ROUGH CUT OF PROVEN WORK ON $\mathscr{H}$.

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1. Introduction

Let S_{2n+r} be the symmetric group on $2n+r$ indices with $2n+r \geq 2$, let $\mathscr{H} = \mathscr{H}_{k,q}(S_{2n+r})$ be the corresponding Hecke algebra over field k with parameter $q \in k^\times$ having square root $q^{1/2}$, and let $\{T_i\}$ be the reflections generating $\mathscr{H}$. Let $[m]_q = 1+q+\cdots+q^{m-1}$ be the q-number of m. Let e be the smallest positive integer such that $[e]_q = 0$, and set $e = \infty$ if no such integer exists. Either $q = 1$ and e is the characteristic of k, or $q \neq 1$ and q is a primitive eth root of unity.

Let $S^{(n+r,n)'}$ be the Specht module corresponding to the young diagram with two columns with height difference r, and let $D^{(n+r,n)'}$ be the corresponding irreducible quotient. The purpose of this writing is to characterize these representation via an isomorphism with two graphical representations of $\mathscr{H}$.

1.1. **Crossingless Matchings.**

Definition 1.1. A *crossingless matching on* $2n+r$ *indices with* r *anchors* is a partition of $\{1,\ldots,2n+r\}$ into n parts of size 2 and r of size 1 such that no two parts of size two"cross", i.e. there are no parts (a,a') and (b,b') such that $a < b < a' < b'$, and no parts of size one are "inside" of a part of size two, i.e. there are no $c,(a,a')$ such that $a < c < a'$. We will call these arcs and anchors, respectively. Then, define W^r_{2n+r} to be the k-vector space with basis the set of generalized crossingless matchings on $2n+r$ indices with r anchors.

In order for this to be a $\mathscr{H}$-module, endow this with the action given by Figure 1; if a "loop" is created, scale by $q+1$, if a loop is not created and the action involves fewer than 2 anchors, deform into a new crossingless maching and scale by $q^{1/2}$, and if it involves two anchors, scale by 0. We verify that this is well-defined in appendix A.1.

Let the length of an arc (i,j) be $l(i,j) := j-i+1$. Note that the crossingless matchings on $2n$ indices with no anchros can all be identified with a list of n integers describing the lengths of the arcs from left to right; using this, we may order the crossingless matchings with no anchors in increasing lexicographical order in order to obtain an order on the subbasis containing a particular set of anchors; let the basis be ordered first by the position of the anchors in increasing lexicographical order, then increasing for the matchings between each anchor. Let this basis be $\{w_i\}$. This basis is illustrated for W^1_5 in Figure 2.

We will prove that $W := W^r_{2n+r}$ and $S := S^{(n+r,n)'}$ are isomorphic as representations in the case that $e > n+r+1$. Note that, when $r = 0$, these have the same dimension given by the nth catalan number C_n.

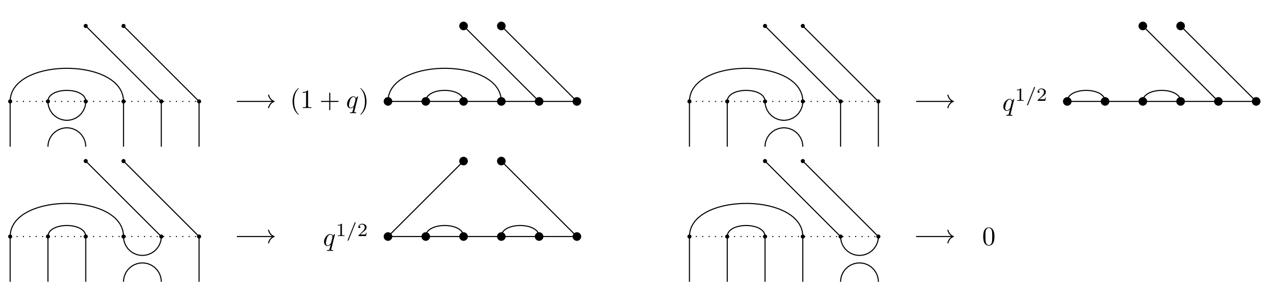

Figure 1. Illustration of the actions $(1+T_i)w_{|W^2_6|}$. In general, we act by deleting loops, deforming into a new crossingless matching, and scaling by either $q^{1/2}$, $(q+1)$, or 0.

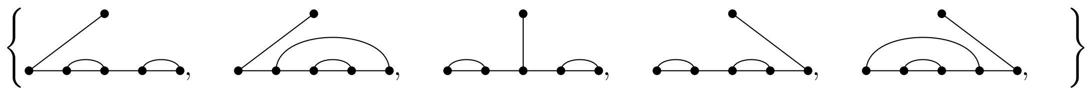

Figure 2. The basis for W_5^1.

1.2. **Fibonacci Representation.** Now suppose that $k = \mathbb{C}$ and $q = \exp(2\pi i\ell/5)$ is a primitive 5th root of unity. Let V^m be a k-vector space with basis given by the strings $\{*, p\}^{n+1}$ such that the character $*$ never appears twice in a row. We will surpress the superscript whenever it is clear from context.

We wish to endow this with a $\mathscr{H}$-action which acts on a basis vector only dependent on characters $i, i+1, i+2$, sending each basis vector to a combination of the other basis vectors having the same characters $1, \ldots, i, i+2, \ldots, n+1$ as follows:

$$\begin{aligned} T_1(*pp) &:= \alpha(*pp) \\ T_1(pp*) &:= \alpha(pp*) \\ T_1(*p*) &:= \beta(*p*) \\ T_1(p*p) &:= \gamma(p*p) + \delta(ppp) \\ T_1(ppp) &:= \delta(p*p) + \varepsilon(ppp) \end{aligned} \tag{1}$$

for constants

$$\begin{aligned} \alpha &= -1 \\ \beta &= q \\ \gamma &= \tau(q\tau - 1) \\ \delta &= \tau^{3/2}(q+1) \\ \varepsilon &= \tau(q-\tau) \\ \tau &= \begin{cases} \frac{1}{2}\left(\sqrt{5}-1\right) & \ell \equiv 1, 4 \pmod 5 \\ \frac{1}{2}\left(\sqrt{5}+1\right) & \ell \equiv 2, 3 \pmod 5 \end{cases} \end{aligned} \tag{2}$$

with T_i acting similarly on the substring $i, i+1, i+2$. We will verify that this is a representation of $\mathscr{H}$ in Appendix A.2

This contains 4 subrepresentations based on the first and last character of the string, which are not modified by $\mathscr{H}$. Label the subrepresentation of strings $(* \cdots *)$ by V_{**}, and similar for the other 3. It is easy to see that $V_{*p} \simeq V_{p*}$, so that

$$V \simeq 2V_{*p} \oplus V_{**} \oplus V_{pp}.$$

We will show that $V_{pp} \simeq V_{*p} \oplus V_{**}$, and give the following isomorphisms with irreducible quotients of specht modules depending on the parity of the number of indices in $\mathscr{H}$:

$$\begin{aligned} F_{**}^{2n} &\simeq D^{(n,n)'} \\ F_{**}^{2n-1} &\simeq D^{(n+1,n-2)'} \\ F_{*p}^{2n} &\simeq D^{(n+1,n-1)'} \\ F_{*p}^{2n-1} &\simeq D^{(n,n-1)'}. \end{aligned} \tag{3}$$

2. Crossingless Matchings and Specht Modules

3. The Fibonacci Representation and Specht Modules

4. Explicit Relationships

Appendix A. Compatibility of Representations with the Relations

In general, we define representations above for the free algebra on generators $\{T_i\}$. Recall that we may give a presentation of $\mathscr{H}$ having generators T_i and relations

$$(T_i - q)(T_i + 1) = 0 \tag{4}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{5}$$

$$T_i T_j = T_j T_i \qquad |i-j| > 1. \tag{6}$$

We call (4) the *quadratic relation* and (5), (6) the *braid relations.* It is easily seen that a representation of $\mathscr{H}$ is equivalent to a representation of the free algebra $k\langle T_i \rangle$ which acts as 0 on the relations (henceforth referred to as *compatibility* with the relations). We will prove in the following sections that V and W are compatible with the Hecke algebra relations.

A.1. **The Crossingless Matchings Representaiton.** Take some basis vector w_i. We will first check (4) by case work:

- Suppose there is an arc $(i, i+1)$. Then, $(T_i - q)(T_i + 1)w = (1+q)\left[(1+T_i)w - (1+q)w\right] = 0$, giving (4).
- Suppose there is no arc $(i, i+1)$ and $i, i+1$ do not both have anchors; then $(T_i + 1)w = q^{1/2}w''$ for some basis vector w' having arc $(i, i+1)$, and the computation follows as above for (4).
- Suppose $i, i+1$ are anchrors; then $(T_i + 1)w = 0$, giving (4).

Now we verify (5). Let $h := (1+T_i)(1+T_{i+1})(1+T_i)$, and let $g := (1+T_{i+1})(1+T_i)(1+T_{i+1})$. Note the following expansion:

$$\begin{aligned} hw &= 1 + 2T_i + T_i^2 + T_{i+1} + T_i T_{i+1} + T_{i+1} T_i + T_i T_{i+1} T_i \\ &= 1 + (1+q)T_i + T_{i+1} + T_i T_{i+1} + T_{i+1} T_i + T_i T_{i+1} T_i. \end{aligned}$$

An analogous formula gives an analogous equality in g. Hence we have

$$(h-g)w = q(T_i - T_{i+1}) + T_i T_{i+1} T_i - T_{i+1} T_i T_{i+1}.$$

Hence we may equivalently check that $(h-g)w = q(T_i - T_{i+1})$. This is illustrated in Figure 3.

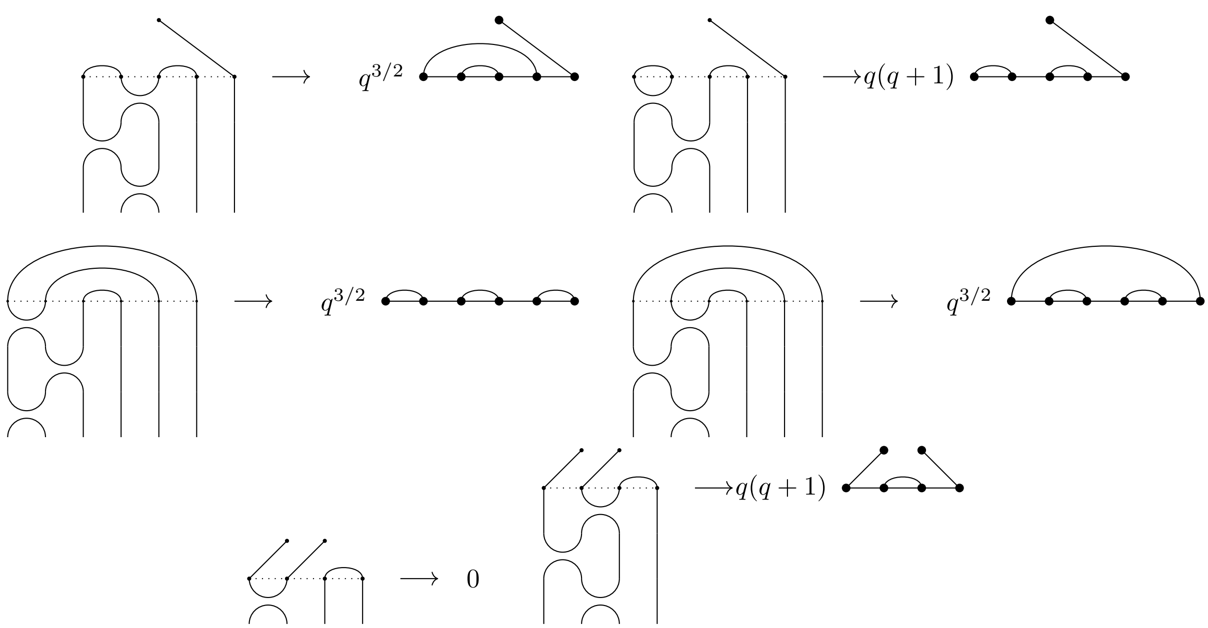

Figure 3. Here we verify in small cases that $hw = qT_i$ and $gw = qT_{i+1}$. These 6 cases cover the situations that there is an arc among the indices $i, i+1, i+2$, that there isn't and there are not two arcs, and that there are two arcs.

Lastly, we have the equation

$$(1+T_i)(1+T_j)-(1+T_j)(1+T_i)=T_iT_j-T_jT_i$$

and hence we simply need to verify that $(1+T_i)$ and $(1+T_j)$ commute, which the reader may easily check.

A.2. **The Fibonacci Representation.** Similar to before, the reader may verify that (6) follows easily, and the others may be verified on strings of length 3 and 4. By considering the coefficients in order of (1), the quadratic relation (4) gives the following quadratics:

$$\begin{aligned}(\alpha-q)(\alpha+1)&=0\\(\beta-q)(\beta+1)&=0\\\gamma\delta+\delta\varepsilon&=(q-1)\delta\\\gamma^2+\delta^2&=(q-1)\gamma+q\\\varepsilon^2+\delta^2&=(q-1)\varepsilon+q\end{aligned}\tag{7}$$

The first two of these are easily verified for any q. Since $\delta\neq 0$, the third is equivalently given by

$$(q-1)=\gamma+\varepsilon=t(q\tau-1+q-\tau)=(\tau^2+\tau)(q-1)$$

or that $\left(\tau^2+\tau-1\right)(q-1)=0$. The reader may verify that $\tau^2+\tau-1=0$, so this is true for every q.

The fourth is given by the quadratic

$$\tau^2\left[(q\tau-1)^2-\tau(q+1)\right]=\tau(q-1)(q\tau-1)+q$$

or equivalently,

$$(\tau^2+\tau-1)\left[q\left(qt^2+1\right)+t\right]=0$$

which is true for every q.

The fifth is similarly given by

$$(\tau^2+\tau-1)\left[q\left(qt+1\right)+t^2\right]=0$$

which is true for every q.

We now verify (5). We may order the basis for V^4 as follows:

$$\{(pppp),(*pp*),(ppp*),(*ppp),(*p*p),(p*p*),(pp*p),(p*pp)\}.$$

Then, in verifying the braid relation (5) in this order, we encounter the following quadratics (with tautologies and repetitions omitted):

$$\begin{aligned}\alpha\varepsilon^2+\beta\delta^2&=\alpha^2\varepsilon\\\alpha\delta\varepsilon+\beta\gamma\delta&=\alpha\beta\delta\\\beta\gamma^2+\alpha\delta^2&=\beta^2\gamma\\\alpha\gamma^2+\delta^2\varepsilon&=\alpha^2\gamma\\\delta\varepsilon^2+\alpha\gamma\delta&=\alpha\delta\varepsilon\end{aligned}$$

The reader may verify that each of these are satisfied for q a primitive 5th root of unity and τ as defined.

This highlights the difficulty with deforming our module to $q=1$ at any field; the quadratic relations require that $(\tau^2+\tau-1)(\tau^2+\tau+1)=0$, but neither of these appear in the first braid relation, which reads $\tau(\tau^3-6\tau^2+1)=0$. If we have $\tau^2+\tau\pm 1=0$, then $\tau\neq 0$ and $-7\tau^2\pm\tau+1=0$. Hence $(7\pm 1)\tau+(1\pm 1)=0$, implying that $\tau=\frac{1}{4},0$, neither of which satisfy $\tau^2+\tau\pm 1=0$, a contradiction.

To attempt to deform this to $q=1$ would require that we rewrite $\gamma,\delta,\varepsilon$ entirely, rather than simply modifying τ.