SOME GRAPHICAL REALIZATIONS OF TWO-ROW SPECHT MODULES OF IWAHORI–HECKE ALGEBRAS OF THE SYMMETRIC GROUP

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Abstract. We consider the Iwahori–Hecke algebra of the symmetric group on $2n+r$ letters with parameter q. Let e be the smallest positive integer such that the q-number $[e]_q = 0$, or set $e = \infty$ if none exist. We modify Khovanov's crossingless matchings to include $2n$ "nodes" and r "anchors," and prove in the case $e > n+r+1$ that the associated module is isomorphic to the Specht module $S^{(n+r,n)}$ which corresponds to the partition $(n+r,n) \vdash 2n+r$. We then give heuristics in support of the general case, including explicit composition series for $e = n+r+1$ and for $2n+r \leq 7$. Lastly, when $e = 5$, we prove an isomorphism between the irreducible quotient $D^{(n+r,n)}$ with $r \leq 3$ and some subrepresentations of Jordan–Shor's Fibonacci representation. We provide explicit transition matrices between this representation and the crossingless matchings representation for $2n+r \leq 6$.

Contents

1. Introduction

Let S_{2n+r} be the symmetric group on $2n+r$ letters with $2n+r \geq 2$. Let $\mathscr{H} := \mathscr{H}_{k,q}(S_{2n+r})$ be the corresponding Iwahori-Hecke algebra (henceforth simply Hecke algebra) over a field k with parameter $q \in k^\times$ having a fixed square root $q^{1/2}$. Let $\{T_1, \ldots, T_{2n+r-1}\}$ be the simple reflections generating $\mathscr{H}$.

Let $[m]_q = 1 + q + \cdots + q^{m-1}$ be the q-number of m. Let e be the smallest positive integer such that $[e]_q = 0$, and set $e = \infty$ if no such integer exists. Note that either $q = 1$ and e is the characteristic of k (with 0 replaced by ∞), or $q \neq 1$ and q is a primitive eth root of unity.

When $q = 1$, the Hecke algebra $\mathscr{H}$ is isomorphic to the group algebra $k[S_{2n+r}]$; hence the representation theory of $\mathscr{H}$ generalizes the representation theory of the symmetric group. The Hecke algebra is also well-known to be connected to the representation theory of the general linear group over a finite field [9]. It is a classical result that $\mathscr{H}$ is semisimple precisely when $e > 2n+r$, in which case the irreducible representations of $\mathscr{H}$ are given by *Specht modules* S^λ, which are indexed by the partitions λ of $2n+r$.

For all e, $\mathscr{H}$ admits a cellular basis with cell modules given by S^λ. In particular, these admit quotients D^λ such that the modules $\{D^\lambda \mid D^\lambda \neq 0, \lambda \vdash n\}$ are a pairwise-nonisomorphic list of all irreducible $\mathscr{H}$-modules. This set is indexed by the partitions $\lambda \vdash n$ which are e-restricted [11, 12].

These representations D^λ have explicit constructions, but many of their properties are unknown. For instance, the dimension of D^λ is unknown outside of some special cases [9]. However, there does exist an algorithm due to Lascoux–Leclerc–Thibon–Ariki which computes the decomposition matrices of the Specht modules of the Hecke algebra $\mathscr{H}_{\mathbb{C},q}(S_{2n+r})$ for q an eth root of unity [1, 8].

The cellular basis for S^λ and associated basis for D^λ are complicated and often computationally intractable. We aim to give simple graphical realizations of S^λ and D^λ in some cases that $\lambda = (n+r, n)$ is a partition of two parts. These realizations behave in an intuitive and computationally simple way.

Throughout this paper, we analyze the *two-row partitions* $(n+r, n) \vdash 2n+r$ and their corresponding modules $S^{(n+r,n)}$ and $D^{(n+r,n)}$.

Remark. Note that we follow the convention of Murphy–Kleshchev concerning the correspondence $\lambda \leftrightarrow S^\lambda$, which is dual to the conventions of Dipper–James–Mathas; one may translate our results to the latter convention by transposing all partitions [7, 9, 11]. For instance, we refer to the *sign representation* as $S^{(1^{2n+r})}$.

Our first approach uses crossingless matchings, originally defined by Khovanov [6], to realize irreducible Specht modules of two-row partitions.

Crossingless matchings. Literature goes here.

Definition 1.1. Define a *crossingless matching on $2n+r$ nodes and r anchors* to be an isotopy class of $n+r$ non-intersecting paths in the slice $\mathbb{R} \times [0,1]$ connecting $2n+r$ distinct points of $\mathbb{R} \times \{0\}$ and r points of $\mathbb{R} \times \{1\}$ such that none of the latter points are connected. Let M^r_{2n+r} have basis given by these matchings. This is illustrated in Figure 1.

Order the points on $\mathbb{R} \times \{0\}$ via the order $<$ on $\mathbb{R}$, and refer to these as *nodes*. Refer to a path connecting the ath and bth node as an *arc* (a, b), and refer to a path connecting node a to a point in $\mathbb{R} \times \{1\}$ as an *anchor*. Let the length of an arc (i, j) be $j - i + 1$.

We endow M^r_{2n+r} with an action by specifying $(1 + T_i)w_j$ for any basis element w_j of M_{2n+r}, as illustrated in Figure 2. We do so by concatenating in "vertical lines" below each point other than the ith and $(i+1)$st, concatenating paths between the ith and $(i+1)$st points as well as points below them, removing any "loops" this forms, and taking the isotopy class of the resultant diagram; if this is not the isotopy class of a crossingless matching, then there are anchors at $i, i+1$ and we set $(1 + T_i)w_j := 0$; if this is the isotopy class of a crossingless matching w_l and there is a "loop," set $(1 + T_i)w_j := (1 + q)w_j$ and otherwise set $(1 + T_i)w_j := q^{1/2} w_l$.

A more explicit definition is given in Appendix A.1 and we verify that this is well-defined in Appendix A.2.

We will prove the following theorem on irreducibility of M^r_{2n+r}.

Theorem 3.10. *Suppose $e > n$ and $S^{(n+r,n)}$ is irreducible. Then M^r_{2n+r} is irreducible.*

Note that the representations M^r_{0+r} and $S^{(r)}$ are both isomorphic to the sign representation. This and Theorem 3.10 are suggestive; in fact, we will prove the following.

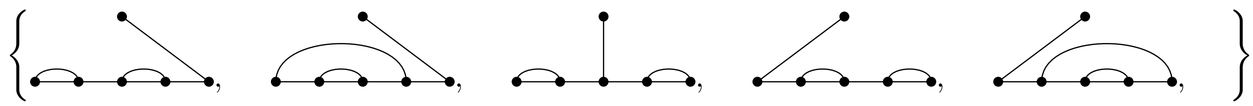

Figure 1. The basis for M_5^1.

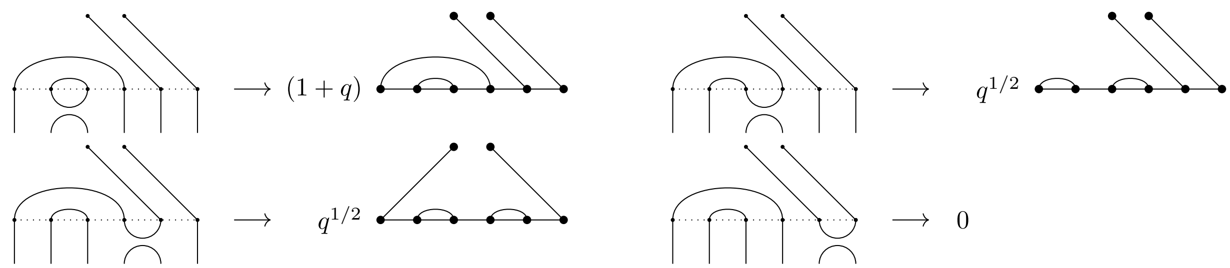

Figure 2. Illustration of the actions $(1+T_i)w_{|M_6^2|}$. In general, we act by deleting loops, taking an isotopy onto a new crossingless matching, and scaling by either $q^{1/2}$, $(q+1)$, or 0.

Theorem 3.13. *Suppose $e > n+r+1$. Then, $M^r_{2n+r} \simeq S^{(n+r,n)}$.*

Each of these are powerful characterizations of the module M^r_{2n+r} in the case that it and $S^{(n+r,n)}$ are irrducible. We will prove the following theorem which characterizes M^r_{2n+r} in the reducible case:

Theorem 4.12. *Suppose $e = n+r+1$. Then, M^r_{2n+r} contains a subrepresentation isomorphic to the sign representation.*

We will go on to prove Corrollary 4.15, which specifies a composition series for M^r_{2n+r} for the case $e = n+r+1$.

This proves a graphical characterization of $S^\lambda \simeq D^\lambda$ in many cases. However, when S^λ is reducible, the crossingless matchings representations cannot provide a graphical realization of the irreducible quotients D^λ; our next goal is to provide a similar graphical realization of the modules $D^{(n+r,n)}$ with $r \leq 3$ when $e = 5$, using a modification of Shor–Jordan's Fibonacci representation of the braid group [13]. It is possible that similar graphical representations can be constructed for other r and e, but we do not attempt to do so here.

Fibonacci representation. Now suppose that $e = 5$ and k contains the algebraic number $\sqrt{q+q^4}$ (for reasons which will be apparent soon). For convenience set $m := 2n+r$. The following is essentially a rescaling of Shor–Jordan's Fibonacci representation of the braid group [13].

Literature goes here.

Definition 1.2. Let V^m be the k-vector space with basis given by the strings $\{*, 0\}^{m+1}$ such that the character $*$ never appears twice consecutively. We will refer to V^m as the *Fibonacci representation.*

We endow V^m with an $\mathscr{H}$-action which acts on a basis vector in a manner which depends only on bits $i, i+1, i+2$, sending each basis vector to a combination of other basis vectors agreeing on characters $1, \ldots, i, i+2, \ldots, n+1$ as follows:

$$\begin{aligned}
T_1\,(\mathtt{*00}) &:= \alpha_1\,(\mathtt{*00})\,,\\
T_1\,(\mathtt{00*}) &:= \alpha_1\,(\mathtt{00*})\,,\\
T_1\,(\mathtt{*0*}) &:= \alpha_2\,(\mathtt{*0*})\,,\\
T_1\,(\mathtt{0*0}) &:= \varepsilon_1\,(\mathtt{0*0}) + \delta\,(\mathtt{000})\,,\\
T_1\,(\mathtt{000}) &:= \delta\,(\mathtt{0*0}) + \varepsilon_2\,(\mathtt{000})\,.
\end{aligned} \tag{1.1}$$

for constants

$$\begin{aligned} \tau &:= q + q^4, \\ \alpha_1 &:= -1, \\ \alpha_2 &:= q, \\ \varepsilon_1 &:= \tau(q\tau - 1), \\ \delta &:= \tau^{3/2}(q+1), \\ \varepsilon_2 &:= \tau(q - \tau). \end{aligned} \tag{1.2}$$

with T_i acting analogously on a basis element dependent on the substring $i, i+1, i+2$. We verify that V^m is a representation of $\mathscr{H}$ in Appendix A.3

Note that the action T_i does not modify characters $i, i+2$, so characters 1 and $n+1$ are preserved by $\mathscr{H}$. Hence the representation V^m contains four subrepresentations spanned by strings with beginning and ending with specified characters. Label the subrepresentation spanned by strings $(* \ldots *)$ by V_{**}, V_{*0}, V_{0*}, and V_{00}. We prove the following theorem.

Theorem 5.7. *We have the following isomorphisms:*

$$\begin{aligned} V_{**}^{2n} &\simeq D^{(n,n)}, \\ V_{**}^{2n-1} &\simeq D^{(n+1,n-2)}, \\ V_{*0}^{2n} &\simeq D^{(n+1,n-1)}, \\ V_{*0}^{2n-1} &\simeq D^{(n,n-1)}. \end{aligned}$$

This provides a graphical characterization of $D^{(n+r,n)}$ for $e = 5$, $r \leq 3$, as well as a combinatorial characterization of the Fibonacci representation in [13].

Overview of paper.

In Section 2 we give corollaries to standard theorems concerning Specht modules. First, James-Mathas provide a sharp characterization of the irreducibility S^λ for $\lambda \vdash 2n + r$ which is e-regular, called the *Carter criterion* [9, Thm. 5.42]. We specialize this to the case that $\lambda = (n + r, n)$ to give a combinatorial condition for irreducibility of $S^{(n+r,n)}$. We note that this irreducibility depends only on e when $e > n$; otherwise it depends on both e and the characteristic of k. Further, we use Kleshchev–Brundan's modular branching rules to prove our first significant statement: if $S^\lambda \simeq D^\lambda$ and $e > n$, then a particular length-2 composition series uniquely determines λ; further, an irreducible restriction to $D^{(n,n-1)}$ determines λ as well [3, 7].

In Section 3, we begin by proving proposition 3.8 concerning sign subrepresentations of M_{2n+r}^r when $e \nmid n + r + 1$; this allows us to prove Theorem 3.10. Following this, we prove the existence of a particular filtration with factors given by other crossingless matchings representations; using irreducibility, this becomes a composition series. This combined with an inductive argument and the branching of Section 2 allow us to prove Theorem 3.13.

In Section 4, we begin by determining an explicit basis for the direct sum K of all sign subrepresentations of M in the case $e = n + r + 1$. We prove in Theorem 4.12 that such K is nontrivial whe $e = n + r + 1$, and thereby provide an explicit composition series for such M in corrolary 4.15. We finish the section by providing several corollaries concerning the structure of M_{2n+r}^r at irreducible cases with $e < n + r + 1$.

In Section 5, we begin by establishing the $2n = 2$ case of Theorem 5.7 for the subrepresentations of V^2, as well as irreducibility of V_{*0}^3. We then use these cases to prove that V_{*0}^m and V_{**}^m are irreducible for all m. From this, we inductively prove Theorem 5.7. Overview of Conjecture and Empirics goes here.

In Appendix A, we begin by giving a precise definition of M_{2n+r}^r. Then, we verify that the crossingless matchings and Fibonacci representations are compatible with the Hecke algebra relations. In Appendix B, we prove a lemma concerning restrictions to various subalgebras of the Hecke algebra. In Appendix C, we give explicit data both supporting the conjectures laid out in Section reference to conjecture section and giving explicit transitions between M_{2n+r}^r and V^{2n+r} and composition series of M_{2n+r}^r in the case that $2n + r \leq 7$.

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2. Preliminaries on Specht modules

For this section and the rest of the paper, assume $n > 0$.

Throughout the rest of the paper, it will be useful to have precise notation; identify each partition with a tuple $\lambda = (\lambda_1^{a_1}, \dots, \lambda_l^{a_l})$ having $\lambda_i > \lambda_{i+1}$, $a_i > 0$, and $\sum_i a_i\lambda_i = 2n + r$. Identify each of these with a subset $[\lambda] \subset \mathbb{N}^2$ as in [7], and define $\lambda(i) = (\lambda_1^{a_1}, \dots, \lambda_{i-1}^{a_{i-1}}, \lambda_i^{a_i - 1}, \lambda_i - 1, \lambda_{i+1}^{a_{i+1}}, \dots, \lambda_l^{a_l})$ to be the partition with the ith row removed. Say that λ is *e-regular* if $\lambda_i - \lambda_{i+1} < e$ for all i and $\lambda_l < e$.

In the following subsection, we cite a theorem of James–Mathas which precisely characterizes the irreducibility of S^λ in the case that λ is e-regular, and we specialize this result to the case of two-row Specht modules. This falls into two cases: either $e > n$, where $S^{(n+r,n)}$ is irreducible iff $e \nmid r + 2, \dots, n + r + 1$, or $e \le n$, where the irreducibility of $S^{(n+r,n)}$ is more complicated and depends also on the characteristic of k. We will focus primarily on the former case.

Following this, we reproduce the branching theorems of Kleshchev–Brundan, which allow us to fully characterize the socle of $\operatorname{Res} D^\lambda$. This and some combinatorial arguments yield the main result of this section, which allows us to determine certain $D^{(n+r,n)}$ via their composition series. This will be instrumental later for characterizing the crossingless matchings representation M_{2n+r}^r as a Specht module, and it will extend to all cases with $e > n + r + 1$.

2.1. Irreducibility of Specht modules.

Let ℓ be the characteristic of k; then, set

$$p := \begin{cases} \ell & \text{if } \ell > 0, \\ \infty & \text{if } \ell = 0. \end{cases}$$

Note that $p = e$ when $q = 1$. For h a natural number, let $\nu_p(h)$ be the p-adic valuation of h. By convention, set $\nu_\infty(h) = 0$ for all h. Define the function $\nu_{e,p} : \mathbb{N} \to \{-1\} \cup \mathbb{N}$ by

$$\nu_{e,p}(h) := \begin{cases} \nu_p(h) & \text{if } e \mid h, \\ -1 & \text{if } e \nmid h. \end{cases}$$

Lastly, let h_{ab}^λ be the hook length of node (a, b) in $[\lambda]$ as defined in [7]. With this language, we may express the following theorem, parts (ii)-(iii) of which are known as the *Carter criterion* in the symmetric group case, due to James–Mathas [9].

Theorem 2.1 (James–Mathas). *The following are equivalent:*

(i) $S^\lambda \simeq D^\lambda$.

(ii) λ *is e-regular and* S^λ *is irreducible.*

(iii) $\nu_{e,p}\left(h_{ab}^\lambda\right) = \nu_{e,p}\left(h_{ac}^\lambda\right)$ *for all nodes* (a, b) *and* (a, c) *in* $[\lambda]$.

Proof. See [9, Thm 5.42]. □

This result gives information solely on e-regular partitions, and the general irreducibility of S^λ away from $p = 2$ is not well understood. We will henceforth specialize slightly to the case that $(n+r, n)$ is e-regular.

Corollary 2.2. *If* $r = 0$*, assume* $e > 2$*.*

(i) Suppose $e > n$*. Then,* $S^{(n+r,n)}$ *is irreducible iff* $e \nmid r + 2, r + 3, \dots, n + r + 1$*.*

(ii) Suppose $e \le n$*. If* $S^{(n+r,n)}$ *is irreducible, then* $e \mid r + 1$*.*

Note that the condition $e \nmid r + 2, r + 3, \dots, n + r + 1$ implies that $e > n$.

$n+r+1$	$n+r$	$\dots$	$r+2$	r	$r-1$	$\dots$	1
n	$n-1$	$\dots$	1				

Figure 3. The young diagram corresponding to the partition $(n+r, n)$. The hook lengths are in the center of the corresponding cells.

Proof. Our initial assumption on e implies that λ is e-regular, which we will use below.

(i) Note that $\nu_p(h) \neq -1$ for all naturals h and only hook lengths in the top row may vanish mod e by Figure 3; hence we may equivalently prove that e divides no hook lengths in the leftmost n columns of the second row by Theorem 2.1. These hook lengths are precisely $r+2, \dots, n+r+1$.

(ii) Note that we have $\nu_{e,p}\left(h^{\lambda}_{2,n-e+1}\right) \neq -1$, and $\nu_{e,p}\left(h^{\lambda}_{2,n+r}\right) = -1$ so $\nu_{e,p}$ acquires at least two values. Suppose that $e \nmid r+1$. Then,

$$\nu_{e,p}\left(h^{\lambda}_{1,n-e+1}\right) = \nu_{e,p}\left(h^{\lambda}_{2,n-e+1} + r + 1\right) = -1,$$

giving $S^{(n+r,n)}$ reducible by Theorem 2.1. □

From part (i) we see that irreducibility at $e > n$ is not dependent on p, and we may cover many modular cases without reference to the characteristic of k. We will finish our discussion of irreducibility of S^{λ} via sharp characterization of the $e \leq n$ case.

Corollary 2.3. *If $r = 0$, assume $e > 2$. Suppose $e \leq n$, and suppose $p > n + r + 1$. Then, $S^{(n+r,n)}$ is irreducible if and only if $e \mid r + 1$.*

Proof. This follows from the proof of Corollary 2.2 part (ii) and the fact that $\nu_p(h) = \nu_p(h')$ for all naturals h, h'. □

2.2. **Branching theorems for Specht modules.** In this section as well as later sections, we will consider the restriction of representations of $\mathscr{H}$ to particular subalgebras isomorphic to $\mathscr{H}_{k,q}\left(S_{2n+r-1}\right)$. We verify in Appendix B that any two subalgebras of $\mathscr{H}$ generated by $2n+r-2$ simple transpositions are canonically isomorphic, and the corresponding restrictions are canonically isomorphic via this isomorphism of algebras. We will hence abuse notation, pick one such subalgebra $\mathscr{H}'$, and notate $\operatorname{Res}^{\mathscr{H}}_{\mathscr{H}'} W$ by $\operatorname{Res} W$ for any $\mathscr{H}$-module W.

Fixing some partition $\lambda \vdash 2n + r$, for $1 \leq i \leq j \leq l$, let $\beta_\lambda(i,j)$ and γ_λ be the quantities

$$\beta_\lambda(i,j) = \lambda_i - \lambda_j + \sum_{t=i}^{j} a_t,$$

$$\gamma_\lambda(i,j) = \lambda_i - \lambda_j + \sum_{t=i+1}^{j} a_t.$$

Note that $\beta_\lambda(i,j)$ is the hook length of cell $(a_1 + \cdots + a_{i-1} + 1, \lambda_j)$.

Results due to Kleshchev and Brundan refer to *normal* and *good* numbers; for these, we will use the facts that 1 is always normal and that j is normal when $\beta_\lambda(i,j) \not\equiv 0 \pmod e$ for all $i \leq j$. Further, we will use that j is good if and only if j is normal and $\gamma_\lambda(j,j') \not\equiv 0 \pmod e$ for all $j' \geq j$ normal [3, 7]. When $\lambda(i) = \mu$ for i normal, write $\mu \xrightarrow{\text{normal}} \lambda$, and similar in the good case.

The following statements, collectively known as *modular branching rules* of D^{λ}, were originally written by Kleshchev for Specht modules of the group algebra $k\left[S_n\right]$, then generalized to the Hecke algebra case by Brundan [3, 7]. They entirely characterize the socle of $\operatorname{Res} D^{\lambda}$, as well as semisimplicity od $\operatorname{Res} D^{\lambda}$.

Theorem 2.4 (Kleshchev-Brundan). *We have the following isomorphisms of vector spaces*

$$\mathrm{Hom}_{\mathscr{H}'}\left(S^\mu, \mathrm{Res}\, D^\lambda\right) \simeq \begin{cases} k & \text{if } \mu \xrightarrow{normal} \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{Hom}_{\mathscr{H}'}\left(D^\mu, \mathrm{Res}\, D^\lambda\right) \simeq \begin{cases} k & \text{if } \mu \xrightarrow{good} \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

and $\mathrm{Res}\, D^\lambda$ *is semisimple if and only if every normal number in* λ *is good.* □

Using this, we immediately see that, for any rectangular partition (m^ℓ), we have

$$\mathrm{Res}\, D^{(m^\ell)} \simeq D^{(m^{\ell-1}, m-1)}.$$

The non-rectangular two-row case is more complicated, but we may still describe it fully as follows.

Corollary 2.5. *Suppose* $r > 0$*. Then, we may characterize the socle of* Res D^λ *as follows:*

$$\mathrm{soc}\left(\mathrm{Res}\ D^{(n+r,n)}\right) \simeq \begin{cases} D^{(n+r-1,n)} & \text{if } e \mid r+2 \\ D^{(n+r,n-1)} & \text{eif } e \nmid r+2,\ e \mid r \\ D^{(n+r-1,n)} \oplus D^{(n+r,n-1)} & \text{if } e \nmid r+2, r \end{cases}$$

Further, when $e \nmid r$ *or* $e \mid r+2$*,* $\mathrm{Res}\, D^{(n+r,n)}$ *is semisimple.*

Proof. This amounts to computations of the hook lengths $\beta(1,2)$ and $\gamma(1,2)$:

$$\beta_\lambda(1,2) = r+2$$
$$\gamma_\lambda(1,2) = r$$

Since 2 is the largest removable number, $D^{(n+r,n-1)} \subset D^{(n+r,n)}$ if and only if $e \nmid r+2$. Further, if $e \nmid r+2$, then $D^{(n+r-1,n)} \subset D^{(n+r,n)}$ if and only if 1 is good; this is equivalent to $e \nmid r$. □

Now that we've characterized how D^λ restrict, we can describe how strongly these restrictions characterize irreducibles. Namely, we will prove that some D^λ having the same composition series as $D^{(n+r,n)}$ is sufficient to determine that $\lambda = (n+r,n)$ in the case that condition (i) of Corollary 2.2 holds and either $r \neq 0$ or $e \neq 4$.

Proposition 2.6. *Let* λ *be an* e*-regular partition of* $2n+r$*.*

(i) Suppose $r > 0$*, suppose* $e \nmid r+1, r+2, \ldots, n+r+1$*, and suppose either* $e \mid r$ *or* $e \nmid r-2$*. If* D^λ *has the composition series*

$$0 \subset D^{(n+r-1,n)} \subset \mathrm{Res}\ D^\lambda \tag{2.1}$$

with factor Res $D^\lambda / D^{(n+r-1,n)} \simeq D^{(n+r,n)}$*, then* $\lambda = (n+r,n)$*.*

(ii) Suppose $r = 0$*, suppose* $e \nmid 4$*, and suppose* $D^{(n,n-1)} \simeq$ Res D^λ*. Then* $\lambda = (n,n)$*.*

Proof. Note that $e > n$. Further, note that the above characterizations are necessary regardless of e-regularity; in the case that μ below fails too be e-regular, this proposition will prove that λ does not satisfy 2.1, a contradiction.

(i) Let $\varpi := (n+r-1, n, 1)$, let $\varsigma := (n+r-1, n+1)$, and let $\mu := (n+r, n)$. Since $D^{(n+r-1,n)} \subset$ Res D^λ, we have $(n+r-1, n) \longrightarrow \lambda$, implying $\lambda \in \{\varpi, \varsigma, \mu\}$. We will show that ϖ, ς do not have socle compatible with (2.1), allowing us to conclude $\lambda = \mu$.

If ϖ or ς are not e-regular, then $D^\varpi = 0$ or $D^\varsigma = 0$, and we may immediately rule these out; henceforth assume that these are each e-regular. First suppose that that $\lambda = \varpi$. We will break into cases with r.

Case 1. Suppose that $r > 1$. Note that $e \nmid r+1 = \beta_\varpi(1,2)$, so 2 is normal. Further,

$$\gamma_\varpi(2,3) = n \not\equiv 0 \pmod{e},$$

so 2 is good and $D^{(n+r-1,n-1,1)} \subset D^\varpi$, which is not a composition factor in (2.1). Hence, by the Jordan-Hölder theorem [4, Thm. 3.7.1], we have $\lambda \neq \varpi$.

Case 2. Suppose that $r = 1$. Then, $\varpi = (n,n,1)$ has

$$\gamma_\varpi(1,2) = n \not\equiv 0 \pmod{e},$$

giving $D^{(n,n-1,1)} \subset D^\varpi$ and hence $\lambda \neq \varpi$ as in the previous case.

Now suppose that $\lambda = \varsigma$. Note that ς is not a partition when $r < 2$, so we may assume that $r \geq 2$. We further break into cases with r:

Case 1. Suppose $r > 2$. Then, by Corollary 2.5, we require that $e \nmid r$ and $e \mid r-2$; these are not satisfied, so $\lambda \neq \varsigma$.

Case 2. Suppose $r = 2$. Then, we have the restriction

$$\text{Res } D^{\varsigma} \simeq D^{(n+1,n)},$$

as $\varsigma = (n+1, n+1)$ has tworows of the same length. This contradicts (2.1).

Hence $\lambda = \mu$, completing the proof.

(ii) Since the socle of D^{λ} is irreducible, we require that 1 is the only normal number and $\lambda(1) = (n, n-1)$. This reduces to the cases of $\varsigma := (n+1, n-1)$ and $\mu := (n, n)$; if $\lambda = \varsigma$, then we have that

$$\beta_{\varsigma}(1, 2) = 4 \equiv 0 \pmod{e},$$

a contradiction. Hence $\lambda = \mu$. ☐

This will be an integral technical tool in proving the correspondence between the crossingless matchings representation M^r_{2n+r} and the Specht module $S^{(n+r,n)}$ in the following section.

3. Crossingless matchings and Specht modules

In this section, we analyze the crossingless matchings representation $M := M^r_{2n+r}$ with the goal of proving $M^r_{2n+r} \simeq S^{(n+r,n)}$ under certain conditions in e. We begin by proving that M^r_{2n+r} contains no subrepresentations isomorphic to the sign representation (henceforth referred to as *sign subrepresentations*) when $e \nmid n+r+1$. Using this, we prove irreducibility of M^r_{2n+r} whenever $e \nmid r+2, r+3, \ldots, n+r+1$; when $e > n$, this is true if and only if $S^{(n+r,n)}$ is irreducible by Corollary 2.2.

We will use the following base case to the correspondence throughout:

Lemma 3.1. *Note that $S^{(1^n)}$ is the sign representation and $S^{(n)}$ the trivial representation. We have the following isomorphisms*

(i) $M^0_2 \simeq S^{(2)}$,

(ii) $M^r_r \simeq S^{(1^r)}$.

Each of these are 1-dimensional, so they are irreducible.

Proof. (i) follows from the fact that $(1+T_1)w = (1+q)w$, and hence $T_1 w = qw$ for any nonzero vector $w \in M^0_2$. Similarly, (ii) follows from the fact that $(1+T_i)w = 0$, so $T_i w = -w$ for any i and any $w \in M^r_r$. ☐

In particular, Lemma 3.1 is the base case in an inductive proof that $M^r_{2n+r} \simeq S^{(n+r,n)}$ whenever $e > n+r+1$.

3.1. Sign subrepresentations when $e \nmid n+r+1$.

Define $K^r_{2n+r} := \bigcap_{i=1}^{2n+r-1} \ker(1+T_i) = \ker \bigoplus_{i=1}^{2n+r-1}(1+T_i)$. When the limits are clear, we will simply denote this kernel by K. Note that K is the direct sum of all sign subrepresentations of M^r_{2n+r}. In this section, we prove K is trivial for $e \nmid n+r+1$, thus M^r_{2n+r} has no sign subrepresentations.

For compactness, in this section we use $\sim$ to denote "proportional to." For convenience, define M^0_0 and M^1_1 to be the zero representation. Note that in this subsection, as well as section 4, we do not assume $n > 0$.

Definition 3.2. We will use the following notation extensively while exploring the structure of the kernel. Examples are given in figure 4. Fix some basis element $\psi \in M^r_{2n+r}$.

- For $1 \leq a, b \leq 2n+r$ define $\psi(a) := b$ if a and b are matched in ψ, and $\psi(a) := a$ if a is an anchor in ψ.
- Let r' be the number of anchors in ψ in the range $a, \ldots, b$. Suppose $1 \leq a \leq b \leq 2n+r$ and $\psi(i) \in \{a, \ldots, b\}$ for all $i \in \{a, \ldots, b\}$. Define a *sub-matching* $\psi(a, b)$ of ψ to be the basis element $\sigma \in M^{r'}_{b-a+1}$ specified by $\sigma(i) = \psi(i+a-1) - (a-1)$.
- For a, b satisfying $1 \leq a \leq b \leq 2n+r$, if $\psi(a, b)$ is not a sub-matching, define it to be the ordered set of nodes $(\psi(a), \ldots, \psi(b))$. For any other a, b, we define $\psi(a, b)$ to be an element of the zero representation M^0_0.

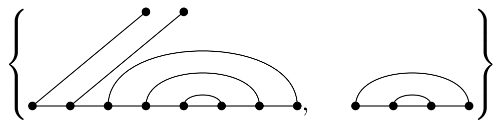


Figure 4. The rainbow element $R \in M_8^2$ is pictured on the left, $\sigma \in M_4^0$ is pictured on the right. The nodes $R(4) = 7$, $R(5) = 6$, so $R(4, 7)$ defines a sub-matching. Specifically, $R(4, 7)$ is a rainbow sub-matching, and $R(4, 7) = \sigma$. Alternatively, $R(3) = 8$, so, for instance $R(2, 7)$ is not a sub-matching, and refers to the ordered set $(2, 8, 7, 6, 5, 4)$. $R(2, 1)$ is an element of the zero representation since $2 > 1$.

- Define the *rainbow element* $R \in M^r_{2n+r}$ to be the basis element specified by $R(i) = 2n + 2r - i + 1$ for $i > r$, $R(i) = i$ for $i \leq r$. In other words, the basis element with all anchors to the left followed by a "rainbow" to the right.
- Define a *rainbow sub-matching* to be a sub-matching $\psi(a, b)$ corresponding to some rainbow element and satisfying the following property: if a is an anchor then $b = 2n + r$. This definition is motivated by the isotopy mentioned illustrated in diagram 3.4.

The following proposition lets us begin to characterize the coordinate vector of any element in K, and will serve as the starting point for all following characterizations.

Proposition 3.3. *Let $w \in M^r_{2n+r}$. If $w \in K$ is nonzero, then the coordinate of the rainbow element R in w is nonzero.*

Proof. Let W be the set of basis elements with nonzero coordinate in w. First, we will prove that W contains an element with all anchors to the far left. Then we will apply the same argument to prove W contains the rainbow element.

Let z be the maximal number of anchors to the far left in any element $\sigma \in W$. Formally, z is the greatest positive integer such that there exists some element $\sigma \in W$ satisfying $\sigma(1) - 1 = ... = \sigma(z) - z = 0$. For any positive integer j define $W^j \subset W$ to be the set of all basis elements in W with j anchors to the far left.

Suppose $z < r$. Then for each $\psi \in W^z$, exactly z anchors are positioned to the far left, so we may define i_ψ to be the position of the next leftmost anchor in ψ. In other words, i_y is the position of the $z + 1$st anchor from the left in ψ. Fix some $\widehat{\psi} \in W^z$ such that $i_{\widehat{\psi}} \leq i_\psi$ for all ψ.

Define the basis element $\psi' := q^{-1/2}(1 + T_{i_{\widehat{\psi}}-1})\widehat{\psi}$. First, note that $i_\psi > z + 1$, so $i_\psi - 1$ is not an anchor and $\psi' \neq 0$. This follows from the definition of z and W^z, which require that the node $z + 1$ is not an anchor. We will prove that ψ' has nonzero coordinate in $(1 + T_{i_{\widehat{\psi}}-1})w$, implying $w \notin K$.

It suffices to show that $\widehat{\psi}$ is the only basis element in W brought to ψ' by the action of $(1 + T_{i_{\widehat{\psi}}-1})$. Formally, we will show that if $\sigma \in W$ and $(1 + T_{i_{\widehat{\psi}}-1})\sigma \sim \psi'$, then $\sigma = \widehat{\psi}$.

Immediately, we may intuit a few properties of ψ':

(i) ψ' still has z anchors on the left.
(ii) Defining $i_{\psi'}$ to be the position of the next leftmost anchor, $i_{\psi'} < i_{\widehat{\psi}}$

Together, (i) and (ii) imply $\psi' \notin W$. So, if $(1 + T_{i_{\widehat{\psi}}-1})\sigma \sim \psi'$, $q^{-1/2}(1 + T_{i_{\widehat{\psi}}-1})\sigma = \psi'$.

Define $\sigma' := q^{-1/2}(1 + T_{i_{\widehat{\psi}}-1})\sigma$. Note that the action of $(1 + T_{i_{\widehat{\psi}}-1})$ changes exactly two matches in σ, and these matches must involve the nodes $i_{\widehat{\psi}} - 1, i_{\widehat{\psi}}$. It follows that if $\sigma \notin W^z$, σ' will have z anchors at the far left only if $\sigma' \in W^{z-1}$ and the zth leftmost anchor is at position $i_{\widehat{\psi}}$. But then the position of the $z + 1$st anchor is unchanged by the action, and must be at position greater than $i_{\widehat{\psi}}$, so from (ii) $\sigma' \neq \psi'$. If $\sigma \in W^z$, σ' will have anchor at $i_{\psi'}$ if and only if $i_\sigma = i_{\widehat{\psi}}$ and $\sigma(i_{\widehat{\psi}} - 1) = \widehat{\psi}(i_{\widehat{\psi}} - 1)$. Since these are the only three indices altered by action of $(1 + T_{i_{\widehat{\psi}}-1})$ on σ, if $(1 + T_{i_{\widehat{\psi}}-1})\sigma \sim \psi'$ this implies $\sigma = \widehat{\psi}$ as desired. So, if $z < r$, w is not in the desired kernel.

We have proven that there exists an element in W with all anchors to the far left, or equivalently that W^r is nonempty. Now we must prove that $R \in W^r$. The proof is analogous to the previous case. First we

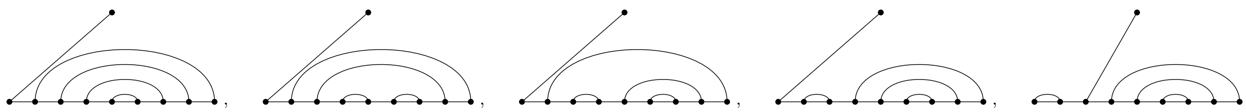


Figure 5. $R_{L,0}, ..., R_{L,4}$ pictured from left to right. If $w \in K_9^1$ and $R = R_{L,0}$ has coordinate c in w, then proposition 3.6 claims $R_{L,1}$ has coordinate Q_1, $R_{L,2}$ has coordinate Q_2, etc.

define subsets $W_j^r \subset W^r$ that have the top j arcs of the rainbow, and define t to be the most arcs any element has. Previously, we showed that if we move the next left-most anchor further left, the image of w under that transposition will be nonzero. Now, we will show that if we expand the next-largest arc, the image of w under that transposition will be nonzero.

Suppose $z = r$ but $R \notin W^r$ (so $R \notin W$). Let us define a sequence of subsets $W_j^r \subset W^r$ to be the set of elements in W^r with the top j arcs of the rainbow. Formally, $W_0^r := W^r$, $W_{j+1}^r := \{\upsilon \in W_j^r | \upsilon(r+j+1) = 2n+r-j\}$. Since $R \notin W^r$, there exists some positive integer t less than $n-1$ such that $W_{t+1}^r = \varnothing$; t is the largest number of contiguous top level rainbow arcs that any element in W^r has.

Choose $\widehat{\upsilon} \in W_t^r$ such that $\widehat{\upsilon}(r+t+1) \geq \upsilon(r+t+1)$ for all $\upsilon \in W_t^r$. Define the basis element $\upsilon' := q^{-1/2}(1 + T_{\widehat{\upsilon}(r+t+1)})\widehat{\upsilon}$.

First, note that $\widehat{\upsilon}(r+t+1) < 2n+r-t$, so:

[i] $\upsilon'(r+t+1) > \widehat{\upsilon}(r+t+1)$

[ii] For $0 \leq j \leq t$, $\upsilon'(r+j) = \widehat{\upsilon}(r+j) = 2n+r-j+1$.

These properties are analogous to properties (i) and (ii) used earlier in this proof. As before, these imply that $\upsilon' \notin W$, so, for $\sigma \in W$, if $(1 + T_{\widehat{\upsilon}(r+t+1)})\sigma \sim \upsilon'$, $q^{-1/2}(1 + T_{\widehat{\upsilon}(r+t+1)})\sigma = \upsilon'$.

We will prove that υ' has nonzero coordinate in $(1+T_{\widehat{\upsilon}(r+t+1)})w$, implying $w \notin K$. Again, it is sufficient to show that, for $\sigma \in W$, $q^{-1/2}(1 + T_{\widehat{\upsilon}(r+t+1)})\sigma = \upsilon'$ implies $\sigma = \widehat{\upsilon}$.

Define $\sigma' := q^{-1/2}(1 + T_{\widehat{\upsilon}(r+t+1)})\sigma$. Suppose $\sigma \notin W_t^r$. To satisfy [ii], we must have $\sigma \in W_{t-1}^r$ and $\sigma(r+t) = \widehat{\upsilon}(r+t+1)$. But then $\sigma(r+t+1) < \widehat{\upsilon}(r+t+1)$ is unchanged by action of $(1 + T_{\widehat{\upsilon}(r+t+1)})$, and σ' does not satisfy [i], so $\sigma' \neq \upsilon'$. If $\sigma \in W_t^r$, to satisfy $\sigma'(r+t+1) = \upsilon'(r+t+1)$, we must have $\sigma(r+t+1) = \widehat{\upsilon}(r+t+1)$ and $\sigma(\widehat{\upsilon}(r+t+1)+1) = \widehat{\upsilon}(\widehat{\upsilon}(r+t+1)+1)$. Since these are the only two matchings altered by the transposition, if $q^{-1/2}(1 + T_{\widehat{\upsilon}(r+t+1)})\sigma = \upsilon'$ we must have that $\sigma = \widehat{\upsilon}$ as desired. Thus, if $w \in K$ is nonzero, R has nonzero coordinate in w. ☐

Definition 3.4. Given a rainbow element R, define the *shifted rainbow elements* $R_{R,i}, R_{L,i}$ to be those where you "move the 'length two middle arc' across i lines to the right or left, respectively." This is made clear in figure 5. Formally:

$$R_{R,i} := q^{-i/2}(1 + T_{r+n+i}) \cdots (1 + T_{r+n+1})R$$

,

$$R_{L,i} := q^{-i/2}(1 + T_{r+n-i}) \cdots (1 + T_{r+n-1})R$$

.

Definition 3.5. The following q-number deformation will be integral to this subsection and section 4:

$$Q_n := (-1)^n \frac{q^n + \ldots + 1}{q^{n/2}} \text{ for } n \text{ a nonnegative integer}$$

.

The following proposition applies to any basis element y with a rainbow sub-matching. It says that if y has coordinate c in $w \in K$, then those elements θ_i, ϕ_i, where you replace the rainbow sub-matching with a "shifted rainbow sub-matching" $R_{R,i}$ or $R_{L,i}$, have coordinate $Q_i c$ in w. An example is given in figure 5.

Proposition 3.6. *Let $w \in K_{2n+r}^r$. Let y be a basis element with coordinate c in w. Suppose that there exist a, b such that $y(a,b)$ is a rainbow sub-matching. Define the basis elements θ_i, ϕ_i to be those elements where we replace the rainbow sub-matching with a shifted rainbow sub-matching. Formally:*

$$\theta_i(1, a-1) := \phi(1, a-1) := y(1, a-1) \ ,$$

$$\theta_i(b+1,2n) := \phi(b+1,2n) := y(b+1,2n)\ ,$$
$$\theta_i(a,b) = R_{R,i}\ ,$$
$$\phi_i(a,b) = R_{L,i}.$$

We restrict i in θ_i, ϕ_i to be such that $R_{R,i}, R_{L,i}$ are defined, respectively. Then the coordinates of ϕ_i and θ_i in w are both $Q_i c$.

We will use the following algebraic lemma to prove this proposition. The lemma will also be useful in section 4.

Lemma 3.7. *$Q_1 Q_n - Q_{n-1} = Q_{n+1}$ for $n > 0$.*

Proof of lemma. We compute:

$$\begin{aligned} Q_1Q_n - Q_{n-1} &= \left(-\frac{(q+1)}{q^{1/2}}\right)\left((-1)^n\frac{(q^n+\cdots+1)}{q^{n/2}}\right) - (-1)^{n-1}\frac{(q^{n-1}+\cdots+1)}{q^{(n-1)/2}} \\ &= (-1)^{n+1}\frac{(q^{n+1}+2q^n+\cdots+2q+1)}{q^{(n+1)/2}} - (-1)^{n+1}\frac{(q^n+\cdots+q)}{q^{(n+1)/2}} \\ &= (-1)^{n+1}\frac{(q^{n+1}+\cdots+1)}{q^{(n+1)/2}} \\ &= Q_{n+1} \end{aligned}$$

□

Now let us prove the proposition.

Note: This proof relies on the simple fact that the action of a transposition $(1+T_i)$, assuming it does not send an element to zero, will update exactly two matches if it is changes the element. This follows directly from our definition of the representation, and will be integral to many proofs in this document.

Proof. First, we will show that if we act by the transposition corresponding to the moved arc in ϕ_i, the only basis elements sent to ϕ_i are ϕ_{i-1}, ϕ_i, and ϕ_{i+1}. We will show that the analogous statement is true for θ_i, and use this to finish the proof by induction. Formally:

Claim. *Suppose $\psi \in M^r_{2n+r}$ is a basis element. Take i such that ϕ_i, ϕ_{i-1}, and ϕ_{i+1} are all defined. Let $2n' + r'$, r' be the number of nodes and anchors in $y(a,b)$, respectively. Then $(1 + T_{a-1+r'+n'/2-i})\psi \sim \phi_i$ implies $\psi = \phi_i$, ϕ_{i-1}, or ϕ_{i+1}.*

Similarly, taking i such that θ_i, θ_{i-1}, and θ_{i+1} are all defined, $(1 + T_{a-1+r'+n'/2+i})\psi \sim \phi_i$ implies $\psi = \phi_i$, ϕ_{i-1}, or ϕ_{i+1}.

Note that $a - 1 + r' + n'/2 - i$ is simply the starting node of the moved arc in ϕ_i.

For the entirety of the proof of this claim, it will be instructive to reference Figure 6 as an example.

It is easy to see that the action of $(1+T_{a-1+r'+n'/2-i})$ will bring $\phi_{i-1}, \phi_i, \phi_{i+1}$ to ϕ. Suppose there exists another basis element ψ sent to ϕ_i by the given transposition. Consider the arcs (or anchors) starting just inside and just outside of the moved arc in ϕ_i: these are the arcs $(a-1+r'+n'/2-i-1, \phi_i(a-1+r'+n'/2-i-1))$ and $(a-1+r'+n'/2-i+2, \phi_i(a-1+r'+n'/2-i+2))$.

If ψ contains both of these arcs, it must contain the arc $(a-1+r'+n'/2-i, a-1+r'+n'/2-i+1)$ to be a crossingless matching. This follows from our definition of rainbow sub-matching in definition 3.2. Thus ψ is fixed by the action of $(1 + T_{a-1+r'+n'/2-i})$, so $\psi = \phi_i$.

If ψ does not contain the left arc/anchor and $(1+T_{a-1+r'+n'/2-i})\psi \sim \phi_i$, the action of $(1+T_{a-1+r'+n'/2-i})$ must create that arc/anchor. Thus $\psi(a-1+r'+n'/2-i-1) = a-1+r+n/2-i$ and $\psi(a-1+r'+n'/2-i+1) = \phi_i(a-1+r'+n'/2-i-1)$ in the case of an arc or $a-1+r+n/2-i+1$ is an anchor. These matchings are identical to those of ϕ_{i+1}, and no other matchings are changed by the action, so $\psi = \phi_{i+1}$. Likewise, if ψ does not contain the right arc/anchor, that arc/anchor must be created by the action of $(1 + T_{a-1+r'+n'/2-i})$, so $\psi = \phi_{i-1}$.

Lastly, note that the above argument was completely symmetric with respect to the parity of i, and hence an analogous argument without anchors handles the θ_i case. Thus the claim is proved.

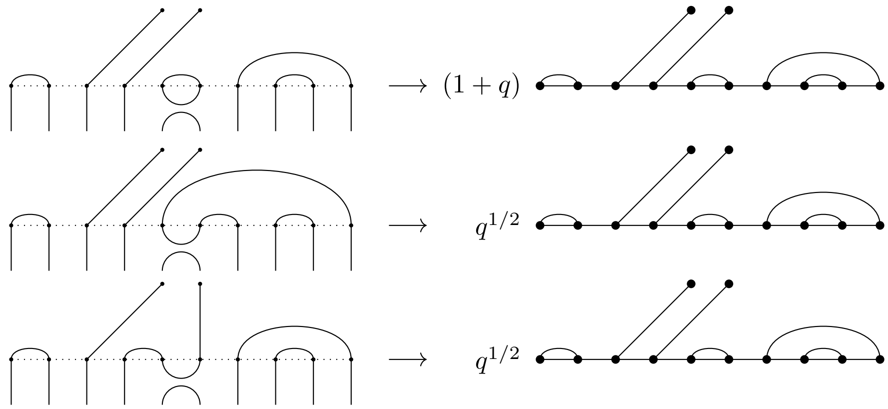


Figure 6. The action of $(1+T_{a-1+r+n/2-i})$ on $\phi_i, \phi_{i-1}, \phi_{i+1}$ (ordered from top to bottom), shown as the case where $y(3,10)$ is the rainbow vector in M_8^2 and $i=2$. When both the first anchor and top level rainbow arc are intact, the arc of the transposition must also exist. When one of these is broken, there is only one way to fix it.

As an edge case related to the claim, note that the only basis element sent to $y=\phi_0=\theta_0$ other than itself by the transposition acting on the middle of the rainbow is $\theta_1=\phi_1$ if they both exist. Otherwise, just ϕ_1 exists and is sent to y, or the rainbow element has two nodes and neither exist.

We finish the proof of the proposition through induction:

Acting by $(1+T_{a-1+r+n/2})$ on w, the new coordinate of $\phi_0=y$ is $(q+1)c+q^{1/2}c_{\phi_1}$ where c_{ϕ_1} is the coordinate of ϕ_1 in w. Since w is in the kernel, we have $(q+1)c+q^{1/2}c_{\phi_1}=0 => c_{\phi_1}=Q_1c$. Since $\phi_1=\theta_1$, this gives us all our base cases.

Acting by $(1+T_{a-1+r+n/2-i})$ on w, the new coordinate of ϕ_i is $q^{1/2}c_{\phi_{i+1}}+q^{1/2}c_{\phi_{i-1}}+(q+1)c_{\phi_i}=0$. By the inductive hypothesis, $q^{1/2}c_{\phi_{i+1}}+q^{1/2}Q_{i-1}c+(q+1)Q_ic=0$ so $c_{\phi_{i+1}}=Q_1Q_ic-Q_{i-1}c=Q_{i+1}c$ by lemma 3.7. The case of θ_i is an identical proof, so the proposition follows. $\square$

We are now ready to prove the central proposition of this section.

Proposition 3.8. *Let W^r_{2n+r} be a generalized crossingless matchings representation. Suppose e does not divide $n+r+1$. Then $K=0$.*

Proof. Suppose $K\neq 0$. Take nonzero $w\in K$. By Proposition 3.3, the coordinate of the rainbow vector R is nonzero; suppose the coordinate is c. By proposition 3.6, the coordinates of the basis elements $R_{L,n+r-1}$ and $R_{L,n+r-2}$ are $Q_{n+r-1}c$ and $Q_{n+r-2}c$ respectively.

Consider the coordinate of $R_{L,n+r-1}$ in $(1+T_1)w$. By the exact same logic as in the proof of proposition 3.6, we note that the only basis elements sent to $R_{L,n+r-1}$ by action of $(1+T_1)$ are itself and $R_{L,n+r-2}$. Thus the desired coordinate is equal to $(1+q)Q_{n+r-1}c+q^{1/2}Q_{n+r-2}c=-q^{1/2}Q_{n+r}c$ by lemma 3.7.

Since $w\in K$, we must have $-q^{1/2}Q_{n+r}c=0$. We have that c is nonzero, and we assume q nonzero, and Q_{n+r} is zero iff q is a root of $q^{n+r}+...+1$, implying $e|n+r+1$. Thus we have arrived at contradiction, and $K=0$. $\square$

3.2. **Irreducibility of the crossingless matchings representation.** We will first give a lemma on cyclic vectors. Then, we will use triviality of K to "project down" onto a copy of $M:=M^r_{2n+r}$ with strictly fewer nodes; this will prove inductively that M is irreducible when $e>n+r+$.

We refer to a vector $w\in M$ satisfying $\mathscr{H}w=M$ as *cyclic*. It is a classical result that a representation M is irreducible if and only if every nonzero element of M is cyclic [4]. We will prove irreducibility by showing that every nonzero $w\in M$ is cyclic; the following lemma is integral in showing this.

Lemma 3.9. *Every basis vector in M^r_{2n+r} is cyclic.*

Proof. I will address the missing part tomorrow. We have already proven this in the $r=0$ case, so suppose that $r>0$.

Note that, between anchors at indices $a < a'$ having no anchor at index b with $a < b < a'$, the $M^0_{a'-a}$ case allows us to generate the basis vector with all length-2 arcs between a, a' and identical arcs/anchors outside of this sub-matching. At the ends, we apply the M^0_a case or the M^0_{2n+r-a} case in the same way for the first a or last $2n+r-a$ indices.

Applying this between each arc gives us a vector with anchors and length-2 arcs, and we may use the appropriate $(1+T_i)$ to move anchors to any positions. Then, we may use the reverse process from above to generate the correct matchings between arcs and generate any other basis vector. □

Now we will prove irreducibility.

Theorem 3.10. *Suppose that $e \nmid r+2, r+3, \dots, n+r+1$. Then the representation M^r_{2n+r} is irreducible.*

Proof. We proceed by induction on n. The base case the base case $n = 0, r \neq 0$ and $n = 1, r = 0$ follow from Lemma 3.1.

Take an arbitrary vector $w \in M$. By Proposition 3.8 there exists some $(1+T_i) \in \mathscr{H}$ such that $(1+T_i)w \neq 0$. Note that

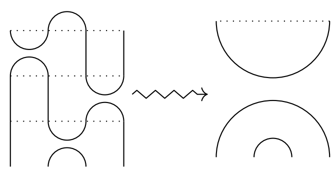
$$\operatorname{im}(1+T_i) = \operatorname{Span}\{w_j \mid w_j \text{ contains the arc } (i, i+1)\}.$$

Hence, as vector spaces, there is an isomorphism $\varphi : \operatorname{im}(1+T_i) \to M^r_{2(n-1)+r}$ "deleting" the arc $(i, i+1)$.

We will show that, for every element $(1+T'_j) \in \mathscr{H}(S_{2(n-1)+r})$, there is some element $h_j \in \mathscr{H}(S_{2n+r})$ such that the following commutes:

$$\begin{tikzcd} \operatorname{im}(1+T_i) \arrow[r, "\varphi", "\sim"'] \arrow[d, "h_j"] & M^r_{2(n-1)+r} \arrow[d, "1+T'_j"] \\ \operatorname{im}(1+T_i) \arrow[r, "\varphi", "\sim"'] & M^r_{2(n-1)+r} \end{tikzcd} \tag{3.1}$$

Indeed, when $i+1 \neq j$ this is given by $h_j = 1+T_j$, and we have $h_{i+1} = q^{-1}(1+T_i)(1+T_{i+1})(1+T_{i-1})$. This is illustrated by Figure 7, and can alternately be seen as a consequence of the following isotopy.



This may be intuitively viewed as a "large" version of the action connecting nodes $i-1$ and $i+2$; it preserves the arc $(i, i+1)$ and the factor $q^{1/2}$ in h_{i+1} gives the correct scaling.

Note that, by definition, $e \nmid r+2, \dots, n+r$ as well, and hence $(n+r-1, n-1)$ satisfies the hypotheses of the proposition as well. Then, by the inductive hypothesis, there is some element $h' \in \mathscr{H}(S_{2(n-1)+r})$ sending $\varphi((1+T_i)w)$ to the image of a basis vector of M^r_{2n+r} via φ; then, by (3.1) the action $\mathscr{H}$ generates the endomorphism $\varphi^{-1}h'\varphi$ of M, which sends $(1+T_i)w$ to a basis vector in M^r_{2n+r}. This implies that w is cyclic, and hence M^r_{2n+r} is irreducible. □

3.3. **Correspondence with Specht modules.** The following theorem due to Mathas [10, Thm. 5.5] generalizes the classical branching theorem of the symmetric group. This result is not be necessary for our present proof of the correspondence in the case $e > n+r+1$, but the analogy with M is suggestive.

Theorem 3.11 (Characteristic-free classical branching theorem)**.** *Let λ be a partition of m with ℓ removable nodes. Then, $\operatorname{Res} S^\lambda$ has an $\mathscr{H}_{k,q}(S_{m-1})$-module filtration*

$$0 = S^{0,\lambda} \subset S^{1,\lambda} \subset \cdots \subset S^{\ell,\lambda} = \operatorname{Res} S^\lambda$$

such that $S^{t,\lambda}/S^{t-1,\lambda} \simeq S^{\lambda(t)}$ for all $1 \leq t \leq \ell$. □

In particular, this holds in cases where S^λ fails to be irreducible. If we replace S^λ with the appropriate M^r_{2n+r} above, we find the statement of the following proposition.

Proposition 3.12. *Suppose that $n > 0$.*

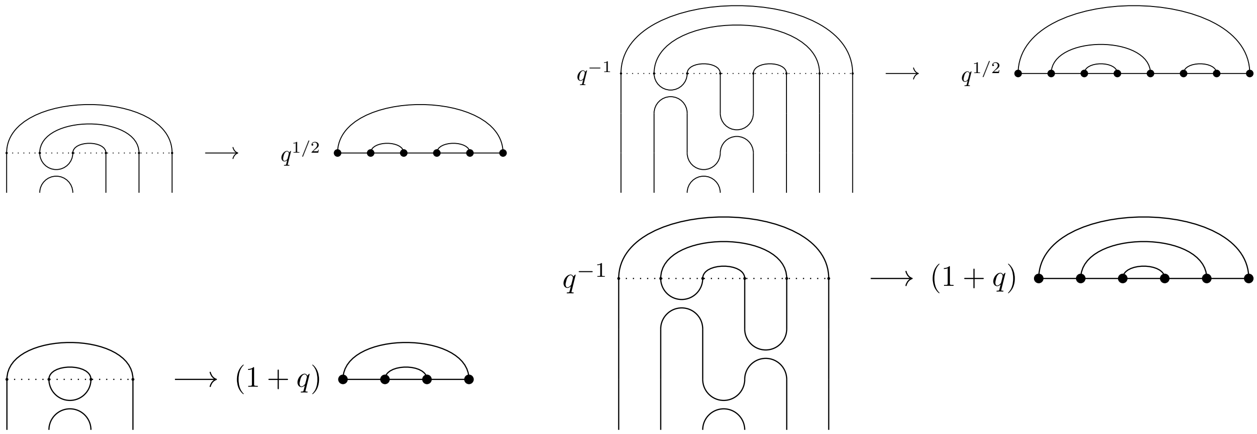


Figure 7. The correspondence between the action of $(1+T_2)$ on $w_5' \in M_6^0$ and the action of $q^{-1}(1+T_3)(1+T_4)(1+T_2)$ on the corresponding vector in M_8^0 having arc $(3,4)$ first, then on $w_2' \in M_4^0$. This demonstrates that the action works with or without creating a loop.

(i) Suppose that $r > 0$. Then, a filtration of $\operatorname{Res} M^r_{2n+r}$ *is given by*

$$0 \subset M^{r-1}_{2n+r-1} \subset \operatorname{Res} M^r_{2n+r}, \tag{3.2}$$

with $\operatorname{Res} M^r_{2n+r}/M^{r-1}_{2n+r-1} \simeq M^{r+1}_{2n+r-1}$.

(ii) We have the following isomorphism of representations:

$$M^1_{2n-1} \simeq \operatorname{Res} M^0_{2n}. \tag{3.3}$$

In the case that $e \nmid r+1, \dots, n+r+1$, (3.2) and (3.3) are composition series.

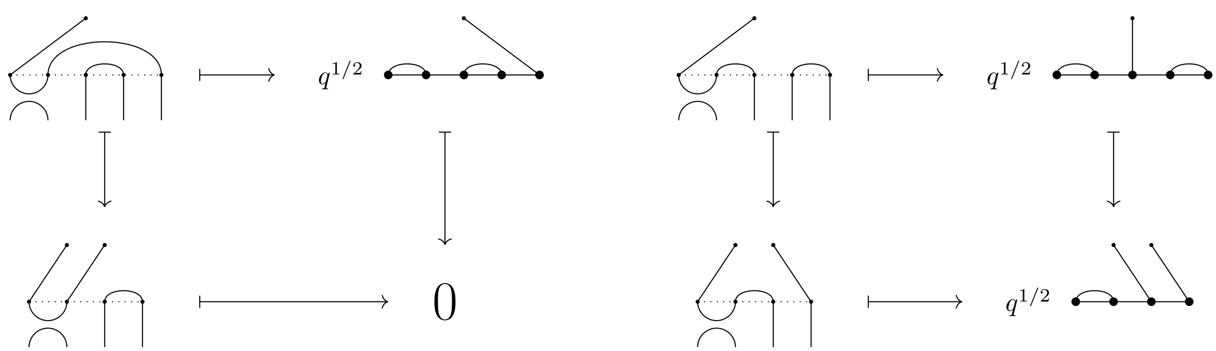
Proof. **(i)** Note that the span of the matchings having an anchor at index $2n+r$ is a subrepresentation of $\operatorname{Res} M^r_{2n+r}$; this subrepresentation is easily verified to be isomorphic to M^{r-1}_{2n+r-1}, and we will henceforth refer to it as such.

Let $U := \operatorname{Res} M^r_{2n+r}/M^{r-1}_{2n+r-1}$, and let $\pi : \operatorname{Res} M^r_{2n+r} \twoheadrightarrow U$ be the associated projection to U. Let $\phi : U \to M^{r+1}_{2n+r-1}$ be the k-linear map which regards the arc $(i, 2n+r)$ in U as an anchor at i in M^{r+1}_{2n+r-1}. It is not hard to verify that this is a well-defined isomorphism of vector spaces, so we must show that it is $\mathscr{H}$-linear.

We begin by showing commutativity of the following diagram:

$$\begin{tikzcd} M^r_{2n+r} \arrow[r, "1+T_i"] \arrow[d, "\phi\circ\pi"] & M^r_{2n+t} \arrow[d, "\phi\circ\pi"] \\ M^{r+1}_{2(n-1)+r+1} \arrow[r, "1+T_i"] & M^{r+1}_{2(n-1)+r+1} \end{tikzcd} \tag{3.4}$$

We illustrate this compatibility in the following graphic.



In both the left and right square, we see that ϕ may be treated as an isotopy which "unfolds" the last node into an anchor. This isotopy may equivalently be pre- or post-composed with the isotopy defining the action of $1 + T_i$, implying commutativity of (3.4).

Now, using (3.4), we have

$$\begin{aligned}\phi(T_j w_i) &= \phi(-w_j + (1 + T_j)w) \\ &= -\phi(w_j) + (1 + T_j)\phi(w_j) \\ &= T_j \phi(w_j).\end{aligned}$$

Hence ϕ is an isomorphism of representations, and the statement is proven.

(ii) This follows from an analogous proof: now, $\phi : \operatorname{Res} M^0_{2n} \to M^1_{2(n-1)+1}$ is an isomorphism of representations, which is proven to be $\mathscr{H}$-linear by the same logic. □

We've now assembled the basic pieces necessary to prove our correspondence in the case $e > n + r + 1$.

Theorem 3.13. *Suppose $e > n + r + 1$. Then, $M^r_{2n+r} \simeq S^{(n+r,n)}$.*

Proof. The case $n = 0$ is already proven, by lemma 3.1, so suppose $n > 0$. In order to use Proposition 2.6, suppose for now that either $e \nmid 4$ or $r > 0$.

We will prove this inductively; the base case $2n + r = 2$ is implied by 3.1, so suppose that $M^s_{2n+s} \simeq S^{(m+s,s)}$ whenever $2m + s < 2n + r$ and $m + s \le n + r$, so that $e > m + s + 1$.

Suppose $r > 0$. By Theorem 3.10, we know that $M^r_{2n+r} \simeq D^\lambda$ for some e-restricted partition λ. By the inductive hypothesis and Corollary 2.2, we have a composition series given by the short exact sequence

$$0 \longrightarrow D^{(n+r-1,n)} \longrightarrow \text{Res } D^\lambda \longrightarrow D^{(n+r,n-1)} \longrightarrow 0 \tag{3.5}$$

Hence the theorem is given by Proposition 2.6 (i).

Now suppose $r = 0$ and $e \neq 4$. Similarly, by Theorem 3.10, we know that $M^r_{2n+r} \simeq D^\lambda$ for some e-restricted partition λ, and by the inductive hypothesis and Corollary 2.2, we have the irreducible restriction $\operatorname{Res} D^\lambda \simeq D^{(n,n-1)}$. Then, the theorem is given by Proposition 2.6 (ii).

Now, suppose $e = 4$ and $r = 0$; then $4 > n + 1$, so $n \le 2$. We've already proven the $n = 1$ case via the trivial representation, so suppose $n = 2$. Then, from the proof of Proposition 2.6, we know that $M^0_4 \simeq D^\lambda$, where $\lambda \in \{(n, n), (n + 1, n - 1)\}$. We have already proven that $M^2_4 \simeq D^{(n+1,n-1)}$, and we may verify that $\dim M^0_4 = 2 \neq 3 = \dim M^2_4$, so we have that $\lambda = (n, n)$ and the theorem is proven for $e = 4$. □

This entirely characterizes M^r_{2n+r} in the case that $e > n+r+1$. In the next section, we will give weaker characterizations of M^r_{2n+r} in the case $e = n + r + 1$, where the representation s M^r_{2n+r} and $S^{(n+r,n)}$ are reducible.

4. Sign subrepresentations and possible extensions

Recall $K := K^r_{2n+r}$, defined in section 3.1, is the direct sum of all sign subrepresentations. In the previous section, we were not able to prove $M^r_{2n+r} \simeq S^{(n+r,n)}$ for $e = n + r + 1$ because we could not assume $K = 0$. In particular, our proof of irreducibility failed.

In this section, we prove that there is exactly one sign subrepresentation when $e = n + r + 1$, thus M^r_{2n+r} is reducible. We give an explicit basis for K, and explore how our approach may generalize to finding subrepresentations for other e.

Note that we will use notation from section 3.1 freely.

4.1. Kernel basis.

Here we determine an explicit basis for K when $e = n + r + 1$, assuming $K \neq 0$. In the next section, we prove $K \neq 0$.

First let us formalize a useful property of sub-matchings. Namely, acting on sub-matchings is the same as acting on the corresponding matching.

Definition 4.1. Given a basis element $\psi \in M^r_{2n+r}$, specify some sub-matching $\psi(a, b)$. Let $\operatorname{Res}^{\mathscr{H}_{2n+r}(q)}_{\mathscr{H}_{b-a+1}(q)} M^r_{2n+r}$ be the restriction to the sub-algebra generated by transpositions $T_a, ..., T_{b-1}$. Define $Y_\psi \subset \operatorname{Res}^{\mathscr{H}_{2n+r}(q)}_{\mathscr{H}_{b-a+1}(q)} M^r_{2n+r}$ to be the subrepresentation generated by the set of basis elements $\{\sigma | \sigma(1, a-1) = \psi(1, a-1),\ \sigma(b+1, 2n+r) = \psi(b + 1, 2n + r)\}$.

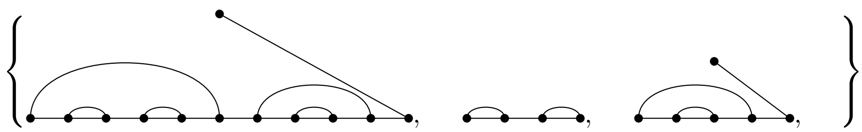


Figure 8. Suppose the second and third elements have coordinates $x_2(q_1)$ and $x_3(q_2)$ in their respective kernel elements, where for q_1, $e = 3$ and for q_2 $e = 4$. The coordinate of the first element is $x(q) = x_2(q)x_3(q)\frac{Q_5Q_4Q_3}{Q_1Q_2}$, where for q, $e = 7$

Lemma 4.2. *Take a basis element $\psi \in M^r_{2n+r}$. Suppose ψ has some sub-matching $\psi(a, b)$ with r' anchors. Define Y_ψ with respect to this sub-matching.*

The map $\rho : Y_\psi \to M^{r'}_{b-a+1}$ defined by

$$\rho(\sigma) = \sigma(a, b)$$

is an isomorphism of representations.

Proof. The map is clearly bijective. Thus it is sufficient to prove the following:

$$\rho((1 + T_{i+a-1})\sigma) = (1 + T_i)\rho(\sigma)$$

As mentioned in the previous section, the action of a transposition T_i can change at most 4 nodes, so we need to show that the transpositions end up changing the same nodes in the same way in $\rho((1 + T_{i+a-1})\sigma)$ and $(1 + T_i)\rho(\sigma)$.

Suppose $\sigma(i + a - 1) = s, \sigma(i + a) = t$. Then $((1 + T_{i+a-1})\sigma)(i + a - 1) = i + a$, $((1 + T_{i+a-1})\sigma)(s) = t$, so $\rho((1 + T_{i+a-1})\sigma)(i) = i + 1$, $\rho((T_{i+a-1})\sigma)(s - a + 1) = t - a + 1$. Separately, $\rho(\sigma)(i) = s - a + 1$ and $\rho(\sigma)(i + 1) = t - a + 1$, so $(1 + T_i)\rho(\sigma)(i) = i + 1$ and $(1 + T_i)\rho(\sigma)(s - a + 1) = t - a + 1$ as desired. So the map is an isomorphism and the lemma is proved. □

The lemma above motivates a recursive characterization of the kernel. To do this, it will be convenient to define some notation.

Definition 4.3. Recall $Q_i := (q^i + ... + q + 1)/q^{i/2}(-1)^i$ (lemma 3.7). For $b \geq 0$, define $\mathfrak{Q}^0_b := 1$. For $b \geq a > 0$ define

$$\mathfrak{Q}^a_b := \frac{Q_{b-1}...Q_{b-a}}{Q_1...Q_{a-1}}$$

Definition 4.4. For $\psi \in M^0_0$, define the function $x_\psi(q) := 1$.

For all other basis elements $\psi \in M^r_{2n+r}$, we define x_ψ recursively:

$$x_\psi(q) := x_{\psi(2,a-1)}(q)x_{\psi(a+1,2n+r)}(q)\mathfrak{Q}^{\lfloor a/2 \rfloor}_{n+r}$$

I will refer to x_ψ as the *coordinate function* of ψ.

The following proposition states the forward direction of our characterization.

Proposition 4.5. *Let M^r_{2n+r} be a crossingless matchings representation, and suppose $Q_1, ...Q_{n+r-1}, \neq 0$. Let $w \in \cap \ker(1 + T_i)$. WLOG the rainbow element R has coordinate 1 in w (by proposition 3.3). Then the coordinate of any basis element $\psi \in M^r_{2n+r}$ in w is $x_\psi(q)$.*

An illustration of this proposition is shown in figure 8.

Proof. Suppose $\psi(1) = a$. The proof is structured as follows: use proposition 3.6 to find the coefficient of the basis element with $\lfloor a/2 \rfloor$ length two arcs then a rainbow element; use the same proposition in a reversed manner to find the coefficient of the basis element consisting of the rainbow for the first a nodes, then the rainbow for the final $2n + r - a$ nodes; finally, we finish the proof through induction using lemma 4.2.

By proposition 3.6 the element $R_1 := R_{L,n+r-1}$ has coordinate Q_{n+r-1} in w. Then $R_1(3, 2n+r)$ defines a rainbow sub-matching, and is the rainbow element in $M^r_{2(n-1)+r}$. Thus we may define the element R_2 by $R_2(1, 2) := R_1(1, 2)$, $R_2(3, 2n + r) := R_{L,n+r-2} \in M^r_{2(n-1)+r}$. By the same proposition, R_2 has coordinate

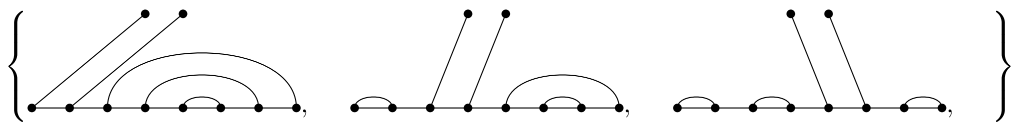


Figure 9. In order, the rainbow element, R_1, and R_2. The coordinate of the rainbow element is 1. The coordinate of R_1 is Q_4. The coordinate of R_2 is Q_4Q_3. Generally, R_i is the element with i length two arcs then a rainbow element, and has coordinate $Q_{n+r-1}...Q_{n+r-i}$.

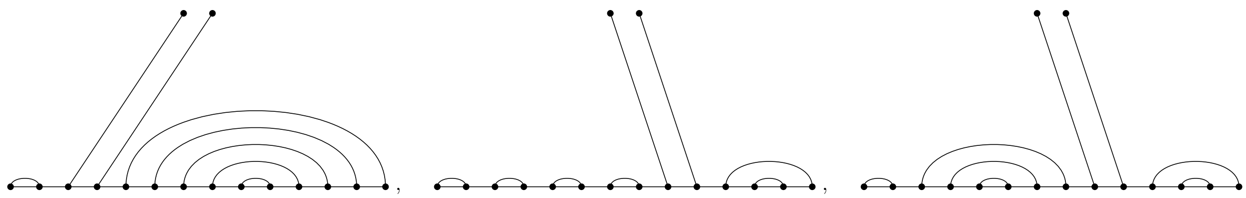


Figure 10. The figure on the left has sub-matching R ignoring the first two nodes. The middle figure has submatching R_3 ignoring the first two nodes. The figure on the right has sub-matching E_3 also ignoring the first two nodes. Since the first two nodes have the same structure for all elements, if the coordinate of the first element is c, the coordinate of the second is $Q_6Q_5Q_4c$, and the coordinate of the third is $\frac{Q_6Q_5Q_4}{Q_1Q_2}c$.

$Q_{n-1}Q_{n-2}$ in w. Generally, define R_i by $R_i(1,2(i-1)) := R_{i-1}(1,2(i-1))$, $R_i(2i-1,2n+r) := R_{L,n+r-i} \in M^r_{2(n-i+1)+r}$. Then the coefficient of R_i is $Q_{n+r-1}...Q_{n+r-i}$. These elements are shown in figure 9.

Now define basis elements E_i to be those basis elements that can be split into two rainbow sub-representations between nodes $2i$ $2i+1$, the first of which has no anchors. Formally, they are defined by $E_i(2i+1,2n+r) := R_i(2i+1,2n+r)$, $E_i(1,2i) := R \in M^0_{2i}$. By the same argument as above, if E_i has coordinate c in w, R_i has coordinate $Q_{i-1}...Q_1c$. To see this, note that E_i by definition has rainbow sub-matchings $E_i(1,2i)$, so we may apply proposition 3.6 to that sub-matching and, as above, sequentially move middle arcs out of the rainbow. We will eventually reach the point where our rainbow sub-matching only has two nodes, and the basis element that we arrive at has all length two arcs and then a rainbow element starting at node $2i+1$. This basis element is R_i. It is easiest to see this in figure 10.

Since we assume $Q_i \neq 0$ for $i < n+r$, this implies the coefficient of E_i is $\frac{Q_{n+r-1}...Q_{n+r-i}}{Q_1...Q_{i-1}} = \mathfrak{Q}^i_{n+r} = x_{E_i}$. In particular, returning to our desired basis element ψ, the coordinate of $E_{\lfloor a/2 \rfloor}$ is $\mathfrak{Q}^{\lfloor a/2 \rfloor}_{n+r} = x_{E_{\lfloor a/2 \rfloor}}$.

Note that the above logic only uses proposition 3.6, which requires only that the basis element has a rainbow sub-matching. So, suppose some basis element σ has rainbow sub-matching $\sigma(s,t) = R$ with n' nodes and r' anchors, and that the coordinate of σ in w is c. Then it follows that the basis element ϑ_i defined by $\vartheta_i(1,s-1) := \sigma(1,s-1)$, $\vartheta_i(t+1,2n+r) := \sigma(t+1,2n+r)$, and $\vartheta_i(s,t) := E_i$ has coefficient $\mathfrak{Q}^i_{n'+r'}c$. In other words, defining Y_σ with respect to the sub-matching $\sigma(s,t)$, the operation of finding the coordinate of E_i given the coordinate of $R = \sigma(s,t)$ commutes with the isomorphism to Y_σ. An example is given in figure 10.

The above technique specifies an algorithm for determining the coordinate of ψ.

As a base case, for the zero element have the algorithm return 1.

Suppose inductively that the algorithm returns the coordinate for any $\sigma \in M^{r'}_{2n'+r'}$, $2n'+r' < 2n+r$, and that that coordinate is equal to the coordinate function x_σ. Also suppose that the algorithm commutes with any isomorphism defined by lemma 4.2. These statements are clearly true for the base case.

Given $\psi \in M^r_{2n+r}$, if $\psi(1) = a$, we may find the coordinate of $E_{\lfloor a/2 \rfloor}$ as before. Note that this operation commutes with any isomorphism defined by lemma 4.2. Me may define $Y_{E_{\lfloor a/2 \rfloor}}$ with respect to the sub-matching $E_{\lfloor a/2 \rfloor}(2,a-1)$. By the inductive hypothesis, we may apply the algorithm to this sub-matching and commute with the isomorphism with $Y_{E_{\lfloor a/2 \rfloor}}$. In this way, we find that the coordinate of $\widehat{\psi}$ defined by $\widehat{\psi}(1,a) := \psi(1,a)$ and $\widehat{\psi}(a+1,2n+r) = R$ is $x_{\psi(2,a-1)}(q)\mathfrak{Q}^{\lfloor a/2 \rfloor}_{n+r} = x_{\widehat{\psi}}(q)$. Similarly, define $Y_{\widehat{\psi}}$ with respect to the sub-matching $\widehat{\psi}(a+1,2n+r)$, and commute the algorithm with the isomorphism. In the same way, we obtain that the coordinate of $\psi \in Y_{\widehat{\psi}}$ is $x_{\psi(2,a-1)}(q)x_{\psi(a+1,2n+r)}(q)\mathfrak{Q}^{\lfloor a/2 \rfloor}_{n+r} = x_\psi(q)$ as desired.

Note that we only added a single operation to the algorithm in the inductive step, which also commutes with any isomorphism defined by lemma 4.2. Thus the inductive step holds and the proposition is proved. □

The following few corollaries will help to simplify some later arguments.

Corollary 4.6. *Let $w \in \cap \ker(1+T_i)$, $w \neq 0$. Suppose $\psi(1,a)$ is a sub-matching with no anchors. Then:*

$$x_\psi = x_{\psi(1,a)}(q) x_{\psi(a+1,2n+r)}(q) \mathfrak{Q}_{n+r}^{a/2}$$

Proof. Define $a_1 = \psi(1)$, $a_i = \psi(a_{i-1}+1)$. Then for some j we have $a_j = a$. If $j = 1$, the statement is the same as the proposition. Suppose that the statement is true for any matching with $a_v = a$, $v < j$. Then the statement holds for the sub-matching $\psi(a_1+1, 2n+r)$, and we have:

$$\begin{aligned}
x_\psi(q) =& x_{\psi(1,a_1)}(q) x_{\psi(a_1+1,2n+r)}(q) \mathfrak{Q}_{n+r}^{a_1/2} \\
=& x_{\psi(1,a_1)}(q) x_{\psi(a_1+1,a)}(q) x_{\psi(a+1,2n+r)}(q) \mathfrak{Q}_{n+r}^{a_1/2} \mathfrak{Q}_{n+r-a_1/2}^{a/2-a_1/2} \\
=& x_{\psi(1,a_1)}(q) x_{\psi(a_1+1,a)}(q) x_{\psi(a+1,2n+r)}(q) \mathfrak{Q}_{n+r}^{a_1/2} \mathfrak{Q}_{n+r-a_1/2}^{a/2-a_1/2} \left(\frac{Q_{a/2-1} \dots Q_{a/2-a_1/2}}{Q_{a/2-1} \dots Q_{a/2-a_1/2}} \right) \\
=& x_{\psi(1,a_1)}(q) x_{\psi(a_1+1,a)}(q) x_{\psi(a+1,2n+r)}(q) \mathfrak{Q}_{a/2}^{a_1/2} \mathfrak{Q}_{n+r}^{a/2} \\
=& x_{\psi(1,a)}(q) x_{\psi(a+1,2n+r)}(q) \mathfrak{Q}_{n+r}^{a/2}
\end{aligned}$$

□

Corollary 4.7. *If $\psi \in M_{2n+r}^r$, then $x_\psi(q) \neq 0$ if $e > n+r$.*

Proof. For our base cases, if $2n+r = 2$ all coefficients are 1, which is nonzero for any q.

Assume the statement is true for all $2n'+r' < 2n+r$. Given $\psi(1) = a$ we have

$$x_\psi(q) = x_{\psi(2,a-1)}(q) x_{\psi(a+1,2n+r)}(q) \mathfrak{Q}_{n+r}^{\lfloor a/2 \rfloor}$$

If $e > n+r$, non of the Q_i term appearing in $\mathfrak{Q}_{n+r}^{\lfloor a/2 \rfloor}$ are zero, and $n'+r' < n+r < e$ for any of the sub-matchings that appear, so those coordinates are nonzero and the corollary holds. □

The proposition fully characterizes any possible kernel element when $Q_1 \dots Q_{n+r-1} \neq 0$. In particular, the following corollary holds:

Corollary 4.8. *When $Q_1 .. Q_{n+r-1} \neq 0$ and the kernel is nontrivial, the kernel is one dimensional.*

This corollary follows from the fact that we may write the coordinate of any basis element as proportional to the coordinate of the rainbow basis element.

4.2. **Nontrivial kernel.** To verify the kernel element, we will need to know exactly which basis elements are mapped to a specific basis element by a given $(1+T_i)$. The next two lemmas help address this question.

Move lemmas to a new appendix \section.

Lemma 4.9. *Take some basis element $\psi \in M_{2n+r}^r$.*

(i) Suppose $\psi(a) = b$ for some $b > a+1$, and that $(1+T_i)\psi = (1+q)\psi$ for some $a < i < b-1$. Me then have a subrepresentation $\psi(a,b)$ and define Y_ψ with respect to this subrepresentation. Then for all basis elements σ such that $(1+T_i)\sigma = q^{1/2}\psi$, we have that

$$\sigma \in Y_\psi$$

.

(ii) Suppose ψ has some anchor at position u, and $(1+T_i)\psi = (1+q)\psi$ for some $i > u$, we again have a subrepresentation $\psi(u, 2n+r)$ and define Y_ψ with respect to this subrepresentation. Then for all basis elements σ such that $(1+T_i)\sigma = q^{1/2}\psi$, we have that $\sigma \in Y_\psi$ again.

Proof. This lemma follows from an observation I made in section 3.1: a transposition can only create two arcs or an arc and an anchor.

(i) If $\sigma \notin Y_\psi$ either $\sigma(1, a-1) \neq \psi(1, a-1)$ or $\sigma(b+1, 2n+r) \neq \psi(b+1, 2n+r)$. Suppose it is the first case. Then for some $s, t \in [1, a-1]$, $s < t$, we have $\psi(s) = t$ and $\sigma(s) \neq t$. To have $(1+T_i)\sigma = q^{1/2}\psi$ we must have $\sigma(t) = i+1$, $\sigma(s) = i$. But then $\sigma(a) \neq b$ and $\sigma(a) \neq i$ or $i+1$, so $((1+T_i)\sigma)(a) \neq b$ and $(1+T_i)\sigma \neq q^{1/2}\psi$. The same argument proves the $\sigma(b+1, 2n+r) \neq \psi(b+1, 2n+r)$ case.

(ii) An analogous argument proves the anchor case. Specifically, the anchor cannot exist at position u and is not created by action of $(1+T_i)$ if $\sigma(s) = i$ and $\sigma(t) = i+1$.

□

It is important to note that lemma 4.9 only references cases where a transposition acts under an arc or to the right of an anchor. An example is given in figure 11.

The next lemma characterizes cases where the transposition is not under any arcs and all anchors are to the right.

Essentially, this lemma states that the only elements sent to the same element are those which break at most one of the top level arcs to the left of the leftmost anchor, or that break the leftmost anchor. An illustration is given in figure 12.

Lemma 4.10. *Take a basis element $\psi \in M^r_{2n+r}$. Suppose the leftmost anchor in ψ is at index b, or let $b = 2n+r+1$ if there is no anchor. Define a_j such that $\psi(a_j) = a_{j-1}+1$ and $\psi(a_1) = 1$ for all j such that $a_j < b$.*

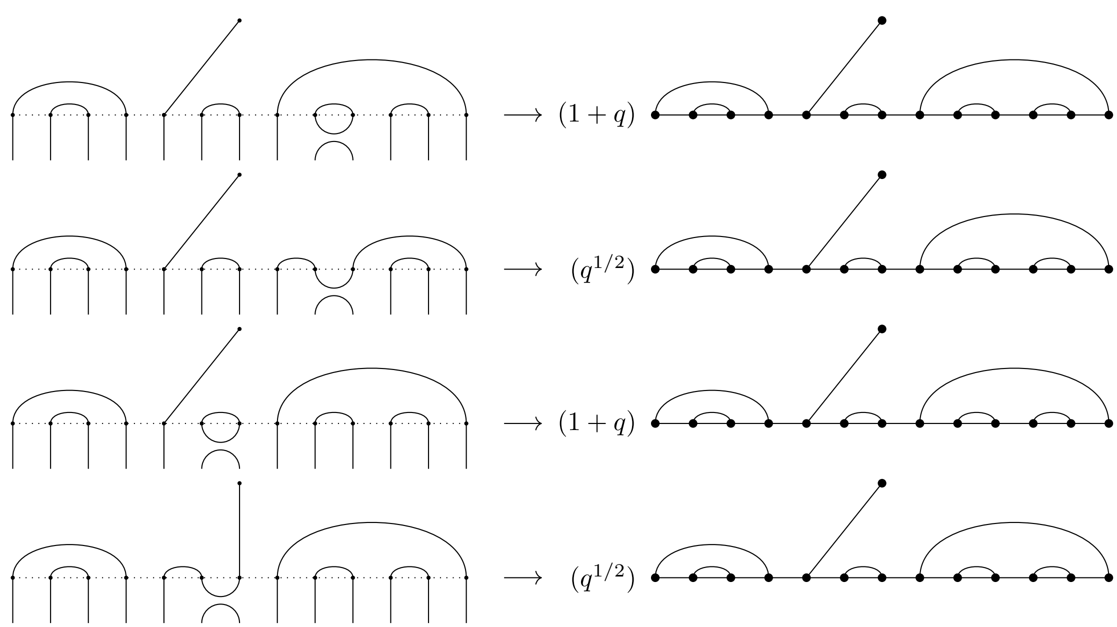


Figure 11. In the first line we act under an arc, so if another element without that arc is sent to that element, it must fix the arc as shown in the second line. In the third line we act to the right of an anchor, so if another element without that anchor is sent to that element, it must fix the anchor as shown in the fourth line.

Suppose $(1+T_i)\psi = (1+q)\psi$ *for some* $i < b-1$ *where* $\nexists s,t$ *such that* $\psi(s) = t$ *and* $s < i, t > i+1$. *Suppose there is some basis element* σ *such that* $(1+T_i)\sigma = q^{1/2}\psi$. *Then:*

(i) $\psi(a_{j-1}+2, a_j-1) = \sigma(a_{j-1}+2, a_j-1)$ *for all* j.
(ii) $\psi(b+1, 2n+r) = \sigma(b+1, 2n+r)$
(iii) If b *is not an anchor in* σ, $\psi(a_j) = \sigma(a_j)$ *for all* j *such that* $a_j \neq i+1$.
(iv) If b *is an anchor in* σ, *there exists exactly one value of* j *such that* $\sigma(a_j) \neq \psi(a_j)$ *and* $a_j \neq i+1$

Proof. (i) Suppose that, for some j there exists $s,t \in [a_{j-1}+2, a_j-1]$ such that $\psi(s) = t$ but $\sigma(s) \neq t$. Then if $(1+T_i)\sigma = q^{1/2}\psi$ we must have $\sigma(i) = s$ or t and $\sigma(i+1) = s$ or t. But, by definition, $i, i+1 \notin [a_{j-1}+1, a_j]$, so this implies $\sigma(a_j) \neq a_{j-1}+1, i, i+1$, so $((1+T_i)\sigma)(a_j) \neq a_{j-1}+1$ and $(1+T_i)\sigma \neq q^{1/2}\psi$. So (i) is proved.

(ii) The proof of (ii) is analogous to the proof of (i). Me cannot have $\psi(b+1, 2n+r) \neq \sigma(b+1, 2n+r)$ and $\psi(b+1, 2n+r) = ((1+T_i)\sigma)(b+1, 2n+r)$ if $((1+T_i)\sigma)(b) = b$.

(iii) If b is not an anchor in σ and $(1+T_i)\sigma = q^{1/2}\psi$, we must have i an anchor in σ, and $\sigma(i+1) = b$. No other nodes in σ are changed, so this proves (iii).

(iiii) From (i)-(iii) we have that the only remaining matchings that can differ are the $(a_{j-1}+1, a_j)$ matchings. If one of them differs, by the same argument as before it must be fixed by the action of $(1+T_i)$, and no other nodes are changed, so (iiii) is proved. ∎

Lastly, we will need a small combinatorial result.

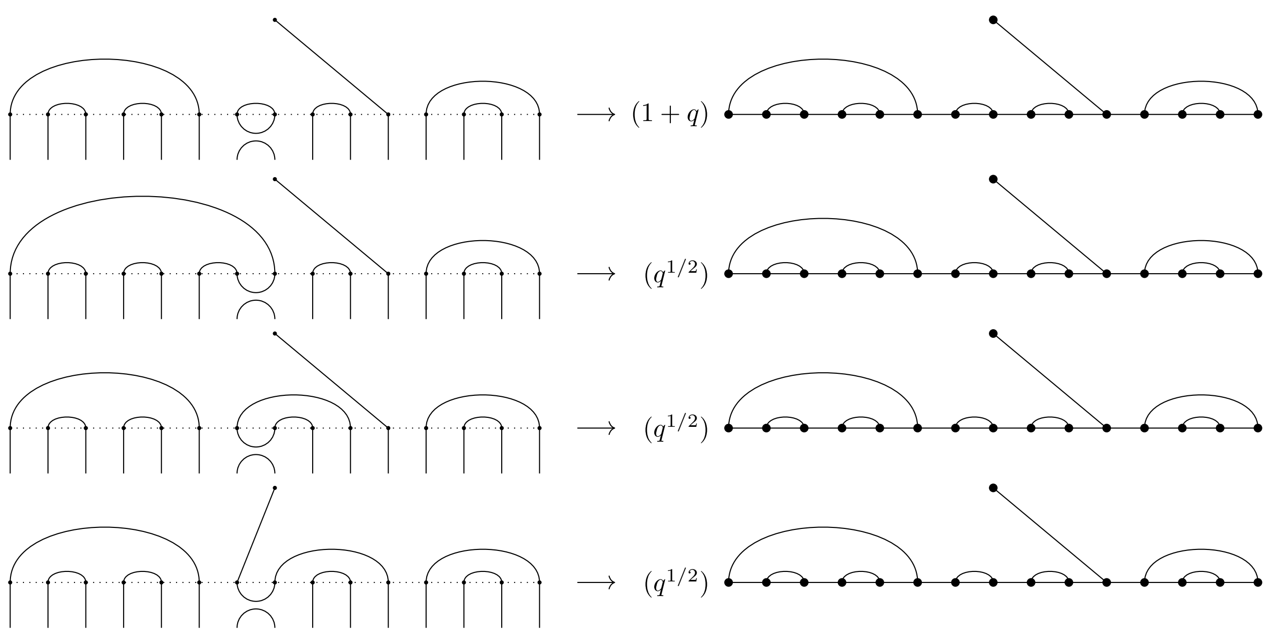


Figure 12. The action of $(1+T_7)$ fixes the first basis element. Shown are all the basis vectors sent to the same element by the same transposition. Note that in all of them nodes 2-5 and 12-15 are the same. This illustrates (i) and (ii) in lemma 4.10. Note that in the last case where the anchor is in a different place, 1,6 and 9,10 are still matched. This illustrates (iii). In the middle two cases where the anchor is in the same place, only one of 1,6 or 9,10 are not paired. This illustrates (iiii).

Lemma 4.11. *Suppose $n > b \geq a > 0$ and $e > n$. Then*
$$Q_{n-a}Q_b - Q_{n-b-1}Q_{a-1} = Q_nQ_{b-a}$$

Proof. If $b = 1$, the only possibility for a is 1, in which reduces to lemma 3.7.

Suppose the lemma is true for all $\widehat{b} < b + 1$. Then for $a < b$ we have
$$\begin{aligned} Q_{n-a}Q_b - Q_{n-b-1}Q_{a-1} =& Q_nQ_{b-a} \\ Q_1Q_{n-a}Q_b - Q_1Q_{n-b-1}Q_{a-1} =& Q_1Q_nQ_{b-a} \end{aligned}$$
from lemma 3.7, we have
$$Q_{n-a}(Q_{b+1} + Q_{b-1}) - (Q_{n-b} + Q_{n-b-2})Q_{a-1} = Q_n(Q_{b-a-1} + Q_{b-a+1})$$
and from the inductive hypothesis we have
$$Q_{n-a}Q_{b+1} - Q_{n-b}Q_{a-1} = Q_nQ_{b-a+1}$$
as desired.

For $a = b$ we have
$$\begin{aligned} Q_{n-b}Q_b - Q_{n-b-1}Q_{b-1} =& Q_n \\ Q_1Q_{n-b}Q_b - Q_1Q_{n-b-1}Q_{b-1} =& Q_1Q_n \end{aligned}$$
from lemma 3.7, we have
$$\begin{aligned} Q_{n-b}(Q_{b+1} + Q_{b-1}) - (Q_{n-b} + Q_{n-b-2})Q_{b-1} =& Q_1Q_n \\ Q_{n-b}Q_{b+1} - Q_{n-b-2}Q_{b-1} =& Q_1Q_n \end{aligned}$$
as desired.

For $a = b + 1$, we continue:
$$\begin{aligned} Q_1Q_{n-b}Q_{b+1} - Q_1Q_{n-b-2}Q_{b-1} =& Q_1Q_1Q_n \\ (Q_{n-b-1} + Q_{n-b+1})Q_{b+1} - Q_{n-b-2}(Q_b + Q_{b-2}) =& (1 + Q_2)Q_n \end{aligned}$$
So by the inductive hypothesis
$$Q_{n-b-1}Q_{b+1} - Q_{n-b-2}Q_b = Q_n$$
as desired, and the proof is finished by induction. □

Me are now ready to prove existence of a kernel element. To prove this, we will show that if $w \in M^r_{2n+r}$ is as characterized above, the coordinate of any basis element in $(1 + T_i)w$ is zero. This will split into various cases related to the previous lemmas.

Theorem 4.12. *Suppose $e = n + r + 1$. Then $\cap \ker(1 + T_i) \neq 0$.*

Proof. As a base case, when $2n' + r' \leq 2$, the representation is at most one dimensional. If the one basis element has only anchors, it is sent to zero by any $(1 + T_i)$, and is in the kernel. If the single basis element is a single arc, it is sent to $(1 + q)$ times itself, and we take $e = n + r + 1 = 2$ so $1 + q = 0$ and the base case holds.

Assume inductively that the statement holds for all $M^{r'}_{2n'+r'}$ where $2n' + r' < 2n + r$. Take w as defined by proposition 4.5.

Given $\psi \in (1 + T_i)M^r_{2n+r}$ let $E_\psi \subset M^r_{2n+r}$ be the pre-image of ψ under the action of $(1 + T_i)$. To prove w is in the kernel, we must show the following:

$$(1 + q)x_\psi(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2}x_\sigma(q) = 0 \text{ for all basis elements } \psi \tag{4.1}$$

Inductively, we assume this equation holds for basis elements in smaller representations $M^{r'}_{2n'+r'}$, but only for q such that $e = n' + r' + 1$. Clearly this is true in the base case. For the following proof we will need a slightly stronger inductive assumption. Take $\psi' \in M^{r'}_{2n'+r'}$, and suppose either that $\psi'(1) = 2n' + r'$, and that $T_i\psi' = (1+q)\psi'$, $1 < i < 2n' + r' - 1$, or that 1 is an anchor in ψ. Defining $E_{\psi'}$ as before, we assume

$$(1+q)x_{\psi'} + \sum_{\sigma \in E_{\psi'}, \sigma \neq \psi'} q^{1/2} x_\sigma = 0 \text{ for any } q \text{ with } e > n' + r' \tag{4.2}$$

Note that 4.2 does not apply in the base case. Our proof of the inductive step will be split into cases, and each case will only depend on sub-cases in which certain inductive hypotheses apply, so this will not lead to any problems.

Before exploring the cases, let us formally define E_ψ to be the pre-image of ψ under the action of $(1+T_i)$, and $E_{\psi(a,b)}$ to be the pre-image of $\psi(a,b)$ under action of $(1 + T_{i-a+1})$:

case 1: Suppose $\psi \in (1+T_i)M^r_{2n+r}$ for some i, and that $\exists s, t$ such that $s < i < t-1$, $s > 1$ or $t < 2n + r$, and $\psi(s) = t$. Also suppose the leftmost anchor is at some index $u > t$, or that there are no anchors. Then we have a sub-matching $\psi(s,t)$, and by lemma 4.9 $E_\psi \subset Y_\psi$. Then, using corollary 4.6, the following equality holds:

$$\begin{aligned}
&(1+q)x_\psi(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2} x_\sigma(q) \\
&= \left(x_{\psi(1,s-1)}(q)\mathfrak{Q}^{(s-1)/2}_{n+r}\right)\left((1+q)x_{\psi(s,2n+r)}(q) + \sum_{\sigma \in E_\psi, \sigma\neq\psi} q^{1/2} x_{\sigma(s,2n+r)}(q)\right) \\
&= \left(x_{\psi(1,s-1)}(q)\mathfrak{Q}^{(s-1)/2}_{n+r}\right)\left(x_{\psi(t+1,2n+r)}(q)\mathfrak{Q}^{(t-s+1)/2}_{n+r-(s-1)/2}\right)\left((1+q)x_{\psi(s,t)}(q) + \sum_{\sigma \in E_\psi, \sigma\neq\psi} q^{1/2} x_{\sigma(s,t)}(q)\right)
\end{aligned}$$

Me have that $e > j$ for any Q_j term appearing in the equation above, and $e > n' + r'$ for any sub-matching coordinate appearing above, so by corollary 4.7:

$$(1+q)x_\psi(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2} x_\sigma(q) = 0$$

if and only if

$$(1+q)x_{\psi(s,t)}(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2} x_{\sigma(s,t)}(q) = 0$$

Note that $(\psi(s,t))(1) = t - s + 1$. So by our inductive hypothesis (ii), we have

$$(1+q)x_{\psi(s,t)}(q) + \sum_{\sigma \in E_{\psi(s,t)}, \sigma \neq \psi(s,t)} q^{1/2} x_\sigma(q) = 0$$

By lemma 4.2, if $\sigma \in Y_\psi$, $(1+T_i)\sigma = q^{1/2}\psi$ if and only if $(1+T_{i-s+1})\sigma(s,t) = q^{1/2}\psi(s,t)$, so the previous equation implies

$$(1+q)x_{\psi(s,t)}(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2} x_{\sigma(s,t)}(q) = 0$$

as desired, and this case is proved.

case 2: Again take $\psi \in (1+T_i)M^r_{2n+r}$ for some i, but suppose the leftmost anchor is at some position u where $1 < u < i$. Then, as before, we have a sub-matching $\psi(u, 2n+r)$ and by lemma 4.9 $E_\psi \subset Y_\psi$.

Note that both corollary 4.6 and our inductive hypothesis 4.2 still apply in this case, where we consider a left anchor instead of a matching. This allows the exact same logic from the proof of the first case to prove this case.

It is important to note that, for both case 1 and case 2, the inductive hypothesis depends only on cases in which 4.2 holds. Thus, if we show these cases rely on valid base cases, case 1 and 2 follow. This will be done in case 4.

case 3: Suppose $\psi \in (1+T_i)M^r_{2n+r}$ for some i, the leftmost anchor is at a position $u > i+1$ or there are no anchors, and $\nexists s, t$ such that $\psi(s) = t$ and $s < i < t-1$. Lemma 4.10 characterizes all $\sigma \in E_\psi$. Me would like to prove the following for arbitrary q where $e > n+r$:

$$(1+q)x_\psi(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2} x_\sigma(q) = -q^{1/2} x_{\psi(1,i-1)}(q) x_{\psi(i+2,2n+r)}(q) \mathfrak{Q}^{(i+1)/2}_{n+r+1}$$

See figure 13 for an example of this equality. Note that if $e = n+r+1$, Q_{n+r} is the only zero component in the right side of this equation, so proving this equation is sufficient to prove case three.

Me will prove this equality through yet another inductive proof, this time inducting on the number of top level arcs, including the leftmost anchor.

Formally, as we have in earlier lemmas, we will define a_j by $a_1 := \psi(1)$, $a_j := \psi(a_{j-1}+1)$. Then define b_ψ such that $a_{b_\psi} = u$ if there is an anchor or $a_{b_\psi} = 2n+r$ otherwise. Me induct on b_ψ.

If $b_\psi = 1$, we must be in M^0_2 to be in case 3 (otherwise $s < i < t-1$ for some s, t where $\psi(s) = t$), which is trivially satisfied. Thus the base case holds.

Suppose for all basis elements σ such that $b_\sigma < b_\psi$, the equality holds. Suppose $i \neq 1$. Then $a_1 < i$ and lemma 4.10 gives that there is a unique $\upsilon \in E_\psi$ such that $\upsilon(1) \neq a_1$. Thus we have the following equality:

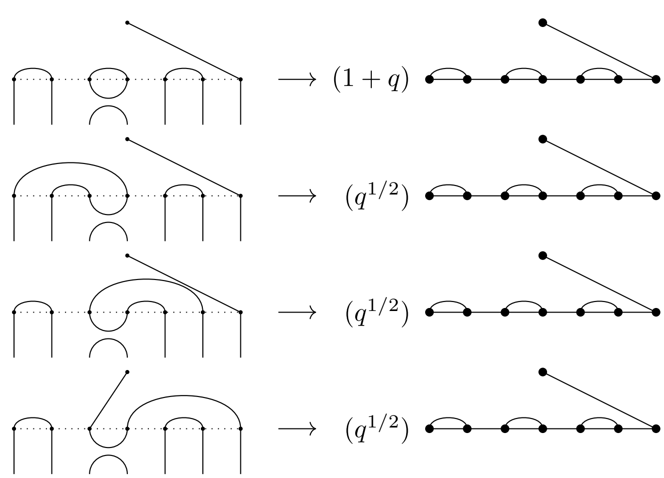


Figure 13. The four elements sent to the first element by $(1+T_3)$ are listed. The coordinate of the first element is $Q_3Q_2Q_1$. The coordinate of the second is Q_3Q_2. The coordinate of the third is Q_3Q_2. The coordinate of the fourth is Q_3. Call the first element ψ. Then $x_{\psi(1,2)} = 1$, $x_{\psi(5,7)} = Q_1$, so $-q^{1/2}x_{\psi(1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}^{(i+1)/2}_{n+r+1} = -q^{1/2}Q_4Q_3$. Me also have $(1+q)x_\psi(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2}x_\sigma(q) = (q+1)(Q_3Q_2Q_1) + q^{1/2}(Q_3Q_2 + Q_3Q_2 + Q_3) = -q^{1/2}(Q_3Q_2Q_1^2 - 2Q_3Q_2 + Q_3) = -q^{1/2}Q_4Q_3$ as desired (one can verify the last equality by hand or simplify using lemma 4.11).

$$(1+q)x_\psi(q) + \sum_{\sigma\in E_\psi,\sigma\neq\psi,\upsilon} q^{1/2}x_\sigma(q)$$
$$= \left(x_{\psi(1,a_1)}(q)\mathfrak{Q}_{n+r}^{a_1/2}\right)\left((1+q)x_{\psi(a_1+1,2n+r)}(q) + \sum_{\sigma\in E_\psi,\sigma\neq\psi,\upsilon} q^{1/2}x_{\sigma(a_1+1,2n+r)}(q)\right)$$

Define Y_ψ with respect to the sub-matching $\psi(a_1+1,2n+r)$. Then $\sigma\in E_\psi$, $\sigma\neq\upsilon$ implies $\sigma\in Y_\psi$. By our inductive hypothesis, we have that

$$(1+q)x_{\psi(a_1+1,2n+r)}(q) + \sum_{\sigma\in E_{\psi(a_1+1,2n+r)},\sigma\neq\psi(a_1+1,2n+r)} q^{1/2}x_\sigma(q)$$
$$= -\,q^{1/2}x_{\psi(a_1+1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}_{n+r-a_1/2+1}^{(i+1-a_1)/2}$$

By lemma 4.2, $\sigma\subset Y_\psi$, $\sigma\in E_\psi$ if and only if $\sigma(a_1+1,2n+r)\in E_{\psi(a_1+1,2n+r)}$. This implies:

$$(1+q)x_{\psi(a_1+1,2n+r)}(q) + \sum_{\sigma\in E_\psi,\sigma\neq\psi,\upsilon} q^{1/2}x_{\sigma(a_1+1,2n+r)}(q)$$
$$= -\,q^{1/2}x_{\psi(a_1+1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}_{n+r-a_1/2+1}^{(i+1-a_1)/2}$$

So, combining with the aforementioned equality, we have

$$(1+q)x_\psi(q) + \sum_{\sigma\in E_\psi,\sigma\neq\psi,\upsilon} q^{1/2}x_\sigma(q)$$
$$= \left(x_{\psi(1,a_1)}(q)\mathfrak{Q}_{n+r-a_1/2+1}^{(i+1-a_1)/2}\right)\left(-q^{1/2}x_{\psi(a_1+1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}_{n+r-a_1/2+1}^{(i+1-a_1)/2}\right)$$
$$= \left(x_{\psi(1,a_1)}(q)\mathfrak{Q}_{n+r-a_1/2+1}^{(i+1-a_1)/2}\right)\left(-q^{1/2}x_{\psi(a_1+1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}_{n+r-a_1/2+1}^{(i+1-a_1)/2}\right)\left(\frac{Q_{(i-1)/2-1}\cdots Q_{(i-1-a_1)/2}}{Q_{(i-1)/2-1}\cdots Q_{(i-1-a_1)/2}}\right)$$
$$= -\,q^{1/2}x_{\psi(1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}_{n+r+1}^{(i+1)/2}\left(\frac{Q_{n+r-a_1/2}Q_{(i-1)/2}}{Q_{n+r}Q_{(i-1-a_1)/2}}\right)$$

Separately, note that υ is defined by $\upsilon(2,a_1-1)=\psi(2,a_1-1)$, $\upsilon(a_1+1,i-1)=\psi(a_1+1,i-1)$, $\upsilon(i+2,2n+r)=\psi(i+2,2n+r)$, and $\upsilon(1)=i+1$, $\upsilon(a_1)=i$. Thus we may determine x_υ, again utilizing corollary 4.6:

$$\begin{aligned}
x_\upsilon =& x_{\psi(i+2,2n+r)}x_{\upsilon(2,i)}\mathfrak{Q}_{n+r}^{(i+1)/2}\\
=& x_{\psi(i+2,2n+r)}\left(x_{\psi(2,a_1-1)}x_{\upsilon(a_1,i)}\mathfrak{Q}_{(i-1)/2}^{(a_1-2)/2}\right)\mathfrak{Q}_{n+r}^{(i+1)/2}\\
=& x_{\psi(i+2,2n+r)}\left(x_{\psi(1,a_1)}x_{\psi(a_1+1,i-1)}\mathfrak{Q}_{(i-1)/2}^{(a_1-2)/2}\right)\mathfrak{Q}_{n+r}^{(i+1)/2}\\
=& x_{\psi(i+2,2n+r)}x_{\psi(1,i-1)}\mathfrak{Q}_{n+r+1}^{(i+1)/2}\left(\frac{Q_{n+r-(i+1)/2}Q_{a_1/2-1}}{Q_{n+r}Q_{(i-1-a_1)/2}}\right)
\end{aligned}$$

Adding this into our previous equation, we have:

$$(1+q)x_\psi(q) + \sum_{\sigma\in E_\psi,\sigma\neq\psi} q^{1/2}x_\sigma(q)$$
$$= -\,q^{1/2}x_{\psi(1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}_{n+r+1}^{(i+1)/2}\left(\frac{Q_{n+r-a_1/2}Q_{(i-1)/2} - Q_{n+r-(i+1)/2}Q_{a_1/2-1}}{Q_{n+r}Q_{(i-1-a_1)/2}}\right)$$

Applying lemma 4.11 to the portion of the equation above in parenthesis, the above is equivalent to

$$(1+q)x_\psi(q) + \sum_{\sigma\in E_\psi,\sigma\neq\psi} q^{1/2}x_\sigma(q) = -q^{1/2}x_{\psi(1,i-1)}(q)x_{\psi(i+2,2n+r)}(q)\mathfrak{Q}_{n+r+1}^{(i+1)/2}$$

as desired. Note that if $e \geq n+r+1$ the only term above that can be zero is Q_{n+r} (by corollary 4.7). Thus we have proved the inductive step for the case where $i \neq 1$.

If $i = 1$, we instead look at the sub-matchings $\psi(1, a_{(b_\psi - 1)})$, $\psi(a_{(b_\psi-1)} + 1, 2n+r)$. Again lemma 4.10 gives that there is a unique $\upsilon \in E_\psi$ such that $\upsilon(a_{(b_\psi-1)}+1) \neq \psi(a_{(b_\psi-1)}+1)$. Taking Y_ψ with respect to the sub-matching $\psi(a_{(b_\psi-1)}+1, a_{b_\psi})$ again we have that $\sigma \in E_\psi$, $\sigma \neq \upsilon$ implies $\sigma \in Y_\psi$. Thus, following the same logic as before, we arrive at the following equality:

$$\begin{aligned}
&(1+q)x_\psi(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi, \upsilon} q^{1/2} x_\sigma(q) \\
&= \left(x_{\psi(a_{(b_\psi-1)}+1, 2n+r)}(q) \mathfrak{Q}_{n+r}^{a_{(b_\psi-1)}/2} \right) \left(-q^{1/2} x_{\psi(3, a_{(b_\psi-1)})}(q) Q_{a_{(b_\psi-1)}/2} \right) \\
&= -\, q^{1/2} x_{\psi(3,2n+r)} \frac{Q_{n+r-1} Q_{a_{(b_\psi-1)}/2}}{Q_{a_{(b_\psi-1)}/2-1}}
\end{aligned}$$

Again, we know the structure of υ from lemma 4.10. Suppose for now that a_{b_ψ} is not an anchor, so it is $2n+r$. Then υ is defined by $\upsilon(3, a_{(b_\psi-1)}) = \psi(3, a_{(b_\psi-1)})$, $\upsilon(a_{(b_\psi-1)} + 2, 2n+r-1) = \psi(a_{(b_\psi-1)} + 2, 2n+r-1)$, and $\upsilon(1) = 2n+r$, $\upsilon(2) = a_{(b_\psi-1)} + 1$. So we may again find x_υ:

$$\begin{aligned}
x_\upsilon = x_{\upsilon(2,2n+r-1)} &= x_{\psi(3, a_{(b_\psi-1)})} x_{\psi(a_{(b_\psi-1)}+2, 2n+r-1)} \mathfrak{Q}_{n+r-1}^{a_{(b_\psi-1)}/2} \\
&= x_{\psi(3, a_{(b_\psi-1)})} x_{\psi(a_{(b_\psi-1)}+1, 2n+r)} \mathfrak{Q}_{n+r-1}^{a_{(b_\psi-1)}/2} \\
&= x_{\psi(3,2n+r)} \frac{Q_{n-r-1-a_{(b_\psi-1)}/2}}{Q_{a_{(b_\psi-1)}/2-1}}
\end{aligned}$$

Alternatively, if a_{b_ψ} is an anchor, the definition of υ is now $\upsilon(3, a_{b_\psi} - 1) = \psi(3, a_{b_\psi} - 1)$, $\upsilon(a_{b_\psi} + 1, 2n+r) = \psi(a_{b_\psi} + 1, 2n+r)$, and $\upsilon(1) = 1$, $\upsilon(2) = a_{b_\psi}$, so we have:

$$x_\upsilon = x_{\upsilon(2,2n+r)} = x_{\psi(3, a_{(b_\psi-1)})} x_{\psi(a_{b_\psi}+1, 2n+r)} \mathfrak{Q}_{n+r-1}^{a_{(b_\psi-1)}/2} = x_{\psi(3,2n+r)} \frac{Q_{n-r-1-a_{(b_\psi-1)}/2}}{Q_{a_{(b_\psi-1)}/2-1}}$$

so for our purposes x_υ is the same in either case.
Incorporating into the above equation, we have:

$$(1+q)x_\psi(q) + \sum_{\sigma \in E_\psi, \sigma \neq \psi} q^{1/2} x_\sigma(q) =$$

$$-q^{1/2} x_{\psi(3,2n+r)} \frac{Q_{n+r-1} Q_{a_{(b_\psi-1)}/2} - Q_{n-r-1-a_{(b_\psi-1)}/2}}{Q_{a_{(b_\psi-1)}/2-1}}$$

By lemma 4.11, this is simply $-q^{1/2} x_{\psi(3,2n+r)} Q_{n+r}$ as desired, and we have finished proving case 3.

case 4: The only cases we have not yet dealt with are those where either 1 is an anchor or $\psi(1) = 2n+r$. These are those cases related to our inductive hypothesis (ii).

To not be in case 1 or 2, we must have that there are no anchors between index 1 and i, and that there is no integer s such that $1 < s < i < \psi(s) - 1$. It follows from the same argument that proved lemma 4.9 that there exists exactly one $\upsilon \in E_\psi$ such that $\upsilon(1) \neq \psi(1)$. Define N to be $2n+r$ if 1 is an anchor, or $2n+r-1$ if 1 is not an anchor. Then, defining Y_ψ with respect to the sub-matching $\psi(2, N)$, we have that $\sigma \in E_\psi, \sigma \neq \upsilon$ if and only if $\sigma(2, N) \in E_{\psi(2,N)}$. Note that for $E_{\psi(2,N)}$ we may apply the inductive hypothesis from case 3, so we have:

$$\begin{aligned}&(1+q)x_{\psi}(q)+\sum_{\sigma\in E_{\psi},\sigma\neq\psi,\upsilon}q^{1/2}x_{\sigma}(q)\\ =&(1+q)x_{\psi(2,N)}(q)+\sum_{\sigma\in E_{\psi(2,N)},\sigma\neq\psi(2,N)}q^{1/2}x_{\sigma}(q)\\ =&-q^{1/2}x_{\psi(2,i-1)}x_{\psi(i+2,N)}\mathfrak{Q}^{i/2}_{n+r}\end{aligned}$$

As in case 3, we can also determine x_υ. υ is defined by $\upsilon(2,i-1)=\psi(2,i-1)$, $\upsilon(i+2,N)=\psi(i+2,N)$, $\upsilon(1)=i$, and $\upsilon(i+1)=2n+r$ if 1 is not an anchor or $i+1$ if 1 is an anchor, and we have:

$$x_\upsilon=x_{\psi(2,i-1)}x_{\upsilon(i+2,N)}\mathfrak{Q}^{i/2}_{n+r}$$

Thus we have

$$(1+q)x_{\psi}(q)+\sum_{\sigma\in E_{\psi},\sigma\neq\psi}q^{1/2}x_{\sigma}(q)=$$

$$-q^{1/2}x_{\psi(2,i-1)}x_{\psi(i+2,N)}\mathfrak{Q}^{i/2}_{n+r}\,(1-1)=0$$

as desired, and the last case is proved. Note that this only relies on the inductive hypothesis from case 3, for which we showed the base case holds.

Thus our inductive hypotheses have all been proven, and those that apply in the base case hold in the base case, so by induction the theorem is proved.

□

Corollary 4.13. *If $e=n+r+1$, M^r_{2n+r} is reducible, and has a unique sign subrepresentation.*

Due to the argument present in Theorem 3.10, sign subrepresentations provide a surprisingly strong characterization of some M^r_{2n+r} as follows.

Proposition 4.14. *This proposition can be strengthened, right? Like we just need a sequence of $n-n'$ actions such that no two of them are closer than two positions apart, right? Fix some $M:=M^r_{2n+r}$. Suppose there is some natural number n' with $n'\leq n$ such that $K^r_{2n'+r}\neq 0$. Let $\pi: M^r_{2n+r}\twoheadrightarrow M^r_{2n'+r}$ be the linear map*

$$\pi:=\begin{cases}(1+T_1)(1+T_3)\cdots(1+T_{n-n'}) & \textit{if } n<n,'\\ \mathrm{id}_M & \textit{if } n=n',\end{cases}$$

where id_M is the identity map on M. Then, $\pi^{-1}(K)$ contains any proper subrepresentations of M.

Proof. It suffices to show that any vector $w\in M-\pi^{-1}(K)$ is cyclic; then, any subrepresentation containing w also contains all of M, implying the proposition.

In fact, since $\pi(w)\notin K$, there is some $(1+T_i)\in\mathscr{H}$ such that $(1+T_i)\pi(w)\neq 0$. Note that $e>(n'-1)+r+1$, so $M^r_{2(n-1)+r}$ is irreducible; then, we may use diagram (3.1) and an analogous argument to Theorem 3.10 to argue that $(1+T_i)\pi(w)$ is cyclic, and hence w is cyclic. □

Corollary 4.15. *Suppose $e=n+r+1$. Then M^r_{2n+r} has a composition series given by*

$$0\subset K\subset M.$$

This is suggestive; the preimage $\pi^{-1}(K)$ may be integral in characterizing the representation M^r_{2n+r} in other reducible cases. The following section poses several conjectures involving this structure.

4.3. **Conjectures on sign subrepresentations.** If $\pi^{-1}(K)$ is itself a subrepresentation of M, then we will have identified the radical of M^r_{2n+r} via Propostion 4.14. Observing decomposition matrices for $S_{(n+r,n)}$ for small n, we note that $S^{(n+r,n)}$ appears to have composition series of length at-most two by [9]; the same appears to be true of M^r_{2n+r} by Figures **??** through **??**. This motivates our first conjecture:

Conjecture 4.16. *In Proposition 4.14, $\pi^{-1}(K)$ is an irreducible subrepresentation of M.*

Could we also say the quotient $M/\pi^{-1}(K)$ is irreducible? We believe that right?

We can posit more sturcutre than this; by observing Figure 17, we note that the kernel is nontrivial in more cases than $e = n + r + 1$, and appears to follow a pattern:

Conjecture 4.17. *Suppose $p = \infty$. Then, $K \subset M^r_{2n+r}$ is nontrivial if and only if $e \mid n + r + 1$ and $e < n$. in which case it is one-dimensional.*

Then, assuming that Conjectures 4.16 and 4.17 are true, we reach the following corollary.

Corollary 4.18. *Suppose e, n, r are such that M^r_{2n+r} is irreducible and $e \mid n' + r + 1$ for some $e < n' < n$. Then, $M := M^r_{2n+r}$ has a composition series given by*

$$0 \subset \pi^{-1}(K) \subset M. \tag{4.3}$$

Proof. By 4.17, we have the filtration (4.3). By 4.16, the first factor is irreducible, and by 4.14, the second factor is irreducible. □

These conjectures hope to characterize much of the structure of M^r_{2n+r}. Due to this structure, the heuristics listed above, their isomorphism in the $e > n + r + 1$ case, and the equality of their dimension, we pose the following conjecture.

Conjecture 4.19. *For any e, n, r, we have $M^r_{2n+r} \simeq S^{(n+r,n)}$.*

5. Fibonacci representations and quotients of Specht modules

The irreducibility of $S^{(n+r,n)}$ is integral to arguments of Section 3; without irreducibility of M^r_{2n+r} (which is conjecturally equivalent to irreducibility of $S^{(n+r,n)}$), M is not necessarily isomorphic to any Specht module S^λ or quotient D^λ, preventing the arguments of Section 2 from being used. Further, when $S^{(n+r,n)}$ is reducible, we have $e \mid l$ for some $r + 2 \le l \le n + r + 1$, and hence the filtration provided in Proposition 3.2 fails to be a composition series. In summary, for $S^{(n+r,n)}$, nearly every argument in Sections 2 and 3 break down, and hence it is difficult to characterize the Specht module $S^{(n+r,n)}$ via crossingless matchings.

Due to these difficulties, we additionally seek a graphical realization of the irreducible quotient $D^{(n+r,n)}$. In doing so, we may study their branching via Corollary 2.5, which gives an interesting recurrence in their dimension.

Example. Suppose $e = 5$. If m is even, let t be such that $2t = m$; if m is odd, let t' be such that $2t' + 3 = m$. Then, we may define the following quantity:

$$d^{0,3}_m := \begin{cases} \dim D^{(t,t)} & \text{if } m \text{ is even,} \\ \dim D^{(t'+3,t')} & \text{if } m \text{ is odd.} \end{cases}$$

Define $d^{1,2}_m$ similarly. Then, by Corollary 2.5, we have

$$\begin{aligned} d^{0,3}_m &= d^{1,2}_{m-1}, \\ d^{1,2}_m &= d^{1,2}_{m-1} + d^{0,3}_{m-1} \\ &= d^{1,2}_{m-1} + d^{1,2}_{m-2}. \end{aligned}$$

Further, by Lemma 3.1, we have that $d^{0,3}_2 = d^{1,2}_3 = 1$; hence $d^{1,2}_m = d^{0,3}_{m+1} = f_m$, where f_m is the nth *Fibonacci number.*

In this section, we henceforth restrict to the case $e = 5$ and $r \le 3$. We note that Shor–Jordan [13] have conveniently used complex representations of the braid group on m strands in [13] having dimensions f_m and f_{m-1}. In fact, the *Fibonacci representation* of Definition 1.2 with $k = \mathbb{C}$ and $q = e^{-3\pi i/5}$ is a rescaling of Shor–Jordan's Fibonacci representation.

Note that Shor–Jordan does not characterize this representation any more than the definition and decomposition into four subrepresentations; we will give a characterization of this representation which is stronger than presented in [13].

We will start our study of the Fibonacci representation $V := V^m$ by studying low-dimensional cases. Recall that we have decomposed V into a direct sum of the subrepresentations V_{**}, V_{*0}, V_{0*}, and V_{00}, indexed by the first and last bit of the character of the strings in V.

Proposition 5.1. *We have the following isomorphisms of representations:*

$$V^2_{**} \simeq D^{(1^2)}$$
$$V^2_{*0} \simeq D^{(2)}$$
$$V^2_{00} \simeq V^2_{**} \oplus V_{*0}.$$

Proof. The first isomorphism follows via identification with the trivial representation, and the second with the sign representation.

Further, V^2_{00} is a 2-dimensional representation of a semisimple commutative algebra, and hence decomposes into a direct sum of two 1-dimensional subrepresentations. In particular, we may fix basis $\{(0{*}0), (000)\}$ for V^2_{00} and note that T_1 acts by the matrix

$$\rho_{T_1} = \begin{bmatrix} \varepsilon_1 & \delta \\ \delta & \varepsilon_2 \end{bmatrix},$$

which has characteristic polynomial

$$\lambda^2 - (\varepsilon_1 + \varepsilon_2)\lambda + (\varepsilon_1\varepsilon_2 - \delta^2).$$

We may verify that, for $\lambda = -1$, this evaluates to

$$-((-1+q+q^2)(1+q^3+q^4+q^5+2q^6+q^7))\,[5]_q = 0,$$

and for $\lambda = q$ this evaluates to

$$-(q^2(-1+q+q^2)(1+q+q^2+q^3+2q^4+q^5))\,[5]_q = 0.$$

Hence ρ_{T_1} has eigenvalues -1 and q.

The eigenspaces with eigenvalues -1 and q are subrepresentations isomorphic to the sign and trivial representation, hence V_{00} is isomorphic to a direct sum of the trivial and sign representations, as required. □

We may further study the low-dimensional Fibonacci representations with the following proposition:

Proposition 5.2. *The representation V^3_{*0} is irreducible.*

Proof. Fix the basis $\{({*}0{*}0), ({*}000)\}$ for V^3_{*0}. Then, T_1 and T_2 act by the following matrices:

$$\rho_{T_1} = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{bmatrix}; \qquad \rho_{T_2} = \begin{bmatrix} \varepsilon_1 & \delta \\ \delta & \varepsilon_2 \end{bmatrix}.$$

A proper nontrivial subrepresentation of V^3_{*0} must be one-dimensional, and hence an eigenspace of each of these matrices; since $\alpha_2 \neq \alpha_1$, the ρ_{T_1} has two independent eigenspaces given by the spans each basis element; since $\delta \neq 0$, neither basis element is an eigenvector of ρ_{T_2}. Hence V^3_{*0} is irreducible. □

These propositions establish the low-dimensional behavior of V^m that we will use in our analysis of general V^m below. We will proceed first by proving that V^m_{*0} and V^m_{**} are irreducible; then, we will use combinatorial arguments to prove that these are isomorphic to the desired irreducible quotients two-row Specht modules.

Proposition 5.3. *The representation V^m_{*0} is irreducible.*

Proof. We will prove this inductively in m. We've already proven irreducibility of V^2_{*0} and V^3_{*0}, so suppose that V^{m-2}_{*0} is irreducible.

Let $\{v_i\}$ be the basis for V_{*0}. Then each v_i is cyclic; indeed, we can transform every basis vector into $({*}0\ldots0)$ via action by the appropriate $\frac{1}{\delta}(T_i - \varepsilon_1)$, and we can transform $({*}0\ldots0)$ into any basis vector via action by the appropriate $\frac{1}{\delta}(T_i - \varepsilon_2)$. Hence it is sufficient to show that each $v \in V_{*0}$ generates some basis element.

Let v_j be the basis element $(\texttt{*0*0}\ldots\texttt{0})$, which is many copies of $\texttt{*0}$, followed by an extra $\texttt{0}$ if m is odd. We will show that each $v \in V_{\texttt{*0}}^m$ generates v_j. Then, each v will be cyclic, implying the proposition.

Suppose that no basis elements beginning $(\texttt{*0*0})$ have nonzero coefficient in v; then, there is some basis element v_i beginning $(\texttt{*000})$ having nonzero coefficient in v, and the basis element having all other characters identical to v_i except for the beginning $(\texttt{*0*0})$ has nonzero coefficient in T_3v. Hence we may assume that at least one element beginning $(\texttt{*0*0})$ has nonzero coefficient in v.

Note that

$$\operatorname{im}(T_1 - \alpha_1) = \operatorname{Span}\{\text{Basis vectors beginning } (\texttt{*0*0})\}$$

and $(T_1 - \alpha_1)v \neq 0$. Further, note that we may consider $\operatorname{im}(T_1 - \alpha_1)$ to be a subrepresentation of $\operatorname{Res}^{\mathscr{H}(S_m)}_{\mathscr{H}(S_{m-2})} V_{\texttt{*0}}^m$; this yields that $\operatorname{im}(T_1 - \alpha_1) \simeq V_{\texttt{*0}}^{m-2}$ as representations. Hence irreducibility of $V_{\texttt{*0}}^{m-2}$ implies that v_j is generated by $(T_1 - \alpha_1)v$, and v is cyclic, as desired. □

Note that the structure of this proof is parallel to the structure of Theorem 3.10: we project down to the analogous representation on fewer letters, and we inductively lift irreducibility from this smaller representation to our original one.

Now we may begin considering restrictions of Fibonacci representations:

Lemma 5.4. *The following branching rules hold:*

$$V_{\texttt{00}}^{m-1} \simeq \operatorname{Res} V_{\texttt{*0}}^m \simeq V_{\texttt{**}}^{m-1} \oplus V_{\texttt{*0}}^{m-1},$$
$$V_{\texttt{0*}}^{m-1} \simeq \operatorname{Res} V_{\texttt{**}}^m \simeq V_{\texttt{*0}}^{m-1}.$$

Proof. By the results of Appendix B, any two restrictions to distinct subalgebras of $\mathscr{H}$ generated each by $m-2$ characters are isomorphic. Using this fact, the leftmost isomorphism on each line follows by considering the restrictions to the subalgebra of $\mathscr{H}$ generated by $\{T_2, \ldots, T_{m-1}\}$. Further, the rightmost isomorphism on each line follows by considering the restrictions to the subalgebra of $\mathscr{H}$ generated by $\{T_1, \ldots, T_{m-2}\}$. □

Corollary 5.5. *The representation $V_{\texttt{**}}^m$ is irreducible.* □

Proof. By Lemma 5.4 and Proposition 5.3, we have $\operatorname{Res} V_{\texttt{**}}^m \simeq V_{\texttt{*0}}^{m-1}$ is irreducible, implying that $V_{\texttt{**}}^m$ is also irreducible. □

Recall that we have the decomposition

$$V^m \simeq V_{\texttt{**}}^m \oplus V_{\texttt{*0}}^m \oplus V_{\texttt{0*}}^m \oplus V_{\texttt{00}}^m.$$

Using this, we may now decompose V into a direct sum of irreducible representations.

Corollary 5.6. *The representation V^m decomposes into a direct sum of irreducible representations as follows:*

$$V^m \simeq 3V_{\texttt{*0}}^m \oplus 2V_{\texttt{**}}^m.$$

□

Remark. As in [13], we may be consider a representation $\tilde{V}^m$ of the braid group on m strands via Definition 1.2. In fact, Proposition 5.3, Lemma 5.4, and Corollaries 5.5 and 5.6 hold in reference to $\tilde{V}^m$ by analogous arguments.

Now we may use these in order to characterize V via irreducible quotients of Specht modules.

Theorem 5.7. *We have the following isomorphisms:*

$$V_{\texttt{**}}^{2n} \simeq D^{(n,n)},$$
$$V_{\texttt{**}}^{2n-1} \simeq D^{(n+1,n-2)},$$
$$V_{\texttt{*0}}^{2n} \simeq D^{(n+1,n-1)},$$
$$V_{\texttt{*0}}^{2n-1} \simeq D^{(n,n-1)}.$$

Proof. We will prove this by induction on $m = 2n$; we have already proven the base case V^2, so suppose that we have proven these isomorphisms for V^{2n-2}. We will prove the isomorphisms for V^{2n-1} and V^{2n}.

By Proposition 5.3, $V_{\texttt{**}}^{2n-1} \simeq D^\lambda$ and $V_{\texttt{*0}}^{2n-1} \simeq D^\mu$ for some partitions $\lambda, \mu \vdash 2n-1$. We will show that $\lambda = (n+1, n-2)$ and $\mu = (n, n-1)$.

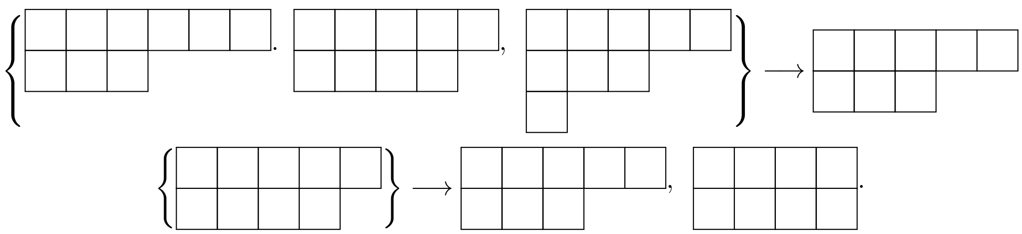


Figure 14. Illustration of the partitions of 9 which can, via removal of cells, yield $(n, n-2)$ alone, or both $(n, n-2)$ and $(n-1, n-1)$.

First, by Lemma 5.4 and the inductive hypothesis, we have

$$\text{Res } D^{\lambda} \simeq D^{(n,n-2)} \simeq \text{Res } D^{(n+1,n-2)}$$

and

$$\text{Res } D^{\mu} \simeq D^{(n,n-2)} \oplus D^{(n-1,n-1)} \simeq \text{Res } D^{(n,n-1)}.$$

By irreducibility of $\operatorname{Res} D^{\lambda}$, the only normal number in λ is 1.[3, 7] Further, the only diagrams which can be transformed into $(n, n-2)$ by removing a cell are $(n+1, n-2)$, $(n, n-1)$, and $(n, n-2, 1)$ as illustrated in Figure 14; we have already seen that $D^{(n,n-1)}$ does not have irreducible restriction via Corollary 2.5, so we are left with $(n+1, n-2)$ and $\varsigma := (n, n-2, 1)$. We may directly check that ς doesn't satisfy this, as we have

$$\begin{aligned}
\beta_{\varsigma}(1,2) &= 3 - 2 + (n-2) = n-1,\\
\beta_{\varsigma}(1,3) &= 3 - 1 + n = n+2,\\
\beta_{\varsigma}(2,3) &= 2 - 1 + 3 = 4.
\end{aligned}$$

At least one of $\beta_{\varsigma}(1,2)$ and $\beta_{\varsigma}(1,3)$ is nonzero, since $\beta_{\varsigma}(1,3) - \beta_{\varsigma}(1,2) = 3 \not\equiv 0 \pmod{e}$. Hence at least one of 2 or 3 is normal in ς, and $\lambda = (n+1, n-2)$.

For μ, we immediately see from Figure 14 that the only option is $(n, n-1)$.

We can perform a similar argument for the V^{2n} case; for $V^{2n}_{**} \simeq D^{\lambda'}$ and $V^{2n}_{*0} \simeq D^{\mu'}$, we have

$$\text{Res } D^{\lambda'} \simeq D^{(n,n-1)} \simeq \text{Res } D^{(n,n)}$$

and

$$\text{Res } D^{\mu'} \simeq D^{(n,n-1)} \oplus D^{(n+1,n-2)} \simeq \text{Res } D^{(n+1,n-1)}.$$

Through a similar process, we see that $\mu' = (n+1, n-1)$. We narrow down λ' to one of (n, n) or $\varpi := (n, n-1, 1)$, and note that

$$\begin{aligned}
\beta_{\varpi}(1,2) &= 3 - 2 + (n-1) = n,\\
\beta_{\varpi}(1,3) &= 3 - 1 + n = n+2,\\
\beta_{\varpi}(2,3) &= 2 - 1 + 2 = 3.
\end{aligned}$$

and hence at least one of 2 or 3 is normal, $\operatorname{Res} D^{\varpi}$ is not irreducible, and $\lambda' = (n, n)$, finishing our proof. ☐

Corollary 5.8. *We have the following isomorphisms of representations:*

$$\begin{aligned}
V^{2n} &\simeq 3D^{(n+1,n-1)} \oplus 2D^{(n,n)},\\
V^{2n-1} &\simeq 3D^{(n,n-1)} \oplus 2D^{(n+1,n-2).}
\end{aligned}$$

Proof. This follows from Corollary 5.6 and Theorem 5.7. ☐

Hence we have entirely characterized Shor–Jordan's Fibonacci representation [13] as a direct sum of irreducible quotients of Specht modules, and we have given graphical realizations of $D^{(n+r,n)}$ for $e = 5$ and $r \leq 3$.

Appendix A. Compatibility of Representations with the Relations

In general, we defined the representations $V := V^{2n+r}$ and $M := M_{2n+r}$ for the free algebra on generators $\{T_1, \ldots, T_{2n+r-1}\}$. Recall that we may give a presentation of $\mathscr{H}$ having generators T_i and relations

$$(T_i - q)(T_i + 1) = 0 \tag{A.1}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{A.2}$$

$$T_i T_j = T_j T_i \qquad |i - j| > 1. \tag{A.3}$$

We call (A.1) the *quadratic relation* and (A.2), (A.3) the *braid relations*. It is easily seen that a representation of $\mathscr{H}$ is equivalent to a representation of the free algebra $k\langle T_i\rangle$ which are compatible with the relations. We will prove in the following sections that V and M are compatible with the Hecke algebra relations.

A.1. **Explicit definition of crossingless matchings.** We will give a more careful definition of the crossingless matchings representation here.

Definition A.1. A *crossingless matching on $2n + r$ indices with r anchors* is a partition of $\{1, \ldots, 2n + r\}$ into n parts of size 2 and r of size 1 such that no two parts (a, a') and (b, b') satisfy $a < b < a' < b'$, and no parts $(c), (a, a')$ satisfy $a < c < a'$. We will call these arcs and anchors, respectively. Then, define M^r_{2n+r} to be the k-vector space with basis the set of crossingless matchings on $2n + r$ indices with r anchors. If basis element w_j contains arc (a, b), say $w_j(a) := b$ and $w_k(b) := a$.

In order to endow M^r_{2n+r} with an $\mathscr{H}$-action, consider some basis element w_j and some element $(1 + T_i)$ of $\mathscr{H}$. The elements $\{1\} \cup \{1 + T_i | 1 \leq i < 2n + r\}$ generate $\mathscr{H}$, so it is sufficient to define the action of $1 + T_i$ on w_j.

If w_j has arc $(i, i + 1)$, define $(1 + T_i)w_j := (1 + q)w_j$. If w_j has anchors $w_j(i) = i$ and $w_j(i + 1) = i + 1$, define $(1 + T_i)w_j := 0$. If w_j has anchor $w_j(i) = i$ and arc $w_j(i + 1) = b$, define $(1 + T_i)w_j := q^{1/2}w_l$, where $w_l(i) = i + 1$, $w_l(b) = b$, and all other arcs agree with w_j. If w_j has arcs $w_j(i) = a$ and $w_j(i + 1) = b$, then define $(1 + T_i)w_j := q^{1/2}w_l$, where $w_l(i) = i + 1$,$w(a) = b$, and all other acts agree with w_j. We verify that this is well-defined in Appendix A.2.

We may alternately sharpen our topological definition;

Definition A.2. Fix $2n + r$ distinct points $a_1, \ldots, a_{2n+r}$ points along $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ and r distinct points $b_1, \ldots, b_r$ along $\mathbb{R} \times \{1\}$. Then, define M^r_{2n+r} to have basis given by the isotopy classes of $n + r$ paths connecting the points $a_1, \ldots, a_{2n+r}, b_1, \ldots, b_r$ such that no distinct b_i, b_j are connected by a path.

We will take some basis element $w_j \in M^r_{2n+r}$ and define the action $(1+T_i)w_j$. To do so, map w_j through the natural embedding $\mathbb{R} \times [0, 1] \hookrightarrow \mathbb{R} \times \left[\frac{1}{2}, 1\right]$, and form the figure w^i_j by adjoining the lines connecting a_l and $a_l + \left(0, \frac{1}{2}\right)$ for all $l \neq i, i + 1$ as well as paths from a_i to a_{i+1} and $a_i + \left(0, \frac{1}{2}\right)$ to $a_{i+1} + \left(0, \frac{1}{2}\right)$. This has either 0 or 1 path components which do not intersect $\mathbb{R} \times \{0, 1\}$; these form "loops."

Take the figure $\tilde{w}^i_j$ without this component. If $\tilde{w}^i_j$ is not isotopic to some w_l, then define $(1 + T_i)w_j := 0$. If $\tilde{w}^i_j$ is isotopic to some w_l, define $(1 + T_i)w_j := (1 + q)w_l$ if w^i_j has a loop and $(1 + T_i)w_j := q^{1/2}w_l$ otherwise. This process is illustrated in Figure 2.

Let the length of an arc (i, j) be $l(i, j) := j - i + 1$. Note that the crossingless matchings on $2n$ indices with no anchors can all be identified with a list of n integers describing the lengths of the arcs from left to right; The basis on M^1_5 induced by the quotient $M^0_6 \twoheadrightarrow M^1_5$ is illustrated in Figure 1, and we call this the *increasing lexicographic basis.*

Remark. This definition gives a graphical calculus for working with our module. It should be clear that, if w^i_j has a loop then $w_l(i) = i + 1$ and $w_l = w_j$. Further, this easily defines an arbitrary composition:

$$(1 + T_{i_1}) \cdots (1 + T_{i_\ell})w_j = q^{(\ell-t)\frac{1}{2}}(1 + q)^t w_l$$

if the figure we make via $(1 + T_{i_1}) \cdots (1 + T_{i_\ell})$ is isotopic to w_l after removing t loops.

Note that the crossingless matchings on $2n$ indices with no anchors can all be identified with a list of n integers describing the lengths of the arcs from left to right; call this the *increasing lexicographical order basis.* Further, we have a surjection $M^0_{2n+2r} \twoheadrightarrow M^r_{2n+r}$ which takes basis elements to basis elements; this induces an

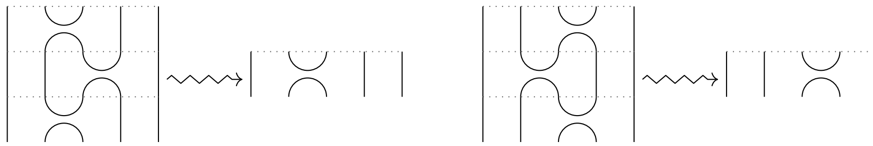


Figure 15. The above give visual intuition for isotopies giving rise to equalities between $(1+T_i)(1+T_{i-1})(1+T_i)w_j$ and $q(1+T_i)$, and between $(1+T_{i+1})(1+T_i)(1+T_{i+1})w_j$ and $q(1+T_i)$.

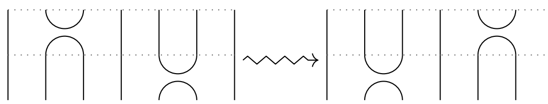


Figure 16. The above give visual intuition for isotopies giving rise to equalities between $(1+T_i)(1+T_j)w_l$ and $(1+T_j)(1+T_i)w_l$.

order on the basis for M^r_{2n+r}, which we will henceforth refer to as the *induced lexicographical order basis.* This basis is illustrated for M^1_5 in Figure 1.

A.2. **Compatibility for the crossingless matchings representations.** We verify the relations on the crossingless matchings representation M. Take some basis vector $w_i \in M$. We will first check (A.1) by case work:

- Suppose there is an arc $(i, i+1)$. Then,

$$(T_i - q)(T_i + 1)w = (1+q)\left((1+T_i)w - (1+q)w\right) = 0,$$

  giving (A.1).
- Suppose there is no arc $(i, i+1)$ and indices $i, i+1$ do not both have anchors; then $(T_i+1)w = q^{1/2}w'$ for some basis vector w' having arc $(i, i+1)$, and the computation follows as above for (A.1).
- Suppose $i, i+1$ are anchors; then $(T_i+1)w = 0$, giving (A.1).

Now we verify (A.2). Let $h := (1+T_i)(1+T_{i+1})(1+T_i)$, and let $g := (1+T_{i+1})(1+T_i)(1+T_{i+1})$. Note the following expansion:

$$\begin{aligned} hw &= 1 + 2T_i + T_i^2 + T_{i+1} + T_iT_{i+1} + T_{i+1}T_i + T_iT_{i+1}T_i \\ &= 1 + (1+q)T_i + T_{i+1} + T_iT_{i+1} + T_{i+1}T_i + T_iT_{i+1}T_i. \end{aligned}$$

This equality, with i and $i+1$ interchanged, holds for g. Hence we have

$$(h-g)w = q(T_i - T_{i+1}) + T_iT_{i+1}T_i - T_{i+1}T_iT_{i+1}.$$

Hence we may equivalently check that $(h-g)w = q(T_i - T_{i+1})$. In fact, $hw = q(1+T_i)$ and $gw = q(1+T_{i+1})$ by Figure 15, giving compatibility.

Lastly, we have the equation

$$(1+T_i)(1+T_j) - (1+T_j)(1+T_i) = T_iT_j - T_jT_i$$

and hence we simply need to verify that $(1+T_i)$ and $(1+T_j)$ commute, as shown in Figure 16

A.3. **Compatibility for the fibonacci representations.** We verify the relations on the Fibonacci representation V. Note that (A.3) follows easily from the "local" nature of V, and the others may be verified explicitly on strings of length 3 and 4. By considering the coefficients in order of (1.1), the quadratic relation

(A.1) gives the following polynomials in q:

$$\begin{aligned}(\alpha_1 - q)(\alpha_1+1) &= 0,\\ (\alpha_2 - q)(\alpha_2+1) &= 0,\\ \varepsilon_1\delta + \delta\varepsilon_2 &= (q-1)\delta,\\ \varepsilon_1^2+\delta^2 &= (q-1)\varepsilon_1 + q,\\ \varepsilon_2^2+\delta^2 &= (q-1)\varepsilon_2 + q\end{aligned} \tag{A.4}$$

The first two of these are easily verified. Since $\delta \neq 0$, the third is equivalently given by

$$(q-1) = \varepsilon_1 + \varepsilon_2 = \tau(q\tau - 1 + q - \tau) = (\tau^2+\tau)(q-1)$$

or that $\left(\tau^2+\tau-1\right)(q-1) = 0$. One may verify that

$$\tau^2+\tau-1 = q^6+2q^5+q^4+q^3+q^2-1 = (-1+q+q^2)\,[5]_q = 0.$$

The fourth is given by the quadratic

$$\tau^2\left[(q\tau-1)^2 - \tau(q+1)\right] = \tau(q-1)(q\tau-1)+q$$

or equivalently,

$$(\tau^2+\tau-1)\left[q\left(qt^2+1\right)+t\right] = 0.$$

The fifth is similarly given by

$$(\tau^2+\tau-1)\left[q\left(qt+1\right)+t^2\right] = 0.$$

All of these vanish for $e=5$, giving compatibility with (A.1).

We now verify (A.2). We may order the basis for V^4 as follows:

$$\left\{(\mathtt{0000}),(\mathtt{*00*}),(\mathtt{000*}),(\mathtt{*000}),(\mathtt{*0*0}),(\mathtt{0*0*}),(\mathtt{00*0}),(\mathtt{0*00})\right\}.$$

Then, in verifying the braid relation (A.2) in this order, we encounter the following quadratics (with tautologies and repetitions omitted):

$$\begin{aligned}\alpha_1\varepsilon_2^2+\alpha_2\delta^2 &= \alpha_1^2\varepsilon_2\\ \alpha_1\delta\varepsilon_2+\alpha_2\varepsilon_1\delta &= \alpha_1\alpha_2\delta\\ \alpha_2\varepsilon_1^2+\alpha_1\delta^2 &= \alpha_2^2\varepsilon_1\\ \alpha_1\varepsilon_1^2+\delta^2\varepsilon_2 &= \alpha_1^2\varepsilon_1\\ \delta\varepsilon_2^2+\alpha_1\varepsilon_1\delta &= \alpha_1\delta\varepsilon_2\end{aligned}$$

Substituting in τ and dividing by δ whenever possible, these are equivalent to the vanishing of the following polynomials in q:

$$\begin{aligned}-q(1+q)(1+q^2+q^3)(2+q+3q^2+2q^3)\,[5]_q &= 0\\ (1+2q+q^3+q^4)\,[5]_q &= 0\\ (1+q)^2(1+q^2+q^3)(1+3q^3-q^4+q^6)\,[5]_q &= 0\\ (1+q)^2(1+q^2+q^3)(1+5q+5q^2+3q^3+3q^4+3q^5+q^6)\,[5]_q &= 0\\ (1+q)(1+q^2+q^3)(-1+2q+q^2+q^3+q^4)\,[5]_q &= 0.\end{aligned}$$

Notably, each of these vanish when $e=5$, giving compatibility with (A.2).

Appendix B. Restrictions to conjugate subalgebras

Throughout the text, for some representation V, we refer to $\mathrm{Res}^{\mathscr{H}(S_m)}_{\mathscr{H}(S_l)} V$ without specifying exactly which subalgebra $\mathscr{H}(S_l)$. For instance, in section 5, we explicitly state that the subrepresentations $V_{*0} \oplus V_{**}$ and V_{00} are isomorphic because they both may be characterized by such a restriction. We will verify that this is justified, using a more general fact about resctrictions to conjugate subalgebras.

Proposition B.1. *Suppose B, B' are subalgebras of a k-algebra A with $B = uB'u^{-1}$ for some unit $u \in A^\times$, and let V be a left A-module. Let $\phi : V \to V$ be the linear automorphism specified by $v \mapsto uv$. Then, the following commutes for any $b \in B$:*

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "b"] & V \arrow[d, "ubu^{-1}"] \\ V \arrow[r, "\phi"] & V \end{tikzcd}$$

Hence, through the identification of B and B' via conjugation by u, we have $\operatorname{Res}^A_B V \simeq \operatorname{Res}^A_{B'} V$

Proof. It suffices to note that $(ubu^{-1})uv = ubv$. ☐

Corollary B.2. *Suppose $\mathscr{H}', \mathscr{H}''$ are two subalgebras of $\mathscr{H}(S_m)$ generated by l simple reflections and V is a representation of $\mathscr{H}$. Then, $\operatorname{Res}^{\mathscr{H}}_{\mathscr{H}'} V \simeq \operatorname{Res}^{\mathscr{H}}_{\mathscr{H}''} V$.*

Proof. Let $\mathscr{H}'$ and $\mathscr{H}''$ be the subalgebras of $\mathscr{H}(S_m)$ generated by the reflections $\{T_{i_1}, \dots, T_{i_l}\}$ and $\left\{T_{i_1}, \dots, T_{i_{j-1}}, T_{i_j+1}, T_{i_{j+1}}, \dots, T_{i_l}\right\}$ for $1 \le i_1 < \cdots < i_{j-1} < i_j + 1 < i_{j+1} < \cdots < i_l \le n$. It is sufficient to prove that $\mathscr{H}'$ and $\mathscr{H}''$ are conjugate; then transitivity gives conjugacy of any $S_l \subset S_m$, and the previous proposition gives isomorphisms of the representations.

We will show that $\mathscr{H}'' = T_{i_j}\mathscr{H}'T_{i_j}^{-1}$. It suffices to show that $T_{i_j}T_wT_{i_j}^{-1} \in \mathscr{H}''$ for w a word generated by simple transpositions $s_{i_1}, \dots, s_{i_l} \in S_m$. First, note that $l(w) < l(s_{i_j}w)$, implying $T_{i_j}T_w = T_{s_{i_j}w}$ by lemma 1.12 in [9]. Further, by the same lemma, we have

$$\begin{aligned} T_{s_{i_j}w}T_{i_j}^{-1} &= q^{-1}\left(T_{s_{i_j}w}T_{i_j} + (1-q)T_{s_{i_j}w}\right) \\ &= q^{-1}\left(T_{qs_{i_j}ws_{i_j}} + (q-1)T_{s_{i_j}w} + (1-q)T_{s_{i_j}w}\right) \\ &= T_{s_{i_j}ws_{i_j}} \end{aligned}$$

which is in $\mathscr{H}''$. ☐

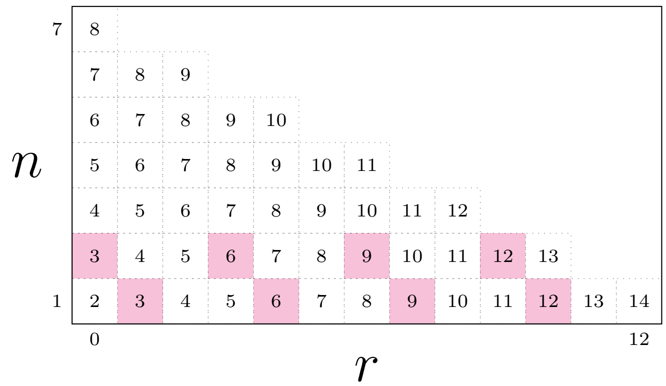
Appendix C. Heuristics

In this section we aim both to support conjecture with data and to provide transition matrices wherever possible between our two graphical representations.[5] The computations were made via a combination of the Python [14] and Magma [2] languages.

Remark. The data on composition series and transition matrices $M/K \to V$ were computed using our implimentation of the Hecke algebra in the Magma language. This stores the algebra via basis elements and structure constants, and hence the memory required for the structure grows with $(n!)^3$. This is prohibitively large when $m > 7$. An implementation of $\mathscr{H}$ as a quotient of a free algebra is possible, but potentially difficult to work with via the Magma language.

For M, the data throughout this section are specified with repsect to the basis on M^r_{2n+r} inuced by the increasing lexicographic basis on M^0_{2n+2r} and the quotient $M^r_{2n+2r} \twoheadrightarrow M^r_{2n+r}$ as defined in Appendix A.1. For V, hey are specified with respect to the basis on V given by increasing lexicographic order $\mathtt{*} < \mathtt{0}$.

The following data define representation $\varphi^r_{2n+r} : M^r_{2n+r}/K^r_{2n+r} \to V^{2n+r}_s$ where $s = \mathtt{**}$ if $r \in \{0, 3\}$ and $s = \mathtt{*0}$ otherwise. All of such computations use q a primitive 5th root of unity in the algebraic extension of the Cyclotomic field $\mathbb{Q}(\zeta_{10})$ by a root of the polynomial $x^2 - \tau$. They cover all cases $2n + r \le 6$, and they



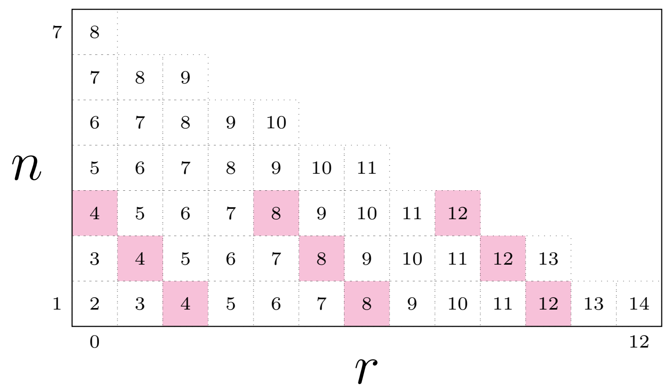


Figure 17. Illustration of the modules M^r_{2n+r} having sign submodules for $p = \infty$ and $e = 3, 4$ respectively. The value $n + r + 1$ is filled in the squares, and modules having sign submodules are colored magenta. For $p = \infty$ and $2n + r \leq 14$, it has been verified, through a combination of theorems here and empirical computations, that K^r_{2n+r} is nontrivial if and only if $e | n + r + 1$ and $e < n$.

include the cases $n = r = 2$ and $n = 1, r = 3$ where $K^r_{2n+r} \neq 0$.

$$\varphi^0_6, \varphi^1_5 = \begin{bmatrix} 0 & 0 & -q^{3/2}+1 & 0 & 0 \\ 0 & -q^{3/2}+1 & [4]_{q^{1/2}} & 0 & 0 \\ 0 & 0 & [4]_{q^{1/2}} & 0 & -q^{3/2}+1 \\ -[4]_{q^{1/2}} & [4]_{q^{1/2}} & q^{1/2}\left(q^{1/2}+1\right) & 0 & [4]_{q^{1/2}} \\ [4]_{q^{1/2}} & 0 & [4]_{q^{1/2}} & -[4]_{q^{1/2}} & 0 \end{bmatrix}$$

$$\varphi^2_6 = \begin{bmatrix} 0 & 0 & -q-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q-1 & -q^{1/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & q+1 & 0 & q^{1/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{1/2} & 0 & 0 & 0 & -q^2-1 & 0 \\ q^{1/2} & -q^{1/2} & -[3]_{q^{1/2}} & 0 & 0 & 0 & -q^{1/2} & 0 \\ -q^{1/2} & q^{1/2} & 0 & [3]_{q^{1/2}} & 0 & 0 & 0 & q^{1/2} \\ -q^{1/2} & 0 & -q^{1/2} & 0 & q^{1/2} & 0 & 0 & 0 \\ q^{1/2} & 0 & 0 & 0 & -q^{1/2} & -[3]_{q^{1/2}} & 0 & -q^{1/2} \end{bmatrix}$$

$$\varphi_5^3 = \begin{bmatrix} -q^{1/2}\left(q^{1/2}+1\right) & [4]_{q^{1/2}} & q^{1/2} \\ q^{1/2} & q^{3/2} & 0 \\ 0 & -q-1 \end{bmatrix} \qquad \varphi_4^0 = \begin{bmatrix} 0 & -[4]_{q^{1/2}} \\ -1 & 1 \end{bmatrix}$$

$$\varphi_3^1 = \begin{bmatrix} 0 & -[4]_{q^{1/2}} \\ q^{1/2}\left(q^{1/2}+1\right) & -q^{1/2}\left(q^{1/2}+1\right) \end{bmatrix} \qquad \varphi_4^2 = \begin{bmatrix} 0 & [4]_{q^{1/2}} & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

We give in Figure 17 some data supporting a conjecture concerning sign subrepresentations of M_{2n+r}^r. The computations to support this were done over $\mathbb{C}$ with q a primitive 5th root of unity.

It is known that, for small $2n+r$, each Specht module $S^{(n+r,n)}$ has a composition series of length 2.[9] Heuristically, M_{2n+r}^r does as well; let this series be

$$0 \subset U_{2n+r}^r \subset M_{2n+r}.$$

In the following data, we specify the inclusion maps map $\iota_{e,2n+r}^r : U_{2n+r}^r \hookrightarrow M_{2n+r}^r$, which conjecturally illustrates the inclusion of the first composition factor of $S^{(n+r,n)}$ into $S^{(n+r,n)}$ for all $2n+r \leq 7$.

$$\iota_{3,3}^1 = \iota_{3,4}^0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\intercal$$

$$\iota_{3,5}^1 = \iota_{3,6}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}^\intercal$$

$$\iota_{3,6}^4 = \begin{bmatrix} -1 & -1 & 0 & 1 & 1 \end{bmatrix}^\intercal$$

$$\iota_{3,7}^3 = \begin{bmatrix} 1 & 0 & -1 & -1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^\intercal$$

$$\iota_{3,7}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^\intercal$$

$$\iota_{4,4}^2 = \begin{bmatrix} 1 & \alpha & 1 \end{bmatrix}^\intercal$$

$$\iota_{4,4}^1 = \begin{bmatrix} 1 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & \frac{1}{2}\alpha \end{bmatrix}^\intercal$$

$$\iota_{4,6}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & -\alpha & 0 \\ 0 & 0 & 1 & 0 & 0 & \alpha & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -\alpha & -1 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & \frac{1}{2}\alpha \end{bmatrix}^\intercal$$

$$\iota^0_{4,6} = \begin{bmatrix}\alpha & 1 & 1 & \alpha & 1\end{bmatrix}^\intercal$$

$$\iota^1_{4,7} = \begin{bmatrix}1 & 0 & 0 & 0 & \frac{1}{2}\alpha & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 1 & 1/2 & \alpha & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -1 & 0 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & 0 & \alpha & \frac{1}{2}\alpha & -1 & \frac{1}{2}\alpha \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & \alpha & 1 & 0 & -\alpha & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & \alpha & 0 & 0 & -1 & -\alpha & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}\alpha & \frac{1}{2}\alpha & 0 & 0 & 0 & \frac{1}{2}\alpha & -1 & \frac{1}{2}\alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}\alpha & \frac{1}{2}\alpha & 1 & \frac{1}{2}\alpha\end{bmatrix}^\intercal$$

$$\iota^3_{5,5} = \begin{bmatrix}1 & \beta & \beta & 1\end{bmatrix}^\intercal$$

$$\iota^2_{5,6} = \begin{bmatrix}\beta & \beta & 1 & 1 & \beta+1 & \beta & \beta & \beta & 1\end{bmatrix}^\intercal$$

$$\iota^3_{5,7} = \begin{bmatrix}1 & \beta & \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \beta & \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & \beta & 1 & 0 & 0 & 0 \\ \beta & \beta & 1 & 0 & 1 & \beta+1 & \beta & 0 & \beta & \beta & 0 & 1 & 0 & 0 \\ -\beta-1 & -\beta & -\beta & 0 & -\beta & -\beta-1 & -\beta-1 & 0 & -\beta-1 & -\beta-1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \beta & 0 & 0 & \beta & 0 & 0 & 0 & 1\end{bmatrix}^\intercal$$

$$\iota^1_{5,7} = \begin{bmatrix}1 & \alpha & \alpha & \alpha & -\beta & \alpha & -\beta & \alpha & -\beta & 1 & \alpha & \alpha & \alpha & -\beta\end{bmatrix}^\intercal$$

$$\iota^4_{6,6} = \begin{bmatrix}1 & \gamma & 2 & \gamma & 1\end{bmatrix}^\intercal$$

$$\iota^3_{6,7} = \begin{bmatrix}\gamma & 2 & \gamma & 1 & 1 & 2\gamma & 3 & \gamma & 2 & 2\gamma & 2 & 2 & \gamma & 1\end{bmatrix}^\intercal$$

$$\iota^5_{6,7} = \begin{bmatrix}1 & \delta & \varepsilon & \varepsilon & \delta & 1\end{bmatrix}^\intercal$$

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