THE UNEVEN-HEIGHT TWO-COLUMN SPECHT MODULES OF THE HECKE ALGEBRA OF S_n

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1. Introduction

Let S_{2n+r} be the symmetric group on $2n+r$ indices with $2n+r \geq 2$, let $\mathscr{H} = \mathscr{H}_{k,q}(S_{2n+r})$ be the corresponding Hecke algebra over field k with parameter $q \in k^\times$, and let $\{T_i\}$ be the reflections generating $\mathscr{H}$. Let $[m]_q = 1 + q + \cdots + q^{m-1}$ be the q-number of m. Let e be the smallest positive integer such that $[e]_q = 0$, and set $e = \infty$ if no such integer exists. Either $q = 1$ and e is the characteristic of k, or $q \neq 1$ and q is a primitive eth root of unity.

Let $S^{(n+r,n)'}$ be the Specht module corresponding to the young diagram with two columns with height difference r. The purpose of this writing is to characterize this representation via an isomorphism with another representation of $\mathscr{H}$.

Definition 1.1. A *generalized crossingless matching* on $2n+r$ indices with r anchors is a partition of $\{1, \ldots, 2n+r\}$ into n parts of size 2 and r of size 1 such that no two parts of size two"cross", i.e. there are no parts (a, a') and (b, b') such that $a < b < a' < b'$, and no parts of size one are "inside" of a part of size two, i.e. there are no $c, (a, a')$ such that $a < c < a'$. We will call these arcs and anchors, respectively. Then, define W^r_{2n+r} to be the k-vector space with basis the set of generalized crossingless matchings on $2n+r$ indices with r anchors.

In order for this to be a $\mathscr{H}$-module, endow this with the action given by Figure 1; if this involves no anchors, act as in W^0_{2n}; if it involves one anchor, deform to another generalized crossingless matching and scale by $q^{1/2}$, and otherwise scale by 0.

Let the length of an arc (i, j) be $l(i, j) := j - i + 1$. Note that the crossingless matchings can all be identified with a list of n integers describing the lengths of the arcs from left to right; using this, we may order the crossingless matchings with 0 hooks in increasing lexicographical order in order to obtain an order on the subbasis containing a particular set of anchors; let the basis be ordered first by the position of the anchors in increasing lexicographical order, then increasing for the matchings between each anchor. Let this basis be $\{w_i\}$. This basis is illustrated for W^1_5 in Figure 2.

We will prove that $W := W^r_{2n+r}$ and $S := S^{(n+r,n)'}$ are isomorphic as representations in the case that $e > n + r + 1$ is semisimple.

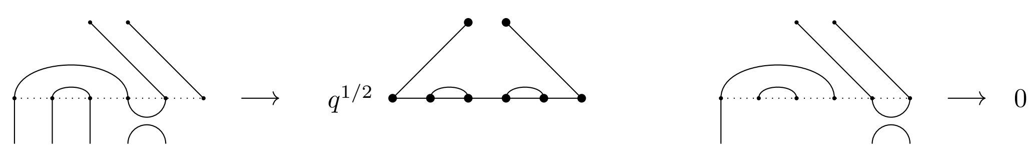

Figure 1. Illustration of the actions $(1+T_4)w_{|W^2_6|}$ and $(1+T_2)w_{|W^2_6|}$ in W^2_6. In general, we act on basis elements away from anchors as we did for W, at one anchor we act by deforming and scaling by $q^{1/2}$, and at two anchors we send the element to zero.

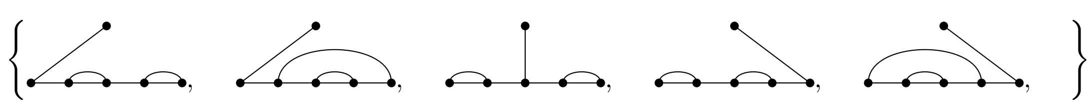

Figure 2. The basis for W^1_5.

2. Correspondence

Proposition 2.1. *Suppose that $n, r > 0$. Then, a filtration of $Res W^r_{2n+r}$ is given by*

$$0 \subset W^{r-1}_{2n+r-1} \subset Res\, W^r_{2n+r} \tag{1}$$

with $Res\, W^r_{2n+r}/W^{r-1}_{2n+r-1} \simeq W^{r+1}_{2n+r-1}$.

Proof. Note that we may identify the subrepresentation of Res W^r_{2n+r} having anchor n with W^{r-1}_{2n+r-1}.

Let $U :=$ Res $W^r_{2n+r}/W^{r-1}_{2n+r-1}$. Let $\phi : U \to W^{r+1}_{2n+r-1}$ be the k-linear map which regards the arc $(i, 2n+r)$ in U as an anchor at i in W^{r+1}_{2n+r-1}. It is not hard to verify that this is a well-defined isomorphism of vector spaces, so we must show that it is $\mathscr{H}$-linear.

Given a basis vector w_j with arc $(i, 2n+r)$, ϕ is clearly compatible with $T_{i'}$ with $i' \neq i, i-1$. Further, it's easy to verify that ϕ is compatible with T_i and T_{i-1}, as actions on one anchor were designed for this deformation. When there are anchors $(i, i+1)$, then $\phi(T_i w_j) = T_i \phi(w_j) = 0$, and similar for T_{i-1}. Hence ϕ is an isomorphism of representations, and the statement is proven. □

Lemma 2.2. *Every basis vector in W^r_{2n+r} is cyclic.*

Proof. We have already proven this in the $r = 0$ case, so suppose that $r > 0$.

Note that, between anchors $a < a'$ having no arc b with $a < b < a'$, the $W^0_{a'-a}$ case allows us to generate the vector with all length-2 arcs between a, a' and identical arcs/anchors outside of this sub-matching.[1]

Applying this between each arc gives us a vector with length-2 arcs and anchors, and we may use the appropriate $(1 + T_i)$ to move anchors to any positions, and the reverse process from above to generate the correct matchings between arcs and generate any other basis vector. □

Proposition 2.3. *The representation W^r_{2n+r} is irreducible when $e > n + r + 1$.*

Proof. We proceed by induction on $2n + r$. Note that, by identification with the trivial and sign representations, the base case $2n + r = 2$ is already prove, so suppose we have proven this for each $2m + s < 2n + r$.

We will prove this in essentially the same way as before; the inductive argument continues to apply assuming that $K := \ker \bigoplus(1 + T_i)$ is trivial, and we will prove that this is trivial when $e \not| n + r + 1$ using a similar argument. This argument is very long, and I will recreate it later. □

Corollary 2.4. *Other than W^1_3, the representation W^r_{2n+r} is irreducible when $e > n + 1$.*

The next piece in our puzzle is to characterize the restrictions of W to $\mathscr{H}' := \mathscr{H}_{k,q}(S_{2n+r-1}) \subset \mathscr{H}$. Recall that, when $r, n > 0$ and $\mathscr{H}$ is semisimple, $\mathrm{Res} S^{(n+r,n)'} \simeq S^{(n+r-1,n)'} \oplus S^{(n+r,n-1)'}$. Further, note that $S^{(n+r,n)'}$ is the unique irreducible having this restriction.

Next, note that we have already proven the correspondence for W^0_{2n}; for W^r_{0+r}, this is the sign representation, which is given correctly by $S^{(r)}$. Hence, pending information on restrictions, we may prove this via induction on $2n + r$.

Corollary 2.5. *Suppose $n, r > 0$ and $e > n + 1$. Then, the sequence* (1) *is a composition series of $Res\, W^r_{2n+r}$.* □

Corollary 2.6. *Suppose $n, r > 0$ and $\mathscr{H}$ is semisimple. Then, Res $W^r_{2n+r} \simeq W^{r-1}_{2n+r-1} \oplus W^{r+1}_{2n+r-1}$.* □

Proposition 2.7. *Suppose $e > n + r + 1$, and let λ be a partition of $2n + r$.*

- *(i) Suppose $n, r > 0$. If $Res\, S^\lambda$ admits a filtration $0 \subset S^{(n+r-1,n)'} \subset Res\, S^\lambda$ with $Res\, S^\lambda / S^{(n+r-1,n)'} \simeq S^{(n+r,n-1)'}$ then $\lambda = (n + r, n)'$.*
- *(ii) Suppose $r = 0$. If $Res\, S^\lambda \simeq S^{(n,n-1)'}$, then $\lambda = (n, n)'$.*

Proof. **(i)** Recall that the characteristic-free classical branching theorem gives that every specht module S^λ admits a filtration

$$0 = M_0 \subset \cdots \subset M_l = S^\lambda \tag{2}$$

with $M_i/M_{i-1} \simeq S^{\lambda(i)}$. Hence $\lambda = (n + r, n)'$ satisfies the above formula.

[1] At the ends, we apply the W^0_a case or the W^0_{2n+r-a} case in the same way for the first a or last $2n + r - a$ indices.

Note that, when $e > n + r + 1$, this filtration gives a composition series. Hence the sequence (2) is at most length-2.

Suppose it is length-1. Then λ is rectangle-shaped; to have rows removed to the above we then need that that it be $\lambda = \left(\frac{2n+r}{2}, \frac{2n+r}{2}\right)'$, so that $\mathrm{Res} S^{\lambda} \simeq S^{\left(\frac{2n+r}{2}, \frac{2n+r}{2}-1\right)'}$. However, since $e > n+r+1 \geq n+\frac{r}{2}+1$, $\mathrm{Res}\, S^{\lambda}$ is irreducible, contradicting the existence of a length-2 composition series in the hypothesis.

Now suppose that (2) is length 2, so that it is also a composition series. Then, by the Jordan-Hölder theorem, there is a rearranement of the composition factors of 2 to give $S^{(n+r-1,n)'}$ and $S^{(n+r,n-1)'}$. This implies that both $(n+r-1, n)'$ and $(n+r, n-1)'$ can be acquired by removing rows of λ, giving that $\lambda = (n+r, n)'$.

(ii) Note that, since $e > n+1$, $\mathrm{Res} S^{\lambda}$ is irreducible. Hence the sequence (2) is a composition series of length 1, and $\lambda(1) = (n, n-1)'$. The first implies that λ is rectangular, and the second implies that $\lambda = (n, n)'$. $\square$

Corollary 2.8. *If $e > n + r + 1$, then $W^r_{2n+r} \simeq S^{(n+r,n)'}$.* $\square$