THE TWO-COLUMN SPECHT MODULE OF THE HECKE ALGEBRA OF S_n

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1. Introduction

Let S_{2n} be the symmetric group on $2n$ indices, let $\mathscr{H} = \mathscr{H}_{k,q}(S_{2n})$ be the corresponding Hecke algebra over field k with parameter $q \in k$, and let $\{T_i\}$ be the simple transpositions generating $\mathscr{H}$. Let $[m]_q = 1 + q + \cdots + q^{m-1}$ be the q-number of m. Let e be the smallest positive integer such that $[e]_q = 0$, and set $e = \infty$ if no such integer exists. Either $q = 1$ and e is the characteristic of k, or $q \neq 1$ and q is a primitive eth root of unity.

Let $V := S^{(2,\dots,2)}$ be the Specht module corresponding to the young diagram with rows of length 2. The purpose of this writing is to characterize this representation via an isomorphism with another representation of $\mathscr{H}$.

Definition 1.1. A *crossingless matching* on $2n$ indices is a partition of $\{1, \dots, 2n\}$ into parts of size 2 such that no two parts "cross", i.e. there are no parts (a, a') and (b, b') such that $a < b < a' < b'$. We will call these arcs. Then, define W_{2n} to be the k-vector space with basis the set of crossingless matchings on 2n indices.

In order for this to be a $\mathscr{H}$-module, endow this with the action given by Figure 1; if this creates a loop, simply scale by $(1 + q)$, and otherwise deform into a crossingless matching and scale by $q^{1/2}$.

Let the length of an arc (i, j) be $l(i, j) := j - i + 1$. Note that the crossingless matchings can all be identified with a list of n integers describing the lengths of the arcs from left to right; using this, we may order the crossingless matchings in increasing lexicographical order in order to obtain an order on the basis. Let C_n be the nth catalan number, and let the resulting basis for W_{2n} be $\{w_1, \dots, w_{C_n}\}$ as illustrated in Figure 2.

We will prove that $W := W_{2n}$ is isomorphic to V as representations in the case that $\mathscr{H}$ is semisimple. To do so, we will prove that W has an irreducible restriction to $S_{2n-1} \subset S_{2n}$; using the branching theorem, this implies that W is isomorphic to a Specht module corresponding to a rectangular young diagram.

We will move on to prove that these modules have unique dimension up to transposition of the diagram; then, we will show that $\dim V = \dim W$ so that W corresponds to an $n \times 2$ or $2 \times n$ diagram. We will then do a short character computation to prove that $V \cong W$.

Figure 1. Illustration of the actions $(1 + T_4)w_3$ and $(1 + T_2)w_3$ in W_6. In general, we act on basis elements by simple transpositions by deleting loops, deforming into a crossingless matching, and scaling based on whether a loop was deleted.

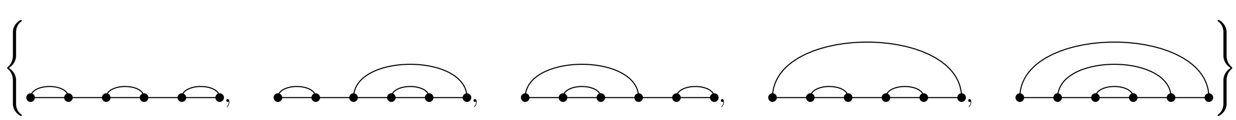

Figure 2. The increasing lexicgraphical basis for W_6.

2. Irreducibility

We can now begin by proving that $\mathrm{Res}^{\mathscr{H}_{k,p}(S_{2n})}_{\mathscr{H}_{k,p}(S_{2n-1})} W$ is irreducible; in partuclar, this implies that W itself is irreducible, as an $\mathscr{H}$-subrepresentation is a $\mathscr{H}_{k,p}(S_{2n-1})$-subrepresentation.

Proposition 2.1. *Set $\mathscr{H}' := \mathscr{H}_{k,p}(S_{2n-2}) \subset \mathscr{H}$. Then, $Res^{\mathscr{H}}_{\mathscr{H}'} W$ is irreducible if $e > n+1$.*

Proof. We will prove the equivalent condition that each vector in $w \in W$ is *cyclic*, i.e. $\mathscr{H}'w = W$.

We will first prove that w_1 is cyclic, for which it is sufficient to prove that every basis vector of W is in Aw_1. fix wome basis vector w_k, and suppose that it contains arc $(1, j)$, Then, the vector

$$w' := (1 + T_2)(1 + T_4) \dots (1 + T_{j-2})w_1$$

contains an arc $(1, j)$ and all other arcs are of the form $(a, a + 1)$ for some a. We may separately act on the subset of arcs with $1 < a < j$ and with $a > j$; this process gave our base case of W_4, and allows us to recurse to W_{2m} with $m < n$, outlining explicit vectors $h \in \mathscr{H}'$ with $hw_1 = w_i$. Hence it is sufficient to prove that w_1 is generated by every $w \neq 0 \in W$.

Since the image of $(1 + T_i)$ has arc $(i, i + 1)$, it is isomorphic as a vector space to W_{2n} with $2n - 2$ vertices.[1] Further, all actions of $(1+T'_j) \in \mathscr{H}_{k,p}(S_{2n-4})$ act identically (through the isomorphism) with simple transpositions other than $1 + T'_i$, which acts equivalently to $q^{-1}(1 + T_i)(1 + T_{i+1})(1 + T_{i-1})$ as illustrated in Figure 3; hence, if the image of $(1 + T_i)w$ in W_{2n-2} generates an ideal containing w'_1.

Then $(1 + T_i)w$ generates an ideal containing the pre-image of w'_1, which may either be w_1 if i is odd or the basis vector containing $(i - 1, i + 2)$, $(i, i + 1)$, and all other arcs length 2 if i is even.

We are now ready to make the central claim in our proof:

Claim. *Suppose $n > 1$. Then, the intersection $K := \bigcap_{i=1}^{2n-2} ker(1 + T_i)$ is trivial if e does not divide $n + 1$.*

This claim is necessary for irreducibility, as K is a proper subrepresentation of W.

Suppose this claim is true. We will use induction on n to prove irreducibility; the base case $n = 1$ is clear, so suppose W_{2n-2} is irreducible for all $e \geq n$ or, and pick some $w \in W_{2n}$. Then, pick some $1 + T_i$ such that $(1 + T_i)w \neq 0$, and pick some action $h' \in W_{2n-2}$ which takes the image of $(1 + T_i)w$ in W_{2n-2} to w'_1; this pulls back to an action $h \in W_{2n}$ such that $h(1 + T_i)w$ is the preimage of w'_1.

If i is odd then we have $h(1+T_i)w = w_1$ and we are done. Otherwise, $h(1+T_i)w$ is a basis vector with the arc $(i - 1, i + 2)$ and all other arcs having length 2; hence $q^{-1/2}(1 + T_{i-1})h(1 + T_i)w = w_1$ and we are done.

[1]This ismorphism "ignores" the arc $(i, i + 1)$.

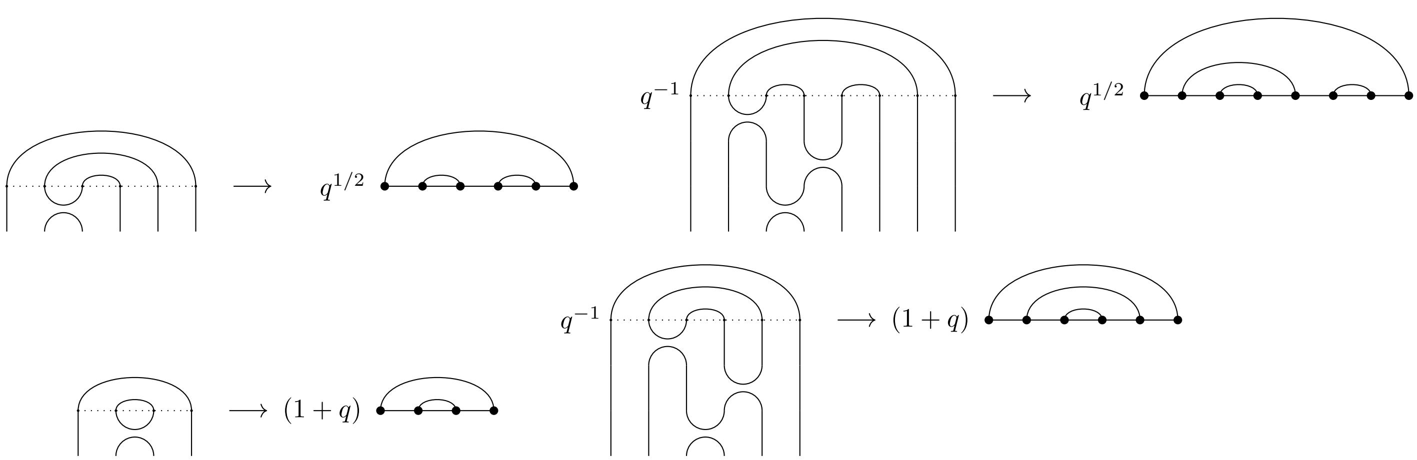

Figure 3. The correspondence between the action of $(1 + T_2)$ on $w'_5 \in W_6$ and the action of $q^{-1}(1 + T_3)(1 + T_4)(1 + T_2)$ on the corresponding vector in W_8 having arc $(3, 4)$ first, then on $w'_2 \in W_4$. This demonstrates that the action works with and without creating a loop.

Proof of claim. Note that $\bigcap \ker(1+T_i) = \ker\left(\bigoplus(1+T_i)\right)$. Further, we may postcompose this with the isomorphism $\operatorname{im}(1+T_i) \simeq W_{2n-2}$ in order to arrive at a $(n-2)C_{n-1} \times C_n$ matrix A whose kernel is K. Further, we may apply elementary row operations to this matrix, postcomposing with a change of basis, and preserve the kernel of this matrix as well.

We may characterize these rows as follows: the row corresponding to $(1+T_i)$ and mapping into the element $w_k \in W$ is of the form $[a_1, \ldots, a_{C_n}]$ where $a_k = 1+q$, $a_j = q^{1/2}$ whenever $(1+T_i)w_j = q^{1/2}w_k$, and 0 otherwise. In particular, we may characterize some particular rows with few nonzero entries.

First, the row of A corresponding to $(1+T_n)$ and w_{C_n} will be of the form $[0, \ldots, 0, q^{1/2}, q+1]$. Similarly, row corresponding to $(1+T_{n-1})$ and w_{C_n-1} will be of the form $[0, \ldots, 0, q^{1/2}, 0, q+1, q^{1/2}]$. An inductive process is illustrated in Figure 4. in order to construct a row corresponding to each $(1+T_{n-i})$ with nonzero components appearing in the order $q^{1/2}, q+1, q^{1/2}$ and having the last two of these appear above the first two in the row corresponding to $(1+T_{n-i-1})$. This process works for $1 \leq i \leq n-1$, and two-nonzero-element rows, which are in column-alignment with the appropriate elements in $(1+T_2)$ and $(1+T_{n-1})$.

This construction yields a $n \times C_n$ submatrix of A which has nonzero entries on the rightmost column, and has (by removing zero columns) the same column space as the following:

$$\begin{bmatrix} q+1 & q^{1/2} & & & & & \\ q^{1/2} & q+1 & q^{1/2} & & & \text{\Large 0} & \\ & q^{1/2} & q+1 & q^{1/2} & & & \\ & & \ddots & \ddots & & & \\ & \text{\Large 0} & & & q^{1/2} & q+1 & q^{1/2} \\ & & & & & q^{1/2} & q+1 \end{bmatrix}$$

We may recursively compute that this matrix has determinent $[n+1]_q$, which vanishes exactly when $e \mid n+1$; henceforth assume that this does not vanish. Then, this matrix is invertible, so we may use elementary row operations to transform it into the identity; these correspond to elementary row operations on A which yield a row $[0, \ldots, 0, 1]$. Hence any vector $w \in K$ has a zero coefficient corresponding to w_{C_n}. We will now prove that every nonzero vector $w \in K$ must have a nonzero coefficient corresponding to w_{C_n}, giving that $K = 0$. $\square$

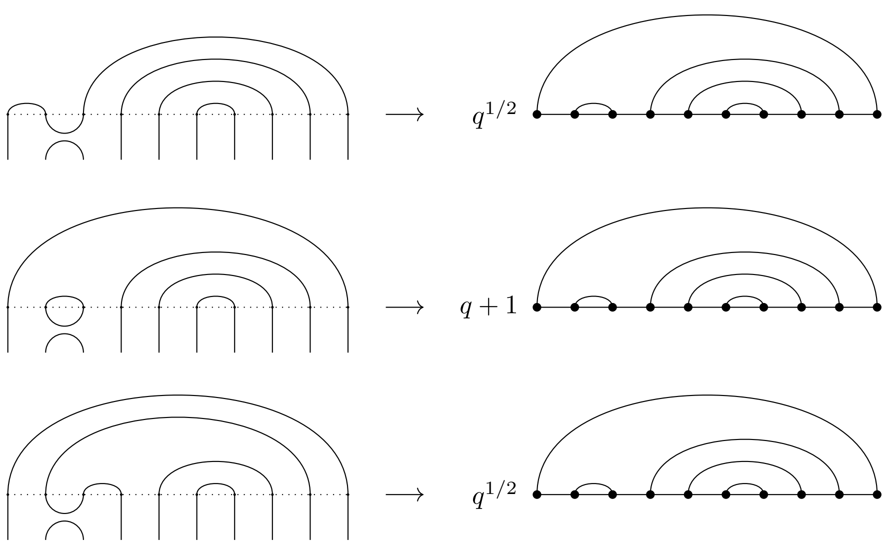

Figure 4. Illustrated is the row constructed for transposition $(1+T_2)$; clearly these are the only basis elements mapping to multiples of the desired element, and they relate to each other. In general, the basis elements considered are the preimages of $w_{C_{n-1}}$ in the isomorphism $\operatorname{im}(1+T_i) \simeq W_{2n-2}$.

3. Correspondence with the Specht Module

Suppose $\mathscr{H}$ is semisimple; then W is a Specht module corresponding to a rectangular region. We may now use dimension, for which one should note that there is an easy bijection between standard tableaus on $n = 2 + \cdots + 2$ and the Dyck paths on $2n$ points by specifying that the number in the $(i, 1)$ box of the tableau is the index of the ith "up" path. Hence $V_{(2,\ldots,2)}$ has dimension the nth catalan number C_n.

Similarly, we may biject the Dyck paths on $2n$ points with the crossingless matchings, by making the value at point i the number of "open crossings" at that point, i.e. the number of arcs (a, b) with $a \leq i < b$. Hence V and W have the same dimension. This pins the shape of the diagram corresponding to W as follows.

Proposition 3.1. *Let V_1 and V_2 be two specht modules corresponding to $a_1 \times b_1$ and $a_2 \times b_2$ rectangular young diagrams. Then, $dimV_1 = dimV_2$ if and only if the diagram of V_1 is the same as or a transposition of the diagram of V_2.*

Proof. Recall that, if they have the same diagram then $V_1 \cong V_2$, and if they're transposed from each other, $V_1 \cong V_2 \otimes U$ where U is the alternating representation; hence $\dim V_1 = \dim V_2 \cdot \dim U = \dim V_2$.

Suppose WLOG that $a_1 < a_2 < b_2 < b_1$. By the hook-length formula, it is sufficient to give a bijective correspondence between the boxes in V_2 and V_1 such that the hook-length in V_2 is larger than the hool-length in V_1, and at least once strictly larger. We can give this correspondence by listing the boxes beginning at the bottom right corner, then increasing up left-to-right diagonals, and note that this satisfies our conditions. $\square$

Let $V' := V_{(n,n)}$. Then, W is isomorphic to exactly one of V and V'. It is hence sufficient to prove that W is not isomorphic to V', and we may do this via a character computation.

4. Summary of Results

What follows is a color-coded list of propositions on crossingless matchings. Those which are blue are confirmed, those which are olive are in progress/soon to be proven, and those which are red are conjectural.

Proposition 4.1. *Suppose $n > 1$. Then, the intersection $K := \bigcap_{i=1}^{2n-2} ker(1 + T_i)$ is trivial if e does not divide $n + 1$.*

Proposition 4.2. *Define the subalgebra $\mathscr{H}' := \mathscr{H}_{k,p}(S_{2n-2}) \subset \mathscr{H}$. Then, $Res^{\mathscr{H}}_{\mathscr{H}'} W$ is irreducible if $e \geq n + 1$.*

Proposition 4.3. *Suppose $\mathscr{H}$ is semisimple, i.e. $e > 2n$. Then, $V \simeq W$.*

Proposition 4.4. *The intersection $K := \bigcap_{i=1}^{2n-2} ker(1 + T_i)$ is nontrivial when $e = n + 1$.*

Proposition 4.5. *W is reducible when $e \leq n + 1$.*

Proposition 4.6. *Even if $e \leq 2n$, we have $V \simeq W$.*