Kan Seminar Notes

Natalie Stewart

Fall 2021

This will be a rough collection of live-LaTeXed notes covering the Kan seminar talks given in Fall 2021. I'll make no promises that the contents of this are readable, or without significant clerical error. Exercise skepticism, and don't use these as a replacement for the papers. Last update: October 4, 2021.

Contents

1 Gabrielle Li: Serre, Cohomologie modulo 2 des complexes d'Eilenberg Mac Lane (i) 3
1.1 Steenrod operations . . . 3
1.2 Borel's theorem . . . 3
1.3 Performing the calculation . . . 4

2 Weixiao Lu: Serre, Cohomologie modulo 2 des complexes d'Eilenberg Mac Lane (ii) 5
2.1 Preliminaries . . . 5
2.2 Poincaré series . . . 6
2.3 Applications . . . 7

3 Zihong Chen: Moore, Semi-simplicial complexes and Postnikov systems 8
3.1 Review of simplicial sets . . . 8
3.2 Postnikov systems . . . 9
3.3 Principally twisted cartesian products . . . 9

4 Dylan Pentland: Borel, La cohomologie modulo 2 de certains espaces homogenes 11
4.1 Motivation and preprequisites . . . 11
4.2 Cohomology of $\mathrm{BO}(n)$ . . . 12

5 Mikayel Mkrtchyan: Milnor, The Steenrod algebra and its dual 14
5.1 Refresher on the Steenrod algebra and Hopf algebras . . . 14
5.2 Coalgebra structure on the Steenrod algebra . . . 14
5.3 Housekeeping . . . 15
5.4 Algebra structure on the dual Steenrod algebra . . . 16
5.5 Positive-degree homogeneous elements of the Steenrod algebra are nilpotent . . . 17
5.6 A sketch of the $p > 2$ case . . . 17

6 Preston Cranford: Thom, Quelques proprietes globales des varieties differentiables (i) 18
6.1 Motivation and overview . . . 18
6.2 Thom spaces and realizability . . . 18

7 Swapnil Garg: Thom, Quelques proprietes globales des varieties differentiables (ii) 20
7.1 Transversality and tubular neighborhoods . . . 20
7.2 Cobordism and L-equivalence . . . 21
7.3 Other remarks . . . 21

8 Haoshuo Fu: Hirzebruch, Topological methods in algebraic geometry **22**
8.1 The signature of a manifold . . . 22
8.2 Review of the structure of the cobordism ring . . . 23
8.3 Multiplicative sequences . . . 24

9 Natalie Stewart: Milnor, On manifolds homeomorphic to the 7-sphere **26**
9.1 Motivation: the generalized Poincaré conjecture . . . 26
9.2 Milnor's λ invariant . . . 26
9.3 The construction of Milnor's exotic spheres . . . 28
9.4 M_k is homeomorphic to S^7: a Morse theoretic sketch . . . 29
9.5 Completing the proof: calculating λ invariants . . . 29

1 Gabrielle Li: Serre, Cohomologie modulo 2 des complexes d'Eilenberg Mac Lane (i)

This talk was delivered September 15, 2021 by Gabrielle Li. Throughout, $H^*(-) := H^*(-; \mathbb{F}_2)$.

1.1 Steenrod operations

The *Steenrod operations* are a family of cohomology operations $\mathrm{Sq}^n : H^*(X) \to H^{*+n}(X)$ such that:

(1) Each Sq^n is natural in X.

(2) Each Sq^n is stable: $\mathrm{Sq}^n(\Sigma X) = \Sigma \mathrm{Sq}^n(X)$.

(3) When $|x| = n$, $\mathrm{Sq}^n(x) = x \cup x$.

(4) $\mathrm{Sq}^0 = \mathrm{id}$.

We give a basis for these:

Definition 1.1. A sequence $I = (i_1) \subset \mathbb{Z}_{>0}$ is *admissable* if $i_k \geq 2i_{k+1}$ for each k. We define the *degree* $n(I) := \sum i_k$ and the *excess* $e(I) = \sum(i_k - i_{k+1}) = 2i_i - n(I)$ (padding with zeros).

1.2 Borel's theorem

Let $F \hookrightarrow E \to B$ be a Serre fibration. Recall that, in the cohomological Serre spectral sequence, we have transgression morphisms $\tau : E_r^{0,r-1} \to E_r^{r,0}$, whose domain is a subset of $H^{r-1}(F)$ and whose codomain is a quotient of $H^r(B)$. This is an additive relation between $H^{r-1}(F)$ and $H^r(B)$. We say that $x \in H^{r-1}(F)$ is *transgressive* if it survives to the r page.

We hold off on proving the following proposition until the next talk:

Proposition 1.2. *τ commutes with Steenrod operations.*

We need a bit more language to use this:

Definition 1.3. For a space X, an ordered family of elements $(x_i) \subset H^*(X)$ is a *simple system of generators* if:

(1) Each x_i is homogeneous.

(2) The increasing products $x_{i_1} \cdots x_{i_j}$ (for $i_k < i_{k+1}$) form a $\mathbb{F}_2$-basis of $H^*(X)$.

The following examples are important:

> **Example 1.4:**
> $\mathbb{F}_2[x_1, x_2, \ldots]$ has simple system of generators $(x_j^{2^i})$. Similar systems apply to the exterior algebra $E[x]$ and the truncated poltnomial algebra $\mathbb{F}_2[x]/(x^{2^i})$.

We're finally ready to state our theorem:

Theorem 1.5 (Borel)**.** *Given a fibration $F \hookrightarrow E \to B$ satisfying the following properties:*

(1) $E_2^{s,t} = H^s(B) \otimes H^t(F)$ (for instance, when B is 1-connected and $H^*(B), H^*(F)$ are f.g.).

(2) $H^i(E) = 0$ for $i > 0$.

(3) $H^*(F)$ have a simple system of transgressive generators (x_i).

Then, $H^*(B)$ is a polynomial algebra generated (independently) by the any choice of representatives $y_i \in H^*(B)$ which map to $\tau(x_i)$ in $E_*^{*,0}$.

Note that, whenever $H^*(F)$ is a polynomial algebra generated by z_i, we know that $H^*(F)$ has a simple system of generators $z_i^{2^r}$. In order to use this, we introduce a bit of notation:

Notation. $L(a,r) := \left\{2^{r-1}a, 2^{r-2}a, \cdots, 2a, a\right\}$.

Note that $z_i^{2^r} = \mathrm{Sq}^{L(n_i,r)}(z_i)$. Hence

$$\tau\left(z_i^{2^r}\right) = \mathrm{Sq}^{L(n_i,r)}\, t_i$$

where $t_i := \tau(z_i)$. Hence $H^*(B)$ is a polynomial algebra generated by $\mathrm{Sq}^{L(n_i,r)}(z_i)$.

1.3 Performing the calculation

We will use Borel's theorem soon, but first, a lemma:

Lemma 1.6. *An admissible sequence $J = \{j_1, \dots, j_k\}$ with $e(J) < q-1$. Then, we may define a sequence*

$$J' := \left\{2^{r-1}s_J, 2^{r-2}s_J, \dots, s_J, j_1, j_2, \dots, j_k\right\},$$

where $s_J = q - 1 + n(J)$. Then, J' is admissible, with $e(J') < q$; furthermore, all admissible sequences of excess $< q$ arise this way.

The reversal is surprisingly easy; simply take the longest prefix satisfying $j_1 = 2j_2 = \cdots = 2^i j_i$.

We will need a few more constructions to prepare for the calculation:

(1) There is a fibration $K(\mathbb{F}_2, q-1) \hookrightarrow E \to K(\mathbb{F}_2, q)$ where E is contractible.

(2) By Hurewicz, $H^q(K(\mathbb{F}_2, q)) = \mathbb{F}_2$, with a generator that we call u_q.

Theorem 1.7. *$H^*(K(\mathbb{Z}/2, q), \mathbb{Z}/2)$ is a polynomial algebra (independently) generated by $\mathrm{Sq}^I(u_q)$ where I runs over the admissible sequences of excess $e(I) < q$.*

Proof. We prove this via induction. The $q = 1$ case is easy, as we have $K(\mathbb{F}_2, 1) = \mathbb{RP}^\infty$, and $H^*(\mathbb{RP}^\infty) = \mathbb{F}_2[u_q]$ via the usual computation.

For the inductive step, assume we've proven the theorem for $q-1$. We use the fibration from (1). For an admissible sequence J, let

$$S_J := \left|\mathrm{Sq}^J(u_{q-1})\right| = q - 1 + n(J).$$

We have transgression additive relation $H^{q-1}(K(\mathbb{F}_2, q-1)) \rightsquigarrow H^q(K(\mathbb{F}_2, q))$. Note that the transgression sends $\tau(u_{q-1}) = u_q$ (this will be justified later). Using our trick,

$$\tau(\mathrm{Sq}^J(u_{q-1})) = \mathrm{Sq}^J u_q.$$

By Borel, the $H^*(K(\mathbb{F}_2, q))$ is generated by $\mathrm{Sq}^{L(s_J,r)}\,\mathrm{Sq}^J u_q = \mathrm{Sq}^{L(s_j,r)J}\, u_q = \mathrm{Sq}^I u_q$, where I is an admissible sequence with $e(I) < q$, and all such I are generated this way. ∎

The other computations are routine and similar.

2 Weixiao Lu: Serre, Cohomologie modulo 2 des complexes d'Eilenberg Mac Lane (ii)

This talk was delivered September 17, 2021 by Weixiao Lu. We'll first cover some preliminaries.

2.1 Preliminaries

Theorem 2.1 (Serre spectral sequence)**.** *Let $F \hookrightarrow E \xrightarrow{p} B$ be a Serre fibration. Then, there is a spectral sequence*

$$E_2^{s,t} = H^s(B; H^t(p^{-1}(-); G)) \implies H^{s+t}(E; G).$$

If $\pi_1(B)$ acts trivially on $H^n(p^{-1}(-))$, then

$$E_2^{2,t} = H^s(B; H^t(F; G)).$$

Proof sketch. If F^*C^* is a filtered cochain complex, we have an SS,

$$E_0^{s,t} = \mathrm{gr}^s(C^{s+t}) \implies H^{s+t}(C^*).$$

Assume B is a CW complex with n-skeleton B^n. Then, $E_n := p^{-1}(B^n)$. We have $F^sS^*(E) = S^*(E, E_{s-1}) = \ker(S^*(E) \to S^*(E_{s-1}))$, which gives the right E_0 page. $\square$

In any upper-right quadrant SS, we have a transgression morphism $d^n : E^{0,n-1} \to E_n^{n,0}$. Note that $E_n^{0,n-1} \subset E_{n-1}^{0,n-1} \subset \cdots \subset H^{n-1}(F)$. The transgressive elements of $H^{n-1}(F)$ map to some quotient of $H^n(B)$.

We can create a diagram

$$\begin{tikzcd}
H^n(B,b) \arrow[r, "p^*"] \arrow[d, "\sim"] & H^n(E,F) & \\
H^n(B) \arrow[ur] & & H^{n-1}(F) \arrow[ul, "\partial"']
\end{tikzcd}$$

Theorem 2.2 (Transgression theorem)**.** *The transgression relation coincides with this diagram.*

This comes down to how the Serre SS was constructed.

Proposition 2.3. *The Steenrod square Sq_i "commutes" with transgression in the sense that any $x \in H^{n-1}(F; \mathbb{Z}/2)$ transgressive has $\mathrm{Sq}^i x$ transgressive, and $\tau(\mathrm{Sq}^i x) = \mathrm{Sq}^i(\tau x)$.*

Proof. Recall that a functor is stable iff it commutes with coboundary operators, so Sq_1 commutes with coboundary operators. Further, recall that it's natural. Hence the following diagram commutes, so Sq^i "commutes with the transgression relation" (is a morphism of cospans):

$$\begin{tikzcd}
& H^{n+i}(E,F) & \\
H^{n+i}(B) \arrow[ur, "p^*"] & & H^{n+i-1}(F) \arrow[ul, "\partial"'] \\
& H^n(E,F) \arrow[uu, "\mathrm{Sq}^i"] & \\
H^n(B \arrow[uu, "\mathrm{Sq}^i"] \arrow[ur, "p^*"] & & H^{n-1}(F) \arrow[uu, "\mathrm{Sq}^i"'] \arrow[ul, "\partial"']
\end{tikzcd}$$

$\square$

Reclal that for G a f.g. Abelian group,

1. $H^*(K(G \times H; q)) = H^*(K(G; q)) \otimes H^*(K(H; q))$.
2. $H^*(K(\mathbb{F}_2; q)) = \mathbb{F}_2[\mathrm{Sq}^I u_q \mid I \text{ admissible, s.t. } e(I) < q]$.
3. $H^*(K(\mathbb{F}_2; q)) = \mathbb{F}_2[\mathrm{Sq}^I u_q \mid I \text{ admissible, s.t. } e(I) < q, 1 \text{ does not appear in } i]$.
4. $H^*(K(\mathbb{F}_2^h; q)) = \mathbb{F}_2[\mathrm{Sq}^I u_q, \mathrm{Sq}^J k_{q+1}]$ where $k_{q+1} \in H^{q+1}(K(\mathbb{F}_{2^h}, q))$ for admissibles $e(I) < q$, $e(J) \leq q$ where no Sq^1 term appears in both Sq^I and Sq^J. This comes from a fibration fill in from notes later .
5. $H^*(K(\mathbb{F}_{p^h}; q)) = \mathbb{Z}/2$ for p odd with $q > 0$.

Remark. We have a different choice of generators related to universal classes, but as graded $\mathbb{F}_2$-algebras,

$$H^*(K(\mathbb{F}_{2^h}; q)) \simeq H^*(K(\mathbb{F}_2; q)).$$

We will aim towards the following theorem:

Theorem 2.4. *For all $n > 1$, there are infinitely many indices i at which $\pi_i(S^n)$ has nonzero 2-torsion.*

Our tool will be Poincaré series. The accents in Poincaré's name are to be understood from here on out.

2.2 Poincaré series

For L_* a finite type graded k-vector space, define the series

$$L(t) = \sum_{n \in \mathbb{N}} \dim L^n t^n \in \mathbb{Z}[[t]].$$

This is called the *Poincare series*, called $\theta(G; q; t)$ in the case of $H^*(K(G; q))$.

Example 2.5:
For $L^* = \mathbb{Z}/2[u]$, we have

$$L(t) = \frac{1}{1 - t^m}.$$

Note that $(N^* \otimes M^*)(t) = L(t)M(t)$. Hence $L^{'*} = k[u_1, \dots]$ with finite type has

$$L(t) = \prod_{n \geq 1} \frac{1}{1 - t^{\deg u_i}}$$

which converges t-adically.

Hence

$$\theta(\mathbb{F}_2, q, t) = \prod_{e(I) < q} \frac{1}{1 - t^{\deg(\mathrm{Sq}^I u_q)}} = \prod_{e(I) < q} \frac{1}{1 + t^{q + n(I)}}.$$

We can give this another combinatorial description:

Proposition 2.6.

$$\theta(\mathbb{F}_2, q, t) = \prod_{n_1 \geq n_2 \geq \cdots \geq n_{q-1} \geq 0} \frac{1}{1 - t^{2^{n_1} + \cdots 2^{n_{q-1}} + 1}}.$$

The radius of convergence of this is 1 considered as a complex power series. We can continue to analyze this series along these lines:

Theorem 2.7.

$$\lim_{x \to \infty} \frac{\log_2 \theta\left(\mathbb{F}_2, q, 1 - 2^{-x}\right)}{x^q / q!} = 1.$$

In general there is an essential singularity at 1. Serre used this replacement to reign it in, but we won't work with it very explicitly.

2.3 Applications

Theorem 2.8. *Suppose X is a 1-connected space satisfying the following conditions:*

1. *$H_*(X;\mathbb{Z})$ is of finite type.*
2. *$H_i(X;\mathbb{F}_2) = 0$ for $i \gg 0$.*

Then, for infinitely many indices i, $\pi_i(X)$ has a subspace isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/2$.

This directly implies Theorem 2.4 once you know that only finitely many homotopy groups of spheres are infinite.

We see this using a whitehead tower

$$\begin{tikzcd}
\cdots \arrow[d] & \\
X_{n+1} \arrow[d] \arrow[dr] & \\
X_n \arrow[r] \arrow[d] & X \\
X_{n-1} \arrow[d] \arrow[ur] & \\
\cdots &
\end{tikzcd}$$

where X_n is n-connected, and a π_i iso to X and X_{n-1} for $i > n$. We'll use another piece of machinery, seen by the Serre SS directly.

Lemma 2.9. *For $F \hookrightarrow E \to B$ a Serre fibration with B simply connected, $B(t)F(t) \geq E(t)$.*

Proof of Theorem 2.8. Otherwise, there is some largest q with $\pi_q(X) \otimes \mathbb{Z}/2 \neq 0$. Then, there is some j smallest such that $H_j(X;\mathbb{Z}/2) \neq 0$. Then, $\pi_j(X) \otimes \mathbb{Z}/2 \neq 0$.

In the whitehead tower, $X_q \to X_{q-1}$ is trivial on $\pi_*(-) \otimes \mathbb{Z}/2$, so $H^*(X_q, \mathbb{Z}/2)$ is trivial. Using the fibration $X_q \hookrightarrow X_{q-1} \to K(\pi_q(X), q)$ from the whitehead tower, we must have $H^*(X_{q-1}) = H^*(K(\pi_q(X), q))$. Then,

$$X_{q-1}(t) = \theta(\pi_q(x), q, t).$$

Further, the fibbrations in the whitehead series imply that

$$X_{i+1}(t) \leq X_i(t)\theta(\pi_{i+1}(X), i, t)$$

for each i, Chaining these together forever, what we get is

$$\theta(\pi_q(X), q, t) \leq X_1(t)\theta(\pi_2(X), 1, t) \cdots \theta(\pi_{q-1}(x), q-2, t).$$

Note that $X_1(t)$ is a polynomial, so bounded on $[0, 1]$. Applying our asymptotic bound on θ yields a contradiction. $\square$

3 Zihong Chen: Moore, Semi-simplicial complexes and Postnikov systems

This talk was delivered September 20, 2021 by Zihong (Peter) Chen.

3.1 Review of simplicial sets

The talk began with a very brief review of simplicial sets: let Δ be the category of finite ordered sets and order preserving maps. Recall that such maps are generated by distinguished maps $\delta_i : [n] \to [n+1]$ and $s_i : [n+1] \to [n]$, called the *face and degeneracy maps*.

Definition 3.1. A *simplicial set* is a functor $X : \Delta^n \to \mathbf{Set}$.

The morphism set is completely characterised by their images on face and degeneracy maps, which must satisfy a collection of combinatorial relations, which I won't write down here.

> **Example 3.2:**
> The *standard n-simplex* is given by the representable functor $\Delta[n] := \mathrm{Hom}(-, [n])$.

By Yoneda's lemma, $X_n = \mathrm{Hom}(\Delta[n], X)$, where $X_n = X([n])$.

> **Example 3.3:**
> If $X \in \mathbf{Top}$, the singular simplicial set $\mathrm{Sing}(X)$ is familiar. It participates in an adjunction, with left adjoint $|\cdot|$ the *Geometric realization.*

> **Example 3.4:**
> Define the *ith face* $\delta_i : \Delta[n-1] \to \Delta[n]$. The *ith horm* is $\bigvee_i^n := \cup_{k \neq i} \delta_i$. The *boundary* is $\partial\Delta[n] = \bigcup_i \delta_i$.

This allows us to define the combinatorial equivalent of a topological space:

Definition 3.5. A simplicial set X is a *Kan complex* if every morphism $\bigvee_k^n \to X$ factors through $\Delta[n] \to X$; you can *fill any horn* (not necessarily uniquely).

A morphism $p : E \to B$ is a *Kan fibration* if it has the right lifting property against horn inclusions:

$$\begin{tikzcd} \bigvee_k^n \arrow[r] \arrow[d] & E \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Examples of this include $\mathrm{Sing}(X)$, and any simplicial group (which we won't prove).

Definition 3.6. For X a Kan complex, define the *path components* $\pi_0(X) = X_0/\sim$ where $x \sim y$ if there exists some p with $d_1 p = x$ and $d_0 p = y$.

This is in fact an equivalence relation: you can do this via horn filling, which was drawn on the board, but which I will not spell out. We can define higher homotopy groups after defining the internal hom:

Definition 3.7. For $A \subset X$ and $B \subset Y$, define the *mapping object*

$$\mathrm{Map}((X, A), (Y, B)) = \mathrm{Hom}(\Delta[n] \times (X, A), (Y, B))$$

i.e. the maps $\Delta[n] \times X \to Y$ restricting to a map $\Delta[n] \times A \to B$. The maps $\Delta[i] \to \mathbf{Set}$ form a covariant functor, so this is a contravariant functor, i.e. a simplicial set.

We use the following Theorem of Kan:

Theorem 3.8 (Kan)**.** *If Y, B are Kan complexes, then so is* $\mathrm{Map}((X, A), (Y, B))$.

We finally define homotopy groups.

Definition 3.9. If X is a Kan complex, define $\pi_n(X, x) := \pi_0(\mathrm{Map}((\Delta[n], \partial\Delta[n]), (X, x)))$. A Kan complex is $K(\Pi, n)$ if $\pi_q(X, x) = \Pi$ when $q = n$ and 0 otherwise.[1]

We will use these to decompose Kan complexes.

3.2 Postnikov systems

Let $\Delta[q]_n$ be the n-skeleton of $\Delta[q]$. For X a Kan complex, define the complex $X^{(n)}$ via

$$X_q^{(n)} = X_q/\sim \qquad\qquad x \sim y \iff x|_{\Delta[q]_n} = y|_{\Delta[q]_n}.$$

The maps are induced by X. We have the following properties:

1. $X^{(n)}$ is a Kan complex.
2. There is a quotient Kan fibration $X^{(n)} \xrightarrow{p} X^{(k)}$ if $n > k$.
3. $\pi_q(X^{(n)}, x) = 0$ if $q > n$.
4. $p_* : \pi_q(X^{(n)}, x) \xrightarrow{\sim} \pi_q(X^{(k)}, x)$ is an iso if $n \geq k \geq q$.

As in topology, Kan fibrations induce LES of homotopy groups; hence the fiber $F^{(n+1)} \hookrightarrow X^{(n+1)} \xrightarrow{p} X^{(n)}$ is a $K(\pi_{n+1}(X), x+1)$. We finally give this a name:

Definition 3.10. $(X^0, X^{(1)}, \dots)$ is called the *natural Postnikov system* of X.

This motivates a question: How far is X from $\prod_n K(\pi_n, n)$? It's always a colimit, but we'll measure how complex it is in the following section.

The idea is that $K(\pi_{n+1}, n+1) \hookrightarrow X^{(n+1)} \to X^{(n+1)}$ will be seen as something like a "principal $K(\pi_{n+1}, n+1)$-bundle." We will construct something like a "classifying space" $\overline{W}(K(\pi_{n+1}, n+1)) = K(\pi_{n+1}, n+2)$, and derive algebraic invariants from this. Let's actually do this now:

3.3 Principally twisted cartesian products

Definition 3.11. A *principally twisted Cartesian product* (PTCP) with simplicial group G and base G is written

$$E(T) = G \times_T B$$

where $E(T)_n = G_n \times B_n$ with degeneracy maps all the same, except that

$$\partial_0(g, b) = (T(b) \cdot d_0 g, d_0 b)$$

and T is a *twisting function* $B_q \to G_{q-1}$ for $q \geq 1$.

This is a combinatorial version of *holonomy*, as per a comment from Prof. Miller.

Definition 3.12. A PTCP is *of type (W)* if $B_0 = \{b_0\}$ and

$$\partial_0|_{\{e_q\} \times B_q} : [e_q] \times B_q \xrightarrow{\sim} E(T)_{q-1}$$

is an iso. Let $\int$ be its inverse.

Theorem 3.13. *If $G \times_T B$, $G' \times_{T'} B'$, and $\gamma : G \to G'$ is a morphism of simplicial group, then there exists a unique γ-equivariant map $\theta : G \times_T B \to G' \times_{T'} B'$ and Some condition holds of θ–fill in later .*

I couldn't follow this part; use $\int$ to construct this "upwards" from b_0, or something like that.

Corollary 3.14. *A PTCP of type (W) with group G is unique, if it exists.*

[1]This *actually* has a requirement of minimality, but we handwave this away.

Theorem 3.15. *If $E(T)$ is PTCP of type (W), it is contractible.*

They do exist! We can construct them by $B := \overline{W}(G)$, $W(G) = G \times_{T(G)} \overline{W(G)}$, where $\overline{W}_n(G) = G_{n-1} \times \cdots \times G_0$ for $n \geq 1$, and terminal for $n = 0$. put face and degen maps here . It has twisting function

$$T(G)[g_n, \ldots, g_0] = g_n.$$

It can be checked explicitly that this is type (W).[2]

Corollary 3.16. *Every PTCP with group G is by*

$$B \xrightarrow{\pi} \overline{W}(G)$$

with $\pi(b) = \left[T(b), T(\partial_0 b), \ldots, T(\partial_0^{n-1} b)\right]$.

This is a simplicial version of the bar construction??

This allows us to explicitly construct $K(\pi, n)$! Define $K(\pi, 0)$ to be π in each degree and $\partial_i s_i$ all identity. Define $K(\pi, n) = \overline{W}(K(\pi, n-1))$ inductively. We can see this is in fact a $K(\pi_1)$ via a fibration

$$K(\pi, n) \to W(K(\pi, n)) \to \overline{W}(K(\pi, n)),$$

where we know $W(*)$ to be contractible.

The main technical result follows:

Lemma 3.17. *Suppose there is no nontrivial morphism $\pi_1 \to \mathrm{Aut}(\pi_n)$. Then, $X^{(n)}$ is a PTCP with group $K(\pi_{n+1}, n+1)$.*[3]

To handwave, the idea for this is that minimal Kan fibrations are fiber bundles. Given the π_1 assumption, the structure group is $K(\pi_{n+1}, n+1)$. Then, a "principal G-bundle" is the same thing as a PTCP, in some intuitive way.

We can define the k-invariants via the fibrations $K(\pi_{n+1}, n+1) \hookrightarrow X^{(n+1)} \to X^{(n)}$: there is a universal class

$$u \in H^{n+2}(K(\pi_{n+1}, n+2))$$

and via the map $X^{(n+1)} \xrightarrow{f^{n+2}} \overline{W}(K(\pi_{n+1}, n+1)) = K(\pi_{n+1,n+2})$, we can define k-invariants as $\left(f^{n+2}\right)^* u = k^{n+2}$.

[2] This was written down in class.

[3] Per a comment of Prof. Miller, we only need simplicity, not total nontriviality of morphisms $\pi_1 \to \mathrm{Aut}(\pi_n)$.

4 Dylan Pentland: Borel, La cohomologie modulo 2 de certains espaces homogenes

This talk was delivered September 22, 2021 by Dylan Pentland.

4.1 Motivation and preprequisites

Characteristic classes We have a functor

$$\mathrm{Bun}_{O(n)} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$$

sending X to the isomorphism classes of principal $O(n)$ bundles mod isomorphisms. We know that this is representable, i.e. expressable as $\mathrm{Bun}_{O(n)}(-) = \mathrm{Hom}(-, \mathrm{BO}(n))$ (in the homotopy category).

Definition 4.1. A *characteristic class* is a natural transformation $\mathrm{Bun}_{O(n)} \implies H^i(-)$, where coefficients are understood mod 2. By the Yoneda lemma, this is the same thing as an element of $H^i(\mathrm{BO}(n))$.

We're going to characterize these via a cohomology computation. The main theorem is as follows: let $Q(n) \subset O(n)$ be the diagonal matrices. From this inclusion, we get a projection $\mathrm{BQ}(n) \xrightarrow{\rho} \mathrm{BO}(n)$, wich yields an induced map

$$\rho^* : H^*(\mathrm{BO}(n)) \to H^*(\mathrm{BQ}(n)) \simeq \mathbb{F}_2[x_1, \dots, x_n].$$

Theorem 4.2. *The map ρ^* satisfies the following properties:*

- *ρ^* is injective.*
- *the image of ρ^* is $\mathbb{F}_2[x_1, ; x_n]^{\Sigma_{in}}$.*
- *$p^*(w_i) = e_i$.*

The splitting principle We give a modern POV on this:

Theorem 4.3. *Let X be paracompact and $E \to X$ a bundle. There is an undiced bundle $f : \mathrm{Fl}(E) \to X$ so that*

$$f^* : H^*(X) \to H^*(\mathrm{Fl}(E))$$

is injective, and f^*E splits into a direct sum of line bundles.

This winds up telling you the injectivity of Theorem 4.2, but not the image statement (only a containment). Either way, the proof is not much easier.

Spectral sequences breaking down We'll keep some assumptions about finite type cohomology. Dylan stated the requirement of simply connected spaces or principal G-bundle.[4]

Theorem 4.4. *Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration. The associated Serre spectral sequence (SSS) is trivial if and only if $H^*(E) \to H^*(F)$ is surjective. In this case, we say that F is* totally non-homologous to zero*, and we have the following properties:*

- *p^* is injective.*
- *$P(E) = P(B) \cdot P(F)$.*

The condition is called totally non-homologous to zero because the dual condition $H_*(F) \hookrightarrow H_*(E)$ makes sense for this name.

[4]Haynes had some comments about this; principality is not enough in general. There's secretly some connectedness condition.

4.2 Cohomology of $\mathrm{BO}(n)$

Outline of the proof of Theorem 4.2 Let $F_n = O(N)/Q(N)$. We will use the fibration

$$F_n \hookrightarrow \mathrm{BQ}(n) \xrightarrow{\rho} \mathrm{BO}(n).$$

We call this fibration $(\star)$. We follow the following steps:

(1) $H^*(F_n) = \langle H^1(F_n)\rangle$ so that $P(F_n) = (1-t)\cdots(1-t)^n \cdot (1-t)^{-n}$.

(2) The SSS for $(\star)$ is trivial, so $P(\mathrm{BQ}(n)) = P(\mathrm{BO}(n))P(F_n)$, giving injectivity.

(3) $\operatorname{im}\rho^* \subset \mathbb{F}_2[x_1,\dots,x_n]^{\Sigma_{in}}$, and dimensions yield that this is an equality.

General rule of theorem: every spectral sequence written today will be trivial.

Step 1: cohomology of F_n. We use induction via the fibration

$$F_{n-1} \hookrightarrow F_n \to \mathbb{P}^{n-1}$$

Lemma 4.5. $\dim H^1(F_n) \geq n-1$.

Write ${}^nE_r = \bigoplus_{s+t=n} E_r^{s,t}$.

Proof. Recall the fibration $F_n \hookrightarrow \mathrm{BSQ}(n) \to \mathrm{BSO}(n)$. The base space is simply connected, so

$$E_2^{1,0} = H^1(\mathrm{BSO}(n), H^0(F_n)) = 0.$$

Hence

$$\dim{}^1 E_2 \geq \dim{}^1 E_\infty = \dim H^1(\mathrm{BSQ}(n)) = n-1.$$

This implies that $E_2^{0,1} = H^1(F_n)$. $\square$

Proposition 4.6. *$P(F_n) = (1-t)\cdots(1-t^n)(1-t)^{-n}$ and $H^*(F_n) = \langle H^1(F_n)\rangle$.*

Proof. Return to the fibration from the beginning of the fibration. We know the Poincaré polynomial for projective space, and we just have to prove that the SSS is trivial.[5] Write $H^*(F_n) \xrightarrow{i^*} H^*(F_{n-1})$. Assume both claims for $n-1$, so $\dim H^1(F_{n-1}) = n-2$. Note the following:

- $\dim E_2^{1,0} = \dim H^1(\mathbb{P}^{n-1}) = 1$.
- $\dim E_2^{0,1} = \dim H^0(\mathbb{P}^{n-1}, H^1(F_n)) \leq n-2$.

Look at $\dim H^1(F_n) = {}^1E_\infty \leq n-1$; combined with our previous bound, we have $\dim H^1(F_n) = n-1$. This implies that ${}^1E_n ={}^1 E_\infty$ since they have equal dimensions. This implies that $E_2^{0,1}$ are cocycles for differentials.

Further, note that $\operatorname{im} i^*|_{\deg 1} = E_\infty^{0,1} = H^1(F_{n-1})$. Since cohomology of the codomain is generated in degree 1, this implies that i^* is surjective, so the SSS is trivial. This implies the Poincaré polynomial is as we said it is, by a familiar technique. $\square$

Step 2: triviality of the SSS of $(\star)$.

Proposition 4.7. *The SSS for $(\star)$ is trivial.*

Proof. Note that $\dim{}^1 E_2 = \dim H^1(\mathrm{BO}(n), H^0(F_n)) + \dim H^0(\mathrm{BO}(n), H^1(F_n))$. The first is equal to 1, and the second is $\leq n+1$, and the second is $\leq n-1$, so the total is $\leq n$.

Now look at $\dim{}^1E_\infty \leq \dim{}^1 E_2$, which is an equality for dimension reasons. We have $\dim{}^1 E_2 \geq \dim{}^1 E_\infty$, and hence $H^0(\mathrm{BO}(n), H^1(F_n)) = H^1(F_n)$. For reasons relating to generation t degree 1, we also have $H^0(\mathrm{BO}(n), H^k(F_n)) = H^k(F_n)$. Hence $H^*(\mathrm{BQ}(n)) \twoheadrightarrow H^*(F_n)$. Hence the SSS is trivial. $\square$

This allows us to immediately compute the Poincaré polynomial

$$P(\mathrm{BO}(n)) = \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)}.$$

[5] In particular, $P(\mathbb{P}^{n-1}) = \frac{1-t^n}{1-t}$, so since passing to the associated graded preserves graded dimension, triviality implies that the Poincaré series are multiplicative, and we can prove the Poincaré series computation inductively. I'll skip this.

Step 3: containment of the image of ρ^* in the symmetric polynomials. Combinatorics exists:

Lemma 4.8. $P(\mathbb{F}_2[x_1, \ldots, x_n]^{\Sigma_{in}}) = P(\mathrm{BO}(n))$.

Use Schur polynomials. Back to the topology.

Proposition 4.9. in $\rho^* \subset \mathbb{F}_2[x_1, \ldots, x_n]^{\Sigma_{in}}$.

Proof. $\Sigma_{in} = N_n/Q(n)$. Write down the classifying space fibration:

$$Q(n) \hookrightarrow EQ(n) \to BQ(n)$$

N_n acts on this, and acts on polynomials by permuting the generators in $\mathbb{F}_2[x_1, \ldots, x_n]$. The normalizer N_n also acts on

$$O(n) \to EO(n) \to \mathrm{BO}(n).$$

the action on $\mathrm{BO}(n)$ is homotopically trivial[6], which we could use... Instead, we know the groups, so we can check concretely that this acts trivially on the cohomology, which gives the image containment. □

Hidden step 4: talking about $p^*(w_i) = e_i$. Once we know the $p^*(w_i) = e_i$ statement, universal relations on Steifel-Whitney classes come down to relations on $H^*(\mathrm{BO}(n))$.

[6] This is a general fact stated by Haynes, which comes down to some categories thing I didn't catch...

5 Mikayel Mkrtchyan: Milnor, The Steenrod algebra and its dual

This talk was delivered September 24, 2021 by Mikayel Mkrtchyan.

5.1 Refresher on the Steenrod algebra and Hopf algebras

For the next 48 minutes or so, we set $p = 2$ and work with the mod-2 Steenrod algebra. Recall that the *Steenrod algebra* $\mathcal{A}^*$ is a graded commutative algebra of mod-2 cohomology operations generated as an algebra by $\mathrm{Sq}^n \in \mathcal{A}^n$ for $n \geq 1$. For a finite sequence $I = (i_1, \ldots, i_r)$, define

$$\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_r}.$$

Recall that a sequence I is admissible if $a_i \geq 2a_{i-1}$. We have a basis made of these:

Theorem 5.1 (Serre-Cartan). *The set $\left\{\mathrm{Sq}^I \mid I \text{ admissible}\right\}$ is an $\mathbb{F}_2$-basis for $\mathcal{A}^*$.*

This is proved via the following relation:

Theorem 5.2 (Adem relation). *For all $0 < n < 2m$,*

$$\mathrm{Sq}^n \mathrm{Sq}^m = \sum_{k=1}^{n/2} \binom{m-k-1}{n-2k} \mathrm{Sq}^{n+m-k} \mathrm{Sq}^k,$$

and these generate all relations in a presentation of $\mathcal{A}^*$.[7]

As a preview of what's to come, define $\mathcal{A}_*$ to be the dual coalgebra.

Theorem 5.3. *$\mathcal{A}^*$ is a graded connected Hopf algebra, and its dual satisfies $\mathcal{A}_* \simeq \mathbb{F}_2[\zeta_1, \zeta_2, \ldots]$.*

As an application, we'll show that all elements in $\mathcal{A}^{>0}$ are nilpotent.
The base field of $\mathbb{F}_2$ is to be understood.

Definition 5.4. A connected graded Hopf algebra is a graded associative algebra B^* s.t. $B_0 = \mathbb{F}_2$, endowed with a coassociative *comultiplication* map

$$B^* \xrightarrow{\psi} B^* \otimes B^*$$

s.t. $\psi(b) = b \otimes 1 + 1 \otimes b + \sum b_i' \otimes b_i''$ for all $b \in B^{>0}$.

Projecting to the 0th graded part is the "augmentation" (counit), and you can define the antipode uniquely given this data.

5.2 Coalgebra structure on the Steenrod algebra

We want to define a map $\psi : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$ extending the map

$$\psi(\mathrm{Sq}^n) = \sum_{i+j=n} \mathrm{Sq}^i \otimes \mathrm{Sq}^j.$$

We will prove that this is well defined using the Cartan formula

$$\mathrm{Sq}^n(a \times b) = \sum_{i+j=n} \mathrm{Sq}^i(a) \times \mathrm{Sq}^j(b).$$

We'll also use the following

[7] The presentation statement was said verbally but not written.

Lemma 5.5. *Fix some n. There exists a space U with finite-type cohomology and a class $u \in H^*(U)$ such that*

$$\sigma : \mathcal{A}^* \to H^*(U)$$

given by $a \mapsto a \cdot u$ is injective on $\mathcal{A}^{\leq n}$.

Proof sketch. This is given by $U = K(\mathbb{Z}/2, n+1)$ with $u \in H^{n+1}(K(\mathbb{Z}/2, n+1))$. Use the result of Gabi's talk. □

Lemma 5.6. *There exists a lift*

$$\begin{tikzcd}
T(\{\mathrm{Sq}^n\}) \arrow[r, two heads] \arrow[d, "\psi"] & \mathcal{A}^* \arrow[dl, dashed] \\
\mathcal{A}^* \otimes \mathcal{A}^* &
\end{tikzcd}$$

Proof. Choose (U, u, σ) as in the lemma. Form the diagram

$$\begin{tikzcd}
T(\{\mathrm{Sq}^n\}) \arrow[rr, two heads] \arrow[d, "\psi"] & & \mathcal{A}^* \arrow[dll, dashed] \arrow[d, "\text{act on } u \times u"] \\
\mathcal{A}^* \otimes \mathcal{A}^* \arrow[r, "\sigma\otimes\sigma"] & H^*(U) \otimes H^*(U) \simeq & H^*(U \times U).
\end{tikzcd}$$

Then, note that $\sigma \otimes \sigma$ is injective at degrees $\leq n$, and this commutes, hence we can choose a lift in lower degrees; this allows you to define it degreewise by picking high enough n. □

Corollary 5.7. *$\mathcal{A}^*$ is a cocommutative Hopf algebra.*[8]

5.3 Housekeeping

Suppose X is a finite CW complex. This is routine, formal, and not talked about explicitly.

1. We have an action
$$\mathcal{A}^* \otimes H^* \to H^*.$$
2. This yields a dual operation
$$H_* \otimes \mathcal{A}* \to H_*.$$
3. We can dualize this:
$$\lambda : H^* \to H^* \otimes \mathcal{A}_*,$$
since the homology and $\mathcal{A}$ are both finite type.
4. Note that λ makes H^* into a $\mathcal{A}_*$-comodule.
5. The following proof was omitted:

   Lemma 5.8. *λ is an $\mathbb{F}_2$-algebra homomorphism.*

Let's work an example.

> **Example 5.9:**
> Let $X := \mathbb{RP}^\infty = K(\mathbb{Z}/2, 1)$, with $u \in H^1(X)$.[a]
> **Lemma 5.10.**
> $$\mathrm{Sq}^n(x^{2^m}) = \begin{cases} x^{2^{m+1}} & n = 2^m \\ 0 & \textit{otherwise} \end{cases}$$
>
> *Proof sketch.* Define $\mathrm{Sq} := \sum_i \mathrm{Sq}_i$. Note that $\mathrm{Sq}(u) = u + u^2$, so $\mathrm{Sq}(u^{2^m}) = u^{2^m} + u^{2^{m+1}}$. □

[8]He called it coassociative, but I omit this as this is the convention for Hopf algebras in general.

Corollary 5.11. *$\lambda : H^*(X) \to H^*(X) \otimes \mathcal{A}_*$ is given by*

$$\lambda(u) = \sum_k u^{2^j} \otimes \zeta_k$$

where $\langle \zeta_i, \mathrm{Sq}^I \rangle = 0$ unless $I = I_k := (2^{i-1}, 2^{k-2}, \ldots, 1, 0)$.

I'm lagging a bit behind, so expect this next bit to be choppy.

[a] We'll see why we don't have to care that X is finite.

5.4 Algebra structure on the dual Steenrod algebra

Let I be an admissible sequence, and define

$$\gamma(I) = (i_1 - 2i_2, i_2 - 2i_3, \ldots, i_r, 0).$$

Let R be a sequence.

Proposition 5.12. *Let I, J be admissible sequences of the same degree. Then,*

$$\langle \zeta^{\gamma(J)}, \mathrm{Sq}^I \rangle = \begin{cases} 1 & I = J \\ 0 & I < J \end{cases}$$

where $<$ denotes the lexicographic order.[9]

Proof. We prove this by induction. Let $J = (aj_1, \ldots, a_k, 0)$ and similar for I and b. define

$$J' = (a_1 - 2^{k-1}, a_2 - 2^{k-2}, \ldots, 0).$$

Then,

$$\gamma(J) = \gamma(J') + (\text{a } 1 \text{ in the } k\text{th spot.}).$$

Hence

$$\zeta^{\gamma(J)} = \zeta^{\gamma(J')} \cdot \zeta_k,$$

so that

$$\langle (\zeta^{\gamma(J)}, \mathrm{Sq}^I \rangle = \langle \zeta^{\gamma(J)} \otimes \zeta_k, \psi(\mathrm{Sq}^I) \rangle = \langle \zeta^{\gamma(J')} \otimes \zeta_k, \sum \mathrm{Sq}^{I_1} \otimes \mathrm{Sq}^{I_2} \rangle.$$

If you work out the nitty gritty, this concludes the proof by induction. □

Corollary 5.13. $\mathcal{A}_* \simeq \mathbb{F}_2[\zeta_1, \zeta_2, \ldots]$.

Proof. The last proposition proved that $\zeta^{\gamma(J)}$ form an $\mathbb{F}_2$-basis, which is exactly equivalent to $\mathcal{A}_*$ being a polynomial algebra in ζ_i. □

We now characterize the comultiplication.

Theorem 5.14. *The comultiplication map $\varphi_* : \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ is given by*

$$\zeta_n \mapsto \sum_{k \geq 0} \zeta_{n-k}^{2^k} \otimes \zeta_k.$$

This is some measure of the basis we gave being nice.

Proof. We have coassociativity:

$$\begin{tikzcd} H^* \arrow[r] \arrow[d] & H^* \otimes \mathcal{A}_* \arrow[d, "\lambda\otimes\mathrm{id}"] \\ H^* \otimes \mathcal{A}_* \arrow[r, "\mathrm{id}\otimes\varphi_*"] & H^* \otimes \mathcal{A}_* \otimes \mathcal{A}_* \end{tikzcd}$$

We perform a diagram chase for $X := \mathbb{RP}^\infty$. □

[9] It hasn't been mentioned what happens when $I > J$.

5.5 Positive-degree homogeneous elements of the Steenrod algebra are nilpotent

Define $J_n \subset \mathcal{A}_*$ by $(\zeta_1^{2k}, \zeta_2^{k-1}, \ldots, \zeta_{k-1}^2, \zeta_{k+1}, \ldots)$. Observe that $\varphi_*(J_n) \subset J_n \otimes \mathcal{A}_*$ by our characterization of the Milnor diagonal, and hence $\mathcal{A}_*/J_n$ is a Hopf algebra quotient of $\mathcal{A}_*$ of finite dimension. By duality, this corresponds with a f.d. Hopf subalgebra, and expanding n threatens to swallow $\mathcal{A}_*$:

Corollary 5.15. *$\mathcal{A}^*$ is the union of its finite dimensional Hopf subalgebras.*

By degree arguments, a positive dimension homogeneous element either is nilpotent or spans an infinite dimensional Hopf subalgebra, so this gives the nilpotency statement.

5.6 A sketch of the $p > 2$ case

In the odd p case, we have Lens spaces instead of $\mathbb{RP}^\infty$, and there are more cohomology elements:

Theorem 5.16.

$$\mathcal{A}_*^p = \mathbb{F}_p[\zeta_1, \ldots, \zeta_i] \otimes \bigwedge{}^* [\tau_0, \tau_1, \ldots] \,.$$

6 Preston Cranford: Thom, Quelques proprietes globales des varieties differentiables (i)

This talk was delivered September 29, 2021 by Preston Cranford.[10]

6.1 Motivation and overview

We're motivated by the following question

Problem 6.1 (Steenrod's problem)**.** *Let K be a finite polyhedron. Given $z \in H_r(K)$ (over $\mathbb{F}_2$ or $\mathbb{Z}$), does there exist compact M and map $f : M \to K$ s.t. $f_*[M] = Z$?*

If so, say that z is *realized* by M. We will use the notation V^n for the n-manifold, and W^p the manifold realizing a class. We henceforth restrict to $\mathbb{F}_2$ coefficients. Our result will be that whenever $2p \leq n$, alL z are realizable.

We'll follow the following outline:

- Thom spaces, classes,. . .
- We show z is realizable iff another class of a Thom space is realizable.
- We will studdy $\mathrm{MO}(k)$ and show this has the homotopy type of a product of $\mathbb{F}_2$-Eilenberg Mac Lane spaces.

6.2 Thom spaces and realizability

We'll use the following:

Definition 6.2. Let $G \subset O(n)$ be a distinguished subgroup with a faithful representation. A G-strcture on M is a principal G-subbundle of the frame bundle on M. The $O(k)$ structure is associated with a metric on M.

Recall that we have a universal bundle $\mathrm{EO}(k) \xrightarrow{p} \mathrm{BO}(k)$ where $\mathrm{EO}(k)$ is weakly contractible. Let $\mathrm{AO}(k)$ be the mapping cylinder, and let $\mathrm{MO}(k)$ be $\mathrm{AO}(k)/(\mathrm{EO}(k) \sim *)$.[11] See footnote.

Recall the following: construct the map

$$\varphi^* : H^{r-k}(\mathrm{BO}(k)) \to H^r(\mathrm{AO}(k), \mathrm{EO}(k)) \simeq H^r(\mathrm{MO}(k)).$$

Theorem 6.3 (Thom isomorphism theorem)**.** *The map φ^* is an isomorphism.*

For unit class $w \in H^0(\mathrm{BO}(k))$, call $\varphi^*(w) = u$ the *fundamental class of* $\mathrm{MO}(k)$.

We'll prove the following:

Theorem 6.4. *Let V^n be a closed n-manifold. Then, $z \in H_{n-k}(V^n)$ is realizable by W^{n-k} if and only if its Poincaré dual $u \in H^k(V^n)$ is induced by some $f : U^n \to \mathrm{MO}(k)$.*

Proof. ($\implies$) Suppose we have $z \in H_{n-k}(V^n)$ realized by $i : W^{n-k} \to V^n$. Let $N \to W^{n-k}$ be a normal tubular neighborhood of W^{n-k}, which has an $O(k)$-structure.[12] Define a filler via pullback[13]

$$\begin{tikzcd} N \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{VO}(k) \arrow[d] \\ W^{n-k} \arrow[r] & \mathrm{BO}(k) \end{tikzcd}$$

[10] I came in a bit late, so expect the beginning of these to be choppy and potentially subtly wrong.

[11] This was corrected by Haynes to more modern notation, noting that this is wrong. $\mathrm{EO}(k)$ here was supposed to be the universal sphere bundle. Haynes suggested the Thom space as the quotient of the universal disk bundle by the universal sphere bundle, and that the homotopical definition here was more general than necessary.

[12] He went over how to construct this locally, but I won't repeat it, since it's routine and I'm catching up a bit.

[13] There's notational confusion; $\mathrm{AO}(k)$ henceforth means the mapping cone of the universal sphere bundle, and $\mathrm{VO}(k)$ is the total space of the universal vector bundle.

Explicitly, this takes geodesic ball fibers to ball fibers. Taking cohomology, we have

$$\begin{tikzcd}
H^k(N, \partial N) & H^n(\mathrm{VO}(n), \mathrm{EO}(n)) \arrow[l] \\
H^0(W^{n-k}) \arrow[u] & H^0(\mathrm{BO}(n)) \arrow[l] \arrow[u]
\end{tikzcd}$$

We have a collapse map, where a is the point at infinity:

$$\begin{tikzcd}
& N \arrow[r] & W^{n-k} \arrow[r] & \mathrm{BO}(n) \arrow[r] & \mathrm{MO}(k) & \\
& \partial n \arrow[u, hook] \arrow[rrr] & & & \{a\} \arrow[u, hook] & \\
V^n/N \arrow[uur, hook] \arrow[rrrrr] & & & & & \{a\} \arrow[uul, hook]
\end{tikzcd}$$

I am very confused. The following diagram was on the board, but I can't decipher what it means:

$$\begin{tikzcd}
\text{p.d. to } z \arrow[r, equal] & \varphi^*(w_0) & U \arrow[l] \\
& w_0 \arrow[u] & w_{O(k)} \arrow[l] \arrow[u]
\end{tikzcd}$$

He commented on the other direction after this. □

We can identify the cohomology of Thom space as an ideal: $H^*(\mathrm{MO}(k)) = w_k H^*(\mathrm{BO}(k))$, recalling that $H^*(\mathrm{BO}(k)) = \mathbb{F}_2[w_1, \ldots, w_k]$.

Recall that we have the bundle $\mathrm{VO}(n) \to \mathrm{BO}(n)$, and this expresses the fundamental group

$$\pi_1(\mathrm{MO}(k)) = \pi_1(\mathrm{BO}(k))/\operatorname{im}(\pi_1(\mathrm{VO}(k)) \to \pi_1(\mathrm{BO}(k))) = \mathbb{F}_2/\mathbb{F}_2 = 0.$$

Hence $\mathrm{MO}(k)$ is simply connected. Recall the following:

Theorem 6.5. *If X, Y are simply connected and $f : X \to Y$ is a map s.t. f^* is an iso for $r < k$ and monic at $r = k$, then there is a map $g : X_k \to Y_k$ that is a homotopy equivalence.*

This allows you to prove the following theorme:

Theorem 6.6. *There is a $2k$-equivalence* $\mathrm{MO}(k) \to K(\mathbb{F}_2, k)^{e_1} \times \cdots \times K(\mathbb{F}_2, 2k)^{e_{2R}}$.

7 Swapnil Garg: Thom, Quelques proprietes globales des varieties differentiables (ii)

This talk was delivered October 1, 2021 by Swapnil Garg.

7.1 Transversality and tubular neighborhoods

For the duration of this talk, V^n is a smooth manifold, and N^{p-q} is a smooth, compact manifold.

Definition 7.1. A map $f : V^n \to M^p$ is transversal to $N^{p-q} \subset M^p$ at a point $y \in N^{p-q}$ if for all $x \in f^{-1}(y)$, the map $DF : T_x V^n \to T_y M^p \to T_y M^p / T_y N^{p-q}$.

If d is tranverse at all Y, then we say f is *transversal to* N^{p-q}.

In this case, $f^{-1}(p-q)$ is a smooth manifold with normal bundle f^{-1}(normal bundle of N^{p-q}).

Let $p : T \to N^{p-q}$ be a tubular neighborhood.

Definition 7.2. Let H be the group of diffeomorphisms $A \in \operatorname{Aut} T$ s.t.

(a) A is the identity on ∂T.

(b) A preserves $p^{-1}(y)$ for all $y \in N^{p-q}$.

Defining the distance between diffeomorphisms by taking some supremum over the maps, derivatives, inverse, etc., H is a metrizable space.

Let $y \in N^{p-q}$ be a point, and $y \in X, X'$ balls of radius r, r' around y in N^{p-q}. These can be chosen small enough to trivialize the normal bundle. Let $D = p^{-1}(X) = X \times B^q$ and similarly for D'.

There is a diagram

$$\begin{tikzcd}
V^n \arrow[r, "f"] & M \\
 & N \arrow[u, hook] \\
V|_{f^{-1}(D)} \arrow[r, "f"] & D' \arrow[u, hook] \arrow[r, "k"] & B^q
\end{tikzcd}$$

If $A \circ f$ has a critical value at y, i.e. the derivative is non-surjective, then $k \circ A \circ f$ has a critical value at $k(y) = 0$ (the center of B^q).

Let $\sigma_i \subset H$ be the set of automorphisms A s.t. $k \circ A \circ f|_{f^{-1}(D) \cap K_i}$ has a critical value at 0. We'll characterize this:

Lemma 7.3. *The set σ_i is closed and has no interior points in H.*

Hence the generic diffeomorphism is regular on K_i.

Proof. Closedness. For $A \notin \sigma_i$, we want to construct a ball around A. If the absolute value of the determinant of the $q \times q$ Jacobian of A is bounded below by 2ε, then under perturbing A s.t. the Jacobian changes by at most ε, the map remains regular. Hence σ_i is closed.

Empty interior. We invoke Sard's thoemre. Take $A \in \sigma_i$. Assuming f is C^n, there exists a regular value c of $K \circ A \circ f$ which is arbitrarily close to 0. Take G_1 to be the identity on B^q, G_0 a diffeomorphism of B^q with $G_0(c) = 0$, and G_t to be a homotopy between them with varying t. Let $E(y, z) = (y, G_{d(y)}(z))$ with $d(x) = 0$, $d(\partial x') = 1$ and $d \in C^\infty$. Take $k \circ E \circ A \circ f$; this has a regular value at 0, which allows you to conclude.[14] □

Varying K_i, the countable intersection of $(\sigma_i)^c$ is dense. We get a meaningful result out of this:

Theorem 7.4. *A C^n map $f : V^n \to M^p$ can be perturbed to be transversal to $N^{p-q} \subset M^p$.*

We'll use this to talk about cobordism.

[14] Not sure how this finishes.

7.2 Cobordism and L-equivalence

Definition 7.5. Let V, V' be oriented compact manifolds of dimension k. The manifolds V and V' are *cobordant* if there exists a $k+1$-manifold X with $\partial(X) = V' \coprod -V$.[15][16]

Suppose $W^{n-k} \subset V^n \subset \mathbb{R}^{n+m}$. For $x \in W^{n-k}$, let $H(kx)$ be the k-dimensional vetor space tangent to V^n and tangent to W^{n-k} inside $\mathbb{R}^{n+m}$. This yields a map $W^{n-k} \to \mathrm{Gr}_k(\mathbb{R}^{n+m})$. Taking $m \to \infty$, we have a map $\mathrm{Gr}_k(\mathbb{R}^{n+m}) \to \mathrm{BO}(k)$.

For N a tubular neighborhood of W^{n-k} in V^n, we get a map

$$\begin{tikzcd} N \arrow[r] \arrow[d] & D(\mathrm{EO}(k)) \arrow[d] \\ W^{r-k} \arrow[r, "g"] & \mathrm{BO}(k) \end{tikzcd}$$

Collapsing $V - N$ to a point gives a map

$$V \xrightarrow{f} D(\mathrm{EO}(k))/S(\mathrm{EO}(k)) = \mathrm{MO}(k).$$

whose homotopy class turns out to be independent of the Riemannian metric chosen to define it.

We define a stronger equivalence:

Definition 7.6. Let $W_i^{n-k} \subset V^n$ be submanifolds. We say that they are *L-equivalent* with respect to V^n if there exists a submanifold $X^{n-k+1} \subset v^n \times [0,1]$ such that $\partial X^{n-k+1} = W_0^{n-k} \sqcup W_1^{n-k}$.[17]

If W_i^{n-k} are L-equivalent, they generate homotopic maps $V^n \to \mathrm{MO}(k)$. The main theorem of Thom's in this are shows that this is a bijection:

Theorem 7.7. *The above map $L_{n-k}(V^n) \to [V^n, \mathrm{MO}(k)]$ is a bijection.*[18]

There is some confusion about hos this depends on V^n. Taking V^n the sphere spectrum, taking a limit yields that the Cobrodism ring corresponds with the stable homotopy groups of the Thom spectrum:

Lemma 7.8. *$L_k(S^n) \simeq \mathfrak{N}^k$ is a bijection if $n \geq 2k+2$. A similar statement holds in the oriented case.*

Theorem 7.9. *$\mathfrak{N}^k \simeq L_k(S^{n+k}) \simeq \pi_{n+k}(\mathrm{MO}(n)$ for $n \geq k+2$. Similarly, $\Omega^k \simeq \pi_{n+k}(\mathrm{MSO}(n))$.*

7.3 Other remarks

Note that we know the stable homotopy groups of $\mathrm{MO}(n)$; they are $\mathbb{F}_2^{d(k)}$, as per Preston's talk.

We can further argue that cobordant manifold have the same Steifel-Whitney numbers. We argue via a big diagram:

$$\begin{tikzcd} & S^{n+k} \arrow[r, "f_{W'}"] & \mathrm{MO}(n) \arrow[r, "F"] \arrow[d, "\mathrm{proj}"] & K(n+k, \mathbb{F}_2) \\ & V_\omega \arrow[r] & \mathrm{Gr}_k & \end{tikzcd}$$

we can trace elements of the cohomology and make an argument there. I couldn't quite follow.

---

[15] The sign indicates reversal of orientation.

[16] He drew the pair of pants cobordism as an example here.

[17] He drew a helpful picture; $V^n \times [0,1]$ is pictured as a solid cylinder, whose ends are disk copies of V^n, in which W^i are embedded circles. An L-equivalence is pictured as a cylinder bounded by each W_i, on opposite disk faces.

[18] I use my own notation here' Swapnil followed Thom's notation.

8 Haoshuo Fu: Hirzebruch, Topological methods in algebraic geometry

This talk was delivered on October 4, 2021 by Haoshuo Fu. The topic of this talk will be the index theorem.

8.1 The signature of a manifold

Definition 8.1. Let M be an oriented closed manifold of dimension $4k$. There is a symmetric bilinear form

$$\begin{aligned}\phi_M : H^{2k}(M,\mathbb{R}) \otimes H^{2k}(M,\mathbb{R}) &\to \mathbb{R}\\ x \otimes y &\mapsto \langle x \cup y, [M]\rangle.\end{aligned}$$

The *signature of* M is the signature

$$\tau(M) = (\#\text{ pos. eigenvals. of } \phi_M) - (\#\text{ neg. eigenvals. of } \phi_M)\,.$$

> **Example 8.2:**
> In the case $M = \mathbb{CP}^{2k}$, then $H^{2k}(\mathbb{CP}^{2k}) = \mathbb{R}$, we have and $\varphi = I$, so $\tau(\mathbb{CP}^{2k}) = 1$.

We can characterize these:

Proposition 8.3. *The following hold:*

- $\tau(-M) = -\tau(M)$.
- $\tau(M \sqcup N) = \tau(M) + \tau(N)$.
- $\tau(M^{4k} \times N^{4\ell}) = \tau(M^{4k}) \cdot \tau(N^{4\ell})$.
- *If* $M^{4k} = \partial W^{4k+1}$, *then* $\tau(M^{4k}) = 0$.

Proof sketch. For the third bullet, note that[19]

$$h^{2k+2\ell}(M^{4k} \times N^{4\ell}) = \bigoplus_{i\in\mathbb{Z}} H^{2k+i}(M^{4k}) \oplus H^{2k-i}(N^{4\ell}) =: \bigoplus_{i\in\mathbb{Z}} V(i).$$

When $i = 0$, we have $\phi_{M\times N}|_{V(0)} = \phi_M \otimes \phi_N$. When $i > 0$, letting A be the pairing of $V(i)$ and $V(-i)$, we have

$$\phi_{M\times N}|_{V(i)\oplus V(-i)} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}.$$

Noting that $\det(\lambda I - \phi_{M\times N}|_{V(i)\oplus V(-i)} = \det(\lambda^2 I - A^2)$ allows one to conclude.

The last bullet it proved via Poincaré duality; given the morphism of exact sequences

$$\begin{tikzcd} H^{2k}(w) \arrow[r, "i^*"] \arrow[d] & H^{2k}(M) \arrow[r, "\delta"] \arrow[d] & H^{2k+1}(W,M) \arrow[d] \\ H_{2k+1}(W,M) \arrow[r] & H_{2k}(M \arrow[r] & H_{2k}(W) \end{tikzcd}$$

the snake lemma yields a short exact sequence

$$0 \to \operatorname{im} i_* \to H^{2k}(M) \to \operatorname{im}\delta \to 0.$$

Note that $\dim \operatorname{im}\delta = \dim \operatorname{im} i^*$, so $\operatorname{im} i^*$ is a subspace of $H^{2k}(M)$ of half dimension. We have $\phi_M|_{\operatorname{im} i^* \otimes \operatorname{im} i^*} = 0$, and subsequently, $\phi_M = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, gviving the fourth bullet point. $\square$

Corollary 8.4. τ *yields a homomorphism* $\tau : \Omega_{4*} \to \mathbb{Z}$.

[19] I omit the proof of the first two, as they're obvious.

8.2 Review of the structure of the cobordism ring

Take notation from the previous two talks, on Thom. Note that there is a (pulled back) map

$$\begin{tikzcd} \xi_n \oplus e \arrow[r] \arrow[d] & \xi_{n+1} \arrow[d] \\ \mathrm{BSO}(n) \arrow[r] & \mathrm{BSO}(n+1). \end{tikzcd}$$

Taking Thom spaces yields a morphism $\Sigma\,\mathrm{MSO}(n) = \mathrm{Th}(\xi_n \oplus e) \to \mathrm{MSO}(n+1)$. This yields a prespectrum MSO.[20]

Previous results yield an iso of graded Abelian grooups

$$\pi_*(\mathrm{MSO}) = \operatorname{colim} \pi_{*+n)}(\mathrm{MSO}(n)) = \Omega_*.$$

We can give MSO a ring spectrum structure via maps

$$\begin{tikzcd} \xi_m \oplus \xi_n \arrow[r] \arrow[d] & \xi_{m+n} \arrow[d] \\ \mathrm{BSO}(m) \times \mathrm{BSO}(n) \arrow[r] & \mathrm{BSO}(m+n) \end{tikzcd}$$

and a similar trick to before.

Recall the rational Hurewicz theorem:

Theorem 8.5 (Rational Hurewicz theorem)**.** *Let X be a simply connected space with $\pi_i(X) \otimes \mathbb{Q} = 0$ for all $0 \le i \le$. Then, the Hurewicz map*

$$h : \pi_i(X) \otimes \mathbb{Q} \to \widetilde{H}_i(X; \mathbb{Q})$$

is an isomorphism for $0 \le i \le 2i$.

Applying this to the Thom spectrum, noting that we have we have

$$\widetilde{H}^i(\mathrm{MSO}(n); \mathbb{Q}) = 0 \qquad\qquad i \le n$$

we can compute rational stable homotopy groups of MSO by computing homology in degree $\le 2n$:[21]

$$\pi_*(\mathrm{MSO}) \otimes \mathbb{Q} \xrightarrow{\sim} H_*(\mathrm{MSO}; \mathbb{Q}).$$

We may combine this with the Thom isomorphism

$$H^i(\mathrm{BSO}(n); \mathbb{Q}) \xrightarrow{\sim} H^{i+n}(\mathrm{MSO}(n); \mathbb{Q})$$

by taking a limit in order to yield an iso

$$H^*(\mathrm{BSO}(n); \mathbb{Q}) \xrightarrow{\sim} H^*(\mathrm{MSO}; \mathbb{Q}).$$

Taking udals yields an iso

$$H_*(\mathrm{MSO}; \mathbb{Q}) \xrightarrow{\sim} H_*(\mathrm{BSO}(n); \mathbb{Q}).$$

Hence the signature actually yields a homomorphism

$$\tau : H_*(\mathrm{BSO}; \mathbb{Q}) \to \mathbb{Q},$$

i.e. an element $L \in H^*(\mathrm{BSO}; \mathbb{Q})$. We can write this in homogeneous parts as $L_n \in H^{4n}(\mathrm{BSO}; \mathbb{Q})$.

A diagram chase yields that

$$\langle L_N(\tau_M), [M] \rangle = \tau(M).$$

[20] I would just call this a sequential spectrum.

[21] This is true of any ring spectrum, according to Haynes.

This is powerful, if we can determine the structure of L_N. We do so in a little while, but first, a bit more structure.

There is a commutative diagram

$$\begin{tikzcd} H_*(\mathrm{BSO};\mathbb{Q}) \otimes H(\mathrm{BSO};\mathbb{Q}) \arrow[r] \arrow[d] & H_*(\mathrm{BSO};\mathbb{Q}) \arrow[d] \\ \mathbb{Q} \otimes \mathbb{Q} \arrow[r, equal] & \mathbb{Q} \end{tikzcd}$$

yielding a comultiplication

$$H^*(\mathrm{BSO};\mathbb{Q}) \to H^*(\mathrm{BSO};\mathbb{Q}) \otimes H^*(\mathrm{BSO};\mathbb{Q})$$

sending $L \mapsto L \otimes L$. The Whitney sum formula yields that $p \mapsto p \otimes p$ as well.

8.3 Multiplicative sequences

We henceforth fix B a unital ring, P_i indeterminants with $P_0 = 1$ and $\deg P_i = i$.

Definition 8.6. A sequence $(K_n(P_1, \ldots, P_n) \in B[P_1, \ldots]_n)$ of polynomials of degree n is called a *multiplicative sequence*, or an *m-sequence*, if in the notation

$$\sum_{i=0}^{\infty} P_i \zeta^i = \left(\sum P'_j \zeta^j\right) \left(\sum P''_k \zeta^k\right)$$

we have

$$\sum_{i=0}^{\infty} K_i(P_1, \ldots, P_n)\zeta^i = \left(\sum_j K_j(P'_1, \ldots, P'_j)\zeta^j\right) \left(\sum_k K_k(P''_1, \ldots, P''_k)\zeta^k\right).$$

Note that (L_n) is an m-sequence.[22] We will use computational tools for multiplicative sequences to work with this:

Theorem 8.7. *The sequence (K_n) is uniquely determined by*

$$Q(\xi) = K(1 + \xi) = \sum_{i=0}^{\infty} b_i \zeta^i$$

where $b_i = K_i(1, 0, \ldots, 0)$.

Proof. Note that

$$\sum_{i=0} P_i \zeta^i = \prod_{i=1}^{m} (1 + \beta_i \zeta)$$

for some β_i via the splitting principle. Hence

$$\sum_{i=0}^{m} K_i(P_1, \ldots, P_i)\zeta^i \prod_{i=1}^{m} Q(\beta_i \zeta).$$

This determines each K_i. ☐

Let's work an example.

Example 8.8:
Recall that $H^*(\mathbb{CP}^{2k};\mathbb{Q}) = \mathbb{Q}[h]/h^{2k+1}$. We can use some algebraic geometry[a] to show that

$$\sum L_i \zeta^i = Q(h^i \zeta)^{2k+1}.$$

[22] He justified this, but I couldn't follow it.

This shows that the coefficient of ζ^k in $Q(\zeta^{2k+1})$ is 1. We will conclude with this:

Claim. *$Q(\zeta)$ is unique and $Q(\zeta) = \frac{\sqrt{\zeta}}{\sinh\sqrt{\zeta}}$.*

Proof. Note that, in the notation $Q(\zeta) = \sum_{i=0}^{\infty} b_i \zeta_i$, we have $(2k+1)b_k + \text{ lower terms } = 1$. By induction, b_k is unique, and

$$\int \frac{1}{\zeta^{k+1}} \left(\frac{\sqrt{\zeta}}{\sinh\sqrt{\zeta}} \right)^{2k+1} d\zeta = 1.$$

□

[a] I'm not going to copy the alg geo out of laziness.

This allows us to conclude the index theorem:

Theorem 8.9 (Hirzubruch's signature theorem)**.** *If (L_n) is the m-sequence related to $\frac{\sqrt{\zeta}}{\sinh\sqrt{\zeta}}$, then*

$$\langle L_n(\tau_M), \sqrt{M}\rangle = \tau(M) \in \mathbb{Z}.$$

In particular, we have

$$L_1 = \frac{1}{3}p_i \qquad L_2 = \frac{1}{45}(7p_2 - p_1^2).$$

9 Natalie Stewart: Milnor, On manifolds homeomorphic to the 7-sphere

This talk will be delivered on October 6, 2021 by Natalie Stewart. The notes here are currently being prepared in advance.

9.1 Motivation: the generalized Poincaré conjecture

The following conjecture was proposed by Poincaré in 1904:

Conjecture 9.1 (Poincaré conjecture)**.** *Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.*

This problem alluded solution for nearly a century. Before then, analogous conjectures were proved; that closed 2-manifolds homotopy equivalent to S^2 (henceforth *topological homotopy 2-spheres*) must be homeomorphic to the 2-sphere followed from classification of surfaces. That homotopy n-spheres are homeomorphic to S^n was proved for $n \geq 5$ in 1966 by M. H. A. Newman using a technique pioneered by Stephen Smale called *PL engulfing* [6].

This motivates a class of conjectures;

Conjecture 9.2 (Generalized Poincaré conjecture)**.** *Given a category of manifolds $\mathcal{C}$, all homotopy n-spheres in $\mathcal{C}$ are isomorphic.*

In the case where $\mathcal{C} = \mathbf{Diff}$ is the category of smooth manifolds, the first known counterexample was due to Milor, who constructed a family of at least 7 pairwise nondiffeomorphic smooth structures on the topological 7-sphere [4]. Those which are not diffeomorphic to the usual smooth structure on S^7 are called *exotic 7-spheres*. We will focus on his original construction in this note and the corresponding lecture, culminating in a proof of the following theorem:

Theorem 9.3. *There at least 6 diffeomorphism classes of exotic spheres.*

Later work of Milnor and Kervaire computed that there are exactly 27 diffeomorphism classes of exotic 7-spheres, and that exotic spheres exist in many dimensions ≥ 7, by using h-cobordism [3]. This breaks down in dimension ≤ 4, as the h-cobordism theorem fails, so this fails to classify smooth 4 or 3 spheres. The following chart summarizes the status of the generalized Poincaré conjecture:

category	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n \geq 7$
Top	True	True	True	True	True	True
PL	True	Open	True	True	True	True
Diff	True	Open ($\mathbf{PL}_{\leq 4} \simeq \mathbf{Diff}_{\leq 4}$)	True	True	False	Often false

With that context in mind, we now move on to define the invariants Milnor used in his construction.

9.2 Milnor's λ invariant

Throughout this note, set

$$H^*(X) := H^*(X; \mathbb{Z}).$$

For the duration of this section, fix a closed compact orientable 7-manifold M satisfying

$$H^3(M) = H^4(M) = 0. \tag{1}$$

By Thom [7], any closed compact oriented 7-manifold bounds a compact oriented 8-manifold (with boundary);[23] suppose that M is bounded by B, with orientation class $[B] \in H_8(B)$. Condition (1) implies that the inclusion

$$\iota : H^4(B, M) \to H^4(B)$$

[23] This comes down to the identification $\Omega^* = \pi_*(\mathrm{MSO})$ and explicit computations of the low-degree homotopy groups of this spectrum via the $2k$-equivalence of its k-th space with products of Eilenberg Mac Lane spaces.

is an isomorphism, so we may define the first Pontryagin number of B in this setting to be

$$q(B) := \langle [B], (\iota^{-1} p_1)^2 \rangle$$

Recall that the *index* $\tau(B)$ is defined to be the index of the quadratic form $\alpha \mapsto \langle [B], \alpha^2 \rangle$ on $H^4(B, M)/(\text{torsion})$, i.e. the number of positive terms minus the number of negative terms in a real diagonalization of the form.

We will build an invariant on M out of $q(B)$ and $\tau(B)$, then prove that it's independent of choice of B. First, we take a quick digression into properties of closed 8-manifolds, which may fail in the non-closed case.

Recall the following corollary of the Hirzrebruch signature theorem [2]:

Theorem 9.4 (Hirzrebruch signiature theorem, 8-dimensional case)**.** *For C a closed oriented 8-manifold and $[C] \in H_8(C)$ a fundamental class, we have*

$$\tau(C) = \langle [C], \frac{1}{45}\left(7p_2(C) - p_1^2(C)\right) \rangle.$$

In particular, the signature theorem implies that

$$2q(C) - \tau(C) = 2(q(C) + 45\tau(C)) = 2 \cdot 7p_2(C) \equiv 0 \pmod 7.$$

Since B bounds M, a nontrivial space, there is no guarantee that the analogue to this equality holds in our setting. In fact, failure of this equality is our main tool:

Definition 9.5. For M a closed 7-manifold bounded by an 8-manifold B, define the invariant

$$\lambda_B(M) := q(B) - \tau(B) \in \mathbb{F}_7.$$

This should worry the reader; it appears to depend on B. We assuage this fear via the following theorem:

Theorem 9.6. *The residue class $\lambda_B(M) \in \mathbb{F}_7$ doesn't depend on the choice of the manifold B.*

We henceforth simplify notation to

$$\lambda(M) := \lambda_B(M).$$

We will this to prove Theorem 9.3; for each $z \in \mathbb{F}_7$, we will construct a homeomorphism 7-sphere M with $\lambda(M) = z$.

Proof of Theorem 9.6. Suppose B, B' both bound M. Let B'' be the oriented manifold given by B' with opposite orientation. Defined the closed 3-manifold

$$C := B \cup_M B''$$

It is enough to prove the following equations:

$$\tau(C) = \tau(B) - \tau(B') \tag{2}$$
$$q(C) = q(B) - q(B'), \tag{3}$$

since then the different choices of $\lambda(M)$ differ by $q(C) - \tau(C) \equiv 0 \pmod 7$.

We first prove (2). The co-inclusion morphisms yield a commutative square

$$\begin{array}{ccc} H^n(C, M) & \xrightarrow{h} & H^n(B, M) \oplus H^n(B'', M) \\ \downarrow{\scriptstyle \iota_C} & & \downarrow{\scriptstyle \iota_B \oplus \iota_{B''}} \\ H^n(C) & \xrightarrow{k} & H^n(B) \oplus H^n(B'') \end{array}$$

compatible with the Kronecker pairing. The top horizontal arrow is always an isomorphism. The vertical arrows are clearly isos when $n = 4$ by (1), so k is an isomorphism in that case as well. Let $\alpha := \iota_C h^{-1}(\alpha, \alpha'') \in H^4(C)$ be an element. Then, we may diagram chase:

$$\begin{aligned} \langle [C], \alpha^2 \rangle_{H^8(C)} &= \langle [C], \iota_C h^{-1}(\alpha^2, \alpha''^2) \rangle_{H^8(C)} \\ &= \langle h\iota_C^{-1}[C], (\alpha^2, \alpha''^2) \rangle_{H^8(B,M) \oplus H^8(B'',M)} \\ &= \langle ([B], [B'']), (\alpha^2, \alpha''^2) \rangle_{H^8(B,M) \oplus H^8(B'',M)} \\ &= \langle [B], \alpha^2 \rangle_{H^8(B,M)} + \langle [B''], \alpha''^2 \rangle_{H^8(B'',M)}. \end{aligned}$$

Hence the quadratic form of C is the direct sum of that of B and B'', and hence it's the direct sum of that of B and the negative of that of B'. This yields equation (2).

For equation (3), note[24] that

$$kp_1(C) = p_1(B) \oplus p_1(B'')$$

Hence $q(C) = q(B) - q(B'')$ by an analogous argument to (2). $\square$

Now that we know that λ is well defined, note that Pontryagin classes and indices both switch sign when reversing orientation; this yields the following technical lemma:

Lemma 9.7. *Reversing the orientation of M multiplies $\lambda(M)$ by -1. Hence any M possessing $\lambda(M) \neq 0$ has no orientation reversing diffeomorphism onto itself.*

We want to use λ; our strategy will begin with the construction of a convenient family of spaces with easily computable λ invariants.

9.3 The construction of Milnor's exotic spheres

One candidate for exotic spheres is the restriction of 4-plane bundles over S^4 to their associated 3-sphere bundles; these are always 7-dimensional manifolds, and we can classify them explicitly via the *clutching construction*:

Construction 9.8 (the clutching construction)**.** Consider S^4 as the union of the upper and lower hemispheres D_+ and D_- along the equator $S^3 \subset S^4$. For a map $f : S^3 \to \mathrm{SO}(4)$, construct the 4-plane bundle B_f by gluing $D_+ \times \mathbb{R}^4$ to $D_- \times \mathbb{R}^4$ along $(x, v)_+ \sim (x, f(x)(v))_-$ for $x \in S^3$. The following theorem is well known [1]:

Theorem 9.9. *This identification descends to an isomorphism $\pi_3(\mathrm{SO}(4)) \xrightarrow{\sim} \mathrm{Vec}_{\mathbb{R},4}(S^3)$.*

The group $\mathrm{SO}(4)$ has universal cover $\pi : S^3 \to \mathrm{SO}(4)$ given by $\pi(u, w)(v) = uvw$, written using quaternionic multiplication.[25] Hence there is an isomorphism $\mathrm{Vec}_{\mathbb{R},4}(S^3) \simeq \pi_3(\mathrm{SO}(4)) \simeq \mathbb{Z}^2$.

Remark. This construction is easy to picture topologically, but not very good for determining the differentiable structure as written. Taking a hemmed gluing is more suitable; we can instead replace D_+, D_- with $\mathbb{R}^4$, glued along $\mathbb{R}^4 - \{0\}$ via a modified stereographic projection

$$(u, v) \mapsto (u', v') = \left(\frac{u}{|u|^2}, \frac{u^h v u^j}{|u|^{i+j}} \right)$$

The associated 4-plane bundle of this is isomorphic to the previously described bundle. Further, this describes the differentiable structure explicitly, and restricting to the associated sphere bundle, the transition function of the differentiable structure has the same formula.

Let $f_{h,j} : S^3 \to \mathrm{SO}(4)$ correspond with the pair $(h, -j)$. Let

$$\begin{tikzcd} S^3 \arrow[r, hook] & E_{h,j} \arrow[d, "\xi_{h,j}"] \\ & S^4 \end{tikzcd}$$

be the corresponding 3-sphere bundle. For each odd integer k, let M_k be the total space of $E_{h,j}$ where h and j are determined by the equations $h + j = 1$ and $h - j = k$. These will be our candidates; we will show that they are homeomorphic to S^7, usually with nontrivial λ invariant. First we tackle the homeomorphism, via techniques from Morse theory.

[24] One can see this via representability; pushing forward the iso

$$[C, \mathrm{BU}(n)] \simeq [B \cup_M B'', \mathrm{BU}(n)] \simeq [B, \mathrm{BU}(n)] \coprod_{[M, \mathrm{BU}(n)]} [B'', \mathrm{BU}(n)]$$

along the second Chern class morphism $\mathrm{BU}(n) \to K(\mathbb{Z}, n)$ and applying this to the complexification of tangent bundles yields the desired statement after noting that the second Chern class of a bundle on M is trivial by (1).

[25] This convention is nonstandard, but agrees with Milnor.

9.4 M_k is homeomorphic to S^7: a Morse theoretic sketch

Consider the following hypothesis on a closed manifold M:

- (H) *There exists a differentiable function $f : M \to \mathbb{R}$ having only two critical points, where each are nondegenerate.*

We prove the following:

Theorem 9.10. *A manifold M satisfying hypothesis (H) is homeomorphic to S^7.*

Proposition 9.11. *M_k satisfies hypothesis (H), and hence it is homeomorphic to S^7.*

The proofs of these are largely irrelevant to each other, so I'll sketch the concrete statement first.

Proof sketch for Proposition 9.11. We can define the function f in local coordinates, compatibly with transition functions:

$$f(u, v) = \frac{\Re(v)}{\sqrt{1 + |u|^2}}$$
$$f(u', v') = \frac{\Re(u'/v')}{\sqrt{1 + |u'/v'|^2}}$$

where $\Re(\cdot)$ is the real part of a quaternion. The reader can verify that this has exactly two critical points, each nondegenerate, at $(u, v) = (0, \pm 1)$. □

Proof sketch for Theorem 9.10. Suppose $f : M \to \mathbb{R}$ is a function witnessing hypothesis (H). Normalize f so that $f(x_0) = 0$ and $f(x_1) = 1$. According to Morse [5], one can take local coordinates $v_1, \ldots, v_n$ in a neighborhood V of x_0 so that

$$f(x) = v_1^2 + \cdots + v_n^2 \qquad \text{on } V.$$

One may define a Riemannian metric on V via $ds^2 = dv_1^2 + \cdots + dv_n^2$, and extend this to one on all of M.

The gradient of f defines a vector field on M; consider the differential equation

$$\frac{dx}{dt} = \frac{\nabla f}{|\nabla f|^2}.$$

This equation has solutions. Suppose $x_a(t)$ is a solution equation; note that

$$f(x_a(t)).$$

so that $x_a(t) = (a_1^{1/2}(t), \ldots, a_n^{1/2}(t))$ on V. Map the unit sphere of $\mathbb{R}^n$ into M via the map

$$a \mapsto x_a(t).$$

This defines a diffeomorphism of the open n-cell onto $M - \{x_1\}$. Adding a single point yields the theorem. □

9.5 Completing the proof: calculating λ invariants

We refer to elements of $H^4(S^4)$ as elements of $\mathbb{Z}$, but we will find that signs don't matter, as we always square these elements when using them.. We first show that these generate all of the possible λ invariants:

Lemma 9.12. *The λ invariant of M_k is as follows:*

(i) $p_1(\xi_{h,j}) = \pm 2(h - j)\iota$.

(ii) $\lambda(M_k) = k^2 - 1$.

Proof. Part (i). First, note that $\xi_{h,j} \oplus \xi_{h',j'}$ is stably isomorphic to $\xi_{h+h',j+j'}$: writing local coordinates[26] of $\xi_{h+h',j+j'} \oplus e$ as (u, v, w) on the preimage of D^+ with transition function

$$(u, v, w) \longmapsto (u', v', w') = \left(u, u^{h'} u^h v u^h u^{j'}, w\right)$$

and the coordinates on $\xi_{h,j} \oplus \xi_{h',j'}$ similarly as (x, y, z) with transition function

$$(x, y, z) \longmapsto (x', y', z') = \left(x, x^h y x^j, x^{h'} y x^{j'}\right),$$

there is an iso $\xi_{h+h',j+j'} \oplus e \to \xi_{h,j} \oplus \xi_{h',j'}$ given by the identity above D^+ and the map

$$(u', v', w') \mapsto \left(x', x'^{-h'} v' x'^{-j'}, x'^{h'} w' x'^{j'}\right)$$

over D_-. We can check that this is compatible with transition functions:

$$\begin{tikzcd}
(\xi_{h+j} \oplus e)|_{D_+ \cap D_-} \arrow[rrr] \arrow[ddd] & & & (\xi_{h+j} \oplus e)|_{D_+ \cap D_-} \arrow[ddd] \\
& (u, v, w) \arrow[r, mapsto] \arrow[d, mapsto] & (u, u^{h+h'} v u^{j+j'}, w) \arrow[d, mapsto] & \\
& (x, y, z) \arrow[r, mapsto] & (x, x^h y x^j, x^{h'} y x^{j'}) & \\
\xi_{h+h',j+j'}|_{D_+ \cap D_-} \arrow[rrr] & & & \xi_{h+h',j+j'}|_{D_+ \cap D_-}
\end{tikzcd}$$

Hence the Whitney sum formula yields $p_1(\xi_{h+h',j+j'}) = p_1(\xi_{h,j} + \xi_{h',j'})$, i.e. $p_1(\xi_{i,j})$ is linear in h and j. Further, there is an isomorphism $\xi_{h,j} \to \xi_{-j,-h}$; hence $p_1(\xi_{h,j}) = ah + bj = -bh - aj$ for some constants a, b, i.e. $p_1(\xi_{h,j}) = c(h - j)$ for some constant c.

To compute the constant c, it suffices to compute an example. The case $(h, j) = (1, 0)$ corresponds with the ordinary quaternionic Hopf fibration, i.e. the sphere bundle associated with the tautological quaternionic line bundle

$$\begin{tikzcd}
S^3 \arrow[r, hook] \arrow[d, hook] & S^7 \arrow[d, hook] \\
\mathbb{H} \arrow[r, hook] & \mathbb{H}^2 \arrow[d, "T_{\mathbb{H}}"] \\
& \mathbb{HP} \arrow[d, equal] \\
& S^4
\end{tikzcd}$$

For any complex vector bundle ξ, one has $\xi \otimes \mathbb{C} = \xi \oplus \overline{\xi}$, and $c_i(\overline{\xi}) = (-1)^i c_i(\xi)$. In the case that ξ is 2-dimensional with vanishing first Chern class (e.g. if $H^2(B(\xi)) = 0$), the Whitney sum formula then yields

$$1 + p_1(\xi) = (1 + c_2(\xi))^2 = 1 + 2c_2(\xi)$$

and hence $p_1(\xi) = 2c_2(\xi)^2$. Hence it is enough to compute $c_2(T_{\mathbb{H}})$. We compute this via the comparison map

$$g : \mathbb{CP}^2 \to \mathbb{HP}^1$$

sending $[a + bi : c + di] \mapsto [a + bi + cj + dk]$. This yields a pullback square

$$\begin{tikzcd}
\mathbb{H}^2 \times_{\mathbb{HP}^1} \mathbb{CP}^2 \arrow[r] \arrow[d, "g^* T_{\mathbb{H}}"] & \mathbb{H}^2 \arrow[d, "T_{\mathbb{H}}"] \\
\mathbb{CP}^2 \arrow[r, "g"] & \mathbb{HP}^1
\end{tikzcd}$$

[26]Here and elsewhere, e refers to a trivial 1-dimensional vector bundle.

In fact, there is a diagram of real vector bundles

$$\begin{tikzcd}
\mathbb{C}^4 \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "T_{\mathbb{C}}^2"'] & & \\
& g^*(\mathbb{H}^2) \arrow[r] \arrow[d, "g^*T_{\mathbb{H}}"] & \mathbb{H}^2 \arrow[d, "T_{\mathbb{H}}"] \\
& \mathbb{CP}^2 \arrow[r, "g"] & \mathbb{HP}^1
\end{tikzcd}$$

where

$$\phi([a+bi : c+di], w+xi, y+zi) = ([a+bi+cj+dk], w+xi+yj+zk)$$

Note that ϕ is fiberwise-injective, and hence the canonical map $\mathbb{C}^4 \to g^*(\mathbb{H}^2)$ is fiberwise-injective. Both have fiber of real dimension 4, so this must be an isomorphism of real vector bundles, i.e. $g^*T_{\mathbb{H}} = T_{\mathbb{C}}^2$. Then, naturality of Chern classes implies that

$$g^*c_2(T_H) = c_2(T_{\mathbb{C}}^2) = c_1(T_{\mathbb{C}})^2$$

is a generator of $H^4(\mathbb{CP}^2)$. In turn, since $g : \mathbb{CP}^2 \to \mathbb{HP}^1$ is a $\mathbb{CP}^1 = S^2$ bundle, the Gysin sequence yields

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(\mathbb{HP}^1) \arrow[r] \arrow[d, equal] & H^4(\mathbb{HP}^1) \arrow[r, "g^*"] & H^4(\mathbb{CP}^2) \arrow[r] & H^2(\mathbb{HP}^1) \arrow[r] \arrow[d, equal] & \cdots \\
& 0 & & & 0 &
\end{tikzcd}$$

and g^* is an isomorphism. Hence $c_2(T_{\mathbb{H}})$ is a generator of $H^4(S^4)$, i.e. $p_1(\xi_{1,0}) = p_1(T_{\mathbb{H}}) = \pm 2$, determining the constant c.

Part (ii). Note that, for any smooth fiber bundle $F \hookrightarrow E \to B$, one has $TE = (TB)^* \oplus (TF)_*$, by decomposition of the tangent space at a point into vectors tangent and normal to the copy of F containing it.

In particular, applying this to $TE_{h,j}$ and along with the Whitney sum formula yields

$$p_1(E_{h,j}) = \xi_{h,j}^* p_1(\xi_{h,j}) + \iota_* p_1(S^3) = \xi_{h,j}^* p_1(\xi_{h,j}).$$

Again applying the Gysin sequence yields an isomorphism

$$\begin{tikzcd}
\cdots \arrow[r] & H^7(S^4) \arrow[r] \arrow[d, equal] & H^7(E_{h,j}) \arrow[r, "\xi_{h,j}^*"] & H^4(S^4) \arrow[r] & H^8(E_{h,j}) \arrow[r] \arrow[d, equal] & \cdots \\
& 0 & & & 0 &
\end{tikzcd}$$

i.e. $p_1(E_{h,j}) = \pm 2(h-j)$. Pick an orientation of $E_{h,j}$ (and hence B_k) so that this is positive. Then, $\tau(B_k) = 1$, and

$$q(B_k) = \langle [B_k], 4k^2 \rangle = 4k^2.$$

Hence we have

$$\lambda(M_k) = 8k^2 - 1 = k^2 - 1 \pmod 7,$$

as desired. □

This allows us to finally conclude Theorem 9.3:

Proof of Theorem 9.3. Whenever $k \neq k' \pmod 7$, we have $\lambda(M_k) \neq \lambda(M_{k'})$, and hence M_k is not diffeomorphic to $M_{k'}$. Hence $\{M_k\}$ yields at least 7 diffeomorphism classes of spaces; by Proposition 9.11, these are all homeomorphism 7-spheres, so there are at least 7 diffeomorphism classes of homeomorphism 7-spheres. □

References

[1] A. Hatcher. *Vector Bundles and K-Theory*. http://www.math.cornell.edu/~hatcher. 2003.

[2] Friedrich Hirzebruch. *Topological methods in algebraic geometry*. Classics in Mathematics. Translated from the German and Appendix One by R. L. E. Schwarzenberger, With a preface to the third English edition by the author and Schwarzenberger, Appendix Two by A. Borel, Reprint of the 1978 edition. Springer-Verlag, Berlin, 1995, pp. xii+234. ISBN: 3-540-58663-6.

[3] Michel A. Kervaire and John W. Milnor. "Groups of homotopy spheres. I". In: *Ann. of Math. (2)* 77 (1963), pp. 504–537. ISSN: 0003-486X. DOI: 10.2307/1970128. URL: https://doi-org.ezp-prod1.hul.harvard.edu/10.2307/1970128.

[4] John Milnor. "On manifolds homeomorphic to the 7-sphere". In: *Ann. of Math. (2)* 64 (1956), pp. 399–405. ISSN: 0003-486X. DOI: 10.2307/1969983. URL: https://doi-org.ezp-prod1.hul.harvard.edu/10.2307/1969983.

[5] Marston Morse. "Relations between the critical points of a real function of n independent variables". In: *Trans. Amer. Math. Soc.* 27.3 (1925), pp. 345–396. ISSN: 0002-9947. DOI: 10.2307/1989110. URL: https://doi-org.ezp-prod1.hul.harvard.edu/10.2307/1989110.

[6] Stephen Smale. "Generalized Poincaré's conjecture in dimensions greater than four". In: *Ann. of Math. (2)* 74 (1961), pp. 391–406. ISSN: 0003-486X. DOI: 10.2307/1970239. URL: https://doi-org.ezp-prod1.hul.harvard.edu/10.2307/1970239.

[7] René Thom. "Quelques propriétés globales des variétés différentiables". In: *Comment. Math. Helv.* 28 (1954), pp. 17–86. ISSN: 0010-2571. DOI: 10.1007/BF02566923. URL: https://doi-org.ezp-prod1.hul.harvard.edu/10.1007/BF02566923.