V^{m-2}_{*p}$ as representations. Hence irreducibility of V^{m-2}_{*p} implies that v' is generated by $(T_2-\alpha)v$, and V^m_{*p} is irreducible. $\square$

Knowing this, the restriction statements are clear; $\operatorname{Res}V^m_{*p} \simeq V^{m-1}_{pp}$ by considering the last $m-2$ transpositions, and $\operatorname{Res}V^{m-1}_{*p} \simeq V^{m-1}_{*p} \oplus V^{m-1}_{**}$ by considering the first $m-2$. Similarly, $\operatorname{Res}V^m_{**} \simeq V^{m-1}_{*p}$ by considering the first $m-2$ transpositions. This gives that $V \simeq 3V_{*p} \oplus 2V_{**}$.

Now we may move on and use Young Tableau to characterize V. Recall that the socle of D^λ is given by $\bigoplus_{\mu \xrightarrow{\text{good}} \lambda} D^\mu$, and that D^λ is semisimple iff every $\mu \xrightarrow{\text{normal}} \lambda$ is good.

Theorem 3.2. *The irreducible components of V are given by the following isomorphisms:*

$$\begin{aligned} V^{2n}_{**} &\simeq D^{(n,n)'} \\ V^{2n-1}_{**} &\simeq D^{(n+1,n-2)'} \\ V^{2n}_{*p} &\simeq D^{(n+1,n-1)'} \\ V^{2n-1}_{*p} &\simeq D^{(n,n-1)'}. \end{aligned}$$

Proof. We will prove this by induction on n; we have already proven the base case V^2, so suppose that we have proven these isomorphisms for V^{2n-2}. We will prove the isomorphisms for V^{2n-1} and V^{2n}.

By irreducibility, $V^{2n-1}_{**} \simeq D^{\lambda_{**}}$ and $V^{2n-1}_{*p} \simeq D^{\lambda_{*p}}$ for some diagrams λ_{**} and λ_{*p}. We will show that $\lambda_{**} = (n+1,n-2)'$ and $\lambda_{*p} = (n+1,n-1)'$.

First, note that we have

$$\operatorname{Res} D^{\lambda_{**}} \simeq D^{(n,n-2)'} \simeq \operatorname{Res} D^{(n+1,n-2)'}$$

and

$$\operatorname{Res} D^{\lambda_{*p}} \simeq D^{(n,n-2)} \oplus D^{(n-1,n-1)} \simeq \operatorname{Res} D^{(n,n-1)'}.$$

By semisimplicity of $\operatorname{Res}D^{\lambda_{**}}$ and $\operatorname{Res}D^{\lambda_{*p}}$, every normal cell in λ_{**} and λ_{*p} is good, and every good cell is removed in a summand of the restriction. In particular, the only normal number in λ_{**} is 1.

For λ_{**}, the only tableaux which can remove a cell to yield $D^{(n,n-2)'}$ are $(n+1,n-2)'$, $(n,n-1)'$, and $(n,n-2,1)'$ as illustrated in Figure 5; we have already seen that $D^{(n,n-1)'}$ does not have irreducible restriction, so we are left with $(n+1,n-2)'$ and $(n,n-2,1)'$. We may directly check that $(n,n-2,1)'$ doesn't satisfy this, as we have the following:

$$\begin{aligned} \beta_\lambda(1,2) &= 3-2+(n-2) = n-1 \\ \beta_\lambda(1,3) &= 3-1+n = n+2 \\ \beta_\lambda(2,3) &= 2-1+3 = 4. \end{aligned}$$

At least one of $\beta(1,2)$ and $\beta(1,3)$ is nonzero, since $\beta_\lambda(1,3)-\beta_\lambda(1,2) = 3 \not\equiv 0 \pmod e$, and hence at least one of M_2 and M_3 is empty. Hence at least one of 2 or 3 is normal in $(n,n-2,1)'$, and $\lambda_{**} = (n+1,n-2)$.

For λ_{*p}, we immediately see from Figure 5 that the only option is $(n,n-1)$.

We can perform a similar argument for the V^{2n} case, finding now that

$$\text{Res } D^{\mu_{**}} \simeq D^{(n,n-1)'} \simeq \text{Res } D^{(n,n)'}$$

and

$$\text{Res } D^{\mu_{*p}} \simeq D^{(n,n-1)'} \oplus D^{(n+1,n-2)'} \simeq \text{Res } D^{(n+1,n-1)'}.$$

Through a similar process, we see that $\mu_{*p} = (n+1, n-1)'$. We narrow down μ_{**} to one of $(n,n)'$ or $(n, n-1, 1)'$, and note that

$$\beta_\mu(1,2) = 3 - 2 + (n-1) = n$$
$$\beta_\mu(1,3) = 3 - 1 + n = n + 2$$
$$\beta_\mu(2,3) = 2 - 1 + 2 = 3$$

and hence at least one of 2 or 3 is normal, $\text{Res}D^{(n,n-1,1)'}$ is not irreducible, and $\mu_{**} = (n,n)'$, finishing our proof. □

Corollary 3.3. *We have the following isomorphisms of representations:*

$$V^{2n} \simeq 3D^{(n+1,n-1)'} \oplus 2D^{(n,n)'} V^{2n-1} \simeq 3D^{(n,n-1)'} \oplus 2D^{(n+1,n-2)'}$$

4. Explicit Relationships

Appendix A. Compatibility of Representations with the Relations

In general, we define representations above for the free algebra on generators $\{T_i\}$. Recall that we may give a presentation of $\mathscr{H}$ having generators T_i and relations

$$(T_i - q)(T_i + 1) = 0 \tag{7}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{8}$$

$$T_i T_j = T_j T_i \qquad |i - j| > 1. \tag{9}$$

We call (7) the *quadratic relation* and (8), (9) the *braid relations.* It is easily seen that a representation of $\mathscr{H}$ is equivalent to a representation of the free algebra $k\langle T_i\rangle$ which acts as 0 on the relations (henceforth referred to as *compatibility* with the relations). We will prove in the following sections that V and W are compatible with the Hecke algebra relations.

A.1. **The Crossingless Matchings Representaiton.** Take some basis vector w_i. We will first check (7) by case work:

- Suppose there is an arc $(i, i+1)$. Then, $(T_i - q)(T_i + 1)w = (1+q)\left[(1 + T_i)w - (1+q)w\right] = 0$, giving (7).
- Suppose there is no arc $(i, i+1)$ and $i, i+1$ do not both have anchors; then $(T_i + 1)w = q^{1/2}w''$ for some basis vector w' having arc $(i, i+1)$, and the computation follows as above for (7).
- Suppose $i, i+1$ are anchrors; then $(T_i + 1)w = 0$, giving (7).

Now we verify (8). Let $h := (1 + T_i)(1 + T_{i+1})(1 + T_i)$, and let $g := (1 + T_{i+1})(1 + T_i)(1 + T_{i+1})$. Note the following expansion:

$$\begin{aligned} hw &= 1 + 2T_i + T_i^2 + T_{i+1} + T_i T_{i+1} + T_{i+1} T_i + T_i T_{i+1} T_i \\ &= 1 + (1+q)T_i + T_{i+1} + T_i T_{i+1} + T_{i+1} T_i + T_i T_{i+1} T_i. \end{aligned}$$

An analogous formula gives an analogous equality in g. Hence we have

$$(h - g)w = q(T_i - T_{i+1}) + T_i T_{i+1} T_i - T_{i+1} T_i T_{i+1}.$$

Hence we may equivalently check that $(h - g)w = q(T_i - T_{i+1})$. This is illustrated in Figure 6.

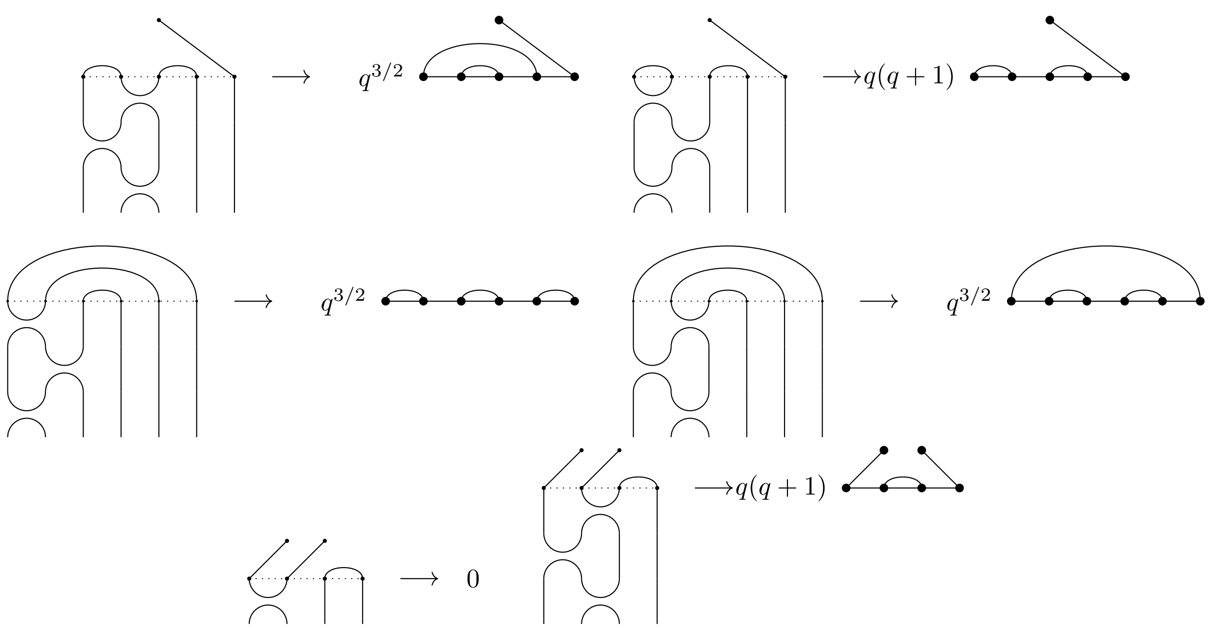

Figure 6. Here we verify in small cases that $hw = qT_i$ and $gw = qT_{i+1}$. These 6 cases cover the situations that there is an arc among the indices $i, i+1, i+2$, that there isn't and there are not two arcs, and that there are two arcs.

Lastly, we have the equation

$$(1+T_i)(1+T_j)-(1+T_j)(1+T_i)=T_iT_j-T_jT_i$$

and hence we simply need to verify that $(1+T_i)$ and $(1+T_j)$ commute, which the reader may easily check.

A.2. **The Fibonacci Representation.** Similar to before, the reader may verify that (9) follows easily, and the others may be verified on strings of length 3 and 4. By considering the coefficients in order of (1), the quadratic relation (7) gives the following quadratics:

$$\begin{aligned}(\alpha-q)(\alpha+1)&=0\\(\beta-q)(\beta+1)&=0\\\gamma\delta+\delta\varepsilon&=(q-1)\delta\\\gamma^2+\delta^2&=(q-1)\gamma+q\\\varepsilon^2+\delta^2&=(q-1)\varepsilon+q\end{aligned}\tag{10}$$

The first two of these are easily verified for any q. Since $\delta\neq 0$, the third is equivalently given by

$$(q-1)=\gamma+\varepsilon=t(q\tau-1+q-\tau)=(\tau^2+\tau)(q-1)$$

or that $\left(\tau^2+\tau-1\right)(q-1)=0$. One may verify that

$$\tau^2+\tau-1=q^6+2q^5+q^4+q^3+q^2-1=(-1+q+q^2)\,[5]_q=0.$$

The fourth is given by the quadratic

$$\tau^2\left[(q\tau-1)^2-\tau(q+1)\right]=\tau(q-1)(q\tau-1)+q$$

or equivalently,

$$(\tau^2+\tau-1)\left[q\left(qt^2+1\right)+t\right]=0$$

which is true for every q.

The fifth is similarly given by

$$(\tau^2+\tau-1)\left[q\left(qt+1\right)+t^2\right]=0$$

which is true for every q.

We now verify (8). We may order the basis for V^4 as follows:

$$\left\{(pppp),(*pp*),(ppp*),(*ppp),(*p*p),(p*p*),(pp*p),(p*pp)\right\}.$$

Then, in verifying the braid relation (8) in this order, we encounter the following quadratics (with tautologies and repetitions omitted):

$$\begin{aligned}\alpha\varepsilon^2+\beta\delta^2&=\alpha^2\varepsilon\\\alpha\delta\varepsilon+\beta\gamma\delta&=\alpha\beta\delta\\\beta\gamma^2+\alpha\delta^2&=\beta^2\gamma\\\alpha\gamma^2+\delta^2\varepsilon&=\alpha^2\gamma\\\delta\varepsilon^2+\alpha\gamma\delta&=\alpha\delta\varepsilon\end{aligned}$$

Substituting in τ and dividing by δ whenever possible, these are equivalent to the vanishing of the following polynomials in q:

$$\begin{aligned}-q(1+q)(1+q^2+q^3)(2+q+3q^2+2q^3)\,[5]_q&=0\\(1+2q+q^3+q^4)\,[5]_q&=0\\(1+q)^2(1+q^2+q^3)(1+3q^3-q^4+q^6)\,[5]_q&=0\\(1+q)^2(1+q^2+q^3)(1+5q+5q^2+3q^3+3q^4+3q^5+q^6)\,[5]_q&=0\\(1+q)(1+q^2+q^3)(-1+2q+q^2+q^3+q^4)\,[5]_q&=0.\end{aligned}$$

Notably, each of these vanish when $e=5$.

Appendix B. Miscellaneous Algebra Facts

Throughout the text, for some representation V, we refer to $\mathrm{Res}^{\mathscr{H}(S_m)}_{\mathscr{H}(S_l)} V$ without specifying exactly which subalgebra $\mathscr{H}(S_l)$.

Proposition B.1. *Suppose B, B' are subalgebras of the k-algebra A with $B = uB'u^{-1}$, and let V be a representation of A. Then, the linear isomorphism $V \xrightarrow{\phi} V$ given by $v \mapsto uv$ causes the following to commute for any $b \in B$:*

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "b"] & V \arrow[d, "ubu^{-1}"] \\ V \arrow[r, "\phi"] & V \end{tikzcd}$$

Hence, through the identification of B and B' via conjugation, we have $Res^A_B V \simeq Res^A_{B'} V$

Proof. This is simply given by $(ubu^{-1})uv = ubv$. $\square$

Corollary B.2. *Suppose $\mathscr{H}', \mathscr{H}''$ are two subalgebras of $\mathscr{H}(S_m)$ generated by l reflections and V is a representation of $\mathscr{H}$. Then, $Res^{\mathscr{H}}_{\mathscr{H}'} V \simeq Res^{\mathscr{H}}_{\mathscr{H}''} V$.*

Proof. Let $\mathscr{H}'$ and $\mathscr{H}''$ be the subalgebras of $\mathscr{H}(S_m)$ generated by the reflections $\{T_{i_1}, \dots, T_{i_l}\}$ and $\left\{T_{i_1}, \dots, T_{i_{j-1}}, T_{i_j+1}, T_{i_{j+1}}, \dots, T_{i_l}\right\}$ for $1 \leq i_1 < \cdots < i_{j-1} < i_j + 1 < i_{j+1} < \cdots < i_l \leq n$. It is sufficient to prove that $\mathscr{H}'$ and $\mathscr{H}''$ are conjugate; then transitivity gives conjugacy of any $S_l \subset S_m$, and the previous proposition gives isomorphisms of the representations.

In fact, the reader can verify that $\mathscr{H}'' = T_{i_j} \mathscr{H}' T_{i_j}^{-1}$. $\square$