SOME GRAPHICAL REALIZATIONS OF TWO-ROW SPECHT MODULES OF IWAHORI-HECKE ALGEBRAS OF THE SYMMETRIC GROUP

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Abstract. We consider the Iwahori-Hecke algebra of the symmetric group on $2n+r$ letters with parameter $q \in k^{\times}$. Let e be the smallest integer such that $[e]_q = 0$, or set $e = \infty$ if none exist. We modify Khovanov's Crossingless Matchings to include $2n$ nodes and r anchors, and prove that the corresponding module is isomorphic to the Specht module $S^{(n+r,n)}$ when the specht module is irreducible and $e > n$. Additionally, we prove heuristics in support of the general case. Lastly, when $e = 5$, we prove an isomorphism between $D^{(n+r,n)}$ with $r \leq 3$ and a subrepresentations of Jordan-Shor's fibonacci representation.

Contents

1. Introduction

Let S_{2n+r} be the symmetric group on $2n+r$ indices with $2n+r \geq 2$, let $\mathscr{H} = \mathscr{H}_{k,q}(S_{2n+r})$ be the corresponding Hecke algebra over field k with parameter $q \in k^\times$ having square root $q^{1/2}$, and let $\{T_i\}$ be the reflections generating $\mathscr{H}$. Let $[m]_q = 1+q+\cdots+q^{m-1}$ be the q-number of m. Let e be the smallest positive integer such that $[e]_q = 0$, and set $e = \infty$ if no such integer exists. Either $q = 1$ and e is the characteristic of k (with 0 replaced by ∞), or $q \neq 1$ and q is a primitive eth root of unity.

Throughout the text, we will refer to partitions of $2n+r$; identify each partition with a tuple $\lambda = (\lambda_1^{a_1}, \ldots, \lambda_l^{a_l})$ having $\lambda_i > \lambda_{i+1}$, $a_i > 0$, and $\sum_i a_i\lambda_i = 2n+r$. Identify each of these with a subset $[\lambda] \subset \mathbb{N}^2$ as defined in Kleshchev, and define $\lambda(i) = (\lambda_1^{a_1}, \ldots, \lambda_{i-1}^{a_{i-1}}, \lambda_i^{a_i-1}, \lambda_i - 1, \lambda_{i+1}^{a_{i+1}}, \ldots, \lambda_l^{a_l})$ to be the partition with the ith row removed.

Fixing some partition λ, for $1 \leq i \leq j \leq l$, let $\beta(i,j)$ be the hook length

$$\beta(i,j) = \lambda_i - \lambda_j + \sum_{t=i}^{j} a_t.$$

Then, adopting Kleshchev's terminology, j is normal in λ if $\beta(i,j) \not\equiv 0 \pmod{e}$ for all $i < j$, and j is good if it is the largest normal number (these are stronger conditions than generally necessary).

Let $S^{(n+r,n)'}$ be the Specht module corresponding to the young diagram with two columns with height difference r, and let $D^{(n+r,n)'}$ be the corresponding irreducible quotient. The purpose of this writing is to characterize these representation via an isomorphism with two graphical representations of $\mathscr{H}$.

Crossingless Matchings.

Definition 1.1. A *crossingless matching on $2n+r$ indices with r anchors* is a partition of $\{1, \ldots, 2n+r\}$ into n parts of size 2 and r of size 1 such that no two parts of size two"cross", i.e. there are no parts (a, a') and (b, b') such that $a < b < a' < b'$, and no parts of size one are "inside" of a part of size two, i.e. there are no $c, (a, a')$ such that $a < c < a'$. We will call these arcs and anchors, respectively. Then, define W_{2n+r}^r to be the k-vector space with basis the set of generalized crossingless matchings on $2n+r$ indices with r anchors.

In order for this to be a $\mathscr{H}$-module, endow this with the action given by Figure 1; if a "loop" is created, scale by $q+1$, if a loop is not created and the action involves fewer than 2 anchors, deform into a new crossingless maching and scale by $q^{1/2}$, and if it involves two anchors, scale by 0. We verify that this is well-defined in appendix A.1.

Let the length of an arc (i,j) be $l(i,j) := j - i + 1$. Note that the crossingless matchings on $2n$ indices with no anchros can all be identified with a list of n integers describing the lengths of the arcs from left to right; using this, we may order the crossingless matchings with no anchors in increasing lexicographical order in order to obtain an order on the subbasis containing a particular set of anchors; let the basis be ordered first by the position of the anchors in decreasing lexicographical order, then increasing for the matchings between each anchor. Let this basis be $\{w_i\}$. This basis is illustrated for W_5^1 in Figure 2.

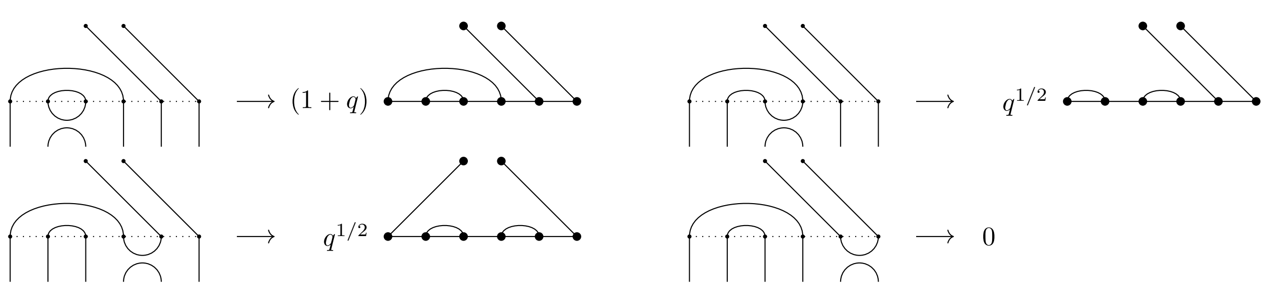

Figure 1. Illustration of the actions $(1+T_i)w_{|W_6^2|}$. In general, we act by deleting loops, deforming into a new crossingless matching, and scaling by either $q^{1/2}$, $(q+1)$, or 0.

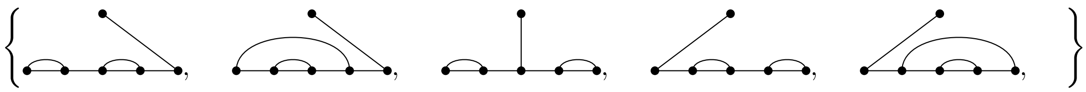

Figure 2. The basis for W_5^1.

Note that the representations W_{0+r}^r and $S^{(r)}$ are isomorphic to the sign representation; we will prove that $W := W_{2n+r}^r$ and $S := S^{(n+r,n)'}$ are isomorphic as representations in the case that $e > n$ and $S^{(n+r,n)'}$ is irreducible. Note that, when $r = 0$, these have the same dimension given by the nth catalan number C_n.

Fibonacci Representation. Now suppose that $e = 5$ and k contains the algebraic number $(-1-q^2-q^3)^{3/2}$. Let V^m be a k-vector space with basis given by the strings $\{\mathtt{*}, \mathtt{0}\}^{n+1}$ such that the character $\mathtt{*}$ never appears twice in a row. We will suppress the superscript whenever it is clear from context.

We wish to endow this with a $\mathscr{H}$-action which acts on a basis vector only dependent on characters $i, i+1, i+2$, sending each basis vector to a combination of the other basis vectors having the same characters $1, \ldots, i, i+2, \ldots, n+1$ as follows:

$$\begin{aligned} T_1\ (\mathtt{*00}) &:= \alpha_1\,(\mathtt{*00}) \\ T_1\ (\mathtt{00*}) &:= \alpha_1\,(\mathtt{00*}) \\ T_1\ (\mathtt{*0*}) &:= \alpha_2\,(\mathtt{*0*}) \\ T_1\ (\mathtt{0*0}) &:= \varepsilon_1\,(\mathtt{0*0}) + \delta\,(\mathtt{000}) \\ T_1\ (\mathtt{000}) &:= \delta\,(\mathtt{0*0}) + \varepsilon_2\,(\mathtt{000}) \end{aligned} \tag{1.1}$$

for constants

$$\begin{aligned} \alpha_1 &= -1 \\ \alpha_2 &= q \\ \varepsilon_1 &= \tau(q\tau - 1) \\ \delta &= \tau^{3/2}(q+1) \\ \varepsilon_2 &= \tau(q-\tau) \\ \tau &= -1-q^2-q^3 \end{aligned} \tag{1.2}$$

with T_i acting similarly on the substring $i, i+1, i+2$. We will verify that this is a representation of $\mathscr{H}$ in Appendix A.2

This contains 4 subrepresentations based on the first and last character of the string, which are not modified by $\mathscr{H}$. Label the subrepresentation spanned by strings $(\mathtt{*}\ldots\mathtt{*})$ by $V_{\mathtt{**}}$, and similar for the other 3. It is easy to see that $V_{\mathtt{*0}} \simeq V_{\mathtt{0*}}$, so that

$$V \simeq 2V_{\mathtt{*0}} \oplus V_{\mathtt{**}} \oplus V_{\mathtt{00}}.$$

We will show that $V_{\mathtt{00}} \simeq V_{\mathtt{*0}} \oplus V_{\mathtt{**}}$, and give the following isomorphisms with irreducible quotients of specht modules depending on the parity of the number of indices in $\mathscr{H}$:

$$\begin{aligned} V_{\mathtt{**}}^{2n} &\simeq D^{(n,n)'} \\ V_{\mathtt{**}}^{2n-1} &\simeq D^{(n+1,n-2)'} \\ V_{\mathtt{*0}}^{2n} &\simeq D^{(n+1,n-1)'} \\ V_{\mathtt{*0}}^{2n-1} &\simeq D^{(n,n-1)'}. \end{aligned} \tag{1.3}$$

Acknowledgements.

2. Preliminaries on Specht Modules

2.1. **Irreducibility of S^λ.** Let k have characteristic ℓ; then, set

$$p := \begin{cases} \ell & \ell > 0 \\ \infty & \ell = 0 \end{cases}.$$

For h a natural number, let $\nu_p(h)$ be the p-adic evaluation of h. As a convention, set $\nu_\infty(h) = 0$ for all h. Define the function $\nu_{e,p} : \mathbb{N} \to \{-1\} \cup \mathbb{N}$ by

$$\nu_{e,p}(h) := \begin{cases} \nu_p(h) & e \mid h \\ -1 & e \nmid h \end{cases}.$$

Lastly, let h^λ_{ab} be the hook length of node (a, b) in $[\lambda]$.

Theorem 2.1 (James-Mathas)**.** *The following are equivalent:*

(1) $S^\lambda \simeq D^\lambda$.
(2) λ *is e-restricted and* S^λ *is irreducible.*
(3) $\nu_{e,p}\left(h^\lambda_{ab}\right) = \nu_{e,p}\left(h^\lambda_{ac}\right)$ *for all nodes* (a, b) *and* (a, c) *in* $[\lambda]$.

□

Corollary 2.2. *Assume* $e > 2$ *and* $n > 0$.

(i) Suppose $e > n$. *Then,* $S^{(n+r,n)'}$ *is irreducible iff* $e \nmid l$ *for all* $r + 2 \le l \le n + r + 1$.
(ii) Suppose $e \le n$. *Then,* $S^{(n+r,n)'}$ *is irreducible iff* $e|r + 1$ *and* $\nu_{e,p}(h^\lambda_{ab})$ *acquires exactly two values over rows* $a \le n$.

Proof. **(i)** Note that $\nu_p(l) \ne -1$ for all l and only hook lengths in the first column may vanish mod e; hence we may equivalently prove that e divides no hook lengths in the first n rows of the first column. These hook lengths are precisely $r + 2, \dots, n + r + 1$.

(ii) Note that we have $\nu_{e,p}\left(h^\lambda_{n-e+1,2}\right) \ne -1$. Suppose that $e \nmid r + 1$. Then,

$$\nu_{e,p}\left(h^\lambda_{n-e+1,1}\right) = \nu_{e,p}\left(h^\lambda_{n-e,2} + r + 1\right) = -1,$$

giving $S^{(n+r,n)'}$ reducible.

Note that ν acquires at least two values since e is a hook length. Now, suppose that

$$0 \le \nu_{e,p}(h^\lambda_{ab}) < \nu_{e,p}\left(h^\lambda_{a'b'}\right)$$

and $S^{(n+r,n)'}$ is irreducible. If $a = a'$ then we have reducibility, so assume $a \ne a'$. Further, if $b = b'$, then we may replace b with the other column; hence we may assume WLOG that $b \ne b'$ as well.

Note that p-adic valuation is monotonic, so $h^\lambda_{ab} < h^\lambda_{a'b'}$. If $b < b'$, then $\nu_p(h_{ab}) = \nu_p(h_{ab'}) < \nu(h_{a'b'})$ while $h_{ab'} > h_{a'b'}$, a contradiction. If $b > b'$, then $\nu_p(h_{ab}) = \nu_p(h_{ab'})$ and $h_{ab} < h_{ab'}$, so there is some $c < b$ with $\nu_{e,b}(h_{ab}) = \nu_{e,p}(h_{ac})$; we may replace b with c, and repeat until $b = b'$ to reach contradiction.

Finally, assume $e|r + 1$ and $\nu_{e,p}$ acquires two values over the top n rows. Since $e|r + 1$, $e \mid h^\lambda_{a1}$ iff $e \mid h_{a2}$; since $\nu_{e,p}$ acquires only two values, this proves that $\nu_{e,p}(h_{a1}) = \nu_{e,p}(h_{a2})$ across these rows, giving the lemma. □

In the case that $\lambda = (n + r, n)$ and e satisfy hypothesis (2.2).(i), say that λ is *e-top-indivisible.*

This is complicated and annoying, and the second amounts to a condition on p. We will illustrate that irreducibility interpolates between $p = 2$ and p large:

Corollary 2.3. *Assume* $2 < e \le n$.

(i) Suppose $p > n + r + 1$. *Then,* $S^{(n+r,n)'}$ *is irreducible iff* $e|r + 1$.
(ii) Suppose $p = 2$. *Then,* $S^{(n+r,n)'}$ *is not irreducible.*

Proof. (i) is clear. (ii) is given by noting that $e + r + 1 \ge 2e$; then, $h_{n-e+1,1} - h_{n-e+1,2} \ge 1$, giving reducibility. □

2.2. **Branching.**

Theorem 2.4 (Kleshchev-Brundan). *We have the following isomorphisms of vector spaces*

$$Hom_{\mathscr{H}'}\left(S^\mu, ResD^\lambda\right) \simeq \begin{cases} k & \mu \xrightarrow{normal} \lambda \\ 0 & otherwise \end{cases}$$

$$Hom_{\mathscr{H}'}\left(D^\mu, ResD^\lambda\right) \simeq \begin{cases} k & \mu \xrightarrow{good} \lambda \\ 0 & otherwise \end{cases}$$

and $ResD^\lambda$ is semisimple if and only if every normal number in λ is good. □

Corollary 2.5. *Suppose $n, r > 0$. Then, we may characterize the socle of $Res\, D^\lambda$ as follows:*

$$soc\left(Res\, D^{(n+r,n)'}\right) \simeq \begin{cases} D^{(n+r-1,n)'} & e \mid r+2 \\ D^{(n+r,n-1)'} & e \nmid r+2,\ e \mid r \\ D^{(n+r-1,n)'} \oplus D^{(n+r,n-1)} & e \nmid r+2, r \end{cases}$$

Proof. This amounts to computations of the hook lengths $\beta(1,2)$ and $\gamma(1,2)$:

$$\beta_\lambda(1,2) = r+2$$
$$\gamma_\lambda(1,2) = r$$

Since 2 is the largest removable number, $D^{(n+r,n-1)'} \subset D^{(n+r,n)'}$ iff $e \nmid r+2$. Further, if $e \nmid r+2$, then $D^{(n+r-1,n)'} \subset D^{(n+r,n)'}$ iff 1 is good iff $e \nmid r$ □

Proposition 2.6. *Suppose that e, n, r satisfy criterion (i) of Proposition 2.2. Let λ be a partition of $2n+r$.*

(i) Suppose $r > 0$. If D^λ has the composition series

$$0 \subset D^{(n+r-1,n)'} \subset Res\, D^\lambda \tag{2.1}$$

with factor $Res\, D^\lambda / D^{(n+r-1,n)'} \simeq D^{(n+r,n)'}$, then $\lambda = (n+r,n)'$.

(ii) Suppose $r = 0$ and $D^{(n,n-1)'} \simeq Res\, D^\lambda$. Then $\lambda = (n,n)'$.

Proof. **(i)** Let $\varpi := (n+r-1, n, 1)$, let $\varsigma := (n+r-1, n+1)'$, and let $\mu := (n+r, n)'$. Since $D^{(n+r-1,n)'} \subset \operatorname{Res} D^\lambda$, we have $(n+r-1,n)' \longrightarrow \lambda$, implying $\lambda = \varpi, \varsigma, \mu$.

We will show that ϖ, ς do not have socle compatible with (2.1); then, we will have $\lambda = \mu$.

First, we focus on ϖ; Suppose $r > 1$ and $e \mid n+1$; then, $\beta_\varpi(2,3) = n+1 \equiv 0 \pmod e$, so 3 is not normal and $D^{(n+r-1,n)'}$ is not in the socle of D^ϖ, giving $\lambda \neq \varpi$.

Now suppose $r > 1$ and $e \nmid n+1$. Then $S^{(n+r-2,n,1)'} \simeq D^{(n+r-2,n,1)'}$. Then, we have $D^{(n+r-2,n,1)'} \simeq S^{(n+r-2,n,1} \subset \operatorname{Res} D^\varpi$ since 1 is normal; this is not a composition factor in (2.1), so $\lambda \neq \varpi$.

Similarly, assume $r = 1$; then, by the same logic, $D^{(n+r-1,n)'} \not\subset \operatorname{Res} D^\varpi$ or $D^{(n,n-1,1)} \subset \operatorname{Res} D^\varpi$, also implying $\lambda \neq \varpi$.

Now, focus on ς; if $r < 2$, then ς is not a partition. If $r = 2$, then $\operatorname{Res} D^\varsigma \simeq D^{(n+1,n)'}$ is irreducible, contradicting (2.1). If $r > 2$, then $D^{(n+r-2,n+1)'}$ is irreducible, so $D^{(n+r-2,n+1)'} \simeq S^{(n+r-2,n+1)'} \subset D^\varsigma$, contradicting (2.1).

(ii) This result is fundamentally similar to the previous result; since the socle of D^λ is irreducible, we require that 1 is the only normal number, which pins λ. □

3. Crossingless Matchings and Specht Modules

3.1. **Irreducibility of M.**

Lemma 3.1. *Every basis vector in W^r_{2n+r} is cyclic.*

Proof. We have already proven this in the $r = 0$ case, so suppose that $r > 0$.

Note that, between anchors $a < a'$ having no arc b with $a < b < a'$, the $W^0_{a'-a}$ case allows us to generate the vector with all length-2 arcs between a, a' and identical arcs/anchors outside of this sub-matching.[1]

[1] At the ends, we apply the W^0_a case or the W^0_{2n+r-a} case in the same way for the first a or last $2n+r-a$ indices.

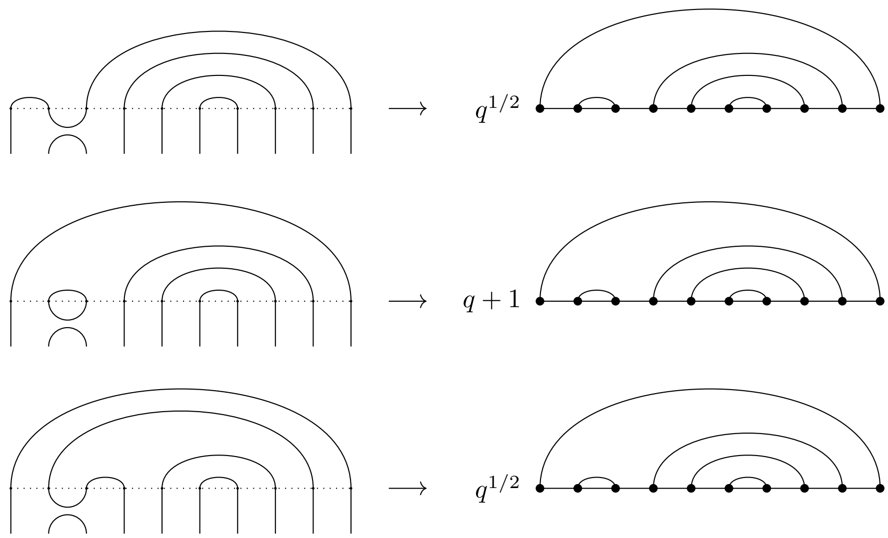

Figure 3. Illustrated is the row constructed for transposition $(1+T_2)$; clearly these are the only basis elements mapping to multiples of the desired element, and they relate to each other. replacing the outermost and/or innermost arc with an anchor typifies the rows constructed with three nonzero coefficients.

Applying this between each arc gives us a vector with length-2 arcs and anchors, and we may use the appropriate $(1+T_i)$ to move anchors to any positions, and the reverse process from above to generate the correct matchings between arcs and generate any other basis vector. $\square$

Let $K := \bigcap_{i=1}^{2n+r-1} \ker(1+T_i) = \ker \bigoplus_{i=1}^{2n+r-1}(1+T_i)$. This will be a large technical tool in our proof of irreducibility.

Lemma 3.2. *Suppose $e \nmid n+r+1$. Then, $K = 0$.*

Proof 1. Consider the matrix $A = \bigoplus(1+T_i)$ having kernel K. It is sufficient by lemma 3.6 to show that A includes a row $[0,\dots,0,1,0,\dots,0]$ with a nonzero entry only on the column j.

Now, we may characterize the rows of A as follows; if the row corresponding to $(1+T_i)$ and mapping onto the element $w_l \in W$ is nonzero, then it is of the form $\left[a_1,\dots,a_{|W|}\right]$ where $a_l = 1+q$, $a_m = q^{1/2}$ whenever $(1+T_i)w_m = q^{1/2}w_l$, and $a_m = 0$ otherwise.

Seeing this, the row corresponding to $(1+T_{n+r})$ and w_j has nonzero entries $q^{1/2}$ at w_j and $(1+q)$ at the vector w agreeing with w_j at all indices except having arcs at $(n+r-1, n+r)$ and $(n+r+1, n+r+2)$. Similar justification leads the row corresponding to $(1+T_{n+r-1})$ at w to have nonzero entries $q^{1/2}$ at w and $(1+q)$ at w_j and the vector with anchors $1,\dots,r$, arc $(n+r-3, n+r-2)$, and all other arcs maximum length.

We may iterate this process as illustrated in Figure 3, eventually ending at a row with two nonzero entries, either an arc $(1,2)$ or an arc $(2,3)$, and all anchors otherwise left-aligned and arcs of maximum length. These rows together form an $(n+r) \times \left|W_{2n+r}^r\right|$ submatrix of A which has a nonzero column in the row corresponding to j, and has (by removing zero columns) the same column space as the following square matrix:

$$B_{n+r} := \begin{bmatrix} q+1 & q^{1/2} & & & & \\ q^{1/2} & q+1 & q^{1/2} & & \text{\Large 0} & \\ & q^{1/2} & q+1 & q^{1/2} & & \\ & & \ddots & \ddots & & \\ & \text{\Large 0} & & q^{1/2} & q+1 & q^{1/2} \\ & & & & q^{1/2} & q+1 \end{bmatrix}. \tag{3.1}$$

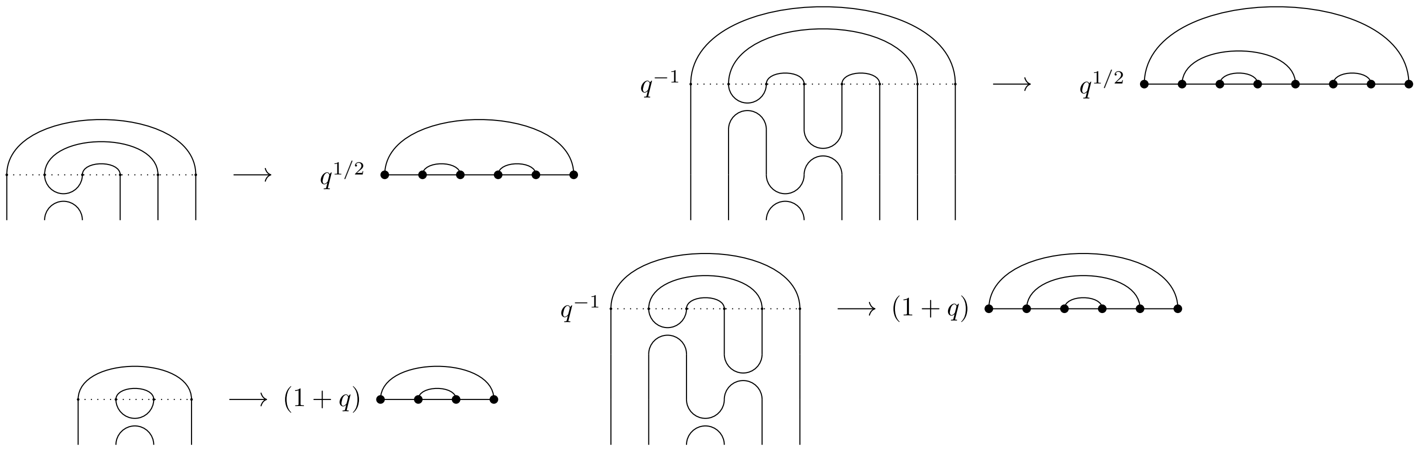

Figure 4. The correspondence between the action of $(1+T_2)$ on $w_5' \in W_6^0$ and the action of $q^{-1}(1+T_3)(1+T_4)(1+T_2)$ on the corresponding vector in W_8^0 having arc $(3,4)$ first, then on $w_2' \in W_4^0$. This demonstrates that the action works with and without creating a loop.

We will show that this matrix is invertible; then, a sequence of elementary row operations will yield the identity, and in particular, when applied to A, will yield a row with a nonzero entry only on column j, giving $K = 0$.

We may prove invertibility of this matrix by proving that $\det B_{n+r} = [n+r+1]_q$ inductively on $n+r$. This is satisfied for our base case $n+r=1$, so suppose that it is true for each $m < n+r$. Then,

$$\begin{aligned}\det B_{n+r} &= (q+1)\det B_{n+r-1} - q\det B_{n+r-2} \\ &= (q+1)(1+\cdots+q^{n+r-1}) - (q+\cdots+q^{n+r-1}) \\ &= 1+\cdots+q^{n+r} \\ &= [n+r+1]_q\,.\end{aligned}$$

Hence $\det B_{n+r} \neq 0$, and $K = 0$. $\square$

Proposition 3.3. *The representation W_{2n+r}^r is irreducible when $e > n+r+1$.*

Proof. We proceed by induction on $2n+r$. Note that, by identification with the trivial and sign representations, the base case $2n+r=2$ is already prove, so suppose we have proven this for each $2m+s < 2n+r$.

Take some $w \in W$ and some $(1+T_i)$ not annhialating w. Note that

$$\operatorname{im}(1+T_i) = \operatorname{Span}\{w_j \mid w_j \text{ contains arc } (i,i+1)\}\,.$$

Hence, as vector spaces, there is an isomorphism $\varphi : \operatorname{im}(1+T_i) \to W_{2(n-1)+r}^r$ "deleting" the arc $(i,i+1)$. This sends every basis vector to another basis vector.

We will show that, for every action $(1+T_j') \in \mathscr{H}(S_{2(n-1)+r})$, there is some action $h_j \in W_{2n+r}^r$ such that the following commutes:

$$\begin{tikzcd} \operatorname{im}(1+T_i) \arrow[r, "\varphi", "\sim"'] \arrow[d, "h_j"] & W_{2(n-1)+r}^r \arrow[d, "1+T_j"] \\ \operatorname{im}(1+T_i) \arrow[r, "\varphi", "\sim"'] & W_{2(n-1)+r}^r \end{tikzcd}$$

Indeed, when $i \neq j$ this is given by $h_j = 1+T_j$, and we have $h_i = q^{-1}(1+T_i)(1+T_{i+1})(1+T_{i-1})$, as given by Figure 4.

Due to the inductive hypothesis, there is some action $h' \in \mathscr{H}(S_{2(n-1)+r})$ sending $\varphi((1+T_i)w)$ to a basis vector; then, the action $\mathscr{H}$ generates the endomorphism $\varphi^{-1}h'\varphi$ sending $(1+T_i)w$ to a basis vector, giving w cyclic and hence W_{2n+r}^r irreducible. $\square$

3.2. **Correspondence.**

Proposition 3.4.

(i) Suppose that $n, r > 0$. Then, a filtration of $Res W^r_{2n+r}$ is given by

$$0 \subset W^{r-1}_{2n+r-1} \subset Res\, W^r_{2n+r} \tag{3.2}$$

with $Res\, W^r_{2n+r}/W^{r-1}_{2n+r-1} \simeq W^{r+1}_{2n+r-1}$.

(ii) We have the following isomorphism of representations:

$$W^1_{2n-1} \simeq Res\, W^0_{2n} \tag{3.3}$$

When the case is type (i) from the irreduibility lemma, this is a composition series.

Proof. **(i)** Note that we may identify the subrepresentation of Res W^r_{2n+r} having anchor n with W^{r-1}_{2n+r-1}.

Let $U :=$ Res $W^r_{2n+r}/W^{r-1}_{2n+r-1}$. Let $\phi : U \to W^{r+1}_{2n+r-1}$ be the k-linear map which regards the arc $(i, 2n+r)$ in U as an anchor at i in W^{r+1}_{2n+r-1}. It is not hard to verify that this is a well-defined isomorphism of vector spaces, so we must show that it is $\mathscr{H}$-linear.

Given a basis vector w_j with arc $(i, 2n+r)$, ϕ is clearly compatible with $T_{i'}$ with $i' \neq i, i-1$. Further, it's easy to verify that ϕ is compatible with T_i and T_{i-1}, as actions on one anchor were designed for this deformation. When there are anchors $(i, i+1)$, then $\phi(T_i w_j) = T_i \phi(w_j) = 0$, and similar for T_{i-1}. Hence ϕ is an isomorphism of representations, and the statement is proven.

(ii) This follows with the above proof, defining $W^{-1}_{2n-1} := 0$ □

Theorem 3.5. *Suppose $e > n + r + 1$. Then, $W^r_{2n+r} \simeq S^{(n+r,n)'}$.*

Proof. The case $n = 0$ is already proven, so suppose $n > 0$.

By irreducibility, we know that $W^r_{2n+r} \simeq D^\lambda$ for some e-restricted partition λ. We will prove this inductively; by identification with the trivial and sign representations, the $2n + r = 2$ caseholds, so suppose this is true for W^s_{2m+s} whenever $2m + s < 2n + r$ and $m + s \leq n + r$ (i.e. $e > m + s + 1$).

By the inductive hypothesis and irreducibility, we have a composition series given by

$$0 \longrightarrow D^{(n+r-1,n)'} \longrightarrow \mathrm{Res}\, D^\lambda \longrightarrow D^{(n+r,n-1)'} \longrightarrow 0 \tag{3.4}$$

Hence the theorem is given by Proposition 2.6. □

3.3. **Kernels and Further Work.**

Lemma 3.6. *Let w_j be the basis vector with anchors $1, \ldots, r$ and all arcs of maximal length. Suppose $w \in K \setminus \{0\}$. Then, w_j is represented in w.*

Proposition 3.7. *Suppose $e = n + r + 1$. Then K is nontrivial.*

4. The Fibonacci Representation and Quotients of Specht Modules

We can start our study of V by studying low-dimensional cases. First, note that V^2_{*0} is the sign representation $D^{(2)}$ and V^2_{*0} is the trivial representation $D^{(1)^2}$.

V^2_{00} is a 2-dimensional representation of a semisimple commutative algebra, and hence decomposes into a direct sum of two subrepresentations. In particular, we can use the basis $\{(0{*}0), (000)\}$ and explicitly write the matrix

$$\rho_{T_1} = \begin{bmatrix} \varepsilon_1 & \delta \\ \delta & \varepsilon_2 \end{bmatrix}$$

having characteristic polynomial $(\varepsilon_1 - \lambda)(\varepsilon_2 - \lambda) - \delta^2 = \lambda^2 - (\varepsilon_1 + \varepsilon_2)\lambda + (\varepsilon_1\varepsilon_2 - \delta^2)$. We may verify that, for $\lambda = -1$, this evaluates to

$$-((-1 + q + q^2)(1 + q^3 + q^4 + q^5 + 2q^6 + q^7))\, [5]_q = 0$$

and for $\lambda = q$ this evaluates to

$$-(q^2(-1 + q + q^2)(1 + q + q^2 + q^3 + 2q^4 + q^5))\, [5]_q = 0$$

hence ρ_{T_1} has eigenvalues -1 and q.

The eigenspaces with eigenvalues -1 and q are subrepresentations isomorphic to the sign and trivial representation, hence V_{00} is isomorphic to a direct sum of the trivial and sign representations: $V^2_{00} \simeq V^2_{*0} \oplus V^2_{**}$.

Now let's prove that V_{**}^3 is irreducible; this has basis $\{\texttt{*0*0}\}, \{\texttt{*000}\}$, and the following matrices:

$$\rho_{T_1} = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{bmatrix}; \qquad \rho_{T_2} = \begin{bmatrix} \varepsilon_1 & \delta \\ \delta & \varepsilon_2 \end{bmatrix}.$$

A subrepresentation must be one-dimensional, and hence an eigenspace of each of these matrices; since $\alpha_2 \neq \alpha_1$, the first has eigenspaces given by the spans of basis elements, and since $\delta \neq 0$, these are not eigenspaces of the second. Hence V_{**}^3 is irreducible. Now we may move on to the general case.

Proposition 4.1. *The representation $V_{*0} := V_{*0}^m$ is irreducible.*

Proof. We will prove this inductively in m. We've already proven it for V_{*0}^2 and V_{*0}^3, so suppose that V_{*0}^{m-2} is irreducible.

Let $\{v_i\}$ be the basis for V_{*0}. Then, each v_i is cyclic; indeed, we can transform every basis vector into $(\texttt{*0}\dots\texttt{0})$ by multiplying by the appropriate $\frac{1}{\delta-\varepsilon_1}(T_i - \varepsilon_1)$, and we can transform $(\texttt{*0}\dots\texttt{0})$ into any basis vector by multiplying be the appropriate $\frac{1}{\delta-\varepsilon_2}(T_i - \varepsilon_2)$. Hence it is sufficient to show that each $v \in V_{*0}$ generate some basis element.

Let v' be the basis element $(\texttt{*0*0}\dots\texttt{0})$, which is many copies of $\texttt{*0}$, followed by an extra $\texttt{0}$ if m is odd. We will show that each $v \in F$ generates v'.

Suppose that no elements beginning $(\texttt{*0*0})$ are represented in v_i; then, all such elements are represented in $T_3 v$, so we may assume that at least one is represented in v.

Note that $\operatorname{im}(T_2 - \alpha_1) = \operatorname{Span}\{\text{Basis vectors beginning } (\texttt{*0*0})\}$ and $(T_2 - \alpha_1)v \neq 0$. Further, note that $\operatorname{Res}^{\mathscr{H}(S_m)}_{\mathscr{H}(S_{m-2})} \operatorname{im}(T_2 - \alpha_1) \simeq V_{*0}^{m-2}$ as representations. Hence irreducibility of V_{*0}^{m-2} implies that v' is generated by $(T_2 - \alpha_1)v$, and V_{*0}^m is irreducible. $\square$

Knowing this, the restriction statements are clear; $\operatorname{Res} V_{*0}^m \simeq V_{00}^{m-1}$ by considering the last $m-2$ transpositions, and $\operatorname{Res} V_{*0}^{m-1} \simeq V_{*0}^{m-1} \oplus V_{**}^{m-1}$ by considering the first $m-2$. Similarly, $\operatorname{Res} V_{**}^m \simeq V_{*0}^{m-1}$ by considering the first $m-2$ transpositions. This gives that $V \simeq 3V_{*0} \oplus 2V_{**}$.

Now we may move on and use Young Tableau to characterize V. Recall that the socle of D^λ is given by $\bigoplus_{\mu \xrightarrow{\text{good}} \lambda} D^\mu$, and that D^λ is semisimple iff every $\mu \xrightarrow{\text{normal}} \lambda$ is good.

Theorem 4.2. *The irreducible components of V are given by the following isomorphisms:*

$$\begin{aligned} V_{**}^{2n} &\simeq D^{(n,n)'} \\ V_{**}^{2n-1} &\simeq D^{(n+1,n-2)'} \\ V_{*0}^{2n} &\simeq D^{(n+1,n-1)'} \\ V_{*0}^{2n-1} &\simeq D^{(n,n-1)'}. \end{aligned}$$

Proof. We will prove this by induction on n; we have already proven the base case V^2, so suppose that we have proven these isomorphisms for V^{2n-2}. We will prove the isomorphisms for V^{2n-1} and V^{2n}.

By irreducibility, $V_{**}^{2n-1} \simeq D^{\lambda_{**}}$ and $V_{*0}^{2n-1} \simeq D^{\lambda_{*0}}$ for some diagrams λ_{**} and λ_{*0}. We will show that $\lambda_{**} = (n+1, n-2)'$ and $\lambda_{*0} = (n+1, n-1)'$.

First, note that we have

$$\operatorname{Res} D^{\lambda_{**}} \simeq D^{(n,n-2)'} \simeq \operatorname{Res} D^{(n+1,n-2)'}$$

and

$$\operatorname{Res} D^{\lambda_{*0}} \simeq D^{(n,n-2)} \oplus D^{(n-1,n-1)} \simeq \operatorname{Res} D^{(n,n-1)'}.$$

By semisimplicity of $\operatorname{Res} D^{\lambda_{**}}$ and $\operatorname{Res} D^{\lambda_{*0}}$, every normal cell in λ_{**} and λ_{*0} is good. In particular, the only normal number in λ_{**} is 1.

For λ_{**}, the only tableaux which can remove a cell to yield $D^{(n,n-2)'}$ are $(n+1, n-2)'$, $(n, n-1)'$, and $(n, n-2, 1)'$ as illustrated in Figure 5; we have already seen that $D^{(n,n-1)'}$ does not have irreducible restriction, so we are left with $(n+1, n-2)'$ and $\lambda = (n, n-2, 1)'$. We may directly check that λ doesn't

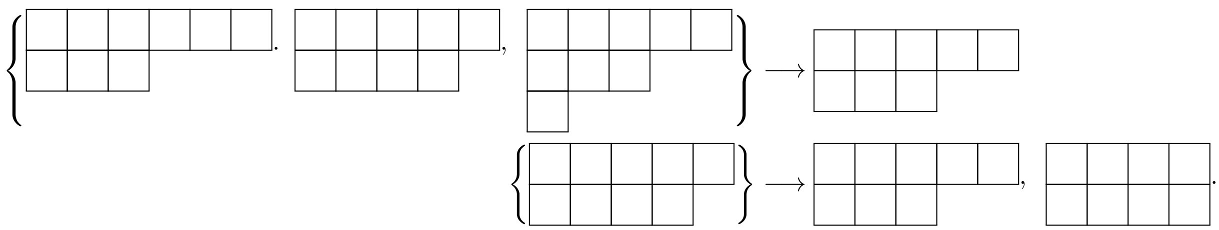

Figure 5. Illustration of the partitions of 9 which can, via row removal, yield $(n, n-2)'$ alone, or both $(n, n-2)'$ and $(n-1, n-1)'$.

satisfy this, as we have the following:

$$\beta_\lambda(1,2) = 3 - 2 + (n-2) = n-1$$
$$\beta_\lambda(1,3) = 3 - 1 + n = n+2$$
$$\beta_\lambda(2,3) = 2 - 1 + 3 = 4.$$

At least one of $\beta_\lambda(1,2)$ and $\beta_\lambda(1,3)$ is nonzero, since $\beta_\lambda(1,3) - \beta_\lambda(1,2) = 3 \not\equiv 0 \pmod e$, and hence at least one of M_2 and M_3 is empty. Hence at least one of 2 or 3 is normal in $(n, n-2, 1)'$, and $\lambda_{**} = (n+1, n-2)$.

For λ_{*0}, we immediately see from Figure 5 that the only option is $(n, n-1)$.

We can perform a similar argument for the V^{2n} case, finding now that

$$\text{Res } D^{\mu_{**}} \simeq D^{(n,n-1)'} \simeq \text{Res } D^{(n,n)'}$$

and

$$\text{Res } D^{\mu_{*0}} \simeq D^{(n,n-1)'} \oplus D^{(n+1,n-2)'} \simeq \text{Res } D^{(n+1,n-1)'}.$$

Through a similar process, we see that $\mu_{*0} = (n+1, n-1)'$. We narrow down μ_{**} to one of $(n,n)'$ or $\mu := (n, n-1, 1)'$, and note that

$$\beta_\mu(1,2) = 3 - 2 + (n-1) = n$$
$$\beta_\mu(1,3) = 3 - 1 + n = n+2$$
$$\beta_\mu(2,3) = 2 - 1 + 2 = 3$$

and hence at least one of 2 or 3 is normal, $\text{Res} D^{(n,n-1,1)'}$ is not irreducible, and $\mu_{**} = (n,n)'$, finishing our proof. ☐

Corollary 4.3. *We have the following isomorphisms of representations:*

$$V^{2n} \simeq 3D^{(n+1,n-1)'} \oplus 2D^{(n,n)'}$$
$$V^{2n-1} \simeq 3D^{(n,n-1)'} \oplus 2D^{(n+1,n-2)'}$$

5. Conjecture

Recall that $K^r_{2n+r} := K$ is the direct sum of all copies of the sign representation in W. Hence the following characterises sign subrepresentations of W completely:

Proposition 5.1. *$K \subset W^r_{2n+r}$ is trivial when $e \neq n+r+1$, and $\dim K = 1$ when $e = n+r+1$.* ☐

Proposition 5.2. *Suppose $e < n+r+1$, and suppose n' is such that $e = n'+r+1$. Note that $h := (1+T_1)(1+T_3)\dots(1+T_{n-n'})$ maps W^r_{2n+r} onto $W^r_{2n'+r}$. Then, the preimage $h^{-1}(K^r_{2n+r})$ is a subrepresentation of W^r_{2n+r}, and the series*

$$0 \subset h^{-1}(K^r_{2n+r}) \subset W^r_{2n+r}$$

is a composition series of W^r_{2n+r}. ☐

Proposition 5.3. *Denote the composition factor $W^r_{2n+r}/h^{-1}(K^r_{2n+r})$ by U^r_{2n+r}. Then, there exist some naturals m, s satisfying $2m+s = 2n+r$ and $m+s > n+r$ such that the following is an isomorphism of $\mathscr{H}$-modules*

$$h^{-1}\left(K^r_{2n+r}\right) \simeq U^s_{2m+s}$$

. □

Proposition 5.4. *For the same m, s as above, we have the following composition series of specht modules:*
$$0 \longrightarrow D^{(m+s,m)'} \longrightarrow S^{(n+r,n)'} \longrightarrow D^{(n+r,n)'} \longrightarrow 0.$$

Proposition 5.5. $W^r_{2n+r} \simeq S^{(n+r,n)'}$ *and* $U^s_{2m+s} \simeq D^{(m+s,m)'}$.

6. Empirical Results

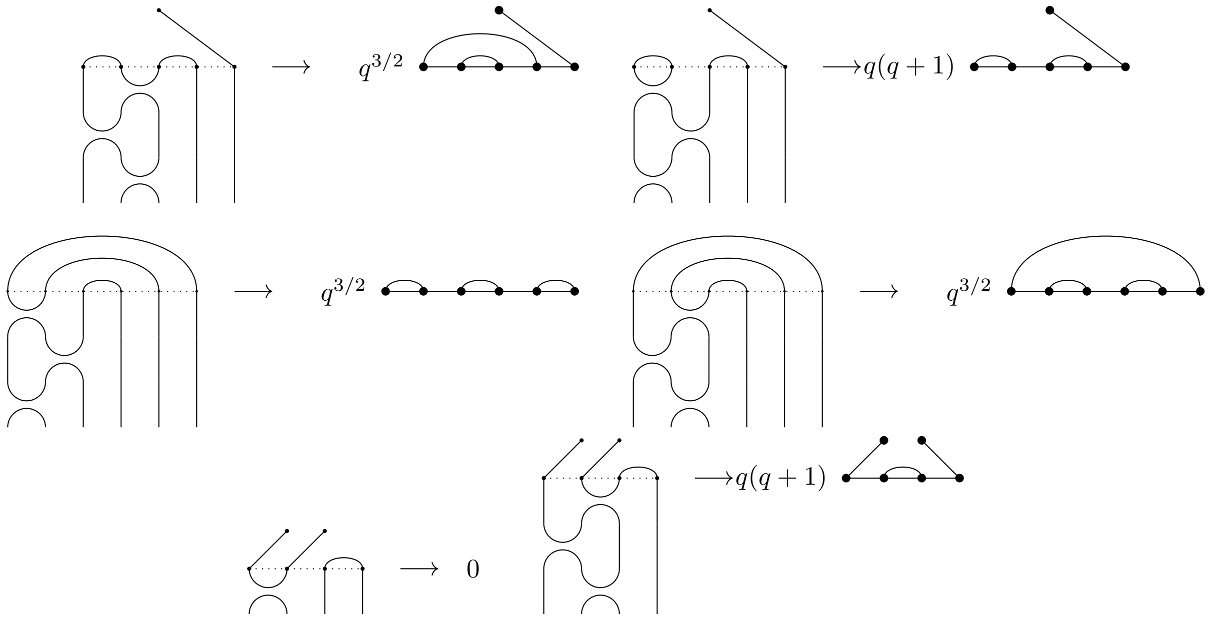

Figure 6. Here we verify in small cases that $hw = qT_i$ and $gw = qT_{i+1}$. These 6 cases cover the situations that there is an arc among the indices $i, i+1, i+2$, that there isn't and there are not two arcs, and that there are two arcs.

Appendix A. Compatibility of Representations with the Relations

In general, we define representations above for the free algebra on generators $\{T_i\}$. Recall that we may give a presentation of $\mathscr{H}$ having generators T_i and relations

$$(T_i - q)(T_i + 1) = 0 \tag{A.1}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{A.2}$$

$$T_i T_j = T_j T_i \qquad |i - j| > 1. \tag{A.3}$$

We call (A.1) the *quadratic relation* and (A.2), (A.3) the *braid relations*. It is easily seen that a representation of $\mathscr{H}$ is equivalent to a representation of the free algebra $k\langle T_i \rangle$ which acts as 0 on the relations (henceforth referred to as *compatibility* with the relations). We will prove in the following sections that V and W are compatible with the Hecke algebra relations.

A.1. **The Crossingless Matchings Representaiton.** Take some basis vector w_i. We will first check (A.1) by case work:

- Suppose there is an arc $(i, i+1)$. Then, $(T_i - q)(T_i + 1)w = (1+q)\left[(1 + T_i)w - (1+q)w\right] = 0$, giving (A.1).
- Suppose there is no arc $(i, i+1)$ and $i, i+1$ do not both have anchors; then $(T_i + 1)w = q^{1/2}w''$ for some basis vector w' having arc $(i, i+1)$, and the computation follows as above for (A.1).
- Suppose $i, i+1$ are anchrors; then $(T_i + 1)w = 0$, giving (A.1).

Now we verify (A.2). Let $h := (1 + T_i)(1 + T_{i+1})(1 + T_i)$, and let $g := (1 + T_{i+1})(1 + T_i)(1 + T_{i+1})$. Note the following expansion:

$$\begin{aligned} hw &= 1 + 2T_i + T_i^2 + T_{i+1} + T_i T_{i+1} + T_{i+1} T_i + T_i T_{i+1} T_i \\ &= 1 + (1+q)T_i + T_{i+1} + T_i T_{i+1} + T_{i+1} T_i + T_i T_{i+1} T_i. \end{aligned}$$

An analogous formula gives an analogous equality in g. Hence we have

$$(h - g)w = q(T_i - T_{i+1}) + T_i T_{i+1} T_i - T_{i+1} T_i T_{i+1}.$$

Hence we may equivalently check that $(h - g)w = q(T_i - T_{i+1})$. This is illustrated in Figure 6.

Lastly, we have the equation

$$(1+T_i)(1+T_j)-(1+T_j)(1+T_i)=T_iT_j-T_jT_i$$

and hence we simply need to verify that $(1+T_i)$ and $(1+T_j)$ commute, which the reader may easily check.

A.2. **The Fibonacci Representation.** Similar to before, the reader may verify that (A.3) follows easily, and the others may be verified on strings of length 3 and 4. By considering the coefficients in order of (1.1), the quadratic relation (A.1) gives the following quadratics:

$$\begin{aligned}(\alpha_1-q)(\alpha_1+1)&=0\\(\alpha_2-q)(\alpha_2+1)&=0\\\varepsilon_1\delta+\delta\varepsilon_2&=(q-1)\delta\\\varepsilon_1^2+\delta^2&=(q-1)\varepsilon_1+q\\\varepsilon_2^2+\delta^2&=(q-1)\varepsilon_2+q\end{aligned}\tag{A.4}$$

The first two of these are easily verified for any q. Since $\delta\neq 0$, the third is equivalently given by

$$(q-1)=\varepsilon_1+\varepsilon_2=t(q\tau-1+q-\tau)=(\tau^2+\tau)(q-1)$$

or that $\left(\tau^2+\tau-1\right)(q-1)=0$. One may verify that

$$\tau^2+\tau-1=q^6+2q^5+q^4+q^3+q^2-1=(-1+q+q^2)\,[5]_q=0.$$

The fourth is given by the quadratic

$$\tau^2\left[(q\tau-1)^2-\tau(q+1)\right]=\tau(q-1)(q\tau-1)+q$$

or equivalently,

$$(\tau^2+\tau-1)\left[q\left(qt^2+1\right)+t\right]=0$$

which is true for every q.

The fifth is similarly given by

$$(\tau^2+\tau-1)\left[q\left(qt+1\right)+t^2\right]=0$$

which is true for every q.

We now verify (A.2). We may order the basis for V^4 as follows:

$$\{(\mathtt{0000}),(\mathtt{*00*}),(\mathtt{000*}),(\mathtt{*000}),(\mathtt{*0*0}),(\mathtt{0*0*}),(\mathtt{00*0}),(\mathtt{0*00})\}.$$

Then, in verifying the braid relation (A.2) in this order, we encounter the following quadratics (with tautologies and repetitions omitted):

$$\begin{aligned}\alpha_1\varepsilon_2^2+\alpha_2\delta^2&=\alpha_1^2\varepsilon_2\\\alpha_1\delta\varepsilon_2+\alpha_2\varepsilon_1\delta&=\alpha_1\alpha_2\delta\\\alpha_2\varepsilon_1^2+\alpha_1\delta^2&=\alpha_2^2\varepsilon_1\\\alpha_1\varepsilon_1^2+\delta^2\varepsilon_2&=\alpha_1^2\varepsilon_1\\\delta\varepsilon_2^2+\alpha_1\varepsilon_1\delta&=\alpha_1\delta\varepsilon_2\end{aligned}$$

Substituting in τ and dividing by δ whenever possible, these are equivalent to the vanishing of the following polynomials in q:

$$\begin{aligned}-q(1+q)(1+q^2+q^3)(2+q+3q^2+2q^3)\,[5]_q&=0\\(1+2q+q^3+q^4)\,[5]_q&=0\\(1+q)^2(1+q^2+q^3)(1+3q^3-q^4+q^6)\,[5]_q&=0\\(1+q)^2(1+q^2+q^3)(1+5q+5q^2+3q^3+3q^4+3q^5+q^6)\,[5]_q&=0\\(1+q)(1+q^2+q^3)(-1+2q+q^2+q^3+q^4)\,[5]_q&=0.\end{aligned}$$

Notably, each of these vanish when $e=5$.

Appendix B. Miscellaneous Algebra Facts

Throughout the text, for some representation V, we refer to $\mathrm{Res}^{\mathscr{H}(S_m)}_{\mathscr{H}(S_l)} V$ without specifying exactly which subalgebra $\mathscr{H}(S_l)$.

Proposition B.1. *Suppose B, B' are subalgebras of the k-algebra A with $B = uB'u^{-1}$, and let V be a representation of A. Then, the linear isomorphism $V \xrightarrow{\phi} V$ given by $v \mapsto uv$ causes the following to commute for any $b \in B$:*

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "b"] & V \arrow[d, "ubu^{-1}"] \\ V \arrow[r, "\phi"] & V \end{tikzcd}$$

Hence, through the identification of B and B' via conjugation, we have $Res^A_B V \simeq Res^A_{B'} V$

Proof. This is simply given by $(ubu^{-1})uv = ubv$. $\square$

Corollary B.2. *Suppose $\mathscr{H}', \mathscr{H}''$ are two subalgebras of $\mathscr{H}(S_m)$ generated by l reflections and V is a representation of $\mathscr{H}$. Then, $Res^{\mathscr{H}}_{\mathscr{H}'} V \simeq Res^{\mathscr{H}}_{\mathscr{H}''} V$.*

Proof. Let $\mathscr{H}'$ and $\mathscr{H}''$ be the subalgebras of $\mathscr{H}(S_m)$ generated by the reflections $\{T_{i_1}, \ldots, T_{i_l}\}$ and $\left\{T_{i_1}, \ldots, T_{i_{j-1}}, T_{i_j+1}, T_{i_{j+1}}, \ldots, T_{i_l}\right\}$ for $1 \leq i_1 < \cdots < i_{j-1} < i_j + 1 < i_{j+1} < \cdots < i_l \leq n$. It is sufficient to prove that $\mathscr{H}'$ and $\mathscr{H}''$ are conjugate; then transitivity gives conjugacy of any $S_l \subset S_m$, and the previous proposition gives isomorphisms of the representations.

In fact, the reader can verify that $\mathscr{H}'' = T_{i_j} \mathscr{H}' T_{i_j}^{-1}$. $\square$