SOME GRAPHICAL REALIZATIONS OF TWO-ROW SPECHT MODULES OF IWAHORI–HECKE ALGEBRAS OF THE SYMMETRIC GROUP

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Abstract. We consider the Iwahori–Hecke algebra of the symmetric group on $2n+r$ letters with parameter q. Let e be the smallest positive integer such that the q-number $[e]_q = 0$, or set $e = \infty$ if none exist. We modify Khovanov's crossingless matchings to include $2n$ "nodes" and r "anchors," and prove in the case $e > n + r + 1$ that the associated module is isomorphic to the Specht module $S^{(n+r,n)}$ which corresponds to the partition $(n + r, n) \vdash 2n + r$. We then give heuristics in support of the general case, including explicit composition series for $e = n + r + 1$ and for $2n + r \leq 7$. Lastly, when $e = 5$, we prove an isomorphism between the irreducible quotient $D^{(n+r,n)}$ with $r \leq 3$ and some subrepresentations of Jordan–Shor's Fibonacci representation. We provide explicit transition matrices between this representation and the crossingless matchings representation for $2n + r \leq 6$.

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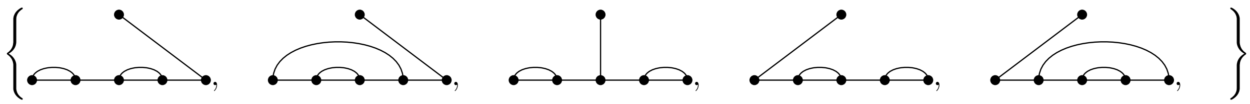

Figure 1. The basis for M_5^1.

1. Introduction

Let S_{2n+r} be the symmetric group on $2n+r$ letters with $2n+r \geq 2$, let $\mathscr{H} := \mathscr{H}_{k,q}(S_{2n+r})$ be the corresponding Hecke algebra over a field k with parameter $q \in k^\times$ having a fixed square root $q^{1/2}$, and let $\{T_i\}$ be the simple reflections generating $\mathscr{H}$.

Let $[m]_q = 1 + q + \cdots + q^{m-1}$ be the q-number of m. Let e be the smallest positive integer such that $[e]_q = 0$, and set $e = \infty$ if no such integer exists. Note that either $q = 1$ and e is the characteristic of k (with 0 replaced by ∞), or $q \neq 1$ and q is a primitive eth root of unity.

When $q = 1$, the Hecke algebra $\mathscr{H}$ is isomorphic to the group algebra $k[S_{2n+r}]$; hence the representation theory of $\mathscr{H}$ generalizes the representation theory of the symmetric group. The Hecke algebra is also well known to be connected to the representation theory of the general linear group over a finite field.[**?**]

It is a classical result that $\mathscr{H}$ is semisimple precisely when $e > 2n+r$, in which case the irreducible representations of $\mathscr{H}$ are given by *Specht modules* S^λ, which are indexed by the partitions λ of $2n+r$. Further, $\mathscr{H}$ admits a cellular basis with cell modules given by S^λ. In particular, these admit quotients D^λ such that the set $\{D^\lambda \mid D^\lambda \neq 0, \lambda \vdash n\}$ is a complete set of pairwise-nonisomorphic irreducible $\mathscr{H}$-modules. This set is indexed by the partitions $\lambda \vdash n$ which are e-restricted.[**? ?**]

These representations D^λ have explicit constructions, but many of their properties are unknown. For instance, the dimension of these modules is unknown outside of some special cases.[**?**] However, there does exist an algorithm due to Lascoux–Leclerc–Thibon–Ariki which computes the decomposition matrices of the Iwahori–Hecke Algebra $\mathscr{H}_{\mathbb{C},q}(S_{2n+r})$ for q an eth root of unity.[**? ?**]

The cellular basis for S^λ and associated construction for D^λ tend to be complicated and often computationally intractable. We aim to give simple graphical realizations of S^λ and D^λ in some cases that $\lambda = (n+r, n)$. These realizations expose the structure of irreducibility and branching in an intuitive and computationally simple way.

Note that we follow the convention of Murphy and Kleshchev, which is dual to the conventions of Dipper, James, and Mathas; one may translate our results to the latter convention by transposing all partitions.[**? ? ?**]

Throughout this paper, we analyze the *two-row partitions* $(n+r, n) \vdash 2n+r$ and their corresponding modules $S^{(n+r,n)}$ and $D^{(n+r,n)}$.

Crossingless matchings. The following definition modifies the crossingless matchings defined by Khovanov.[**?**]

Definition 1.1. Define a *crossingless matching on $2n+r$ nodes and r anchors* to be an isotopy class of $n+r$ nonintersecting paths in the slice $\mathbb{R} \times [0,1]$ connecting $2n+r$ disctinct points of $\mathbb{R} \times \{0\}$ and r points of $\mathbb{R} \times \{1\}$ such that none of the latter points are conected. Let M_{2n+r}^r have basis given by these matchings. This is illustrated in Figure 1.

Let the length of an arc (i,j) be $j - i + 1$.

We endow M_{2n+r}^r with an action by specifying $(1+T_i)w_j$ for some basis element w_j of M_{2n+r}. We do so by concatenating in "vertical lines" below each point other than the ith and $i+1$st, concatenating paths between the ith and $i+1$st points as well as points below them, removing any "loops" this forms, and taking isotopy to some matching w_l if possible; if this is not possible, then there are anchors at $i, i+1$ and we set $(1+T_i)w_j := 0$; if this is possible and there is a "loop," set $(1+T_i)w_j := (1+q)w_j$ and otherwise set $(1+T_i)w_j := q^{1/2}w_l$. This action is illustrated in Figure 2, and we verify that this is well-defined in Appendix A.2.

Note that the representations M_{0+r}^r and $S^{(r)}$ are isomorphic to the sign representation. In fact, we will prove that $M_{2n+r}^r \simeq S^{(n+r,n)}$ whenever $e > n+r+1$. Additionally, we will prove that M_{2n+r}^r is

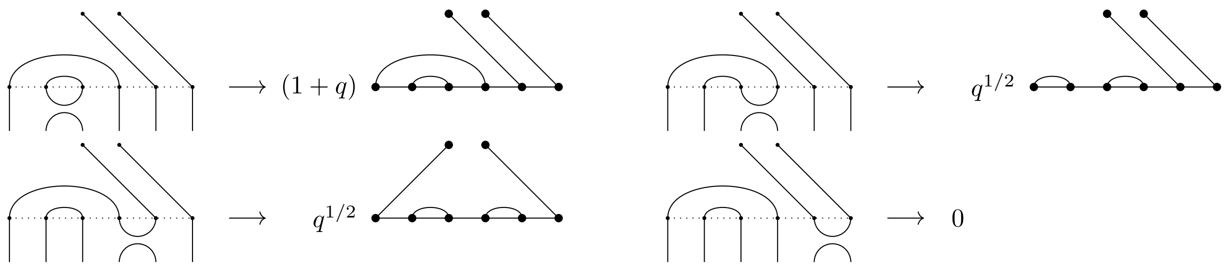

Figure 2. Illustration of the actions $(1+T_i)w_{|M_6^2|}$. In general, we act by deleting loops, isotoping onto a new crossingless matching, and scaling by either $q^{1/2}$, $(q+1)$, or 0.

irreducible everywhere that $e > n$ and $S^{(n+r,n)}$ is irreducible. We will conjecture that $M^r_{2n+r} \simeq S^{(n+r,n)}$ with e unrestricted as well.

This proves a graphical characterization of S^λ in many cases, and hence D^λ in the cases where $S^\lambda \simeq D^\lambda$. However, the representation theory of $\mathscr{H}$ is tied to the modules D^λ even in cases where S^λ is not irreducible; we will present a characterization of the module $D^{(n+r,n)}$ when $e = 5$ and $r \leq 3$.

Fibonacci representation. Now suppose that $e = 5$ and k contains the algebraic number $\sqrt{q+q^4}$ for reasons which will be apparent soon. For convenience set $m := 2n + 4$. The following is a modification of Shor–Jordan's Fibonacci representation of the braid group.[**?**]

Definition 1.2. Let V^m be the k-vector space with basis given by the strings $\{*, 0\}^{n+1}$ such that the bit $*$ never appears twice consecutively. We will refer to V^m as the *Fibonacci representation* and suppress the superscript whenever it is clear from context.

We wish to endow this with a $\mathscr{H}$-action which acts on a basis vector only dependent on bits $i, i+1, i+2$, sending each basis vector to a combination of the other basis vectors having the same bits $1, \ldots, i, i+2, \ldots, n+1$ as follows:

$$\begin{aligned} T_1\,(*00) &:= \alpha_1\,(*00) \\ T_1\,(00*) &:= \alpha_1\,(00*) \\ T_1\,(*0*) &:= \alpha_2\,(*0*) \\ T_1\,(0*0) &:= \varepsilon_1\,(0*0) + \delta\,(000) \\ T_1\,(000) &:= \delta\,(0*0) + \varepsilon_2\,(000) \end{aligned} \tag{1.1}$$

for constants

$$\begin{aligned} \tau &:= q + q^4 \\ \alpha_1 &:= -1 \\ \alpha_2 &:= q \\ \varepsilon_1 &:= \tau(q\tau - 1) \\ \delta &:= \tau^{3/2}(q+1) \\ \varepsilon_2 &:= \tau(q - \tau) \end{aligned} \tag{1.2}$$

with T_i acting similarly on the substring $i, i+1, i+2$. We will verify that this is a representation of $\mathscr{H}$ in Appendix A.3

This contains four subrepresentations spanned by strings with beginning and ending with specified bits. Label the subrepresentation spanned by strings $(* \ldots *)$ by V_{**}, V_{*0}, V_{0*}, and V_{00}; these bits are preserved by the action (1.1). We will show that these are isomorphic to particular D^λ.

Overview of paper.

In Section 2 we give corollaries to standard theorems concerning Specht modules. First, James-Mathas provides a sharp characterization of the irreducibility S^λ for $\lambda \vdash 2n + r$ which is e-regular, called the

Carter criterion.[?] We specialize this to the case that $\lambda = (n+r, n)$ to give a combinatorial condition for irreducibility of $S^{(n+r,n)}$. We note that this irreducibility depends only on e when $e > n$; otherwise it depends on both e and the characteristic of the field. Further, we use Kleshchev–Brundan's modular branching rules to prove our first significant statement: if $S^\lambda \simeq D^\lambda$ and $e > n$, then a particular length-2 composition series uniquely determines λ; further, an irreducible restriction to $D^{(n,n-1)}$ determines λ as well.[? ?]

In Section 3, we first prove that every basis vector in M is cyclic, then that M contains no sign subrepresentation. Then, we prove the following theorem.

Theorem 3.4. *Suppose $e > n$ and $S^{(n+r,n)}$ is irreducible. Then M^r_{2n+r} is irreducible.*

Following this, we prove the existence of a particular filtration with factors given by other crossingless matchings representations; using irreducibility, this becomes a composition series, and an inductive argument combined with the branching of Section 2 allow us to prove the following:

Theorem 3.7. *Suppose $e > n + r + 1$. Then, $M^r_{2n+r} \simeq S^{(n+r,n)}$.*

Last, we finish the section by proving more general statements concerning sign subrepresentations of M, which contribute to conjectures later in the paper.Fill in with specific information later.

In Section 4, we begin by establishing isomorphisms as in 4.5 for the subrepresentations of V^2, as well as irreducibility of V^3_{*0}. We then use these cases to prove that V^m_{*0} is irreducibile for all m, which implies that that V^m_{**} is irreducible. From this, we inductively prove the following theorem:

Theorem 4.5. *The following isomorphisms characterize the subrepresentations of the Fibonacci representation:*

$$\begin{aligned} V^{2n}_{**} &\simeq D^{(n,n)} \\ V^{2n-1}_{**} &\simeq D^{(n+1,n-2)} \\ V^{2n}_{*0} &\simeq D^{(n+1,n-1)} \\ V^{2n-1}_{*0} &\simeq D^{(n,n-1)}. \end{aligned}$$

Overview of Conjecture and Empirics goes here.

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2. Preliminaries on Specht modules

For this section and the rest of the paper, assume $n > 0$.

Throughout the rest of the paper, it will be useful to have precise notation; identify each partition with a tuple $\lambda = (\lambda_1^{a_1}, \dots, \lambda_l^{a_l})$ having $\lambda_i > \lambda_{i+1}$, $a_i > 0$, and $\sum_i a_i\lambda_i = 2n + r$. Identify each of these with a subset $[\lambda] \subset \mathbb{N}^2$ as in [?], and define $\lambda(i) = (\lambda_1^{a_1}, \dots, \lambda_{i-1}^{a_{i-1}}, \lambda_i^{a_i-1}, \lambda_i - 1, \lambda_{i+1}^{a_{i+1}}, \dots, \lambda_l^{a_l})$ to be the partition with the ith row removed. Say that λ is *e-regular* if $\lambda_i - \lambda_{i+1} < e$ for all i and $\lambda_l < e$.

In the following subsection, we cite a theorem of James–Mathas which precisely characterizes the irreducibility of S^λ in the case that λ is e-regular, and we specialize this result to the case of two-row Specht modules. This falls into two cases: either $e > n$, where $S^{(n+r,n)}$ is irreducible iff $e \nmid r + 2, \dots, n + r + 1$, or $e \le n$, where the irreducibility of $S^{(n+r,n)}$ is more complicated and depends also on the characteristic of k. We will focus primarily on the former case.

Following this, we reproduce the branching theorems of Kleshchev–Brundan, which allow us to fully characterize the socle of $\operatorname{Res} D^\lambda$. This and some combinatorial arguments yield the main result of this section, which allows us to determine certain $D^{(n+r,n)}$ via their composition series. This will be instrumental later for characterizing the crossingless matchings representation M^r_{2n+r} as a Specht module, and it will extend to all cases with $e > n + r + 1$.

$n+r+1$	$n+r$	$\dots$	$r+2$	r	$r-1$	$\dots$	1
n	$n-1$	$\dots$	1				

Figure 3. The young diagram corresponding to the partition $(n+r, n)$. The hook lengths are in the center of the corresponding cells.

2.1. **Irreducibility of S^λ.** Let ℓ be the characteristic of k; then, set

$$p := \begin{cases} \ell & \text{if } \ell > 0, \\ \infty & \text{if } \ell = 0. \end{cases}$$

Note that $p = e$ when $q = 1$. For h a natural number, let $\nu_p(h)$ be the p-adic valuation of h. By convention, set $\nu_\infty(h) = 0$ for all h. Define the function $\nu_{e,p} : \mathbb{N} \to \{-1\} \cup \mathbb{N}$ by

$$\nu_{e,p}(h) := \begin{cases} \nu_p(h) & e\text{if } \mid h \\ -1 & e\text{if } \nmid h \end{cases}.$$

Lastly, let h^λ_{ab} be the hook length of node (a, b) in $[\lambda]$ as defined in [**?**]. With this language, we may express the following theorem, parts (ii)-(iii) of which are known as the *Carter criterion* in the symmetric group case, due to James–Mathas.[**?**]

Theorem 2.1 (James–Mathas)**.** *The following are equivalent:*

(i) $S^\lambda \simeq D^\lambda$.
(ii) λ *is* e*-regular and* S^λ *is irreducible.*
(iii) $\nu_{e,p}\left(h^\lambda_{ab}\right) = \nu_{e,p}\left(h^\lambda_{ac}\right)$ *for all nodes* (a, b) *and* (a, c) *in* $[\lambda]$. □

Proof. See [**?**] theorem 5.42. □

This result gives information solely on e-regular partitions, and the general irreducibility of S^λ away from $p = 2$ is not well understood. We will henceforth specialize slightly to the case that $(n + r, n)$ is e-regular.

Corollary 2.2. *If* $r = 0$*, assume* $e > 2$*.*

(i) Suppose $e > n$*. Then,* $S^{(n+r,n)}$ *is irreducible iff* $e \nmid r + 2, r + 3, \dots, n + r + 1$.
(ii) Suppose $e \le n$*. If* $S^{(n+r,n)}$ *is irreducible, then* $e \mid r + 1$.

Note that the condition $e \nmid r + 2, r + 3, \dots, n + r + 1$ implies that $e > n$.

Proof. Our initial assumption on e implies that λ is e-regular, which we will use below.

(i) Note that $\nu_p(h) \neq -1$ for all naturals h and only hook lengths in the top row may vanish mod e by Figure 3; hence we may equivalently prove that e divides no hook lengths in the leftmost n columns of the second row by Theorem 2.1. These hook lengths are precisely $r + 2, \dots, n + r + 1$.

(ii) Note that we have $\nu_{e,p}\left(h^\lambda_{2,n-e+1}\right) \neq -1$, and $\nu_{e,p}\left(h^\lambda_{2,n+r}\right) = -1$ so $\nu_{e,p}$ acquires at least two values. Suppose that $e \nmid r + 1$. Then,

$$\nu_{e,p}\left(h^\lambda_{1,n-e+1}\right) = \nu_{e,p}\left(h^\lambda_{2,n-e+1} + r + 1\right) = -1,$$

giving $S^{(n+r,n)}$ reducible by Theorem 2.1. □

From part (i) we see that irreducibility at $e > n$ is not dependent on p, and we may cover many modular cases without reference to the characteristic of k. We will finish our discussion of irreducibility of S^λ via sharp characterization of the $e \le n$ case.

Corollary 2.3. *If* $r = 0$*, assume* $e > 2$*. Suppose* $e \le n$*, and suppose* $p > n + r + 1$*. Then,* $S^{(n+r,n)}$ *is irreducible if and only if* $e \mid r + 1$.

Proof. This follows from the proof of Corollary 2.2 part (ii) and the fact that $\nu_p(h) = \nu_p(h')$ for all naturalls h, h'. □

2.2. **Branching theorems of Specht modules.** In this section as well as later sections, we will consider the restriction of representations of $\mathscr{H}$ to particular subalgebras isomorphic to $\mathscr{H}_{k,q}(S_{2n+r-1})$. We verify in Appendix B that any two subalgebras of $\mathscr{H}$ generated by $2n+r-2$ simple transpositions are canonically isomorphic, and the corresponding restrictions are canonically isomorphic via this isomorphism of algebras. We will hence abuse notation, pick one such subalgebra $\mathscr{H}'$, and notate $\operatorname{Res}^{\mathscr{H}}_{\mathscr{H}'} W$ by $\operatorname{Res} W$ for any $\mathscr{H}$-module W.

Fixing some partition $\lambda \vdash 2n+r$, for $1 \le i \le j \le l$, let $\beta_\lambda(i,j)$ and γ_λ be the quantities

$$\beta_\lambda(i,j) = \lambda_i - \lambda_j + \sum_{t=i}^{j} a_t$$
$$\gamma_\lambda(i,j) = \lambda_i - \lambda_j + \sum_{t=i+1}^{j} a_t.$$

Note that $\beta_\lambda(i,j)$ is the hook length of cell $(a_1 + \cdots + a_{i-1} + 1, \lambda_j)$.

Results due to Kleshchev and Brundan refer to *normal* and *good* numbers; for these, we will use the facts that 1 is always normal and that j is normal when $\beta_\lambda(i,j) \not\equiv 0 \pmod e$ for all $i \le j$. Further, we will use that j is good if and only if j is normal and $\gamma_\lambda(j,j') \not\equiv 0 \pmod e$ for all $j' \ge j$ normal.[**?** **?**] When $\lambda(i) = \mu$ for i normal, write $\mu \xrightarrow{\text{normal}} \lambda$, and similar in the good case.

The following statements, collectively known as *modular branching rules* of D^λ, were originally written by Kleshchev for Specht modules of the group algebra $k[S_n]$, then generalized to the Hecke algebra case by Brundan.[**?** **?**] They entirely characterize the socle of $\operatorname{Res} D^\lambda$, as well as semisimplicity od $\operatorname{Res} D^\lambda$.

Theorem 2.4 (Kleshchev-Brundan)**.** *We have the following isomorphisms of vector spaces*

$$\operatorname{Hom}_{\mathscr{H}'}\left(S^\mu, \operatorname{Res} D^\lambda\right) \simeq \begin{cases} k & \text{if } \mu \xrightarrow{\text{normal}} \lambda, \\ 0 & \text{otherwise.} \end{cases}$$
$$\operatorname{Hom}_{\mathscr{H}'}\left(D^\mu, \operatorname{Res} D^\lambda\right) \simeq \begin{cases} k & \text{if } \mu \xrightarrow{\text{good}} \lambda \\ 0 & \text{otherwise} \end{cases}$$

and $\operatorname{Res} D^\lambda$ *is semisimple if and only if every normal number in* λ *is good.* □

Using this, we immediately see that, for any rectangular partition (m^ℓ), we have

$$\operatorname{Res} D^{(m^\ell)} \simeq D^{(m^{\ell-1}, m-1)}.$$

The non-rectangular two-row case is more complicated, but we may still describe it fully as follows.

Corollary 2.5. *Suppose $r > 0$. Then, we may characterize the socle of* $\operatorname{Res} D^\lambda$ *as follows:*

$$\operatorname{soc}\left(\operatorname{Res} D^{(n+r,n)}\right) \simeq \begin{cases} D^{(n+r-1,n)} & \text{if } e \mid r+2 \\ D^{(n+r,n-1)} & \text{eif } e \nmid r+2,\ e \mid r \\ D^{(n+r-1,n)} \oplus D^{(n+r,n-1)} & \text{if } e \nmid r+2, r \end{cases}$$

Further, when $e \nmid r$ or $e \mid r+2$, $\operatorname{Res} D^{(n+r,n)}$ *is semisimple.*

Proof. This amounts to computations of the hook lengths $\beta(1,2)$ and $\gamma(1,2)$:

$$\beta_\lambda(1,2) = r+2$$
$$\gamma_\lambda(1,2) = r$$

Since 2 is the largest removable number, $D^{(n+r,n-1)} \subset D^{(n+r,n)}$ if and only if $e \nmid r+2$. Further, if $e \nmid r+2$, then $D^{(n+r-1,n)} \subset D^{(n+r,n)}$ if and only if 1 is good; this is equivalent to $e \nmid r$. □

Now that we've characterized how D^λ restrict, we can describe how strongly these restrictions characterize irreducibles. Namely, we will prove that some D^λ having the same composition series as $D^{(n+r,n)}$ is sufficient to determine that $\lambda = (n+r,n)$ in the case that condition (i) of Corollary 2.2 holds and either $r \neq 0$ or $e \neq 4$.

Proposition 2.6. *Let λ be an e-regular partition of $2n+r$.*

(i) Suppose $r > 0$, suppose $e \nmid r+1, r+2, \ldots, n+r+1$, and suppose either $e \mid r$ or $e \nmid r-2$. If D^λ has the composition series

$$0 \subset D^{(n+r-1,n)} \subset \text{Res } D^\lambda \tag{2.1}$$

with factor Res $D^\lambda / D^{(n+r-1,n)} \simeq D^{(n+r,n)}$, *then $\lambda = (n+r,n)$.*

(ii) Suppose $r = 0$, suppose $e \nmid 4$, and suppose $D^{(n,n-1)} \simeq$ Res D^λ. *Then $\lambda = (n,n)$.*

Proof. Note that $e > n$. Further, note that the above characterizations are necessary regardless of e-regularity; in the case that μ below fails too be e-regular, this proposition will prove that λ does not satisfy 2.1, a contradiction.

(i) Let $\varpi := (n+r-1, n, 1)$, let $\varsigma := (n+r-1, n+1)$, and let $\mu := (n+r, n)$. Since $D^{(n+r-1,n)} \subset \text{Res } D^\lambda$, we have $(n+r-1, n) \longrightarrow \lambda$, implying $\lambda \in \{\varpi, \varsigma, \mu\}$. We will show that ϖ, ς do not have socle compatible with (2.1), allowing us to conclude $\lambda = \mu$.

If ϖ or ς are not e-regular, then $D^\varpi = 0$ or $D^\varsigma = 0$, and we may immediately rule these out; henceforth assume that these are each e-regular.

First suppose that that $\lambda = \varpi$. We will break into cases with r.

- Suppose that $r > 1$. Note that $e \nmid r+1 = \beta_\varpi(1,2)$, so 2 is normal. Further, $\gamma_\varpi(2,3) = n \not\equiv 0 \pmod e$, so 2 is good and $D^{(n+r-1,n-1,1)} \subset D^\varpi$, which is not a composition factor in (2.1). Hence, by the Jordan-Hölder theorem [?], we have $\lambda \neq \varpi$.
- Suppose that $r = 1$. Then, $\varpi = (n,n,1)$ has $\gamma_\varpi(1,2) = n \not\equiv 0 \pmod e$, giving $D^{(n,n-1,1)} \subset D^\varpi$ and hence $\lambda \neq \varpi$ as in the previous case.

Now suppose that $\lambda = \varsigma$ and break into cases with r:

- Suppose $r > 2$. Then, by Corollary 2.5, we require that $e \nmid r$ and $e \mid r-2$; these are not satisfied, so $\lambda \neq \varsigma$.
- Suppose $r = 2$. Then Res $D^\varsigma \simeq D^{(n+1,n)}$ is irreducible, as $\varsigma = (n+1, n+1)$ has rows of the same length; this contradicts (2.1).
- Suppose $r < 2$. Then ς is not a partition.

This completes the proof.

(ii) Since the socle of D^λ is irreducible, we require that 1 is the only normal number and $\lambda(1) = (n, n-1)$. This reduces to the cases of $\varsigma := (n+1, n-1)$ and $\mu := (n,n)$; if $\lambda = \varsigma$, then we have that $e \mid \beta_\varsigma(1,2) = 4$, a contradiction. Hence $\lambda = \mu$. □

3. Crossingless matchings and Specht modules

In this section, we will analyze the crossingless matchings representation $M := M^r_{2n+r}$ with the goal of proving $M^r_{2n+r} \simeq S^{(n+r,n)}$ under certain conditions in e. We begin by proving irreducibility of M^r_{2n+r} whenever $e \nmid r+2, r+3, \ldots, n+r+1$; when $e > n$, this is true if and only if $S^{(n+r,n)}$ is irreducible by Corollary 2.2. This is proven via an inductive process; if $e \nmid n+r+1$, then M contains no sign subrepresentation, and this allows us to "project" down to the case $M^r_{2(n-1)+r}$ and deduce irreducibility of M from irreducibility of this representation.

We will use the following base case to the correspondence throughout:

Lemma 3.1. *Note that $S^{(1^n)}$ is the sign representation and $S^{(n)}$ the trivial representation. We have the following isomorphisms*

(i) $M^0_2 \simeq S^{(2)}$,

(ii) $M^r_r \simeq S^{(1^r)}$.

Each of these are 1-dimensional, so they are irreducible.

Proof. (i) follows from the fact that $(1+T_1)w = (1+q)w$, and hence $T_1 w = qw$ for any nonzero vector $w \in M^0_2$. Similarly, (ii) follows from the fact that $(1+T_i)w = 0$, so $T_i w = -w$ for any i and any $w \in M^r_r$. □

3.1. **Irreducibility of M.** We refer to a vector $w \in M$ satisfying $\mathscr{H} w = M$ as *cyclic*. It is a classical result that a representation W is irreducible if and only if every nonzero element of W is cyclic.[**?**] We will prove irreducibility by showing that every nonzero $w \in M$ is cyclic; the following lemma is integral in showing this.

Lemma 3.2. *Every basis vector in M^r_{2n+r} is cyclic.*

Proof. We have already proven this in the $r = 0$ case, so suppose that $r > 0$.

Note that, between anchors at indices $a < a'$ having no arc at index b with $a < b < a'$, the $M^0_{a'-a}$ case allows us to generate the basis vector with all length-2 arcs between a, a' and identical arcs/anchors outside of this sub-matching. At the ends, we apply the M^0_a case or the M^0_{2n+r-a} case in the same way for the first a or last $2n + r - a$ indices.

Applying this between each arc gives us a vector with anchors and length-2 arcs, and we may use the appropriate $(1 + T_i)$ to move anchors to any positions. Then, we may use the reverse process from above to generate the correct matchings between arcs and generate any other basis vector. □

Let $K^r_{2n+r} := \bigcap_{i=1}^{2n+r-1} \ker(1 + T_i) = \ker \bigoplus_{i=1}^{2n+r-1} (1 + T_i)$. This will be an important technical tool in our proof of irreducibility.

We give two proofs of the following proposition. The first, below, intuits the necessary graphical intuition to construct a sub-matrix with nonzero determinant and perform row reduction. The second, given in section 3.3.1, formalizes the graphical intuition and gives results necessary for the following sections.

Proposition 3.3. *Suppose $e \nmid n + r + 1$. Then, $K := K^r_{2n+r} = 0$.*

Proof 1. Consider the matrix $A = \bigoplus(1 + T_i)$ having kernel K. It is sufficient by lemma 3.9 to show that A includes a row $[0, \dots, 0, 1, 0, \dots, 0]$ with a nonzero entry only on the row j.

Now, we may characterize the rows of A as follows; if the row corresponding to $(1 + T_i)$ and mapping onto the element $w_l \in W$ is nonzero, then it is of the form $\left[a_1, \dots, a_{|W|}\right]$ where $a_l = 1 + q$, $a_m = q^{1/2}$ whenever $(1 + T_i)w_m = q^{1/2} w_l$, and $a_m = 0$ otherwise.

Seeing this, the row corresponding to $(1 + T_{n+r})$ and w_j has nonzero entries $q^{1/2}$ at w_j and $(1 + q)$ at the vector w agreeing with w_j at all indices except having arcs at $(n+r-1, n+r)$ and $(n+r+1, n+r+2)$. Similar justification leads the row corresponding to $(1 + T_{n+r-1})$ at w to have nonzero entries $q^{1/2}$ at w and $(1 + q)$ at w_j and the vector with anchors $1, \dots, r$, arc $(n + r - 3, n + r - 2)$, and all other arcs maximum length.

We may iterate this process as illustrated in Figure 4, eventually ending at a row with two nonzero entries, either an arc $(1, 2)$ or an arc $(2, 3)$, and all anchors otherwise left-aligned and arcs of maximum length. These rows together form an $(n + r) \times \left|M^r_{2n+r}\right|$ submatrix of A which has a nonzero row in the row corresponding to j, and has (by removing zero rows) the same row space as the following square matrix:

$$B_{n+r} := \begin{bmatrix} q+1 & q^{1/2} & & & & & \\ q^{1/2} & q+1 & q^{1/2} & & & \text{\Large 0} & \\ & q^{1/2} & q+1 & q^{1/2} & & & \\ & & \ddots & \ddots & & & \\ & \text{\Large 0} & & & q^{1/2} & q+1 & q^{1/2} \\ & & & & & q^{1/2} & q+1 \end{bmatrix}. \tag{3.1}$$

We will show that this matrix is invertible; then, a sequence of elementary row operations will yield the identity, and in particular, when applied to A, will yield a row with a nonzero entry only on row j, giving $K = 0$.

We may prove invertibility of this matrix by proving that $\det B_{n+r} = [n + r + 1]_q$ inductively on $n + r$. This is satisfied for our base case $n + r = 1$, so suppose that it is true for each $m < n + r$. Then,

$$\begin{aligned} \det B_{n+r} &= (q + 1) \det B_{n+r-1} - q \det B_{n+r-2} \\ &= (q + 1)(1 + \cdots + q^{n+r-1}) - (q + \cdots + q^{n+r-1}) \\ &= 1 + \cdots + q^{n+r} \\ &= [n + r + 1]_q . \end{aligned}$$

Hence $\det B_{n+r} \neq 0$, and $K = 0$. □

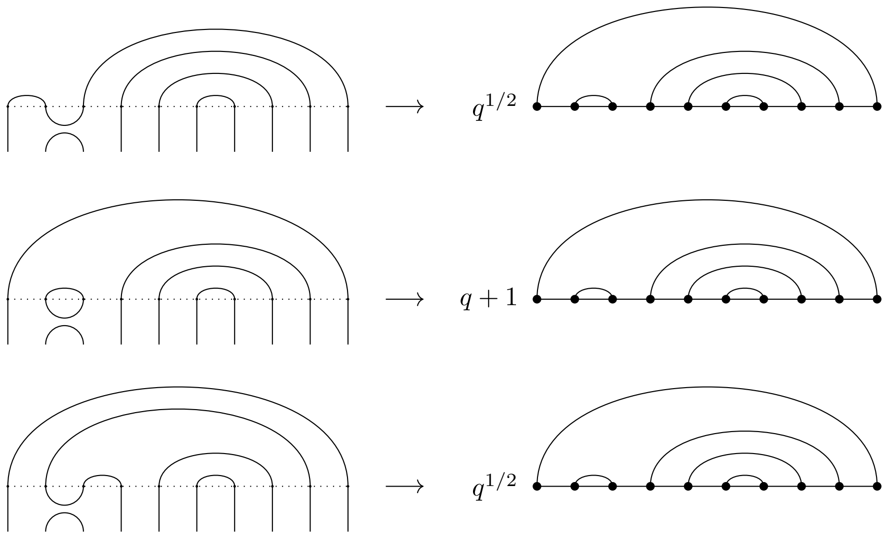


Figure 4. Illustrated is the row constructed for transposition $(1+T_2)$; clearly these are the only basis elements mapping to multiples of the desired element, and they relate to each other. replacing the outermost and/or innermost arc with an anchor typifies the rows constructed with three nonzero coefficients.

Theorem 3.4. *Suppose that $e \nmid r+2, r+3, \ldots, n+r+1$. Then the representation M^r_{2n+r} is irreducible.*

Proof. We proceed by induction on n. The base case the base case $n=0, r \neq 0$ and $n=1, r=0$ follow from Lemma 3.1.

Take an arbitrary vector $w \in M$. By Proposition 3.3 there exists some $(1+T_i) \in \mathscr{H}$ such that $(1+T_i)w \neq 0$. Note that

$$\operatorname{im}(1+T_i) = \operatorname{Span}\{w_j \mid w_j \text{ contains the arc } (i,i+1)\}.$$

Hence, as vector spaces, there is an isomorphism $\varphi : \operatorname{im}(1+T_i) \to M^r_{2(n-1)+r}$ "deleting" the arc $(i,i+1)$.

We will show that, for every element $(1+T'_j) \in \mathscr{H}(S_{2(n-1)+r})$, there is some element $h_j \in \mathscr{H}(S_{2n+r})$ such that the following commutes:

$$\begin{tikzcd}
\operatorname{im}(1+T_i) \arrow[r, "\varphi", "\sim"'] \arrow[d, "h_j"] & M^r_{2(n-1)+r} \arrow[d, "1+T'_j"] \\
\operatorname{im}(1+T_i) \arrow[r, "\varphi", "\sim"'] & M^r_{2(n-1)+r}
\end{tikzcd}$$

Indeed, when $i+1 \neq j$ this is given by $h_j = 1+T_j$, and we have $h_{i+1} = q^{-1}(1+T_i)(1+T_{i+1})(1+T_{i-1})$, as given by Figure 5.

Note that, by definition, $e \nmid r+2, \ldots, n+r$ as well, and hence $(n+r-1, n-1)$ satisfies the hypotheses of the proposition as well. Then, by the inductive hypothesis, there is some element $h' \in \mathscr{H}(S_{2(n-1)+r})$ sending $\varphi((1+T_i)w)$ to the image of a basis vector of M^r_{2n+r} via φ; then, the action $\mathscr{H}$ generates the endomorphism $\varphi^{-1}h'\varphi$ of M, which sends $(1+T_i)w$ to a basis vector in M^r_{2n+r}. This implies that w is cyclic, and hence M^r_{2n+r} is irreducible. $\square$

3.2. **Correspondence with Specht modules.** The following theorem due to Mathas (theorem 5.5 in [?]) generalizes the classical branching theorem of the symmetric group. It will not be necessary for our present proof of the correspondence, but analogy with M is suggestive.

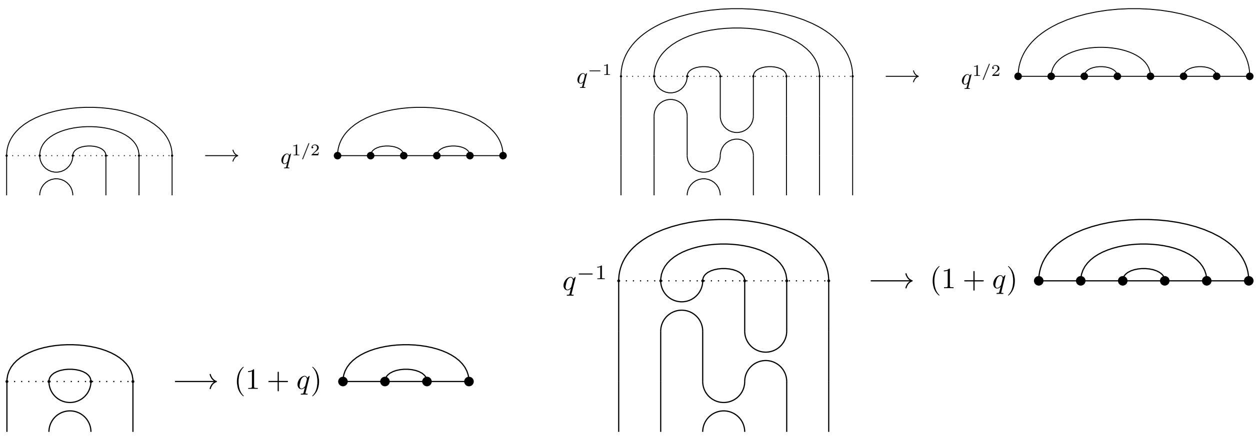


Figure 5. The correspondence between the action of $(1 + T_2)$ on $w_5' \in M_6^0$ and the action of $q^{-1}(1 + T_3)(1 + T_4)(1 + T_2)$ on the corresponding vector in M_8^0 having arc $(3, 4)$ first, then on $w_2' \in M_4^0$. This demonstrates that the action works with or without creating a loop.

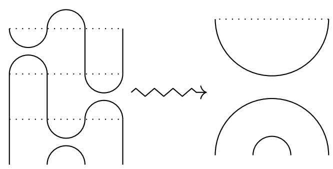


Figure 6. The isotopy demonstrating the correspondence between h_{i+1} and T_i'; the scaling issue with having three simple reflections is handled by the q^{-1} factor in h_{i+1}.

Theorem 3.5 (Characteristic-free classical branching theorem)**.** *Let λ be a partition of m with ℓ removable nodes. Then,* $\operatorname{Res} S^\lambda$ *has an* $\mathscr{H}_{k,q}(S_{m-1})$*-module filtration*

$$0 = S^{0,\lambda} \subset S^{1,\lambda} \subset \cdots \subset S^{\ell,\lambda} = \operatorname{Res} S^\lambda$$

such that $S^{t,\lambda}/S^{t-1,\lambda} \simeq S^{\lambda(t)}$ *for all* $1 \leq t \leq \ell$. □

In particular, this holds in cases where S^λ fails to be irreducible. If we replace S^λ with the appropriate M^r_{2n+r} above, we find the statement of the following proposition.

Proposition 3.6. *Suppose that $n > 0$.*

(i) Suppose that $r > 0$. Then, a filtration of $\operatorname{Res} M^r_{2n+r}$ *is given by*

$$0 \subset M^{r-1}_{2n+r-1} \subset \operatorname{Res} M^r_{2n+r}, \tag{3.2}$$

with $\operatorname{Res} M^r_{2n+r}/M^{r-1}_{2n+r-1} \simeq M^{r+1}_{2n+r-1}$.

(ii) We have the following isomorphism of representations:

$$M^1_{2n-1} \simeq \operatorname{Res} M^0_{2n}. \tag{3.3}$$

In the case that $e \nmid r + 1, \dots, n + r + 1$, 3.2 and 3.3 are composition series.

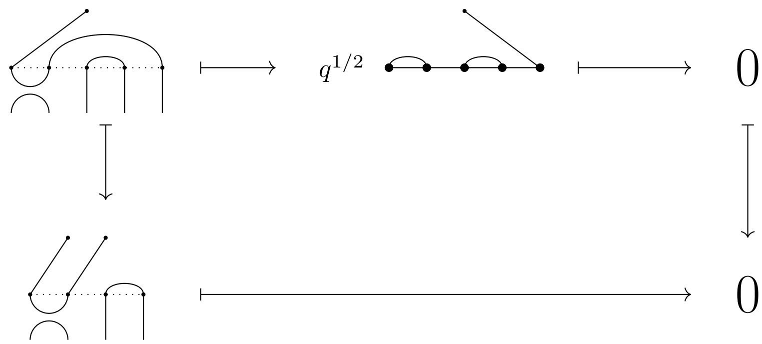
Proof. **(i)** Note that we may identify the subrepresentation of $\operatorname{Res} M^r_{2n+r}$ having anchor at index $2n+r$ with M^{r-1}_{2n+r-1}.

Let $U := \operatorname{Res} M^r_{2n+r}/M^{r-1}_{2n+r-1}$, and let $\pi : \operatorname{Res} M^r_{2n+r} \twoheadrightarrow U$ be the associated projection to U. Let $\phi : U \to M^{r+1}_{2n+r-1}$ be the k-linear map which regards the arc $(i, 2n + r)$ in U as an anchor at i in M^{r+1}_{2n+r-1}. It is not hard to verify that this is a well-defined isomorphism of vector spaces, so we must show that it is $\mathscr{H}$-linear.

Given a basis vector w_j containing arc $(i, 2n+r)$, ϕ is clearly compatible with $(1+T_{i'})$ with $i' \neq i, i-1$. Further, it is easy to verify that ϕ is compatible with T_i by definition of the relevant modules, as well as with $(1+T_{i-1})$ when w_j does not have anchor $i-1$. The case that w_j has anchor $i-1$ is illustrated in the following commutative diagram:

$$\begin{tikzcd}
M^r_{2n+r} \arrow[r, "1+T_i"] \arrow[d, "\phi\circ\pi"] & M^r_{2n+t} \arrow[r, "\pi"] & U \arrow[d, "\phi"] \\
M^{r+1}_{2(n-1)+r+1} \arrow[rr, "1+T_i"] & & M^{r+1}_{2(n-1)+r+1}
\end{tikzcd}$$

which we may diagram chase as follows



In each of these cases, linearity of $\mathscr{H}$-action and ϕ give

$$\begin{aligned}
\phi(T_j w_i) &= \phi(-w_j + (1+T_j)w) \\
&= -\phi(w_j) + (1+T_j)\phi(w_j) \\
&= T_j\phi(w_j).
\end{aligned}$$

Hence ϕ is an isomorphism of representations, and the statement is proven.

(ii) This follows from an analogous proof: now, $\phi : \operatorname{Res} M^0_{2n} \to M^1_{2(n-1)+1}$ is an isomorphism of representaitons, which is proven to be $\mathscr{H}$-linear by the same proof. □

We've now assembled the basic pieces necessary to prove our correspondence in the case $e > n+r+1$.

Theorem 3.7. *Suppose $e > n+r+1$. Then, $M^r_{2n+r} \simeq S^{(n+r,n)}$.*

Proof. The case $n = 0$ is already proven, by lemma 3.1, so suppose $n > 0$. In order to use Proposition 2.6, suppose for now that either $e \nmid 4$ or $r > 0$.

We will prove this inductively; the base case $2n + r = 2$ is implied by 3.1, so suppose that $M^s_{2n+s} \simeq S^{(m+s,s)}$ whenever $2m + s < 2n + r$ and $m + s \le n + r$, so that $e > m + s + 1$.

Suppose $r > 0$. By Theorem 3.4, we know that $M^r_{2n+r} \simeq D^\lambda$ for some e-restricted partition λ. By the inductive hypothesis and Corollary 2.2, we have a composition series given by the short exact sequence

$$0 \longrightarrow D^{(n+r-1,n)} \longrightarrow \operatorname{Res} D^\lambda \longrightarrow D^{(n+r,n-1)} \longrightarrow 0 \tag{3.4}$$

Hence the theorem is given by Proposition 2.6 (i).

Now suppose $r = 0$ and $e \neq 4$. Similarly, by Theorem 3.4, we know that $M^r_{2n+r} \simeq D^\lambda$ for some e-restricted partition λ, and by the inductive hypothesis and Corollary 2.2, we have the irreducible restriction $\operatorname{Res} D^\lambda \simeq D^{(n,n-1)}$. Then, the theorem is given by Proposition 2.6 (ii).

Now, suppose $e = 4$ and $r = 0$; then $4 > n+1$, so $n \le 2$. We've already proven the $n = 1$ case via the trivial representation, so suppose $n = 2$. Then, from the proof of Proposition 2.6, we know that $M^0_4 \simeq D^\lambda$, where $\lambda \in \{(n,n), (n+1,n-1)\}$. We have already proven that $M^2_4 \simeq D^{(n+1,n-1)}$, and we may verify that $\dim M^0_4 = 2 \neq 3 = \dim M^2_4$, so we have that $\lambda = (n,n)$ and the theorem is proven for $e = 4$. □

3.3. **Kernels and further work.** Recall $K^r_{2n+r} := \bigcap_{i=1}^{2n+r-1} \ker(1+T_i) = \ker \bigoplus_{i=1}^{2n+r-1}(1+T_i)$ defined before proposition 3.3. When the limits are clear, we will simply denote this kernel by K. In this section, we prove K is trivial for $e \nmid n+r+1$ and give a one-dimensional basis for $e = n+r+1$.

For compactness, in this section we use $\sim$ to denote "proportional to." For convenience, define M^0_0 and M^1_1 to be the zero representation. Note that in this section we do not assume $n > 0$.

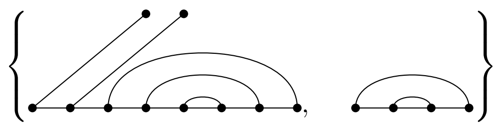


Figure 7. The rainbow element $R \in M_8^2$ is pictured on the left, $\sigma \in M_4^0$ is pictured on the right. The nodes $R(4) = 7$, $R(5) = 6$, so $R(4,7)$ defines a sub-matching. Specifically, $R(4,7) = \sigma$. Alternatively, $R(3) = 8$, so, for instance $R(2,7)$ is not a sub-matching, and refers to the ordered set $(2, 8, 7, 6, 5, 4)$. $R(2,1)$ is an element of the zero representation since $2 > 1$.

3.3.1. *Proposition 3.3.*

Definition 3.8. We will use the following notation extensively while exploring the structure of the kernel. Examples are given in figure 7. Fix some basis element $\psi \in M^r_{2n+r}$.

- For $1 \leq a, b \leq 2n+r$ define $\psi(a) := b$ if a and b are matched in ψ, and $\psi(a) := a$ if a is an anchor in ψ.
- Let r' be the number of anchors in ψ in the range $a, \ldots, b$. Suppose $1 \leq a \leq b \leq 2n+r$ and $\psi(i) \in \{a, \ldots, b\}$ for all $i \in \{a, \ldots, b\}$. Define a *sub-matching* $\psi(a,b)$ of ψ to be the basis element $\sigma \in M^{r'}_{b-a+1}$ specified by $\sigma(i) = \psi(i + a - 1) - (a - 1)$.
- For a, b satisfying $1 \leq a \leq b \leq 2n+r$, if $\psi(a,b)$ is not a sub-matching, define it to be the ordered set of nodes $(\psi(a), \ldots, \psi(b))$. For any other a, b, we define $\psi(a,b)$ to be an element of the zero representation M_0^0.
- Define the *rainbow element* $R \in M^r_{2n+r}$ to be the basis element specified by $R(i) = 2n + 2r - i + 1$ for $i > r$, $R(i) = i$ for $i \leq r$. In other words, the basis element with all anchors to the left followed by a "rainbow" to the right.

The following proposition lets us begin to characterize the coordinate vector of any element in K, and will serve as the starting point for all following characterizations.

Proposition 3.9. *Let $w \in M^r_{2n+r}$. If $w \in K$ is nonzero, then the coordinate of the rainbow element R in w is nonzero.*

Proof. Let W be the set of basis elements with nonzero coordinate in w. Let z be the maximal number of anchors to the far left in any element $\psi \in W$. Formally, z is the greatest positive integer such that there exists some element $\psi \in W$ satisfying $\psi(1) - 1 = \ldots = \psi(z) - z = 0$. For some positive integer j define $W^j \subset W$ to be the set of all basis elements in W with j anchors to the far left.

Suppose $z < r$. Then for each $\psi \in W^z$, exactly z anchors are positioned to the far left, so we may define i_ψ to be the position of the next leftmost anchor in ψ. Fix some $\widehat{\psi} \in W^z$ such that $i_{\widehat{\psi}} \leq i_\psi$ for all ψ.

Define the basis element $\psi' := q^{-1/2}(1 + T_{i_{\widehat{\psi}}-1})\widehat{\psi}$. First, note that $i_\psi > z+1$, so $i_\psi - 1$ is not an anchor and $\psi' \neq 0$. This follows from the definition of z and W^z, which require that the node $z+1$ is not an anchor. We will prove that ψ' has nonzero coordinate in $(1 + T_{i_{\widehat{\psi}}-1})w$, implying $w \notin K$.

It suffices to show that $\widehat{\psi}$ is the only basis element in W brought to ψ' by the action of $(1 + T_{i_{\widehat{\psi}}-1})$. Formally, we will show that if $\sigma \in W$ and $(1 + T_{i_{\widehat{\psi}}-1})\sigma \sim \psi'$, then $\sigma = \widehat{\psi}$.

Immediately, we may intuit a few properties of ψ':

(i) ψ' still has z anchors on the left.
(ii) Defining $i_{\psi'}$ to be the next leftmost anchor, $i_{\psi'} < i_{\widehat{\psi}}$

Together, (i) and (ii) imply $\psi' \notin W$. So, if $(1 + T_{i_{\widehat{\psi}}-1})\sigma \sim \psi'$, $q^{-1/2}(1 + T_{i_{\widehat{\psi}}-1})\sigma = \psi'$.

Define $\sigma' := q^{-1/2}(1 + T_{i_{\widehat{\psi}}-1})\sigma$.If $\sigma \notin W^z$, σ' will have z anchors at the far left only if $\sigma' \in W^{z-1}$ and the zth leftmost anchor is at position $i_{\widehat{\psi}}$. But then the position of the $z+1$st anchor is unchanged by the action, and must be at position greater than $i_{\widehat{\psi}}$, so from (ii) $\sigma' \neq \psi'$. If $\sigma \in W^z$, σ' will have anchor at $i_{\psi'}$ if and only if $i_\sigma = i_{\widehat{\psi}}$ and $\sigma(i_{\widehat{\psi}} - 1) = \widehat{\psi}(i_{\widehat{\psi}} - 1)$. Since these are the only three indices altered by action of $(1 + T_{i_{\widehat{\psi}}-1})$ on σ, if $(1 + T_{i_{\widehat{\psi}}-1})\sigma \sim \psi'$ this implies $\sigma = \widehat{\psi}$ as desired. So if $z < r$ w is not in the desired kernel.

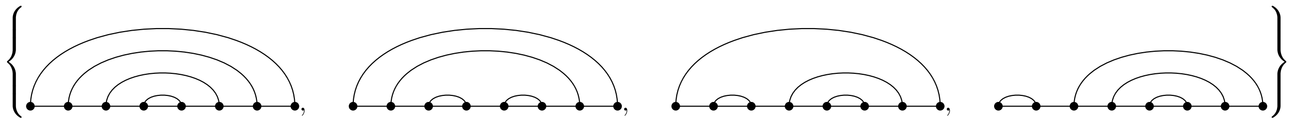


Figure 8. $R_{L,0}, \ldots, R_{L,3}$ pictured from left to right

We have proven that there exists an element in W with all anchors to the far left, or equivalently that W^r is nonempty. Now we must prove that $R \in W^r$. The proof is analogous to the previous case. First we define subsets $W_j^r \subset W^r$ that have the top j arcs of the rainbow, and define y to be the most arcs any element has. In the previous proof, we showed that if we move the next left-most anchor further left, the image of w under that transposition will be nonzero. In this proof, we will show that if we expand the next-largest arc, the image of w under that transposition will be nonzero.

Suppose $z = r$ but $R \notin W^r$ (so $R \notin W$). Let us define a sequence of subsets $W_i^r \subset W^r$ to be the set of elements in W^r with the top i humps of the rainbow. Formally, $W_0^r := W^r$, $W_{i+1}^r := \{\upsilon \in W_i^r | \upsilon(r+i+1) = 2n+r-i\}$. Since $R \notin W^r$, there exists some positive integer y less than $n-1$ such that $W_{y+1}^r = \varnothing$; y is the largest number of contiguous top level rainbow humps that any element in W^r has.

Choose $\widehat{\upsilon} \in W_t^r$ such that $\widehat{\upsilon}(r+t+1) \geq \upsilon(r+t+1)$ for all $\upsilon \in W_t^r$. Define the basis element $\upsilon' := q^{-1/2}(1 + T_{\widehat{\upsilon}(r+t+1)})\widehat{\upsilon}$.

First, note that $\widehat{\upsilon}(r+t+1) < 2n+r-t$, so:

[i] $\upsilon'(r+t+1) > \widehat{\upsilon}(r+t+1)$

[ii] For $0 \leq j \leq t$, $\upsilon'(r+j) = \widehat{\upsilon}(r+j) = 2n+r-j+1$.

These properties are analogous to properties (i) and (ii) used earlier in this proof. As before, these imply that $\upsilon' \notin W$, so, for $\sigma \in W$, if $(1 + T_{\widehat{\upsilon}(r+t+1)})\sigma \sim \upsilon'$, $q^{-1/2}(1 + T_{\widehat{\upsilon}(r+t+1)})\sigma = \upsilon'$.

We will prove that υ' has nonzero coordinate in $(1+T_{\widehat{\upsilon}(r+t+1)})w$, implying $w \notin K$. Again, it is sufficient to show that, for $\sigma \in W$, $q^{-1/2}(1 + T_{\widehat{\upsilon}(r+t+1)})\sigma = \upsilon'$ implies $\sigma = \widehat{\upsilon}$.

Define $\sigma' := q^{-1/2}(1 + T_{\widehat{\upsilon}(r+t+1)})\sigma$. Suppose $\sigma \notin W_t^r$. To satisfy [ii], we must have $\sigma \in W_{t-1}^r$ and $\sigma(r+t) = \widehat{\upsilon}(r+t+1)$. But then $\sigma(r+t+1) < \widehat{\upsilon}(r+t+1)$ is unchanged by action of $(1 + T_{\widehat{\upsilon}(r+t+1)})$, and σ' does not satisfy [i], so $\sigma' \neq \upsilon'$. If $\sigma \in W_t^r$, to satisfy $\sigma'(r+t+1) = \upsilon'(r+t+1)$, we must have $\sigma(r+t+1) = \widehat{\upsilon}(r+t+1)$ and $\sigma(\widehat{\upsilon}(r+t+1)+1) = \widehat{\upsilon}(\widehat{\upsilon}(r+t+1)+1)$. Since these are the only two matchings altered by the transposition, if $q^{-1/2}(1 + T_{\widehat{\upsilon}(r+t+1)})\sigma = \upsilon'$ we must have that $\sigma = \widehat{\upsilon}$ as desired. Thus, if $w \in K$ is nonzero, R has nonzero coordinate in w. $\square$

Given a rainbow element R, define the basis elements $R_{R,i}, R_{L,i}$ to be those where you move the middle hump across i lines to the right or left, respectively. Examples are pictured in 8. Formally, $R_{R,i} := q^{-i/2}(1 + T_{r+n+i})\ldots(1 + T_{r+n+1})R$, $R_{L,i} := q^{-i/2}(1 + T_{r+n-i})\ldots(1 + T_{r+n-1})R$.

Define $Q_n := (q^n + \ldots + 1)/q^{n/2}(-1)^n$ for $n \in \{0, 1, \ldots\}$. The following proposition says that, for any element in the kernel, if some basis element y has coordinate c in that element, and if y has a rainbow sub-matching, the basis elements where you replace that sub-matching by the shifted rainbow matchings $R_{L,i}$ or $R_{R,i}$ both have coordinate $Q_i c$ in the kernel element.

Proposition 3.10. *Let w be an element in the kernel intersection $\cap(1+T_z)$ in some generalized crossingless matchings representation. Let y be a basis element with coordinate c in w. Suppose $\exists a, b$ such that $y(a, b) = R$, the rainbow element. Define the basis elements θ_i, ϕ_i by $\theta_i(1, a-1) = \phi(1, a-1) = y(1, a-1)$, $\theta_i(b+1, 2n) = \phi(b+1, 2n) = y(b+1, 2n)$, $\theta_i(a, b) = R_{R,i}$, $\phi_i(a, b) = R_{L,i}$ (leave θ_i or ϕ_i undefined for any i where $R_{R,i}, R_{L,i}$ are undefined, respectively). The coordinates of ϕ_i and θ_i in w are both $Q_i c$.*

Proof of this proposition requires a simple algebraic fact that will be used throughout this document, so I state it as a lemma.

Lemma 3.11. $Q_1 Q_n - Q_{n-1} = Q_{n+1}$

Proof of lemma.

$$\begin{aligned} Q_1Q_n - Q_{n-1} =& \frac{-(q+1)}{q^{1/2}}\frac{(-1)^n(q^n+...+1)}{q^{n/2}} - \frac{(-1)^{n-1}(q^{n-1}+...+1)}{q^{(n-1)/2}} \\ =& \frac{(-1)^{n+1}(q^{n+1}+2q^n+...+2q+1)}{q^{(n+1)/2}} - \frac{(-1)^{n+1}(q^n+...+q)}{q^{(n+1)/2}} \\ =& \frac{(-1)^{n+1}(q^{n+1}+...+1)}{q^{(n+1)/2}} = Q_{n+1} \end{aligned}$$

Now let us prove the proposition.

Proof. Consider acting on w by an element $(1+T_z)$. The coordinate of ϕ_i in $(1+T_z)w$ will be a linear combination of the coordinates of basis elements sent to ϕ_i by the element $(1+T_z)$. Specifically, it will be $(1+q)c\iota + (q^1/2)\sum c_\psi$ where $\iota = 1$ if $y(z) = z+1$, $\iota = 0$ otherwise, and c_ψ are the coordinates of all basis elements ψ where $(1+T_z)\psi \sim y$.

Let $n := a+b-1$ and r be the number of anchors in $y(a,b)$. Consider the coordinate of ϕ_i in $(1+T_{a-1+r+n/2-i})w$. This is the transposition that acts on the "moved middle hump" in $\phi_i(a,b) = R_{L,i}$, as shown in 9. I claim the following:

claim: The only basis elements ψ where $(1+T_{a-1+r+n/2-i})\psi \sim \phi_i$ are ϕ_i and ϕ_{i-1}, ϕ_{i+1} when they exist[1].

Note that the action of any $(1+T_z)$ on a basis element ψ creates exactly two lines: an arc of length two connecting z and $z+1$, and either an anchor or an arc of length ≥ 2 connecting $\psi(z)$ and $\psi(z+1)$.[2] The easiest way to see the claim is to see that the given transposition is surrounded by arcs on both sides, so any basis element sent to the same element can vary from ϕ_i by at most one of those arcs and nothing else.

Let us prove the claim formally: It is easy to see that the action of $(1+T_{a-1+r+n/2-i})$ will bring $\phi_{i-1}, \phi_i, \phi_{i+1}$ to $\sim \phi$, as shown in 9. Suppose there was another basis element ψ sent to ϕ_i by the given transposition. Me note that if ψ contains the arcs or anchors directly to the right and left of the arc $(a-1+r+n/2-i, a-1+r+n/2-i+1)$ in ϕ_i (formally, it contains the arc $(a-1+r+n/2-i-1, a-1+r+n/2+i+2)$ or an anchor at $a-1+r+n/2-i-1$ and the arc $(a-1+r+n/2-i+2, a-1+r+n/2+i+1)$ or an anchor at $a-1+r+n/2-i+2)$, it must contain the arc $(a-1+r+n/2-i, a-1+r+n/2-i+1)$ to be a crossingless matching. Thus, if ψ contains both of these arcs/anchors, $(1+T_{a-1+r+n/2-i})$ acts as the constant $(1+q)$, so $(1+T_{a-1+r+n/2-i})\psi \sim \phi_i => \psi \sim \phi$. If ψ does not contain the left arc/anchor and $(1+T_{a-1+r+n/2-i})\psi \sim \phi_i$, the action of $(1+T_{a-1+r+n/2-i})$ must create that arc/anchor, so $\psi(a-1+r+n/2-i-1) = a-1+r+n/2-i$ and $\psi(a-1+r+n/2-i+1) = a-1+r+n/2+i+2$ in the case of an arc or $a-1+r+n/2-i+1$ is an anchor. All other matchings remain unchanged, so this implies $\psi = \phi_{i+1}$. Likewise, if the right arc $((a+b-1)/2-i+2, (a+b-1)/2+i+1)$ does not exist, $\psi = \phi_{i-1}$. For boundary cases, note that for $\phi_0 = \theta_0$, the only other basis element sent to this by the middle transposition is $\phi_1 = \theta_1$. Also note that at the edge case ϕ_{n+r-1} there is not necessarily a left arc, so other elements may be sent to ϕ_{n+r-1} by the given transposition, and this case gives no new information. Lastly, note that our argument was completely symmetric and thus applies to the θ_i case, except that for θ_i we do not have to deal with anchors. Thus the claim is proved.

Given this claim and lemma 3.11, the proposition follows quickly through induction:

Acting by $(1+T_{a-1+r+n/2})$ on w, the new coordinate of $\phi_0 = y$ is $(q+1)c + q^{1/2}c_{\phi_1}$ where c_{ϕ_1} is the coordinate of ϕ_1 in w. Since w is in the kernel, we have $(q+1)c + q^{1/2}c_{\phi_1} = 0 => c_{\phi_1} = Q_1 c$. $\phi_1 = \theta_1$ so this gives us all our base cases.

Acting by $(1+T_{a-1+r+n/2-i})$ on w, the new coordinate of ϕ_i is $q^{1/2}c_{\phi_{i+1}} + q^{1/2}c_{\phi_{i-1}} + (q+1)c_{\phi_i} = 0$. By the inductive hypothesis, $q^{1/2}c_{\phi_{i+1}} + q^{1/2}Q_{i-1}c + (q+1)Q_i = 0$ so $c_{\phi_{i+1}} = Q_1Q_i - Q_{i-1} = Q_{i+1}$ by lemma 3.11. θ_i is an identical proof, so the proposition follows.

□

We are now ready to prove proposition 3.3.

[1]we defined $R_{L,i}$ as far out as we can move the hump, so for $0 \leq i < n+r$, and take the analogous domain for ϕ_i

[2]Although this fact follows directly from our definition of the representation, it will be used throughout the document, so it is important that the reader understands it.

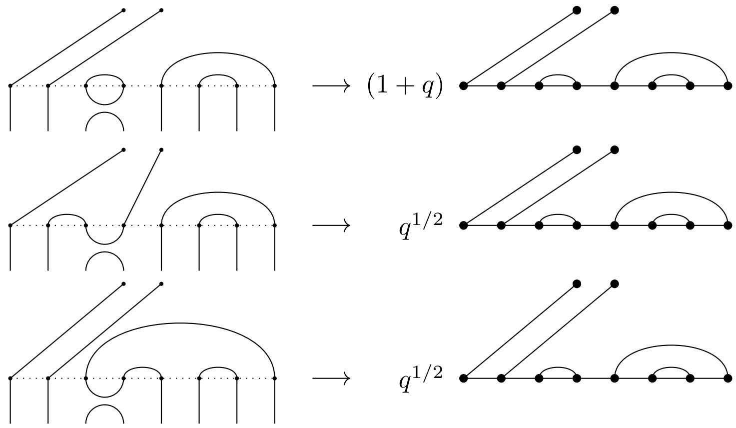


Figure 9. The action of $(1+T_{a-1+r+n/2-i})$ on $\phi_i, \phi_{i=1}, \phi_{i+1}$ (ordered from top to bottom), shown as the case where y is the rainbow vector in M_8^2 and $i = 2$.

Proof. Suppose $\cap \ker(1+T_i) = K \neq 0$. Take nonzero $w \in K$. By Proposition 3.9, the coordinate of the rainbow vector R is nonzero; suppose the coordinate is c. By proposition 3.10, the coordinates of the basis elements $R_{L,n+r-1}$ and $R_{L,n+r-2}$ are $Q_{n+r-1}c$ and $Q_{n+r-2}c$ respectively.

Consider the coordinate of $R_{L,n+r-1}$ in $(1+T_1)w$. Using the same logic as in the proof of proposition 3.10, we note that if a basis element ψ has no anchor at position 3 and is not equal to $R_{L,n+r-2}$, $(1+T_1)\psi \not\sim R_{L,n+r-1}$. Thus the desired coordinate is equal to $(1+q)Q_{n+r-1}c + q^{1/2}Q_{n+r-2}c = -q^{1/2}Q_{n+r}c$ by lemma 3.11. Since $w \in K$, we must have $-q^{1/2}Q_{n+r}c = 0$. Me have that c is nonzero, and we assume q nonzero, and Q_{n+r} is zero iff q is a root of $q^{n+r} + ... + 1$, implying $e|n+r+1$. Thus we have arrived at contradiction, and $K = 0$. $\square$

3.3.2. *Kernel Basis.* In this section, we determine an explicit basis for K when $e = n+r+1$, assuming $K \neq 0$. In the next section, we prove $K \neq 0$.

First let us formalize a useful property of sub-matchings.

Definition 3.12. Given a basis element $\psi \in M^r_{2n+r}$, specify some sub-matching $\psi(a,b)$. Let $\mathrm{Res}^{\mathscr{H}_{2n+r}(q)}_{\mathscr{H}_{b-a+1}(q)} M^r_{2n+r}$ be the restriction to the sub-algebra generated by transpositions $T_a, ..., T_{b-1}$. Define $Y_\psi \subset \mathrm{Res}^{\mathscr{H}_{2n+r}(q)}_{\mathscr{H}_{b-a+1}(q)} M^r_{2n+r}$ to be the subrepresentation generated by the set of basis elements $\{\sigma | \sigma(1, a-1) = \psi(1, a-1),\ \sigma(b+1, 2n+r) = \psi(b+1, 2n+r)\}$.

Lemma 3.13. *Take a basis element $\psi \in M^r_{2n+r}$. Suppose ψ has some sub-matching $\psi(a,b)$ with r' anchors. Define Y_ψ with respect to this sub-matching.*

The map $\rho : Y_\psi \to M^{r'}_{b-a+1}$ defined by

$$\rho(\sigma) = \sigma(a,b)$$

is an isomorphism of representations.

Proof. The map is clearly bijective. Thus it is sufficient to prove the following:

$$\rho(T_{i+a-1}\sigma) = T_i\rho(\sigma)$$

As mentioned in the previous section, the action of a transposition T_i can change at most 4 nodes, so we need to show that the transpositions end up changing the same nodes in the same way in $\rho(T_{i+a-1}\sigma)$ and $T_i\rho(\sigma)$.

Suppose $\sigma(i+a-1) = s, \sigma(i+a) = t$. Then $(T_{i+a-1}\sigma)(i+a-1) = i+a$, $(T_{i+a-1}\sigma)(s) = t$, so $\rho(T_{i+a-1}\sigma)(i) = i+1$, $\rho(T_{i+a-1}\sigma)(s-a+1) = t-a+1$. Separately, $\rho(\sigma)(i) = s-a+1$ and $\rho(\sigma)(i+1) = t-a+1$, so $T_i\rho(\sigma)(i) = i+1$ and $T_i\rho(\sigma)(s-a+1) = t-a+1$ as desired. So the map is an isomorphism and the lemma is proved. $\square$

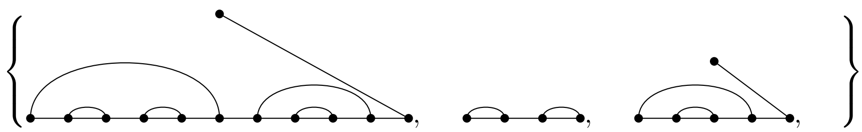


Figure 10. Suppose the second and third elements have coordinates $x_2(q_1)$ and $x_3(q_2)$ in their respective kernel elements, where for q_1, $e = 3$ and for q_2 $e = 4$. The coordinate of the first element is $x(q) = x_2(q)x_3(q)\frac{Q_5Q_4Q_3}{Q_1Q_2}$, where for q, $e = 7$

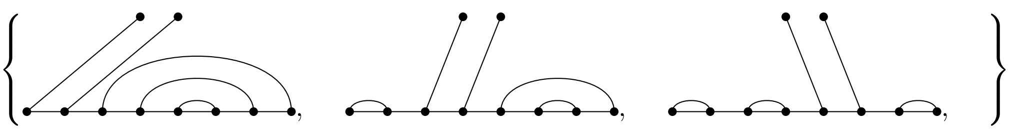


Figure 11. In order, the rainbow element, R_1, and R_2. The coordinate of the rainbow element is 1. The coordinate of R_1 is Q_4. The coordinate of R_2 is Q_4Q_3. Generally, R_i is the element with i humps then a rainbow element, and has coordinate $Q_{n+r-1}...Q_{n+r-i}$.

The lemma above motivates a recursive characterization of the kernel. To do this, it will be convenient to define some notation.

Definition 3.14. Recall $Q_i := (q^i + ... + q + 1)/q^{i/2}(-1)^i$ (lemma 3.11). For $a > 0$ define $Q(0, b) := 1$. For $b > a > 0$ define

$$\mathfrak{Q}_b^a := \frac{Q_{b-1}...Q_{b-a}}{Q_1...Q_{a-1}}$$

Definition 3.15. For $\psi \in M_0^0$, define the function $x_\psi(q) := 1$.

For all other basis elements $\psi \in M_{2n+r}^r$, we define x_ψ recursively:

$$x_\psi(q) := x_{\psi(2,a-1)}(q)x_{\psi(a+1,2n+r)}(q)\mathfrak{Q}_{n+r}^{\lfloor a/2 \rfloor}$$

I will refer to x_ψ as the **coordinate function** of ψ.

The following proposition states the forward direction of our characterization.

Proposition 3.16. *Let M_{2n+r}^r be a crossingless matchings representation, and suppose $Q_1, ...Q_{n+r-1}, \neq 0$. Let $w \in \cap \ker(1 + T_i)$. MLOG the rainbow element R has coordinate 1 in w (by proposition 3.9). Then the coordinate of any basis element $\psi \in M_{2n+r}^r$ in w is $x_\psi(q)$.*

An illustration of this proposition is shown in figure 10.

Proof. Suppose $\psi(1) = a$. The proof is structured as follows: use proposition 3.10 to find the coefficient of the basis element with $\lfloor a/2 \rfloor$ humps then a rainbow element; use the same proposition in a reversed manner to find the coefficient of the basis element consisting of the rainbow for the first a nodes, then the rainbow for the final $2n + r - a$ nodes; finally, we finish the proof through induction using lemma 3.13.

By proposition 3.10 the element $R_1 := R_{L,n+r-1}$ has coordinate Q_{n+r-1} in w. Then $R_1(3, 2n + r)$ is the rainbow element in $M_{2(n-1)+r}^r$, so the element R_2 defined by $R_2(1, 2) := R_1(1, 2)$, $R_2(3, 2n + r) := R_{L,n+r-2} \in M_{2(n-1)+r}^r$ has coordinate $Q_{n-1}Q_{n-2}$. Generally, define R_i by $R_i(1, 2(i-1)) := R_{i-1}(1, 2(i-1))$, $R_i(2i-1, 2n+r) := R_{L,n+r-i} \in M_{2(n-i+1)+r}^r$. Then the coefficient of R_i is $Q_{n+r-1}...Q_{n+r-i}$. These elements are shown in figure 5.

Now define basis elements E_i by $E_i(2i+1, 2n+r) := R_i(2i+1, 2n+r)$, $E_i(1, 2i) := R$, the appropriate rainbow element. By the same argument as above, if E_i has coordinate c in w, R_i has coordinate $Q_{i-1}...Q_1c$. One way to make this more clear is to consider intermediate basis elements $\sigma_j^{E_i}$ defined by $\sigma_j^{E_i}(2i+1, 2n+r) := E_i(2i+1, 2n+r)$ and $\sigma_j^{E_i}(1, 2i) := R_{L,j}$. Then the coordinates of $\sigma_j^{E_i}(2i+1, 2n+r)$ in terms of the coordinate c of E_i are $Q_{i-1}...Q_{i-j}$, and $R_i = \sigma_{i-1}^{E_i}$.

Since we assume $Q_i \neq 0$ for $i < n+r$, this implies the coefficient of E_i is $\frac{Q_{n+r-1}...Q_{n+r-i}}{Q_1...Q_{i-1}} = \mathfrak{Q}_{n+r}^i = x_{E_i}$. In particular, returning to our desired basis element ψ, the coordinate of $E_{\lfloor a/2 \rfloor}$ is $\mathfrak{Q}_{n+r}^{\lfloor a/2 \rfloor} = x_{E_{\lfloor a/2 \rfloor}}$.

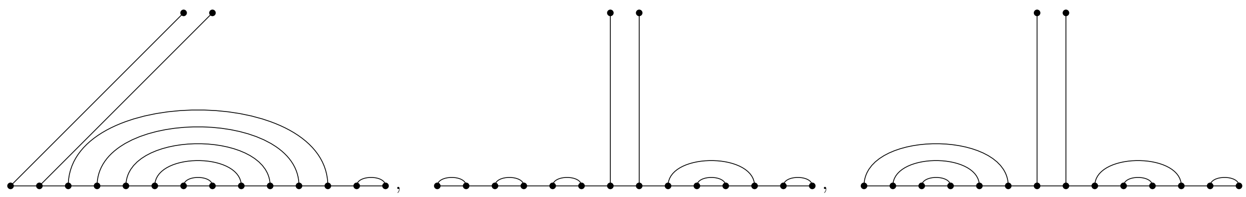


Figure 12. The figure on the left has sub-matching R ignoring the last two nodes. The middle figure has submatching R_3 ignoring the last two nodes. The figure on the right has sub-matching E_3 also ignoring the last two nodes. Since the last two nodes have the same structure for all elements, if the coordinate of the first element is c, the coordinate of the second is $Q_6Q_5Q_4c$, and the coordinate of the third is $\frac{Q_6Q_5Q_4}{Q_1Q_2}c$.

Note that the above logic only uses proposition 3.10, which requires only that a sub-matching be a rainbow element. So, suppose some basis element σ has sub-matching $\sigma(s,t) = R$ with n' nodes and r' anchors, and that the coordinate of σ in w is c. Then it follows that the basis element θ_i defined by $\theta_i(1, s-1) := \sigma(1, s-1)$, $\theta_i(t+1, 2n+r) := \sigma(t+1, 2n+r)$, and $\theta_i(s,t) := E_i$ has coefficient $\mathfrak{Q}^i_{n'+r'}c$. In other words, defining Y_ψ with respect to the sub-matching $\sigma(s,t)$, the operation of finding the coordinate of E_i given the coordinate of $R = \sigma(s,t)$ commutes with the isomorphism to Y_ψ. An example is given in figure 12.

The above technique specifies an algorithm for determining the coordinate of ψ.

As a base case, for the zero element have the algorithm return 1.

Suppose inductively that the algorithm returns the coordinate for any $\sigma \in M^{r'}_{2n'+r'}$, $2n'+r' < 2n+r$, and that that coordinate is equal to the coordinate function x_σ. Also suppose that the algorithm commutes with any isomorphism defined by lemma 3.13. These statements are clearly true for the base case.

Given $\psi \in M^r_{2n+r}$, if $\psi(1) = a$, we may find the coordinate of $E_{\lfloor a/2 \rfloor}$ as before. Note that this operation commutes with any isomorphism defined by lemma 3.13. Me may define $Y_{E_{\lfloor a/2 \rfloor}}$ with respect to the sub-matching $E_{\lfloor a/2 \rfloor}(2, a-1)$. By the inductive hypothesis, we may apply the algorithm to this sub-matching and commute with the isomorphism with $Y_{E_{\lfloor a/2 \rfloor}}$. In this way, we find that the coordinate of $\widehat{\psi}$ defined by $\widehat{\psi}(1,a) := \psi(1,a)$ and $\widehat{\psi}(a+1, 2n+r) = R$ is $x_{\psi(2,a-1)}(q)\mathfrak{Q}^{\lfloor a/2 \rfloor}_{n+r} = x_{\widehat{\psi}}(q)$. Similarly, define $Y_{\widehat{\psi}}$ with respect to the sub-matching $\widehat{\psi}(a+1, 2n+r)$, and commute the algorithm with the isomorphism. In the same way, we obtain that the coordinate of $\psi \in Y_{\widehat{\psi}}$ is $x_{\psi(2,a-1)}(q)x_{\psi(a+1,2n+r)}(q)\mathfrak{Q}^{\lfloor a/2 \rfloor}_{n+r} = x_\psi(q)$ as desired.

Note that we only added a single operation to the algorithm in the inductive step, which also commutes with any isomorphism defined by lemma 3.13. Thus the inductive step holds and the proposition is proved. □

The following few corollaries will help to simplify some later arguments.

Corollary 3.17. *Let $w \in \cap \ker(1+T_i)$, $w \neq 0$. Suppose $\psi(1,a)$ is a sub-matching with no anchors. Then:*

$$x_\psi = x_{\psi(1,a)}(q)x_{\psi(a+1,2n+r)}(q)\mathfrak{Q}^{a/2}_{n+r}$$

Proof. Define $a_1 = \psi(1)$, $a_i = \psi(a_{i-1}+1)$. Then for some j we have $a_j = a$. If $j = 1$, the statement is the same as the proposition. Suppose that the statement is true for any matching with $a_v = a$, $v < j$. Then the statement holds for the sub-matching $\psi(a_1+1, 2n+r)$, and we have:

$$\begin{aligned}
x_\psi(q) =& x_{\psi(1,a_1)}(q)x_{\psi(a_1+1,2n+r)}(q)\mathfrak{Q}_{n+r}^{a_1/2} \\
=& x_{\psi(1,a_1)}(q)x_{\psi(a_1+1,a)}(q)x_{\psi(a+1,2n+r)}(q)\mathfrak{Q}_{n+r}^{a_1/2}\mathfrak{Q}_{n+r-a_1/2}^{a/2-a_1/2} \\
=& x_{\psi(1,a_1)}(q)x_{\psi(a_1+1,a)}(q)x_{\psi(a+1,2n+r)}(q)\mathfrak{Q}_{n+r}^{a_1/2}\mathfrak{Q}_{n+r-a_1/2}^{a/2-a_1/2}\left(\frac{Q_{a/2-1}\dots Q_{a/2-a_1/2}}{Q_{a/2-1}\dots Q_{a/2-a_1/2}}\right) \\
=& x_{\psi(1,a_1)}(q)x_{\psi(a_1+1,a)}(q)x_{\psi(a+1,2n+r)}(q)\mathfrak{Q}_{a/2}^{a_1/2}\mathfrak{Q}_{n+r}^{a/2} \\
=& x_{\psi(1,a)}(q)x_{\psi(a+1,2n+r)}(q)\mathfrak{Q}_{n+r}^{a/2}
\end{aligned}$$

□

Corollary 3.18. *If $\psi \in M_{2n+r}^r$, then $x_\psi(q) \neq 0$ if $e > n+r$.*

Proof. For our base cases, if $2n+r=2$ all coefficients are 1, which is nonzero for any q.

Assume the statement is true for all $2n'+r' < 2n+r$. Given $\psi(1)=a$ we have

$$x_\psi(q) = x_{\psi(2,a-1)}(q)x_{\psi(a+1,2n+r)}(q)\mathfrak{Q}_{n+r}^{\lfloor a/2 \rfloor}$$

If $e > n+r$, non of the Q_i term appearing in $\mathfrak{Q}_{n+r}^{\lfloor a/2 \rfloor}$ are zero, and $n'+r' < n+r < e$ for any of the sub-matchings that appear, so those coordinates are nonzero and the corollary holds.

□

The proposition fully characterizes any possible kernel element when $Q_1 \dots Q_{n+r-1} \neq 0$. In particular, the following corollary holds:

Corollary 3.19. *When $Q_1 .. Q_{n+r-1} \neq 0$ and the kernel is nontrivial, the kernel is one dimensional.*

This corollary follows from the fact that we may write the coordinate of any basis element as proportional to the coordinate of the rainbow basis element.

3.3.3. *Nontrivial Kernel.* To verify the kernel element, we will need to know exactly which basis elements are mapped to a specific basis element by a given $(1+T_i)$. The next two lemmas help address this question.

Lemma 3.20. *Take some basis element $\psi \in M_{2n+r}^r$.*

(i) Suppose $\psi(a) = b$ for some $b > a+1$, and that $(1+T_i)\psi = (1+q)\psi$ for some $a < i < b-1$. Me then have a subrepresentation $\psi(a,b)$ and define Y_ψ with respect to this subrepresentation. Then for all basis elements σ such that $(1+T_i)\sigma = q^{1/2}\psi$, we have that

$$\sigma \in Y_\psi$$

.

(ii) Suppose ψ has some anchor at position u, and $(1+T_i)\psi = (1+q)\psi$ for some $i > u$, we again have a subrepresentation $\psi(u, 2n+r)$ and define Y_ψ with respect to this subrepresentation. Then for all basis elements σ such that $(1+T_i)\sigma = q^{1/2}\psi$, we have that $\sigma \in Y_\psi$ again.

Proof. This lemma follows from an observation I made in section 2: a transposition can only create two arcs or an arc and an anchor.

(i) If $\sigma \notin Y_\psi$ either $\sigma(1,a-1) \neq \psi(1,a-1)$ or $\sigma(b+1,2n+r) \neq \psi(b+1,2n+r)$. Suppose it is the first case. Then for some $s,t \in [1,a-1]$, $s<t$, we have $\psi(s)=t$ and $\sigma(s) \neq t$. To have $(1+T_i)\sigma = q^{1/2}\psi$ we must have $\sigma(t) = i+1$, $\sigma(s) = i$. But then $\sigma(a) \neq b$ and $\sigma(a) \neq i$ or $i+1$, so $((1+T_i)\sigma)(a) \neq b$ and $(1+T_i)\sigma \neq q^{1/2}\psi$. The same argument proves the $\sigma(b+1,2n+r) \neq \psi(b+1,2n+r)$ case.

(ii) An analogous argument proves the anchor case. Specifically, the anchor cannot exist at position u and is not created by action of $(1+T_i)$ if $\sigma(s) = i$ and $\sigma(t) = i+1$.

□

It is important to note that lemma 3.20 only references cases where a transposition acts under an arc or to the right of an anchor. An example is given in figure 13.

The next lemma characterizes cases where the transposition is not under any arcs and all anchors are to the right.

Essentially, this lemma states that the only elements sent to the same element are those which break at most one of the top level arcs to the left of the leftmost anchor, or that break the leftmost anchor. An illustration is given in figure 14.

Lemma 3.21. *Take a basis element $\psi \in M^r_{2n+r}$. Suppose the leftmost anchor in ψ is at index b, or let $b = 2n+r+1$ if there is no anchor. Define a_j such that $\psi(a_j) = a_{j-1}+1$ and $\psi(a_1) = 1$ for all j such that $a_j < b$.*

Suppose $(1+T_i)\psi = (1+q)\psi$ for some $i < b-1$ where $\nexists s,t$ such that $\psi(s) = t$ and $s < i, t > i+1$. Suppose there is some basis element σ such that $(1+T_i)\sigma = q^{1/2}\psi$. Then:

(i) $\psi(a_{j-1}+2, a_j-1) = \sigma(a_{j-1}+2, a_j-1)$ *for all j.*
(ii) $\psi(b+1, 2n+r) = \sigma(b+1, 2n+r)$
(iii) If b is not an anchor in σ, $\psi(a_j) = \sigma(a_j)$ for all j such that $a_j \neq i+1$.
(iv) If b is an anchor in σ, there exists exactly one value of j such that $\sigma(a_j) \neq \psi(a_j)$ and $a_j \neq i+1$

Proof. (i) Suppose that, for some j there exists $s,t \in [a_{j-1}+2, a_j-1]$ such that $\psi(s) = t$ but $\sigma(s) \neq t$. Then if $(1+T_i)\sigma = q^{1/2}\psi$ we must have $\sigma(i) = s$ or t and $\sigma(i+1) = s$ or t. But, by definition, $i, i+1 \notin [a_{j-1}+1, a_j]$, so this implies $\sigma(a_j) \neq a_{j-1}+1, i, i+1$, so $((1+T_i)\sigma)(a_j) \neq a_{j-1}+1$ and $(1+T_i)\sigma \neq q^{1/2}\psi$. So (i) is proved.

(ii) The proof of (ii) is analogous to the proof of (i). Me cannot have $\psi(b+1, 2n+r) \neq \sigma(b+1, 2n+r)$ and $\psi(b+1, 2n+r) = ((1+T_i)\sigma)(b+1, 2n+r)$ if $((1+T_i)\sigma)(b) = b$.

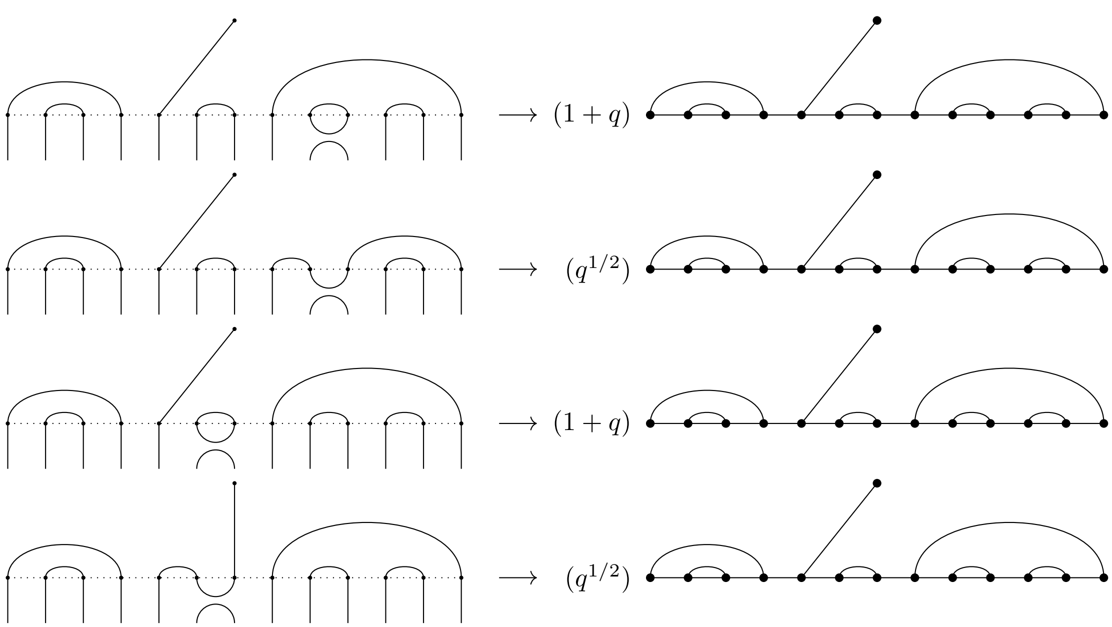


Figure 13. In the first line we act under an arc, so if another element without that arc is sent to that element, it must fix the arc as shown in the second line. In the third line we act to the right of an anchor, so if another element without that anchor is sent to that element, it must fix the anchor as shown in the fourth line.

(iii) If b is not an anchor in σ and $(1+T_i)\sigma = q^{1/2}\psi$, we must have i an anchor in σ, and $\sigma(i+1) = b$. No other nodes in σ are changed, so this proves (iii).

(iiii) From (i)-(iii) we have that the only remaining matchings that can differ are the $(a_{j-1}+1, a_j)$ matchings. If one of them differs, by the same argument as before it must be fixed by the action of $(1+T_i)$, and no other nodes are changed, so (iiii) is proved.

$\square$

Lastly, we will need a small combinatorial result.

Lemma 3.22. *Suppose $n > b \geq a > 0$ and $e > n$. Then*

$$Q_{n-a}Q_b - Q_{n-b-1}Q_{a-1} = Q_nQ_{b-a}$$

Proof. If $b = 1$, the only possibility for a is 1, in which reduces to lemma 3.11.

Suppose the lemma is true for all $\widehat{b} < b+1$. Then for $a < b$ we have

$$\begin{aligned} Q_{n-a}Q_b - Q_{n-b-1}Q_{a-1} &= Q_nQ_{b-a} \\ Q_1Q_{n-a}Q_b - Q_1Q_{n-b-1}Q_{a-1} &= Q_1Q_nQ_{b-a} \end{aligned}$$

from lemma 3.11, we have

$$Q_{n-a}(Q_{b+1} + Q_{b-1}) - (Q_{n-b} + Q_{n-b-2})Q_{a-1} = Q_n(Q_{b-a-1} + Q_{b-a+1})$$

and from the inductive hypothesis we have

$$Q_{n-a}Q_{b+1} - Q_{n-b}Q_{a-1} = Q_nQ_{b-a+1}$$

as desired.

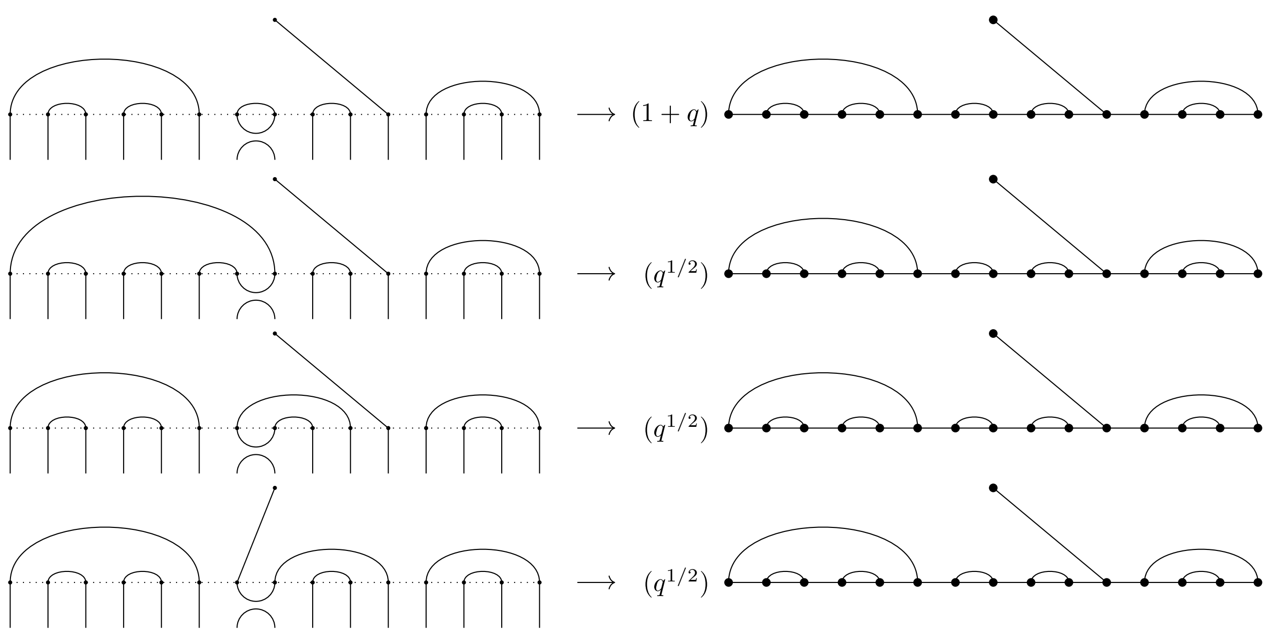


Figure 14. The action of $(1+T_7)$ fixes the first basis element. Shown are all the basis vectors sent to the same element by the same transposition. Note that in all of them nodes 2-5 and 12-15 are the same. This illustrates (i) and (ii) in lemma 3.21. Note that in the last case where the anchor is in a different place, 1,6 and 9,10 are still matched. This illustrates (iii). In the middle two cases where the anchor is in the same place, only one of 1,6 or 9,10 are not paired. This illustrates (iiii).

For $a = b$ we have

$$\begin{aligned} Q_{n-b}Q_b - Q_{n-b-1}Q_{b-1} =& Q_n \\ Q_1Q_{n-b}Q_b - Q_1Q_{n-b-1}Q_{b-1} =& Q_1Q_n \end{aligned}$$

from lemma 3.11, we have

$$\begin{aligned} Q_{n-b}(Q_{b+1} + Q_{b-1}) - (Q_{n-b} + Q_{n-b-2})Q_{b-1} =& Q_1Q_n \\ Q_{n-b}Q_{b+1} - Q_{n-b-2}Q_{b-1} =& Q_1Q_n \end{aligned}$$

as desired.

For $a = b + 1$, we continue:

$$\begin{aligned} Q_1Q_{n-b}Q_{b+1} - Q_1Q_{n-b-2}Q_{b-1} =& Q_1Q_1Q_n \\ (Q_{n-b-1} + Q_{n-b+1})Q_{b+1} - Q_{n-b-2}(Q_b + Q_{b-2}) =& (1 + Q_2)Q_n \end{aligned}$$

So by the inductive hypothesis

$$Q_{n-b-1}Q_{b+1} - Q_{n-b-2}Q_b = Q_n$$

as desired, and the proof is finished by induction. □

Me are now ready to prove existence of a kernel element. To prove this, we will show that if $w \in M^r_{2n+r}$ is as characterized above, the coordinate of any basis element in $(1+T_i)w$ is zero. This will split into various cases related to the previous lemmas.

Theorem 3.23. *Suppose $e = n + r + 1$. Then $\cap \ker(1 + T_i) \neq 0$.*

Proof. As a base case, when $2n' + r' \leq 2$, the representation is at most one dimensional. If the one basis element has only anchors, it is sent to zero by any $(1 + T_i)$, and is in the kernel. If the single basis element is a single arc, it is sent to $(1 + q)$ times itself, and we take $e = n + r + 1 = 2$ so $1 + q = 0$ and the base case holds.

Assume inductively that the statement holds for all $M^{r'}_{2n'+r'}$ where $2n'+r' < 2n+r$. Take w as defined by proposition 3.16.

Given $\psi \in (1+T_i)M^r_{2n+r}$ let $E_\psi \subset M^r_{2n+r}$ be the pre-image of ψ under the action of $(1+T_i)$. To prove w is in the kernel, we must show the following:

$$(1 + q)x_\psi(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2}x_\sigma(q) = 0 \text{ for all basis elements } \psi \tag{3.5}$$

Inductively, we assume this equation holds for basis elements in smaller representations $M^{r'}_{2n'+r'}$, but only for q such that $e = n' + r' + 1$. Clearly this is true in the base case. For the following proof we will need a slightly stronger inductive assumption. Take $\psi' \in M^{r'}_{2n'+r'}$, and suppose either that $\psi'(1) = 2n' + r'$, and that $T_i\psi' = (1 + q)\psi'$, $1 < i < 2n' + r' - 1$, or that 1 is an anchor in ψ. Defining $E_{\psi'}$ as before, we assume

$$(1 + q)x_{\psi'} + \sum_{\sigma \in E_{\psi'}, \sigma \neq \psi'} q^{1/2}x_\sigma = 0 \text{ for any } q \text{ with } e > n' + r' \tag{3.6}$$

Note that 3.6 does not apply in the base case. Our proof of the inductive step will be split into cases, and each case will only depend on sub-cases in which certain inductive hypotheses apply, so this will not lead to any problems.

Before exploring the cases, let us formally define E_ψ to be the pre-image of ψ under the action of $(1 + T_i)$, and $E_{\psi(a,b)}$ to be the pre-image of $\psi(a, b)$ under action of $(1 + T_{i-a+1})$:

case 1: Suppose $\psi \in (1+T_i)M^r_{2n+r}$ for some i, and that $\exists s, t$ such that $s < i < t-1$, $s > 1$ or $t < 2n+r$, and $\psi(s) = t$. Also suppose the leftmost anchor is at some index $u > t$, or that there are no anchors. Then we have a sub-matching $\psi(s,t)$, and by lemma 3.20 $E_\psi \subset Y_\psi$. Then, using corollary 3.17, the following equality holds:

$$\begin{aligned}
&(1+q)x_\psi(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2} x_\sigma(q) \\
&= \left(x_{\psi(1,s-1)}(q)\mathfrak{Q}^{(s-1)/2}_{n+r}\right)\left((1+q)x_{\psi(s,2n+r)}(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2} x_{\sigma(s,2n+r)}(q)\right) \\
&= \left(x_{\psi(1,s-1)}(q)\mathfrak{Q}^{(s-1)/2}_{n+r}\right)\left(x_{\psi(t+1,2n+r)}(q)\mathfrak{Q}^{(t-s+1)/2}_{n+r-(s-1)/2}\right)\left((1+q)x_{\psi(s,t)}(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2} x_{\sigma(s,t)}(q)\right)
\end{aligned}$$

Me have that $e > j$ for any Q_j term appearing in the equation above, and $e > n' + r'$ for any sub-matching coordinate appearing above, so by corollary 3.18:

$$(1+q)x_\psi(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2} x_\sigma(q) = 0$$

if and only if

$$(1+q)x_{\psi(s,t)}(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2} x_{\sigma(s,t)}(q) = 0$$

Note that $(\psi(s,t))(1) = t - s + 1$. So by our inductive hypothesis (ii), we have

$$(1+q)x_{\psi(s,t)}(q) + \sum_{\sigma \in E_{\psi(s,t)}, \sigma \neq \psi(s,t)} q^{1/2} x_\sigma(q) = 0$$

By lemma 3.13, if $\sigma \in Y_\psi$, $(1+T_i)\sigma = q^{1/2}\psi$ if and only if $(1+T_{i-s+1})\sigma(s,t) = q^{1/2}\psi(s,t)$, so the previous equation implies

$$(1+q)x_{\psi(s,t)}(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2} x_{\sigma(s,t)}(q) = 0$$

as desired, and this case is proved.

case 2: Again take $\psi \in (1+T_i)M^r_{2n+r}$ for some i, but suppose the leftmost anchor is at some position u where $1 < u < i$. Then, as before, we have a sub-matching $\psi(u, 2n+r)$ and by lemma 3.20 $E_\psi \subset Y_\psi$.

Note that both corollary 3.17 and our inductive hypothesis 3.6 still apply in this case, where we consider a left anchor instead of a matching. This allows the exact same logic from the proof of the first case to prove this case.

It is important to note that, for both case 1 and case 2, the inductive hypothesis depends only on cases in which 3.6 holds. Thus, if we show these cases rely on valid base cases, case 1 and 2 follow. This will be done in case 4.

case 3: Suppose $\psi \in (1+T_i)M^r_{2n+r}$ for some i, the leftmost anchor is at a position $u > i+1$ or there are no anchors, and $\nexists s, t$ such that $\psi(s) = t$ and $s < i < t-1$. Lemma 3.21 characterizes all $\sigma \in E_\psi$. Me would like to prove the following for arbitrary q where $e > n + r$:

$$(1+q)x_\psi(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2} x_\sigma(q) = -q^{1/2} x_{\psi(1,i-1)}(q) x_{\psi(i+2,2n+r)}(q) \mathfrak{Q}^{(i+1)/2}_{n+r+1}$$

See figure 15 for an example of this equality. Note that if $e = n+r+1$, Q_{n+r} is the only zero component in the right side of this equation, so proving this equation is sufficient to prove case three.

Me will prove this equality through yet another inductive proof, this time inducting on the number of top level humps, including the leftmost anchor.

Formally, as we have in earlier lemmas, we will define a_j by $a_1 := \psi(1)$, $a_j := \psi(a_{j-1}+1)$. Then define b_ψ such that $a_{b_\psi} = u$ if there is an anchor or $a_{b_\psi} = 2n+r$ otherwise. Me induct on b_ψ.

If $b_\psi = 1$, we must be in M_2^0 to be in case 3 (otherwise $s < i < t-1$ for some s, t where $\psi(s) = t$), which is trivially satisfied. Thus the base case holds.

Suppose for all basis elements σ such that $b_\sigma < b_\psi$, the equality holds. Suppose $i \neq 1$. Then $a_1 < i$ and lemma 3.21 gives that there is a unique $\upsilon \in E_\psi$ such that $\upsilon(1) \neq a_1$. Thus we have the following equality:

$$
\begin{aligned}
&(1+q)x_\psi(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi, \upsilon} q^{1/2} x_\sigma(q) \\
&= \left(x_{\psi(1,a_1)}(q)\mathfrak{Q}_{n+r}^{a_1/2}\right)\left((1+q)x_{\psi(a_1+1,2n+r)}(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi, \upsilon} q^{1/2} x_{\sigma(a_1+1,2n+r)}(q)\right)
\end{aligned}
$$

Define Y_ψ with respect to the sub-matching $\psi(a_1+1, 2n+r)$. Then $\sigma \in E_\psi$, $\sigma \neq \upsilon$ implies $\sigma \in Y_\psi$. By our inductive hypothesis, we have that

$$
\begin{aligned}
&(1+q)x_{\psi(a_1+1,2n+r)}(q) + \sum_{\sigma \in E_{\psi(a_1+1,2n+r)}, \sigma \neq \psi(a_1+1,2n+r)} q^{1/2} x_\sigma(q) \\
&= -q^{1/2} x_{\psi(a_1+1,i-1)}(q) x_{\psi(i+2,2n+r)}(q) \mathfrak{Q}_{n+r-a_1/2+1}^{(i+1-a_1)/2}
\end{aligned}
$$

By lemma 3.13, $\sigma \subset Y_\psi$, $\sigma \in E_\psi$ if and only if $\sigma(a_1+1, 2n+r) \in E_{\psi(a_1+1,2n+r)}$. This implies:

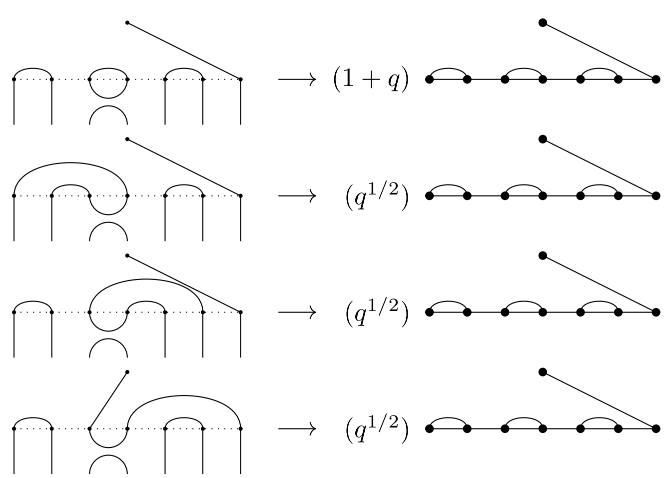


Figure 15. The four elements sent to the first element by $(1+T_3)$ are listed. The coordinate of the first element is $Q_3Q_2Q_1$. The coordinate of the second is Q_3Q_2. The coordinate of the third is Q_3Q_2. The coordinate of the fourth is Q_3. Call the first element ψ. Then $x_{\psi(1,2)} = 1$, $x_{\psi(5,7)} = Q_1$, so $-q^{1/2}x_{\psi(1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}_{n+r+1}^{(i+1)/2} = -q^{1/2}Q_4Q_3$. Me also have $(1+q)x_\psi(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2}x_\sigma(q) = (q+1)(Q_3Q_2Q_1) + q^{1/2}(Q_3Q_2 + Q_3Q_2 + Q_3) = -q^{1/2}(Q_3Q_2Q_1^2 - 2Q_3Q_2 + Q_3) = -q^{1/2}Q_4Q_3$ as desired (one can verify the last equality by hand or simplify using lemma 3.22).

$$(1+q)x_{\psi(a_1+1,2n+r)}(q) + \sum_{\sigma\in E_\psi,\sigma\neq\psi,\upsilon} q^{1/2}x_{\sigma(a_1+1,2n+r)}(q)$$
$$= -\, q^{1/2}x_{\psi(a_1+1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}^{(i+1-a_1)/2}_{n+r-a_1/2+1}$$

So, combining with the aforementioned equality, we have

$$(1+q)x_\psi(q) + \sum_{\sigma\in E_\psi,\sigma\neq\psi,\upsilon} q^{1/2}x_\sigma(q)$$
$$= \left(x_{\psi(1,a_1)}(q)\mathfrak{Q}^{(i+1-a_1)/2}_{n+r-a_1/2+1}\right)\left(-q^{1/2}x_{\psi(a_1+1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}^{(i+1-a_1)/2}_{n+r-a_1/2+1}\right)$$
$$= \left(x_{\psi(1,a_1)}(q)\mathfrak{Q}^{(i+1-a_1)/2}_{n+r-a_1/2+1}\right)\left(-q^{1/2}x_{\psi(a_1+1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}^{(i+1-a_1)/2}_{n+r-a_1/2+1}\right)\left(\frac{Q_{(i-1)/2-1}\ldots Q_{(i-1-a_1)/2}}{Q_{(i-1)/2-1}\ldots Q_{(i-1-a_1)/2}}\right)$$
$$= -\, q^{1/2}x_{\psi(1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}^{(i+1)/2}_{n+r+1}\left(\frac{Q_{n+r-a_1/2}Q_{(i-1)/2}}{Q_{n+r}Q_{(i-1-a_1)/2}}\right)$$

Separately, note that υ is defined by $\upsilon(2,a_1-1)=\psi(2,a_1-1)$, $\upsilon(a_1+1,i-1)=\psi(a_1+1,i-1)$, $\upsilon(i+2,2n+r)=\psi(i+2,2n+r)$, and $\upsilon(1)=i+1$, $\upsilon(a_1)=i$. Thus we may determine x_υ, again utilizing corollary 3.17:

$$\begin{aligned} x_\upsilon =& x_{\psi(i+2,2n+r)}x_{\upsilon(2,i)}\mathfrak{Q}^{(i+1)/2}_{n+r} \\ =& x_{\psi(i+2,2n+r)}\left(x_{\psi(2,a_1-1)}x_{\upsilon(a_1,i)}\mathfrak{Q}^{(a_1-2)/2}_{(i-1)/2}\right)\mathfrak{Q}^{(i+1)/2}_{n+r} \\ =& x_{\psi(i+2,2n+r)}\left(x_{\psi(1,a_1)}x_{\psi(a_1+1,i-1)}\mathfrak{Q}^{(a_1-2)/2}_{(i-1)/2}\right)\mathfrak{Q}^{(i+1)/2}_{n+r} \\ =& x_{\psi(i+2,2n+r)}x_{\psi(1,i-1)}\mathfrak{Q}^{(i+1)/2}_{n+r+1}\left(\frac{Q_{n+r-(i+1)/2}Q_{a_1/2-1}}{Q_{n+r}Q_{(i-1-a_1)/2}}\right) \end{aligned}$$

Adding this into our previous equation, we have:

$$(1+q)x_\psi(q) + \sum_{\sigma\in E_\psi,\sigma\neq\psi} q^{1/2}x_\sigma(q)$$
$$= -\, q^{1/2}x_{\psi(1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}^{(i+1)/2}_{n+r+1}\left(\frac{Q_{n+r-a_1/2}Q_{(i-1)/2}-Q_{n+r-(i+1)/2}Q_{a_1/2-1}}{Q_{n+r}Q_{(i-1-a_1)/2}}\right)$$

Applying lemma 3.22 to the portion of the equation above in parenthesis, the above is equivalent to

$$(1+q)x_\psi(q) + \sum_{\sigma\in E_\psi,\sigma\neq\psi} q^{1/2}x_\sigma(q) = -q^{1/2}x_{\psi(1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}^{(i+1)/2}_{n+r+1}$$

as desired. Note that if $e\geq n+r+1$ the only term above that can be zero is Q_{n+r} (by corollary 3.18). Thus we have proved the inductive step for the case where $i\neq 1$.

If $i=1$, we instead look at the sub-matchings $\psi(1,a_{(b_\psi-1)})$, $\psi(a_{(b_\psi-1)}+1,2n+r)$. Again lemma 3.21 gives that there is a unique $\upsilon\in E_\psi$ such that $\upsilon(a_{(b_\psi-1)}+1)\neq\psi(a_{(b_\psi-1)}+1)$. Taking Y_ψ with respect to the sub-matching $\psi(a_{(b_\psi-1)}+1,a_{b_\psi})$ again we have that $\sigma\in E_\psi$, $\sigma\neq\upsilon$ implies $\sigma\in Y_\psi$. Thus, following the same logic as before, we arrive at the following equality:

$$(1+q)x_\psi(q) + \sum_{\sigma\in E_\psi,\sigma\neq\psi,\upsilon} q^{1/2}x_\sigma(q)$$
$$= \left(x_{\psi(a_{(b_\psi-1)}+1,2n+r)}(q)\mathfrak{Q}^{a_{(b_\psi-1)}/2}_{n+r}\right)\left(-q^{1/2}x_{\psi(3,a_{(b_\psi-1)})}(q)Q_{a_{(b_\psi-1)}/2}\right)$$
$$= -\, q^{1/2}x_{\psi(3,2n+r)}\frac{Q_{n+r-1}Q_{a_{(b_\psi-1)}/2}}{Q_{a_{(b_\psi-1)}/2-1}}$$

Again, we know the structure of υ from lemma 3.21. Suppose for now that a_{b_ψ} is not an anchor, so it is $2n+r$. Then υ is defined by $\upsilon(3, a_{(b_\psi-1)}) = \psi(3, a_{(b_\psi-1)})$, $\upsilon(a_{(b_\psi-1)}+2, 2n+r-1) = \psi(a_{(b_\psi-1)}+2, 2n+r-1)$, and $\upsilon(1) = 2n+r$, $\upsilon(2) = a_{(b_\psi-1)}+1$. So we may again find x_υ:

$$\begin{aligned}
x_\upsilon = x_{\upsilon(2,2n+r-1)} =& x_{\psi(3,a_{(b_\psi-1)})} x_{\psi(a_{(b_\psi-1)}+2,2n+r-1)} \mathfrak{Q}_{n+r-1}^{a_{(b_\psi-1)}/2} \\
=& x_{\psi(3,a_{(b_\psi-1)})} x_{\psi(a_{(b_\psi-1)}+1,2n+r)} \mathfrak{Q}_{n+r-1}^{a_{(b_\psi-1)}/2} \\
=& x_{\psi(3,2n+r)} \frac{Q_{n-r-1-a_{(b_\psi-1)}/2}}{Q_{a_{(b_\psi-1)}/2-1}}
\end{aligned}$$

Alternatively, if a_{b_ψ} is an anchor, the definition of υ is now $\upsilon(3, a_{b_\psi}-1) = \psi(3, a_{b_\psi}-1)$, $\upsilon(a_{b_\psi}+1, 2n+r) = \psi(a_{b_\psi}+1, 2n+r)$, and $\upsilon(1) = 1$, $\upsilon(2) = a_{b_\psi}$, so we have:

$$x_\upsilon = x_{\upsilon(2,2n+r)} = x_{\psi(3,a_{(b_\psi-1)})} x_{\psi(a_{b_\psi}+1,2n+r)} \mathfrak{Q}_{n+r-1}^{a_{(b_\psi-1)}/2} = x_{\psi(3,2n+r)} \frac{Q_{n-r-1-a_{(b_\psi-1)}/2}}{Q_{a_{(b_\psi-1)}/2-1}}$$

so for our purposes x_υ is the same in either case.

Incorporating into the above equation, we have:

$$\begin{gathered}
(1+q)x_\psi(q) + \sum_{\sigma\in E_\psi, \sigma\neq\psi} q^{1/2} x_\sigma(q) = \\
-q^{1/2} x_{\psi(3,2n+r)} \frac{Q_{n+r-1} Q_{a_{(b_\psi-1)}/2} - Q_{n-r-1-a_{(b_\psi-1)}/2}}{Q_{a_{(b_\psi-1)}/2-1}}
\end{gathered}$$

By lemma 3.22, this is simply $-q^{1/2} x_{\psi(3,2n+r)} Q_{n+r}$ as desired, and we have finished proving case 3.

case 4: The only cases we have not yet dealt with are those where either 1 is an anchor or $\psi(1) = 2n+r$. These are those cases related to our inductive hypothesis (ii).

To not be in case 1 or 2, we must have that there are no anchors between index 1 and i, and that there is no integer s such that $1 < s < i < \psi(s) - 1$. It follows from the same argument that proved lemma 3.20 that there exists exactly one $\upsilon \in E_\psi$ such that $\upsilon(1) \neq \psi(1)$. Define N to be $2n+r$ if 1 is an anchor, or $2n+r-1$ if 1 is not an anchor. Then, defining Y_ψ with respect to the sub-matching $\psi(2, N)$, we have that $\sigma \in E_\psi, \sigma \neq \upsilon$ if and only if $\sigma(2, N) \in E_{\psi(2,N)}$. Note that for $E_{\psi(2,N)}$ we may apply the inductive hypothesis from case 3, so we have:

$$\begin{aligned}
&(1+q)x_\psi(q) + \sum_{\sigma\in E_\psi, \sigma\neq\psi,\upsilon} q^{1/2} x_\sigma(q) \\
=&(1+q)x_{\psi(2,N)}(q) + \sum_{\sigma\in E_{\psi(2,N)}, \sigma\neq\psi(2,N)} q^{1/2} x_\sigma(q) \\
=& - q^{1/2} x_{\psi(2,i-1)} x_{\psi(i+2,N)} \mathfrak{Q}n + r^{i/2}
\end{aligned}$$

As in case 3, we can also determine x_υ. υ is defined by $\upsilon(2, i-1) = \psi(2, i-1)$, $\upsilon(i+2, N) = \psi(i+2, N)$, $\upsilon(1) = i$, and $\upsilon(i+1) = 2n+r$ if 1 is not an anchor or $i+1$ if 1 is an anchor, and we have:

$$x_\upsilon = x_{\psi(2,i-1)} x_{\upsilon(i+2,N)} \mathfrak{Q}n + r^{i/2}$$

Thus we have

$$\begin{gathered}
(1+q)x_\psi(q) + \sum_{\sigma\in E_\psi, \sigma\neq\psi} q^{1/2} x_\sigma(q) = \\
-q^{1/2} x_{\psi(2,i-1)} x_{\psi(i+2,N)} \mathfrak{Q}n + r^{i/2} (1-1) = 0
\end{gathered}$$

as desired, and the last case is proved. Note that this only relies on the inductive hypothesis from case 3, for which we showed the base case holds.

Thus our inductive hypotheses have all been proven, and those that apply in the base case hold in the base case, so by induction the theorem is proved.

□

Corollary 3.24. *If $e = n + r + 1$, M^r_{2n+r} is reducible, and has a unique sign subrepresentation.*

4. Fibonacci representations and quotients of Specht modules

We have difficulty characterizing the Specht module above when it is irreducible. We may instead attempt to characterize the irreducible quotient D^λ of the Specht module. In particular, note that, whenever $e \leq r+2$, the restriction $\operatorname{Res} D^{(n+r,n)}$ decomposes into a direct sum of partitions $(m+s, m)$ such that $e \leq s+2$ and $2m + s = 2n + r - 1$. This forms a recurrence among these representations, allowing a combinatorial description of their dimension. In this section we henceforth assume $e = 5$ and note that this recurrence resembles the Fibonacci recurrence; we follow this to characterize the restriction $D^{(n+r,n)}$ with $r \leq 3$.

Remark. Let $d^{0,3}_m$ be the dimension $\dim D^{(n,n)}$ with $2n = m$ when m is even, and $\dim D^{(n+3,n)}$ with $2n+3 = m$ when m is odd. Similarly define $d^{1,2}_m$. Then, Corollary 2.5 gives

$$\begin{aligned} d^{0,3}_m &= d^{1,2}_{m-1} \\ d^{1,2}_m &= d^{1,2}_{m-1} + d^{0,3}_{m-1} \\ &= d^{0,3}_{m-1} + d^{0,3}_{m-2}. \end{aligned}$$

Carefully following this and noting the base cases $d^{0,3}_2 = d^{1,2}_2 = 1$, one may note that this recurrences proves that $d^{1,2}_m = d^{0,3}_{m+1} = f_n$, where f_n is the nth *Fibonacci number*. This matches the dimension our Fibonacci subrepresentations, motivating their definition. □

We can start our study of V by studying low-dimensional cases. First, note that V^2_{*0} is the sign representation $D^{(1^2)}$ and V^2_{*0} is the trivial representation $D^{(2)}$.

V^2_{00} is a 2-dimensional representation of a semisimple commutative algebra, and hence decomposes into a direct sum of two subrepresentations. In particular, we can use the basis $\{(0{*}0), (000)\}$ and explicitly write the matrix

$$\rho_{T_1} = \begin{bmatrix} \varepsilon_1 & \delta \\ \delta & \varepsilon_2 \end{bmatrix}$$

having characteristic polynomial $(\varepsilon_1 - \lambda)(\varepsilon_2 - \lambda) - \delta^2 = \lambda^2 - (\varepsilon_1 + \varepsilon_2)\lambda + (\varepsilon_1\varepsilon_2 - \delta^2)$. We may verify that, for $\lambda = -1$, this evaluates to

$$-((-1 + q + q^2)(1 + q^3 + q^4 + q^5 + 2q^6 + q^7))\,[5]_q = 0$$

and for $\lambda = q$ this evaluates to

$$-(q^2(-1 + q + q^2)(1 + q + q^2 + q^3 + 2q^4 + q^5))\,[5]_q = 0$$

hence ρ_{T_1} has eigenvalues -1 and q.

The eigenspaces with eigenvalues -1 and q are subrepresentations isomorphic to the sign and trivial representation, hence V_{00} is isomorphic to a direct sum of the trivial and sign representations: $V^2_{00} \simeq V^2_{*0} \oplus V^2_{**}$.

Now we may prove that V^3_{*0} is irreducible; this has basis $\{({*}0{*}0), ({*}000)\}$, and the following matrices:

$$\rho_{T_1} = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{bmatrix}; \qquad \rho_{T_2} = \begin{bmatrix} \varepsilon_1 & \delta \\ \delta & \varepsilon_2 \end{bmatrix}.$$

A proper nontrivial subrepresentation of V^3_{*0} must be one-dimensional, and hence an eigenspace of each of these matrices; since $\alpha_2 \neq \alpha_1$, the first has eigenspaces given by the spans of basis elements, and since $\delta \neq 0$, these are not eigenspaces of the second. Hence V^3_{*0} is irreducible.

These establish the low-dimensional behavior that we will use as base cases below. The rest of this section will proceed first by proving that V_{*0} and V_{**} are irreducible; then, we will use combinatorial arguments to prove that these are isomorphic to the correct two-row Specht module quotients.

Proposition 4.1. *The representation $V_{*0} := V^m_{*0}$ is irreducible.*

Proof. We will prove this inductively in m. We've already proven it for V_{*0}^2 and V_{*0}^3, so suppose that V_{*0}^{m-2} is irreducible.

Let $\{v_i\}$ be the basis for V_{*0}. Then, each v_i is cyclic; indeed, we can transform every basis vector into $(*0\dots0)$ by multiplying by the appropriate $\frac{1}{\delta-\varepsilon_1}(T_i-\varepsilon_1)$, and we can transform $(*0\dots0)$ into any basis vector by multiplying be the appropriate $\frac{1}{\delta-\varepsilon_2}(T_i-\varepsilon_2)$. Hence it is sufficient to show that each $v\in V_{*0}$ generate some basis element.

Let v' be the basis element $(*0*0\dots0)$, which is many copies of $*0$, followed by an extra 0 if m is odd. We will show that each $v\in F$ generates v'.

Suppose that no elements beginning $(*0*0)$ are represented in v_i; then, all such elements are represented in T_3v, so we may assume that at least one is represented in v.

Note that $\operatorname{im}(T_2-\alpha_1)=\operatorname{Span}\{\text{Basis vectors beginning }(*0*0)\}$ and $(T_2-\alpha_1)v\neq 0$. Further, note that $\operatorname{Res}^{\mathscr{H}(S_m)}_{\mathscr{H}(S_{m-2})}\operatorname{im}(T_2-\alpha_1)\simeq V_{*0}^{m-2}$ as representations. Hence irreducibility of V_{*0}^{m-2} implies that v' is generated by $(T_2-\alpha_1)v$, and V_{*0}^m is irreducible. □

Now, we may begin considering restrictions:

Lemma 4.2. *The following branching rules hold:*

$$V_{00}^{m-1}\simeq \operatorname{Res}V_{*0}^m\simeq V_{**}^{m-1}\oplus V_{*0}^{m-1}$$
$$\operatorname{Res}V_{**}^m\simeq V_{*0}^{m-1}.$$

Proof. The first line follows by considering the last two $m-2$ transpositions for the left isomorphism, then the first two for the right isomorphism. This is well-behaved by Appendix B.

Similarly, the second line follows by considering the last $m-2$ transpositions. □

This immediately gives a rather strong characterization of V.

Corollary 4.3. *The representation V_{**} is irreducible.* □

Corollary 4.4. *The representation V decomposes into a direct sum of irreducible representations as follows:*

$$V\simeq 3V_{*0}\oplus 2V_{**}.$$

□

Now we may use these in order to apply Young Tableau to characterize V.

Theorem 4.5. *The irreducible components of V are given by the following isomorphisms:*

$$V_{**}^{2n}\simeq D^{(n,n)}$$
$$V_{**}^{2n-1}\simeq D^{(n+1,n-2)}$$
$$V_{*0}^{2n}\simeq D^{(n+1,n-1)}$$
$$V_{*0}^{2n-1}\simeq D^{(n,n-1)}.$$

Proof. We will prove this by induction on n; we have already proven the base case V^2, so suppose that we have proven these isomorphisms for V^{2n-2}. We will prove the isomorphisms for V^{2n-1} and V^{2n}.

By irreducibility, $V_{**}^{2n-1}\simeq D^\lambda$ and $V_{*0}^{2n-1}\simeq D^\mu$ for some diagrams λ and μ. We will show that $\lambda=(n+1,n-2)$ and $\mu=(n+1,n-1)$.

First, note that we have

$$\operatorname{Res}\ D^\lambda\simeq D^{(n,n-2)}\simeq \operatorname{Res}\ D^{(n+1,n-2)}$$

and

$$\operatorname{Res}\ D^\mu\simeq D^{(n,n-2)}\oplus D^{(n-1,n-1)}\simeq \operatorname{Res}\ D^{(n,n-1)}.$$

By irreducibility of $\operatorname{Res}D^\lambda$, the only normal number in λ is 1. Further, the only tableaux which can remove a cell to yield $D^{(n,n-2)}$ are $(n+1,n-2)$, $(n,n-1)$, and $(n,n-2,1)$ as illustrated in Figure 16; we

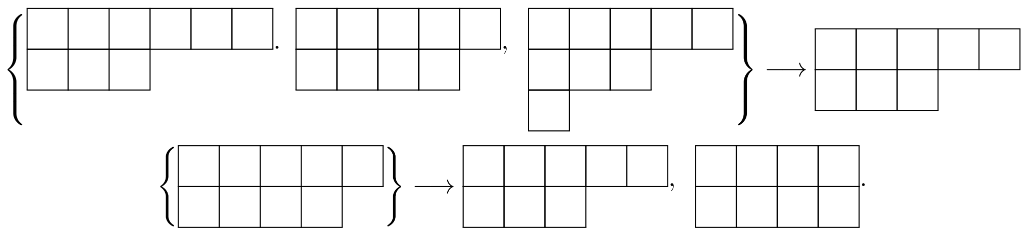


Figure 16. Illustration of the partitions of 9 which can, via row removal, yield $(n, n-2)$ alone, or both $(n, n-2)$ and $(n-1, n-1)$.

have already seen that $D^{(n,n-1)}$ does not have irreducible restriction, so we are left with $(n+1, n-2)$ and $\varsigma = (n, n-2, 1)$. We may directly check that ς doesn't satisfy this, as we have the following:

$$\begin{aligned}\beta_\varsigma(1,2) &= 3-2+(n-2) = n-1\\ \beta_\varsigma(1,3) &= 3-1+n = n+2\\ \beta_\varsigma(2,3) &= 2-1+3 = 4.\end{aligned}$$

At least one of $\beta_\varsigma(1,2)$ and $\beta_\varsigma(1,3)$ is nonzero, since $\beta_\varsigma(1,3)-\beta_\varsigma(1,2) = 3 \not\equiv 0 \pmod e$, and hence at least one of M_2 and M_3 is empty. Hence at least one of 2 or 3 is normal in ς, and $\lambda = (n+1, n-2)$.

For μ, we immediately see from Figure 16 that the only option is $(n, n-1)$.

We can perform a similar argument for the V^{2n} case, finding now that

$$\text{Res } D^{\lambda'} \simeq D^{(n,n-1)} \simeq \text{Res } D^{(n,n)}$$

and

$$\text{Res } D^{\mu'} \simeq D^{(n,n-1)} \oplus D^{(n+1,n-2)} \simeq \text{Res } D^{(n+1,n-1)}.$$

Through a similar process, we see that $\mu' = (n+1, n-1)$. We narrow down λ' to one of (n,n) or $\varpi := (n, n-1, 1)$, and note that

$$\begin{aligned}\beta_\varpi(1,2) &= 3-2+(n-1) = n\\ \beta_\varpi(1,3) &= 3-1+n = n+2\\ \beta_\varpi(2,3) &= 2-1+2 = 3\end{aligned}$$

and hence at least one of 2 or 3 is normal, Res D^ϖ is not irreducible, and $\lambda' = (n,n)$, finishing our proof. $\square$

Corollary 4.6. *We have the following isomorphisms of representations:*

$$\begin{aligned}V^{2n} &\simeq 3D^{(n+1,n-1)} \oplus 2D^{(n,n)}\\ V^{2n-1} &\simeq 3D^{(n,n-1)} \oplus 2D^{(n+1,n-2)}\end{aligned}$$

5. Conjecture

Recall that $K^r_{2n+r} := K$ is the direct sum of all copies of the sign representation in M. Hence the following characterises sign subrepresentations of M completely:

Proposition 5.1. *$K \subset M^r_{2n+r}$ is trivial when $e \neq n+r+1$, and* $\dim K = 1$ *when $e = n+r+1$.* $\square$ *We know this is not accurate anymore, right?*

Proposition 5.2. *Suppose $e < n+r+1$, and suppose n' is such that $e = n'+r+1$. Note that $h := (1+T_1)(1+T_3)\dots(1+T_{n-n'})$ maps M^r_{2n+r} onto $M^r_{2n'+r}$. Then, the preimage $h^{-1}(K^r_{2n+r})$ is a subrepresentation of M^r_{2n+r}, and the series*

$$0 \subset h^{-1}(K^r_{2n+r}) \subset M^r_{2n+r}$$

is a composition series of M^r_{2n+r}. $\square$

Proposition 5.3. *Denote the composition factor* $M^r_{2n+r}/h^{-1}(K^r_{2n+r})$ *by* U^r_{2n+r}. *Then, there exist some naturals* m, s *satisfying* $2m+s = 2n+r$ *and* $m+s > n+r$ *such that the following is an isomorphism of* $\mathscr{H}$*-modules*

$$h^{-1}\left(K^r_{2n+r}\right) \simeq U^s_{2m+s}$$

. □

Proposition 5.4. *For the same* m, s *as above, we have the following composition series of Specht modules:*

$$0 \longrightarrow D^{(m+s,m)} \longrightarrow S^{(n+r,n)} \longrightarrow D^{(n+r,n)} \longrightarrow 0.$$

Proposition 5.5. $M^r_{2n+r} \simeq S^{(n+r,n)}$ *and* $U^s_{2m+s} \simeq D^{(m+s,m)}$.

Appendix A. Compatibility of Representations with the Relations

In general, we defined the representations $V := V^{2n+r}$ and $M := M_{2n+r}$ for the free algebra on generators $\{T_1, \dots, T_{2n+r-1}\}$. Recall that we may give a presentation of $\mathscr{H}$ having generators T_i and relations

$$(T_i - q)(T_i + 1) = 0 \tag{A.1}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{A.2}$$

$$T_i T_j = T_j T_i \qquad |i - j| > 1. \tag{A.3}$$

We call (A.1) the *quadratic relation* and (A.2), (A.3) the *braid relations*. It is easily seen that a representation of $\mathscr{H}$ is equivalent to a representation of the free algebra $k\langle T_i \rangle$ which are compatible with the relations. We will prove in the following sections that V and M are compatible with the Hecke algebra relations.

A.1. Explicit Definition of M.

We will give a more careful definition of the crossingless matchings representation here.

Definition A.1. A *crossingless matching on $2n + r$ indices with r anchors* is a partition of $\{1, \dots, 2n + r\}$ into n parts of size 2 and r of size 1 such that no two parts (a, a') and (b, b') satisfy $a < b < a' < b'$, and no parts $(c), (a, a')$ satisfy $a < c < a'$. We will call these arcs and anchors, respectively. Then, define M^r_{2n+r} to be the k-vector space with basis the set of crossingless matchings on $2n + r$ indices with r anchors.

In order to endow M^r_{2n+r} with an $\mathscr{H}$-action, consider some basis element w_j and some element $(1+T_i)$ of $\mathscr{H}$. The elements $\{1\} \cup \{1 + T_i | 1 \le i < 2n + r\}$ generate $\mathscr{H}$, so it is sufficient to define the action of $1 + T_i$ on w_j.

If w_j has arc $(i, i+1)$, define $(1+T_i)w_j := (1+q)w_j$. If w_j has anchors $W(i) = i$ and $W(i+1) = i+1$, define $(1 + T_i)w_j := 0$. If w_j has anchor $W(i) = i$ and arc $W(i + 1) = b$, define $(1 + T_i)w_j := q^{1/2}w_l$, where $w_l(i) = i + 1$, $w_l(b) = b$, and all other arcs agree with w_j. If w_j has arcs $W(i) = a$ and $W(i + 1) = b$, then define $(1 + T_i)w_j := q^{1/2}w_l$, where $w_l(i) = i + 1$,$w(a) = b$, and all other acts agree with w_j. We verify that this is well-defined in Section A.2.

We may alternately sharpen our topological definition;

Definition A.2. Fix $2n + r$ distinct points $a_1, \dots, a_{2n+r}$ points along $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ and r distinct points $b_1, \dots, b_r$ along $\mathbb{R} \times \{1\}$. Then, define M^r_{2n+r} to have basis given by the isotopy classes of $n + r$ paths connecting the points $a_1, \dots, a_{2n+r}, b_1, \dots, b_r$ such that no distinct b_i, b_j are connected by a path.

We will take some basis element $w_j \in M^r_{2n+r}$ and define the action $(1+T_i)w_j$. To do so, map w_j through the natural embedding $\mathbb{R} \times [0, 1] \hookrightarrow \mathbb{R} \times \left[\frac{1}{2}, 1\right]$, and form the figure w^i_j by adjoining the lines connecting a_l and $a_l + \left(0, \frac{1}{2}\right)$ for all $l \ne i, i + 1$ as well as paths from a_i to a_{i+1} and $a_i + \left(0, \frac{1}{2}\right)$ to $a_{i+1} + \left(0, \frac{1}{2}\right)$. This has either 0 or 1 path components which do not intersect $\mathbb{R} \times \{0, 1\}$; these form "loops."

Take the figure $\tilde{w}^i_j$ without this component. If $\tilde{w}^i_j$ is not isotopic to some w_l, then define $(1+T_i)w_j := 0$. If $\tilde{w}^i_j$ is isotopic to some w_l, define $(1+T_i)w_j := (1+q)w_l$ if w^i_j has a loop and $(1+T_i)w_j := q^{1/2}w_l$ otherwise. This process is illustrated in Figure 2.

Let the length of an arc (i, j) be $l(i, j) := j - i + 1$. Note that the crossingless matchings on $2n$ indices with no anchors can all be identified with a list of n integers describing the lengths of the arcs from left to right; This basis is illustrated for M^1_5 in Figure 1.

Remark. This definition gives a graphical calculus for working with our module. It should be clear that, if w^i_j has a loop then $w_l(i) = i + 1$ and $w_l = w_j$. Further, this easily defines an arbitrary composition:

$$(1 + T_{i_1}) \cdots (1 + T_{i_\ell})w_j = q^{(\ell-t)\frac{1}{2}}(1 + q)^t w_l$$

if the figure we make via $(1 + T_{i_1}) \cdots (1 + T_{i_\ell})$ is isotopic to w_l after removing t loops.

Note that the crossingless matchings on $2n$ indices with no anchors can all be identified with a list of n integers describing the lengths of the arcs from left to right; call this the *increasing lexicographical order basis*. Further, we have a surjection $M^0_{2n+2r} \twoheadrightarrow M^r_{2n+r}$ which takes basis elements to basis elements; this induces an order on the basis for M^r_{2n+r}, which we will henceforth refer to as the *induced lexicographical order basis.* This basis is illustrated for M^1_5 in Figure 1.

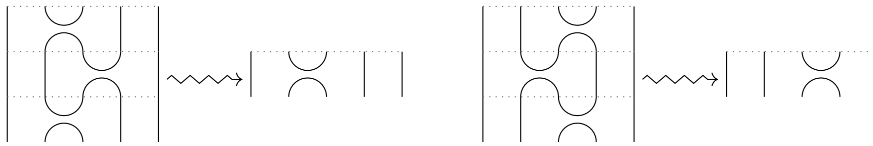


Figure 17. The above give visual intuition for isotopies giving rise to equalities between $(1 + T_i)(1 + T_{i-1})(1 + T_i)w_j$ and $q(1 + T_i)$, and between $(1 + T_{i+1})(1 + T_i)(1 + T_{i+1})w_j$ and $q(1 + T_i)$.

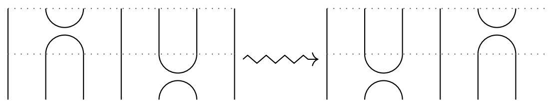


Figure 18. The above give visual intuition for isotopies giving rise to equalities between $(1 + T_i)(1 + T_j)w_l$ and $(1 + T_j)(1 + T_i)w_l$.

A.2. **Compatibility for the Crossingless Matchings Representations.** We verify the relations on the crossingless matchings representation M. Take some basis vector $w_i \in M$. We will first check (A.1) by case work:

- Suppose there is an arc $(i, i + 1)$. Then,
$$(T_i - q)(T_i + 1)w = (1 + q)\left((1 + T_i)w - (1 + q)w\right) = 0,$$
giving (A.1).
- Suppose there is no arc $(i, i+1)$ and indices $i, i+1$ do not both have anchors; then $(T_i+1)w = q^{1/2}w'$ for some basis vector w' having arc $(i, i + 1)$, and the computation follows as above for (A.1).
- Suppose $i, i + 1$ are anchors; then $(T_i + 1)w = 0$, giving (A.1).

Now we verify (A.2). Let $h := (1 + T_i)(1 + T_{i+1})(1 + T_i)$, and let $g := (1 + T_{i+1})(1 + T_i)(1 + T_{i+1})$. Note the following expansion:
$$\begin{aligned} hw &= 1 + 2T_i + T_i^2 + T_{i+1} + T_iT_{i+1} + T_{i+1}T_i + T_iT_{i+1}T_i \\ &= 1 + (1 + q)T_i + T_{i+1} + T_iT_{i+1} + T_{i+1}T_i + T_iT_{i+1}T_i. \end{aligned}$$
This equality, with i and $i + 1$ interchanged, holds for g. Hence we have
$$(h - g)w = q(T_i - T_{i+1}) + T_iT_{i+1}T_i - T_{i+1}T_iT_{i+1}.$$
Hence we may equivalently check that $(h-g)w = q(T_i -T_{i+1})$. In fact, $hw = q(1+T_i)$ and $gw = q(1+T_{i+1})$ by Figure 17, giving compatibility.

Lastly, we have the equation
$$(1 + T_i)(1 + T_j) - (1 + T_j)(1 + T_i) = T_iT_j - T_jT_i$$
and hence we simply need to verify that $(1 + T_i)$ and $(1 + T_j)$ commute, as shown in Figure 18

A.3. **Compatibility for the Fibonacci Representations.** We verify the relations on the Fibonacci representation V. Note that (A.3) follows easily from the "local" nature of V, and the others may be verified explicitly on strings of length 3 and 4. By considering the coefficients in order of (1.1), the quadratic relation (A.1) gives the following polynomials in q:
$$\begin{aligned} (\alpha_1 - q)(\alpha_1 + 1) &= 0, \\ (\alpha_2 - q)(\alpha_2 + 1) &= 0, \\ \varepsilon_1\delta + \delta\varepsilon_2 &= (q - 1)\delta, \\ \varepsilon_1^2 + \delta^2 &= (q - 1)\varepsilon_1 + q, \\ \varepsilon_2^2 + \delta^2 &= (q - 1)\varepsilon_2 + q \end{aligned} \tag{A.4}$$

The first two of these are easily verified. Since $\delta \neq 0$, the third is equivalently given by

$$(q-1) = \varepsilon_1 + \varepsilon_2 = \tau(q\tau - 1 + q - \tau) = (\tau^2 + \tau)(q-1)$$

or that $\left(\tau^2 + \tau - 1\right)(q-1) = 0$. One may verify that

$$\tau^2 + \tau - 1 = q^6 + 2q^5 + q^4 + q^3 + q^2 - 1 = (-1 + q + q^2)\,[5]_q = 0.$$

The fourth is given by the quadratic

$$\tau^2\left[(q\tau - 1)^2 - \tau(q+1)\right] = \tau(q-1)(q\tau - 1) + q$$

or equivalently,

$$(\tau^2 + \tau - 1)\left[q\left(qt^2 + 1\right) + t\right] = 0.$$

The fifth is similarly given by

$$(\tau^2 + \tau - 1)\left[q\left(qt + 1\right) + t^2\right] = 0.$$

All of these vanish for $e = 5$, giving compatibility with (A.1).

We now verify (A.2). We may order the basis for V^4 as follows:

$$\{(\mathtt{0000}), (\mathtt{*00*}), (\mathtt{000*}), (\mathtt{*000}), (\mathtt{*0*0}), (\mathtt{0*0*}), (\mathtt{00*0}), (\mathtt{0*00})\}.$$

Then, in verifying the braid relation (A.2) in this order, we encounter the following quadratics (with tautologies and repetitions omitted):

$$\begin{gathered}
\alpha_1\varepsilon_2^2 + \alpha_2\delta^2 = \alpha_1^2\varepsilon_2 \\
\alpha_1\delta\varepsilon_2 + \alpha_2\varepsilon_1\delta = \alpha_1\alpha_2\delta \\
\alpha_2\varepsilon_1^2 + \alpha_1\delta^2 = \alpha_2^2\varepsilon_1 \\
\alpha_1\varepsilon_1^2 + \delta^2\varepsilon_2 = \alpha_1^2\varepsilon_1 \\
\delta\varepsilon_2^2 + \alpha_1\varepsilon_1\delta = \alpha_1\delta\varepsilon_2
\end{gathered}$$

Substituting in τ and dividing by δ whenever possible, these are equivalent to the vanishing of the following polynomials in q:

$$\begin{aligned}
-q(1+q)(1+q^2+q^3)(2+q+3q^2+2q^3)\,[5]_q &= 0 \\
(1+2q+q^3+q^4)\,[5]_q &= 0 \\
(1+q)^2(1+q^2+q^3)(1+3q^3-q^4+q^6)\,[5]_q &= 0 \\
(1+q)^2(1+q^2+q^3)(1+5q+5q^2+3q^3+3q^4+3q^5+q^6)\,[5]_q &= 0 \\
(1+q)(1+q^2+q^3)(-1+2q+q^2+q^3+q^4)\,[5]_q &= 0.
\end{aligned}$$

Notably, each of these vanish when $e = 5$, giving compatibility with (A.2).

Appendix B. Restrictions to Conjugate Subalgebras

Throughout the text, for some representation V, we refer to $\operatorname{Res}^{\mathscr{H}(S_m)}_{\mathscr{H}(S_l)} V$ without specifying exactly which subalgebra $\mathscr{H}(S_l)$. For instance, in section 4, we explicitly state that the subrepresentations $V_{*0} \oplus V_{**}$ and V_{00} are isomorphic because they both may be characterized by such a restriction. We will verify that this is justified, using a more general fact about resctrictions to conjugate subalgebras.

Proposition B.1. *Suppose B, B' are subalgebras of a k-algebra A with $B = uB'u^{-1}$ for some unit $u \in A^\times$, and let V be a left A-module. Let $\phi : V \to V$ be the linear automorphism specified by $v \mapsto uv$. Then, the following commutes for any $b \in B$:*

$$\begin{tikzcd}
V \arrow[r, "\phi"] \arrow[d, "b"] & V \arrow[d, "ubu^{-1}"] \\
V \arrow[r, "\phi"] & V
\end{tikzcd}$$

Hence, through the identification of B and B' via conjugation by u, we have $\operatorname{Res}^A_B V \simeq \operatorname{Res}^A_{B'} V$

Proof. It suffices to note that $(ubu^{-1})uv = ubv$. □

Corollary B.2. *Suppose $\mathscr{H}', \mathscr{H}''$ are two subalgebras of $\mathscr{H}(S_m)$ generated by l simple reflections and V is a representation of $\mathscr{H}$. Then,* $\operatorname{Res}^{\mathscr{H}}_{\mathscr{H}'} V \simeq \operatorname{Res}^{\mathscr{H}}_{\mathscr{H}''} V$.

Proof. Let $\mathscr{H}'$ and $\mathscr{H}''$ be the subalgebras of $\mathscr{H}(S_m)$ generated by the reflections $\{T_{i_1}, \dots, T_{i_l}\}$ and $\{T_{i_1}, \dots, T_{i_{j-1}}, T_{i_j+1}, T_{i_{j+1}}, \dots, T_{i_l}\}$ for $1 \le i_1 < \cdots < i_{j-1} < i_j + 1 < i_{j+1} < \cdots < i_l \le n$. It is sufficient to prove that $\mathscr{H}'$ and $\mathscr{H}''$ are conjugate; then transitivity gives conjugacy of any $S_l \subset S_m$, and the previous proposition gives isomorphisms of the representations.

We will show that $\mathscr{H}'' = T_{i_j} \mathscr{H}' T_{i_j}^{-1}$. It suffices to show that $T_{i_j} T_w T_{i_j}^{-1} \in \mathscr{H}''$ for w a word generated by simple transpositions $s_{i_1}, \dots, s_{i_l} \in S_m$. First, note that $l(w) < l(s_{i_j} w)$, implying $T_{i_j} T_w = T_{s_{i_j} w}$ by lemma 1.12 in [**?**]. Further, by the same lemma, we have

$$\begin{aligned}
T_{s_{i_j} w} T_{i_j}^{-1} &= q^{-1} \left(T_{s_{i_j} w} T_{i_j} + (1-q) T_{s_{i_j} w} \right) \\
&= q^{-1} \left(T_{q s_{i_j} w s_{i_j}} + (q-1) T_{s_{i_j} w} + (1-q) T_{s_{i_j} w} \right) \\
&= T_{s_{i_j} w s_{i_j}}
\end{aligned}$$

which is in $\mathscr{H}''$. □

$$\varphi_6^0, \varphi_5^1 = \begin{bmatrix} 0 & 0 & -q^{3/2}+1 & 0 & 0 \\ 0 & -q^{3/2}+1 & [4]_{q^{1/2}} & 0 & 0 \\ 0 & 0 & [4]_{q^{1/2}} & 0 & -q^{3/2}+1 \\ -[4]_{q^{1/2}} & [4]_{q^{1/2}} & q^{1/2}\left(q^{1/2}+1\right) & 0 & [4]_{q^{1/2}} \\ [4]_{q^{1/2}} & 0 & [4]_{q^{1/2}} & -[4]_{q^{1/2}} & 0 \end{bmatrix}$$

$$\varphi_6^2 = \begin{bmatrix} 0 & 0 & -q-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q-1 & -q^{1/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & q+1 & 0 & q^{1/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{1/2} & 0 & 0 & 0 & -q^2-1 & 0 \\ q^{1/2} & -q^{1/2} & -[3]_{q^{1/2}} & 0 & 0 & 0 & -q^{1/2} & 0 \\ -q^{1/2} & q^{1/2} & 0 & [3]_{q^{1/2}} & 0 & 0 & 0 & q^{1/2} \\ -q^{1/2} & 0 & -q^{1/2} & 0 & q^{1/2} & 0 & 0 & 0 \\ q^{1/2} & 0 & 0 & 0 & -q^{1/2} & -[3]_{q^{1/2}} & 0 & -q^{1/2} \end{bmatrix}$$

$$\varphi_5^3 = \begin{bmatrix} -q^{1/2}\left(q^{1/2}+1\right) & [4]_{q^{1/2}} & q^{1/2} \\ q^{1/2} & q^{3/2} & 0 \\ 0 & -q-1 & \end{bmatrix} \qquad \varphi_4^0 = \begin{bmatrix} 0 & -[4]_{q^{1/2}} \\ -1 & 1 \end{bmatrix}$$

$$\varphi_3^1 = \begin{bmatrix} 0 & -[4]_{q^{1/2}} \\ q^{1/2}\left(q^{1/2}+1\right) & -q^{1/2}\left(q^{1/2}+1\right) \end{bmatrix} \qquad \varphi_4^2 = \begin{bmatrix} 0 & [4]_{q^{1/2}} & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Figure 19. We define representations $\varphi^r_{2n+r} : M^r_{2n+r}/K^r_{2n+r} \to V_s^{2n+r}$, where $s = \texttt{**}$ if $r = 0, 3$ and $s = \texttt{*0}$ otherwise. These are given with respect to the basis on M^r_{2n+r} induced by the increasing lexicographic order basis on M^0_{2n+2r} and the quotient $M^0_{2n+2r} \twoheadrightarrow M^r_{2n+r}$. These are also given with respect to the basis on V given by increasing lexicographic order $\texttt{*} < \texttt{0}$.

Appendix C. Data

In this section we aim both to support conjecture with data and to provide transition matrices wherever possible between our two graphical representations.[**?**]

Remark. These results are restricted by two things; first and with less recourse, this was computed in Magma, a closed source language with bugs that could not be resolved in certain cases. Second, the memory necessary to store the Hecke algebra via basis elements and structure constants grows with $(n!)^3$; this is prohibitively large when $m > 7$. An implementation as a quotient by a free algebra or only specifying multiplication

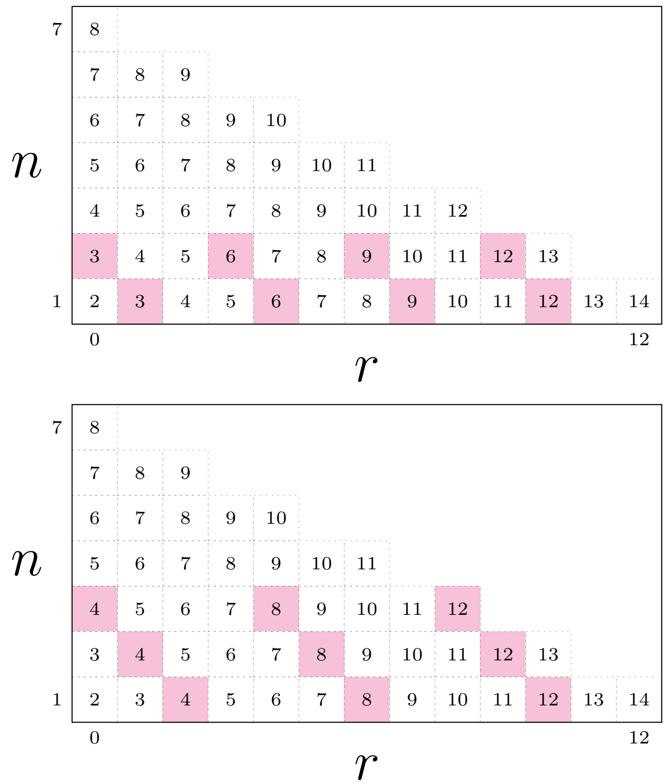


Figure 20. Illustration of the modules M^r_{2n+r} having sign submodules for $p = \infty$ and $e = 3, 4$ respectively. The value $n + r + 1$ is filled in the squares, and modules having sign submodules are colored magenta. For $p = \infty$ and $2n + r \leq 14$, it has been verified, through a combination of theorems here and empirical computations, that K^r_{2n+r} is nontrivial if and only if $e | n + r + 1$ and $e < n$.

by generators may fix this, however, this is not possible with the Magma language, and the closed source renders modification of the language impossible.

We give in Figure 19 some isomorphisms between M/K and V for $e \geq n + r + 1$ with $p = \infty$; all but one of these are cases with $e > n + r + 1$, so $K = 0$ and this is an isomorphism with our crossingless matchings representation. The case $n = 1$, $r = 3$ gives an example of an isomorphism not proven in general, but proven via our computation. All of these computations are done for q a primitive 5th root of unity in the algebraic extension of the Cyclotomic field $\mathbb{Q}(\zeta_{10})$ by a root of the polynomial $x^2 - \tau$.

We give in Figure 20 some data supporting a conjecture concerning sign subrepresentations of M^r_{2n+r}. The computations to support this were done over $\mathbb{C}$ with q a primitive 5th root of unity.

It is known that, for small $2n + r$, each Specht module $S^{(n+r,n)}$ has a composition series of length 2.[**?**] We give in Figures 21 through 23 the map $\iota^r_{e,2n+r} : U^r_{2n+r} \hookrightarrow M^r_{2n+r} \twoheadrightarrow U^r_{2n+r}$, which conjecturally illustrates the inclusion of the first composition factor of $S^{(n+r,n)}$ into $S^{(n+r,n)}$ for all $2n + r \leq 7$.

$$\iota^1_{3,3} = \iota^0_{3,4} = \begin{bmatrix} 1 & 1 \end{bmatrix}^\intercal$$

$$\iota^1_{3,5} = \iota^0_{3,6} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}^\intercal$$

$$\iota^4_{3,6} = \begin{bmatrix} -1 & -1 & 0 & 1 & 1 \end{bmatrix}^\intercal$$

$$\iota^3_{3,7} = \begin{bmatrix} 1 & 0 & -1 & -1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^\intercal$$

$$\iota^1_{3,7} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}^\intercal$$

Figure 21. The maps of representations $\iota^r_{e,2n+r} : U^r_{2n+r} \hookrightarrow M^r_{2n+r}$ for $e = 3$.

$$\iota^2_{4,4} = \begin{bmatrix} 1 & \alpha & 1 \end{bmatrix}^\intercal$$

$$\iota^1_{4,4} = \begin{bmatrix} 1 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & \frac{1}{2}\alpha \end{bmatrix}^\intercal$$

$$\iota^2_{4,6} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & -1 & -\alpha & 0 \\
0 & 0 & 1 & 0 & 0 & \alpha & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & -\alpha & -1 \\
0 & 0 & 0 & 0 & 1 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & \frac{1}{2}\alpha
\end{bmatrix}^\intercal$$

$$\iota^0_{4,6} = \begin{bmatrix} \alpha & 1 & 1 & \alpha & 1 \end{bmatrix}^\intercal$$

$$\iota^1_{4,7} = \begin{bmatrix}
1 & 0 & 0 & 0 & \frac{1}{2}\alpha & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 1 & 1/2 & \alpha & \frac{1}{2} \\
0 & 1 & 0 & 0 & -1 & 0 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & 0 & \alpha & \frac{1}{2}\alpha & -1 & \frac{1}{2}\alpha \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & \alpha & 1 & 0 & -\alpha & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & \alpha & 0 & 0 & -1 & -\alpha & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 0 & 0 & 0 & \frac{1}{2}\alpha & -1 & \frac{1}{2}\alpha \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & \frac{1}{2}\alpha
\end{bmatrix}^\intercal$$

Figure 22. The maps of representations $\iota^r_{e,2n+r} : U^r_{2n+r} \hookrightarrow M^r_{2n+r}$ for $e = 4$.

$$\iota_{5,5}^{3} = \begin{bmatrix} 1 & \beta & \beta & 1 \end{bmatrix}^{\mathsf{T}}$$

$$\iota_{5,6}^{2} = \begin{bmatrix} \beta & \beta & 1 & 1 & \beta+1 & \beta & \beta & \beta & 1 \end{bmatrix}^{\mathsf{T}}$$

$$\iota_{5,7}^{3} = \begin{bmatrix} 1 & \beta & \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \beta & \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & \beta & 1 & 0 & 0 & 0 \\ \beta & \beta & 1 & 0 & 1 & \beta+1 & \beta & 0 & \beta & \beta & 0 & 1 & 0 & 0 \\ -\beta-1 & -\beta & -\beta & 0 & -\beta & -\beta-1 & -\beta-1 & 0 & -\beta-1 & -\beta-1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \beta & 0 & 0 & \beta & 0 & 0 & 0 & 1 \end{bmatrix}^{\mathsf{T}}$$

$$\iota_{5,7}^{1} = \begin{bmatrix} 1 & \alpha & \alpha & \alpha & -\beta & \alpha & -\beta & \alpha & -\beta & 1 & \alpha & \alpha & \alpha & -\beta \end{bmatrix}^{\mathsf{T}}$$

$$\iota_{6,6}^{4} = \begin{bmatrix} 1 & \gamma & 2 & \gamma & 1 \end{bmatrix}^{\mathsf{T}}$$

$$\iota_{6,7}^{3} = \begin{bmatrix} \gamma & 2 & \gamma & 1 & 1 & 2\gamma & 3 & \gamma & 2 & 2\gamma & 2 & 2 & \gamma & 1 \end{bmatrix}^{\mathsf{T}}$$

$$\iota_{6,7}^{5} = \begin{bmatrix} 1 & \delta & \varepsilon & \varepsilon & \delta & 1 \end{bmatrix}^{\mathsf{T}}$$

Figure 23. The maps of representations $\iota_{e,2n+r}^{r} : U_{2n+r}^{r} \hookrightarrow M_{2n+r}^{r}$ for $e = 5, 6$.