SOME GRAPHICAL REALIZATIONS OF TWO-ROW SPECHT MODULES OF IWAHORI-HECKE ALGEBRAS OF THE SYMMETRIC GROUP

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Abstract. We consider the Iwahori–Hecke algebra of the symmetric group on $2n+r$ letters with parameter q. Let e be the smallest positive integer such that the q-number $[e]_q = 0$, or set $e = \infty$ if none exist. We modify Khovanov's crossingless matchings to include $2n$ "nodes" and r "anchors," and prove in the case $e > n + r + 1$ that the associated module is isomorphic to the Specht module $S^{(n+r,n)}$ which corresponds to the partition $(n + r, n) \vdash 2n + r$. We then give heuristics in support of the general case, including explicit composition series for $e = n + r + 1$ and for $2n + r \leq 7$. Lastly, when $e = 5$, we prove an isomorphism between the irreducible quotient $D^{(n+r,n)}$ with $r \leq 3$ and some subrepresentations of Jordan–Shor's Fibonacci representation. We provide explicit transition matrices between this representation and the crossingless matchings representation for $2n + r \leq 6$.

Contents

1. Introduction

Let S_{2n+r} be the symmetric group on $2n+r$ indices with $2n+r \geq 2$, let $\mathscr{H} := \mathscr{H}_{k,q}(S_{2n+r})$ be the corresponding Hecke algebra over field k with parameter $q \in k^\times$ having square root $q^{1/2}$, and let $\{T_i\}$ be the reflections generating $\mathscr{H}$.

Let $[m]_q = 1 + q + \cdots + q^{m-1}$ be the q-number of m. Let e be the smallest positive integer such that $[e]_q = 0$, and set $e = \infty$ if no such integer exists. Either $q = 1$ and e is the characteristic of k (with 0 replaced by ∞), or $q \neq 1$ and q is a primitive eth root of unity.

When $q = 1$, the Hecke algebra $\mathscr{H}$ is isomorphic to the group algebra $k[S_{2n+r}]$. For q a prime power, let $G := \mathrm{GL}_{2n+r}(q)$ be the general linear group over a field with q elements, let $B := B(q)$ be the Borel subgroup of G, and let 1_B be the trivial representation of $k[B]$. Then, it is a well known result that the endomorphism algebra $\mathrm{End}_{k[G]}\left(\mathrm{Ind}_B^G(1_B)\right)$ is isomorphic to the Hecke algebra $\mathscr{H}$.[8] Hence the representation theory of the Hecke algebra generalizes the representation theory of the symmetric group, and informs the representation theory of the general linear group.

It is a classical result that $\mathscr{H}$ is semisimple precisely when $e > 2n+r$, in which case the irreducible representations of $\mathscr{H}$ are given by *Specht modules* S^λ, which are indexed by the partitions λ of $2n+r$. Further, $\mathscr{H}$ admits a cellular basis with cell modules given by S^λ. In particular, these admit quotients D^λ such that the set $\left\{D^\lambda \mid D^\lambda \neq 0, \lambda \vdash n\right\}$ is a complete set of pairwise-nonisomorphic irreducible $\mathscr{H}$-modules. This set is indexed by the partitions $\lambda \vdash n$ which are e-restricted.[10,11]

These representations D^λ are well-defined, but many properties of them are unknown. For instance, the dimension of these modules is unknown outside of some special cases.[8] However, there does exists an algorithm due to Lascoux, Leclerc, and Thibon (verified by Ariki) which produces the decomposition matrices of the Iwahori-Hecke Algebra $\mathscr{H}_{\mathbb{C},q}(S_{2n+r})$ for q an eth root of unity.[1,7]

Further, there are theorems, called *modular branching theorems*, characterizing the socle and semisimplicity of the restriction $\mathrm{Res}D^\lambda$ to a subalgebra generated by $2n+r-2$ simple transpositions. These are originally due to Kleshchev in the group algebra case, and were generalized to the Hecke algebra by Brundan.[2,6]

These constructions tend to be well-defined but complicated and often intractable. We aim to give simple realizations of some particular S^λ and D^λ which expose this structure while being intuitive and simple to manipulate.

Throughout the text, we will refer to partitions of $2n+r$; identify each partition with a tuple $\lambda = (\lambda_1^{a_1}, \ldots, \lambda_l^{a_l})$ having $\lambda_i > \lambda_{i+1}$, $a_i > 0$, and $\sum_i a_i \lambda_i = 2n+r$. Identify each of these with a subset $[\lambda] \subset \mathbb{N}^2$ as is standard, and define $\lambda(i) = (\lambda_1^{a_1}, \ldots, \lambda_{i-1}^{a_{i-1}}, \lambda_i^{a_i - 1}, \lambda_i - 1, \lambda_{i+1}^{a_{i+1}}, \ldots, \lambda_l^{a_l})$ to be the partition with the ith row removed.[6]

Note that we follow the convention of Murphy and Kleshchev, which is dual to the conventions of Dipper, James, and Mathas; one may translate our results to the latter convention by transposing all partitions.[6,8,10]

Fixing some partition λ, for $1 \leq i \leq j \leq l$, let $\beta_\lambda(i,j)$ and γ_λ be the quantities

$$\beta_\lambda(i,j) = \lambda_i - \lambda_j + \sum_{t=i}^{j} a_t$$

$$\gamma_\lambda(i,j) = \lambda_i - \lambda_j + \sum_{t=i+1}^{j} a_t.$$

Results due to Kleshchev and Brundan refer to *normal* and *good* numbers; note that 1 is always normal, and that j is normal when $\beta_\lambda(i,j) \not\equiv 0 \pmod{e}$ for all $i \leq j$. Recall that j is good iff j is normal and $\gamma_\lambda(j,j') \not\equiv 0 \pmod{e}$ for all $j' \geq j$ normal.[2,6] These are the combinatorial properties necessary to state and use branching theorems.

Throughout this paper, we analyze the *two-row partitions* $(n+r,n) \vdash 2n+r$ and their corresponding modules $S^{(n+r,n)}$ and $D^{(n+r,n)}$.

Crossingless Matchings. The following definition modifies the crossingless matchings defined by Khovanov.[5]

Definition 1.1. A *crossingless matching on* $2n+r$ *indices with* r *anchors* is a partition of $\{1,\dots,2n+r\}$ into n parts of size 2 and r of size 1 such that no two parts of size two"cross", i.e. there are no parts (a,a') and (b,b') such that $a<b<a'<b'$, and no parts of size one are "inside" of a part of size two, i.e. there are no $c,(a,a')$ such that $a<c<a'$. We will call these arcs and anchors, respectively. Then, define M^r_{2n+r} to be the k-vector space with basis the set of generalized crossingless matchings on $2n+r$ indices with r anchors.

For convenience, given a basis element $w_j \in M^r_{2n+r}$ and integers $1 \le a,b \le 2n+r$, define the $w_j(a) := b$ if (a,b) is an arc in M, and $w_j(a) := a$ if a is an anchor in M.

In order to endow this with an $\mathscr{H}$-action, consider some basis element w_j and some element $(1+T_i)$ of $\mathscr{H}$. The elements $\{1\}\cup\{1+T_i|1\le i<2n+r\}$ generate $\mathscr{H}$, so it is sufficient to define the action of $1+T_i$ on w_j.

If w_j has arc $(i,i+1)$, define $(1+T_i)w_j := (1+q)w_j$. If w_j has anchors $W(i)=i$ and $W(i+1)=i+1$, define $(1+T_i)w_j := 0$. If w_j has anchor $W(i)=i$ and arc $W(i+1)=b$, define $(1+T_i)w_j := q^{1/2}w_l$, where $w_l(i)=i+1$, $w_l(b)=b$, and all other arcs agree with w_j. If w_j has arcs $W(i)=a$ and $W(i+1)=b$, then define $(1+T_i)w_j := q^{1/2}w_l$, where $w_l(i)=i+1$,$w(a)=b$, and all other acts agree with w_j. We verify that this is well-defined in appendix A.1.

We may alternately define M^r_{2n+r} topologically; fix $2n+r$ distinct points $a_1,\dots,a_{2n+r}$ points along $\mathbb{R}\times\{0\}\subset\mathbb{R}^2$ and r distinct points $b_1,\dots,b_r$ along $\mathbb{R}\times\{1\}$, and define M^r_{2n+r} to have basis given by the isotopy classes of $n+r$ paths connecting the points $a_1,\dots,a_{2n+r},b_1,\dots,b_r$ such that no distinct b_i,b_j are connected by a path.

We will take some basis element $w_j \in M^r_{2n+r}$ and define the action $(1+T_i)w_j$. To do so, map w_j through the natural embedding $\mathbb{R}\times[0,1]\hookrightarrow\mathbb{R}\times\left[\frac{1}{2},1\right]$, and form the figure w^i_j by adjoining the lines connecting a_l and $a_l+\left(0,\frac{1}{2}\right)$ for all $l\neq i,i+1$ as well as paths from a_i to a_{i+1} and $a_i+\left(0,\frac{1}{2}\right)$ to $a_{i+1}+\left(0,\frac{1}{2}\right)$. This has either 0 or 1 path components which do not intersect $\mathbb{R}\times\{0,1\}$; these form "loops."

Take the figure $\tilde{w}^i_j$ without this component. If $\tilde{w}^i_j$ is not isotopic to some w_l, then define $(1+T_i)w_j := 0$. If $\tilde{w}^i_j$ is isotopic to some w_l, define $(1+T_i)w_j := (1+q)w_l$ if w^i_j has a loop and $(1+T_i)w_j := q^{1/2}w_l$ otherwise. This process is illustrated in Figure 1.

Remark. This definition gives a graphical calculus for working with our module. It should be clear that, if w^i_j has a loop then $w_l(i)=i+1$ and $w_l=w_j$. Further, this easily defines an arbitrary composition:

$$(1+T_{i_1})\cdots(1+T_{i_\ell})w_j = q^{(\ell-t)\frac{1}{2}}(1+q)^t w_l$$

if the figure we make via $(1+T_{i_1})\cdots(1+T_{i_\ell})$ is isotopic to w_l after removing t loops.

Let the length of an arc (i,j) be $l(i,j) := j-i+1$. Note that the crossingless matchings on $2n$ indices with no anchors can all be identified with a list of n integers describing the lengths of the arcs from left to right; using this, we may order the crossingless matchings with no anchors in increasing lexicographical order in order to obtain an order on the sub-basis containing a particular set of anchors; let the basis be ordered first by the position of the anchors in decreasing lexicographical order, then increasing for the matchings between each anchor. Let this basis be $\{w_i\}$. This basis is illustrated for M^1_5 in Figure 2.

Note that the representations M^r_{0+r} and $S^{(r)}$ are isomorphic to the sign representation; we will prove that M^r_{2n+r} and $S^{(n+r,n)}$ are isomorphic as representations in the case that $e>n+r+1$. Additionally, we

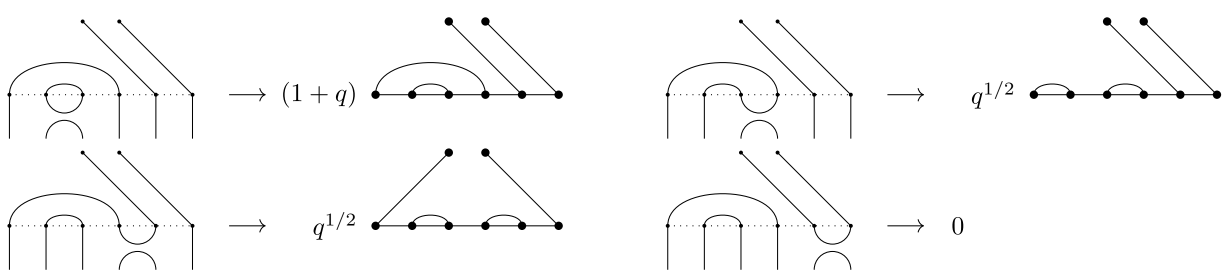

Figure 1. Illustration of the actions $(1+T_i)w_{|M^2_6|}$. In general, we act by deleting loops, isotoping onto a new crossingless matching, and scaling by either $q^{1/2}$, $(q+1)$, or 0.

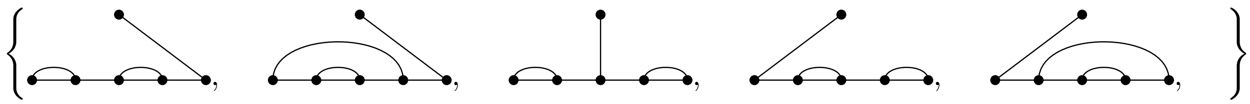

Figure 2. The basis for M_5^1.

will prove that M^r_{2n+r} is irreducible everywhere that $e > n$ and $S^{(n+r,n)}$ is irreducible. We will conjecture that $M^r_{2n+r} \simeq S^{(n+r,n)}$ with e unrestricted as well.

Fibonacci Representation. Now suppose that $e = 5$ and k contains the algebraic number $(-1-q^2-q^3)^{3/2}$. The following is a modification of Shor–Jordan's Fibonacci representation of the braid group.

Definition 1.2. Let V^m be a k-vector space with basis given by the strings $\{*, 0\}^{n+1}$ such that the bit $*$ never appears twice in a row. We will refer to this as the *Fibonacci representation* and suppress the superscript whenever it is clear from context.

We wish to endow this with a $\mathscr{H}$-action which acts on a basis vector only dependent on characters $i, i+1, i+2$, sending each basis vector to a combination of the other basis vectors having the same characters $1, \dots, i, i+2, \dots, n+1$ as follows:

$$
\begin{aligned}
T_1\ (*00) &:= \alpha_1\,(*00)\\
T_1\ (00*) &:= \alpha_1\,(00*)\\
T_1\ (*0*) &:= \alpha_2\,(*0*)\\
T_1\ (0*0) &:= \varepsilon_1\,(0*0) + \delta\,(000)\\
T_1\ (000) &:= \delta\,(0*0) + \varepsilon_2\,(000)
\end{aligned} \tag{1.1}
$$

for constants

$$
\begin{aligned}
\alpha_1 &= -1\\
\alpha_2 &= q\\
\varepsilon_1 &= \tau(q\tau - 1)\\
\delta &= \tau^{3/2}(q+1)\\
\varepsilon_2 &= \tau(q-\tau)\\
\tau &= -1 - q^2 - q^3
\end{aligned} \tag{1.2}
$$

with T_i acting similarly on the substring $i, i+1, i+2$. We will verify that this is a representation of $\mathscr{H}$ in Appendix A.2

This contains 4 subrepresentations based on the first and last character of the string, which are not modified by $\mathscr{H}$. Label the subrepresentation spanned by strings $(*\dots*)$ by V_{**}, and similar for the other 3. It is easy to see that $V_{*0} \simeq V_{0*}$, so that

$$V \simeq 2V_{*0} \oplus V_{**} \oplus V_{00}.$$

We will show that $V_{00} \simeq V_{*0} \oplus V_{**}$, and give the following isomorphisms with irreducible quotients of specht modules depending on the parity of the number of indices in $\mathscr{H}$:

$$
\begin{aligned}
V^{2n}_{**} &\simeq D^{(n,n)}\\
V^{2n-1}_{**} &\simeq D^{(n+1,n-2)}\\
V^{2n}_{*0} &\simeq D^{(n+1,n-1)}\\
V^{2n-1}_{*0} &\simeq D^{(n,n-1)}.
\end{aligned} \tag{1.3}
$$

Overview of Paper. In Section 2 we give corollaries to standard theorems concerning Specht modules. First, James-Mathas provides a sharp characterization of the irreducibility of e-regular, called the *Carter criterion.*[8] We specialize this to the case that $\lambda = (n+r, n)$ to give a tractable combinatorial expression for irreducibility of $S^{(n+r,n)}$. We note that this irreducibility depends only on e when $e > n$; otherwise it depends on the characteristic of the field. Next, we specialize theorems introduced by Kleshchev and generalized by Brundan describing $\operatorname{Hom}\left(W, \operatorname{Res} D^\lambda\right)$ where W is either S^μ or D^μ for μ a partition of $2n+r-1$ and our restriction is taken to the subalgebra generated $2n+r-2$ simple transpositions.[6] [2] We verify that this is unambiguous in Appendix B. These allow us to prove our first significant statement: if $S^\lambda \simeq D^\lambda$ and $e > n$, then a particular length-2 composition series uniquely determines λ; further, an irreducible restriction to $D^{(n,n-1)}$ determines λ as well.

In Section 3, we begin by attempting to prove that $M := M^r_{2n+r}$ is irreducible whenever $e > n + r + 1$. We do this by first proving that every vector in our basis for M is cyclic, then that M contains no sign subrepresentation. We then use these facts to inductively prove irreducibility. Following this, we prove a particular filtration with factors given by other crossingless matchings representations; this becomes a composition series, and an inductive argument combined with the branching of Section 2 allows us to prove $M \simeq S^{(n+r,n)}$. Last, we finish the section by proving more general statements concerning sign subrepresentations, which contribute to conjectures later in the paper.

In Section 4, we begin by establishing isomorphisms of the subrepresentations of V^m in the case that $m = 2$, as well as irreducibility of V^3_{*0}. We then use these cases to prove that V_{*0} is irreducibility, having as a corollary that V_{**} is irreducible. From this, we inductively prove isomorphisms between the representations V_{**}, V_{*0} and $D^{(n+r,n)}$ with r depending on the subscript of V and the parity of m.

Overview of Conjecture and Empirics goes here.

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2. Preliminaries on Specht Modules

For this section and the rest of the paper, assume $n > 0$.

In the following section, we cite a theorem of James-Mathas which precisely characterizes the irreducibility of S^λ in the case that λ is e-regular, and we specialize this result to the case of two-row specht modules. This falls into two cases: either $e > n$, where $S^{(n+r,n)}$ is irreducible iff $e \nmid r+2, \ldots, n+r+1$, or $e \le n$, where the irreducibility of $S^{(n+r,n)}$ is complicated and depends on the characteristic of k. We will focus primarily on the former case.

Following this, we cite the branching theorems of Kleshchev-Brundan, which allow us to fully characterize the Socle of $\operatorname{Res} S^\lambda$. This and some combinatorial arguments allow us to come to the main result of this section, which may loosely be stated as follows: the composition series of $D^{(n+r,n)}$ is restrictive enough in many cases with $S^{(n+r,n)}$ irreducible and $e > n$ case that such a composition series uniquely characterizes irreducibles. This will be immensely useful later for characterizing M via Specht modules, as it will include all cases with $e > n+r+1$.

2.1. Irreducibility of S^λ.

Let k have characteristic ℓ; then, set

$$p := \begin{cases} \ell & \ell > 0 \\ \infty & \ell = 0 \end{cases}.$$

Note that $p = e$ when $q = 1$. For h a natural number, let $\nu_p(h)$ be the p-adic evaluation of h. As a convention, set $\nu_\infty(h) = 0$ for all h. Define the function $\nu_{e,p} : \mathbb{N} \to \{-1\} \cup \mathbb{N}$ by

$$\nu_{e,p}(h) := \begin{cases} \nu_p(h) & e \mid h \\ -1 & e \nmid h \end{cases}.$$

Lastly, let h^λ_{ab} be the hook length of node (a,b) in $[\lambda]$. With this language, we may express the following theorem, part (ii) of which is named the *carter criterion*, due to James-Mathas.[8]

Theorem 2.1 (James-Mathas). *The following are equivalent:*

(i) $S^\lambda \simeq D^\lambda$.
(ii) λ *is e-regular and* S^λ *is irreducible.*
(iii) $\nu_{e,p}\left(h^\lambda_{ab}\right) = \nu_{e,p}\left(h^\lambda_{ac}\right)$ *for all nodes* (a,b) *and* (a,c) *in* $[\lambda]$. □

This result gives information solely on e-regular partitions, and the general irreducibility of S^λ away from $p=2$ is not well understood. We will henceforth specialize slightly to the case that $(n+r,n)$ is e-restricted.

Corollary 2.2. *If* $r=0$, *assume* $e>2$.
(i) Suppose $e>n$. *Then,* $S^{(n+r,n)}$ *is irreducible iff* $e \nmid l$ *for all* $r+2 \le l \le n+r+1$.
(ii) Suppose $e \le n$. *Then,* $S^{(n+r,n)}$ *is irreducible iff* $e|r+1$ *and* $\nu_{e,p}(h^\lambda_{ab})$ *acquires exactly two values.*

In the case that $\lambda = (n+r,n)$ and $e \nmid r+2, r+3, \ldots, n+r+1$, say that λ is *e-top-indivisible*. Since no multiples of e are in an interval of length n in $\mathbb{N}$, we have $e>n$, so this is precisely case (i) above.

Proof. The beginning condition implies that λ is e-regular, which we will use below.

(i) Note that $\nu_p(l) \neq -1$ for all l and only hook lengths in the top row may vanish mod e; hence we may equivalently prove that e divides no hook lengths in the leftmost n columns of the second column. These hook lengths are precisely $r+2, \ldots, n+r+1$.

(ii) Note that we have $\nu_{e,p}\left(h^\lambda_{n-e+1,2}\right) \neq -1$. Suppose that $e \nmid r+1$. Then,

$$\nu_{e,p}\left(h^\lambda_{n-e+1,1}\right) = \nu_{e,p}\left(h^\lambda_{n-e,2} + r + 1\right) = -1,$$

giving $S^{(n+r,n)}$ reducible.

Now, suppose that ν acquires at least three values, and

$$0 \le \nu_{e,p}(h^\lambda_{ab}) < \nu_{e,p}\left(h^\lambda_{a'b'}\right)$$

and $S^{(n+r,n)}$ is irreducible. If $b=b'$ then we have reducibility, so assume $b \neq b'$. Further, if $a=a'$, then we may replace a with the other column; hence we may assume WLOG that $a \neq a'$ as well.

Note that p-adic valuation is monotonic, so $h^\lambda_{ab} < h^\lambda_{a'b'}$. If (a,b) is in the rightmost r columns, then we may $0 \le \nu_{e,p}\left(h^\lambda_{a''b''}\right) \le \nu_{e,p}\left(h^\lambda_{ab}\right)$ for some (a'',b'') in the rightmost n columns; this has the same hook length as a cell in the leftmost n columns and bottom row, so we may assume (a,b) is not in the rightmost r columns. Similar logic allows us to assume (a',b') is in the leftmost n columns as well.

If $b<b'$, then $\nu_p(h_{ab}) = \nu_p(h_{ab'}) < \nu(h_{a'b'})$ while $h_{ab'} > h_{a'b'}$, a contradiction. If $b>b'$, then $\nu_p(h_{ab}) = \nu_p(h_{ab'})$ and $h_{ab} < h_{ab'}$, so there is some $c<b$ with $\nu_{e,b}(h_{ab}) = \nu_{e,p}(h_{ac})$; we may replace b with c, and repeat until $b=b'$ to reach contradiction.

Finally, assume $e|r+1$ and $\nu_{e,p}$ acquires two values. Since $e|r+1$, $e \mid h^\lambda_{a1}$ iff $e \mid h_{a2}$; since $\nu_{e,p}$ acquires only two values, this proves that $\nu_{e,p}(h_{a1}) = \nu_{e,p}(h_{a2})$ across these rows, giving the lemma. □

From condition (ii) we see that irreducibiliy at $e>n$ is not dependent on p, and we may cover many modular cases without reference to the characteristic of k. We will finish our discussion of irreducibilty of S^λ via the following special cases of (ii) above.

Corollary 2.3. *If* $r=0$, *assume* $e>2$. *Suppose* $e \le n$.
(i) Suppose $p > n+r+1$. *Then,* $S^{(n+r,n)}$ *is irreducible iff* $e|r+1$.
(ii) Suppose $p=2$. *Then,* $S^{(n+r,n)}$ *is not irreducible.*

Proof. (i) is clear. (ii) is given by noting that $e+r+1 \ge 2e$; then, $h_{n-e+1,1} - h_{n-e+1,2} \ge 1$, giving reducibility. □

2.2. **Branching.** In this section as well as later sections, we will consider the restruction of representations of $\mathscr{H}$ to particular subalgebras isomorphic to $\mathscr{H}_{k,q}(S_{2n+r-1}$. We verify in Appendix B that any two subalgebras of $\mathscr{H}$ generated by $2n+r-2$ simple transpositions give isomorphic restrictions of representations (through a particular isomorphism of the subalgebras). We will hence abuse notation, pick one such subalgebra $\mathscr{H}'$, and notate $\mathrm{Res}^{\mathscr{H}}_{\mathscr{H}'} W$ by $\mathrm{Res} W$ for any $\mathscr{H}$-module W.

The following statements, collectively called *modular branching rules* of D^λ, were originally written by Kleshchev for Specht modules of the group algebra $k\left[S_n\right]$, then generalized to the Hecke algebra case by

Brundan.[6] [2] They entirely characterize the Socle of ResD^λ, as well as the condition that soc $\left(\mathrm{Res}D^\lambda\right) \simeq \mathrm{Res}D^\lambda$.

Theorem 2.4 (Kleshchev-Brundan)**.** *We have the following isomorphisms of vector spaces*

$$Hom_{\mathscr{H}'}\left(S^\mu, ResD^\lambda\right) \simeq \begin{cases} k & \mu \xrightarrow{normal} \lambda \\ 0 & otherwise \end{cases}$$

$$Hom_{\mathscr{H}'}\left(D^\mu, ResD^\lambda\right) \simeq \begin{cases} k & \mu \xrightarrow{good} \lambda \\ 0 & otherwise \end{cases}$$

and $ResD^\lambda$ is semisimple if and only if every normal number in λ is good. □

Using this, we immediately see that, for any rectangular partition (m^ℓ), we have

$$\mathrm{Res}D^{(m^\ell)} \simeq D^{(m^{\ell-1},m-1)}.$$

The non-rectangular two-row case is more complicated, but we may still describe it fully as follows.

Corollary 2.5. *Suppose $r > 0$. Then, we may characterize the socle of $Res\, D^\lambda$ as follows:*

$$soc\left(Res\, D^{(n+r,n)}\right) \simeq \begin{cases} D^{(n+r-1,n)} & e \mid r+2 \\ D^{(n+r,n-1)} & e \nmid r+2,\ e \mid r \\ D^{(n+r-1,n)} \oplus D^{(n+r,n-1)} & e \nmid r+2, r \end{cases}.$$

Further, when $e \nmid r$ or $e \mid r+2$, $ResD^{(n+r,n)}$ is semisimple.

Proof. This amounts to computations of the hook lengths $\beta(1,2)$ and $\gamma(1,2)$:

$$\beta_\lambda(1,2) = r+2$$
$$\gamma_\lambda(1,2) = r$$

Since 2 is the largest removable number, $D^{(n+r,n-1)} \subset D^{(n+r,n)}$ iff $e \nmid r+2$. Further, if $e \nmid r+2$, then $D^{(n+r-1,n)} \subset D^{(n+r,n)}$ iff 1 is good iff $e \nmid r$ □

Now that we've characterized how these restrict, we can describe how strongly these restrictions characterize irreducibles. Namely, we will prove that a composition series consistent with $(n+r,n)$ and e top-indivisibility of $(n+r-1,n+1)$ is sufficient to determine that an irreducible is $D^{(n+r,n)}$ when either $r \neq 0$ or $e \neq 4$.

Proposition 2.6. *Let λ be an e-regular partition of $2n+r$.*

(i) Suppose $r > 0$, suppose $e \nmid r+1, r+2, \ldots, n+r+1$, and suppose either $e \mid r$ or $e \nmid r$. If D^λ has the composition series

$$0 \subset D^{(n+r-1,n)} \subset Res\, D^\lambda \tag{2.1}$$

with factor $Res\, D^\lambda / D^{(n+r-1,n)} \simeq D^{(n+r,n)}$, then $\lambda = (n+r,n)$.

(ii) Suppose $r = 0$, suppose $e \nmid 4$, and suppose $D^{(n,n-1)} \simeq Res\, D^\lambda$. Then $\lambda = (n,n)$.

Proof. Note that $e > n$.

(i) Let $\varpi := (n+r-1,n,1)$, let $\varsigma := (n+r-1,n+1)$, and let $\mu := (n+r,n)$. Since $D^{(n+r-1,n)} \subset \mathrm{Res}\, D^\lambda$, we have $(n+r-1,n) \longrightarrow \lambda$, implying $\lambda = \varpi, \varsigma, \mu$. We will show that ϖ, ς do not have socle compatible with (2.1); then, we will have $\lambda = \mu$.

Suppose that $\lambda = \varpi$. We will break into cases with r.

- Suppose that $r > 1$. Note that $e \nmid r+1 = \beta_\varpi(1,2)$, so 2 is normal. Further, $\gamma_\varpi(2,3) = n \not\equiv 0 \pmod e$, so 2 is good and $D^{(n+r-1,n-1,1)} \subset D^\varpi$, which is not a composition factor in (2.1). Hence, by Jordan-Hölder, $\lambda \neq \varpi$.[3]
- Suppose that $r = 1$. Then, $\varpi = (n,n,1)$ has $\gamma_\varpi(1,2) = n \not\equiv 0 \pmod e$, giving $D^{(n,n-1,1)} \subset D^\varpi$ and hence $\lambda \neq \varpi$.

Now suppose that $\lambda = \varsigma$ and break into cases with r:

- Suppose $r > 2$. Then, by Corollary 2.5, we require that $e \nmid r$ and $e \mid r-2$; these are not satisfied, so $\lambda \neq \varsigma$.
- Suppose $r = 2$. Then $\operatorname{Res} D^{\varsigma} \simeq D^{(n+1,n)}$ is irreducible, contradicting (2.1).
- Suppose $r < 2$. Then ς is not a partition.

(ii) This result is easier than the previous result; since the socle of D^{λ} is irreducible, we require that 1 is the only normal number and $\lambda(1) = (n, n-1)$. This reduces to the cases of $\varsigma := (n+1, n-1)$ and $\mu := (n,n)$; if $\lambda = \varsigma$, then we have that $e \mid \beta_{\varsigma}(1,2) = 4$, a contradiction. Hence $\lambda = \mu$. □

3. Crossingless Matchings and Specht Modules

In the following section, we will analyze the crossingless matchings representation M^r_{2n+r} with the goal of proving $M^r_{2n+r} \simeq S^{(n+r,n)}$ whenever possible. We begin by proving irreducibility of M^r_{2n+r} in all cases that $(n+r,n)$ is e top-indivisible; when $p=2$, this is all cases where $S^{(n+r,n)}$ is irreducible. This is proven via an inductive process; if $e \nmid n+r+1$, then M contains no sign subrepresentation, and this allows us to "project" down to the case $M^r_{2(n-1)+r}$, and prove irreducibility of M from irreducibility of this space.

Next, we

3.1. Irreducibility of M.

Lemma 3.1. *Every basis vector in M^r_{2n+r} is cyclic.*

Proof. We have already proven this in the $r=0$ case, so suppose that $r > 0$.

Note that, between anchors $a < a'$ having no arc b with $a < b < a'$, the $M^0_{a'-a}$ case allows us to generate the vector with all length-2 arcs between a, a' and identical arcs/anchors outside of this sub-matching.[1]

Applying this between each arc gives us a vector with length-2 arcs and anchors, and we may use the appropriate $(1+T_i)$ to move anchors to any positions, and the reverse process from above to generate the correct matchings between arcs and generate any other basis vector. □

Let $K := \bigcap_{i=1}^{2n+r-1} \ker(1+T_i) = \ker \bigoplus_{i=1}^{2n+r-1}(1+T_i)$. This will be a large technical tool in our proof of irreducibility.

Lemma 3.2. *Suppose $e \nmid n+r+1$. Then, $K = 0$.*

Proof 1. Consider the matrix $A = \bigoplus(1+T_i)$ having kernel K. It is sufficient by lemma 3.7 to show that A includes a row $[0, \ldots, 0, 1, 0, \ldots, 0]$ with a nonzero entry only on the row j.

Now, we may characterize the rows of A as follows; if the row corresponding to $(1+T_i)$ and mapping onto the element $w_l \in W$ is nonzero, then it is of the form $\left[a_1, \ldots, a_{|W|}\right]$ where $a_l = 1+q$, $a_m = q^{1/2}$ whenever $(1+T_i)w_m = q^{1/2}w_l$, and $a_m = 0$ otherwise.

Seeing this, the row corresponding to $(1+T_{n+r})$ and w_j has nonzero entries $q^{1/2}$ at w_j and $(1+q)$ at the vector w agreeing with w_j at all indices except having arcs at $(n+r-1, n+r)$ and $(n+r+1, n+r+2)$. Similar justification leads the row corresponding to $(1+T_{n+r-1})$ at w to have nonzero entries $q^{1/2}$ at w and $(1+q)$ at w_j and the vector with anchors $1, \ldots, r$, arc $(n+r-3, n+r-2)$, and all other arcs maximum length.

We may iterate this process as illustrated in Figure 3, eventually ending at a row with two nonzero entries, either an arc $(1,2)$ or an arc $(2,3)$, and all anchors otherwise left-aligned and arcs of maximum length. These rows together form an $(n+r) \times \left|M^r_{2n+r}\right|$ submatrix of A which has a nonzero row in the row corresponding to j, and has (by removing zero rows) the same row space as the following square matrix:

$$B_{n+r} := \begin{bmatrix} q+1 & q^{1/2} & & & & & \\ q^{1/2} & q+1 & q^{1/2} & & & \text{\Large 0} & \\ & q^{1/2} & q+1 & q^{1/2} & & & \\ & & \ddots & \ddots & & & \\ & \text{\Large 0} & & & q^{1/2} & q+1 & q^{1/2} \\ & & & & & q^{1/2} & q+1 \end{bmatrix}. \tag{3.1}$$

[1] At the ends, we apply the M^0_a case or the M^0_{2n+r-a} case in the same way for the first a or last $2n+r-a$ indices.

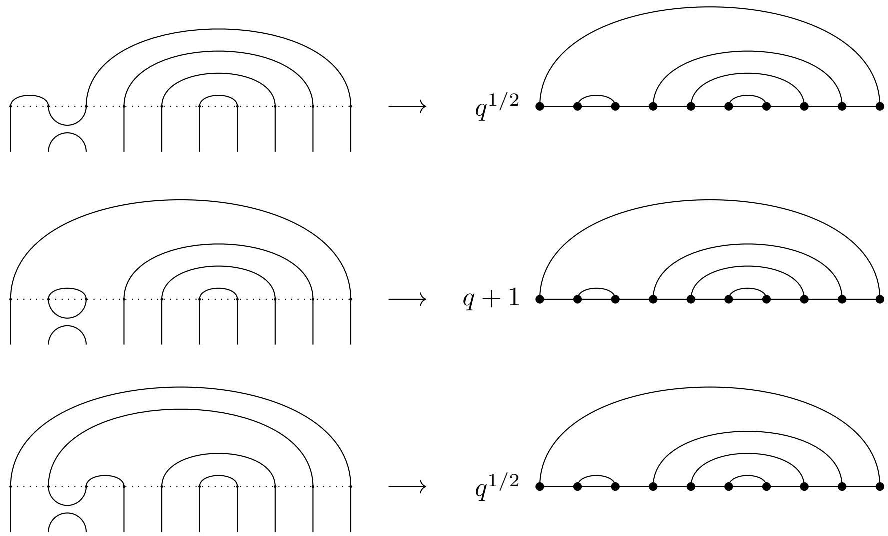


Figure 3. Illustrated is the row constructed for transposition $(1+T_2)$; clearly these are the only basis elements mapping to multiples of the desired element, and they relate to each other. replacing the outermost and/or innermost arc with an anchor typifies the rows constructed with three nonzero coefficients.

We will show that this matrix is invertible; then, a sequence of elementary row operations will yield the identity, and in particular, when applied to A, will yield a row with a nonzero entry only on row j, giving $K = 0$.

We may prove invertibility of this matrix by proving that $\det B_{n+r} = [n+r+1]_q$ inductively on $n+r$. This is satisfied for our base case $n+r=1$, so suppose that it is true for each $m < n+r$. Then,

$$\begin{aligned}\det B_{n+r} &= (q+1)\det B_{n+r-1} - q\det B_{n+r-2}\\ &= (q+1)(1+\cdots+q^{n+r-1}) - (q+\cdots+q^{n+r-1})\\ &= 1+\cdots+q^{n+r}\\ &= [n+r+1]_q\,.\end{aligned}$$

Hence $\det B_{n+r} \neq 0$, and $K = 0$. □

Proposition 3.3. *Suppose that $(n+r,n)$ is e top-indivisible. Then, the representation M^r_{2n+r} is irreducible.*

Proof. We proceed by induction on n. Note that, by identification with the trivial and sign representation, the base case $n=0$ is already proven.

Take an arbitrary vector $w \in W$. By Lemma 3.2 there exists some $(1+T_i) \in \mathscr{H}$ such that $(1+T_i)w$. Note that

$$\operatorname{im}(1+T_i) = \operatorname{Span}\{w_j \mid w_j \text{ contains arc } (i,i+1)\}\,.$$

Hence, as vector spaces, there is an isomorphism $\varphi : \operatorname{im}(1+T_i) \to M^r_{2(n-1)+r}$ "deleting" the arc $(i,i+1)$.

We will show that, for every action $(1+T'_j) \in \mathscr{H}(S_{2(n-1)+r})$, there is some action $h_j \in M^r_{2n+r}$ such that the following commutes:

$$\begin{array}{ccc}\operatorname{im}(1+T_i) & \xrightarrow[\sim]{\varphi} & M^r_{2(n-1)+r}\\ \downarrow{\scriptstyle h_j} & & \downarrow{\scriptstyle 1+T_j}\\ \operatorname{im}(1+T_i) & \xrightarrow[\sim]{\varphi} & M^r_{2(n-1)+r}\end{array}$$

Indeed, when $i \neq j$ this is given by $h_j = 1+T_j$, and we have $h_i = q^{-1}(1+T_i)(1+T_{i+1})(1+T_{i-1})$, as given by Figure 4.

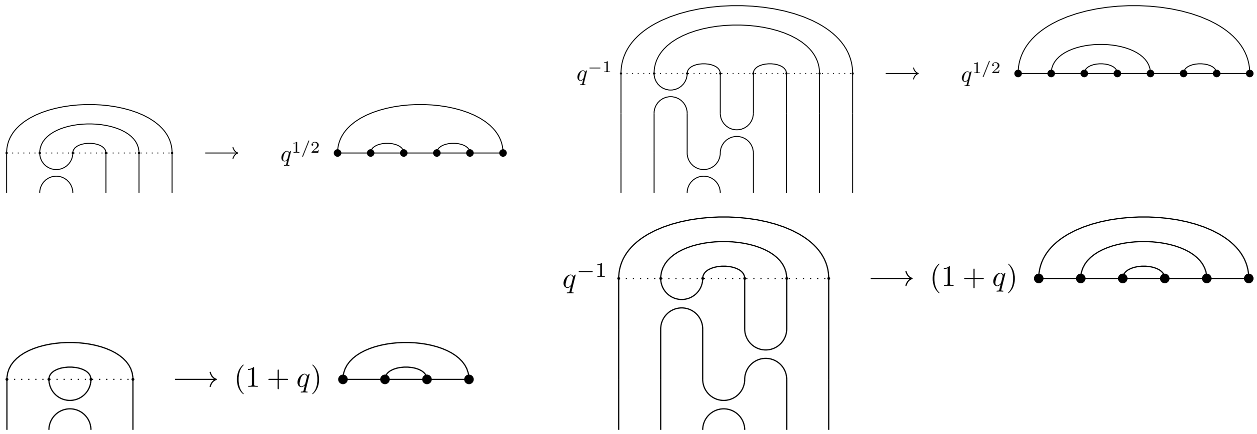


Figure 4. The correspondence between the action of $(1 + T_2)$ on $w'_5 \in M^0_6$ and the action of $q^{-1}(1 + T_3)(1 + T_4)(1 + T_2)$ on the corresponding vector in M^0_8 having arc $(3, 4)$ first, then on $w'_2 \in M^0_4$. This demonstrates that the action works with and without creating a loop.

Note that, by e top-indivisibility of $(n+r, n)$, we have $e \nmid r+2, \dots, n+r$, and hence $(n+r-1, n-1)$ is e top-indivisible as well. Then, due to the inductive hypothesis, there is some action $h' \in \mathscr{H}(S_{2(n-1)+r})$ sending $\varphi((1+T_i)w)$ to the image of a basis vector; then, the action $\mathscr{H}$ generates the endomorphism $\varphi^{-1}h'\varphi$ sending $(1 + T_i)w$ to a basis vector, giving w cyclic and hence M^r_{2n+r} irreducible. □

3.2. **Correspondence.** The following theorem of Mathas generalizes the characteristic-free version of the classical branching theorem.[9] It will not be necessary for our present proof of the correspondence, but analogy with M is suggestive.

Theorem 3.4 (Characteristic-Free Classical Branching Theorem)**.** *Let λ be a partition of m with ℓ removable nodes. Then, Res S^λ has an $\mathscr{H}_{k,q}(S_{m-1})$-module filtration*

$$0 = S^{0,\lambda} \subset S^{1,\lambda} \subset \cdots \subset S^{\ell,\lambda} = \mathit{Res} S^\lambda$$

such that $S^{t,\lambda}/S^{t-1,\lambda} \simeq S^{\lambda(t)}$ for all $1 \le t \le \ell$. □

If we replace S^λ with the appropriate M^r_{2n+r} above, we find the statement of the following proposition.

Proposition 3.5. *Suppose that $n > 0$.*

(i) Suppose that $r > 0$. Then, a filtration of $\mathit{Res} M^r_{2n+r}$ is given by

$$0 \subset M^{r-1}_{2n+r-1} \subset \mathit{Res}\, M^r_{2n+r} \tag{3.2}$$

with $\mathit{Res}\, M^r_{2n+r}/M^{r-1}_{2n+r-1} \simeq M^{r+1}_{2n+r-1}$.

(ii) We have the following isomorphism of representations:

$$M^1_{2n-1} \simeq \mathit{Res}\, M^0_{2n} \tag{3.3}$$

When the case is type (i) from the irreduibility lemma, this is a composition series.

Proof. **(i)** Note that we may identify the subrepresentation of Res M^r_{2n+r} having anchor n with M^{r-1}_{2n+r-1}.

Let $U :=$ Res $M^r_{2n+r}/M^{r-1}_{2n+r-1}$. Let $\phi : U \to M^{r+1}_{2n+r-1}$ be the k-linear map which regards the arc $(i, 2n+r)$ in U as an anchor at i in M^{r+1}_{2n+r-1}. It is not hard to verify that this is a well-defined isomorphism of vector spaces, so we must show that it is $\mathscr{H}$-linear.

Given a basis vector w_j with arc $(i, 2n + r)$, ϕ is clearly compatible with $T_{i'}$ with $i' \neq i, i-1$. Further, it's easy to verify that ϕ is compatible with T_i and T_{i-1}, as actions on one anchor were designed for this deformation. When there are anchors $(i, i+1)$, then $\phi(T_i w_j) = T_i\phi(w_j) = 0$, and similar for T_{i-1}. Hence ϕ is an isomorphism of representations, and the statement is proven.

(ii) This follows with the above proof, defining $M^{-1}_{2n-1} := 0$ □

We've now assembled the basic pieces necessary to prove our correspondence in the case $e > n + r1$. Due to some problematic phenomena in the $r = 0$ case, we will initially make an assumption that $e \neq 4$ or $r \neq 0$ so that we may use the combinatorics of Proposition 2.6. We will then assume the remaining case $e = 4, r = 0$, and prove our correspondence using process of elimination via the other surrounding cases for $e = 4$.

Theorem 3.6. *Suppose $e > n + r + 1$. Then, $M^r_{2n+r} \simeq S^{(n+r,n)}$.*

Proof. The case $n = 0$ is already proven, so suppose $n > 0$. In order to use Proposition 2.6, suppose for now that either $e \nmid 4$ or $r \neq 0$.

We will prove this inductively; by identification with the trivial and sign representations, the $2n+r = 2$ caseholds, so suppose this is true for M^s_{2m+s} whenever $2m+s < 2n+r$ and $m+s \leq n+r$ (i.e. $e > m+s+1$).

By irreducibility, we know that $M^r_{2n+r} \simeq D^\lambda$ for some e-restricted partition λ. By the inductive hypothesis and irreducibility, we have a composition series given by the exact sequence

$$0 \longrightarrow D^{(n+r-1,n)} \longrightarrow \operatorname{Res} D^\lambda \longrightarrow D^{(n+r,n-1)} \longrightarrow 0 \tag{3.4}$$

Hence the theorem is given by Proposition 2.6.

Now, suppose $e = 4$ and $r = 0$; then $4 > n + 1$, so $n \leq 2$. We've already proven the $n = 1$ case via the trivial representation, so suppose $n = 2$. Then, from the proof of Proposition 2.6, we know that $M^0_4 \simeq D^\lambda$, where $\lambda \in \{(n, n), (n + 1, n - 1)\}$. We have already proven that $M^2_4 \simeq D^{(n+1,n-1)}$, and we may verify that $\dim M^0_4 = 2 \neq 3 = \dim M^2_4$, so we have that $\lambda = (n, n)$ and the theorem is proven for $e = 4$. □

3.3. Kernels and Further Work.

Proposition 3.7. *Let w be an arbitrary vector in W^r_{2n+r}. I claim that if $w \in \cap ker(1 + T_i)$, the coordinate c of the rainbow element R in w is nonzero.*

Proof. Let Y be the set of basis elements with nonzero coordinate in w. Let k be the greatest integer such that there exists $y \in Y$ where $y(1) = ... = y(k) = 0$ should this be $y(1) - 1 = \cdots = y(k) - k = 0$? Also, we should avoid using k as an integer, as it's used elsewhere as a field., and let $U \subset Y$ be the set of such y. In other words, U is the set of basis elements in Y which have the most anchors to the far left.

Suppose $k < r$. Then for each $y \in U$ there exists a minimal $i_y > k + 2$ such that $y(i_y) = 0$. In other words, i_y is the position of the next leftmost anchor in y. Fix $\tilde{y}$ such that $i_{\tilde{y}} \leq i_y$ for all y. Then I claim the basis element $y' := q^{-1/2}(1+T_{i_{\tilde{y}}-1})\tilde{y}$ has nonzero coordinate in $(1+T_{i_{\tilde{y}}-1})w$, implying $w \notin \cap\ker(1+T_i)$. To see this, we can show that $\tilde{y}$ is the only element in Y such that $q^{-1/2}(1+T_{i_{\tilde{y}}-1})\tilde{y} \sim y'$. y' still has k anchors on the left, and $i_{y'} < i_{\tilde{y}}$, so $y' \notin Y$. If $x \in Y, \notin U$, the basis element proportional to $(1 + T_{i_{\tilde{y}}-1})x$ will have k anchors at the far left only if the next anchor is at a position $i_{x'} > i_{\tilde{y}}$, so it cannot be y'. If $x \in U$ the basis element proportional to $(1 + T_{i_{\tilde{y}}-1})x$ will have anchor at $i_{y'}$ if and only if $i_x = i_{\tilde{y}}$ and $x(i_{\tilde{y}}) = \tilde{y}(i_{\tilde{y}})$. Since this is the only match altered by action $(1 + T_{i_{\tilde{y}}-1})$ on x, if $(1+T_{i_{\tilde{y}}-1})x \sim y'$ this implies $x = \tilde{y}$. So if $k < r$ w is not in the desired kernel.

Suppose $k = r$ but $R \notin U$ (so $R \notin Y$). Let us define a sequence of subsets of U in the following way: $U_0 := U$, $U_{i+1} := \{u \in U_i | u(r + i + 1) = 2n + 2r - i + 1\}$. Since $R \notin U$, $\exists t < n - 1$ such that $U_{t+1} = \varnothing$. Choose $\tilde{u} \in U_t$ such that $\tilde{u}(r + t + 1) \geq u(r + t + 1)$ for all $u \in U_t$. Consider the basis element $u' := q^{-1/2}(1 + T_{\tilde{u}(r+t+1)})\tilde{u}$. I claim that $\tilde{u}$ is the only element in Y such that $(1 + T_{\tilde{u}(r+t+1)})\tilde{u} \sim u'$, again implying that w is not in the desired kernel. u' still has k anchors on the left, $u'(r + i) = 2n + 2r - i + 2$, $1 \leq i \leq t$, and $u'(r + t + 1) > \tilde{u}(r + t + 1)$, so $u' \notin Y$. If $x \in Y, \notin U$, the basis element x' proportional to $(1 + T_{\tilde{u}(r+t+1)})x$ will have r leftmost anchors only if $x'(r + t + 1) < \tilde{u}(r + t + 1)$, so $x' \neq u'$. Similarly, if $x \in U, \notin U_t$, the basis element x' will have the property $x'(r+t) = 2n+2r-t+2$ only if $x'(r+t+1) < \tilde{u}(r+t+1)$, so $x' \neq u'$. If $x \in U_t$, $x'(r + t + 1) = u'(r + t + 1)$ if and only if $x(r + t + 1) = \tilde{u}(r + t + 1)$ and $x(x(r + t + 1) + 1) = \tilde{u}(\tilde{u}(r + t + 1) + 1)$ (since $u' \notin Y$). These are the only matches altered by the action $(1+T_{\tilde{u}(r+t+1)})$, so this implies $x = \tilde{u}$. Thus we have proved that if $R \notin Y$, w is not in the desired kernel. □

Necessary characterization of the Kernel goes here.

Proposition 3.8. *Suppose $e = n + r + 1$. Then K is nontrivial.*

4. Fibonacci Representations and Quotients of Specht Modules

We have difficulty characterizing the Specht module above when it is irreducible. We may instead attempt to characterize the irreducible quotient D^λ of the Specht module. In particular, note that, whenever $e \le r+2$, the restriction $\mathrm{Res}D^{(n+r,n)}$ decomposes into a direct sum of partitions $(m+s,m)$ such that $e \le s+2$ and $2m+s = 2n+r-1$. This forms a recurrence among these representations, allowing a combinatorial description of their dimension. In this section we henceforth assume $e = 5$ and note that this recurrence resembles the Fibonacci recurrence; we follow this to characterize the restriction $D^{(n+r,n)}$ with $r \le 3$.

Remark. Let $d_m^{0,3}$ be the dimension $\dim D^{(n,n)}$ with $2n = m$ when m is even, and $\dim D^{(n+3,n)}$ with $2n+3 = m$ when m is odd. Similarly define $d_m^{1,2}$. Then, Corollary 2.5 gives

$$\begin{aligned} d_m^{0,3} &= d_{m-1}^{1,2} \\ d_m^{1,2} &= d_{m-1}^{1,2} + d_{m-1}^{0,3} \\ &= d_{m-1}^{0,3} + d_{m-2}^{0,3}. \end{aligned}$$

Carefully following this and noting the base cases $d_2^{0,3} = d_2^{1,2} = 1$, one may note that this recurrences proves that $d_m^{1,2} = d_{m+1}^{0,3} = f_n$, where f_n is the nth *Fibonacci number*. This matches the dimension our Fibonacci subrepresentations, motivating their definition. □

We can start our study of V by studying low-dimensional cases. First, note that V_{*0}^2 is the sign representation $D^{(1^2)}$ and V_{*0}^2 is the trivial representation $D^{(2)}$.

V_{00}^2 is a 2-dimensional representation of a semisimple commutative algebra, and hence decomposes into a direct sum of two subrepresentations. In particular, we can use the basis $\{(\mathtt{0*0}), (\mathtt{000})\}$ and explicitly write the matrix

$$\rho_{T_1} = \begin{bmatrix} \varepsilon_1 & \delta \\ \delta & \varepsilon_2 \end{bmatrix}$$

having characteristic polynomial $(\varepsilon_1 - \lambda)(\varepsilon_2 - \lambda) - \delta^2 = \lambda^2 - (\varepsilon_1 + \varepsilon_2)\lambda + (\varepsilon_1\varepsilon_2 - \delta^2)$. We may verify that, for $\lambda = -1$, this evaluates to

$$-((-1+q+q^2)(1+q^3+q^4+q^5+2q^6+q^7))\,[5]_q = 0$$

and for $\lambda = q$ this evaluates to

$$-(q^2(-1+q+q^2)(1+q+q^2+q^3+2q^4+q^5))\,[5]_q = 0$$

hence ρ_{T_1} has eigenvalues -1 and q.

The eigenspaces with eigenvalues -1 and q are subrepresentations isomorphic to the sign and trivial representation, hence V_{00} is isomorphic to a direct sum of the trivial and sign representations: $V_{00}^2 \simeq V_{*0}^2 \oplus V_{**}^2$.

Now we may prove that V_{*0}^3 is irreducible; this has basis $\{(\mathtt{*0*0}), (\mathtt{*000})\}$, and the following matrices:

$$\rho_{T_1} = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{bmatrix}; \qquad \rho_{T_2} = \begin{bmatrix} \varepsilon_1 & \delta \\ \delta & \varepsilon_2 \end{bmatrix}.$$

A proper nontrivial subrepresentation of V_{*0}^3 must be one-dimensional, and hence an eigenspace of each of these matrices; since $\alpha_2 \ne \alpha_1$, the first has eigenspaces given by the spans of basis elements, and since $\delta \ne 0$, these are not eigenspaces of the second. Hence V_{*0}^3 is irreducible.

These establish the low-dimensional behavior that we will use as base cases below. The rest of this section will proceed first by proving that V_{*0} and V_{**} are irreducible; then, we will use combinatorial arguments to prove that these are isomorphic to the correct two-row Specht module quotients.

Proposition 4.1. *The representation $V_{*0} := V_{*0}^m$ is irreducible.*

Proof. We will prove this inductively in m. We've already proven it for V_{*0}^2 and V_{*0}^3, so suppose that V_{*0}^{m-2} is irreducible.

Let $\{v_i\}$ be the basis for V_{*0}. Then, each v_i is cyclic; indeed, we can transform every basis vector into $(\mathtt{*0}\ldots\mathtt{0})$ by multiplying by the appropriate $\frac{1}{\delta-\varepsilon_1}(T_i - \varepsilon_1)$, and we can transform $(\mathtt{*0}\ldots\mathtt{0})$ into any basis vector by multiplying be the appropriate $\frac{1}{\delta-\varepsilon_2}(T_i - \varepsilon_2)$. Hence it is sufficient to show that each $v \in V_{*0}$ generate some basis element.

Let v' be the basis element $(\texttt{*0*0}\ldots\texttt{0})$, which is many copies of $\texttt{*0}$, followed by an extra $\texttt{0}$ if m is odd. We will show that each $v \in F$ generates v'.

Suppose that no elements beginning $(\texttt{*0*0})$ are represented in v_i; then, all such elements are represented in $T_3 v$, so we may assume that at least one is represented in v.

Note that $\mathrm{im}(T_2 - \alpha_1) = \mathrm{Span}\,\{\text{Basis vectors beginning } (\texttt{*0*0})\}$ and $(T_2 - \alpha_1)v \neq 0$. Further, note that $\mathrm{Res}^{\mathscr{H}(S_m)}_{\mathscr{H}(S_{m-2})} \mathrm{im}(T_2 - \alpha_1) \simeq V^{m-2}_{\texttt{*0}}$ as representations. Hence irreducibility of $V^{m-2}_{\texttt{*0}}$ implies that v' is generated by $(T_2 - \alpha_1)v$, and $V^m_{\texttt{*0}}$ is irreducible. □

Now, we may begin considering restrictions:

Lemma 4.2. *The following branching rules hold:*

$$V^{m-1}_{00} \simeq \mathit{Res} V^m_{*0} \simeq V^{m-1}_{**} \oplus V^{m-1}_{*0}$$
$$\mathit{Res} V^m_{**} \simeq V^{m-1}_{*0}.$$

Proof. The first line follows by considering the last two $m-2$ transpositions for the left isomorphism, then the first two for the right isomorphism. This is well-behaved by Appendix B.

Similarly, the second line follows by considering the last $m-2$ transpositions. □

This immediately gives a rather strong characterization of V.

Corollary 4.3. *The representation V_{**} is irreducible.* □

Corollary 4.4. *The representation V decomposes into a direct sum of irreducible representations as follows:*

$$V \simeq 3V_{*0} \oplus 2V_{**}.$$

□

Now we may use these in order to apply Young Tableau to characterize V.

Theorem 4.5. *The irreducible components of V are given by the following isomorphisms:*

$$V^{2n}_{**} \simeq D^{(n,n)}$$
$$V^{2n-1}_{**} \simeq D^{(n+1,n-2)}$$
$$V^{2n}_{*0} \simeq D^{(n+1,n-1)}$$
$$V^{2n-1}_{*0} \simeq D^{(n,n-1)}.$$

Proof. We will prove this by induction on n; we have already proven the base case V^2, so suppose that we have proven these isomorphisms for V^{2n-2}. We will prove the isomorphisms for V^{2n-1} and V^{2n}.

By irreducibility, $V^{2n-1}_{\texttt{**}} \simeq D^\lambda$ and $V^{2n-1}_{\texttt{*0}} \simeq D^\mu$ for some diagrams λ and μ. We will show that $\lambda = (n+1, n-2)$ and $\mu = (n+1, n-1)$.

First, note that we have

$$\mathrm{Res}\ D^\lambda \simeq D^{(n,n-2)} \simeq \mathrm{Res}\ D^{(n+1,n-2)}$$

and

$$\mathrm{Res}\ D^\mu \simeq D^{(n,n-2)} \oplus D^{(n-1,n-1)} \simeq \mathrm{Res}\ D^{(n,n-1)}.$$

By irreducibility of $\mathrm{Res} D^\lambda$, the only normal number in λ is 1. Further, the only tableaux which can remove a cell to yield $D^{(n,n-2)}$ are $(n+1, n-2)$, $(n, n-1)$, and $(n, n-2, 1)$ as illustrated in Figure 5; we have already seen that $D^{(n,n-1)}$ does not have irreducible restriction, so we are left with $(n+1, n-2)$ and $\varsigma = (n, n-2, 1)$. We may directly check that ς doesn't satisfy this, as we have the following:

$$\beta_\varsigma(1,2) = 3 - 2 + (n-2) = n - 1$$
$$\beta_\varsigma(1,3) = 3 - 1 + n = n + 2$$
$$\beta_\varsigma(2,3) = 2 - 1 + 3 = 4.$$

At least one of $\beta_\varsigma(1,2)$ and $\beta_\varsigma(1,3)$ is nonzero, since $\beta_\varsigma(1,3) - \beta_\varsigma(1,2) = 3 \not\equiv 0 \pmod{e}$, and hence at least one of M_2 and M_3 is empty. Hence at least one of 2 or 3 is normal in ς, and $\lambda = (n+1, n-2)$.

For μ, we immediately see from Figure 5 that the only option is $(n, n-1)$.

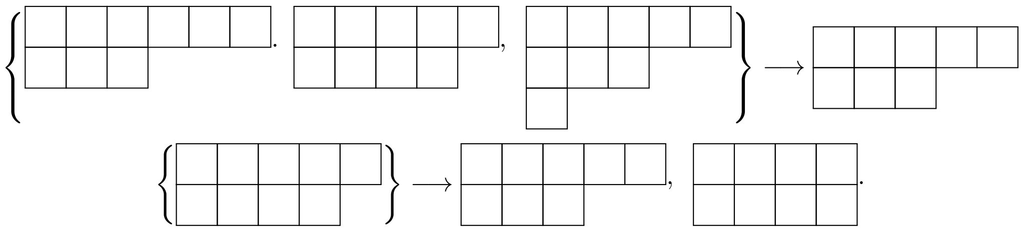


Figure 5. Illustration of the partitions of 9 which can, via row removal, yield $(n, n-2)$ alone, or both $(n, n-2)$ and $(n-1, n-1)$.

We can perform a similar argument for the V^{2n} case, finding now that

$$\text{Res } D^{\lambda'} \simeq D^{(n,n-1)} \simeq \text{Res } D^{(n,n)}$$

and

$$\text{Res } D^{\mu'} \simeq D^{(n,n-1)} \oplus D^{(n+1,n-2)} \simeq \text{Res } D^{(n+1,n-1)}.$$

Through a similar process, we see that $\mu' = (n+1, n-1)$. We narrow down λ' to one of (n, n) or $\varpi := (n, n-1, 1)$, and note that

$$\begin{aligned} \beta_{\varpi}(1,2) &= 3 - 2 + (n-1) = n \\ \beta_{\varpi}(1,3) &= 3 - 1 + n = n+2 \\ \beta_{\varpi}(2,3) &= 2 - 1 + 2 = 3 \end{aligned}$$

and hence at least one of 2 or 3 is normal, $\text{Res}D^{\varpi}$ is not irreducible, and $\lambda' = (n, n)$, finishing our proof. □

Corollary 4.6. *We have the following isomorphisms of representations:*

$$\begin{aligned} V^{2n} &\simeq 3D^{(n+1,n-1)} \oplus 2D^{(n,n)} \\ V^{2n-1} &\simeq 3D^{(n,n-1)} \oplus 2D^{(n+1,n-2)} \end{aligned}$$

5. Conjecture

Recall that $K^r_{2n+r} := K$ is the direct sum of all copies of the sign representation in W. Hence the following characterises sign subrepresentations of W completely:

Proposition 5.1. *$K \subset M^r_{2n+r}$ is trivial when $e \neq n+r+1$, and $\dim K = 1$ when $e = n+r+1$.* □

Proposition 5.2. *Suppose $e < n+r+1$, and suppose n' is such that $e = n'+r+1$. Note that $h := (1+T_1)(1+T_3)\dots(1+T_{n-n'})$ maps M^r_{2n+r} onto $M^r_{2n'+r}$. Then, the preimage $h^{-1}(K^r_{2n+r})$ is a subrepresentation of M^r_{2n+r}, and the series*

$$0 \subset h^{-1}(K^r_{2n+r}) \subset M^r_{2n+r}$$

is a composition series of M^r_{2n+r}. □

Proposition 5.3. *Denote the composition factor $M^r_{2n+r}/h^{-1}(K^r_{2n+r})$ by U^r_{2n+r}. Then, there exist some naturals m, s satisfying $2m+s = 2n+r$ and $m+s > n+r$ such that the following is an isomorphism of $\mathscr{H}$-modules*

$$h^{-1}\left(K^r_{2n+r}\right) \simeq U^s_{2m+s}$$

. □

Proposition 5.4. *For the same m, s as above, we have the following composition series of specht modules:*

$$0 \longrightarrow D^{(m+s,m)} \longrightarrow S^{(n+r,n)} \longrightarrow D^{(n+r,n)} \longrightarrow 0.$$

Proposition 5.5. *$M^r_{2n+r} \simeq S^{(n+r,n)}$ and $U^s_{2m+s} \simeq D^{(m+s,m)}$.*

$$\varphi_6^0, \varphi_5^1 = \begin{bmatrix} 0 & 0 & -q^{3/2}+1 & 0 & 0 \\ 0 & -q^{3/2}+1 & [4]_{q^{1/2}} & 0 & 0 \\ 0 & 0 & [4]_{q^{1/2}} & 0 & -q^{3/2}+1 \\ -[4]_{q^{1/2}} & [4]_{q^{1/2}} & q^{1/2}\left(q^{1/2}+1\right) & 0 & [4]_{q^{1/2}} \\ [4]_{q^{1/2}} & 0 & [4]_{q^{1/2}} & -[4]_{q^{1/2}} & 0 \end{bmatrix}$$

$$\varphi_6^2 = \begin{bmatrix} 0 & 0 & -q-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q-1 & -q^{1/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & q+1 & 0 & q^{1/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{1/2} & 0 & 0 & 0 & -q^2-1 & 0 \\ q^{1/2} & -q^{1/2} & -[3]_{q^{1/2}} & 0 & 0 & 0 & -q^{1/2} & 0 \\ -q^{1/2} & q^{1/2} & 0 & [3]_{q^{1/2}} & 0 & 0 & 0 & q^{1/2} \\ -q^{1/2} & 0 & -q^{1/2} & 0 & q^{1/2} & 0 & 0 & 0 \\ q^{1/2} & 0 & 0 & 0 & -q^{1/2} & -[3]_{q^{1/2}} & 0 & -q^{1/2} \end{bmatrix}$$

$$\varphi_5^3 = \begin{bmatrix} -q^{1/2}\left(q^{1/2}+1\right) & [4]_{q^{1/2}} & q^{1/2} \\ q^{1/2} & q^{3/2} & 0 \\ 0 & -q-1 & \end{bmatrix} \qquad \varphi_4^0 = \begin{bmatrix} 0 & -[4]_{q^{1/2}} \\ -1 & 1 \end{bmatrix}$$

$$\varphi_3^1 = \begin{bmatrix} 0 & -[4]_{q^{1/2}} \\ q^{1/2}\left(q^{1/2}+1\right) & -q^{1/2}\left(q^{1/2}+1\right) \end{bmatrix} \qquad \varphi_4^2 = \begin{bmatrix} 0 & [4]_{q^{1/2}} & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Figure 6. We define representations $\varphi^r_{2n+r} : M^r_{2n+r}/K^r_{2n+r} \to V_s^{2n+r}$, where $s = \mathtt{**}$ if $r = 0, 3$ and $s = \mathtt{*0}$ otherwise. These are given with respect to the basis on M^r_{2n+r} induced by the increasing lexicographic order basis on M^0_{2n+2r} and the quotient $M^0_{2n+2r} \twoheadrightarrow M^r_{2n+r}$. These are also given with respect to the basis on V given by increasing lexicographic order $\mathtt{*} < \mathtt{0}$.

6. Empirical Results

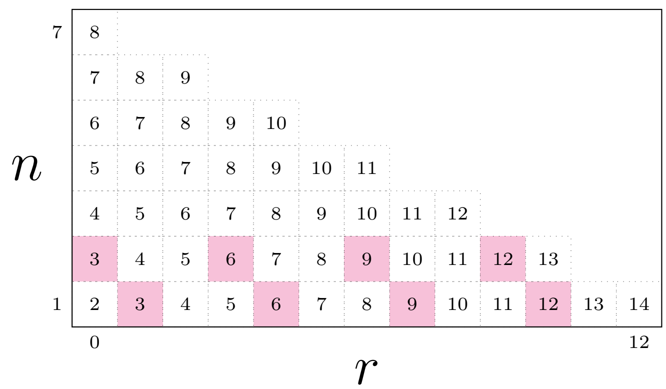
In this section we aim both to support conjecture with data and to provide transition matrices wherever possible between our two graphical representations. [4]

Remark. These results are restricted by two things; first and with less recourse, this was computed in Magma, a closed source language with bugs that could not be resolved in certain cases. Second, the memory necessary to store the Hecke algebra via basis elements and structure constants grows with $(n!)^3$; this is prohibitively large when $m > 7$. An implementation as a quotient by a free algebra or only specifying multiplication by generators may fix this, however, this is not possible with the Magma language, and the closed source renders modification of the language impossible.

We give in Figure 6 some isomorphisms between M/K and V for $e \geq n+r+1$ with $p = \infty$; all but one of these are cases with $e > n+r+1$, so $K = 0$ and this is an isomorphism with our crossingless matchings representation. The case $n = 1$, $r = 3$ gives an example of an isomorphism not proven in general, but proven via our computation. All of these computations are done for q a primitive 5th root of unity in the algebraic extension of the Cyclotomic field $\mathbb{Q}(\zeta_{10})$ by a root of the polynomial $x^2 - \tau$.

We give in Figure 7 some data supporting a conjecture concerning sign subrepresentations of M^r_{2n+r}. The computations to support this were done over $\mathbb{C}$ with q a primitive 5th root of unity.

It is known that, for small $2n+r$, each specht module $S^{(n+r,n)}$ has a composition series of length 2. We give in Figures 8 through 10 the map $M^r_{2n+r} \twoheadrightarrow U^r_{2n+r}$, which conjecturally illustrates the quotient $S^{(n+r,n)} \twoheadrightarrow U^r_{2n+r}$ for all $2n+r \leq 7$.



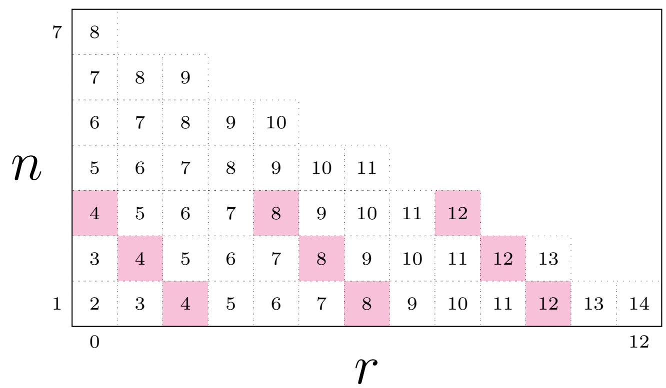


Figure 7. Illustration of the modules M^r_{2n+r} having sign submodules for $p = \infty$ and $e = 3, 4$ respectively. The value $n + r + 1$ is filled in the squares, and modules having sign submodules are colored magenta. For $p = \infty$ and $2n + r \leq 14$, it has been verified, through a combination of theorems here and empirical computations, that K^r_{2n+r} is nontrivial if and only if $e | n + r + 1$ and $e < n$.

Appendix A. Compatibility of Representations with the Relations

In general, we define representations above for the free algebra on generators $\{T_i\}$. Recall that we may give a presentation of $\mathscr{H}$ having generators T_i and relations

$$(T_i - q)(T_i + 1) = 0 \tag{A.1}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{A.2}$$

$$T_i T_j = T_j T_i \qquad |i - j| > 1. \tag{A.3}$$

We call (A.1) the *quadratic relation* and (A.2), (A.3) the *braid relations*. It is easily seen that a representation of $\mathscr{H}$ is equivalent to a representation of the free algebra $k\langle T_i \rangle$ which acts as 0 on the relations (henceforth referred to as *compatibility* with the relations). We will prove in the following sections that V and W are compatible with the Hecke algebra relations.

A.1. **Verifying the Crossingless Matchings Representations.**

Take some basis vector w_i. We will first check (A.1) by case work:

- Suppose there is an arc $(i, i+1)$. Then, $(T_i - q)(T_i + 1)w = (1+q)\left[(1 + T_i)w - (1+q)w\right] = 0$, giving (A.1).
- Suppose there is no arc $(i, i+1)$ and $i, i+1$ do not both have anchors; then $(T_i + 1)w = q^{1/2}w''$ for some basis vector w' having arc $(i, i+1)$, and the computation follows as above for (A.1).
- Suppose $i, i+1$ are anchrors; then $(T_i + 1)w = 0$, giving (A.1).

$$
\begin{aligned}
\pi^1_{3,3} = \pi^0_{3,4} &= \begin{bmatrix} 1 & 1 \end{bmatrix}^\intercal \\
\pi^1_{3,5} = \pi^0_{3,6} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}^\intercal \\
\pi^4_{3,6} &= \begin{bmatrix} -1 & -1 & 0 & 1 & 1 \end{bmatrix}^\intercal \\
\pi^3_{3,7} &= \begin{bmatrix} 1 & 0 & -1 & -1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^\intercal \\
\pi^1_{3,7} &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}^\intercal
\end{aligned}
$$

Figure 8. The maps of representations $\pi^r_{e,2n+r} : M^r_{2n+r} \twoheadrightarrow U^r_{2n+r}$ for $e = 3$.

$$
\begin{aligned}
\pi^2_{4,4} &= \begin{bmatrix} 1 & \alpha & 1 \end{bmatrix}^\intercal \\
\pi^1_{4,4} &= \begin{bmatrix} 1 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & \frac{1}{2}\alpha \end{bmatrix}^\intercal \\
\pi^2_{4,6} &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & -1 & -\alpha & 0 \\
0 & 0 & 1 & 0 & 0 & \alpha & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & -\alpha & -1 \\
0 & 0 & 0 & 0 & 1 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & \frac{1}{2}\alpha
\end{bmatrix}^\intercal \\
\pi^0_{4,6} &= \begin{bmatrix} \alpha & 1 & 1 & \alpha & 1 \end{bmatrix}^\intercal \\
\pi^1_{4,7} &= \begin{bmatrix}
1 & 0 & 0 & 0 & \frac{1}{2}\alpha & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 1 & 1/2 & \alpha & \frac{1}{2} \\
0 & 1 & 0 & 0 & -1 & 0 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & 0 & \alpha & \frac{1}{2}\alpha & -1 & \frac{1}{2}\alpha \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & \alpha & 1 & 0 & -\alpha & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & \alpha & 0 & 0 & -1 & -\alpha & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 0 & 0 & 0 & \frac{1}{2}\alpha & -1 & \frac{1}{2}\alpha \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & \frac{1}{2}\alpha
\end{bmatrix}^\intercal
\end{aligned}
$$

Figure 9. The maps of representations $\pi^r_{e,2n+r} : M^r_{2n+r} \twoheadrightarrow U^r_{2n+r}$ for $e = 4$.

Now we verify (A.2). Let $h := (1 + T_i)(1 + T_{i+1})(1 + T_i)$, and let $g := (1 + T_{i+1})(1 + T_i)(1 + T_{i+1})$. Note the following expansion:

$$
\begin{aligned}
hw &= 1 + 2T_i + T_i^2 + T_{i+1} + T_i T_{i+1} + T_{i+1} T_i + T_i T_{i+1} T_i \\
&= 1 + (1 + q)T_i + T_{i+1} + T_i T_{i+1} + T_{i+1} T_i + T_i T_{i+1} T_i.
\end{aligned}
$$

An analogous formula gives an analogous equality in g. Hence we have

$$(h - g)w = q(T_i - T_{i+1}) + T_i T_{i+1} T_i - T_{i+1} T_i T_{i+1}.$$

$$\pi^3_{5,5} = \begin{bmatrix} 1 & \beta & \beta & 1 \end{bmatrix}^\intercal$$
$$\pi^2_{5,6} = \begin{bmatrix} \beta & \beta & 1 & 1 & \beta+1 & \beta & \beta & \beta & 1 \end{bmatrix}^\intercal$$
$$\pi^3_{5,7} = \begin{bmatrix} 1 & \beta & \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \beta & \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & \beta & 1 & 0 & 0 & 0 \\ \beta & \beta & 1 & 0 & 1 & \beta+1 & \beta & 0 & \beta & \beta & 0 & 1 & 0 & 0 \\ -\beta-1 & -\beta & -\beta & 0 & -\beta & -\beta-1 & -\beta-1 & 0 & -\beta-1 & -\beta-1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \beta & 0 & 0 & \beta & 0 & 0 & 0 & 1 \end{bmatrix}^\intercal$$
$$\pi^1_{5,7} = \begin{bmatrix} 1 & \alpha & \alpha & \alpha & -\beta & \alpha & -\beta & \alpha & -\beta & 1 & \alpha & \alpha & \alpha & -\beta \end{bmatrix}^\intercal$$
$$\pi^4_{6,6} = \begin{bmatrix} 1 & \gamma & 2 & \gamma & 1 \end{bmatrix}^\intercal$$
$$\pi^3_{6,7} = \begin{bmatrix} \gamma & 2 & \gamma & 1 & 1 & 2\gamma & 3 & \gamma & 2 & 2\gamma & 2 & 2 & \gamma & 1 \end{bmatrix}^\intercal$$
$$\pi^5_{6,7} = \begin{bmatrix} 1 & \delta & \varepsilon & \varepsilon & \delta & 1 \end{bmatrix}^\intercal$$

Figure 10. The maps of representations $\pi^r_{e,2n+r} : M^r_{2n+r} \twoheadrightarrow U^r_{2n+r}$ for $e = 5, 6$.

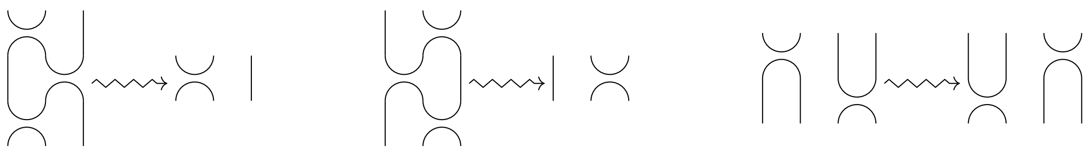


Figure 11. The following give visual intuition for isotopy between $(1+T_i)(1+T_{i-1})(1+T_i)w_j$ and $q(1+T_i)$,between $(1+T_{i+1})(1+T_i)(1+T_{i+1})w_j$ and $q(1+T_i)$, and between $(1+T_i)(1+T_j)w_l$ and $(1+T_j)(1+T_i)w_l$. In all cases, these isotopies imply equality between the elements listed.

Hence we may equivalently check that $(h-g)w = q(T_i - T_{i+1})$. In fact, $hw = q(1+T_i)$ and $gw = q(1+T_j)$ by Figure 11, giving compatibiliy.

Lastly, we have the equation

$$(1+T_i)(1+T_j) - (1+T_j)(1+T_i) = T_iT_j - T_jT_i$$

and hence we simply need to verify that $(1+T_i)$ and $(1+T_j)$ commute, as shown in Figure 11

A.2. **Verifying the Fibonacci Representations.** Similar to before, the reader may verify that (A.3) follows easily, and the others may be verified on strings of length 3 and 4. By considering the coefficients in order of (1.1), the quadratic relation (A.1) gives the following quadratics:

$$\begin{aligned} (\alpha_1 - q)(\alpha_1 + 1) &= 0 \\ (\alpha_2 - q)(\alpha_2 + 1) &= 0 \\ \varepsilon_1\delta + \delta\varepsilon_2 &= (q-1)\delta \\ \varepsilon_1^2 + \delta^2 &= (q-1)\varepsilon_1 + q \\ \varepsilon_2^2 + \delta^2 &= (q-1)\varepsilon_2 + q \end{aligned} \tag{A.4}$$

The first two of these are easily verified for any q. Since $\delta \neq 0$, the third is equivalently given by

$$(q-1) = \varepsilon_1 + \varepsilon_2 = t(q\tau - 1 + q - \tau) = (\tau^2 + \tau)(q-1)$$

or that $\left(\tau^2 + \tau - 1\right)(q-1) = 0$. One may verify that

$$\tau^2 + \tau - 1 = q^6 + 2q^5 + q^4 + q^3 + q^2 - 1 = (-1 + q + q^2)\,[5]_q = 0.$$

The fourth is given by the quadratic

$$\tau^2\left[(q\tau - 1)^2 - \tau(q+1)\right] = \tau(q-1)(q\tau - 1) + q$$

or equivalently,

$$(\tau^2+\tau-1)\left[q\left(qt^2+1\right)+t\right]=0$$

which is true for every q.

The fifth is similarly given by

$$(\tau^2+\tau-1)\left[q\left(qt+1\right)+t^2\right]=0$$

which is true for every q.

We now verify (A.2). We may order the basis for V^4 as follows:

$$\{(\mathtt{0000}),(\mathtt{*00*}),(\mathtt{000*}),(\mathtt{*000}),(\mathtt{*0*0}),(\mathtt{0*0*}),(\mathtt{00*0}),(\mathtt{0*00})\}\,.$$

Then, in verifying the braid relation (A.2) in this order, we encounter the following quadratics (with tautologies and repetitions omitted):

$$\begin{aligned}
\alpha_1\varepsilon_2^2+\alpha_2\delta^2&=\alpha_1^2\varepsilon_2\\
\alpha_1\delta\varepsilon_2+\alpha_2\varepsilon_1\delta&=\alpha_1\alpha_2\delta\\
\alpha_2\varepsilon_1^2+\alpha_1\delta^2&=\alpha_2^2\varepsilon_1\\
\alpha_1\varepsilon_1^2+\delta^2\varepsilon_2&=\alpha_1^2\varepsilon_1\\
\delta\varepsilon_2^2+\alpha_1\varepsilon_1\delta&=\alpha_1\delta\varepsilon_2
\end{aligned}$$

Substituting in τ and dividing by δ whenever possible, these are equivalent to the vanishing of the following polynomials in q:

$$\begin{aligned}
-q(1+q)(1+q^2+q^3)(2+q+3q^2+2q^3)\,[5]_q&=0\\
(1+2q+q^3+q^4)\,[5]_q&=0\\
(1+q)^2(1+q^2+q^3)(1+3q^3-q^4+q^6)\,[5]_q&=0\\
(1+q)^2(1+q^2+q^3)(1+5q+5q^2+3q^3+3q^4+3q^5+q^6)\,[5]_q&=0\\
(1+q)(1+q^2+q^3)(-1+2q+q^2+q^3+q^4)\,[5]_q&=0.
\end{aligned}$$

Notably, each of these vanish when $e=5$.

Appendix B. Conjugate Subalgebras

Throughout the text, for some representation V, we refer to $\mathrm{Res}^{\mathscr{H}(S_m)}_{\mathscr{H}(S_l)}V$ without specifying exactly which subalgebra $\mathscr{H}(S_l)$.

Proposition B.1. *Suppose B,B' are subalgebras of the k-algebra A with $B=uB'u^{-1}$, and let V be a representation of A. Then, the linear isomorphism $V\xrightarrow{\phi}V$ given by $v\mapsto uv$ causes the following to commute for any $b\in B$:*

$$\begin{tikzcd}
V \arrow[r, "\phi"] \arrow[d, "b"] & V \arrow[d, "ubu^{-1}"] \\
V \arrow[r, "\phi"] & V
\end{tikzcd}$$

Hence, through the identification of B and B' via conjugation, we have $Res^A_B V\simeq Res^A_{B'}V$

Proof. This is simply given by $(ubu^{-1})uv=ubv$. □

Corollary B.2. *Suppose $\mathscr{H}',\mathscr{H}''$ are two subalgebras of $\mathscr{H}(S_m)$ generated by l reflections and V is a representation of $\mathscr{H}$. Then, $Res^{\mathscr{H}}_{\mathscr{H}'}V\simeq Res^{\mathscr{H}}_{\mathscr{H}''}V$.*

Proof. Let $\mathscr{H}'$ and $\mathscr{H}''$ be the subalgebras of $\mathscr{H}(S_m)$ generated by the reflections $\{T_{i_1},\dots,T_{i_l}\}$ and $\left\{T_{i_1},\dots,T_{i_{j-1}},T_{i_j+1},T_{i_{j+1}},\dots,T_{i_l}\right\}$ for $1\le i_1<\cdots<i_{j-1}<i_j+1<i_{j+1}<\cdots<i_l\le n$. It is sufficient to prove that $\mathscr{H}'$ and $\mathscr{H}''$ are conjugate; then transitivity gives conjugacy of any $S_l\subset S_m$, and the previous proposition gives isomorphisms of the representations.

In fact, the reader can verify that $\mathscr{H}''=T_{i_j}\mathscr{H}'T_{i_j}^{-1}$. □

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