THE UNEVEN-HEIGHT TWO-COLUMN SPECHT MODULES OF THE HECKE ALGEBRA OF S_n

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1. Introduction

Let S_{2n+r} be the symmetric group on $2n+r$ indices with $2n+r \geq 2$, let $\mathscr{H} = \mathscr{H}_{k,q}(S_{2n+r})$ be the corresponding Hecke algebra over field k with parameter $q \in k^\times$, and let $\{T_i\}$ be the reflections generating $\mathscr{H}$. Let $[m]_q = 1 + q + \cdots + q^{m-1}$ be the q-number of m. Let e be the smallest positive integer such that $[e]_q = 0$, and set $e = \infty$ if no such integer exists. Either $q = 1$ and e is the characteristic of k, or $q \neq 1$ and q is a primitive eth root of unity.

Let $S^{(n+r,n)'}$ be the Specht module corresponding to the young diagram with two columns with height difference r. The purpose of this writing is to characterize this representation via an isomorphism with another representation of $\mathscr{H}$.

Definition 1.1. A *generalized crossingless matching* on $2n+r$ indices with r anchors is a partition of $\{1, \ldots, 2n+r\}$ into n parts of size 2 and r of size 1 such that no two parts of size two"cross", i.e. there are no parts (a, a') and (b, b') such that $a < b < a' < b'$, and no parts of size one are "inside" of a part of size two, i.e. there are no $c, (a, a')$ such that $a < c < a'$. We will call these arcs and anchors, respectively. Then, define W^r_{2n+r} to be the k-vector space with basis the set of generalized crossingless matchings on $2n+r$ indices with r anchors.

In order for this to be a $\mathscr{H}$-module, endow this with the action given by Figure 1; if this involves no anchors, act as in W^0_{2n}; if it involves one anchor, deform to another generalized crossingless matching and scale by $q^{1/2}$, and otherwise scale by 0.

Let the length of an arc (i, j) be $l(i, j) := j - i + 1$. Note that the crossingless matchings can all be identified with a list of n integers describing the lengths of the arcs from left to right; using this, we may order the crossingless matchings with 0 hooks in increasing lexicographical order in order to obtain an order on the subbasis containing a particular set of anchors; let the basis be ordered first by the position of the anchors in increasing lexicographical order, then increasing for the matchings between each anchor. Let this basis be $\{w_i\}$. This basis is illustrated for W^1_5 in Figure 2.

We will prove that $W := W^r_{2n+r}$ and $S := S^{(n+r,n)'}$ are isomorphic as representations in the case that $\mathscr{H}$ is semisimple.

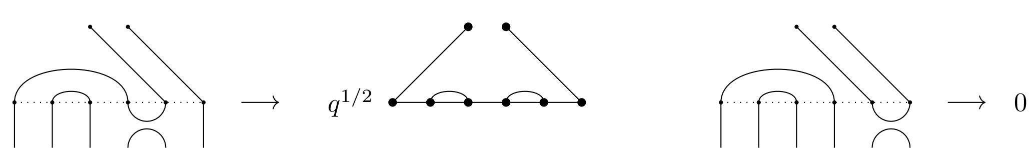

Figure 1. Illustration of the actions $(1+T_4)w_{|W^2_6|}$ and $(1+T_2)w_{|W^2_6|}$ in W^2_6. In general, we act on basis elements away from anchors as we did for W, at one anchor we act by deforming and scaling by $q^{1/2}$, and at two anchors we send the element to zero.

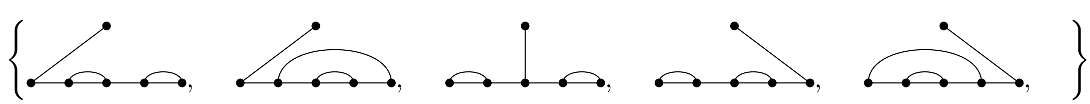

Figure 2. The basis for W^1_5.

2. Correspondence

We can now begin by proving that W is irreducible; then $W \simeq S^\lambda$ for some partition λ of $2n+r$, and we may use branching rules to determne W.

Lemma 2.1. *Every basis vector in W^r_{2n+r} is cyclic.*

Proof. We have already proven this in the $r = 0$ case, so suppose that $r > 0$.

Note that, between anchors $a < a'$ having no arc b with $a < b < a'$, the $W^0_{a'-a}$ case allows us to generate the vector with all length-2 arcs between a, a' and identical arcs/anchors outside of this sub-matching.[1]

Applying this between each arc gives us a vector with length-2 arcs and anchors, and we may use the appropriate $(1 + T_i)$ to move anchors to any positions, and the reverse process from above to generate the correct matchings between arcs and generate any other basis vector. □

Proposition 2.2.

(i) The representation W^r_{2+r} is reducible iff $e \mid r + 2$.
(ii) When $n \neq 1$ and $e > n + 1$, the representation W^r_{2n+r} is irreducible.

Proof. **(i)** Note that $\mathrm{im}(1 + T_i)$ is 1-dimensional for each i, so it is equivalent that

$$K := \bigcap_{i=1}^{r+1} \ker(1 + T_i) = \ker \bigoplus_{i=1}^{r+1}(1 + T_i)$$

is trivial via the lemma. The transformation $\bigoplus(1 + T_i)$ is a linear operator on W^r_{2+r} given by the following matrix:

$$A_{r+1} = \begin{bmatrix} q+1 & q^{1/2} & & & & & \\ q^{1/2} & q+1 & q^{1/2} & & & \text{\huge 0} & \\ & q^{1/2} & q+1 & q^{1/2} & & & \\ & & \ddots & \ddots & & & \\ & \text{\huge 0} & & & q^{1/2} & q+1 & q^{1/2} \\ & & & & & q^{1/2} & q+1 \end{bmatrix}$$

. Hence K is trivial iff the determinant $\det A_{r+1} = [r + 2]_q$ is 0, or equivalently iff $e \mid r + 2$.

(ii) We will prove the equivalent condition that each vector in $w \in W \setminus \{0\}$ is *cyclic*, i.e. $\mathscr{H} w = W$. We will break into case work on r; suppose first that $r = 1$. Then, it is easy to verify that $W^1_{2n+1} \simeq \mathrm{Res} W^0_{2n+1}$, which we have already proven irreducible. We may henceforth assume that $r > 1$.

Overall, we will use induction on $2n + r$; this is easily shown via identification with the sign or trivial representation when $2n + r = 2$, so assume that it is true for all W^s_{2m+s} with $2m + s < 2n + r$.

The proof will proceed in two steps: first we will make sure a particular basis vector is represented with the earliest possible position of the last anchor a_r, then we will use this to generate a nonzero vector representing only vectors with a certain collection of anchors, using the inductive hypothesis to prove that w is cyclic.

Step 1. Let U_{x_r} be the subspace of W containing only anchors at positions $i \leq x_r$. Order these in increasing order; let $U := U_{a_r}$ be the first of these into which w projects to a nonzero vector. If $a_r = n + r$, then w only represents vectors containing anchor r; then, we may use the inductive hypothesis on the first $n + r - 1$ indices to yield a basis vector, and we are done.

Henceforth assume $a_r < n + r$. Then, by our inductive hypothesis, we may use only actions T_i with $i < a_r$ to generate a vector w' which projects to a vector in U representing basis elements with anchors $1, 2, \ldots, r$ and all length-2 arcs at indices $r < i \leq a_r$.

Now, recall that $\mathrm{Res}^{\mathscr{H}(S_{2n+r-a_r})}_{\mathscr{H}(S_{2n+r-a_r-1})} W^0_{2n+r-a_r}$ is irreducible; hence we may only use actions T_i with $i > a_r + 1$ to generate a vector w'' which projects to the basis vector U containing anchors $1, \ldots, r$ and is otherwise all length-2 arcs.[2]

[1] At the ends, we apply the W^0_a case or the W^0_{2n+r-a} case in the same way for the first a or last $2n + r - a$ indices.

[2] Any action by T_i with $i > a_r + 1$ sends basis vectors outside of U to 0 or nonzero vectors outside of U, as they cannot generate a vector which doesn't have an anchor in some position $j > a_r$.

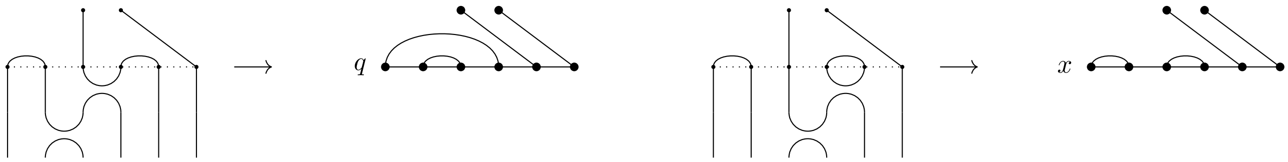

Figure 3. Demonstration of how the transformation $(1 + T_{i+1})(1 + T_i)$ "moves" anchors from positions $i, i + 1$ to position $i + 3$, with constant $x = q^{1/2}(q + 1)$. Iterating this across all elements between $r + 1$ and $2n + r$ via h concentrates all anchors at the beginning or end; if there are at least two anchors in w_j after $r + 1$, then we must act on two anchors eventually, giving $w_j \in \ker h$.

Step 2. Define an element $h \in \mathscr{H}$ by

$$h := (1 + T_{n+r-2})(1 + T_{n+r-1})(1 + T_{n+r-3})(1 + T_{n+r-2}) \dots (1 + T_{r+1})(1 + T_{r+2})$$

Then, every basis vector represented in $hw'' \neq 0$ contains anchors $1, \dots, r - 1$. This is illustrated in Figure 3.

Let U' be the subspace of W having anchors $1, \dots, r - 1$. Note that $hw'' \in U'$. Further, $U' \simeq W^1_{2n+1}$ as vector spaces, and every action in W^1_{2n+1} is reflected by an action of $\mathscr{H}$ on W^r_{2n+r}. Since $r > 1$, we may use the inductive hypothesis to act on indices $i \geq r$ and generate a basis vector, giving w cyclic. □

Corollary 2.3. *Other than W^1_3, the representation W^r_{2n+r} is irreducible when $e > n + 1$.*

The next piece in our puzzle is to characterize the restrictions of W to $\mathscr{H}' := \mathscr{H}_{k,q}(S_{2n+r-1}) \subset \mathscr{H}$. Recall that, when $r, n > 0$ and $\mathscr{H}$ is semisimple, $\mathrm{Res} S^{(n+r,n)'} \simeq S^{(n+r-1,n)'} \oplus S^{(n+r,n-1)'}$. Further, note that $S^{(n+r,n)'}$ is the unique irreducible having this restriction.

Next, note that we have already proven the correspondence for W^0_{2n}; for W^r_{0+r}, this is the sign representation, which is given correctly by $S^{(r)}$. Hence, pending information on restrictions, we may prove this via induction on $2n + r$.

Proposition 2.4. *Suppose that $n, r > 0$. Then, a filtration of $\mathrm{Res} W^r_{2n+r}$ is given by*

$$0 \subset W^{r-1}_{2n+r-1} \subset \mathrm{Res}\, W^r_{2n+r} \tag{1}$$

with $\mathrm{Res}\, W^r_{2n+r}/W^{r-1}_{2n+r-1} \simeq W^{r+1}_{2n+r-1}$.

Proof. Note that we may identify the subrepresentation of Res W^r_{2n+r} having anchor n with W^{r-1}_{2n+r-1}.

Let $U :=$ Res $W^r_{2n+r}/W^{r-1}_{2n+r-1}$. Let $\phi : U \to W^{r+1}_{2n+r-1}$ be the k-linear map which regards the arc $(i, 2n + r)$ in U as an anchor at i in W^{r+1}_{2n+r-1}. It is not hard to verify that this is a well-defined isomorphism of vector spaces, so we must show that it is $\mathscr{H}$-linear.

Given a basis vector w_j with arc $(i, 2n + r)$, ϕ is clearly compatible with $T_{i'}$ with $i' \neq i, i - 1$. Further, it's easy to verify that ϕ is compatible with T_i and T_{i-1}, as actions on one anchor were designed for this deformation. When there are anchors $(i, i + 1)$, then $\phi(T_i w_j) = T_i \phi(w_j) = 0$, and similar for T_{i-1}. Hence ϕ is an isomorphism of representations, and the statement is proven. □

Corollary 2.5. *Suppose $n, r > 0$ and $e > n + 1$. Then, the sequence (1) is a composition series of $\mathrm{Res}\, W^r_{2n+r}$.* □

Corollary 2.6. *Suppose $n, r > 0$ and $\mathscr{H}$ is semisimple. Then, $\mathrm{Res}\, W^r_{2n+r} \simeq W^{r-1}_{2n+r-1} \oplus W^{r+1}_{2n+r-1}$.* □

Corollary 2.7. *When $\mathscr{H}$ is semisimple, $W^r_{2n+r} \simeq S^{(n+r,r)'}$.*

Proof. We may argue by induction on $2n+r$, knowing that we have proven the base case $2n+r = 2$. Assume that we have proven the isomorphism for all W^s_{2n+s} with $2n + s = 2n + r - 2$. We have proven the $n = 0$ and $r = 0$ cases already, so assume $n, r > 0$.

Then, W^r_{2n+r} is the unique irreducible representation of $\mathscr{H}$ having restriction $S^{(n+r-1,n)'} \oplus S^{(n+r,n-1)'}$, implying the desired isomorphism. □