THE UNEVEN-HEIGHT TWO-COLUMN SPECHT MODULES OF THE HECKE ALGEBRA OF S_n

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1. Introduction

Let S_{2n+r} be the symmetric group on $2n+r$ indices, let $\mathscr{H} = \mathscr{H}_{k,q}(S_{2n+r})$ be the corresponding Hecke algebra over field k with parameter $q \in k$, and let $\{T_i\}$ be the simple transpositions generating $\mathscr{H}$. Let $[m]_q = 1 + q + \cdots + q^{m-1}$ be the q-number of m. Let e be the smallest positive integer such that $[e]_q = 0$, and set $e = \infty$ if no such integer exists. Either $q = 1$ and e is the characteristic of k, or $q \neq 1$ and q is a primitive eth root of unity.

Let $V_{2n}^r := S^{(n,n-r)'}$ be the Specht module corresponding to the young diagram with two columns with height difference r. The purpose of this writing is to characterize this representation via an isomorphism with another representation of $\mathscr{H}$.

Definition 1.1. A *generalized crossingless matching* on $2n + r$ indices with r anchors is a partition of $\{1, \ldots, 2n + r\}$ into parts of size 2 or 1 such that no two parts of size two"cross", i.e. there are no parts (a, a') and (b, b') such that $a < b < a' < b'$, and no parts of size one are "inside" of a part of size two, i.e. there are no $c, (a, a')$ such that $a < c < a'$. We will call these arcs and anchors, respectively. Then, define W_{2n+r}^r to be the k-vector space with basis the set of generalized crossingless matchings on $2n + r$ indices with r anchors.

In order for this to be a $\mathscr{H}$-module, endow this with the action given by Figure 1; if this involves no anchors, act as in W; if it involves one loop, deform to another generalized crossingless matching and scale by $q^{1/2}$, and otherwise scale by 0.

Let the length of an arc (i, j) be $l(i, j) := j - i + 1$. Note that the crossingless matchings can all be identified with a list of n integers describing the lengths of the arcs from left to right; using this, we may order the crossingless matchings with 0 hooks in increasing lexicographical order in order to obtain an order on the subbasis containing a particular set of anchors; let the basis be ordered first by the position of the anchors in increasing lexicographical order, then increasing for the matchings between each anchor. Let this basis be $\{w_i\}$. This basis is illustrated for W_5^1 in Figure 2.

We will prove that $W := W_{2n+r}^r$ and $V^{(n+r,n)'}$ are isomorphic as representations in the case that $\mathscr{H}$ is semisimple.

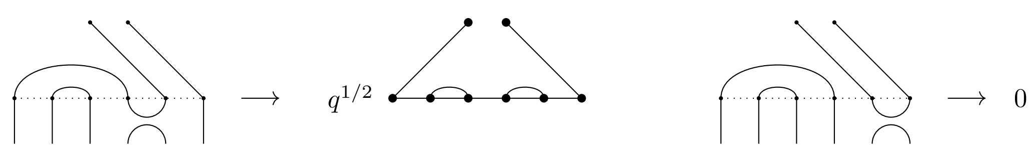

Figure 1. Illustration of the actions $(1+T_4)w_{|W_6^2|}$ and $(1+T_2)w_{|W_6^2|}$ in W_6^2. In general, we act on basis elements away from anchors as we did for W, at one anchor we act by deforming and scaling by $q^{1/2}$, and at two anchors we send the element to zero.

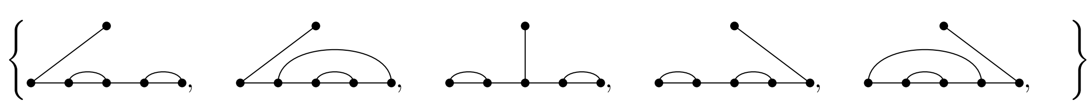

Figure 2. The basis for W_5^1.

2. Correspondence

We can now begin by proving that W is irreducible; then $W \simeq S^\lambda$ for some partition λ of $2n+r$, and we may use branching rules to determne W.

Proposition 2.1. *Suppose $n > 2$. Then, W^r_{2n+r} is irreducible if $e > n+1$.*

Proof. We will prove the equivalent condition that each vector in $w \in W \setminus \{0\}$ is *cyclic*, i.e. $\mathscr{H}w = W$. Note that, similar to the $r = 0$ case, each basis vector is cyclic; we may act between each anchor to have only anchors and length-2 arcs, move the anchors to the desired position, and use irreducibility of W^0_m to act btween arcs in order to generate any other basis element.

Let $U_{x_1 \dots x_r}$ be the subspace of W with anchors $x_1, \dots, x_r$. Order these in increasing lexicographical order; let $U_{a_1 \dots a_r}$ be the first of these on which w projects to a nonzero vector. Then, we may use irreducibility of W^0_m to act between the anchors to generate a vector w' which projects in $U_{a_1 \dots a_r}$ to the basis element of arcs of length 2 between arcs $a_1, \dots, a_r$.

Note that $n + r - a_r$ is even; hence we can apply

$$h = (1 + T_{n+r-3})(1 + T_{n+r-5})(1 + T_{n+r-4})(1 + T_{n+r-7})(1 + T_{n+r-6}) \dots (1 + T_{a_r+2})(1 + T_{a_r+1})$$

and we find that all basis vectors represented in hw' are one in $U_{a_1 \dots a_r}$ containing arc $(2n+r-1, 2n+r)$ or others containing anchor $2n+r$. Hence $(T_{2n+r-1} - q)hw'$ is a nonzero vector (since $q^{1/2} \neq q$ as $e > 3$) represeting a unique basis element, giving that w is cyclic and W^r_{2n+r} is irreducible. $\square$

The next piece in our puzzle is to characterize the restrictions of W to $\mathscr{H}' := \mathscr{H}_{k,q}(S_{2n+r-1}) \subset \mathscr{H}$. Recall that, when $r, n > 0$, $\mathrm{Res} S^{(n+r,n)'} \simeq S^{(n+r-1,n)'} \oplus S^{(n+r,n-1)'}$. Further, note that $S^{(n+r,n)'}$ is the unique irreducible having this restriction.

Next, note that we have already proven the correspondence for W^0_{2n}; for W^r_{0+r}, this is the sign representation, which is given correctly by $S^{(r)}$. Hence, pending information on restrictions, we may prove this via induction on $2n+r$.

Proposition 2.2. *Suppose that $n, r > 0$ and $\mathscr{H}$ is semisimple. Then, Res $W^r_{2n+r} \simeq W^{r-1}_{2n+r-1} \oplus W^{r+1}_{2n+r-1}$.*

Proof. Note that we may identify the subrepresentation of Res W^r_{2n+r} having anchor n with W^{r-1}_{2n+r-1}. By semisimplicity, it is sufficient to prove that $U :=$ Res $W^r_{2n+r}/W^{r-1}_{2n+r-1}$ is isomorphic to W^{r+1}_{2n+r-1}.

Let $\phi : U \to W^{r+1}_{2n+r-1}$ be the k-linear map which regards the arc $(i, 2n+r)$ in U as an anchor at i in W^{r+1}_{2n+r-1}. It is not hard to verify that this is a well-defined isomorphism of vector spaces, so we must show that it is $\mathscr{H}$-linear.

Given a basis vector w_j with arc $(i, 2n+r)$, ϕ is clearly compatible with $T_{i'}$ with $i' \neq i, i-1$. Further, it's easy to verify that ϕ is compatible with T_i and T_{i-1}, as actions on one anchor were designed for this deformation. When there are anchors $(i, i+1)$, then $\phi(T_i w_j) = T_i \phi(w_j) = 0$, and similar for T_{i-1}. Hence ϕ is an isomorphism of representations, and the statement is proven. $\square$

Corollary 2.3. $W^r_{2n+r} \simeq S^{(n+r,r)'}$.

Proof. We may argue by induction on $2n+r$, knowing that we have proven the base case $2n+r = 2$. Assume that we have proven the isomorphism for all W^s_{2n+s} with $2n+s = 2n+r-2$. We have proven the $n = 0$ and $r = 0$ cases already, so assume $n, r > 0$.

Then, W^r_{2n+r} is the unique irreducible representation of $\mathscr{H}$ having restriction $S^{(n+r-1,n)'} \oplus S^{(n+r,n-1)'}$, implying the desired isomorphism. $\square$