

Constructions on ∞ -operads: subcategories, overcategories, envelopes, and tensor products

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Abstract

We describe several methods of constructing ∞ operads and \mathcal{O} -monoidal ∞ -categories from other ones. We first show that a \otimes -closed full subcategory of an \mathcal{O} -monoidal ∞ -category is canonically \mathcal{O} -monoidal. We then state that a slice category of an \mathcal{O} -monoidal ∞ -category over a \mathcal{O} -algebra is canonically \mathcal{O} -monoidal.

We go on to construct coproducts in Op_∞ . We then construct the Boardman-Vogt tensor product for operads in **Set** and for preoperads, and hence we present the symmetric monoidal ∞ -category $\mathrm{Op}_\infty^\otimes$. We sketch the Eckmann Hilton argument in **Set**, and its homotopy-coherent generalization, called Dunn additivity. Time permitting, we describe the constructions of the monoidal envelope and Day convolution.

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These notes are currently rough. We'll cover most of HA §2.2 to some extent.

In what follows, we generally fix \mathcal{O}^\otimes an ∞ -operad and \mathcal{C}^\otimes an \mathcal{O}^∞ -monoidal ∞ -category. Recall that, for \mathcal{M} a symmetric monoidal category, a full subcategory of \mathcal{M} inherits a compatible symmetric monoidal structure iff $\mathrm{ob} \mathcal{M}$ is closed under the monoidal product; in Section [ref](#), we state an analog of this for \mathcal{O}^∞ -monoidal ∞ -categories.

Similarly, recall that, for $A \in \mathcal{M}$ a commutative monoid and M finitely cocomplete, the overcategory $\mathcal{M}_{/A}$ has a symmetric monoidal structure given by a combination of the external tensor product with multiplication:

$$X \otimes Y \rightarrow A \otimes A \xrightarrow{\mu} A.$$

In Section [ref](#), we prove results containing this for ∞ -operads; when the underlying category \mathcal{O} is contractible, $\mathcal{C}_{p/}^\otimes$ and $\mathcal{C}_{/p}^\otimes$ are given the structure of operads with fibrations to \mathcal{O}^\otimes . Assuming one believes that $\mathrm{Fun}(X, Y)$ can be made symmetric monoidal when Y is, this and the previous section will allow us to construct a symmetric monoidal structure on $\mathbf{Alg}_\mathcal{O}(\mathcal{C})$.

In Section [ref](#), we briefly discuss coproducts of ∞ -operads.

There is a forgetful ∞ -functor $\mathbf{Cat}_\infty^\otimes \rightarrow \mathrm{Op}_\infty$, and many constructions of symmetric monoidal ∞ -categories are lifted from constructions of operads. In Section [ref](#), we construct a left adjoint to this forgetful functor, called the \mathcal{O} -monoidal envelope.

The main thrust of this talk is Section [ref](#), as it is the least familiar; using the object-wise tensor product (of functor categories), $\mathbf{Alg}_\mathcal{O}(\mathcal{C})$ is itself a symmetric monoidal ∞ -category, and hence an operad. We define a symmetric monoidal tensor product, called the *Boardman-Vogt tensor product*, on Op_∞ which

endows it with a structure akin to a closed monoidal category; in particular, this is the essentially unique monoidal product on Op_∞ satisfying the adjunction

$$\begin{array}{ccc} \mathbf{Alg}_{\mathcal{O}' \otimes \mathcal{O}}(\mathcal{C}) & \xrightarrow{\sim} & \mathbf{Alg}_{\mathcal{O}'}(\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})) \\ \parallel & & \parallel \\ \underline{\text{Op}}_\infty(\mathcal{O}' \otimes \mathcal{O}, \mathcal{C}) & \xrightarrow{\sim} & \underline{\text{Op}}_\infty(\mathcal{O}', \underline{\text{Op}}_\infty(\mathcal{O}, \mathcal{C})) \end{array}$$

This is the derived monoidal product of a symmetric monoidal model category of preoperads and, conjecturally, a more classical symmetric monoidal model category of either dendroidal sets or topological/simplicial operads. In this language we state *Dunn additivity*, i.e. the computation that

$$\mathbb{E}_n \otimes \mathbb{E}_1 = \mathbb{E}_{n+1}, \quad \text{i.e.} \quad \mathbf{Alg}_{\mathbb{E}_{n+1}}(\mathcal{C}) = \mathbf{Alg}_{\mathbb{E}_1} \mathbf{Alg}_{\mathbb{E}_n}(\mathcal{C}),$$

an unreasonably useful result.

We finish up by discussing the Day convolution monoidal structure on $\text{Fun}(\mathcal{C}, \mathcal{D})$ for \mathcal{C}, \mathcal{D} symmetric monoidal ∞ -categories.

1 Familiar closure properties for monoidal categories

1.1 A \otimes -closed full subcategory of an \mathcal{O} -monoidal ∞ -category is \mathcal{O} -monoidal

Let $p : \mathcal{C}^\otimes \rightarrow \mathcal{O}$ be a coCartesian fibration of ∞ -operads, let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory stable under equivalence, and let $\mathcal{D}^\otimes \subset \mathcal{C}^\otimes$ be the full subcategory of \mathcal{C}^\otimes spanned by objects of the form $D_1 \oplus \cdots \oplus D_n$ where each object belongs to \mathcal{D} . The following proposition is more or less obvious when thinking hard about the definition of ∞ -operads:

Proposition 1.1. *\mathcal{D}^\otimes is an ∞ -operad, and the inclusion $\mathcal{D}^\otimes \hookrightarrow \mathcal{C}^\otimes$ is a map of ∞ -operads.*

The legumes¹ of this section is in the following proposition. [HA 2.2.1.1](#)

Proposition 1.2 (Tensor closed subcategories of \mathcal{O} -monoidal ∞ -categories.). *Let $p : \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$ be a coCartesian fibration of ∞ -operads and let $\mathcal{D}, \mathcal{D}^\otimes$ be as above. Suppose that, for every operation $\theta \in \{X_i\}, Y$, the induced functor*

$$\theta_* : \prod_i \mathcal{C}_{X_i} \rightarrow \mathcal{C}_Y$$

descends to a functor $\prod_i \mathcal{D}_{X_i} \rightarrow \mathcal{D}_Y$. Then,

- (i) *The restricted map $\mathcal{D}^\otimes \rightarrow \mathcal{O}^\otimes$ is a coCartesian fibration of ∞ -operads.*
- (ii) *The inclusion $\mathcal{D}^\otimes \hookrightarrow \mathcal{C}^\otimes$ is an \mathcal{O} -monoidal functor.*
- (iii) *Suppose that for each object $X \in \mathcal{O}$, the inclusion $\mathcal{D}_X \subset \mathcal{C}_X$ admits a right adjoint L_X . Then, there exists a functor $L^\otimes : \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ which is a right adjoint of the inclusion $\mathcal{D}^\otimes \hookrightarrow \mathcal{C}^\otimes$ with counit projecting to degenerate edges of \mathcal{O}^\otimes ; further, L^\otimes is a map of ∞ -operads.*
- (iv) *L^\otimes induces the right adjoint to the inclusion $\mathbf{Alg}_{/\mathcal{O}}(\mathcal{D}) \hookrightarrow \mathbf{Alg}_{/\mathcal{O}}(\mathcal{C})$.*

This has a plain english Corollary:

Corollary 1.3. *Suppose \mathcal{C}^\otimes is a symmetric monoidal ∞ -category and $\mathcal{D} \subset \mathcal{C}$ is a full subcategory closed under equivalence and the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$. Then, there is a canonical sub-symmetric monoidal ∞ -category $\mathcal{D}^\otimes \hookrightarrow \mathcal{C}^\otimes$ whose underlying category is \mathcal{D} ; in particular, \mathcal{D} is canonically symmetric monoidal.*

Suppose $p : \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$ is a coCartesian fibration of ∞ -operads, suppose that \mathcal{C}_X is stable for each X . Suppose further that every operation $\theta \in \text{Mul}_{\mathcal{O}}(\{X_i\}, Y)$ induces an exact functor

$$\theta_* : \prod_{i \in I} \mathcal{C}_{X_i} \rightarrow \mathcal{C}_Y.$$

Suppose even further that we're given a family of t -structures $(\mathcal{C}_{X, \geq 0}, \mathcal{C}_{X, \leq 0})$ preserved by each θ_* . Let $\mathcal{C}_{\leq 0}^\otimes \subset \mathcal{C}^\otimes$ be the full subcategory spanned by objects $C \in \mathcal{C}^\otimes$ such that, for every $C' \in \mathcal{C}_X$ and every inert morphism $C \rightarrow C'$, the object C' belongs to $\mathcal{C}_{X, \geq 0}$.

¹This is an ethical replacement for the “meat” of the section.

Corollary 1.4 (Monoidal structure on connective objects). *The induced map $\mathcal{C}_{\geq 0}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞ -operads.*

We refer to HA §2.2.1 for the proof of the third part; the first and second part follow by the tensor closure, as this essentially says that the coCartesian lifts with chosen domains \mathcal{D}^{\otimes} are contained in \mathcal{D}^{\otimes} (and are hence coCartesian and preserved by the inclusion).

Localization????

1.2 Slices of an \mathcal{O} -monoidal ∞ -category over a commutative algebra are \mathcal{O} -monoidal; slices of an operad over \mathcal{O} are operads over \mathcal{O}

First, let's form a notion of relative slicing:

Definition 1.5. Let $q : X \rightarrow S$ be a map of simplicial sets, and suppose we're given a diagram

$$\begin{array}{ccc} & & X \\ & \nearrow p & \downarrow q \\ S \times K & \longrightarrow & S \end{array}$$

Define the simplicial set $q' : X_{pS/} \rightarrow S$ over S universally such that there is a natural bijection between $\text{Fun}_S(Y, X_{pS/})$ with the commutative diagrams

$$\begin{array}{ccccc} Y \times K & \hookrightarrow & Y \times K^{\triangleright} & \twoheadrightarrow & Y \\ q_Y \times \text{id}_K \downarrow & & \downarrow \text{red} & & \downarrow q_Y \\ S \times K & \xrightarrow{p} & X & \xrightarrow{q_X} & S \end{array}$$

where the top left arrow is inclusion and the top right arrow is projection to the “tip” of K .

We define the relative overcategory $X_{pS/}$ by replacing \triangleright with \triangleleft . This notion recovers traditional overcategories and undercategories when $S = *$.

Remark. The only piece of data supplied in the functor represented by $X_{pS/}$ is the arrow highlighted in red. We may read this definition as saying that a map $\varphi : Y \rightarrow X_{pS/}$ is the same as a map $\tilde{\varphi} : Y \times K^{\triangleright} \rightarrow X$ from “points of Y together with a K -shape extended by a cone point to the right,” satisfying the conditions that

1. The restriction of the map φ to $Y \times K$ factors through the diagram $S \times K \xrightarrow{p} X$.
2. Pushing forward to the cone point yields is compatible with the structure maps to the base simplicial set; this implies that the restricted map $Y \times \{*\} \rightarrow X$ is a functor over S .

For $q : \mathcal{C}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ a fibration of ∞ -operads and $p : K \rightarrow \mathbf{Alg}_{/\mathcal{O}}(\mathcal{C})$ a diagram, there is an adjoint map

$$K \rightarrow \text{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{O}^{\otimes}) \quad \rightsquigarrow \quad K \times \mathcal{O}^{\otimes} \rightarrow \mathcal{C}^{\otimes} \quad \text{over } \mathcal{O}^{\otimes}$$

and hence we may form the slices $\mathcal{C}_{p\mathcal{O}}^{\otimes}$ and $\mathcal{C}_{p\mathcal{O}/}^{\otimes}$. In the case $p = A \in \mathbf{Alg}_{/\mathcal{O}}(\mathcal{C})$, we denote this by $\mathcal{C}_{A\mathcal{O}/}^{\otimes}$.

We can now state the mushrooms and potatoes of this section, from HA §2.2.2.4:

Theorem 1.6. *Let $q : \mathcal{C}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ be a fibration of ∞ operads and let $p : K \rightarrow \mathbf{Alg}_{/\mathcal{O}}(\mathcal{C})$ be a diagram. Then,*

- (1) *the maps $\mathcal{C}_{p\mathcal{O}/}^{\otimes} \rightarrow \mathcal{O}^{\otimes} \leftarrow \mathcal{C}_{p\mathcal{O}}^{\otimes}$ are fibrations of ∞ -operads.*
- (2) *A morphism in $\mathcal{C}_{p\mathcal{O}/}^{\otimes}$ is inert if and only if its image in \mathcal{C}^{\otimes} is inert; the same is true of $\mathcal{C}_{p\mathcal{O}}^{\otimes}$.*
- (3) *If q is a coCartesian fibration, then $\mathcal{C}_{p\mathcal{O}}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ is a coCartesian fibration. If in addition, $p(k) : \mathcal{O}^{\otimes} \rightarrow \mathcal{C}^{\otimes}$ is an \mathcal{O} -monoidal functor for each vertex $k \in K$, then $\mathcal{C}_{p\mathcal{O}/}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞ -operads.*

We get a plain english corollary.

Proposition 1.7. *Suppose \mathcal{C} is an ∞ -operad and $A \in \mathbf{CAlg}(\mathcal{C})$ a commutative algebra object. Then, each of $\mathcal{C}_{A/}^{\otimes}$ and $\mathcal{C}_{/A}^{\otimes}$ are ∞ -operads. If \mathcal{C} is a symmetric monoidal ∞ -category, then $\mathcal{C}_{/A}$ is a symmetric monoidal ∞ -category.*

No proof? *megamind meme*

1.3 An application: the pointwise monoidal structure On $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$

Let \mathcal{C} be a category and \mathcal{D} a symmetric monoidal category. There is a symmetric monoidal structure on the functor category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ given by

$$(F \otimes G)(-) := F(-) \otimes G(-).$$

We refer to this as the *pointwise (symmetric) monoidal structure*. Our goal is to repeat this construction in the ∞ -operadic setting, and use our formalism for restriction and slicing to descend this to a pointwise monoidal structure on $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$.

This is pedagogical in nature; we frequently need to define an ∞ -operad structure on $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$ for \mathcal{C}^{\otimes} an ∞ -operad (not necessarily a symmetric monoidal ∞ -category). Our construction is generalized in HA §3.2.4, which we will cover later in this seminar. As such, proceed with caution; I haven't seen this worked out elsewhere, so mistakes are relatively more likely here than in material covered in HA.

Construction 1.8. Let \mathcal{C}^{\otimes} be a symmetric monoidal ∞ -category and \mathcal{O} an ∞ -category. Let $\theta : N(\mathbf{Fin}_*) \rightarrow \mathbf{Cat}_{\infty}$ be the functor corresponding to $\mathcal{C}^{\otimes} \rightarrow N(\mathbf{Fin}_*)$ via the Grothendieck construction.

Define the functor $\Delta_{\mathcal{O}} : N(\mathbf{Fin}_*) \rightarrow \mathbf{Cat}_{\infty}^{\mathrm{op}}$ to be the functor sending

$$\Delta_{\mathcal{O}}(\langle n \rangle) := \mathcal{O}^n$$

and acting on $f : \langle n \rangle \rightarrow \langle m \rangle$ via the “traditional” diagonal on fibers.

There is a functor

$$\varphi : N(\mathbf{Fin}_*) \xrightarrow{\Delta_{\mathcal{O}} \times \theta} \mathbf{Cat}_{\infty}^{\mathrm{op}} \times \mathbf{Cat}_{\infty} \xrightarrow{\mathbf{Fun}(-, -)} \mathbf{Cat}_{\infty}$$

such that the induced map

$$\begin{array}{ccc} \mathbf{Fun}(\mathcal{O}^n, \mathcal{C}^n) & \xrightarrow{\Pi \rho_i} & \mathbf{Fun}(\mathcal{O}, \mathcal{C}) \\ \parallel & & \parallel \\ \varphi(n) & \longrightarrow & \varphi(1) \end{array}$$

is an equivalence. The *pointwise symmetric monoidal category of functors* is the associated symmetric monoidal ∞ -category, written $\mathbf{Fun}_{\mathrm{ptws}}(\mathcal{O}, \mathcal{C})^{\otimes}$.

The *pointwise symmetric monoidal category of algebras* is the full symmetric monoidal ∞ -subcategory

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes} \subset \mathbf{Fun}_{\mathrm{ptws}}(\mathcal{C}, \mathcal{O})^{\otimes}.$$

formed on the functors over $N(\mathbf{Fin}_*)$ preserving inert morphisms.²

Note that the underlying category of $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ is $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$, avoiding notational and conceptual confusion.

2 New constructions on operads

We first construct the coproduct of ∞ -operads; classically, the BV-tensor product of operads is constructed as a quotient of this construction.

2.1 Coproducts of ∞ -operads

Suppose we have a coproduct functor $\coprod : \mathbf{Op}_{\infty}^{\times 2} \rightarrow \mathbf{Op}_{\infty}$; this is uniquely determined by the identity

$$\begin{aligned} \mathbf{Alg}_{\mathcal{O} \coprod \mathcal{O}'}(\mathcal{C}) &\simeq \mathbf{Op}_{\infty}(\mathcal{O} \coprod \mathcal{O}', \mathcal{C}) \\ &\simeq \mathbf{Op}_{\infty}(\mathcal{O}, \mathcal{C}) \times \mathbf{Op}_{\infty}(\mathcal{O}', \mathcal{C}) \\ &= \mathbf{Alg}_{\mathcal{O}}(\mathcal{C})^{\simeq} \times \mathbf{Alg}_{\mathcal{O}'}(\mathcal{C})^{\simeq}. \end{aligned}$$

That is, we're looking for an operad $\mathcal{O} \coprod \mathcal{O}'$ whose algebras are precisely the pairs of \mathcal{O} -algebras and \mathcal{O}' -algebras; we will construct this as the restriction of a product

$$\boxtimes : \mathbf{Cat}_{\infty, / N(\mathbf{Fin}_*)}^{\times 2} \rightarrow \mathbf{Cat}_{\infty, / N(\mathbf{Fin}_*)}.$$

But first, we need an auxiliary construction.

²Check that this makes sense!

Construction 2.1. The category Sub has:

- objects given by triples $(\langle n \rangle, S, T)$ where $n \in \text{Fin}_*$, and S, T are pointed subsets forming a partition of $\langle n \rangle$, and
- morphisms $(\langle n \rangle, S, T) \rightarrow (\langle n' \rangle, S', T')$ a morphism of triples in Fin_* .

There is a triple of functors $\pi, \pi_-, \pi_+ : \text{Sub} \rightarrow \text{Fin}_*$ given by

$$\pi_-(\langle n \rangle, S, T) = [S] \quad \pi(\langle n \rangle, S, T) = \langle n \rangle \quad \pi_+(\langle n \rangle, S, T) = [T]$$

Note that $\pi_- \times \pi_+ : \text{Sub} \rightarrow \text{Fin}_*^{\times 2}$ is an equivalence.

With this done, we can construct the product \boxtimes ;

Construction 2.2. Given maps $\mathcal{O}^\otimes \rightarrow N(\text{Fin}_*)$ and $\mathcal{O}'^\otimes \rightarrow N(\text{Fin}_*)$, define the map $\mathcal{O}^\otimes \boxtimes \mathcal{O}'^\otimes \rightarrow \text{Fin}$ to be the highlighted composition in the following diagram:

$$\begin{array}{ccc} \mathcal{O}^\otimes \boxtimes \mathcal{O}'^\otimes & \xrightarrow{\sim} & \mathcal{O}^\otimes \times \mathcal{O}'^\otimes \\ \downarrow & \lrcorner & \downarrow \\ N(\text{Sub}) & \xrightarrow[\pi_- \times \pi_+]{} & N(\text{Fin}_*) \times N(\text{Fin}_*) \\ \swarrow \text{red} & \nwarrow \pi & \\ N(\text{Fin}_*) & & \end{array}$$

Choosing a pullback functor, this can be assembled into an honest bifunctor, completing the construction.

Now that we've made the construction, let's simply list the reasons why we care; there is a morphism $\iota_- : \mathcal{O}^\otimes \rightarrow \mathcal{O}^\otimes \boxtimes \mathcal{O}'^\otimes$ given by including \mathcal{O} as the fiber over $*$ of π_+ , and a similar embedding ι_+ of \mathcal{O}' , piecing into a diagram

$$\mathcal{O}^\otimes \hookrightarrow \mathcal{O}^\otimes \boxtimes \mathcal{O}'^\otimes \hookleftarrow \mathcal{O}'^\otimes \quad (1)$$

Theorem 2.3. Let $\mathcal{O}^\otimes, \mathcal{O}'^\otimes$ be ∞ -operads. Then, $\mathcal{O}^\otimes \boxtimes \mathcal{O}'^\otimes$ is an ∞ -operad, and Diagram (1) induces an equivalence of ∞ -categories

$$\text{Alg}_{\mathcal{O}^\otimes \boxtimes \mathcal{O}'^\otimes}(\mathcal{C}) \xrightarrow{\sim} \text{Alg}_{\mathcal{O}}(\mathcal{C}) \times \text{Alg}_{\mathcal{O}'}(\mathcal{C}),$$

natural in \mathcal{C} . In particular, Diagram (1) is a colimit diagram in Op_∞ .

proof here, maybe

2.2 The Boardman-Vogt tensor product in the ∞ -operadic setting, and Dunn additivity

For the sake of staying grounded, let's first work out a closed monoidal structure on the operads in Set .

The BV tensor product of operads in Set . Let \mathcal{O} and \mathcal{O}' be operads in Set . Given an *ideal* in an operad $I \subset \mathcal{O}$, we may describe the quotient operad \mathcal{O}/I in exactly the way one might expect. We use this:

Construction 2.4. The Boardman-Vogt tensor product of \mathcal{O} and \mathcal{O}' is the quotient of the coproduct $\mathcal{O} \amalg \mathcal{O}'$ by the *interchange law*:

$$\mathcal{O} \otimes \mathcal{O}' := \frac{\mathcal{O} \amalg \mathcal{O}'}{(\gamma(\theta; \theta', \dots, \theta')) - \gamma(\theta'; \theta, \dots, \theta) \cdot \sigma_{k, k'}}$$

where γ is composition in the coproduct, $\theta \in \mathcal{O}(k)$, $\theta' \in \mathcal{O}'(k')$, and $\sigma_{k, k'} \in S_{kk'}$ is the permutation which “exchanges rows and columns.”

This was first constructed in [the BV reference](#), at the same time as the instantiation of much of the rest of modern homotopy theory. The following is a classical result, and the internal hom is not too hard to see:

Theorem 2.5. The Boardman-Vogt tensor product endows on Op the structure of a closed monoidal category, with internal hom given by $\text{Alg}_{(-)}(-)$.

With this done, we can give a universal algebraic version of the *Eckmann Hilton argument*:

Theorem 2.6 (Eckmann-Hilton argument).

$$\text{Assoc} \otimes \text{Assoc} \simeq \text{Comm};$$

in particular, if an object in a symmetric monoidal category bears two monoid structures which distribute over each other, then the two structures agree and are commutative, i.e. the natural functor is an equivalence:

$$\text{Comm}(\mathcal{C}) \xrightarrow{\sim} \text{Assoc Assoc}(\mathcal{C})$$

In the more-classical setting, this extends to a notion of an enriched BV tensor product on operads in a suitably nice monoidal model category (say, \mathcal{M} such a category satisfying the monoid axiom and everything else we want). Unfortunately, this does not satisfy the pushout product axiom.³ After introducing a monoidal model structure on preoperads, we'll shout out that the model for dendroidal sets is simply better, and (homotopy-symmetric) strong-monoidally equivalent to our category of preoperads.

The BV tensor product of preoperads Let $\mathcal{C}^\otimes, \mathcal{O}^\otimes, \mathcal{O}'^\otimes$ be ∞ -operads. We'd like to introduce a process akin to passing from the direct sum of vector spaces to the tensor product. To do so, we'll consider the analog of bilinear maps; first, we need an auxiliary construction, which is likely familiar.

Construction 2.7. Suppose \mathcal{C} is a finitely bicomplete symmetric monoidal category with terminal object. Then, the *category of pointed objects* is the slice category

$$\mathcal{C}_* := \mathcal{C}_{I/}.$$

This is endowed with a symmetric monoidal structure given by

$$X \wedge Y := \text{Cofib}_{\mathcal{C}} \left(X \otimes \{*_Y\} \coprod \{*_X\} \otimes Y \rightarrow X \otimes Y \right)$$

In the special case that $\mathcal{C} = \text{Fin}$ is the category of finite sets, we call \wedge the *smash product* of finite pointed sets.

This endows on $N(\text{Fin}_*)$ the structure of a simplicial monoid. We use this:

Definition 2.8. A *bifunctor of operads* is a map $f : \mathcal{O}^\otimes \times \mathcal{O}'^\otimes \rightarrow \mathcal{C}^\otimes$ sending each pair of inert morphisms to an inert morphism, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}^\otimes \times \mathcal{O}'^\otimes & \longrightarrow & \mathcal{C}^\otimes \\ \downarrow & & \downarrow \\ N(\text{Fin}_*) \times N(\text{Fin}_*) & \xrightarrow{\wedge} & N(\text{Fin}_*) \end{array}$$

These form a full subcategory of a functor category, called $\text{BiFun}(\mathcal{O}, \mathcal{O}'; \mathcal{C})$.

Bilinear maps, when corepresented by a bifunctor, form half of a closed monoidal structure on Vect_k , through noting by currying that

$$\text{BiHom}(V, V'; W) \simeq \text{Hom}(V, \text{Hom}(V', W)).$$

We make a bold claim, which we can only currently understand in the case that \mathcal{C} is a symmetric monoidal ∞ -category. The following argument is not in HA as far as I can find, but it is heavily implied that Lurie believes it to be true.

Claim. *There is a natural equivalence of functors*

$$\text{BiFun}(\mathcal{O}, \mathcal{O}'; \mathcal{C}) \simeq \mathbf{Alg}_{\mathcal{O}} \mathbf{Alg}_{\mathcal{O}'}(\mathcal{C})$$

Proof idea. We push through the adjunction: an object in $\mathbf{Alg}_{\mathcal{O}} \mathbf{Alg}_{\mathcal{O}'}(\mathcal{C})$ corresponds with:

$$\varphi : \mathcal{O} \rightarrow \text{Fun}_{N(\text{Fin}_*)}(\mathcal{O}', \mathcal{C}) \quad \rightsquigarrow \quad \tilde{\varphi} : \mathcal{O} \times \mathcal{O}' \rightarrow \mathcal{C} \quad \text{over } N(\text{Fin}_*).$$

The functor φ must preserve inert morphisms and be valued in functors preserving inert morphisms; this is equivalent to $\tilde{\varphi}$ preserving pairs of inert morphisms.⁴ \square

³See the following [math overflow post](#).

⁴I don't know if this is actually true!!!

This gives some hope that we could form a closed symmetric monoidal structure on Op_∞ . The easiest way to go about doing this is to present it as the simplicial localization of a symmetric monoidal model category. Let $\mathcal{P}\mathrm{Op}_\infty \subset \mathbf{sSet}^+_{/N(\mathrm{Fin}_*)}$ be the full subcategory of marked simplicial sets whose markings lie above inert morphisms. We call these ∞ -preoperads. Every ∞ -operad \mathcal{O}^\otimes has an underlying ∞ -preoperad $\mathcal{O}^{\otimes, \natural}$ whose marked morphisms are the inert morphisms. The following is the core of HA §2.1.4.

Theorem 2.9. *There exists a left proper combinatorial simplicial model structure on $\mathcal{P}\mathrm{Op}_\infty$ characterized by the following structure:*

- A morphism $f : \overline{X} \rightarrow \overline{Y}$ in $\mathcal{P}\mathrm{Op}_\infty$ is a weak equivalence iff for every ∞ -operad \mathcal{O}^\otimes , the induced map

$$\mathrm{Map}_{\mathcal{P}\mathrm{Op}_\infty}(\overline{Y}, \mathcal{O}^{\otimes, \natural}) \rightarrow \mathrm{Map}_{\mathcal{P}\mathrm{Op}_\infty}(\overline{X}, \mathcal{O}^{\otimes, \natural})$$

is a weak homotopy equivalence.

- The fibrant objects of $\mathcal{P}\mathrm{Op}_\infty$ are precisely those of the form $\mathcal{O}^{\otimes, \natural}$ for \mathcal{O}^\otimes some ∞ -operad.

This model presents Op_∞ .

Construction 2.10. Let $\overline{X} = (X, M)$ and $\overline{Y} = (Y, N)$ be ∞ -preoperads; then, the preoperad $\overline{X} \otimes \overline{Y}$ is defined by

$$(X \times Y, M \times M').$$

with structure map

$$X \times Y \rightarrow N(\mathrm{Fin}_*) \times N(\mathrm{Fin}_*) \xrightarrow{\wedge} N(\mathrm{Fin}_*).$$

The following theorem is not so new, and the proof is not enlightening.

Theorem 2.11 (\otimes presents the BV tensor product). *The functor \otimes endows $\mathcal{P}\mathrm{Op}_\infty$ with the structure of a monoidal model category; the left-derived monoidal structure on Op restricts to a functor*

$$\otimes : \mathrm{Op}_\infty \times \mathrm{Op}_\infty \rightarrow \mathrm{Op}_\infty$$

such that

$$\mathbf{Alg}_{\mathcal{O}^\otimes \mathcal{O}'}(\mathcal{C}) = \mathrm{BiFun}(\mathcal{O}, \mathcal{O}'; \mathcal{C}).$$

Remark. The monoidal structure on $\mathcal{P}\mathrm{Op}_\infty$ is not symmetric⁵. However, its derived functor satisfies a symmetric universal property, so it must be symmetric.

Corollary 2.12. *There exists a symmetric monoidal ∞ -category $\mathrm{Op}_\infty^\otimes$ with underlying category Op_∞ such that $\otimes : \mathrm{Op}_\infty \times \mathrm{Op}_\infty \rightarrow \mathrm{Op}_\infty$ corepresents BiFun .*

2.3 Day convolution

weeee

2.4 The monoidal envelope of a ∞ -operad

math goes here

⁵ flesh this out