ON TENSOR PRODUCTS OF EQUIVARIANT COMMUTATIVE OPERADS

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Abstract. Let Op_G be Nardin-Shah's ∞-category of $\mathcal{O}_G$-∞-operads (henceforth G-operads). We define the full subcategory $\mathrm{Op}_{G,0} \subset \mathrm{Op}_G$ of G-suboperads of the G-commutative operad, called *weak $\mathcal{N}_\infty$-operads*; we show that these are combinatorially characterized as *weak indexing systems*, and the restriction to Blumberg-Hill's *indexing systems* corresponds with Blumberg-Hill's $\mathcal{N}_\infty$-operads.

We compute the Boardman-Vogt tensor products of unital weak $\mathcal{N}_\infty$-operads as *joins of weak indexing systems*. The restriction of this to $\mathcal{N}_\infty$-operads confirms a conjecture of Blumberg and Hill. In particular, for I, J unital weak indexing systems and $\mathcal{C}$ an $I \vee J$-symmetric monoidal ∞-category, we acquire a canonical $I \vee J$-symmetric monoidal equivalence

$$\underline{\mathrm{CAlg}}_I^{\otimes} \underline{\mathrm{CAlg}}_J^{\otimes} \mathcal{C} \simeq \underline{\mathrm{CAlg}}_{I \vee J}^{\otimes} \mathcal{C}$$

From this we recover derived additivity of the equivariant little disks operads in a variety of infinitary cases.

Along the way, we achieve several structural results concerning G-operads, Boardman-Vogt tensor products, homotopical (incomplete) Mackey functors, and (co)cartesian G-symmetric monoidal ∞-categories, including construction of a canonical lift of the Boardman-Vogt tensor product of G-operads to a presentably symmetric monoidal ∞-category. All such results are presented as equivariant over an atomic orbital ∞-category.

Contents

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Introduction

This paper is a direct sequel to [Ste24a], so we adopt its notation. We make a construction called *arity support*, which turns out to provide a functor

$$A\colon \mathrm{Op}_G \to \mathrm{wIndex}_G,$$

the latter being the poset of *G-weak indexing systems* of [Ste24b]. This possesses a fully faithful right adjoint $\mathcal{N}^{\otimes}_{(-)\infty}$ whose essential image is the *weak $\mathcal{N}_\infty$-operads* of [Ste24a], generalizing the identifications of $\mathcal{N}_\infty$-operads with indexing systems of [BP21; GW18; Rub21a]. We additionally characterize weak $\mathcal{N}_\infty$-operads equivalently as subobjects of the terminal G-operad $\mathrm{Comm}_G^{\otimes}$ or G-0-operads. Bonventre's genuine operadic nerve fully faithfully maps Blumberg-Hill's poset of $\mathcal{N}_\infty$-operads into the poset of weak $\mathcal{N}_\infty$ operads, with image consisting of those admitting a map from $\mathbb{E}_\infty^{\otimes}$.

We go on to characterize *(co)cartesian I-symmetric monoidal ∞-categories* in the case that I is unital in the sense of [Ste24b]; if a G-∞-category $\mathcal{C}$ has I-indexed products, then there is an essentially unique I-symmetric monoidal structure on $\mathcal{C}$ whose indexed tensor products are indexed products. We call this the *cartesian* structure and present the ∞-category of algebras in a cartesian I-symmetric monoidal ∞-category via a generalization of Lurie's *$\mathcal{O}$-monoids* [HA, Prop 2.4.2.5].

There is a dual essentially unique *cocartesian structure* on $\mathcal{C}$, and we show that every object in a cocartesian I-symmetric monoidal ∞-category canonically lifts to an $\mathcal{O}$-algebra (c.f. [HA, Prop 2.4.3.9]). Conversely, when $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal ∞-category, we show that $\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{C})$ underlies a cocartesian I-symmetric monoidal ∞-category. From this we conclude that the unique map $\mathrm{triv}_G^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$ witnesses $\mathcal{N}_{I\infty}^{\otimes}$ as an idempotent object in Op_G in the sense of [HA] and we characterize its associated smashing localization in terms of indexed semiadditivity; we conclude that there is a unique equivalence

$$\mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{J\infty} \simeq \mathcal{N}^{\otimes}_{I\vee J\infty}$$

when I, J are unital weak indexing systems, confirming conjecture 6.27 of [BH15] in this setting. We extend the computation to arbitrary almost essentially unital G-operads, and show that this is as far as we can go; if I is not almost essentially unital, then we show $\mathcal{N}_{I\infty}^{\otimes 2}$ is not connected, so it is not a weak $\mathcal{N}_\infty$-operad.

We use this to develop an infinitary and equivariantly homotopical version of Dunn's additivity theorem [Dun88]; denoting the *little V-disks G-operad* by $\mathbb{E}_V^{\otimes}$ (for V a real orthogonal G-representation), under the assumption either that $V \simeq V \oplus V$ and $W \simeq W \oplus W$ or that $V \oplus W \simeq V$, we prove that the forgetful functors

$$\mathrm{Alg}_{\mathbb{E}_V} \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_W}(\mathcal{C}) \leftarrow \mathrm{Alg}_{\mathbb{E}_{V\oplus W}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_W} \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C})$$

are equivalences. As an application, we show how to define *iterated Real topological Hochschild homology* of $\mathbb{E}_V$-algebras whenever V possesses an $\infty\sigma$-summand.

We now move to a more careful account of the background, motivation, and main results of this paper.

Background and motivation. Let $\mathcal{C}$ be a 1-category with finite products. Recall that a *commutative monoid in $\mathcal{C}$* is the data

$$A \in \mathrm{Ob}(\mathcal{C}); \qquad \text{multiplication } \mu\colon A \times A \to A; \qquad \text{unit } \eta\colon * \to A,$$

subject to the usual unitality, associativity, and commutativity assumptions; more generally, if $\mathcal{C}$ is a symmetric monoidal 1-category, a *commutative algebra in $\mathcal{C}$* is the data of

$$R \in \mathrm{Ob}(\mathcal{C}); \qquad \text{multiplication } \mu\colon R \otimes R \to R; \qquad \text{unit } \eta\colon 1 \to R,$$

satisfying analogous conditions. When $\mathcal{C} = \mathrm{Set}$, this recovers the traditional theory of commutative monoids, and when $\mathcal{C} = \mathrm{Mod}_k$ with the tensor product of k-modules, this recovers the traditional theory of commutative k-algebras. These have been the subject of a great deal of homotopy theory in three guises:

(1) We may define the $(2,1)$-category $\mathrm{Span}(\mathbb{F})$ to have objects the finite sets, morphisms from X to Y the spans of finite sets $X \leftarrow R \to Y$, 2-cells the isomorphisms of spans

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] \arrow[dd, "\sim"] & \\ X & & Y, \\ & R' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

and composition the pullback of spans

$$\begin{tikzcd} & & R_{XZ} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] & & \\ & R_{XY} \arrow[dl] \arrow[dr] & & R_{YZ} \arrow[dl] \arrow[dr] & \\ X & & Y & & Z. \end{tikzcd}$$

If $\mathcal{C}$ is an ∞-category, then we define the *∞-category of commutative monoids in* $\mathcal{C}$ as the models of the associated Lawvere theory; that is, we define the product-preserving functor category

$$\mathrm{CMon}(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}), \mathcal{C}),$$

noting that products in $\mathrm{Span}(\mathbb{F})$ correspond with disjoint unions of finite sets. Indeed, if $\mathcal{C}$ is a 1-category and A a commutative monoid in $\mathcal{C}$, we flesh this out with the dictionary

$$\begin{aligned} ([2] = [2] \to [1]) &\quad\longmapsto\quad \mu\colon A^{\times 2} \to A; \\ (\varnothing = \varnothing \to [1]) &\quad\longmapsto\quad \eta\colon * \simeq A^{\times 0} \to A; \\ ([1] \leftarrow [2] = [2]) &\quad\longmapsto\quad \Delta\colon A \to A^{\times 2} \\ ([1] \leftarrow \varnothing = \varnothing) &\quad\longmapsto\quad !\colon A \to A^{\times 0} \simeq *. \end{aligned}$$

Unitality, associativity, and commutativity are conveniently packaged by functoriality. This turns out to be equivalent to Graeme Segal's *special* Γ *spaces* [Seg74] when $\mathcal{C} = \mathcal{S}$, and for general $\mathcal{C}$, it recovers the anologously defined theory in $\mathcal{C}$ (c.f. [BHS22, Ex 3.1.6, Prop 3.1.16, Pf. of prop 5.2.14]).

(2) We say that an ∞-category is *semiadditive* if it has finite products and coproducts and for all finite sets S, the canonical natural transformation $\coprod_{s \in S}(-) \implies \prod_{s \in S}(-)$ is an equivalence. Then, the full subcategory $\mathrm{Pr}^{L,\oplus} \subset \mathrm{Pr}^L$ of *semiadditive presentable ∞-categories* possesses a localization functor $L_{\oplus}\colon \mathrm{Pr}^L \to \mathrm{Pr}^{L,\oplus}$, which we study.

(3) Let Op denote the ∞-category of operads.[1] Then, there is a terminal operad $\mathrm{Comm}^{\otimes} \simeq \mathbb{E}_{\infty}^{\otimes}$; given $\mathcal{C}$ a symmetric monoidal ∞-category, we may form the ∞-category of *commutative algebra objects*

$$\mathrm{CAlg}(\mathcal{C}) := \mathrm{Alg}_{\mathrm{Comm}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}).$$

We study this and its specialization to the cartesian symmetric monoidal structure.

These perspectives each present the same ∞-category, i.e. [Cra11; GGN15] show that

$$\mathrm{CMon}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C}^{\times}) \simeq L_{\oplus}\mathcal{C}.$$

As a result, translating between these perspectives has proved invaluable; for instance, [GGN15] uses Perspectives 2 and 3 to construct an essentially unique symmetric monoidal structure on $\mathrm{CMon}(\mathcal{C})$ and [CHLL24a] uses Perspectives 1 and 3 to model commutative algebras in $\mathrm{CMon}(\mathcal{C})^{\otimes}$ as models for the Lawvere theory of *commutative semirings*.

Crucially, Perspective 3 may be used to construct homotopical lifts of the *Eckmann-Hilton argument*; for instance, in [HA], it is shown that for *any* reduced operad $\mathcal{O}^{\otimes}$, the forgetful functors

$$\mathrm{CAlg}\mathrm{Alg}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C}) \leftarrow \mathrm{Alg}_{\mathcal{O}}\mathrm{CAlg}^{\otimes}(\mathcal{C}),$$

are equivalences for the "pointwise" symmetric monoidal structure on algebras. Such a task may be accomplished by recognizing the far left and far right side each as algebras over the *Boardman-Vogt tensor product* $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes}$ and each arrow as pullback along the canonical map

$$\mathrm{Comm}^{\otimes} \simeq \mathrm{triv}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes} \xrightarrow{\mathrm{can} \otimes \mathrm{id}} \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes};$$

that the above maps are equivalences is then equivalent to the statement that the object $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes} \in \mathrm{Op}$ is terminal, which is well-known.

This result is used ubiquitously to replace (lax) symmetric monoidal functors $\mathrm{Alg}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathcal{C}^{\otimes}$ with (lax) symmetric monodial endofunctors

$$\mathrm{CAlg}^{\otimes}(\mathcal{C}) \simeq \mathrm{CAlg}^{\otimes}\mathrm{Alg}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathrm{CAlg}^{\otimes}(\mathcal{C});$$

[1] This is unambiguous [HM23], but we will tend to model these as ∞-operads in the sense of [HA].

for instance, this underlies the symmetric monoidal structure on left-modules [HA] and the multiplicative structure on various invariants such as factorization homology [HA, Thm 5.5.3.2], THH, and TC [NS18, § IV.2].

This paper concerns the analog of Perspective 3 in the equivariant theory of algebra stemming from Hill-Hopkins-Ravanel's use of *norms of G-spectra* on the Kervarire invariant one problem, as well as the resulting theory of *indexed tensor products and (co)products* (c.f. [HH16]).

For the rest of this introduction, fix G a finite group. In G-equivariant homotopy theory, the point is replaced with elements of the *orbit category* $\mathcal{O}_G \subset \mathrm{Set}_G$, whose objects are homogeneous G-sets $[G/H]$; indeed, Elmendorf's theorem [Elm83] realizes G-spaces as coefficient systems $\mathcal{S}_G \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S})$.[2] In G-equivariant higher category theory, ∞-categories are thus replaced with G-∞-categories

$$\mathrm{Cat}_G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathrm{Cat}).$$

In G-equivariant higher algebra, following Perspective 1, we may form the effective Burnside 2-category $\mathrm{Span}(\mathbb{F}_G)$ whose objects are finite G-sets, whose morphisms are spans, whose 2-cells are isomorphisms of spans, and whose composition is pullback; the following central definition is the heart of this subject.

Definition. The *∞-category of G-commutative monoids in $\mathcal{C}$* is the product-preserving functor ∞-category

$$\mathrm{CMon}_G(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_G), \mathcal{C});$$

the *∞-category of small G-symmetric monoidal ∞-categories* is

$$\mathrm{Cat}_G^{\otimes} := \mathrm{CMon}_G(\mathrm{Cat}). \qquad \triangleleft$$

These are a homotopical lift of Dress' *Mackey functors* [Dre71] (c.f. [Lin76]). Indeed, given $\mathcal{C}^{\otimes} \in \mathrm{Cat}_G^{\otimes}$ a G-symmetric monoidal ∞-category, the product-preserving functor

$$\iota_H \colon \mathrm{Span}(\mathbb{F}) \xrightarrow{* \mapsto G/H} \mathrm{Span}(\mathbb{F}_G)$$

constructs a symmetric monoidal ∞-category $\mathcal{C}_H^{\otimes} := \iota_H^* \mathcal{C}^{\otimes}$ whose underlying ∞-category $\mathcal{C}_H$ is the *value* of $\mathcal{C}^{\otimes}$ on the orbit $[G/H]$. For all subgroups $K \subset H \subset G$, the covariant and contravariant functoriality of $\mathcal{C}^{\otimes}$ then yield symmetric monoidal *restriction* and *norm* functors

$$\begin{aligned} \mathrm{Res}_K^H &\colon \mathcal{C}_H^{\otimes} \to \mathcal{C}_K^{\otimes}, \\ N_K^H &\colon \mathcal{C}_K^{\otimes} \to \mathcal{C}_H^{\otimes}, \end{aligned}$$

which satisfy a form of Mackey's *double coset formula*.

Example. In Section 1.3, we recall a theorem of [BH21a; CHLL24b]: there exists a unique presentably G-symmetric monoidal ∞-category $\underline{\mathrm{Sp}}_G^{\otimes}$ such that:

- the H-value of $\underline{\mathrm{Sp}}_G^{\otimes}$ is the symmetric monoidal ∞-category $\left(\underline{\mathrm{Sp}}_G^{\otimes}\right)_H \simeq \mathrm{Sp}_H^{\otimes}$ of *genuine H-spectra* under the usual tensor product;
- the restriction functors $\mathrm{Res}_K^H \colon \mathrm{Sp}_H^{\otimes} \to \mathrm{Sp}_K^{\otimes}$ are the usual restriction functors; and
- the norm functors $N_K^H \colon \mathrm{Sp}_K^{\otimes} \to \mathrm{Sp}_H^{\otimes}$ are the *HHR norm* of [HHR16].

In fact, this symmetric monoidal structure is completely determined by its unit object $\mathbb{S}_G \in \mathrm{Sp}_G^{\otimes}$. $\triangleleft$

Fix $\mathcal{C}^{\otimes} \in \mathrm{Cat}_G^{\otimes}$. If $H \subset G$ is a subgroup and $S \in \mathbb{F}_H$ a finite H-set, we may form the induced G-set $\mathrm{Ind}_H^G S \to [G/H]$, and the covariant and contravariant functoriality then yield an *S-indexed tensor product* and *S-indexed diagonal*

$$\bigotimes_K^S \colon \mathcal{C}_S \to \mathcal{C}_H, \qquad\qquad \Delta^S \colon \mathcal{C}_H \to \mathcal{C}_S.$$

where $\mathcal{C}_S := \prod_{[H/K] \in \mathrm{Orb}(S)} \mathcal{C}_K$. Note that N_H^K is the $[H/K]$-indexed tensor product and Res_K^H the $[H/K]$-indexed diagonal. As explained in [Ste24a], functoriality applied to the "collapse" map $\mathrm{Ind}_H^G S \to \coprod_{[H/K] \in \mathrm{Orb}(S)} [G/H] \to$

[2] Maps $[G/K] \to [G/H]$ may equivalently be presented as elements of g such that $gKg^{-1} \subset H$, modulo K; see e.g. [Die09] for details.

$[G/H]$ yields equivalences

$$\bigotimes_K^S X_K \simeq \bigotimes_{[H/K]\in \mathrm{Orb}(S)} N_K^H X_K, \qquad\qquad \Delta^S(X) = \left(\mathrm{Res}_K^H X\right)_{[H/K]\in\mathrm{Orb}(S)},$$

so often reduce arguments to binary tensor products and norms. Similarly, we define the *S-fold tensor power*

$$X_H^{\otimes S} := \bigotimes_K^S \left(\Delta^S X_H\right) \simeq \bigotimes_K^S \mathrm{Res}_H^K X_H \simeq \bigotimes_{[H/K]\in\mathrm{Orb}(S)} N_K^H \mathrm{Res}_K^H X_H.$$

If it exists, the pointwise left-adjoint to Δ^S is the *indexed coproduct*

$$\coprod_K^S X_K \simeq \coprod_{[H/K]\in\mathrm{Orb}(S)} \mathrm{Ind}_K^H S,$$

where Ind_K^H is the left adjoint to the restriction map $\mathcal{C}_H \to \mathcal{C}_K$. The *indexed products* are defined analogously.

Given $H \subset G$ a subgroup, we say that $\mathcal{C}$ is *H-pointed* if $\mathcal{C}_K$ is pointed for all $K \subset H$. Given $S \in \mathbb{F}_H$, we say that S is *$\mathcal{C}$-ambidextrous* if $\mathcal{C}$ is H-pointed, $\mathcal{C}$ admits S-indexed products and coproducts, and *norm* natural transformation

$$\coprod_K^S (-) \implies \prod_K^S (-) \colon \mathcal{C}_S \to \mathcal{C}_H$$

of [Nar16, § 5] is an equivalence (see [Ste24a]). We say that $\mathcal{C}$ is *G-semiadditive* if S is $\mathcal{C}$-ambidextrous for all $S \in \mathbb{F}_H$ and $H \subset G$. More generally, if $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_G$ is a weak indexing system corresponding with the weak indexing category I (see [Ste24b] or our review in Section 1.2), we say that $\mathcal{C}$ is *I-semiadditive* if S is $\mathcal{C}$-ambidextrous whenever $S \in \mathbb{F}_{I,H}$.

In this level of generality, Perspectives 1 and 2 are known to present equivalent ∞-categories of I-commutative monoids; indeed, the *semiadditive closure* theorem of [CLL24, Thm B] demonstrates that $\mathrm{Pr}_G^{L,I-\oplus} \subset \mathrm{Pr}_G^L$ is a smashing localization implemented by

$$L_{I-\oplus}(\mathcal{C}) \simeq \underline{\mathrm{CMon}}_I(\mathcal{C}) := \underline{\mathrm{Fun}}_G^{\times}(\mathrm{Span}_I(\underline{\mathbb{F}}_G), \mathcal{C}),$$

and in particular, when $\mathcal{C}$ is a G-∞-category of coefficient systems

$$\underline{\mathrm{Coeff}}^G(\mathcal{D})_H := \mathrm{Fun}(\mathcal{O}_H^{\mathrm{op}}, \mathcal{D}),$$

[CLL24, Thm C] yields the formula

$$\underline{\mathrm{CMon}}_I(\underline{\mathrm{Coeff}}^G(\mathcal{D}))_H \simeq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_H), \mathcal{D}),$$

where $\mathrm{Span}_I(\mathbb{F}_H) \subset \mathrm{Span}(\mathbb{F}_H)$ is the wide subcategory of spans whose forward maps lie in the restriction of I to $\mathbb{F}_H$. Thus, we set the notation $\mathrm{CMon}_I(\mathcal{D}) := \underline{\mathrm{CMon}}_I(\underline{\mathrm{Coeff}}^G(\mathcal{D}))_G \simeq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_G), \mathcal{D})$ and make the following definition.

Definition. For I is a weak indexing category, the *∞-category of small I-symmetric monoidal ∞-categories* is

$$\mathrm{Cat}_I^{\otimes} := \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_G), \mathrm{Cat}). \qquad \triangleleft$$

Following through on Perspective 3, algebraic objects $X_\bullet$ in a G-symmetric monoidal ∞-category should possess collections of *S-ary operations* $X_H^{\otimes S} \to X_H$ subject to various conditions, controlled by a theory of *genuine equivariant operads*; we use Nardin-Shah's ∞-category Op_G (see [Ste24a]), whose objects we call *G-operads*. There, given $\mathcal{O}^{\otimes} \in \mathrm{Op}_G$ a G-operad, $K \subset H \subset G$ a pair of subgroups, $S \in \mathbb{F}_H$ a finite H-set, and T_i a finite K_i-set for all orbits $[H/K_i] \subset S$, we construct a *space of S-ary operations* $\mathcal{O}(S)$, *operadic composition maps*

$$\gamma \colon \mathcal{O}(S) \otimes \bigotimes_{[H/K_i]\in\mathrm{Orb}(S)} \mathcal{O}(T_i) \to \mathcal{O}\left(\coprod_{[H/K_i]\in\mathrm{Orb}(S)} \mathrm{Ind}_{K_i}^H T_i\right), \tag{1}$$

operadic restriction maps

$$\mathrm{Res} \colon \mathcal{O}(S) \to \mathcal{O}\left(\mathrm{Res}_K^H S\right), \tag{2}$$

and *equivariant symmetric group action*

$$\rho\colon \mathrm{Aut}_H(S)\times\mathcal{O}(S)\to\mathcal{O}(S). \tag{3}$$

We made the following definitions, of which the reader may focus on *having one color* and *unitality.*

Definition. A G-operad $\mathcal{O}^\otimes$

(a) *has at least one color* if $\mathcal{O}(*_H)\neq\varnothing$ for all $H\subset G$,
(b) *has at most one color* if $\mathcal{O}(*_H)\in\{\varnothing,*\}$ for all $H\subset G$,
(c) *has one color* if it has at least one color and at most one color,
(d) *is almost essentially unital* if $\mathcal{O}(\varnothing_H)=*$ whenever there exists some $S\in\mathbb{F}_H-\{*_H\}$ with $\mathcal{O}(S)\neq\varnothing$,
(e) *is unital* if $\mathcal{O}(\varnothing_H)\simeq *$ for all $H\subset G$,
(f) *is almost essentially reduced* if it is almost essentially unital and has color, and
(g) *is reduced* if it is unital and has one color.
(h) *is a G-d-operad* if $\mathcal{O}(S)$ is $(d-1)$-truncated for all $S\in\mathbb{F}_H$.[3]

The corresponding full subcategories are $\mathrm{Op}_G^{\geq\mathrm{oc}},\mathrm{Op}_G^{\leq\mathrm{oc}},\mathrm{Op}_G^{\mathrm{oc}},\mathrm{Op}_G^{aE\mathrm{uni}},\mathrm{Op}_G^{\mathrm{uni}},\mathrm{Op}_G^{aE\,\mathrm{red}},\mathrm{Op}_G^{\mathrm{red}},\mathrm{Op}_{G,d}^{\mathrm{oc}}\subset\mathrm{Op}_G$. ◃

We showed in [Ste24a] that Eqs. (2) and (3) lift to a monadic functor $\mathrm{Op}_G^{\mathrm{oc}}\to\mathrm{Fun}(\mathrm{Tot}\,\Sigma_G,\mathcal{S})$, i.e. one color G-operads are monadic over G-symmetric sequences.

When $\mathcal{O}^\otimes$ has one color, an $\mathcal{O}$-algebra in the G-symmetric monoidal ∞-category $\mathcal{C}^\otimes$ can intuitively be viewed as a tuple $(X_H\in\mathcal{C}_H^{BW_G(H)})_{G/H\in\mathcal{O}_G}$ satisfying $X_K\simeq\mathrm{Res}_K^H X_H$ for all $K\subset H\subset G$, together with $\mathcal{O}(S)$-actions

$$\mu_S\colon\mathcal{O}(S)\otimes X_H^{\otimes S}\to X_H \tag{4}$$

for all $H\subset G$ and $S\in\mathbb{F}_H$, homotopy-coherently compatible with Eqs. (1) to (3).[4]

Example. There exists a terminal G-operad $\mathrm{Comm}_G^\otimes$, which is characterized up to (unique) equivalence by the property that $\mathrm{Comm}_G(S)$ is contractible for all $S\in\mathbb{F}_H$; its algebras are endowed with contractible spaces of maps $X_H^{\otimes S}\to X_H$ for all $S\in\mathbb{F}_H$, as well as coherent homotopies witnessing their compatibility. We call these G*-commutative algberas.*

On one hand, we saw in [Ste24a] that Comm_G-algebras present a homotopical lift of Hill-Hopkins' G*-commutative monoids* [HH16, § 4], though we prefer to reserve this name for the Cartesian case, following the convention of [HA]. On the other hand, our model agrees with that of [CHLL24b], so the recent *homotopical Tambara functor theorem* of Cnossen, Lenz, and Linskens [CHLL24b, Thm B] presents G-commutative algebra objects in $\underline{\mathrm{Sp}}_G^\otimes$ as *spectral G-Tambara functors.* ◃

Example. Let V be a real orthogonal G-representation; then, there is a *little V-disks G-operad* $\mathbb{E}_V^\otimes$ whose structure spaces are *spaces of equivariant configurations*:

$$\mathbb{E}_V(S)\simeq\mathrm{Conf}_S^H(V)$$

(see [Hor19]). This is modelled by the *Steiner graph G-operad*, so e.g. pointed G-spaces of the form $X=\Omega^VY:=\mathrm{Map}_*\left(S^V,Y\right)$ lift to $\mathbb{E}_V$-spaces by composition of loops [GM11]; many $\mathbb{E}_V$-algebras will be able to be constructed in $\underline{\mathrm{Sp}}_G^\otimes$ as equivariant Thom spectra of V-fold loop spaces. ◃

In this paper, we are primarily concerned with homotopy coherently interchanging $\mathcal{O}-$ and $\mathcal{P}$-algebra structures, which are implemented as algebras over *Boardman-Vogt tensor product* $\mathcal{O}^\otimes\overset{\mathrm{BV}}{\otimes}\mathcal{P}^\otimes$ of [Ste24a]. Our main theorem will concern the following.

Example. Given $I\subset\mathbb{F}_G$ a weak indexing category, in [Ste24a] we constructed a G-operad $\mathcal{N}_{I\infty}^\otimes$ which is characterized by its structure spaces

$$\mathcal{N}_{I\infty}(S)\simeq\begin{cases}* & S\in\underline{\mathbb{F}}_I\\ \varnothing & S\notin\underline{\mathbb{F}}_I\end{cases} \tag{5}$$

This recovers the notion from [BH15] when I is an indexing category. ◃

[3] A space is -1-*truncated* if it is either empty or contractible; for all $k\geq0$, a space X is *truncated* if it is a disjoint union of connected spaces $(X_\alpha)_{\alpha\in A}$ such that, for each $\ell>k$ and $\alpha\in A$, the ℓth homotopy group $\pi_\ell(X_\alpha)$ is trivial.

[4] Here, $W_G(H)=N_G(H)/H$ is the *Weyl group* of $H\subset G$, i.e. the automorphism group of the homogeneous G-set $[G/H]$.

If I is an *indexing* category, the structure of an $\mathcal{N}_{I\infty}$-ring spectrum is intuitively viewed as commutative ring structures on each spectrum X_H, connected by multiplicative I-indexed norms, suitably compatible with the restriction and (additive) transfer structures inherent to G-spectra. We refer to $\mathcal{N}_{I\infty}$-algebras in general as *I-commutative algebras* and $\mathcal{N}_{I\infty}$-ring spectra as *I-commutative ring spectra*.

We will construct a pairing $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \to \mathcal{N}_{I\vee J\infty}$, where $I \vee J$ is the join in the poset of indexing categories; intuitively, this says that given an algebra with $I \vee J$-indexed norms, we may separate these into I-indexed norms and J-indexed norms which satisfy an applicable interchange law. Moreover, the transfer system for $I \vee J$ consists of those inclusions $K \subset H$ which can be factored as

$$K \subset K_{I1} \subset K_{J1} \subset K_{I2} \subset \cdots \subset K_{Jn} \subset H$$

where $K_{I\ell} \subset K_{J\ell}$ is in I and $K_{J\ell} \subset K_{I(\ell+1)}$ is in J [Rub21b, Prop 3.1]; intuition would then suggest that we may combine interchanging I- and J-commutative algebra structures to construct an $I \vee J$-commutative algebra structure. Thus Blumberg and Hill conjectured that there is an equivalence $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{I\vee J\infty}^{\otimes}$ [BH15, Conj 6.27]; the main theorem of this paper confirms their conjecture in Op_G.

Summary of main results. Given $\mathcal{O}^{\otimes}$, a G-operad, we define the *arity support* subcategory[5] $A\mathcal{O} \subset \mathbb{F}_G$ by

$$A\mathcal{O} := \left\{ \psi \colon T \to S \;\middle|\; \prod_{[H/K] \in \mathrm{Orb}(S)} \mathcal{O}(T_K) \neq \varnothing \right\} \subset \mathbb{F}_G.$$

In essence, $A\mathcal{O}$ consists of the *equivariant (multi-)arities* over which $\mathcal{O}^{\otimes}$ produces structure on X.

The fact that $\varnothing$ accepts no maps from nonempty spaces obstructs construction of maps matching Eqs. (1) and (2), so $A\mathcal{O}$ can't be an arbitrary subcategory. We use this to show the following.

Theorem A. *The following posets are each equivalent:*

1. *The poset* $\mathrm{Op}_G^{weak-\mathcal{N}_\infty}$ *of* weak $\mathcal{N}_\infty$ G-operads.
2. *The poset* $\mathrm{Sub}_{\mathrm{Op}_G}(\mathrm{Comm}_G)$ *of* sub-commutative G-operads.
3. *The poset* $\mathrm{Op}_{G,0} \subset \mathrm{Op}_G$ *of* G-0-operads.
4. *The essential image* $A(\mathrm{Op}_G) \subset \mathrm{Sub}_{\mathrm{Cat}}(\mathbb{F}_G)$
5. *The embedded sub-poset* $\mathrm{wIndexCat}_G \subset \mathrm{Sub}_{\mathrm{Cat}}(\mathbb{F}_G)$ *spanned by subcategories $I \subset \mathbb{F}_G$ which are closed under base change and automorphisms and satisfy the Segal condition that*

$$T \to S \in I \qquad \Longleftrightarrow \qquad \forall U \in \mathrm{Orb}(S), \quad T \times_S U \to U \in I$$

6. *The sub-poset* $\mathrm{wIndex}_G \subset \mathrm{FullSub}_G(\underline{\mathbb{F}}_G)$ *spanned by full G-subcategories $\mathcal{C} \subset \underline{\mathbb{F}}_G$ which are closed under self-indexed coproducts and have $*_H \in \mathcal{C}_H$ whenever $\mathcal{C}_H \neq \varnothing$.*

Furthermore, there are equalities of sub-posets

$$\mathrm{IndexCat}_G = A\mathrm{Op}_{G,\geq\mathbb{E}_\infty}^{\mathrm{uni}} \subset \mathrm{wIndexCat}_G,$$
$$\mathrm{wIndexCat}_G^{\mathrm{uni}} = A\mathrm{Op}_G^{\mathrm{uni}} \subset \mathrm{wIndexCat}_G$$
$$\mathrm{wIndexCat}_G^{aE\mathrm{uni}} = A\mathrm{Op}_G^{aE\mathrm{uni}} \subset \mathrm{wIndexCat}_G.$$

where $\mathrm{IndexCat}_G \simeq \mathrm{Index}_G$ *denotes the* indexing categories *of [BH15; BP21; GW18; Rub21a] and the remaining notation is that of [Ste24b].*

References. The equivalence between Poset (5) and Poset (6) is handled in [Ste24b]; nevertheless, the composite map from Poset (2) to Poset (6) is shown to be furnished by the *self-indexed symmetric monoidal envelope* in [Ste24a]. We then characterize the image of A, constructing an equivalences between Poset (4) and Poset (5) in Proposition 2.4.

[5] Throughout this paper, we say *subobject* to mean monomorphism in the sense of [HTT, § 5.5.6] and we write $\mathrm{Sub}_{\mathcal{C}}(X)$ for the poset of subobjects of X in $\mathcal{C}$; in the case the ambient ∞-category is a 1-category, this agrees with the traditional notion.

In the case our objects are in the ∞-category Cat of small ∞-categories, we call this a *subcategory*; in the case that the containing ∞-category is a 1-category, this is canonically expressed as a *core-preserving wide subcategory of a full subcategory*, i.e. it is a *replete subcategory*. Hence it is uniquely determined by its morphisms, so we will implicitly identify subcategories of $\mathcal{C}$ a 1-category with their corresponding subsets of $\mathrm{Mor}(\mathcal{C})$.

Poset (3) and Poset (4) are shown to be equivalent in Corollary 2.8 by constructing a fully faithful right adjoint $\mathcal{N}^{\otimes}_{(-)\infty}$ to A:

$$\begin{tikzcd} \mathrm{Op}_G \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathrm{wIndexCat}_G \arrow[l, bend left, "\mathcal{N}^{\otimes}_{I\infty}"] \end{tikzcd} \tag{6}$$

with image the G-0-operads. Along the way, in Proposition 2.6 and Remark 2.7 we show that Poset (1), Poset (2), andPoset (3) are equal full subcategories of Op_G. Finally, the remaining identities follow by Observation 2.9. □

Remark. After this introduction, we replace $\mathcal{O}_G$ with an atomic orbital ∞-category $\mathcal{T}$; we prove Theorem A as well as the other theorems in this introduction in this setting, greatly generalizing the stated results at the cost of ease of exposition. ◁

By Theorem A, a slice category $\mathrm{Op}_{G,/\mathcal{O}^{\otimes}} \to \mathrm{Op}_G$ is a full subcategory if and only if $\mathcal{O}^{\otimes}$ is a weak $\mathcal{N}_\infty$-operad, in which case we write

$$\mathrm{Op}_I := \mathrm{Op}_{G,/\mathcal{N}^{\otimes}_{I\infty}} \simeq A^{-1}(\mathrm{wIndexCat}_{G,\leq I});$$

explicitly, a map $\mathcal{P}^{\otimes} \to \mathcal{N}^{\otimes}_{I\infty}$ is a *property* of $\mathcal{P}^{\otimes}$, and this property is the arity support condition $A\mathcal{P} \leq I$.

We may understand $\mathcal{N}^{\otimes}_{I\infty}$ in a hands-on manner in a number of ways. On one hand, it is constructed explicitly in [Ste24a]. On the other hand, the equivalence between Poset (3) and Poset (6) of Theorem A shows that $\mathcal{N}^{\otimes}_{I\infty}$ is uniquely identified by the structure spaces appearing in Eq. (5).

Example. Given $\mathcal{F} \subset \mathcal{O}_G^{\mathrm{op}}$ a G-family[6], let $\underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}}$ be the (almost essentially unital) weak indexing system

$$\mathbb{F}^{\mathrm{triv}}_{\mathcal{F},H} := \begin{cases} \{*_H\} & H \in \mathcal{F}; \\ \varnothing & H \notin \mathcal{F}. \end{cases}$$

If $I^{\mathrm{triv}}_{\mathcal{F}}$ is the corresponding weak indexing category, then the G-operad $\mathrm{triv}^{\otimes}_{\mathcal{F}} := \mathcal{N}^{\otimes}_{I^{\mathrm{triv}}_{cF}\infty}$ is characterized by a natural equivalence

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}_{\mathcal{F}}}(\mathcal{C}) \simeq \mathrm{Bor}^G_{\mathcal{F}}(\mathcal{C}^{\otimes})$$

in Corollary 2.11, where $\mathrm{Bor}^G_{\mathcal{F}}$ is the *color Borelification* discussed in Section 2.2. ◁

Example. Given $\mathcal{F} \subset \mathcal{O}_G^{\mathrm{op}}$ a G-family, define the (almost-unital) weak indexing system

$$\mathbb{F}^{0}_{\mathcal{F},H} := \begin{cases} \{\varnothing_H, *_H\} & H \in \mathcal{F}; \\ \{*_H\} & H \notin \mathcal{F}. \end{cases}$$

with corresponding weak indexing category $I^0_{\mathcal{F}}$ and weak $\mathcal{N}_\infty$ operad $\mathbb{E}^{\otimes}_{\mathcal{F}0} := \mathcal{N}^{\otimes}_{I^0_{\mathcal{F}}\infty}$. In Section 2.3, $\mathbb{E}^{\otimes}_{\mathcal{F}0}$ is characterized by a natural equivalence

$$\mathrm{Alg}_{\mathbb{E}_{\mathcal{F}0}}(\mathcal{C}) \simeq \left(\Gamma^{\mathcal{F}}\mathcal{C}\right)^{1/} \times_{\Gamma^{\mathcal{F}}\mathcal{C}} \Gamma^G\mathcal{C},$$

where $\Gamma^{\mathcal{F}}\mathcal{C}^{\otimes}$ is the symmetric monoidal ∞-category of $\mathcal{F}$*-objects*

$$\Gamma^{\mathcal{F}}\mathcal{C}^{\otimes} \simeq \lim_{V \in \mathcal{F}^{\mathrm{op}}} \mathcal{C}^{\otimes}_V. \quad ◁$$

Example. Given $\mathcal{F} \subset \mathcal{O}_G^{\mathrm{op}}$ a G-family, define the unital weak indexing system

$$\mathbb{F}^{\infty}_{\mathcal{F},H} := \begin{cases} \{n \cdot *_H \mid n \in \mathbb{N}\} & H \in \mathcal{F}; \\ \{\varnothing_H, *_H\} & H \notin \mathcal{F}. \end{cases}$$

[6] By a *G-family*, we mean a subconjugacy closed family of subgroups. These correspond canonically with full subcategories $\mathcal{F} \subset \mathcal{O}_G$ satisfying the property that for all $V \in \mathcal{F}$ and maps $U \to V$ in $\mathcal{O}_G$, $U \in \mathcal{F}$; we will safely conflate these notions.

with corresponding weak indexing category $I_{\mathcal{F}}^{\infty}$ and weak $\mathcal{N}_\infty$ operad $\mathbb{E}_{\mathcal{F}\infty}^{\otimes} := \mathcal{N}_{I_{\mathcal{F}}^{\infty}\infty}^{\otimes}$. In Section 2.3, $\mathbb{E}_{\mathcal{F}\infty}^{\otimes}$ is characterized by a natural equivalence

$$\mathrm{Alg}_{\mathbb{E}_{\mathcal{F}\infty}}(\mathcal{C}) \simeq \mathrm{CAlg}\left(\Gamma^{\mathcal{F}}\mathcal{C}\right) \times_{(\Gamma^{\mathcal{F}}\mathcal{C})^{1/}} \Gamma^{G}\mathcal{C}^{1/}. \qquad \triangleleft$$

We say a real orthogonal G-representation V is a *weak universe* if it admits an equivalence $V \simeq V \oplus V$.

Example. Given V a weak G-universe, we verify in Section 2.3 that $\mathbb{E}_V^{\otimes}$ is a weak $\mathcal{N}_\infty$-operad whose arity support $\underline{\mathbb{F}}^V := \underline{\mathbb{F}}_{A\mathbb{E}_V}$ is computed by

$$S \in \mathbb{F}_H^V \qquad \iff \qquad \exists\ H-\text{equivariant embedding } S \hookrightarrow V.$$

In particular, if λ is a nontrivial irreducible C_p-representation, we use this to compute $A\mathbb{E}_{\infty\lambda}^{\otimes}$ in Section 2.3, verifying that $\mathbb{E}_{\infty\lambda}^{\otimes}$ is *not* an $\mathcal{N}_\infty$-operad in the sense of [BH15]. Thus $\infty\lambda$-fold loop spaces and their Thom spectra provide a rich topological source of examples of weak $\mathcal{N}_\infty$-algebras which are not $\mathcal{N}_\infty$-algebras. $\triangleleft$

I-symmetric monoidal ∞-categories have underlying I-operads; for $\mathcal{C} \in \mathrm{Cat}_I^{\otimes}$, we define the ∞-category of *I-commutative algebras* in $\mathcal{C}$ as

$$\mathrm{CAlg}_I(\mathcal{C}) := \mathrm{Alg}_{\mathcal{N}_{I\infty}}(\mathcal{C}).$$

We'd like to relate CAlg_I and CMon_I, for which we use the following construction.

Theorem B. *When I is almost-unital, there are fully faithful embeddings $(-)^{I-\sqcup}$ and $(-)^{I-\times}$ making the following commute:*

$$\begin{tikzcd}
\mathrm{Cat}_I^{\sqcup} \arrow[r, hook, "(-)^{I-\sqcup}"] \arrow[dr, "U"'] & \mathrm{Cat}_I^{\otimes} \arrow[d, "U"] & \mathrm{Cat}_I^{\times} \arrow[l, hook', "(-)^{I-\times}"'] \arrow[dl, "U"] \\
& \mathrm{Cat}_G &
\end{tikzcd}$$

The image of $(-)^{I-\sqcup}$ is spanned by the I-symmetric monoidal ∞-categories whose I-indexed tensor products are indexed coproducts and the image of $(-)^{I-\times}$ is spanned by those whose I-indexed tensor products are indexed products.

We refer to I-symmetric monoidal ∞-categories of the form $\mathcal{C}^{I-\times}$ as *cartesian*, and $\mathcal{C}^{I-\sqcup}$ *cocartesian*. In Corollary 1.19, we go on to characterize the *∞-category of I-commutative monoids* in $\mathcal{C}$ a complete ∞-category as an ∞-category of I-commutative algebras, integrating Perspectives 1 to 3:

$$\mathrm{CMon}_I(\mathcal{C}) \simeq \mathrm{CAlg}_I(\mathcal{C}^{I-\times}).$$

In [Ste24a], we verified under an equivariant distributivity assumption that $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is cartesian when $\mathcal{C}$ is, using a monadicity result of [NS22]. Following this, in Section 3.1 we show that I-indexed tensor products in $\underline{\mathrm{CAlg}}_I^{\otimes}\mathcal{C}$ are indexed coproducts (i.e. its underlying I-symmetric monoidal ∞-category is *cocartesian*) and that this completely characterizes $\mathcal{N}_{I\infty}^{\otimes}$. The heart of our stragegy will use the explicit monadic description of [Ste24a] to reduce this to the case $\mathcal{C}^{\otimes} \simeq \underline{\mathcal{S}}_G^{G-\times}$ is the *cartesian G-symmetric monoidal ∞-category of G-spaces*; in this case, we may easily see that the I-symmetric monoidal ∞-category $\underline{\mathrm{CAlg}}_I^{\otimes}(\underline{\mathcal{S}}_G^{G-\times}) \simeq \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_G)^{I-\times}$ is cocartesian, as its underlying G-∞-category is I-semiadditive by [CLL24, Thm B-C]. We conclude the following.

Theorem C. *Let $\mathcal{O}^{\otimes}$ be an almost essentially reduced G-operad. Then, the following conditions are equivalent.*

(a) The G-∞-category $\underline{\mathrm{Alg}}_{\mathcal{O}}\underline{\mathcal{S}}_G$ is $A\mathcal{O}$-semiadditive.

(b) The unique map $\mathcal{O}^{\otimes} \to \mathcal{N}_{A\mathcal{O}\infty}^{\otimes}$ is an equivalence.

Furthermore, for all aE-unital weak indexing categories I and I-symmetric monoidal ∞-categories $\mathcal{C}^{\otimes}$, the I-symmetric monoidal ∞-category $\underline{\mathrm{CAlg}}_I^{\otimes}\mathcal{C}$ is cocartesian.

For the following theorem, we say that an I-operad $\mathcal{O}^{\otimes}$ is *reduced* if, for all $S \in \mathbb{F}_H$ which is empty or contractible, the unique map $\mathcal{O}^{\otimes} \to \mathcal{N}_{I\infty}$ induces an equivalence

$$\mathcal{O}(S) \simeq \mathcal{N}_{I\infty}(S)$$

(c.f. Eq. (5)). We completely characterize algebras in cocartesian I-symmetric monoidal categories in Theorem 3.4, and from this Theorem C entirely characterizes the tensor products of reduced I-operads with $\mathcal{N}_{I\infty}^{\otimes}$ in the almost essentially reduced setting.

Corollary D. *$\mathcal{N}_{I\infty}^{\otimes} \overset{BV}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ is a weak $\mathcal{N}_\infty$-operad if and only if I is almost essentially unital. In this case, if $\mathcal{O}^{\otimes}$ is a reduced I-operad, then the unique map*

$$\mathcal{O}^{\otimes} \otimes \mathcal{N}_{I\infty}^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$$

is an equivalence.

In particular, this implies that whenever I is almost unital (i.e. almost essentially unital and one-color), there exists a map $\mathrm{triv}_G^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$ witnessing $\mathcal{N}_{I\infty}^{\otimes}$ as an idempotent algebra in Op_G. We verified in [Ste24a] that $\mathrm{Env}\colon \mathrm{Op}_G \to \mathrm{Cat}_G^{\otimes}$ is compatible with the unit and tensor products under the mode symmetric monoidal structure on $\mathrm{Cat}_G^{\otimes}$; this yields an idempotent algebra structure on $\underline{\mathbb{F}}_G^{G-\sqcup} = \mathrm{Env}(\mathrm{Comm}_G)$, and hence a symmetric monoidal structure on $\underline{\mathrm{Cat}}^{\otimes}_{G,/\underline{\mathbb{F}}_G^{G-\sqcup}}$. We acquire an equivariantization of a modification of [BS24a].

Corollary E. *There exists a unique symmetric monoidal structure $\underline{\mathrm{Op}}_G^{\otimes}$ on $\underline{\mathrm{Op}}_G$ attaining a (necessarily unique) symmetric monoidal structure on the fully faithful G-functor*

$$\mathrm{Env}^{/\underline{\mathbb{F}}_G^{G-\sqcup}} : \underline{\mathrm{Op}}_G^{\otimes} \to \underline{\mathrm{Cat}}^{\otimes-\mathrm{mode}}_{G,/\underline{\mathbb{F}}_G^{G-\sqcup}}$$

of [BHS22; NS22]; the tensor product of this structure is $\overset{BV}{\otimes}$ and the H-unit is $\mathrm{triv}_H^{\otimes}$.

Idempotent algebras correspond with smashing localizations, i.e. they classifiy $\otimes$-absorptive properties [HA, § 4.8.2]; in view of Corollary D, when $I \le J$ are almost unital, we would like to characterize the smashing localization that $\mathcal{N}_{I\infty}^{\otimes}$ induces on $\mathrm{Op}_J^{\mathrm{red}}$ using the adjunction $- \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes} \dashv \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(-)$. Namely, in Section 2.2, we construct a right adjoint to the natural inclusion $E_I^J : \mathrm{Op}_I \to \mathrm{Op}_J$, called the *$I$-borelification* Bor_I^J and note that the I-indexed tensor products in $\mathcal{C}^{\otimes}$ and $\mathrm{Bor}_I^J \mathcal{C}^{\otimes}$ agree for all $\mathcal{C}^{\otimes} \in \mathrm{Cat}_J^{\otimes}$; thus, in Theorem 3.7, we conclude that the smashing localization corresponding with $\mathcal{N}_{I\infty}^{\otimes} \in \mathrm{Op}_J^{\mathrm{red}}$ classifies the property of *having commutative Borel I-type*:

$$\begin{aligned}
\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{O}^{\otimes} \quad &\iff \quad \mathrm{Bor}_I^J \mathcal{O}^{\otimes} \simeq \mathcal{N}_{I\infty}^{\otimes}, \\
&\iff \quad \forall \mathcal{C}^{\otimes} \in \mathrm{Cat}_J^{\otimes}, \quad \forall S \in \mathbb{F}_{I,V}, \quad \coprod_U^S \simeq \bigotimes_U^S : \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})_S \to \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})_V, \\
&\iff \quad \underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_G) \text{ is } I\text{-semiadditive.}
\end{aligned}$$

Tensor products of idempotents algebras are themselves idempotent algebras, and they classify the conjunction of the properties classified by their factors [CSY20, Prop 5.1.8]. We leverage this to completely characterize indexed tensor products of almost essentially unital weak $\mathcal{N}_\infty$-operads, affirming Conjecture 6.27 of [BH15].

Theorem F. *The functor $\mathcal{N}_{(-)\infty}^{\otimes} : \mathrm{wIndex}_G \to \mathrm{Op}_G$ restricts to a fully faithful symmetric monoidal G-right adjoint*

$$\begin{tikzcd}
\underline{\mathrm{wIndex}}_G^{a\mathrm{Euni}} \arrow[r, hook, bend right, "\mathcal{N}_{(-)\infty}^{\otimes}"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Op}}_G^{a\mathrm{Euni}} \arrow[l, bend right, "A"']
\end{tikzcd}$$

Furthermore, the resulting tensor product of weak $\mathcal{N}_\infty$-operads is computed by the Borelified join

$$\mathcal{N}_I^{\otimes} \overset{BV}{\otimes} \mathcal{N}_J^{\otimes} \simeq \mathcal{N}^{\otimes}_{\mathrm{Bor}^G_{c(I\cap J)}(I \vee J)}.$$

Hence when I, J are unital weak indexing categories and $\mathcal{C}^{\otimes}$ is an $I \vee J$-symmetric monoidal ∞-category, there is a canonical equivalence of $I \vee J$-symmetric monoidal ∞-categories

$$\underline{\mathrm{CAlg}}_I^{\otimes} \underline{\mathrm{CAlg}}_J^{\otimes}(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_{I \vee J}^{\otimes}(\mathcal{C}).$$

This additionally implies that $\mathcal{N}_{(-)\infty}^{\otimes}$ is compatible with products, restriction, and coinduction; hence norms of I-commutative algebras are $\mathrm{CoInd}_H^G I$-commutative algebras.

Remark. The reader interesting in computing tensor products of G-operads may benefit from reading the combinatorial characterization of joins of weak indexing systems in terms of *closures* in [Ste24b]; there, we prove that the join of weak indexing systems $\mathbb{F}_I \vee \mathbb{F}_J$ is computed by closing the union $\mathbb{F}_I \cup \mathbb{F}_J$ under iterated I and J-indexed coproducts. ◁

We conclude an infinitary case of an equivariant homotopical lift of Dunn's additivity theorem [Dun88].

Corollary G (Equivariant infinitary Dunn additivity)**.** *Let V and W be real orthogonal G-representations satisfying at least one of the following conditions:*

(a) V, W are weak G-universes, or
(b) the canonical map $\mathbb{E}_V^{\otimes} \to \mathbb{E}_{V \oplus W}^{\otimes}$ is an equivalence.

Then the canonical map

$$\mathbb{E}_V^{\otimes} \overset{BV}{\otimes} \mathbb{E}_W^{\otimes} \to \mathbb{E}_{V \oplus W}^{\otimes}$$

is an equivalence; equivalently, for any G-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, the pullback functors

$$\mathrm{Alg}_{\mathbb{E}_V} \underline{\mathrm{Alg}}_{\mathbb{E}_W}^{\otimes}(\mathcal{C}) \leftarrow \mathrm{Alg}_{\mathbb{E}_{V \oplus W}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_W} \underline{\mathrm{Alg}}_{\mathbb{E}_V}^{\otimes}(\mathcal{C})$$

are equivalences.

For instance, we may set $\mathcal{C}^{\otimes} \coloneqq \underline{\mathcal{S}}_G^{G-\times}$ to recover a result about $\mathbb{E}_V$-spaces (which are not necessarily grouplike), or we may set $\mathcal{C} \coloneqq \underline{\mathrm{Sp}}_G^{\otimes}$ to conclude additivity of $\mathbb{E}_V$-ring spectra

$$\mathrm{Alg}_{\mathbb{E}_V} \underline{\mathrm{Alg}}_{\mathbb{E}_W}^{\otimes}(\underline{\mathrm{Sp}}_G) \simeq \mathrm{Alg}_{\mathbb{E}_{V \oplus W}}(\underline{\mathrm{Sp}}_G) \simeq \mathrm{Alg}_{\mathbb{E}_W} \underline{\mathrm{Alg}}_{\mathbb{E}_V}^{\otimes}(\underline{\mathrm{Sp}}_G)$$

under either of the assumptions in Corollary G.

Remark. In the thesis [Szc23], an ostensibly-similar result to Corollary G is proved: given D_V the *little Disks graph G-operad*, Szczesny constructs a non-homotopical Boardman-Vogt tensor product $\otimes$ and a canonical map $D_V \otimes D_W \to D_{V \oplus W}$, which he shows to be a weak equivalence of graph G-operads in [Szc23, Thm 4.5.5]. Neither this result nor Corollary G imply each other.

On one hand, Szczesny's result concerns a tensor product with no known homotopical properties, so it is incomparable with results concerning ∞-categories of algebras satisfying *homotopical* universal properties. On the other hand, while Corollary G is homotopical, it only concerns cases where at least one of the representations induces I-symmetric monoidal ∞-categories of algebras whose indexed tensor products are indexed coproducts; this property will not be satisfied for any nontrivial indexed tensor products in the finite-dimensional case, so the range of representations in Szczesny's result is significantly larger. ◁

Along the way, we acquire various corollaries in equivariant higher algebra. For instance, in Section 4.3 we use Corollary G to define iterated Real topological Hochschild homology for $\mathbb{E}_V$-algebras whenever V admits an $\infty\sigma$ summand, and we express it as a S^{σ}-indexed colimit when $V = \infty\rho$. Additionally, Corollary 4.6 uses Theorem F to show that almost essentially reduced k-connected G-operads are closed under tensor products.

Notation and conventions. We assume that the reader is familiar with the technology of higher category theory and higher algebra as developed in [HTT] and [HA, § 2-3], though we encourage the reader to engage with such technologies via a "big picture" perspective akin to that of [Gep19, § 1-2] and [Hau23, § 1-3]. We additionally assume that the reader is familiar with *parameterized* higher category theory over an ∞-category as developed in [Sha22; Sha23]; the material reviewed in the prequel [Ste24a, § 1] will be enough.

Throughout this paper, we frequently describe conditions which may be satisfied by objects parameterized over some ∞-category $\mathcal{T}$. If P is a property, in the instance where there exists Borelification adjunctions

$$E_{\mathcal{F}}^{\mathcal{T}} : \mathcal{C}_{\mathcal{F}} \rightleftarrows \mathcal{C}_{\mathcal{T}} : \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}}$$

along family inclusions $\mathcal{F} \subset \mathcal{T}$, we say that $X \in \mathcal{C}_{\mathcal{T}}$ is E-P when there exists some $\overline{X} \in \mathcal{C}_{\mathcal{F}}$ which is P such that $X \simeq E_{\mathcal{F}}^{\mathcal{T}} \overline{X}$. We say that X is *almost* E-P (or aE-P) if $\mathcal{C}_{\mathcal{F}}$ has a terminal object $*_{\mathcal{F}}$ for all $\mathcal{F}$, and there is a pushout expression

$$X \simeq *_{\mathcal{F}'} \sqcup_{*_{\mathcal{F}}} *_{\mathcal{F}'}$$

for some $\mathcal{F}' \subset \mathcal{F}$; we say that X is *almost* P (or a-P) if it's almost E-P and $\mathcal{F}' = \mathcal{T}$ in the above.

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1. Preliminaries and examples

We begin in Section 1.1 by recalling results of [CLL24; Nar16; NS22; Ste24a; Ste24b] concerning the theory of I-commutative monoids and I-symmetric monoidal ∞-categories. Moving on, in Section 1.2 we recall results of [NS22; Ste24a] concerning $\mathcal{T}$-operads; in either case, all reviewed information was used in the preceding article [Ste24a]. We finish the section in Section 1.3 with a tour through the gamut of existing examples of I-symmetric monoidal ∞-categories, including summarizing the results of Appendix A in the cartesian case.

1.1. Recollections on I-commutative monoids and I-symmetric monoidal ∞-categories.

1.1.1. *Weak indexing systems and semiadditivity.* We will use the following machinery of [Ste24b].

Definition 1.1. A *$\mathcal{T}$-weak indexing category* is a subcategory $I \subset \mathbb{F}_{\mathcal{T}}$ satisfying the following conditions:

(IC-a) (restrictions) I is stable under arbitrary pullbacks in $\underline{\mathbb{F}}_{\mathcal{T}}$;
(IC-b) (segal condition) $T \to S$ and $T' \to S$ are both in I if and only if $T \sqcup T' \to S \sqcup S'$ is in I; and
(IC-c) ($\Sigma_{\mathcal{T}}$-action) if $S \in I$, then all automorphisms of S are in I.

A *$\mathcal{T}$-weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ satisfying the following conditions:

(IS-a) whenever the V-value $\mathbb{F}_{I,V} \coloneqq (\underline{\mathbb{F}}_I)_V$ is nonempty, we have $*_V \in \underline{\mathbb{F}}_{I,V}$; and
(IS-b) $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is closed under $\underline{\mathbb{F}}_I$-indexed coproducts. ◁

We say that a $\mathcal{T}$-weak indexing system $\underline{\mathbb{F}}_I$:

(i) has one color if for all $V \in \mathcal{T}$, we have $\mathbb{F}_{I,V} \neq \varnothing$;
(ii) is almost essentially unital (or aE-unital) if $\underline{\mathbb{F}}_I$ has a non-contractible V-set, $\varnothing_V \in \mathbb{F}_{I,V}$;
(iii) is unital if $\varnothing_V \in \mathbb{F}_{I,V}$ for all $V \in \mathcal{T}$;
(iv) is an *indexing system* if the subcategory $\underline{\mathbb{F}}_{I,V} \subset \mathbb{F}_V$ is closed under finite coproducts for all $V \in \mathcal{T}$.

These occupy embedded sub-posets

$$\mathrm{Index}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}.$$

We denote the *I-admissible V-sets* by

$$\mathbb{F}_{I,V} \coloneqq \left\{ S \in \mathbb{F}_{I,V} \mid \mathrm{Ind}_V^{\mathcal{T}} S \to V \in I \right\} \subset \mathbb{F}_V;$$

these assemble into a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$. In [Ste24b, Thm A] we prove the following and express the conditions of Definition 1.1 in the language of weak indexing categories.

Proposition 1.2. *The assignment $I \mapsto \underline{\mathbb{F}}_I$ implements an equivalence between the posets of $\mathcal{T}$-weak indexing categories and $\mathcal{T}$-weak indexing systems.*

One reason for this is *indexed semiadditivity*. A $\mathcal{T}$-∞-category is said to be *V-pointed* if $\mathcal{C}_U$ is a pointed ∞-category for all $U \to V$. When $S \in \mathbb{F}_V$ is a finite V-set and $\mathcal{C}$ a V-pointed $\mathcal{T}$-∞-category which admits S-indexed products and coproducts, Nardin [Nar16] defined a *norm natural transformation*

$$\mathrm{Nm}_s \colon \coprod_U^S (-) \implies \prod_U^S (-).$$

We say that S is *$\mathcal{C}$-ambidextrous* if $\mathcal{C}$ is V-pointed and Nm_S is an equivalence; given $\underline{\mathbb{F}}_I$ a weak indexing system, we say that $\mathcal{C}$ is *I-semiadditive* if S is $\mathcal{C}$-ambidextrous for all $S \in \underline{\mathbb{F}}_I$. In [Ste24a] we proved that the collection of $\mathcal{C}$-ambidextrous finite V-sets form a weak indexing system and concluded the following important observation.

Proposition 1.3 ([Ste24a])**.** *Let $\vee$ denote the join in* $\mathrm{wIndexCat}_{\mathcal{T}}$*. Then, $\mathcal{C}$ is I-semiadditive and J-semiadditive if and only if $\mathcal{C}$ is $I \vee J$-semiadditive.*

1.1.2. *I-commutative monoids.* In [Bar14], the notion of *adequate* triple was defined, consisting of triples $(\mathcal{C}, \mathcal{C}_b, \mathcal{C}_f)$ with $\mathcal{C}_f, \mathcal{C}_B \subset \mathcal{C}$ a pair of core-preserving wide subcategories satisfying pullback-stability and distributivity conditions; if I is a weak indexing category, then $(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I)$ is an adequate triple.

Adequate triples form a full subcategory $\mathrm{Trip}^{\mathrm{Adeq}} \subset \mathrm{Fun}(\bullet \to \bullet \leftarrow \bullet, \mathrm{Cat})$; [Bar14] constructed a functor

$$\mathrm{Span}_{-,-}(-) \colon \mathrm{Trip}^{\mathrm{Adeq}} \to \mathrm{Cat},$$

called the *effective Burnside category*. In the case that $c(I)$ is a 1-category (e.g. $\mathcal{T}$ has a terminal object), $\mathbb{F}_{c(I)}$ is a 1-category, so the effective Burnside category

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \coloneqq \mathrm{Span}_{\mathbb{F}_{c(I)}, I}(\mathbb{F}_{c(I)})$$

is a $(2,1)$-category with objects agreeing with $\mathbb{F}_{c(I)}$, morphisms the spans $X \leftarrow R \xrightarrow{f} Y$ with f in I, 2-cells the isomorphisms of spans, and composition of morphisms computed by pullbacks in $\mathbb{F}_{c(I)}$ (which are guaranteed to be morphisms in $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ by pullback-stability of I).

Much of the technical work of [Bar14; BGS20] has been extended by [HHLN23], so we generally refer the reader there. At any rate, we recall this in order to define *homotopical incomplete Mackey functors for I*, which we call *I-commutative monoids.*

Definition 1.4. If $\mathcal{C}$ is an ∞-category with finite products, then an *I-commutative monoid in $\mathcal{C}$* is a product-preserving functor $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathcal{C}$ More generally, if $\mathcal{D}$ is a $\mathcal{T}$-∞-category with I-indexed products, then an *I-commutative monoid in $\mathcal{D}$* is an I-indexed product-preserving functor $\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}}) \to \mathcal{D}$. We write

$$\begin{aligned}
\underline{\mathrm{CMon}}(\mathcal{D}) &\coloneqq \underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\times}(\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}}), \mathcal{D}) \\
\mathrm{CMon}(\mathcal{D}) &\coloneqq \Gamma^{\mathcal{T}} \underline{\mathrm{CMon}}(\mathcal{D}) \\
\underline{\mathrm{CMon}}(\mathcal{C}) &\coloneqq \underline{\mathrm{CMon}}(\mathrm{Coeff}^{\mathcal{T}} \mathcal{C}) \\
\mathrm{CMon}(\mathcal{C}) &\coloneqq \Gamma^{\mathcal{T}} \underline{\mathrm{CMon}}(\mathcal{C}).
\end{aligned}$$ ◁

An important result of Cnossen-Lenz-Linskens resolves the notational clash.

Proposition 1.5 ([CLL24, Thm C])**.** *When $\mathcal{C}$ is an ∞-category, restriction furnishes an equivalence*

$$\mathrm{CMon}(\mathcal{C}) \simeq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C}),$$

and more generally, we have $\underline{\mathrm{CMon}}(\mathcal{C})_V \simeq \mathrm{Fun}_V^{\times}(\mathrm{Span}_I(\mathbb{F}_V), \mathcal{C})$ with restriction given by pullback along $\mathrm{Span}_I(\mathbb{F}_V) \to \mathrm{Span}_I(\mathbb{F}_W)$.

Let I be a one-object weak indexing category and let $\mathrm{Cat}_{\mathcal{T}}^{I-\times} \subset \mathrm{Cat}_{\mathcal{T}}$ be the (non-full) subcategory whose objects are $\mathcal{T}$-categories admitting I-indexed products and functors preserving I-indexed products. Let $\mathrm{Cat}_I^{I-\oplus} \subset \mathrm{Cat}_{\mathcal{T}}^{I-\times}$ be the full subcategory spanned by I-semiadditive $\mathcal{T}$-∞-categories. The following result is fundamental in the theory of equivariant semiadditivity and equivariant higher algebra.

Theorem 1.6 ([CLL24, Thm B])**.** $\mathrm{Cat}_{\mathcal{T}}^{I-\oplus} \subset \mathrm{Cat}_{\mathcal{T}}^{I-\times}$ *is a* localizing *subcategory with localization functor* $\underline{\mathrm{CMon}}(-)$*.*

In addition, we verified the following corollary to [CH21, Cor 8.2].

Lemma 1.7. *If $\mathcal{C}$ is an ∞-category and I a one-object weak indexing category, then the* underlying coefficient system *functor* $\mathrm{CMon}_I(\mathcal{C}) \to \Gamma^{\mathcal{T}}\mathcal{C}$ *is conservative; in particular, if a $\mathcal{T}$-symmetric monoidal functor's underlying $\mathcal{T}$-functor is an equivalence, then it is a $\mathcal{T}$-symmetric monoidal equivalence.*

1.2. Recollections on $\mathcal{T}$-operads.

1.2.1. *$\mathcal{T}$-operads and $\mathcal{T}$-symmetric monoidal ∞-categories.* In [Ste24a], we made the following definition.

Definition 1.8. A *$\mathcal{T}$-operad* is a functor $\pi : \mathcal{O}^{\otimes} \to \mathsf{Span}(\mathbb{F}_{\mathcal{T}})$ satisfying the following conditions.

(a) $\mathcal{O}^{\otimes}$ has π-cocartesian lifts for backwards maps in $\mathsf{Span}(\mathbb{F}_{\mathcal{T}})$;
(b) (Segal condition for colors) for every $S \in \mathbb{F}_{\mathcal{T}}$, cocartesian transport along the π-cocartesian lifts lying over the inclusions $(S \leftarrow U = U \mid U \in \mathrm{Orb}(S))$ together induce an equivalence

$$\mathcal{O}_S \simeq \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}_U;$$

(c) (Segal condition for multimorphisms) for every map of orbits $T \to S$ in I and pair of objects $(\mathbf{C}, \mathbf{D}) \in \mathcal{O}_T \times \mathcal{O}_U$, postcomposition with the π-cocartesian lifts $\mathbf{D} \to D_U$ lying over the inclusions $(S \leftarrow U = U \mid U \in \mathrm{Orb}(S))$ induces an equivalence

$$\mathrm{Map}_{\mathcal{O}^{\otimes}}^{T \to S}(\mathbf{C}, \mathbf{D}) \simeq \prod_{U \in \mathrm{Orb}(S)} \mathrm{Map}_{\mathcal{O}^{\otimes}}^{T \leftarrow T_U \to U}(\mathbf{C}, D_U).$$

where $T_U := T \times_S U$.

The corresponding category is a full subcategory $\mathrm{Op}_{\mathcal{T}} \to \mathrm{Cat}^{\mathrm{int-cocart}}_{/\mathsf{Span}(\mathbb{F}_{\mathcal{T}})}$; that is, a morphism of $\mathcal{T}$-operads is a functor $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ sending $\pi_{\mathcal{O}}$-cocartesian morphisms to $\pi_{\mathcal{P}}$-cocartesian morphisms. We also call these *$\mathcal{O}$-algebras in $\mathcal{P}$* and we let

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) := \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathsf{Span}(\mathbb{F}_{\mathcal{T}}}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}) \subset \mathrm{Fun}_{/\mathsf{Span}(\mathbb{F}_{\mathcal{T}})}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes})$$

be the full subcategory spanned by $\mathcal{O}$-algebras in $\mathcal{P}$. ◁

Furthermore, let $\underline{\mathbb{F}}_{\mathcal{T},*} := \mathsf{Span}_{\text{summand inclusion},I}(\underline{\mathbb{F}}_{\mathcal{T}})$. There is an associated map

$$\varphi\colon \mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*} \to \mathsf{Span}_I(\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T}}) \xrightarrow{s} \mathsf{Span}_I(\mathbb{F}_{\mathcal{T}}).$$

Let $\mathrm{Op}_{\mathcal{T},\infty}$ be the $\mathcal{T}$-∞-operads of [NS22, Def 2.1.7]. In [Ste24a] we verified that the argument of [BHS22, § 5.2] lifts to show that pullback along φ furnishes an equivalence $\mathrm{Op}_{\mathcal{T}} \xrightarrow{\sim} \mathrm{Op}_{\mathcal{T},\infty}$. By doing so, we acquired a *conservative* functor

$$\mathrm{Tot}_{\mathcal{T}}\colon \mathrm{Op}_{\mathcal{T}} \simeq \mathrm{Op}_{\mathcal{T},\infty} \subset \mathrm{Cat}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T},*}} \xrightarrow{\mathrm{Cat}} \mathcal{T}$$

taking a $\mathcal{T}$-operad to the total $\mathcal{T}$-∞-category of the pullback fibration of $\mathcal{T}$-∞-categories $\mathrm{Tot}_{\mathcal{T}}\varphi^*\mathcal{O}^{\otimes} \to \underline{\mathbb{F}}_{\mathcal{T},*}$.

Furthermore, we noted that a cocartesian fibration $\pi : \mathcal{O}^{\otimes} \to \mathsf{Span}(\mathbb{F}_{\mathcal{T}})$ is an I-operad if and only if its unstraightening $\mathsf{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathsf{Cat}$ is an I-symmetric monoidal category. [BHS22] and [NS22] thus independently construct an adjunction

$$\begin{tikzcd} \mathrm{Op}_{\mathcal{T}} \arrow[r, bend left, "\mathrm{Env}"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}^{\otimes}_{\mathcal{T}} \arrow[l, bend left, "U"] \end{tikzcd}$$

In [Ste24a] we computed $\mathrm{Env}(\mathrm{Comm}_{\mathcal{T}}) \simeq \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$, i.e. it is the $\mathcal{T}$-∞-category of finite $\mathcal{T}$-sets with indexed tensor products given by indexed coproducts; [BHS22, Prop 4.21] then verifies that the *sliced* left adjoint $\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Cat}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$ is fully faithful and identify its image, i.e. $\mathrm{Op}_{\mathcal{T}}$ is a colocalizing subcategory of $\mathcal{T}$-symmetric monoidal ∞-categories over $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$ consisting of the *equifibrations.*

1.2.2. *The underlying $\mathcal{T}$-symmetric sequence.* From there, we defined an *underlying $\mathcal{T}$-symmetric sequence* functor and proved the following.

Theorem 1.9 ([Ste24a, Thm A]). *The underlying $\mathcal{T}$-symmetric sequence functor* $\mathrm{sseq}\colon \mathrm{Op}_{\mathcal{T}}^{\leq oc} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ *is monadic.*

In particular, it is conservative. The V-objects in $\underline{\Sigma}_{\mathcal{T}} \simeq \underline{\mathbb{F}}_{\mathcal{T}}^{\simeq}$ are finite V-sets; given $S \in \Sigma_V \simeq \mathbb{F}_V^{\simeq}$, writing $\mathcal{O}(S)$ for $\mathrm{sseq}\,\mathcal{O}^{\otimes}(S)$, we remember this as saying that at most one color $\mathcal{T}$-operads are identified conservatively by their S-ary structure spaces. We went on to compute the *free $\mathcal{O}$-algebra monad*; this sends $X \in \Gamma^{\mathcal{T}}\mathcal{C}$ to the $\mathcal{T}$-object $T_{\mathcal{O}}X$ with

$$(T_{\mathcal{O}}X)_V \simeq \coprod_{S \in \mathbb{F}_V} \left(\mathcal{O}(S) \cdot X_V^{\otimes S}\right)_{h\mathrm{Aut}_V(S)}.$$

In particular, given $S \in \mathbb{F}_V$, in [Ste24a] we found a natural splitting $\mathcal{O}(S) \oplus J \simeq (T_{\mathcal{O}}S)_V$. Via a multiple-color version of this argument, we concluded the following.

Proposition 1.10 ([Ste24a]). *A map of $\mathcal{T}$-operads $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is an equivalence if and only if:*

(a) the underlying $\mathcal{T}$-functor $U(\varphi)\colon \mathcal{O} \to \mathcal{P}$ is $\mathcal{T}$-essentially surjective, and
(b) the pullback functor $\varphi^*\colon \mathrm{Alg}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \to \mathrm{Alg}_{\mathcal{O}}\left(\underline{\mathcal{S}}_{\mathcal{T}}\right)$ is an equivalence.

Given a map $U \to V$ in $\mathcal{T}$ and a finite V-set $S \in \mathbb{F}_V$, in [Ste24a] we defined used cocartesian transport to define a *restriction map*

$$\mathcal{O}(S) \to \mathcal{O}(\mathrm{Res}_U^V S) \tag{7}$$

Furthermore, given a finite S-set T, writing $T_U := T \times_S U$, we used composition to define a map

$$\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T_U) \to \mathcal{O}(T) \tag{8}$$

Last, we used cocartesian transport to define a Σ *action*

$$\rho_S : \mathrm{Aut}_V(S) \times \mathcal{O}(S) \longrightarrow \mathcal{O}(S). \tag{9}$$

1.2.3. *Rudiments of weak $\mathcal{N}_\infty$-operads.* In [Ste24a], we constructed a family of $\mathcal{T}$-operads:

Proposition 1.11 ([Ste24a]). *Let $I \subset \mathbb{F}_{\mathcal{T}}$ be a core-full and pullback-stable subcategory. Then,* $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ *presents a $\mathcal{T}$-operad if and only if I is a weak indexing system.*

These are called weak $\mathcal{N}_\infty$-operads; in the case that I is an indexing category, these are called $\mathcal{N}_\infty$-operads. In [Ste24a], we recognized the variant of Definition 1.8 fibered over $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$, called *$I$-operads*, as the slice category $\mathrm{Op}_I \simeq \mathrm{Op}_{\mathcal{T},/\mathcal{N}_{I\infty}^{\otimes}}$; in this paper, we will show that the unslicing functor $\mathrm{Op}_{\mathcal{T},/\mathcal{N}_{I\infty}^{\otimes}} \to \mathrm{Op}_{\mathcal{T}}$ is fully faithful, so *being an I-operad* is purely a condition of a $\mathcal{T}$-operad.

In particular, the notion of weak $\mathcal{N}_\infty$-operads yields a direct construction of a $\mathcal{T}$-operad $\mathrm{triv}_{\mathcal{T}}^{\otimes} = \mathcal{N}_{\mathbb{F}_{\mathcal{T}}^{\simeq}}^{\otimes}$ first defined in [NS22]; they showed that $\mathrm{triv}_{\mathcal{T}}^{\otimes}$ is free on a single color.

Proposition 1.12 ([NS22, Prop 2.5.2]). *The sliced functor* $U_{/\mathrm{triv}_{\mathcal{T}}}\colon \mathrm{Op}_{\mathcal{T},/\mathrm{triv}_{\mathcal{T}}^{\otimes}} \to \mathrm{Cat}_{\mathcal{T},/*_{\mathcal{T}}} \simeq \mathrm{Cat}_{\mathcal{T}}$ *is an equivalence; writing* $\mathrm{triv}_{\mathcal{T}}(-)$ *for* $U_{/\mathrm{triv}_{\mathcal{T}}}^{-1}$, *the composite functor*

$$\mathrm{Cat}_{\mathcal{T}} \xrightarrow{\mathrm{triv}_{\mathcal{T}}(-)} \mathrm{Op}_{\mathcal{T},/\mathrm{triv}_{\mathcal{T}}^{\otimes}} \to \mathrm{Op}_{\mathcal{T}}$$

is left adjoint to U and has algebras

$$\mathrm{Alg}_{\mathrm{triv}_{\mathcal{T}}(\mathcal{C})}(\mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}, \mathcal{D}).$$

This fully faithfully embeds $\mathcal{T}$-categories into $\mathcal{T}$-operads as *trivial $\mathcal{T}$-operads*, which are identified by the *property* that they possess a (necessarily unique) map to $\mathrm{triv}_{\mathcal{T}}^{\otimes}$.

1.2.4. *The Boardman-Vogt tensor product.* In [Ste24a, Thm C], we equipped $\mathrm{Op}_{\mathcal{T}}$ with a closed *Boardman-Vogt tensor product*

$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} := L_{\mathrm{Op}}\left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \longrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \times \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \xrightarrow{\wedge} \mathrm{Span}(\mathbb{F}_{\mathcal{T}})\right),$$

where $L_{\mathrm{Op}} \colon \mathrm{Cat}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})} \to \mathrm{Op}_{\mathcal{T}}$ is the left adjoint to the inclusion [BHS22, Cor 4.2.3]. Its internal hom is denoted $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P})$; its underlying $\mathcal{T}$-∞-category is denoted $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P})$, and it has values

$$\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P})_V \simeq \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{O}}(\mathrm{Res}^{\mathcal{T}}_V \mathcal{P}),$$

where given $V \in \mathcal{T}$, the morphism $\mathrm{Res}^{\mathcal{T}}_V \colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Op}_V := \mathrm{Op}_{\mathcal{T}_{/V}}$ is given by pullback along the morphism of algebraic patterns $\mathrm{Span}(\mathbb{F}_V) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$. We verified several properties in that paper; for instance, $\underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C})$ is an I-symmetric monoidal ∞-category when $\mathcal{C}$ is, functorially for I-symmetric monoidal maps in $\mathcal{C}^{\otimes}$ and $\mathcal{T}$-operad maps in $\mathcal{P}^{\otimes}$. An important one is the following.

Proposition 1.13 ([Ste24a, Thm C.(3)])**.** *$\mathrm{triv}^{\otimes}_{\mathcal{T}}$ is the $\overset{BV}{\otimes}$-unit; hence there is an equivalence of $\mathcal{T}$-operads*

$$\underline{\mathrm{Alg}}_{\mathrm{triv}_{\mathcal{T}}}(\mathcal{O}) \simeq \mathcal{O}^{\otimes}$$

We additionally characterized the interaction of the Boardman-Vogt tensor product and unit.

Proposition 1.14 ([Ste24a, Thm C.(7)])**.** *The $\mathcal{T}$-symmetric monoidal envelope intertwines the mode symmetric monoidal structure with Boardman-Vogt tensor products, i.e.*

$$\mathrm{Env}\left(\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}\right) \simeq \mathrm{Env}\left(\mathcal{O}^{\otimes}\right) \otimes^{\mathrm{Mode}} \mathrm{Env}\left(\mathcal{P}^{\otimes}\right).$$

Furthermore, $\mathrm{Env}\left(\mathrm{triv}^{\otimes}_{\mathcal{T}}\right)$ is the $\otimes^{\mathrm{Mode}}$-unit.

1.3. **Examples of I-symmetric monoidal ∞-categories.** We now review several examples of I-symmetric monoidal ∞-categories of interest.

1.3.1. *(Co)cartesian I-symmetric monoidal ∞-categories.* Fix I a unital weak indexing system in the sense of [Ste24b]. Denote by $\mathrm{Cat}^{I-\sqcup}_I, \mathrm{Cat}^{I-\times}_I \subset \mathrm{Cat}_{\mathcal{T}}$ the non-full subcategories with objects given by $\mathcal{T}$-∞-categories attaining I-indexed coproducts (resp. products) and with morphisms given by $\mathcal{T}$-functors which preserve I-indexed coproducts (products). In Appendix A, we prove the following.

Theorem B'. *There are fully faithful embeddings $(-)^{I-\sqcup}, (-)^{I-\times}$ making the following commute:*

$$\begin{tikzcd}
\mathrm{Cat}^{I-\sqcup}_I \arrow[r, hook, "(-)^{I-\sqcup}"] \arrow[dr, "U"'] & \mathrm{Cat}^{\otimes}_I \arrow[d, dashed, "U"] & \mathrm{Cat}^{I-\times}_I \arrow[l, hook', "(-)^{I-\times}"'] \arrow[dl, "U"] \\
& \mathrm{Cat}_{\mathcal{T}} &
\end{tikzcd}$$

The image of $(-)^{I-\sqcup}$ is spanned by the I-symmetric monoidal ∞-categories whose I-admissible indexed tensor functors $\otimes^S : \mathcal{C}_S \to \mathcal{C}_V$ are left adjoint to the indexed diagonal $\Delta^S : \mathcal{C}_V \to \mathcal{C}_S$ (i.e. whose indexed tensor products are are indexed coproducts), and the image of $(-)^{I-\times}$ is spanned by those whose I-admissible indexed tensor functors $\otimes^S$ are right adjoint to Δ^S.

We call I-symmetric monoidal ∞-categories of the form $\mathcal{C}^{I-\sqcup}$ *cocartesian*, and $\mathcal{C}^{I-\times}$ *cartesian*. Before characterizing the algebras in these, we point out that these are often presentable.

Proposition 1.15. *Suppose $\mathcal{C}$ is a presentable ∞-category*

(1) *$\underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{C}$ is I-presentably symmetric monoidal under the cocartesian structure.*
(2) *If finite products in $\mathcal{C}$ commute with colimits separately in each variable (i.e. it is Cartesian closed), then $\underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{C}$ is I-presentably symmetric monoidal under the cartesian structure.*

Proof. It follows from Hilman's characterization of parameterized presentability [Hil24, Thm 6.1.2] that $\underline{\mathrm{Coeff}}^{\mathcal{T}}$ is presentable, so we're tasked with proving that the $\mathcal{T}$-symmetric monoidal structures are distributive. The first case is just commutativity of colimits with colimits, and the second is [NS22, Prop 3.2.5]. ☐

Additionally, I-indexed tensor products of algebras in cartesian I-symmetric monoidal ∞-categories are indexed products.

Proposition 1.16. $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}\left(\mathcal{C}^{I-\times}\right)$ *is a cartesian I-symmetric monoidal ∞-category.*

Proof. The criterion of Theorem B' were verified in [Ste24a]. ☐

We would like to interpret algebras in $\mathcal{C}^{I-\times}$ purely in terms of $\mathcal{C}$ using the following definition.

Definition 1.17. Fix $\mathcal{O}^{\otimes}$ an I-operad interpreted as a $\mathcal{T}$-∞-category over $\underline{\mathbb{F}}_{I,*}$ (see [Ste24a, Appendix A]) and let $\mathcal{C}$ be a $\mathcal{T}$-∞-category admitting I-indexed products. Then, an *$\mathcal{O}$-monoid in* $\mathcal{C}$ is a $\mathcal{T}$-functor $M : \mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathcal{C}$ satisfying the condition that, for each orbit $V \in \mathcal{T}$, each finite V-set $S \in \mathbb{F}_V$, and each S-tuple $X = (X_U) \in \mathcal{O}_S$, the canonical maps $M(X) \to \mathrm{CoInd}_U^V M(X_U)$ realize $M(X)$ as the indexed product

$$M(X) \simeq \prod_U^S M(X_U). \qquad \triangleleft$$

In Appendix A, we prove the following equivariant lift of [HA, Prop 2.4.2.5].

Proposition 1.18. *The postcomposition functor*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\times}) \to \mathrm{Fun}_{\mathcal{T}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}, \mathcal{C})$$

is fully faithful with image spanned by the $\mathcal{O}$-monoids.

In Section 2.1, we will see that I-operads form a full subcategory $\mathrm{Op}_I \subset \mathrm{Op}_{\mathcal{T}}$ with terminal object $\mathcal{N}_{I\infty}^{\otimes}$, so we set the notation $\underline{\mathrm{CAlg}}_I(\mathcal{C}) \coloneqq \underline{\mathrm{Alg}}_{\mathcal{N}_{I\infty}}(\mathcal{C})$ for the $\mathcal{T}$-∞-category of *I-commutative algebras in $\mathcal{C}$*. Of fundamental importance is the following corollary to Proposition 1.18, which interprets I-commutative monoids as *operad algebras*.

Corollary 1.19 ("CMon = CAlg")**.** *There is a canonical equivalence* $\underline{\mathrm{CMon}}_I(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_I(\mathcal{C}^{I-\times})$ *over $\mathcal{C}$.*

Proof. By Proposition 1.18, I-commutative algebras in $\mathcal{C}^{I-\times}$ are I-semiadditive functors $\underline{\mathbb{F}}_{I,*} \to \mathcal{C}$. Our proof is similar to that of [Nar16, Thm 6.5]; There is a pullback square over $\mathcal{C}$

$$\begin{tikzcd}
\mathrm{CMon}_I(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{CAlg}_I(\mathcal{C}^{I-\times}) \simeq \mathrm{Fun}^{I-\oplus}(\underline{\mathbb{F}}_{I,*}, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}})) \arrow[r] & \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{I-\oplus}(\underline{\mathbb{F}}_{I,*}, \underline{\mathcal{S}}_{\mathcal{T}}))
\end{tikzcd}$$

so it suffices to prove this in the case $\mathcal{C} = \underline{\mathcal{S}}_{\mathcal{T}}$. There, we simply compose equivalences as follows

$$\begin{tikzcd}
\mathrm{CMon}_I(\underline{\mathcal{S}}_{\mathcal{T}}) \arrow[rrr, "\sim"] \arrow[d, "1.5"'] & & & \mathrm{CAlg}_I(\mathcal{C}^{I-\times}) \\
\mathrm{CMon}_I(\mathcal{S}) \arrow[r] & \mathrm{Seg}_{\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}(\mathcal{S}) \arrow[r] & \mathrm{Seg}_{\underline{\mathbb{F}}_{I,*}} \arrow[r] & \mathrm{Fun}_{\mathcal{T}}^{I-\oplus}(\underline{\mathbb{F}}_{I,*}, \underline{\mathcal{S}}_{\mathcal{T}}) \arrow[u, "1.18"']
\end{tikzcd}$$

where each of the bottom arrows are shown to be equivalences in Appendix A of [Ste24a]. ☐

Remark 1.20. As with much of the rest of this subsection, Corollary 1.19 possesses an alternative strategy where both are shown to furnish the I-semiadditive closure, the latter using [CLL24, Thm B]. The above argument was chosen for brevity, as its requisite parts are also needed elsewhere. ◁

Remark 1.21. In the case $\mathcal{C} \simeq \underline{\mathcal{S}}_G$, the analogous result was recently proved in [Mar24] for the ∞-category of algebras over the *graph G-operads* corresponding with indexing systems. To the knowledge of the author, this is one of the first concrete indications that the genuine operadic nerve of [Bon19] may induce equivalences between ∞-categories of algebras. ◁

Using this, we acquire a proof of the following.

Proposition 1.22 (Equivariant [GGN15, Prop 2.3])**.** *Suppose $\mathcal{C}$ is a $\mathcal{T}$-∞-category with I-indexed products and coproducts. Then, the following conditions are equivalent.*

(a) $\mathcal{C}$ *is* I*-semiadditive.*
(b) *There exists an* I*-symmetric monoidal equivalence* $\mathcal{C}^{I-\times} \simeq \mathcal{C}^{I-\sqcup}$ *lifting the identity.*
(c) *The forgetful* $\mathcal{T}$*-functor* $\underline{\mathrm{CMon}}_I(\mathcal{C}) \to \mathcal{C}$ *is an equivalence.*

Proof. Given (a), the I-admissible indexed product maps $\prod_U^S : \mathcal{C}_S \to \mathcal{C}_V$ are *left* adjoint to the restriction map $\Delta^S : \mathcal{C}_V \to \mathcal{C}_S$, so by Proposition A.6, the identity on $\mathcal{C}$ lifts to a symmetric monoidal functor $\mathcal{C}^{I-\times} \to \mathcal{C}^{I-\sqcup}$. We will see in Lemma 1.7 that an I-symmetric monoidal functor is an I-symmetric monoidal equivalence if and only if its underlying $\mathcal{T}$-functor is an equivalence, so this implies (b).

The implication (b) $\implies$ (c) is just Corollary 1.19 and Lemma A.4 and the implication (b) $\implies$ (c) follows from the fact that $\underline{\mathrm{CMon}}_I(\mathcal{C})$ is I-semiadditive [CLL24, Thm B]. ☐

Remark 1.23. We briefly comment on why one may expect Corollary 1.19 in the context of of traditonal equivariant algebra. In order to set this up, recall that the $C_p = \mathbb{Z}/p\mathbb{Z}$-orbit category is the following:

$$\left\langle \begin{tikzcd} {[C_p/e]} \arrow[loop left, "\tau"] \arrow[r, "r"] & *_{C_p} \end{tikzcd} \;\middle|\; \tau^p = \mathrm{id}, \quad r = r\tau \right\rangle;$$

in particular, a C_p-coefficient system of sets is precisely a pair of sets X_e, X_{C_p}, an order-p-permutation of X_e, and a map $X_{C_p} \to X_e^{hC_p}$ which is C_p-equivariant for the trivial action on the codomain.[7] Coinduction in this setting is given by

$$\begin{tikzcd} X^p \arrow[loop left, "\tau^*"] & X \arrow[l, "\Delta"'] \end{tikzcd}$$

where τ^* permutes the factors. One can see this by noting that this presents $\mathrm{Map}(C_p/e, X)$, where C_p acts on the domain. If $Y \in \mathrm{Coeff}^{C_p}\mathrm{Set}$ is a C_p-coefficient system, then a map $Y_{C_p}^{a+b[C_p/e]} \to Y_{C_p}$ has signature

$$Y_{C_p}^a \times Y_e^b \to Y_{C_p}.$$

By [Ste24b, § 2.3], there are six unital C_p-weak indexing categories. For variety, we describe $I = A\lambda$ for λ a nontrivial irredicuble real orthogonal C_p-representation; thus given an $A\lambda$-commutative monoid we have maps $Y_e^n \to Y_e$ for all n, and maps $Y_{C_p}^a \times Y_e^b \to Y_{C_p}$ if and only if $a \le 1$.

Note that the data of a (strict) $A\lambda$-commutative algebra structure on Y is dictated by the unit elements $Y_e \leftarrow * \to Y_{C_p}$, the multiplication map $Y_e^2 \to Y_e$, the transfer map $Y_e \to Y_{C_p}$, and the action map $Y_{C_p} \times Y_e \to Y_{C_p}$. These are subject to the associativity/unitality condition that all maps $Y_{C_p}^a \times Y_e^b \to Y_{C_p}^{a'} \times Y_e^{b'}$ constructed out of composites of products of such maps agree; by closure of $\underline{\mathbb{F}}^\lambda$ under self-indexed coproducts, maps occur in those arities if and only if the map of arities $a + b\left[C_p/e\right] \to a' + b'\left[C_p/e\right]$ is in $A\lambda$. Unwinding definitions, this is exactly the data of an $A\lambda$-commutative monoid. ◁

The cocartesian situation is more simple: the forgetful functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\sqcup}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \mathcal{C})$ is an *equivalence*. We study this more fully in Appendix A and Section 3.1.

1.3.2. *Constructing* I*-symmetric monoidal* ∞*-categories from other* I*-symmetric monoidal* ∞*-categories.* Fix I a one-object weak indexing system; that is, we assume that $*_V$ is I-admissible for all $V \in \mathcal{T}$, so that I-commutative monoids have underlying $\mathcal{T}$-objects. In this subsection, we review some known equivariant lifts to [HA, § 2.2.1].

When $\mathcal{C}^\otimes \subset \mathrm{Op}_I$ is an I-operad and $\mathcal{D} \subset \mathcal{C}$ is a full $\mathcal{T}$-subcategory. let $\mathcal{D}^\otimes \subset \mathcal{C}^\otimes$ be the full subcategory spanned by the objects belonging to

$$\mathcal{D}_S := \prod_{U \in \mathrm{Orb}(S)} \mathcal{D}_U \subset \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_U \simeq \mathcal{C}_S.$$

[7] The notation of homotopy fixed points were placed here to remind the viewer that they are computed as the fixed points of the Borel action on X_e, not due to any nontrivial homotopical considerations; following Elmendorf's theorem, some authors refer to X_{C_p} as the genuine C_p-fixed points of the coefficient system, which is a terminological collision we would like to avoid.

Note that the composite map $\mathcal{D}^\otimes \to \mathcal{C}^\otimes \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ presents an I-operad by construction.

Theorem 1.24 ([NS22, § 2.9])**.** *Let $L^\otimes \colon \mathcal{B}^\otimes \to \mathcal{C}^\otimes$ be an I-symmetric monoidal functor and let $\iota\colon \mathcal{D} \hookrightarrow \mathcal{C}$ be a full $\mathcal{T}$-subcategory. Then,*

(1) (Doctrinal adjunction) Suppose the underlying $\mathcal{T}$-functor L of $L^\otimes$ participates in a $\mathcal{T}$-adjunction

$$L\colon \mathcal{B} \rightleftarrows \mathcal{C}\colon R$$

Then, $L^\otimes$ has a unique lax I-symmetric monoidal right adjoint $R^\otimes$ lifting R.

(2) (Full subcategories) Suppose that, for all $S \in \mathbb{F}_{I,V}$, the S-indexed tensor functor

$$^{\mathcal{C}}\bigotimes^S \colon \mathcal{C}_S \to \mathcal{S}_V$$

restricts to a functor $^{\mathcal{D}}\bigotimes^S\colon \mathcal{D}_S \to \mathcal{D}_V$. Then, the I-operad $\mathcal{D}^\otimes$ constructed above is an I-symmetric monoidal category, and the inclusion $\mathcal{D}^\otimes \hookrightarrow \mathcal{C}^\otimes$ is a symmetric monoidal functor lifting ι; furthermore, $\mathcal{D}^\otimes$ is the unique I-symmetric monoidal category over $\mathcal{C}^\otimes$ extending i.

(3) (Localization) Suppose ι has a left adjoint $L\colon \mathcal{C} \to \mathcal{D}$ such that $^{\mathcal{C}}\otimes^S\colon \mathcal{C}_S \to \mathcal{C}_V$ preserves L-equivalences. Then, $\mathcal{D}$ attains a I-symmetric monoidal structure together with an I-symmetric functor $L^\otimes\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ extending L. Furthermore, the associated lax I-symmetric monoidal structure on i is symmetric monoidal if and only if $\mathcal{D}$ satisfies the conditions of part (2).

In particular, if $\mathcal{D}$ is an I-symmetric monoidal localization, then its indexed tensor functors are computed by

$$^{\mathcal{D}}\bigotimes_U^S X_U \simeq L\left(^{\mathcal{C}}\bigotimes_U^S X_U\right).$$

Proof. (1) follows from [HA, Prop 7.3.2.6] on opposite categories. (2) is [NS22, Prop 2.9.1] and (3) is [NS22, Thm 2.9.2]. The final statement follows by noting that the composite $\mathcal{D}^\otimes \to \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is the identity, hence it is symmetric monoidal. □

1.3.3. *The pointwise $\mathcal{T}$-symmetric monoidal structure.* Once more fix I a one-object weak indexing system. In classical algebra, there are two well-known tensor products of functors $F,G : \mathcal{C} \to \mathcal{D}$: when $\mathcal{D}$ is monoidal, the *pointwise tensor product* sets $F \otimes G(C) \coloneqq F(C) \otimes G(C)$, and when additionally $\mathcal{C}$ is monoidal, the *Day convolution product* sets $F \circledast G(-)$ to be the left Kan extension of the functor $F(-)\otimes G(-) : \mathcal{C}^2 \to \mathcal{D}$ along the tensor functor $\mathcal{C}^2 \to \mathcal{C}$.

[NS22] has equivariantly lifted of both structures. We first review pointwise indexed tensor products.

Theorem 1.25 ([NS22, Thm 3.3.1, 3.3.3])**.** *Let $\mathcal{K}$ be a $\mathcal{T}$-∞-category, and $\mathcal{C}^\otimes$ a $\mathcal{T}$-operad. Then, there exists a unique (functorial) I-operad structure $\mathrm{Fun}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}$ on $\mathrm{Fun}_{\mathcal{T}}(\mathcal{K},\mathcal{C})$ satisfying the universal property*

$$\mathrm{Alg}_{\mathcal{O}}(\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}))$$

for $\mathcal{O} \in \mathrm{Op}_I$. Furthermore, when $\mathcal{C}^\otimes$ is I-symmetric monoidal, $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}$ is I-symmetric monoidal and satisfies the universal property

$$\mathrm{Fun}_{\mathcal{T}}^{I-\otimes}\left(\mathcal{D},\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}\right) \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{K},\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})\right).$$

If S is I-admissible, then the S-indexed tensor product of $(F_U) \in \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})_S^{\otimes-\mathrm{ptws}}$ has values

$$\begin{tikzcd} \mathcal{D}_V \arrow[r, "\Delta^S"] \arrow[rrr, bend right, "\bigotimes_U^S F_U"'] & \mathcal{D}_S \arrow[r, "(F_U)"] & \mathcal{C}_S \arrow[r, "\bigotimes^S"] & \mathcal{C}_V \end{tikzcd}$$

Observation 1.26. Suppose $F : \mathcal{K}' \to \mathcal{K}$ is a functor. Then, the restriction and left Kan extension natural transformations

$$F_! : \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{K}',\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})\right) \rightleftarrows \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{K},\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})\right) : F^*$$

yield I-symmetric monoidal functors $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}',\mathcal{C})^{\otimes-\mathrm{ptws}} \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}$ extending the left Kan extension and restriction functors between functor categories via Yoneda's lemma. In particular, give $X \in \Gamma^{\mathcal{T}}\mathcal{K}$ this yields an I-symmetric monoidal lift $\mathrm{ev}_X : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}} \to \mathcal{C}^\otimes$ of the ordinary evaluation $\mathcal{T}$-functor $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\{X\},\mathcal{C}) \simeq \mathcal{C}$. ◁

1.3.4. *Equivariant Day convolution.* Once again fix I a one-color weak indexing category. The other structure we recall is *Day convolution.*

Definition 1.27. Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ be I-operads. Then, the *Day convolution I-operad* $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\circledast}$, if it exists, is the unique I-operad possessing a natural equivalence

$$\mathrm{Alg}_{\mathcal{Q}}\left(\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\circledast}\right) \simeq \mathrm{Alg}_{\mathcal{Q}\times\mathcal{P}}(\mathcal{O})$$

for all $\mathcal{Q}^{\otimes} \in \mathrm{Op}_I$. ◃

Remark 1.28. $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\circledast}$ is the *exponential object* in I-operads from $\mathcal{P}$ to $\mathcal{O}$; in particular, if $\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes}$ are I-symmetric monoidal ∞-categories, then commutative algebras in $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\circledast}$ correspond with lax I-symmetric monoidal functors $\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \times \mathcal{N}^{\otimes}_{I\infty} \to \mathcal{C}^{\otimes}$. ◃

We recall an omnibus theorem due to Nardin-Shah [NS22, § 3] for Day convolution coming from I-symmetric monoidal ∞-categories; these results may be generalized to construct exponential objects over an arbitrary $\mathcal{T}$-operad $\mathcal{B}^{\otimes}$ under the condition that $\mathcal{O}^{\otimes}$ is $\mathcal{B}^{\otimes}$*-promonoidal*, but this is not necessary at the moment, so we specialize to the I-symmetric monoidal case.

Proposition 1.29. *Suppose $\mathcal{O}^{\otimes}$ is an I-symmetric monoidal ∞-category and $\mathcal{C}^{\otimes}$ is a I-operad. Then, the I-operad $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\circledast}$ exists and satisfies the following properties:*

(1) The functor $\mathcal{C} \mapsto \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\circledast}$ is the right adjoint in an adjoint pair [NS22, Prop 3.1.7]

$$(-)\times\mathcal{O}^{\otimes} : \mathrm{Op}_I \rightleftarrows \mathrm{Op}_I : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},-)^{\circledast};$$

(2) the underlying $\mathcal{T}$-∞-category of $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\circledast}$ is $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})$ [NS22, Prop 3.1.9]

(3) For all $S \in V$ and $\underline{V}$-functors $\mathcal{O}_{\underline{V}} \to \mathcal{C}_{\underline{V}}$, there exists a $\underline{V}$-left Kan extension diagram

$$\begin{tikzcd}
\mathcal{O}_S \arrow[r, "(F_U)"] \arrow[d, "\otimes^S"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{C}_S \arrow[r, "\otimes^S"] & \mathcal{C}_V \\
\mathcal{O}_V \arrow[urr, bend right, "\circledast^S_U F_U"'] & &
\end{tikzcd}$$

where $\circledast^S : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})_S \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})_V$ is the S-indexed tensor functor.

(4) If $\mathcal{C}$ is a presentably I-symmetric monoidal ∞-category, then $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\otimes-\mathrm{Day}}$ is a presentably I-symmetric monoidal ∞-category [NS22, Prop 3.2.2] [Hil24, Thm 6.1.2].

Given $G : \mathcal{C} \to \mathcal{D}$ a lax I-symmetric monoidal functor and $\mathcal{O}^{\otimes}$ an I-symmetric monoidal ∞-category, we may apply Yoneda's lemma to the postcomposition functor $\mathrm{Alg}_{\mathcal{Q}\times\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{Q}\times\mathcal{O}}(\mathcal{D})$ to construct a lax I-symmetric monoidal functor $\widetilde{G} : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\circledast} \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{D})^{\circledast}$ lifting postcomposition with G. We make the following verification.

Proposition 1.30 (Equivariant [BS24b, Prop 3.3])**.** *If G is $\mathcal{T}$-colimit preserving and I-symmetric monoidal, then so is $\widetilde{G}$.*

In order to prove this, we need to understand how to upgrade lax I-symmetric monoidal functors to I-symmetric monoidal functors.

Observation 1.31. Let $G : \mathcal{A}^{\otimes} \to \mathcal{B}^{\otimes}$ be a lax I-symmetric monoidal functor, $S \in \underline{\mathbb{F}}_I$ a finite I-admissible V-set, and $(X_U) \in \mathcal{A}_S$ an S-tuple in $\mathcal{A}$. Then, there are cocartesian arrows $f_S : (X_U) \to \bigotimes_U^S X_U$ and $g_S : (GX_U) \to \bigotimes_U^S GX_U$ in $\mathcal{A}^{\otimes}$ and $\mathcal{B}^{\otimes}$, and the universal property for cocartesian arrows yields a diagram

$$\begin{tikzcd}
(GX_U) \arrow[r, "g_S"] \arrow[dr, "Gf_S"'] & \prod_U^S GX_U \arrow[d, dashed, "h_S"] \\
& G\left(\prod_U^S X_U\right)
\end{tikzcd}$$

Unwinding definitions, an I-symmetric monoidal functor is precisely a lax I-symmetric monoidal functor satisfying the condition that h_S is an equivalence (in $\mathcal{B}$) for all I-admissible S (so that Gf_S is cocartesian for all S). ◃

Proof of Proposition 1.30. The fact that $\widetilde{G}$ is $\mathcal{T}$-colimit preserving follows from the fact that $\mathcal{T}$-colimits in functor $\mathcal{T}$-categories are computed pointwise [Sha23, Prop 9.17], so we're left with verifying that $\widetilde{G}$ is I-symmetric monoidal. Unwinding definitions, the comparison map h_S of Observation 1.31 are implemented by the universal property of $\mathcal{T}_{/V}$-left Kan extension

$$\begin{tikzcd}
 & & \mathcal{D}_S \arrow[dr, "\otimes^S"] & \\
\mathcal{O}_S \arrow[r, "(F_U)"] \arrow[d, "\mu"] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{C}_S \arrow[ur, "G"] \arrow[r, "\otimes^S"] & \mathcal{C}_V \arrow[r, "G"] & \mathcal{D}_V \\
\mathcal{O}_V \arrow[urr, "\circledast^S_U F_U" description] \arrow[urrr, bend right, "\circledast^S_U GF_U"'] & & \arrow[u, phantom, "\Rightarrow"] &
\end{tikzcd}$$

That is, we're left with verifying that $G \circledast^S_U F_U$ is the $\mathcal{T}_{/V}$-left kan extension of $G \circ \otimes^S \circ (F_U)$ along μ. In fact, the pointwise formula for $\mathcal{T}_{/V}$-left Kan extensions [Sha23, Def 10.1] shows that postcomposition with strongly $\mathcal{T}_{/V}$-colimit preserving functors preserves $\mathcal{T}_{/V}$-left Kan extensions, so this is true. $\square$

1.3.5. *The smash product of pointed $\mathcal{T}$-spaces.* Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category possessing a terminal $\mathcal{T}$-object $*$. Then, the under category $\mathcal{C}_* \coloneqq \mathcal{C}_{*/}$ embeds as a localizing $\mathcal{T}$-subcategory

$$\mathcal{C}_* \subset \underline{\mathrm{Fun}}_{\mathcal{T}}(\Delta^1 \times \mathcal{T}^{\mathrm{op}}, \mathcal{C}). \tag{10}$$

In [NS22, Ex 3.2.8], $\Delta^1 \times \mathcal{T}^{\mathrm{op}}$ is given a $\mathcal{T}$-symmetric monoidal structure satisfying the condition that the associated Day convolution structure is compatible with the localization left adjoint to (10). Thus, $\mathcal{C}_*$ possesses a $\mathcal{T}$-symmetric monoidal structure; the author suspects that an analog of the argument of [GGN15] will show that this is uniquely determined by its unit.

In any case, the localization functor $\mathrm{Fun}_{\mathcal{T}}(\Delta^1 \times \mathcal{T}^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}_*$ is computed by the pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
*_{\mathcal{T}} \arrow[r, "Lf"'] \arrow[ur, phantom, "\ulcorner" very near end] & LY
\end{tikzcd}$$

Furthermore, the tensor products in $\mathrm{Fun}_{\mathcal{T}}(\Delta^1 \times \mathcal{T}^{\mathrm{op}}, \mathcal{C})^{\circledast}$ of $f\colon A \to B$ and $g\colon\ : X \to Y$ are computed by the pushout product

$$f \circledast g\colon A \otimes Y \sqcup_{A \otimes X} B \otimes Y \to B \otimes Y;$$

the norms in $\Delta^1 \times \mathcal{T}^{\mathrm{op}}$ are identities, so the norms in $\mathrm{Fun}_{\mathcal{T}}(\Delta^1 \times \mathcal{T}^{\mathrm{op}}, \mathcal{C})^{\circledast}$ are computed simply by the exterior norm

$$N_V^W f : \Delta^1 \xrightarrow{f} \mathcal{C}_V \xrightarrow{N_V^W} \mathcal{C}_W.$$

In particular, if $\mathcal{C}$ is cartesian, we arrive at the formulas

$$X \wedge Y \simeq X \times Y / X \vee Y \qquad\qquad N_V^W X \simeq \mathrm{CoInd}_V^W X.$$

For instance, in the case $\mathcal{T} = \mathcal{O}_G$, we have pointed representation spheres S^V for all real orthogonal G-representations V; the above formulas compute the indexed tensor products

$$\bigwedge_{G/H_i}^{T} S^{V_i} \simeq S^{\bigoplus_{G/H_i}^{T} V_i},$$

which were claimed in [NS22, Ex 3.2.8] without proof.

1.3.6. *The box product of I-commutative monoids and I-spectra.* The *spectral Mackey functor theorem* of [GM11] stipulates that

$$\mathrm{CMon}_G(\mathrm{Sp}) \simeq \lim\left(\cdots \xrightarrow{\Omega^\rho} \mathcal{S}_G \xrightarrow{\Omega^\rho} \mathcal{S}_G\right)$$

whenever G is a finite group. We refer to the result of this as Sp_G. It was noted in [Nar16] that this satisfies a universal property of *G-stability*, which we may generalize to $\mathcal{T}$.

Definition 1.32 (C.f. [CLL23, Def 6.2.2])**.** Let I be an indexing system. Then, a $\mathcal{T}$-∞-category $\mathcal{C}$ is *I-stable* if it is I-semiadditive and its straightening factors as

$$\mathcal{T}^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{St}} \hookrightarrow \mathrm{Cat},$$

i.e. it's *fiberwise-stable.* ◁

If $\mathcal{K}$ consists of I-product diagrams and finite fiberwise limits, then we denote by $\mathrm{Cat}_{\mathcal{T}}^{I-lex} := \mathrm{Cat}_{\mathcal{T}}^{\mathcal{K}-lex}$ the ∞-category of $\mathcal{T}$-∞-categories with finite fiberwise limits and I-products, and $\mathrm{Cat}_{\mathcal{T}}^{I-st} \subset \mathrm{Cat}_{\mathcal{T}}^{I-lex}$ the full subcategory spanned by I-stable $\mathcal{T}$-∞-categories.

We denote by $\mathrm{Sp}\otimes(-)$ the postcomposition functor

$$\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}^{lex}) \to \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}^{st}).$$

The following may then be seen as an orbital generalization of the spectral Mackey functor theorem.

Proposition 1.33 ([CLL23, Cor 6.2.6]; c.f. [BH21b, Def 1.5] or [Nar16, Thm 7.4])**.** *The fully faithful inclusion* $\mathrm{Cat}_{\mathcal{T}}^{I-st} \hookrightarrow \mathrm{Cat}_{\mathcal{T}}^{I-lex}$ *has a* right *adjoint given by* $\mathrm{CMon}_I(\mathrm{Sp}\otimes-)$*.*

In particular, this presents $\mathrm{Sp}_I := \mathrm{CMon}_I(\mathrm{Sp})$ as the *I-stabilization of $\mathcal{T}$-spaces.* We'll endow this with an equivariant symmetric monoidal structure, for which we need a definition. By [HHLN23, Thm 2.18], the functor $\mathrm{Span}_{-,-}(-) : \mathrm{Trip}^{\mathrm{adeq}} \to \mathrm{Cat}$ is compatible with pullbacks. In particular, it sends triples of *I-symmetric monoidal categories* to I-symmetric monoidal categories. Recall the following definition.

Definition 1.34 ([BH22; Ste24b])**.** A pair of $\mathcal{T}$-weak indexing category (I_a, I_m) is *compatible* if $\underline{\mathbb{F}}_{I_a} \subset \underline{\mathbb{F}}_{\mathcal{T}}^{\times}$ is an I_m-symmetric monoidal subcategory. ◁

Remark 1.35. By [Ste24b, § 2.6], the subposet of weak indexing categories I_m such that (I_a, I_m) is compatible is a lowerray $\mathrm{wIndexCat}_{\mathcal{T}, \le m(I_a)}$ for the indexing category $m(I)$ with corresponding indexing system

$$\underline{\mathbb{F}}_{m(I_a),V} = \left\{ S \in \mathbb{F}_V \mid \underline{\mathbb{F}}_{I_a} \subset \underline{\mathbb{F}}_{\mathcal{T}} \text{ closed under } S\text{-indexed products} \right\}.$$ ◁

It follows from this that $(\underline{\mathbb{F}}_{\mathcal{T}}, \underline{\mathbb{F}}_{I_a}, \underline{\mathbb{F}}_{\mathcal{T}}) \subset (\underline{\mathbb{F}}_{\mathcal{T}}, \underline{\mathbb{F}}_{\mathcal{T}}, \underline{\mathbb{F}}_{\mathcal{T}})$ is an I_m-symmetric monoidal sub-adequate triple; hence $\mathrm{Span}_{-,-}(-)$ induces a map of I_m-symmetric monoidal categories.

$$\mathrm{Span}_{I_a}(\underline{\mathbb{F}}_{\mathcal{T}}) \subset \mathrm{Span}(\underline{\mathbb{F}}_{\mathcal{T}}).$$

Observation 1.36. Fix $\mathcal{C}$ a presentably I_m-symmetric monoidal ∞-category. Then, left Kan extension preserves product-preserving functors; hence the I_m-symmetric Day convolution structure preserves the full subcategory

$$\underline{\mathrm{CMon}}_{I_a}(\mathcal{C}) \subset \mathrm{Fun}(\mathrm{Span}_{I_a}(\underline{\mathbb{F}}_{\mathcal{T}}), \mathcal{C}),$$

yielding an I_m-symmetric monoidal Day convolution structure on $\underline{\mathrm{CMon}}_{I_a}(\mathcal{C})$. In analogy to Lewis' unpublished notes on the theory of Green functors [Lew81], we refer to this as the *indexed box product*, and write

$$\underline{\mathrm{CMon}}_{I_a}(\mathcal{C})^{\square} := \underline{\mathrm{CMon}}_{I_a}(\mathcal{C})^{\circledast}; \qquad \square^S := \circledast^S.$$ ◁

Notably, if (I_m, I_a) is a compatible pair of weak indexing systems, then Observation 1.36 constructs an I_m-symmetric monoidal structure on Sp_{I_a}.

Proposition 1.37. *There exists a G-symmetric monoidal equivalence between* $\underline{\mathrm{Sp}}_G^{\square} := \underline{\mathrm{CMon}}_G(\mathrm{Sp})$ *and the G-symmetric monoidal structure* $\underline{\mathrm{Sp}}_G^{\otimes}$ *of [BH21a; CHLL24a].*

Proof. [CHLL24a, Thm 5.4.10] constructs an essentially unique G-symmetric monoidal structure on $\underline{\mathrm{Sp}}_G \simeq \underline{\mathrm{CMon}}_G(\mathrm{Sp})$ with G-unit $\mathbb{S}_G$, so it suffices to verify that $\mathbb{S}_G$ is the $\square$-unit. In fact, since the underlying symmetric monoidal structure is Day convolution, the G-unit is the image of the unit $S_G^0 \in \underline{\mathcal{S}}_G \simeq \mathrm{Coeff}^G \mathcal{S}$ under the free functor $\Sigma_G^\infty : \mathcal{S}_G \to \mathrm{Sp}_G$.[8] In particular, this element is $\mathbb{S}_G = \Sigma_G^\infty S^0$, so [CHLL24a, Thm 5.4.10] constructs the desired equivalences. □

[8] To see that the free functor is modelled by Σ_G^∞, apply [NS22, Thm A.4].

Remark 1.38. When $\mathcal{C} = \underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{D})$, recall that Proposition 1.5 yields an equivalence

$$\mathrm{CMon}_{I_a}(\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{D})) \simeq \mathrm{Fun}^{\times}(\mathrm{Span}_{I_a}(\mathbb{F}_{\mathcal{T}}), \mathcal{D});$$

in particular, unwinding definitions we may express indexed box products via a left Kan extension

$$\begin{tikzcd}
\prod\limits_{U \in \mathrm{Orb}(S)} \mathrm{Span}_{I_a}(\mathbb{F}_U) \arrow[r, "(M_U)"] \arrow[d, "(\mathrm{CoInd}_U^V)_{U \in \mathrm{Orb}(S)}"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{D}^{\times \mathrm{Orb}(S)} \arrow[r, "\otimes"] & \mathcal{D} \\
\mathrm{Span}_{I_a}(\mathbb{F}_V) \arrow[urr, bend right, "\square_U^S M_U"'] & &
\end{tikzcd}$$

Define the V-geometric fixed points of $M \colon \mathrm{Span}_{I_a}(\mathbb{F}_V) \to \mathcal{D}$ to be the left Kan extension of M along the "span of fixed points" functor $r \colon \mathrm{Span}_{I_a}(\mathbb{F}_V) \to \mathrm{Span}(\mathbb{F})$. Composition of left Kan extensions then computes the geometric fixed points formulas

$$\begin{tikzcd}
\mathrm{Span}_{I_a}(\mathbb{F}_U) \arrow[rr, "M"] \arrow[d, "\mathrm{Span}(\mathrm{CoInd}_U^V)"'] \arrow[dd, bend right=60, "r_U"'] \arrow[drr, phantom, "\Downarrow"] & & \mathcal{D} \\
\mathrm{Span}_{I_a}(\mathbb{F}_V) \arrow[urr, "N_U^V M"'] \arrow[d, "r_V"] \arrow[dr, phantom, "\Downarrow"] & & \\
\mathrm{Span}(\mathbb{F}) \arrow[uurr, bend right, "\Phi^V N_U^V M \simeq \Phi^U M"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Span}_{I_a}(\mathbb{F}_V)^{\times 2} \arrow[r, "{(M,N)}"] \arrow[d, "\wedge"'] \arrow[dr, phantom, "\Downarrow"] \arrow[rrr, bend left, equal] & \mathcal{D}^{\times 2} \arrow[r, "\otimes"] & \mathcal{D} \arrow[rrr, bend left, equal] & \mathrm{Span}_{I_a}(\mathbb{F}_V)^{\times 2} \arrow[r, "{(M,N)}"] \arrow[d, "{(r_V,r_V)}"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{D}^{\times 2} \arrow[r, "\otimes"] & \mathcal{D} \\
\mathrm{Span}_{I_a}(\mathbb{F}_V) \arrow[urr, "M \square N"'] \arrow[d, "r_V"'] \arrow[dr, phantom, "\Downarrow"] & & & \mathrm{Span}(\mathbb{F})^{\times 2} \arrow[ur, "{(\Phi^V M, \Phi^V N)}"'] \arrow[d, "\wedge"'] \arrow[dr, phantom, "\Downarrow"] & & \\
\mathrm{Span}(\mathbb{F}) \arrow[uurr, bend right, "\Phi^V(M \square N)"'] \arrow[rrr, bend right, equal] & & & \mathrm{Span}(\mathbb{F}) \arrow[uurr, bend right, "\Phi^V M \otimes \Phi^V N"'] & &
\end{tikzcd}$$

In particular, this yields the formula

$$\Phi^V \square_U^S M_U \simeq \bigotimes_{U \in \mathrm{Orb}(S)} \Phi^U M_U,$$

extending the formulae of [HHR16, Prop B.199, Prop B.209]. ◁

The author expects that this will satisfy an I_m-symmetric monoidal universal property akin to that of spectra developed in [GGN15], but we put such considerations off for forthcoming work. Before then, we recall another result from the literature concerning box products.

Recollection 1.39. If $I \subset \mathbb{F}_{\mathcal{T}}$ is a one-object weak indexing category, then by [Ste24b, § 2.6], $(\mathbb{F}_{\mathcal{T}}, I, \mathbb{F}_{\mathcal{T}})$ is a bispan triple in the sense of [EH23, Def 2.4.3] (and furthermore a semiring context in the sense of [CHLL24b, Def 4.4.1]); hence it possesses an ∞-category of bispans

$$P_I^{\mathcal{T}} := \mathrm{Bispan}_{I,all}(\mathbb{F}_{\mathcal{T}})$$ ◁

The following theorem was proved in the discrete setting for $\mathcal{T} = \mathcal{O}_G$ independently by Chan and Vekemens [Cha24; San23] and in general by [CHLL24b, Thm 4.3.6].

Corollary 1.40. *There is a canonical equivalence*

$$\mathrm{CAlg}_I(\mathrm{CMon}_{\mathcal{T}}^{\square}(\mathcal{C})) \simeq \mathrm{Fun}^{\times}\left(\mathcal{P}_I^{\mathcal{T}}, \mathbb{F}_{\mathcal{T}}\right)$$

over $\mathrm{CMon}_{\mathcal{T}}(\mathcal{C})$.

2. Arity support

Indexing systems were first defined in [BH15], and conjectured to classify the $\mathcal{N}_{I\infty}$-operads. This was separately verified in [BP21; GW18; NS22; Rub21a], each time introducing a different combinatorial expression for indexing systems. These have seen extensive combinatorial study in e.g. [BBR21; BHKKNOPST23; FOOQW22; HMOO22], which we do not repeat here. Instead, we carry out this program for the class of *arbitrary* suboperads of $\mathrm{Comm}_{\mathcal{T}}^{\otimes}$, who may not contain colors above all orbits or contain fold maps for all of its colors; these will be called *weak $\mathcal{N}_\infty$-operads.*

In Section 2.1, we finally define the *arity support* functor $A : \mathrm{Op}_{\mathcal{T}} \to \mathrm{wIndex}_{\mathcal{T}}$. We go on in to finally define weak $\mathcal{N}_\infty$-operads, initially as the class of $\mathcal{T}$-0-operads; we show that they are the image of a fully faithful right adjoint to A in Corollary 2.8. Following these, in Section 2.2 we construct and characterize the *arity-Borelification* and *restriction* adjunctions

$$\begin{tikzcd} \mathrm{Op}_I \arrow[r, bend left, "E_I^J"] & \mathrm{Op}_J \arrow[l, bend left, "\mathrm{Bor}_I^J"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Op}_V \arrow[r, bend left=50, "\mathrm{Ind}_V^W"] \arrow[r, bend right=50, "\mathrm{CoInd}_V^W"'] & \mathrm{Op}_W; \arrow[l, "\mathrm{Res}_V^W" description] \end{tikzcd}$$

along the way, in Proposition 2.14, we compute the arity support of BV tensor products. Finally, we finish the section in Section 2.3 by defining and characterizing a wide variety of I-operads of algebraic interest in equivariant homotopy theory.

2.1. Arity support and weak $\mathcal{N}_\infty$-$\mathcal{T}$-operads.

Construction 2.1. Given $\mathcal{O} \in \mathrm{Op}_{\mathcal{T}}$, the *arity support of $\mathcal{O}$* is the subcategory $A\mathcal{O} \subset \mathbb{F}_{\mathcal{T}}$ defined by

$$A\mathcal{O} := \left\{ \psi : T \to S \mid \mathrm{Mul}_{\mathcal{O}}^{\psi}(T;S) \neq \emptyset \right\} \subset \mathbb{F}_{\mathcal{T}} \qquad \triangleleft$$

In particular, maps of operads $\mathcal{O} \to \mathcal{P}$ are functors over $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$, hence they induce maps $\mathcal{O}(S) \to \mathcal{O}(P)$; this endows A with the structure of a functor

$$A : \mathrm{Op}_{\mathcal{T}} \to \mathrm{Sub}(\mathbb{F}_{\mathcal{T}}),$$

where the codomain is the poset of subcategories of $\mathbb{F}_{\mathcal{T}}$.

Remark 2.2. A product is empty if and only if one of its factors is empty, so $A\mathcal{O}$ is equal to

$$A\mathcal{O} = \left\{ \coprod_i \mathrm{Ind}_V^{\mathcal{T}} T_i \to V_i \,\middle|\, \forall i, \mathcal{O}(T_i) \neq \emptyset \right\} \subset \mathbb{F}_{\mathcal{T}}.$$

as a subcategory of $\mathbb{F}_{\mathcal{T}}$; in particular, this implies that A factors as

$$\mathrm{Op}_{\mathcal{T}} \xrightarrow{\mathrm{sseq}_{\mathcal{T}}} \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}}, \mathcal{S}) \to \mathrm{Sub}(\mathbb{F}_{\mathcal{T}}).$$

However, we will see that A has smaller image than the right functor in Proposition 2.4, so the associated essentially surjective functor will only factor through the essential image of $\mathrm{sseq}_{\mathcal{T}}$, rather than the full ∞-category of $\mathcal{T}$-symmetric sequences. $\triangleleft$

Example 2.3. For all $I \in \mathrm{wIndexCat}_{\mathcal{T}}$, we have $A\mathcal{N}_{I\infty} = I$, so $\mathrm{wIndexCat}_{\mathcal{T}} \subset A(\mathrm{Op}_{\mathcal{T}})$. $\triangleleft$

Proposition 2.4. *For all $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$, the subcategory $A\mathcal{O} \subset \mathbb{F}_{\mathcal{T}}$ is a weak indexing category; hence*

$$A(\mathrm{Op}_{\mathcal{T}}) = \mathrm{wIndexCat}_{\mathcal{T}} \subset \mathrm{Sub}(\mathbb{F}_{\mathcal{T}}).$$

Proof. The second statement follows from the first by Example 2.3, so it suffices to prove that $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$ satisfies Conditions (IC-a) to (IC-c).

Our main trick in characterizing $A\mathcal{O}$ is to transfer *nonemptyness* of the structure spaces of $\mathcal{O}^{\otimes}$ backwards along the $\mathcal{T}$-operad structure maps; indeed, there exists no map of spaces $X_1 \times X_2 \to Y_1 \times Y_2$ if and only if $X_1, X_2 \neq \emptyset$ and $Y_i = \emptyset$ for some i.

Using this, Condition (IC-a) follows by unwinding definitions using existence of the arity restriction map of Eq. (7). Similarly, Condition (IC-b) follows by unwinding definitions using the existence of the operadic composition map of Eq. (8). Lastly, Condition (IC-c) follows by existence of the $\mathrm{Aut}_V(S)$-action of Eq. (9). $\square$

Recall that the mapping fibers of $\mathcal{P}^{\otimes}$ a reduced $\mathcal{T}$-operad over backwards maps of $\mathsf{Span}(\mathbb{F}_{\mathcal{T}})$ are contractible; thus the condition that $\mathcal{P}^{\otimes}$ is a $\mathcal{T}$-0-operad (i.e. $\mathrm{Mul}_{\mathcal{P}}(S;T)$ is (-1)-truncated) is equivalent to the statement that the map $\mathcal{P}^{\otimes} \to \mathsf{Span}(\mathbb{F}_{\mathcal{T}})$ is a subcategory inclusion (see [Ste24a]). By inspecting mapping fibers, we find that $\mathcal{P}^{\otimes} = \mathsf{Span}_{A\mathcal{P}}(\mathbb{F}_{\mathcal{T}})$ as subcategories. We've proved the following.

Proposition 2.5. *If $\mathcal{P}^{\otimes}$ is a $\mathcal{T}$-0-operad, then there is a unique equivalence $\mathcal{P}^{\otimes} \simeq \mathcal{N}^{\otimes}_{A\mathcal{P}\infty}$.*

We use this to recognize weak $\mathcal{N}_{\infty}$-operads as *sub-terminal objects*.

Proposition 2.6. *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-operad and I a weak indexing system. Then there is an equivalence*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{N}_{I\infty}) \simeq \begin{cases} * & A\mathcal{O} \leq I, \\ \varnothing & \text{otherwise.} \end{cases} \tag{11}$$

In particular, there is a unique map $\mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{A\mathcal{O}}$ witnessing a unique equivalence $h_{0,\mathcal{T}}\mathcal{O}^{\otimes} \simeq \mathcal{N}^{\otimes}_{A\mathcal{O}}$.

Proof. All statements of this proposition follow immediately from Eq. (11), so it suffices to prove that statement. By [Ste24a], $\mathrm{Op}_{\mathcal{T},0}$ is a poset and

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{N}^{\otimes}_{I\infty}) \simeq \mathrm{Alg}_{h_{0,\mathcal{T}}\mathcal{O}}(\mathcal{N}^{\otimes}_{I\infty}) \in \{\varnothing, *\}.$$

By Proposition 2.5 it suffices to characterize precisely when there exist maps $\mathcal{N}^{\otimes}_{I\infty} \to \mathcal{N}^{\otimes}_{J\infty}$. In fact, unwinding definitions, we are asking for factorizations of subcategory inclusions

$$\mathsf{Span}_I(\mathbb{F}_{\mathcal{T}}) \subset \mathsf{Span}_J(\mathbb{F}_{\mathcal{T}}) \subset \mathsf{Span}(\mathbb{F}_{\mathcal{T}});$$

this occurs if and only if $I \leq J$. ☐

Remark 2.7. By [Ste24a], the functor $\mathrm{ev}_S : \mathcal{O}^{\otimes} \mapsto \mathcal{O}(S)$ has a left adjoint $\mathrm{Fr}_S(-) : \mathcal{S} \to \mathrm{Op}_{\mathcal{T}}$; applying this to $* \in \mathcal{S}$, we find that $\mathcal{O}(S) \simeq \mathrm{Alg}_{\mathrm{Fr}_S(*)}(\mathcal{O})^{\simeq}$; in particular, if $\mathcal{P}^{\otimes}$ has the property that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \in \{*, \varnothing\}$ for all $\mathcal{O}^{\otimes}$, then $\mathcal{P}^{\otimes}$ must be a weak $\mathcal{N}_{\infty}$-operad.

By [HTT, Rem 5.5.6.12], this demonstrates that the poset of *sub-terminal objects* $\mathrm{Sub}_{\mathrm{Op}_{\mathcal{T}}}(\mathrm{Comm}^{\otimes}_{\mathcal{T}})$ is spanned by the weak $\mathcal{N}_{\infty}$-operads, by Proposition 2.6, we then find that

$$\mathrm{Sub}_{\mathrm{Op}_{\mathcal{T}}}\left(\mathrm{Comm}^{\otimes}_{\mathcal{T}}\right) \simeq \mathrm{wIndex}_{\mathcal{T}}. \qquad \triangleleft$$

The following generalization of the indexing systems theorems of [BP21; GW18; NS22; Rub21a] then immediately follows from Propositions 2.4 and 2.6.

Corollary 2.8. *The functor of admissible maps admits a fully faithful right adjoint*

$$\begin{tikzcd} \mathrm{Op}_{\mathcal{T}} \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathrm{wIndex}_{\mathcal{T}} \arrow[l, bend left, "\mathcal{N}^{\otimes}_{(-)\infty}"] \end{tikzcd} \tag{12}$$

whose image consists of the weak $\mathcal{N}_{\infty}$-operads; furthermore, the following are equal full subcategories of $\mathrm{Op}_{\mathcal{T}}$:

$$\mathrm{Op}_I = \mathrm{Op}_{\mathcal{T},/\mathcal{N}_{I\infty}} = A^{-1}(\mathrm{wIndexCat}_{\mathcal{T},\leq I}).$$

Observation 2.9. Let S be a property in

$$\{\text{at least one-color, almost essentially unital, unital, has finite fold maps}\}.$$

Say that a weak indexing system *has at least one color* if it has one color. Then, note that

$$\mathcal{O}^{\otimes} \text{ has property } S \qquad \iff \qquad A\mathcal{O}^{\otimes} \text{ has property } S.$$

In particular, Corollary 2.8 restricts to an adjunction

$$\begin{tikzcd} \mathrm{Op}^{P}_{\mathcal{T}} \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathrm{wIndex}^{P}_{\mathcal{T}} \arrow[l, bend left, "\mathcal{N}^{\otimes}_{(-)\infty}"] \end{tikzcd}$$

$\triangleleft$

2.2. **Operadic restriction and arity-borelification.** Given $\varphi : \mathcal{T}' \to \mathcal{T}$ a functor of atomic orbital ∞-categories, we verified in [Ste24a, Appendix A] that the associated map of Burnside algebraic patterns $\mathsf{Span}(\mathbb{F}_{\mathcal{T}'}) \to \mathsf{Span}(\mathbb{F}_{\mathcal{T}})$ is a Segal morphism. In this section, we use this to to define various adjunctions between categories of I-operads.

2.2.1. *Arity borelification and its left adjoint.* Given $I \le J$ a related pair of weak indexing sysems, let E_I^J : $\mathrm{wIndexCat}_{\mathcal{T},/I} \to \mathrm{wIndexCat}_{\mathcal{T},/J}$ be the evident inclusion, with right adjoint $\mathrm{Bor}_I^J = (-) \cap \mathbb{F}_I : \mathrm{wIndexCat}_{\mathcal{T},/J} \to \mathrm{wIndexCat}_{\mathcal{T},/I}$. These are push-pull adjunctions; following in form, we write the corresponding *unslicing functor* as

$$E_I^J \colon \mathrm{Op}_I \simeq \mathrm{Op}_{J,/\mathcal{N}_{I\infty}^{\otimes}} \to \mathrm{Op}_J.$$

This has a right adjoint

$$\mathrm{Bor}_I^J \colon \mathrm{Op}_J \to \mathrm{Op}_{J,/\mathcal{N}_{I\infty}^{\otimes}} \simeq \mathrm{Op}_I$$

given by pullback along the unique map $\mathcal{N}_{I\infty}^{\otimes} \to \mathcal{N}_{J\infty}^{\otimes}$. Furthermore, these map to push-pull along the inclusion $\iota_I^J \colon \mathsf{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathsf{Span}_J(\mathbb{F}_{\mathcal{T}})$ along the forgetful functor $\mathrm{Op}_I \to \mathsf{Cat}_{/\mathsf{Span}_I(\mathbb{F}_{\mathcal{T}})}$ and similar for J [BHS22, § 4]. Hence these intertwine with A, i.e.

$$E_I^J A\mathcal{O} = A E_I^J \mathcal{O}; \qquad\qquad \mathrm{Bor}_I^J A\mathcal{O} = A\mathrm{Bor}_I^J \mathcal{O}.$$

Corollary 2.10. *For $I \le J$ weak indexing systems, the functor $E_I^J \colon \mathrm{Op}_I \to \mathrm{Op}_J$ is an inclusion of a colocalizing $\mathcal{T}$-subcategory*

$$\begin{tikzcd} \underline{\mathrm{Op}}_I^{\otimes} \arrow[r, bend left, "E_I^J"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Op}}_J^{\otimes} \arrow[l, bend left, "\mathrm{Bor}_I^J"] \end{tikzcd}$$

whose terminal object is $\mathcal{N}_{I\infty}^{\otimes}$. Furthermore, there is are equivalences

$$E_I^{I'} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{E_I^{I'} J\infty}^{\otimes}$$
$$\mathrm{Bor}_I^{I'} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{\mathrm{Bor}_I^{I'} J\infty}^{\otimes}.$$

Proof. The first sentence follows by the above argument. The computations follow by examining the structure spaces of the resulting $\mathcal{T}$-operads. □

Corollary 2.11 (Color-borelification)**.** *Given $\mathcal{F} \in \mathrm{Fam}_{\mathcal{T}}$ is a $\mathcal{T}$-family, there is a natural equivalence*

$$\mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathcal{O}) \simeq \Gamma^{\mathcal{F}} \mathcal{O};$$

hence there is a natural equivalence

$$\mathrm{triv}_{\mathcal{F}}^{\otimes} \overset{BV}{\otimes} \mathcal{O}^{\otimes} \simeq E_{\mathcal{F}}^{\mathcal{T}} \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{O}^{\otimes}.$$

Proof. The first statement follows by noting that $\mathrm{triv}_{\mathcal{F}}^{\otimes} \simeq E_{\mathcal{F}}^{\mathcal{T}} \mathrm{triv}_{\mathcal{F}}^{\otimes}$, so that

$$\mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathcal{O}) \simeq \mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}}(\mathcal{O})) \simeq \Gamma^{\mathcal{F}} \mathcal{O}$$

by Proposition 1.13. The second statement then follows by Yoneda's lemma, noting that

$$\begin{aligned} \mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}} \otimes \mathcal{O}}(\mathcal{P}) &\simeq \mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}}} \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{P}) \\ &\simeq \Gamma^{\mathcal{F}} \mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \\ &\simeq \mathrm{Alg}_{\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{O}}(\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{P}) \\ &\simeq \mathrm{Alg}_{E_{\mathcal{F}}^{\mathcal{T}} \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{O}}(\mathcal{P}). \end{aligned}$$ □

Given $\mathcal{O} \in \mathrm{Op}_{\mathcal{T}}$, we set $c(\mathcal{O}) := c(A\mathcal{O}) = \{V \mid \mathcal{O}(*_V) \neq \varnothing\}$.

Remark 2.12. As with all smashing localizations, Corollary 2.11 implies that $\mathrm{Im} E_{\mathcal{F}}^{\mathcal{T}} = \left\{\mathcal{O}^{\otimes} \in \mathrm{Op}^{\mathcal{T}} \mid c(\mathcal{O}) \subset \mathcal{F}\right\}$ is a $\otimes$-*ideal*, i.e. if $c(\mathcal{O}) \subset \mathcal{F}$, and $\mathcal{P}^{\otimes}$ is arbitrary, then $c\left(\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{P}\right) \subset \mathcal{F}$. In particular, $\underline{\mathrm{Op}}_I^{\otimes}$ is a *nonunital* symmetric monoidal full subcategory of $\underline{\mathrm{Op}}_J^{\otimes}$. ◁

Observation 2.13. There are natural equivalences

$$\begin{aligned}
\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} &\simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}^{\otimes}_{c\mathcal{O}} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}^{\otimes}_{c\mathcal{P}} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes},\\
&\simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}^{\otimes}_{c\mathcal{O}\cap c\mathcal{P}} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes},\\
&\simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}^{\otimes}_{c\mathcal{O}\cap c\mathcal{P}} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}^{\otimes}_{c\mathcal{O}\cap c\mathcal{P}} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes},\\
&\simeq E^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}} \mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}} \left(\mathcal{O}^{\otimes}\right) \overset{\mathrm{BV}}{\otimes} E^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}} \mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}} \left(\mathcal{P}^{\otimes}\right),\\
&\simeq E^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}} \left(\mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}} \left(\mathcal{O}^{\otimes}\right) \overset{\mathrm{BV}}{\otimes} \mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}} \left(\mathcal{P}^{\otimes}\right)\right).
\end{aligned}$$

The $c\mathcal{O} \cap c\mathcal{P}$-operads $\mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}}(\mathcal{O}^{\otimes})$ and $\mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}}(\mathcal{P}^{\otimes})$ both have at least one color; hence we may compute arbitrary tensor products of $\mathcal{T}$-operads via tensor products of equivariant operads with at least one color. ◁

Having done this, we may compute supports of arbitrary tensor products of $\mathcal{T}$-operads.

Proposition 2.14. *Suppose $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are $\mathcal{T}$-operads. Then,*

$$A\left(\mathcal{O} \overset{BV}{\otimes} \mathcal{P}\right) = E^{\mathcal{T}}_{\mathcal{F}} \mathrm{Bor}^{\mathcal{T}}_{\mathcal{F}} (A\mathcal{O} \vee A\mathcal{P}).$$

Proof. By Observation 2.13, we have equivalences

$$A\left(\mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes}\right) \simeq E^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}} A\left(\mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}}\left(\mathcal{O}^{\otimes}\right) \overset{\mathrm{BV}}{\otimes} \mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}}\left(\mathcal{P}^{\otimes}\right)\right),$$

so it suffices to prove the proposition in the case that $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ have at least one color.

In this case, first note that there exist maps

$$\mathcal{O}^{\otimes} \otimes \mathrm{triv}^{\otimes}_{\mathcal{T}}, \mathrm{triv}^{\otimes}_{\mathcal{T}} \otimes \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes},$$

so that

$$A\mathcal{O} \vee A\mathcal{P} \leq A(\mathcal{O} \vee \mathcal{P}).$$

On the other hand, there exists a composite map

$$\mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes} \to \mathcal{N}^{\otimes}_{A\mathcal{O}\infty} \otimes \mathcal{N}^{\otimes}_{A\mathcal{P}\infty} \to \mathcal{N}^{\otimes}_{A\mathcal{O}\vee A\mathcal{P}\infty} \otimes \mathcal{N}^{\otimes}_{A\mathcal{O}\vee A\mathcal{P}\infty} \to \mathcal{N}^{\otimes}_{A\mathcal{O}\vee A\mathcal{P}\infty},$$

hence $A(\mathcal{O} \vee \mathcal{P}) \leq A\mathcal{O} \vee A\mathcal{P}$. □

Corollary 2.15. $\mathrm{Op}_I \subset \mathrm{Op}_{\mathcal{T}}$ *is closed under binary tensor products.*

Corollary 2.16. *Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ be $\mathcal{T}$-operads.*

(1) Let S be a property in

$$\{\textit{at least one color, almost essentially unital, unital, has finite fold maps}\}.$$

If $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ have property S then $\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}$ has property S.

(2) If $\mathcal{O}^{\otimes}$ is unital and $\mathcal{P}^{\otimes}$ has at least one color, then $\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}$ is unital.

Proof. Each of these properties of $\mathcal{Q}^{\otimes}$ are detected by $A\mathcal{Q}$, so it suffices to verify the corresponding corollary in weak indexing systems. In view of Proposition 2.14, (1) was verified in [Ste24b, § 1.5]. For (2), it suffices to note that unitality of $A\mathcal{Q}$ is equivalent to the relation $A\mathbb{E}_0 \leq A\mathcal{Q}$, so $A\mathbb{E}_0 \leq A\mathcal{O}$ implies $A\mathbb{E}_0 \leq A\mathcal{O} \vee A\mathcal{P}$. □

2.2.2. *Results about reduced $\mathcal{T}$-operads extend to the aE-reduced setting.* Given I an aE-unital weak indexing system, set the notation $\overline{I} := \mathrm{Bor}^{\mathcal{T}}_{\upsilon(I)} I$, where $\upsilon(I) = \{V \mid \varnothing \to V \in I\}$ is the family of *units* of I (c.f. [Ste24b]).

Observation 2.17. For $\mathcal{P}$ an aE-unital $\mathcal{T}$-operad, the following is a pushout diagram:

$$\begin{tikzcd}
E^{\mathcal{T}}_{\upsilon(\mathcal{P})} \mathrm{Bor}^{\mathcal{T}}_{\upsilon(\mathcal{P})} \mathcal{P}^{\otimes} \arrow[r] & \mathcal{P}^{\otimes} \\
E^{\mathcal{T}}_{\upsilon(\mathcal{P})} \mathrm{Bor}^{\mathcal{T}}_{\upsilon(\mathcal{P})} \mathrm{triv}(\mathcal{P})^{\otimes} \arrow[r] \arrow[u] \arrow[ur, phantom, "\ulcorner"] & \mathrm{triv}(\mathcal{P})^{\otimes} \arrow[u]
\end{tikzcd}$$

Applying this for $\mathcal{P}^\otimes := \mathcal{N}^\otimes_{I\infty}$, we have a diagram

$$\begin{tikzcd} \mathcal{N}^\otimes_{\overline{I}\infty} \arrow[r] & \mathcal{N}^\otimes_{I\infty} \\ \mathrm{triv}_{v(I)} \arrow[r] \arrow[u] \arrow[ur, phantom, "\urcorner"] & \mathrm{triv}_{c(I)} \arrow[u] \end{tikzcd} \tag{13}$$

Unwinding definitions, this constructs a pullback diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}_{\mathcal{T}}(\mathcal{P},\mathcal{C}) \arrow[d] \\ \mathrm{Alg}_{\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{P}}\left(\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{C}\right) \arrow[r] & \mathrm{Fun}_{v(\mathcal{P})}\left(\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{P}, \mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{C}\right) \end{tikzcd} \tag{14}$$

for all $\mathcal{T}$-operads (hence $\mathcal{T}$-symmetric monoidal categories) $\mathcal{C}$. In particular, if $\mathcal{P}$ has at most one object (i.e. it is *aE-reduced*, then the above diagram reads as

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Gamma^{\mathcal{T}}\mathcal{C} \arrow[d] \\ \mathrm{Alg}_{\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{P}}\left(\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{C}\right) \arrow[r] & \Gamma^{v(\mathcal{P})}\mathcal{C} \end{tikzcd}$$

In particular, a $\mathcal{P}^\otimes$-algebra structure is seen as simply a $\mathcal{T}$-object together with a (reduced) $\mathrm{Bor}^\otimes_{\nu(\mathcal{P})}\mathcal{P}^\otimes$-algebra structure on its $\nu(\mathcal{P})$-Borelification. $\triangleleft$

Proposition 2.18. *Suppose I is an almost-E-unital weak indexing system. Then, for a $\mathcal{T}$-operad $\mathcal{O}^\otimes$, the map $\mathrm{Bor}^{\mathcal{T}}_{c(I)}\mathcal{O}^\otimes \to \mathcal{N}^\otimes_{\overline{I}\infty} \overset{BV}{\otimes} \mathcal{O}^\otimes$ is an equivalence if and only if the map*

$$\mathrm{Bor}(f) : \mathrm{Bor}^{\mathcal{T}}_{v(I)}\mathcal{O}^\otimes \to \mathcal{N}^\otimes_{\overline{I}\infty} \overset{BV}{\otimes} \mathrm{Bor}^{\mathcal{T}}_{v(I)}\mathcal{O}^\otimes$$

is an equivalence.

Proof. Tensoring Eq. (13) with $\mathcal{O}^\otimes$ yields the following.

$$\begin{tikzcd} E^{\mathcal{T}}_{v(I)}\left(\mathcal{N}^\otimes_{\overline{I}\infty} \overset{\mathrm{BV}}{\otimes} \mathrm{Bor}^{\mathcal{T}}_{v(I)}\mathcal{O}^\otimes\right) \arrow[r, phantom, "\simeq"] & E^{\mathcal{T}}_{\nu(I)}\mathrm{Bor}_{\nu(I)}\left(\mathcal{N}^\otimes_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^\otimes\right) \arrow[r] & \mathcal{N}^\otimes_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^\otimes \\ E^{\mathcal{T}}_{v(I)}\mathrm{Bor}^{\mathcal{T}}_{v(I)}\mathcal{O}^\otimes \arrow[u, "E\mathrm{Bor}(f)"] \arrow[r, phantom, "\simeq"] & E^{\mathcal{T}}_{v(I)}\mathrm{Bor}^{\mathcal{T}}_{v(I)}\mathrm{Bor}^{\mathcal{T}}_{c(I)}\mathcal{O}^\otimes \arrow[u] \arrow[r] \arrow[ur, phantom, "\urcorner"] & \mathrm{Bor}_{c(I)}\mathcal{O}^\otimes \arrow[u, "f"'] \end{tikzcd}$$

In particular, we find that if f is an equivalence, then $\mathrm{Bor}(f)$ is an equivalence, and if $\mathrm{Bor}(f)$ is an equivalence, then $E\mathrm{Bor}(F)$ is an equivalence, so pushout stability of equivalences implies that f is an equivalence. $\square$

2.2.3. *Operadic restriction and (co)induction.* Recall from [Ste24a] that the underlying $\mathcal{T}$-symmetric sequence forms a $\mathcal{T}$-functor $\underline{\mathrm{sseq}} : \underline{\mathrm{Op}}^{\mathrm{red}}_{\mathcal{T}} \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \underline{\mathcal{S}}_{\mathcal{T}})$; in particular, restrictions of $\underline{V}$-operads correspond with restrictions of $\underline{V}$-symmetric sequences; We may use this to upgrade Corollary 2.8 to an adjunction of $\mathcal{T}$-categories.

Proposition 2.19. $\mathrm{Res}^W_V\, \mathcal{N}^\otimes_{I\infty} \simeq \mathcal{N}^\otimes_{\mathrm{Res}^W_V I\infty}$*; more generally, Eq. (12) lifts to a a $\mathcal{T}$-adjunction*

$$\begin{tikzcd} \underline{\mathrm{Op}}_{\mathcal{T}} \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{wIndex}}_{\mathcal{T}} \arrow[l, bend left, "\mathcal{N}^\otimes_{(-)\infty}"] \end{tikzcd}$$

Proof. Restriction compatiblility of the underlying symmetric sequence implies that $\mathrm{Res}_V^W A\mathcal{O} = A\,\mathrm{Res}_V^W \mathcal{O}$, lifting A to a $\mathcal{T}$-functor $\underline{\mathrm{Op}}_{\mathcal{T}} \to \underline{\mathrm{wIndex}}_{\mathcal{T}}$ whose V-value is $A : \mathrm{Op}_V \to \mathrm{wIndex}_V$. The right adjoints $\mathcal{N}^{\otimes}_{(-)\infty}$ uniquely lift to a right $\mathcal{T}$-adjoint to $\mathcal{N}^{\otimes}_{(-)\infty}$ by [HA, Prop 7.3.2.1], completing the proposition. □

Since A is a $\mathcal{T}$-left adjoint, it is compatible with $\mathcal{T}$-colimits. Applying this for indexed coproducts, we immediately acquire the following properties of A.

Corollary 2.20. *If $\mathcal{O}, \mathcal{P}$ are $\mathcal{T}$-operads, then we have*

$$A(\mathcal{O} \sqcup \mathcal{P}) = A\mathcal{O} \vee A\mathcal{P}.$$

If $\mathcal{Q}$ is a V-operad, then we have

$$A\mathrm{Ind}_V^{\mathcal{T}} \mathcal{Q} = \mathrm{Ind}_V^{\mathcal{T}} A\mathcal{Q}.$$

We may compute use an analogous argument to that of [BHS22, Lem 4.1.13] to show that $\underline{\mathrm{Op}}_{\mathcal{T}}$ strongly admits $\mathcal{T}$-limits; since the fully faithful $\mathcal{T}$-functor $\underline{\mathrm{Op}}_{\mathcal{T}} \to \underline{\mathrm{Cat}}^{\mathrm{int-cocart}}_{/\underline{\mathrm{Span}}(\mathbb{F}_{\mathcal{T}})}$ possesses pointwise left adjoints (given by L_{Fbrs}), it possesses a $\mathcal{T}$-left adjoint; in particular, we may compute $\mathcal{T}$-limits of $\mathcal{T}$-operads in $\underline{\mathrm{Cat}}^{\mathrm{int-cocart}}_{/\underline{\mathrm{Span}}(\mathbb{F}_{\mathcal{T}})}$. Then, an analogous argument using [BHS22, Prop 2.3.7] constructs $\mathcal{T}$-limits in $\underline{\mathrm{Cat}}^{\mathrm{int-cocart}}_{/\underline{\mathrm{Span}}(\mathbb{F}_{\mathcal{T}})}$ in $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Span}}(\mathbb{F}_{\mathcal{T}}), \underline{\mathrm{Cat}}_{\mathcal{T}})_{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$, which strongly admits $\mathcal{T}$-limits, as its a slice $\mathcal{T}$-∞-category of a functor $\mathcal{T}$-∞-category into a $\mathcal{T}$-∞-category which strongly admits $\mathcal{T}$-limits. In particular, this implies that $\mathrm{Res}_U^V : \mathrm{Op}_V \to \mathrm{Op}_U$ has a right adjoint, which we call $\mathrm{CoInd}_U^V : \mathrm{Op}_U \to \mathrm{Op}_V$.

Proposition 2.21. *If $\mathcal{O}^{\otimes}$ is a d-truncated V-operad, then $\mathrm{CoInd}_V^W \mathcal{O}^{\otimes}$ is d-truncated.*

Proof. This follows simply by taking right adjoints within the following diagram

$$\begin{tikzcd}
\mathrm{Op}_{W,d} \arrow[r, "\mathrm{Res}_V^W"] \arrow[d, hook] & \mathrm{Op}_{V,d} \arrow[d, hook] \\
\mathrm{Op}_W \arrow[r, "\mathrm{Res}_V^W"] & \mathrm{Op}_V
\end{tikzcd}$$

□

Corollary 2.22. *If $\iota_V^{\mathcal{T}} : \mathrm{Tot}\,\underline{\Sigma}_V \to \mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}}$ is the inclusion, then*

$$\mathrm{sseq}\,\mathrm{CoInd}_V^W \mathcal{O}^{\otimes} \simeq \mathrm{CoInd}_V^W \,\mathrm{sseq}\,\mathcal{O}^{\otimes};$$

in particular, we have

$$A\mathrm{CoInd}_V^W \mathcal{O} = \mathrm{CoInd}_V^W A\mathcal{O}.$$

Proof. The first statement follows by noting that $\mathrm{Fr}\,\mathrm{Res}_V^W = \iota_V^{W*}\,\mathrm{Fr}$ and taking right adjoints. For the second statement, fix some $S \in \mathbb{F}_U$ for $U \to W$. In view of [Ste24b], we're tasked with proving that $\mathcal{O}(S) \neq \varnothing$ if and only if for all $U' \to W$, we have $\mathcal{O}(\mathrm{Res}_{U'}^W \mathrm{Ind}_U^W S) \neq \varnothing$.

The pointwise formula for right Kan extension along $\Sigma_V \to \Sigma_W$ yields

$$\mathcal{O}(S) \simeq \lim_{\mathrm{Ind}_V^W S \xleftarrow{\Sigma_W} T} \mathcal{O}(T) \simeq \lim_{\mathrm{Res}_{U'}^W \mathrm{Ind}_U^W S \simeq T} \mathcal{O}(T) \tag{15}$$

Note that a limit of spaces is nonempty if and only if its factors are nonempty; thus this limit is nonempty if and only if $\mathcal{O}(\mathrm{Res}_{U'}^W \mathrm{Ind}_U^W S)$ is nonempty for all $U' \to W$, as desired. □

We care about $\mathrm{CoInd}_V^{\mathcal{T}} \mathcal{O}^{\otimes}$ because it is a structure borne by *norms of algebras.*

Construction 2.23. Let $\mathcal{P}^{\otimes} \to \mathrm{CoInd}_V^W \mathcal{O}^{\otimes}$ be a functor of one-object I-operads, let $\mathcal{C}$ be a I-symmetric monoidal ∞-category, and let $V \to W$ be a transfer in I. Then, the adjunct map $\varphi : \mathrm{Res}_V^W \mathcal{P} \to \mathcal{O}^{\otimes}$ participates in a commutative diagram of symmetric monoidal functors

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathrm{Res}_V^W \mathcal{C}) \arrow[r, "\varphi^*"] \arrow[d, "U_V"] & \mathrm{Alg}_{\mathrm{Res}_V^W \mathcal{P}}(\mathrm{Res}_V^W \mathcal{C}) \arrow[r, "N_V^W"] \arrow[d, "U_V"] & \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \arrow[d, "U_W"] \\
\mathcal{C}_V \arrow[r, equal] & \mathcal{C}_V \arrow[r, "N_V^W"] & \mathcal{C}_W
\end{tikzcd}$$

Intuitively, we view this situation as saying that $\mathbf{CoInd}_V^W \mathcal{O}^\otimes$ bears the *universal* structure which is naturally endowed on $N_V^W X$ ranging across $X \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. ◁

2.3. **Examples of I-operads.** In this subsection, we survey various examples of I-operads which corepresent notable algebraic theories.

2.3.1. *Basic examples of $\mathcal{N}_{I\infty}^\otimes$ operads.* Fix $\mathcal{F} \subset \mathcal{T}$ be a $\mathcal{T}$-family. In [Ste24a], we introduced the example $\mathrm{triv}_{\mathcal{F}}^\otimes := \mathcal{N}_{I_{\mathrm{triv},\mathcal{F}}}^\otimes \simeq E_{\mathcal{F}}^{\mathcal{T}} \mathrm{triv}_{\mathcal{F}}^\otimes$. It was verified in [NS22, Cor 2.4.5] that this is characterized by the algebras

$$\mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathcal{C}) \simeq \Gamma^{\mathcal{F}} \mathcal{C};$$

i.e. its algebras are $\mathcal{F}$*-objects*. Furthermore, we used this in Corollary 2.11 to verify that $\mathrm{triv}_{\mathcal{F}}$ is $\overset{\mathrm{BV}}{\otimes}$-idempotent, with corresponding localizing subcategory consisting of the image of $E_{\mathcal{F}}^{\mathcal{T}}$ (i.e. $(-) \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{\mathcal{F}}$ implements *color-borelification*).

Example 2.24. Let $\mathcal{F} \subset \mathcal{T}$ be a $\mathcal{T}$-family, and denote by $\underline{\mathbb{F}}_{I_{\mathcal{F}}^0}$ the weak indexing system satisfying

$$S \in \underline{\mathbb{F}}_{I_{\mathcal{F}}^0} \qquad \iff \qquad S = *_V \ \text{ or } \ S \in \{\emptyset_V \mid V \in \mathcal{F}\}.$$

We set the notation $\mathbb{E}_{0,\mathcal{F}}^\otimes := \mathcal{N}_{I_{\mathcal{F}}^0 \infty}^\otimes$. Note that Eq. (13) specializes to a pushout presentation

$$\begin{tikzcd} E_{\mathcal{F}}^{\mathcal{T}} \mathbb{E}_0^\otimes \arrow[r] & \mathbb{E}_{0,\mathcal{F}}^\otimes \\ \mathrm{triv}_{\mathcal{F}}^\otimes \arrow[u] \arrow[r] \arrow[ur, phantom, "\urcorner"] & \mathrm{triv}_{\mathcal{T}}^\otimes \arrow[u] \end{tikzcd} \tag{16}$$

Intuitively, this presents $\mathbb{E}_{0,\mathcal{F}}$-algebras as $\mathcal{T}$-objects together with a distinguished "1-shaped element" of their underlying $\mathcal{F}$-objects; more precisely, the universal property for pushouts yields

$$\mathrm{Alg}_{\mathbb{E}_{0,\mathcal{F}}}(\mathcal{C}) \simeq \Gamma^{\mathcal{T}} \mathcal{C} \times_{\Gamma_{\mathcal{F}} \mathcal{C}} \left(\Gamma^{\mathcal{F}} \mathcal{C}\right)^{1/}.$$ ◁

We prove a generalization of the following in Proposition 3.9.

Corollary 2.25. *$\mathbb{E}_{0,\mathcal{F}}$ is $\overset{BV}{\otimes}$-idempotent, whose corresponding (smashing-)localizing subcategory of $\mathrm{Op}_{\mathcal{T}}$ consists of those whose $\mathcal{F}$-borelification is E-unital. Furthermore, $\mathbb{E}_{0,\mathcal{F}}$ is initial among almost-reduced operads whose $\mathcal{F}$-borelifications are unital.*

Example 2.26. Let $\mathbb{F}_{\mathcal{T}}^\infty$ be the minimal indexing system and I^∞ the corresponding indexing category [Ste24b]. We write $\mathbb{E}_\infty^\otimes := \mathcal{N}_{I^\infty}^\otimes$. ◁

Unwinding definitions, a weak $\mathcal{N}_\infty$-operad is an $\mathcal{N}_\infty$-operad if and only if it admits a map from $\mathbb{E}_\infty^\otimes$. Furthermore, in [Ste24a] we constructed a diagram

$$\begin{tikzcd} \mathrm{Op} \arrow[r, bend left, "\mathrm{Infl}_e^{\mathcal{T}}"] \arrow[r, phantom, "\top"] & \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Op}) \arrow[l, bend left, "\Gamma^{\mathcal{T}}"] & \simeq & \mathrm{Op}_{I^\infty} \arrow[r, bend left, "E_{I^\infty}^{\mathcal{T}}"] \arrow[r, phantom, "\perp"] & \mathrm{Op}_{\mathcal{T}} \arrow[l, bend left, "\mathrm{Bor}_{I^\infty}^{\mathcal{T}}"] \end{tikzcd}$$

whose equivalence $\mathrm{Op}_{I^\infty} \to \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Op})$ is furnished by pullback along $\mathcal{T}^{\mathrm{op}} \times \mathrm{Span}(\mathbb{F}) \to \mathrm{Span}_{I^\infty}(\mathbb{F}_{\mathcal{T}})$ and whose lefthand adjunction is between constant diagrams and limits; hence the fully faithful left adjoint has image spanned by constant functors.

We will simply write $\mathrm{Infl}_e^{\mathcal{T}}$ for the composite functor $\mathrm{Op} \to \mathrm{Op}_{\mathcal{T}}$. Unwinding definitions, $\mathrm{Op} \to \mathrm{Op}_{\mathcal{T}}$ sends weak $\mathcal{N}_\infty$-operads to weak $\mathcal{N}_\infty$-operads identified by

$$\mathbb{F}_{A\mathrm{Infl}_e^{\mathcal{T}} \mathcal{O}^\otimes, V} = \{n \cdot *_V \mid n \cdot * \in \mathbb{F}_{A\mathcal{O}}\}.$$

In particular, we have equivalences

$$\mathrm{triv}_{\mathcal{T}}^{\otimes} \simeq \mathrm{Infl}_e^{\mathcal{T}} \mathrm{triv}^{\otimes};$$
$$\mathbb{E}_0^{\otimes} \simeq \mathrm{Infl}_e^{\mathcal{T}} \mathbb{E}_0^{\otimes};$$
$$\mathbb{E}_\infty^{\otimes} \simeq \mathrm{Infl}_e^{\mathcal{T}} \mathbb{E}_\infty^{\otimes}.$$

This recovers Propositions 1.13 and 3.12 and yields an equivalence

$$\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_\infty}(\Gamma^{\mathcal{T}} \mathcal{C}).$$

2.3.2. *Equivariant little disks/steiner operads.* In [Bon19], a *genuine operadic nerve* 1-categorical functor was constructed between a model of graph-G operads and a model for G-operads. In [Ste24a], we lifted this to a conservative functor of ∞-categories $N^{\otimes}\colon g\mathrm{Op}_G \to \mathrm{Op}_G$. We define the G-operad

$$\mathbb{E}_V := N^{\otimes} D_V,$$

where D_V is the *little V-disks graph G-operad* of [GM17], whose n-ary $G \times \Sigma_n$ space has

$$D_V(n) := \mathrm{Emb}^{\mathrm{Rect.lin.}}(D(V) \times n, D(V)) \simeq \mathrm{Conf}_n(V)$$

by [GM17, Lem 1.2]. The resulting unital G-operad $\mathbb{E}_V$ was studied in [Hor19], who showed for instance that

$$\mathbb{E}_V(S) \simeq \mathrm{Emb}^{\mathrm{Rec.lin}}(D(V) \times S, D(V))^H \simeq \mathrm{Conf}_S^H(V),$$

where

$$\mathrm{Conf}_S^H(V) := \underset{\substack{W \subset V \\ \text{fin.dim}}}{\mathrm{colim}} \mathrm{Conf}_S^H(W)$$

in view of the fact that sseq preserves sifted colimits [Ste24a].

Given V a real orthogonal G-representation, we let $AV := A\mathbb{E}_V$, i.e. AV corresponds with the weak indexing system $\underline{\mathbb{F}}^V = \underline{\mathbb{F}}_{AV}$ of finite H-sets admitting an embedding into V.

Example 2.27. Let p be prime and let λ be an irreducible real orthogonal C_p-representation given by rotating the plane (or line if $p = 2$) by a primitive pth root of unity. Then, we may explicitly describe $A\infty\lambda = A\lambda$ by noting that it has infinitely many orbits of type $\left[C_p/e\right]$ and exactly one orbit of type $*_{C_p}$; this implies that it admits a C_p-equivariant embeddings of the C_p-set $a *_{C_p} + b\left[C_p/e\right]$ if and only if $a \leq 1$.

Moreover, the underlying vector space of λ is positive-dimensional, so it admits embeddings of $a*_e$ for all a. Hence we've completely characterized the weak indexing system, and it matches windex. ◁

A weak form of the following claim appears to be folklore.

Proposition 2.28. *Let G be a topological group, $H \subset G$ a closed subgroup, $S \in \mathbb{F}_H$ a finite H-set admitting an configuration $\iota\colon S \hookrightarrow W$, and V, W real orthogonal G-representations whose associated map*

$$\mathrm{Conf}_S^H(V) \hookrightarrow \mathrm{Conf}_S^H(V \oplus W)$$

is an equivalence. Then, $\mathrm{Conf}_S^H(V)$ is contractible.

Proof. Note that linear interpolation to ι yields a deformation of $\mathrm{Map}^H(S, V \oplus W)$ onto the subspace $\mathrm{Map}^H(S, W)$ consisting of maps whose image has zero projection to V. The path of a point beginning in the subspace $\mathrm{Conf}_S^H(V) \subset \mathrm{Conf}_S^H(V \oplus W)$ consisting of configurations with zero projection to W lands within $\mathrm{Conf}_S^H(V \oplus W)$ at all times; composing this deformation after the deformation retract $\mathrm{Conf}_S^H(V \oplus W) \xrightarrow{\sim} \mathrm{Conf}_H^S(V)$ thus yields a deformation retract of $\mathrm{Conf}_S^H(V \oplus W)$ onto $\{\iota\}$, so it is contractible.[9] By the equivalence $\mathrm{Conf}_S^H(V) \simeq \mathrm{Conf}_S^H(V \oplus W)$, the space $\mathrm{Conf}_S^H(V)$ is contractible as well. □

[9] Said explicitly, let $h\colon [0,1] \to \mathrm{Conf}_S^H(V \oplus W)$ be the deformation retract onto those configurations with zero projection to W. Then, our deformation retract h' onto $\iota(w)$ is computed by

$$h'(t) = \begin{cases} h(2t) & t \leq \frac{1}{2}, \\ (2-2t)\cdot h(1) + (2t-1)\iota & t \geq \frac{1}{2}. \end{cases}$$

Remark 2.29. This argument only produces *contractibility*, whereas the nonequivariant argument using Fadell and Neuwirth's fibration [FN62] sharply characterizes *n-connectivity* of $\mathrm{Conf}_S^H(V)$. In forthcoming work, the author will develop a Fadell-Neuwirth fibration for spaces of equivariant configurations in order to sharply characterize the n for which $\mathbb{E}_V$ is an n-connected G-operad. ◁

We say that V is a *weak universe* if it is a direct sum of infinitely many copies of a collection of irreducible real orthogonal G-representations; equivalently, there is an equivalence $V \simeq V \oplus V$.

Corollary 2.30. *If there exists an equivalence $\mathbb{E}_V^{\otimes} \simeq \mathbb{E}_{V\oplus W}^{\otimes}$, then the canonical map $\mathrm{Bor}_{AW}^{\mathcal{T}} \mathbb{E}_V^{\otimes} \to \mathcal{N}_{AW}^{\otimes}$ is an equivalence; in particular, if V is a weak universe, then the canonical map*

$$\mathbb{E}_V^{\otimes} \to \mathcal{N}_{AV}^{\otimes}$$

is an equivalence.

Observation 2.31. If V is a *universe* (i.e. it is a weak universe admitting a positive-dimensional fixed point locus), then it admits embeddings of all finite sets; hence it is not just a weak $\mathcal{N}_\infty$-operad, but an $\mathcal{N}_\infty$-operad. ◁

Because of the above observation, much study has been dedicated to the less general setting of *universes*; Rubin has given a complete and simple characterization of those indexing systems (equivalently, transfer systems) occurring as the arity-support of an $\mathbb{E}_V$-operad in [Rub19] for G abelian, where they are modelled via *Steiner operads*.

2.3.3. *Equivariant linear isometries.* Let V be a real orthogonal G-representation. The nth space of the *linear isometries operad* $\mathcal{L}(V)$, given by the linear isometries $\mathcal{L}(V^n, V)$, canonically acquires an action of $G \times \Sigma_n$, where G acts on V. Hence it presents a graph G-operad. We refer to the associated G-operad simply as $\mathcal{L}_V$.

Warning 2.32. As noted in [Rub19, Prop 2.7], the composite map of sets $A\mathcal{L}_{(-)} \colon \{C_9 - \mathrm{Universes}\} \to \mathrm{wIndex}_{C_9}$ fails to be monotone. ◁

The following proposition is often asserted in the literature, but the author could not find a proof; instead, she could only references recursively claiming it to be analogous to the nonequivariant case. We find this to be true, but spell it out regardless.

Proposition 2.33. *For any weak G-universe V, $\mathcal{L}_V$ is an $\mathcal{N}_\infty$-operad.*

Note that V being a weak G-universe is equivalent to existence of an equivalence

$$V \simeq V \oplus V;$$

hence it suffices to prove that $\mathcal{L}_V$ is a weak $\mathcal{N}_\infty$-operad. Unwinding definitions, for $S \in \mathbb{F}_H$ we find that its space of S-ary operations are given by the Γ_S-fixed points

$$\mathcal{L}_V(S) \simeq \mathcal{L}(V^{\oplus |S|}, V)^{\Gamma_S} \simeq \mathcal{L}^H(V^{\oplus S}, V),$$

where V^S is the S-fold direct sum $V^{\oplus S} \simeq \bigoplus_{G/H \in \mathrm{Orb}(S)} \mathrm{Ind}_H^G \mathrm{Res}_H^G V$. Thus, it suffices to prove the following.

Lemma 2.34. *If V is a weak G-universe and W a real orthogonal G-representation, then the space of equivariant linear isometric embeddings $\mathcal{L}^G(W, V)$ is either empty or contractible.*

Proof. Assume W embeds into V, and fix ι one such embedding. Unsurprisingly, we perform an analogous swindle to [May77]. Indeed, we write a decomposition $V \simeq V \oplus V$, and we perform a sequence of linear deformation retracts of $\mathcal{L}(W, V) \simeq \mathcal{L}(W, V \oplus V)$; the first deforms linearly onto those linear isometries intersecting trivially with the first summand, and the second deforms linearly onto $\iota \oplus 0$. □

Thus, Theorem F will imply the following.

Corollary 2.35. *Given U, V weak universes, $\mathcal{L}_U^{\otimes} \otimes \mathcal{L}_V^{\otimes}$ is an $\mathcal{N}_\infty$-operad.*

Example 2.36. If V is a weak G-universe with 0-dimensional fixed points, then it only embeds its self-induction from subgroups $H \subset G$ such that $V^H = 0$; indeed, we have $\left(\mathrm{Ind}_H^G \mathrm{Res}_H^G V\right)^G \simeq V^H$.

In particular, if λ is an irreducible C_p representation rotating the plane (or line when $p = 2$) by a primitive pth root of unity, the above argument shows that the canonical map $\mathbb{E}_\infty^{\otimes} \to \mathcal{L}_{\infty\lambda}^{\otimes}$ is an equivalence. ◁

Example 2.37 ([Rub21a, Prop 5.2, Cor 5.4])**.** The following conditions are equivalent:

(a) V is a complete G-universe;
(b) $A\mathcal{L}_V$ contains the transfer $e \subset G$;
(c) $\mathcal{L}_V^\otimes \simeq \mathrm{Comm}_G^\otimes$. ◁

The author is not aware of how to compute $A\mathcal{L}_U$ in general. In fact, we can't even reduce to irreducibles in the obvious way, as shown by the following disturbing fact.

Remark 2.38. We do not attain an equivalence $\mathcal{L}_U^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{L}_V^\otimes \simeq \mathcal{L}_{U\oplus V}^\otimes$. We see this from Example 2.37, since there are exactly 2 C_p-indexing systems, given by $\underline{\mathbb{F}}_{C_p}^\infty$ and $\underline{\mathbb{F}}_{C_p}$. This directly implies that

$$\mathcal{L}_{\lambda_p(i)}^\otimes \simeq \mathbb{E}_\infty^\otimes$$

for all i, where $\lambda_p(i)$ is the 2-dimensional real orthogonal C_p-representation on which a fixed generator acts by rotating by $2\pi/i$; hence the unique map

$$\mathbb{E}_\infty^\otimes \simeq \bigotimes_{i=0}^{p} \mathcal{L}_{\lambda_p(i)}^\otimes \to \mathcal{L}_{\bigoplus_i \lambda_p(i)} = \underline{\mathbb{F}}_{C_p}$$

is not an equivalence. ◁

3. I-commutative algebras

Philosophical remark 3.1. On one hand, it follows from Proposition 1.10 that $\mathcal{T}$-operads are determined conservatively by their theories of *algebras on $\mathcal{T}$-symmetric monoidal categories*; indeed, it suffices to characterize their algebras in the case $\underline{\mathcal{S}}_{\mathcal{T}}^{\mathcal{T}-\times}$.

On the other hand, the right adjoint $\mathrm{Cat}_{\mathcal{T}}^\otimes \to \mathrm{Op}_{\mathcal{T}}$ is full on cores, since automorphisms in a slice category $\mathrm{Cat}_{\mathcal{C}}$ automatically preserve cocartesian morphisms. Hence the associated map of spaces

$$\begin{array}{ccccc} \mathrm{Cat}_{\mathcal{T}}^{\otimes,\simeq} & \longrightarrow & \mathrm{Op}_{\mathcal{T}}^{\simeq} & \longrightarrow & \mathrm{Fun}(\mathrm{Op}_{\mathcal{T}}, \mathrm{Cat})^{\simeq} \\ \Psi & & & & \Psi \\ \mathcal{C}^\otimes & & \longmapsto & & \mathrm{Alg}_{(-)}(\mathcal{C}) \end{array}$$

is a summand inclusion. That is, a $\mathcal{T}$-symmetric monoidal category is determined (functorially on equivalences) by its categories of $\mathcal{O}$-algebras for each $\mathcal{O} \in \mathrm{Op}_{\mathcal{T}}$. ◁

Following along these lines and using Proposition 1.18, we will generally characterize algebraic theories in *arbitrary* $\mathcal{T}$-symmetric monoidal ∞-categories by reducing to the universal case of $\underline{\mathcal{S}}_{\mathcal{T}}^{\mathcal{T}-\times}$, which we study using category theoretic means. Indeed, in Section 3.1 we use this to bootstrap I-semiadditivity of $\underline{\mathrm{CMon}}_I(\mathcal{C})$ to I-cocartesianness of $\underline{\mathrm{CAlg}}_I^\otimes(\mathcal{C})$ for $\mathcal{C}^\otimes$ an arbitrary I-symmetric monoidal ∞-category. Using work from Appendix A, we use this to conclude lifts of Theorem C and Corollary D.

We take this to its logical extreme in Section 3.2, using this to completely characterize the smashing localizations associated with $\otimes$-idempotent weak $\mathcal{N}_\infty$-operads. As promised in the introduction, we use this classification to prove a generalization of Theorem F. Following this, in Section 3.3 we show that our results are sharp; if I is not almost essentially unital, then $\mathcal{N}_{I\infty}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^\otimes$ fails to be connected, so $\mathcal{N}_{I\infty}^\otimes$ is idempotent under $\overset{\mathrm{BV}}{\otimes}$ if and only if I is almost essentially unital.

3.1. Indexed tensor products of I-commutative algebras.

In Lemma A.4, we show that every object in a cocartesian I-symmetric monoidal structure bears a canonical I-commutative algebra algebra structure, i.e. $\underline{\mathrm{CAlg}}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence. In this subsection, we demonstrate the converse, or equivalently, we demonstrate that I-indexed tensor products of I-commutative algebras are indexed coproducts. We go on to use this to completely characterize the smashing localization on $\mathrm{Op}_{\mathcal{T}}$ associated with aE-unital weak $\mathcal{N}_\infty$-operads.

First, we need some prerequisites on unital $\mathcal{T}$-operads, beginning with the following.

Observation 3.2. If $\mathcal{C}^\otimes$ is an I-symmetric monoidal category with unit $\mathcal{T}$-object $1_\bullet$ and $X \in \mathcal{C}_V$, then $\mathrm{Map}_{\mathcal{C}^\otimes}(\varnothing, X) \simeq \mathrm{Map}_{\mathcal{C}_V}(1_V, X)$, so $\mathcal{C}^\otimes$ is unital if and only if $1_\bullet$ is initial; in particular, if $\mathcal{C}^\otimes$ is cartesian, then it is unital if and only if it is pointed. ◁

Using this, in [NS22] unitality was shown to be compatible with algebras, which we recall here.

Proposition 3.3 ([NS22, Thm 5.2.11])**.** *If $\mathcal{O}^\otimes$ is a unital $\mathcal{T}$-operad and $\mathcal{C}^\otimes$ an $\mathcal{O}$-monoidal ∞-category, then $\underline{\mathrm{Alg}}^\otimes_{\mathcal{O}}(\mathcal{C})$ is unital.*

Thus Yoneda's lemma characterizes $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ by its algebras over *unital* $\mathcal{T}$-operads.

Theorem 3.4 (Indexed tensor products of $\mathcal{N}_\infty$-algebras)**.** *The following are equivalent for $\mathcal{C}^\otimes \in \mathrm{Cat}_I^\otimes$.*

(a) For all morphisms $f : S \to T$ in $\mathcal{I}$, the action map $f_\otimes : \mathcal{C}_S \to \mathcal{C}_T$ is left adjoint to $f^* : \mathcal{C}_T \to \mathcal{C}_S$.
(b) There is an I-symmetric monoidal equivalence $\mathcal{C}^\otimes \simeq \mathcal{C}^{I-\sqcup}$ extending the identity on $\mathcal{C}$.
(c) For all unital I-operads $\mathcal{O}^\otimes$, the forgetful functor $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O}, \mathcal{C})$ is an equivalence.
(d) The forgetful functor $\underline{\mathrm{CAlg}}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence.

In order to prove Theorem 3.4, we introduce yet another condition:

(b') There is an I-symmetric monoidal equivalence $\mathcal{C}^\otimes \simeq \mathcal{C}^{I-\sqcup}$.

The implication (b') $\implies$ (c) is precisely the computation Lemma A.4. For the implication (c) $\implies$ (b'), note that $\mathcal{C}^{1/} \simeq \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C}) \to \mathcal{C}$ an equivalence implies that $\mathcal{C}^\otimes$ is unital by Proposition 3.3; hence Yoneda's lemma applied to $\mathrm{Op}_I^{\mathrm{uni}}$ constructs an I-operad equivalence $\mathcal{C}^\otimes \simeq \mathcal{C}^{I-\sqcup}$, which is an I-symmetric monoidal equivalence by Philosophical remark 3.1.

Furthermore, the implication (b') $\implies$ (a) follows by definition, (a) $\implies$ (b) is precisely Theorem B', and the statements (b) $\implies$ (b') and (c) $\implies$ (d) follow by neglect of assumptions. To summarize, we've arrived at the implications

$$\begin{tikzcd}
 & \text{(b)} \arrow[dd, Rightarrow] & & \\
\text{(a)} \arrow[ur, Rightarrow] & & \text{(c)} \arrow[r, Rightarrow] & \text{(d)} \\
 & \text{(b')} \arrow[ul, Rightarrow] \arrow[ur, Leftrightarrow] & &
\end{tikzcd} \tag{17}$$

Our workhorse lemma for closing the gap is the following.

Lemma 3.5. *The following are equivalent for $\mathcal{P}^\otimes \in \mathrm{Op}_I$:*

(e) The $\mathcal{T}$-∞-category $\underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive.
(f) For all $\mathcal{O}^\otimes \in \mathrm{Op}_I^{\mathrm{uni}}$, the forgetful functor

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{Alg}}^\otimes_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \underline{\mathcal{S}}_{\mathcal{T}})$$

is an equivalence.
(g) For all $\mathcal{O}^\otimes \in \mathrm{Op}_I^{\mathrm{uni}}$, the map $\mathrm{triv}^\otimes_{\mathcal{O}} \otimes^{\mathrm{BV}} \mathcal{P}^\otimes \to \mathcal{O}^\otimes \otimes^{\mathrm{BV}} \mathcal{P}^\otimes$ is an equivalence.
(h) For all $\mathcal{O}^\otimes \in \mathrm{Op}_I^{\mathrm{uni}}$ and $\mathcal{C} \in \mathrm{Cat}_I^\otimes$, the forgetful functor

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{Alg}}^\otimes_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \mathcal{C})$$

is an equivalence.

Proof. Since Proposition 1.16 shows that $\underline{\mathrm{Alg}}^\otimes_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is cartesian, the equivalence between (e) $\iff$ (f) is just (a) $\iff$ (c) applied to $\underline{\mathrm{Alg}}^\otimes_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$. (f) $\implies$ (g) follows from Proposition 1.10, and the implications (g) $\implies$ (h) $\implies$ (f) are obvious. □

Proof of Theorem 3.4. After the implications illustrated in Eq. (17), it suffices to prove that $\underline{\mathrm{CAlg}}_I(\mathcal{C})$ satisfies (c) for all $\mathcal{C}^\otimes \in \mathrm{Cat}_I^\otimes$; by Lemma 3.5, it suffices to prove that $\underline{\mathrm{CAlg}}_I(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive. But in fact, by Corollary 1.19 there is an equivalence $\underline{\mathrm{CAlg}}_I(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}})$ and the latter is I-semiadditive by Cnossen-Lenz-Linsken's semiadditive closure theorem Theorem 1.6. □

Rephrasing things somewhat, we've arrive at the following theorem.

Theorem C'. *Let $\mathcal{O}^\otimes$ be an almost-E-reduced $\mathcal{T}$-operad. Then, the following properties are equivalenent.*

(a) *The $\mathcal{T}$-∞-category $\underline{\mathrm{Alg}}_{\mathcal{O}}\underline{\mathcal{S}}_{\mathcal{T}}$ is $A\mathcal{O}$-semiadditive.*
(b) *The unique map $\mathcal{O}^\otimes \to \mathcal{N}^\otimes_{A\mathcal{O}\infty}$ is an equivalence.*

Furthermore, for any almost-E-unital weak indexing system I and I-symmetric monoidal ∞-category $\mathcal{C}^\otimes$, the I-symmetric monoidal ∞-category $\underline{\mathrm{CAlg}}^\otimes_I\mathcal{C}$ is cocartesian.

Proof. By Lemma 3.5 and Theorem 3.4, Condition (a) is equivalent to the condition that $\underline{\mathrm{Alg}}^\otimes_{\mathcal{O}}(\mathcal{C})$ is $A\mathcal{O}$-cocartesian for all $\mathcal{C}$. In fact by Theorem 3.4, this is equivalent to existence of the first equivalence in

$$\mathrm{CAlg}^\otimes_{A\mathcal{O}}\underline{\mathrm{Alg}}^\otimes_{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{CAlg}}_{A\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}),$$

which by Yoneda's lemma is equivalent to the unique map $\mathcal{O}^\otimes \to \mathcal{N}^\otimes_{A\mathcal{O}\infty}$ being an equivalence, i.e. Condition (b). The remaining statement follows immediately from Theorem 3.4. $\square$

Corollary 3.6. *Let $\mathcal{O}^\otimes$ be a reduced I-operad. Then, the canonical map $F : \mathcal{N}^\otimes_{I\infty} \to \mathcal{N}^\otimes_{I\infty} \otimes \mathcal{O}^\otimes$ is an equivalence.*

Proof. By Theorem 3.4, the forgetful map

$$F^* : \mathrm{Alg}_{\mathcal{O}\otimes\mathcal{N}_{I\infty}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}^\otimes_{\mathcal{N}_{I\infty}}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{N}_{I\infty}}(\mathcal{C})$$

is an equivalence for all distributive G-symmetric monoidal categories $\mathcal{C}$; the statement follows by specializing to $\mathcal{C} := \underline{\mathcal{S}}_G$ and applying Proposition 1.10. $\square$

3.2. The smashing localization for $\mathcal{N}^\otimes_{I\infty}$ and the main theorem.

3.2.1. *The smashing localization classified by $\mathcal{N}^\otimes_{I\infty}$.* We would like to prove the following.

Theorem 3.7. *Let I be an aE-unital weak indexing system. Then, an at-most one color $\mathcal{T}$-operad $\mathcal{O}^\otimes$ satisfies $\mathcal{O}^\otimes \overset{BV}{\otimes} \mathcal{N}^\otimes_{I\infty} \simeq \mathcal{O}^\otimes$ if and only if the following conditions are satisfied:*

(a) *$c(\mathcal{O}) \subset c(I)$.*
(b) *$v(\mathcal{O}) \supset v(I)$.*
(c) *The canonical map $\mathrm{Bor}^{\mathcal{T}}_{I\cap c(O)}\mathcal{O}^\otimes \to \mathcal{N}^\otimes_{I\cap c(\mathcal{O})}$ is an equivalence.*

Remark 3.8. Condition (c) of Theorem 3.7 is equivalent to the condition that, for all $\mathcal{P}^\otimes \in \mathrm{Op}_{I\cap c(\mathcal{O})}$ and $\mathcal{C} \in \mathrm{Cat}^\otimes_{\mathcal{T}}$, the forgetful map $\mathrm{Alg}_{\mathcal{P}}\underline{\mathrm{Alg}}^\otimes_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is an equivalence; by Theorem 3.4, this in turn is equivalent to the condition that, for all $\mathcal{C}$ (or just $\mathcal{C} = \underline{\mathcal{S}}_{\mathcal{T}}$) and all I-admissible $c(\mathcal{O})$-sets S, the S-indexed tensor products in $\underline{\mathrm{Alg}}^\otimes_{\mathcal{O}}(\mathcal{C})$ are indexed coproducts. ◁

Note that $c\left(\mathcal{O} \otimes \mathcal{N}^\otimes_{I\infty}\right) \simeq c(\mathcal{O}) \cap c(I)$, so (a) is necessary; in fact, assuming (a), we may apply Proposition 2.18. This reduces Theorem 3.7 to the following proposition.

Proposition 3.9. *Let I be a unital weak indexing system. Then, a one color $\mathcal{T}$-operad $\mathcal{O}^\otimes$ satisfies $\mathcal{O}^\otimes \overset{BV}{\otimes} \mathcal{N}^\otimes_{I\infty} \simeq \mathcal{O}^\otimes$ if and only if the following are true:*

(b) *$\mathcal{O}^\otimes$ is unital.*
(c) *The canonical map $\mathrm{Bor}^{\mathcal{T}}_I \mathcal{O}^\otimes \to \mathcal{N}^\otimes_{I\infty}$ is an equivalence.*

The hard step of this is the following lemma, whose proof we slightly postpone.

Lemma 3.10. *$\mathcal{O}^\otimes \in \mathrm{Op}^{\mathrm{oc}}_{\mathcal{T}}$ satisfies $\mathcal{O}^\otimes \simeq \mathcal{O}^\otimes \overset{BV}{\otimes} \mathbb{E}^\otimes_0$ if and only if it is unital.*

Proof of Proposition 3.9. First assume that $\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}^\otimes_{I\infty} \simeq \mathcal{O}^\otimes$. By Lemma 3.10, we have

$$\mathcal{O}^\otimes \simeq \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}^\otimes_{I\infty} \simeq \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}^\otimes_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathbb{E}^\otimes_0 \simeq \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{E}^\otimes_0,$$

so $\mathcal{O}^\otimes$ is unital. To prove (c), in light of Remark 3.8, it suffices to note that the equivalence $\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}^\otimes_{I\infty} \simeq \mathcal{O}^\otimes$ demonstrates that the canonical map

$$\begin{aligned}
\mathrm{CAlg}_I(\underline{\mathcal{S}}_{\mathcal{T}}) &\xleftarrow{\sim} \mathrm{Alg}_{\mathrm{Bor}_I^{\mathcal{T}}\mathcal{O}} \underline{\mathrm{CAlg}}_I^\otimes(\underline{\mathcal{S}}_{\mathcal{T}}) \\
&\simeq \mathrm{CAlg}_I^\otimes \underline{\mathrm{Alg}}^\otimes_{\mathrm{Bor}_I^{\mathcal{T}}\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}}) \\
&\to \mathrm{Alg}_{\mathrm{Bor}_I^{\mathcal{T}}\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})
\end{aligned}$$

is an equivalence, so Proposition 1.10 proves that $\mathrm{Bor}_I^{\mathcal{T}}\mathcal{O}^\otimes \to \mathcal{N}^\otimes_{I\infty}$ is an equivalence. The converse follows by noting that each of the above arguments works in reverse. □

3.2.2. *The proof of the main theorem.* We are finally ready for Theorem F. We start with the unital case.

Proposition 3.11. *When I and J are unital, there is an equivalence $\mathcal{N}^\otimes_{I\infty} \overset{BV}{\otimes} \mathcal{N}^\otimes_{J\infty} \simeq \mathcal{N}^\otimes_{I\vee J\infty}$.*

Proof. By [CSY20, Prop 5.1.8], $\mathcal{N}^\otimes_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^\otimes_{J\infty}$ is an $\overset{\mathrm{BV}}{\otimes}$-idempotent classifying the conjunction of the properties which are classified by $\mathcal{N}^\otimes_{I\infty}$ and $\mathcal{N}^\otimes_{\infty}$; that is, a unital $\mathcal{T}$-operad $\mathcal{O}^\otimes$ is fixed by $(-) \overset{\mathrm{BV}}{\otimes} \mathcal{N}^\otimes_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^\otimes_{J\infty}$ if and only if $\underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive and J-semiadditive; by Proposition 1.3, this is equivalent to the property that $\underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is $I \vee J$-semiadditive , i.e. $\mathcal{O}^\otimes$ is fixed by $(-) \overset{\mathrm{BV}}{\otimes} \mathcal{N}^\otimes_{I\vee J}$. Thus, we have

$$\mathcal{N}^\otimes_{I\vee J} \simeq \mathcal{N}^\otimes_{I\vee J} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^\otimes_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\overset{\mathrm{BV}}{\otimes}}_{J\infty} \simeq \mathcal{N}^\otimes_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}.$$

□

We may now conclude the full theorem, which we restate in the orbital case.

Theorem F'. *The functor $\mathcal{N}^\otimes_{(-)\infty} : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T}}$ lifts to a fully faithful $\mathcal{T}$-right adjoint*

$$\begin{tikzcd}
\underline{\mathrm{wIndex}}_{\mathcal{T}} \arrow[r, bend right, "\mathcal{N}^\otimes_{(-)\infty}"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Op}}_{\mathcal{T}} \arrow[l, bend right, "A"']
\end{tikzcd}$$

whose restriction $\underline{\mathrm{wIndex}}^{aE\mathrm{uni}}_{\mathcal{T}} \subset \underline{\mathrm{Op}}_{\mathcal{T}}$ is symmetric monoidal. Furthermore, the resulting tensor product on $\mathrm{wIndex}^{aE\mathrm{uni},\otimes}_{\mathcal{T}}$ is computed by the Borelified join

$$I \otimes J = \mathrm{Bor}^{\mathcal{T}}_{\mathrm{cSupp}(I\cap J)}(I \vee J);$$

in particular, when I and J are almost-E-unital weak indexing systems, we have

$$\begin{aligned}
\mathcal{N}^\otimes_{I\infty} \otimes \mathcal{N}^\otimes_{J\infty} &\simeq \mathcal{N}^\otimes_{(I\vee J)\infty} \otimes \mathrm{triv}^\otimes_{c(I\cap J)} \\
\mathcal{N}^\otimes_{I\infty} \times \mathcal{N}^\otimes_{J\infty} &\simeq \mathcal{N}^\otimes_{(I\cap J)\infty} \\
\mathrm{Res}^W_V \mathcal{N}^\otimes_{I\infty} &\simeq \mathcal{N}^\otimes_{\mathrm{Res}^W_V I\infty} \\
\mathrm{CoInd}^W_V \mathcal{N}^\otimes_{I\infty} &\simeq \mathcal{N}^\otimes_{\mathrm{CoInd}^W_V I\infty}.
\end{aligned}$$

Hence W-norms of I-commutative algebras are $\mathrm{CoInd}^W_V I$-commutative algebras, and when I, J are almost-unital, we have

$$\underline{\mathrm{CAlg}}^\otimes_I \underline{\mathrm{CAlg}}^\otimes_J(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_{I\vee J}(\mathcal{C}). \tag{18}$$

Proof of Theorem F'. The $\mathcal{T}$-adjunction is precisely Proposition 2.19, the equations are immediate from the symmetric monoidal adjunction, the statement about norms of I-commutative algebras is Construction 2.23, and Eq. (18) follows immediately from symmetric monoidality of $\mathcal{N}^\otimes_{(-)\infty}$. We are left with proving that the adjunction is symmetric monoidal in the aE-unital case.

In view of Proposition 2.14, to prove that this is a $\mathcal{T}$-symmetric monoidal adjunction with the prescribed tensor product, it suffices to prove that the collection of aE-unital weak $\mathcal{N}_\infty$-operads is $\overset{\mathrm{BV}}{\otimes}$-closed, for which it suffices to prove that for all aE-unital weak indexing systems I and J, the unique map $\varphi : \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \to \mathcal{N}_{I\vee J}^{\otimes}$ is an equivalence. In fact, by Proposition 2.18, it suffices to prove that $\mathrm{Bor}_{v(I\cap J)}^{\mathcal{T}}(\varphi)$ is an equivalence, i.e. we may assume that I and J are unital. Then, the statement is precisely Proposition 3.11. $\square$

3.2.3. *Unitalization.* We now focus on Lemma 3.10, beginning by recalling a result of Nardin-Shah.

Proposition 3.12 ([NS22, Thm 5.2.10])**.** *If $\mathcal{C}$ is a $\mathcal{T}$-symmetric monoidal ∞-category with unit $\mathcal{T}$-object 1, then there is a canonical equivalence $\underline{\mathrm{Alg}}_{\mathbb{E}_0}(\mathcal{C}) \simeq \mathcal{C}^{1/}$.*

In the case that $\mathcal{C}$ is a cartesian I_0-symmetric monoidal category (i.e. the unit is terminal, e.g. it is pulled back from a cartesian $\mathcal{T}$-symmetric monoidal category), this has a more familiar form, as

$$\mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C}^{\times}) \simeq \left(\Gamma^{\mathcal{T}}\mathcal{C}\right)_*.$$

We use this to prove the following strengthening of Lemma 3.10.

Proposition 3.13. *Given a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ with at least one color, the following are equivalent:*

- *(a)* $\mathrm{Bor}_{I_0}^{\mathcal{T}}\mathcal{O}^{\otimes}$ *is unital.*
- *(b)* $\mathcal{O}^{\otimes}$ *is unital.*
- *(c) The ∞-category $\mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is pointed.*
- *(d)* $\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathbb{E}_0^{\otimes}$.
- *(e)* $\mathrm{Bor}_{I_0}^{\mathcal{T}}\mathcal{O}^{\otimes} \simeq \mathbb{E}_0^{\otimes} \overset{BV}{\otimes} \mathrm{Bor}_{I_0}^{\mathcal{T}}\mathcal{O}^{\otimes}$.
- *(f) The ∞-category $\mathrm{Alg}_{\mathrm{Bor}_{I_0}^{\mathcal{T}}\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is pointed*

Proof. (a) $\implies$ (b) follows immediately by definition; (b) $\implies$ (c) follows immeditely by [NS22, Thm 5.2.11]. (c) $\implies$ (d) and (e) $\implies$ (f), since $\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_0}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Mon}_{\mathbb{E}_0}\mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})_*$ over $\mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$. (d) $\implies$ (e) follows by applying Borelification.

What's left is to prove that (f) $\implies$ (a). We argue the contrapositive, writing $\mathcal{P}^{\otimes} := \mathrm{Bor}_{I_0}^{\mathcal{T}}\mathcal{O}^{\otimes}$, assuming that $\mathcal{P}^{\otimes}$ is not unital, and fixing $C \in \mathcal{P}_V$ such that $\mathcal{P}(\varnothing_V; C) \neq *$. We choose the "skyscraper" $\mathcal{P}$-algebra M, with values

$$M(D) = \begin{cases} \mathcal{P}(\varnothing_V, C) & D = C \\ * & \text{otherwise,} \end{cases}$$

gotten by truncating the functor corepresented by $\varnothing$. Then, note that

$$\mathrm{Map}(*_{\mathcal{P}}, M) \simeq \mathcal{P}(\varnothing; C) \not\simeq *,$$

so the unit $*_{\mathcal{P}} \in \mathrm{Alg}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is not initial. By [NS22, Thm 5.2.11] it is terminal, so by contrapositgion we have shown (f) $\implies$ (a). $\square$

Last, we point out a corollary. In Appendix A, given $\mathcal{C}$ a $\mathcal{T}$-category (which may not admit I-indexed coproducts), we construct an I-operad $\mathcal{C}^{I-\sqcup}$ together with an equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\sqcup}) \simeq \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \tag{19}$$

for all unital I-operads $\mathcal{O}$. In particular, this proves the following.

Corollary 3.14. *The restriction $U_{\mathrm{uni}} : \underline{\mathrm{Op}}_{\mathcal{T}}^{\mathrm{uni}} \to \underline{\mathrm{Cat}}_{\mathcal{T}}$ is left $\mathcal{T}$-adjoint to $(-)^{I-\sqcup}$.*

Warning 3.15. Corollary 3.14 shows that no nontrivial $\mathcal{T}$-colimit of one-color $\mathcal{T}$-operads has one color; in particular, no one-color $\mathcal{T}$-operads are the result of a nontrivial induction. ◁

Furthermore, note that Theorem 3.4 yields equivalences

$$\begin{aligned} \mathrm{CAlg}_{\mathcal{T}}\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}^{I-\sqcup}) &\simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{CAlg}}_{\mathcal{T}}^{\otimes}(\mathcal{C}^{I-\sqcup}) \\ &\simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\sqcup}), \end{aligned}$$

for all $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}^{\mathrm{uni}}$. Hence Theorem 3.4 implies the following.

Corollary 3.16. *Suppose $\mathcal{O}^{\otimes}$ is a unital I-operad and $\mathcal{C}$ admits I-indexed coproducts. Then, the I-symmetric monoidal category $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}\left(\mathcal{C}^{I-\sqcup}\right)$ is cocartesian.*

We use this to compute the $\mathcal{T}$-category underlying BV tensor products.

Proposition 3.17. *The underlying category $U|_{\mathrm{uni}} : \mathrm{Op}^{\mathrm{uni}}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}}$ functor sends*

$$U\left(\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}\right) \simeq U(\mathcal{O}^{\otimes}) \times U(\mathcal{P}^{\otimes}).$$

in particular, $\underline{\mathrm{Op}}^{\mathrm{red}}_{\mathcal{T}} \subset \underline{\mathrm{Op}}_{\mathcal{T}}$ is a $\overset{BV}{\otimes}$-closed $\mathcal{T}$-subcategory.

Proof. Corollaries 3.14 and 3.16 together yield a string of equivalences

$$\begin{aligned}
\mathrm{Fun}_{\mathcal{T}}\left(U\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right), \mathcal{C}\right) &\simeq \mathrm{Alg}_{\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{P}}\left(\mathcal{C}^{I-\sqcup}\right) \\
&\simeq \mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}\left(\mathcal{C}^{I-\sqcup}\right) \\
&\simeq \mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{Fun}}_{\mathcal{T}}\left(U(\mathcal{P}^{\otimes}), \mathcal{C}\right)^{I-\sqcup} \\
&\simeq \mathrm{Fun}_{\mathcal{T}}\left(U(\mathcal{O}^{\otimes}), \underline{\mathrm{Fun}}_{\mathcal{T}}\left(U(\mathcal{P}^{\otimes}), \mathcal{C}\right)\right) \\
&\simeq \mathrm{Fun}_{\mathcal{T}}\left((U(\mathcal{O}^{\otimes}) \times U(\mathcal{P}^{\otimes}), \mathcal{C}\right),
\end{aligned}$$

so the result follows by Yoneda's lemma. $\square$

We additionally transport this to the almost essentially unital setting.

Corollary 3.18. *If $\mathcal{O}^{\otimes}$ is an almost essentially unital I-operad for I a unital weak indexing system and $\mathcal{C}$ a $\mathcal{T}$-∞-category, then the forgetful functor*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\sqcup}) \to \mathrm{Fun}_{\mathcal{T}}(U(\mathcal{O}), \mathcal{C})$$

is an equivalence; hence there is an equivalence $U\left(\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}\right) \simeq U\left(\mathcal{O}^{\otimes}\right) \times U\left(\mathcal{P}^{\otimes}\right)$ whenever $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are almost essentially unital.

Proof. In view of Proposition 1.13 and Corollary 3.14, the pullback presentation of Eq. (14) specializes to

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \arrow[rr, "\sim"] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}_{\mathcal{T}}(\mathcal{P}, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}_{v(\mathcal{P})}\left(\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P}}\mathcal{P}, \mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{C}\right) \arrow[rr, "\sim"'] & & \mathrm{Fun}_{v(\mathcal{P})}\left(\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P}}\mathcal{P}, \mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{C}\right)
\end{tikzcd}$$

the first statement follows from the top horizontal equivalence. The remaining statement follows by repeating the proof of Proposition 3.17 verbatim. $\square$

Corollary 3.19. *The full subcategory $\mathrm{Op}^{aE\,\mathrm{red}}_{\mathcal{T}} \subset \mathrm{Op}_{\mathcal{T}}$ is closed under tensor products.*

3.3. n-connected $\mathcal{T}$-operads and the failure of nonunital equivariant Eckmann-Hilton.

Recall from [Ste24a] that a map of $\mathcal{T}$-operads $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is called *n-connected* if the induced map $h_{n+1}\mathcal{O}^{\otimes} \to h_{n+1}\mathcal{P}^{\otimes}$ is an equivalence. We will say that a $\mathcal{T}$-operad with at most one color $\mathcal{O}^{\otimes}$ is *n-connected* if the canonical map $\mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{AO\infty}$ is n-connected, and write the full subcategory of n-connected $\mathcal{T}$-operads as

$$\mathrm{Op}^{\leq \mathrm{oc}}_{\mathcal{T}, \geq n} \subset \mathrm{Op}^{\leq \mathrm{oc}}_{\mathcal{T}}.$$

These are further characterized by the following corollary of a result of [Ste24a].

Proposition 3.20. *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-operad with at most one color. Then, the following are equivalent:*

(a) For all $V \in \mathcal{T}$ and $S \in \mathbb{F}_V$, $\mathcal{O}(S)$ is n-connected.
(b) $\mathcal{O}^{\otimes}$ is n-conncted.
(c) For all $\mathcal{T}$-symmetric monoidal $(n+1)$-categories $\mathcal{C}$, the canonical $\mathcal{T}$-symmetric monoidal functor

$$\underline{\mathrm{CAlg}}^{\otimes}_{AO}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$$

is an equivalence.

(d) The canonical functor

$$\mathrm{CMon}_{A\mathcal{O}}(\mathcal{S}_{\le n+1}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{S}_{\le n+1})$$

is an equivalence.

Proof. A generalization of this for *arbitrary* n-connected maps is in [Ste24a, § 2.6]. □

We conclude from Theorem F' that n-connected $\mathcal{T}$-operads are closed under tensor products.

Corollary 3.21. *When $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ are n-connected almost essentially reduced $\mathcal{T}$-operads, $\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}$ is an n-connected almost essentially reduced $\mathcal{T}$-operad.*

Proof. In view of Theorem F', we have a string of natural equivalences

$$\begin{aligned}\mathrm{Mon}_{\mathcal{O}\otimes\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\le n+1}) &\simeq \mathrm{Mon}_{\mathcal{O}}\underline{\mathrm{Mon}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\le n+1})\\ &\simeq \mathrm{Mon}_{\mathcal{O}}\underline{\mathrm{CMon}}_{A\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\le n+1})\\ &\simeq \mathrm{CMon}_{A\mathcal{O}}\underline{\mathrm{CMon}}_{A\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\le n+1})\\ &\simeq \mathrm{CMon}_{A\mathcal{O}\vee A\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\le n+1})\\ &\simeq \mathrm{CMon}_{A(\mathcal{O}\otimes\mathcal{P})}(\underline{\mathcal{S}}_{\mathcal{T},\le n+1}),\end{aligned}$$

induced by the unique map $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \to \mathcal{N}^{\otimes}_{A\left(\mathcal{O}\overset{\mathrm{BV}}{\otimes}\mathcal{P}\right)}$. By Proposition 3.20, this implies that $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ is n-connected. □

Remark 3.22. The unit object $\mathrm{triv}^{\otimes}_{\mathcal{T}} \in \mathrm{Op}_{\mathcal{T}}$ is n-connected for all n, so n-connected $\mathcal{T}$-operads are closed under k-fold tensor products for all $k \in \mathbb{N}$. ◁

The example $\mathrm{triv}^{\otimes}_{\mathcal{T}} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes}$ demonstrates that this is the best we can say without further assumptions on the $\mathcal{T}$-operads in question; the author hopes to return to this question in forthcoming work, constructing analogues to [SY19] under the assumption that $A\mathcal{O} = A\mathcal{P}$.

Observation 3.23. Fix I a weak indexing system. By Propositions 2.6 and 2.14, there is a contractible space of diagrams of the following form:

$$\mathcal{N}^{\otimes}_{I\infty} \simeq \mathcal{N}^{\otimes}_{I\infty} \otimes^{\mathrm{BV}} \mathrm{triv}^{\otimes}_{\mathrm{cSupp}(I)} \xrightarrow{\mathrm{id}\otimes^{\mathrm{BV}}\mathrm{can}} \mathcal{N}^{\otimes}_{I\infty} \otimes^{\mathrm{BV}} \mathcal{N}^{\otimes}_{I\infty} \to \mathcal{N}^{\otimes}_{I\infty};$$

furthermore, the composite $\mathcal{N}^{\otimes}_{I\infty} \to \mathcal{N}^{\otimes}_{I\infty}$ is homotopic to the identity by Proposition 2.6.

In particular, this implies that there is a canonical natural *split codiagonal* diagram

$$\begin{tikzcd} & \mathrm{CAlg}_I\underline{\mathrm{CAlg}}^{\otimes}_I(-) \arrow[dr, "U"] & \\ \mathrm{CAlg}_I(-) \arrow[ur, "\delta"] \arrow[rr, equal] & & \mathrm{CAlg}_I(-) \end{tikzcd}$$

◁

We will interpret $\mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty}$-algebras as pairs of interchanging I-commutative algebra structures in Observation 3.30, thus δ will take a structure to two interchanging copies of itself, and U will simply forget one of the structures. Hence a weak form of the *Eckmann-Hilton argument* states that the functor U is an equivalence, or equivalently, δ is an equivalence.

Unfortunately, this does not hold for all weak indexing systems I. The following counterexample to nonunital Eckmann-Hilton was pointed out to the author by Piotr Pstrągowski.

Example 3.24. Let R be a nonzero commutative ring. Then, the Abelian group underlying R sports a $\mathrm{Comm}^{\otimes}_{nu} \otimes \mathrm{Comm}^{\otimes}_{nu}$ structure given by the two multiplications

$$\mu(r,s) = rs, \qquad\qquad \mu_0(r,s) = 0,$$

which are easily seen to satisfy interchange but be distinct. In particular, the associated $\mathrm{Comm}^{\otimes}_{nu} \otimes \mathrm{Comm}^{\otimes}_{nu}$-algebra is not in the essential image of the codiagonal

$$\mathrm{Alg}_{\mathrm{Comm}_{nu}}(\mathbf{Ab}) \to \mathrm{Alg}_{\mathrm{Comm}_{nu}}\underline{\mathrm{Alg}}_{\mathrm{Comm}_{nu}}(\mathbf{Ab}),$$

so δ is not an equivalence. ◁

An analogous weak form of the ∞-categorical Eckmann-Hilton argument of [SY19] yields a classification of $\otimes^{\mathrm{BV}}$-idempotent algebras in *reduced* ∞-operads. In fact, Example 3.24 shows that the associated unitality assumption only misses one example among nonequivariant weak $\mathcal{N}_\infty$-operad.

Corollary 3.25. *A weak $\mathcal{N}_\infty$-$*$-operad $\mathcal{O}^\otimes$ possesses a map $\mathrm{triv}^\otimes \to \mathcal{O}^\otimes$ inducing an equivalence*

$$\mathcal{O}^\otimes \xrightarrow{\sim} \mathcal{O}^\otimes \otimes^{\mathrm{BV}} \mathcal{O}^\otimes$$

if and only if $\mathcal{O}^\otimes$ is equivalent to $\mathrm{triv}^\otimes$, $\mathbb{E}_0^\otimes$, or $\mathbb{E}_\infty^\otimes$.

Proof. [SY19, Cor 5.3.4] covers the unital case, so it suffices to assume that $\mathcal{O}(\varnothing) = \varnothing$ and show that $\mathcal{O}^\otimes \simeq \mathrm{triv}^\otimes$. Note that Comm_{nu} is the terminal nonunital $\mathcal{N}_\infty$-$*$-operad, i.e. there exists a map $\mathcal{O}^\otimes \to \mathrm{Comm}_{nu}$, yielding a diagram

$$\begin{tikzcd} \mathcal{O}^\otimes \otimes \mathcal{O}^\otimes \arrow[r] & \mathrm{Comm}_{nu}^\otimes \otimes \mathrm{Comm}_{nu}^\otimes \\ \mathcal{O}^\otimes \arrow[r] \arrow[u] & \mathrm{Comm}_{nu}^\otimes \arrow[u] \end{tikzcd}$$

Pulling back the example of Example 3.24, we find that if $\mathcal{O}(n) = *$ for any $n \neq 1$, then R has a $\mathcal{O}^\otimes \otimes \mathcal{O}^\otimes$-structure that is not in the image of the diagonal; hence $\mathcal{O}(n) = \varnothing$ when $n \neq 1$, i.e. it's equivalent to $\mathrm{triv}^\otimes$. $\square$

By [Ste24b], this is precisely the list of nonempty aE-unital weak indexing systems for $*$. In this section, we introduce an equivariant analogue to this argument in order to prove the following proposition; in order to do so, we say that $\mathcal{O}^\otimes$ is *n-connected* if $\mathcal{O}(S)$ is n-connected for all n, and we say that $\mathcal{O}^\otimes$ is *connected* if it is 0-connected.

Proposition 3.26. *Suppose $\mathcal{N}_{I\infty}^\otimes \overset{BV}{\otimes} \mathcal{N}_{I\infty}^\otimes$ is connected. Then, I aE-unital.*

Thus, given a non-aE-unital weak indexing category I, it will suffices to construct two distinct interchanging I-commutative algebra structures in some $\mathcal{T}$-symmetric monoidal 1-category. We do so by passing to a universal case.

Construction 3.27. Let $\mathcal{F}^\perp \subset \mathcal{T}$ be a $\mathcal{T}$-cofamily Then, define the full subcategory

$$\mathbb{F}_V \supset \mathbb{F}_{\mathcal{F}^\perp-nu,V} = \begin{cases} \mathbb{F}_V - \{\varnothing_V\} & V \in \mathcal{F}^\perp; \\ \mathbb{F}_V & \text{otherwise.} \end{cases}$$

This is evidently closed under restriction, so it defines a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_{\mathcal{F}^\perp-nu} \subset \underline{\mathbb{F}}_{\mathcal{T}}$. Furthermore, it has contractible V-sets and is closed under self-indexed coproducts by inspection. Hence it is a weak indexing system. ◁

Observation 3.28. $\underline{\mathbb{F}}_{\mathcal{F}^\perp-nu}$ is the terminal weak indexing system possessing unit-family $v(I) = \mathcal{F}$; $\underline{\mathbb{F}}_I$ is non-aE-unital if and only if it shares a non-contractible V-set with $\underline{\mathbb{F}}_{v(I)^\perp-nu}$ for some $V \in v(I)^\perp$; thus, to prove Proposition 3.26, it suffices to construct two interchanging $\mathcal{N}_I^{\mathcal{F}^\perp}$-algebra structures who differ in $v(I)^\perp$-arities and apply the analogous argument to Corollary 3.25. ◁

Construction 3.29. Let M be a nontrivial commutative monoid and le $F : \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Set}$ be the functor

$$F(S) := M^{|S|}$$

with functoriality induced by the action maps in M; this is evidently product-preserving, i.e. it's a $\mathcal{T}$-commutative monoid in Set. In particular, since $\mathrm{Comm}_{\mathcal{T}}^\otimes \otimes \mathbb{E}_0^\otimes \simeq \mathrm{Comm}_{\mathcal{T}}^\otimes$, this is in the image of the forgetful functor $\mathrm{CAlg}_{\mathcal{T}}(\mathrm{Set}_*) \to \mathrm{CMon}_{\mathcal{T}}(\mathrm{Set})$, so we replace F with a product preserving functor $F' : \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Set}_*$.

We furthermore modify this, constructing a new functor $G : \mathrm{Span}_{I_{\mathcal{F}^\perp-nu}}(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Set}_*$ via

$$G(S) := \prod_{U \in \mathrm{Orb}(S) \cap \mathcal{F}^\perp} F'(U).$$

This is product-preserving, so it yields an $I_{\mathcal{F}^\perp-nu}$-commutative monoid in Set_*. Last, we let G_0 be the $I_{\mathcal{F}^\perp-nu}$ on the underlying G-coefficient system of pointed sets whose action maps are all zero. ◁

We would like to show that G and G_0 interchange, for which we make the following observation.

Observation 3.30. Let $\mathcal{C}^\otimes$ be a $\mathcal{T}$-symmetric monoidal 1-category, and let $\mathcal{O}^\otimes, \mathcal{P}^\otimes$ be 1-object $\mathcal{T}$-1-operads. The data of a bifunctor of $\mathcal{T}$-operads $\mathcal{O}^\otimes \times \mathcal{P}^\otimes \to \mathcal{C}^\otimes$ maybe viewed as an object $X \in \Gamma^{\mathcal{T}}\mathcal{C}$ (which is the image of intert morphisms of $\mathcal{O}^\otimes \times \mathcal{P}^\otimes$) together with action maps

$$X_H^{\otimes S} \otimes \mathcal{O}(S) \to X_H \qquad\qquad X_H^{\otimes S} \otimes \mathcal{P}(S) \to X_H$$

subject to the functoriality condition that these structures yield an $\mathcal{O}$-algebra, a $\mathcal{P}$-algebra, and these structures satisfy the interchange law

$$\begin{tikzcd}
\bigotimes_U^S X_V^{\otimes \mathrm{Res}_U^V T} \arrow[d, "\bigotimes_U^S \mathrm{Res}_U^V \mu_T"'] & \simeq & X_V^{\otimes S \times T} & \simeq & \bigotimes_W^T X_V^{\otimes \mathrm{Res}_W^V S} \arrow[r, "\bigotimes_W^T \mathrm{Res}_W^V \mu_S"] & X_V^{\otimes T} \arrow[d, "\mu_T"] \\
X_V^{\otimes S} \arrow[rrrrr, "\mu_S"] & & & & & X_V
\end{tikzcd}$$

for all pairs $\mu_S \in \mathcal{O}(S)$ and $\mu_T \in \mathcal{P}(T)$. A morphism of $\mathcal{O}\otimes\mathcal{P}$-algebras is a natural transformation of bifunctors, i.e. a morphism of $\mathcal{T}$-objects $X \to Y$ which is both a $\mathcal{O}$-algebra map and a $\mathcal{P}$-algebra map.

In particular, an $\mathcal{N}_{I\infty} \otimes \mathcal{N}_{I\infty}$-algebra is equivalently a pair of collections of maps $\mu, \mu' : X^{\otimes T} \to X^{\otimes R}$ for all $T \to R$ in I which are separately $\mathcal{N}_{I\infty}$-algebra structures and which satisfy the interchange law

$$\begin{tikzcd}
\bigotimes_U^S X_V^{\otimes \mathrm{Res}_U^V T} \arrow[d, "\bigotimes \mu"'] & \simeq & X_V^{\otimes S \times T} & \simeq & \bigotimes_W^T X_V^{\otimes \mathrm{Res}_W^V S} \arrow[r, "\bigotimes \mu'"] & X_V^{\otimes T} \arrow[d, "\mu"] \\
X_V^{\otimes S} \arrow[rrrrr, "\mu'"] & & & & & X_V
\end{tikzcd}$$

$\triangleleft$

Lemma 3.31. *G and G_0 interchange.*

Proof. It suffices to note that all of the compositions in Observation 3.30 factor through a zero map, and hence they are all zero, making the diagram commute. $\square$

Corollary 3.32. *If $\mathcal{N}_{I\infty}^\otimes$ is not aE-unital, then $\mathcal{N}_{I\infty}^\otimes \overset{BV}{\otimes} \mathcal{N}_{I\infty}^\otimes$ is not connected.*

Proof. Note that

$$\begin{aligned}
\mathcal{N}_{I\infty}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^\otimes \text{ connected} &\iff \tau_{\le 1}\mathcal{N}_{I\infty}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^\otimes \simeq \mathcal{N}_{A\mathcal{N}_{I\infty}\otimes\mathcal{N}_{I\infty}} \simeq \mathcal{N}_{I\infty}^\otimes \\
&\implies \mathrm{CAlg}_I(\mathrm{Set}_*) \to \mathrm{Alg}_{\mathcal{N}_{I\infty}\otimes\mathcal{N}_{I\infty}}(\mathrm{Set}_*) \text{ essentially surjective.}
\end{aligned}$$

Furthermore, Lemma 3.31 constructs an $\mathcal{N}_{v(I)^\perp - nu\infty}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{v(I)^\perp - nu\infty}^\otimes$ satisfying the condition that its two individual structure maps $G(S) \to G(*_V)$ differ whenever $V \in v(I)^\perp$ and $S \neq *_V$. Since I is not aE-unital, it must have some noncontractible $S \in \mathbb{F}_{I,V}$ for $V v(I)^\perp$, so the pullback $\mathcal{N}_{I\infty}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^\otimes$ structure on (G, G_0) has two distinct underlying I-algebra structures, implying it is outside of this essential image. The contrapositive shows that $\mathcal{N}_{I\infty}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\overset{\mathrm{BV}}{\otimes}}$ is not connected. $\square$

By combining Corollaries 3.6 and 3.32, we have the following.

Corollary D'. *$\mathcal{N}_{I\infty}^\otimes \otimes \mathcal{N}_{I\infty}^\otimes$ is a weak $\mathcal{N}_\infty$-operad if and only if I is almost-E-unital. in this case, if $\mathcal{O}^\otimes$ is a reduced I-operad, then the unique map*

$$\mathcal{O}^\otimes \otimes \mathcal{N}_{I\infty}^\otimes \to \mathcal{N}_{I\infty}^\otimes$$

is an equivalence.

Remark 3.33. Using the above argument, one can show that if $\mathcal{O}^\otimes$ is a $\overset{\mathrm{BV}}{\otimes}$-idempotent $\mathcal{T}$-operad, then its nullary spaces $\mathcal{O}(\varnothing_V)$ are nonempty. If additionally $\mathcal{O}(\varnothing_V)$ are assumed to be contractible (i.e. $\mathcal{O}^\otimes$ is aE-unital), then Proposition 3.17 shows that the underlying fixed point catgeories $\mathcal{O}_V$ are all $\times$-idempotent algebras, i.e. they are contractible or empty. Hence $\mathcal{O}^\otimes$ will be shown to be aE-reduced. It is likely that the equivariant analog to [SY19] will demonstrate that such idempotents are all infinitely connected; hence the author believes that the aE-unital weak $\mathcal{N}_\infty$-operads are likely to completely enumerate the $\overset{\mathrm{BV}}{\otimes}$-idempotent algebras in $\mathrm{Op}_{\mathcal{T}}$. ◃

4. Corollaries and future directions

4.1. **Canonical coherences for the $\mathcal{T}$-BV tensor product.** In [Ste24a] we proved the following.

Proposition 4.1. $\mathrm{Env}\left(\mathrm{triv}_{\mathcal{T}}^{\otimes}\right) \in \mathrm{Cat}_{\mathcal{T}}^{\otimes}$ *is the unit under the mode structure, and there is an equivalence*

$$\mathrm{Env}\left(\mathcal{O}^\otimes \overset{BV}{\otimes} \mathcal{P}^\otimes\right) \simeq \mathrm{Env}\left(\mathcal{O}^\otimes\right) \otimes^{\mathrm{Mode}} \mathrm{Env}\left(\mathcal{P}^\otimes\right).$$

We would like to use this to construct coherences for $\overset{\mathrm{BV}}{\otimes}$, but it is currently not know whether Env yields a monomorphism in Cat, so we can not use the exact same strategy as [BS24a]. Instead, we proved in Corollary D' that $\mathrm{Comm}_{\mathcal{T}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}_{\mathcal{T}}^{\otimes}$ is terminal, so the unique map $\mathrm{triv}_{\mathcal{T}}^{\otimes} \to \mathrm{Comm}_{\mathcal{T}}^{\otimes}$ yields an equivalence

$$\mathrm{Comm}_{\mathcal{T}}^{\otimes} \simeq \mathrm{Comm}_{\mathcal{T}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{\mathcal{T}}^{\otimes} \xrightarrow{\sim} \mathrm{Comm}_{\mathcal{T}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}_{\mathcal{T}}^{\otimes};$$

that is, $\mathrm{Comm}_{\mathcal{T}}^{\otimes}$ bears a unique structure as an idempotent $\mathbb{E}_0$-algebra (or *idempotent object* in the sense of [HA, Rmk 4.8.2.1], noting that the definition only depends on an $\mathbb{E}_0$-structure on the ambient ∞-category). In particular, together imply that Env induces an idempotent $\mathbb{E}_0$-structure on $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \in \mathrm{Cat}_{\mathcal{T}}^{\otimes}$ under the mode structure. By [HA, Prop 4.8.2.9] this canonically lifts to an $\mathbb{E}_\infty$-structure, so [HA, Thm 2.2.2.4] constructs a symmetric monoidal structure on $\mathrm{Cat}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$ whose underlying tensor functor has value

$$\mathcal{C} \circledast \mathcal{D} \xrightarrow{\pi_{\mathcal{C}} \circledast \pi_{\mathcal{D}}} \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \circledast \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \xrightarrow{\sim} \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}.$$

and whose unit is

$$\mathrm{Env}\left(\mathrm{triv}_{\mathcal{T}}^{\otimes}\right) \xrightarrow{\eta} \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}.$$

Corollary 4.2. $\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes} \subset \underline{\mathrm{Cat}}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$ *is a symmetric monoidal subcategory under the mode structure, with unit corresponding with* $\mathrm{triv}_{\mathcal{T}}^{\otimes}$ *and tensor bifunctor corresponding with* $\overset{BV}{\otimes}$. *Hence there exists a unique symmetric monoidal $\mathcal{T}$-∞-category lifting* $\overset{BV}{\otimes}$ *such that the $\mathcal{T}$-functor*

$$\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes} \to \underline{\mathrm{Cat}}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$$

admits a symmetric monoidal structure.

Proof. We're tasked with proving that the image of $\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}(-)$ contains the unit and is closed under tensor products. The unit is Proposition 4.1, and for tensor products, the above argument constructs a diagram

$$\begin{tikzcd}
\mathrm{Env}\left(\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes\right) \arrow[r, "\mathrm{Env}(\pi_{\mathcal{O}} \overset{\mathrm{BV}}{\otimes} \pi_{\mathcal{P}})"] \arrow[d, phantom, "\wr"] & \mathrm{Env}\left(\mathrm{Comm}_{\mathcal{T}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}_{\mathcal{T}}^{\otimes}\right) \arrow[d, phantom, "\wr"] & \mathrm{Env}\left(\mathrm{Comm}_{\mathcal{T}}^{\otimes}\right) \arrow[l, "\mathrm{Env}(\mathrm{id}\otimes\eta)"', "\sim"] \arrow[d, phantom, "\wr"] \\
\mathrm{Env}(\mathcal{O}^\otimes) \circledast \mathrm{Env}(\mathcal{P}^\otimes) \arrow[r, "\pi_{\mathrm{Env}(\mathcal{O}^\otimes)} \circledast \pi_{\mathrm{Env}(\mathcal{P}^\otimes)}"'] & \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \circledast \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} & \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \arrow[l, "\sim", "\mathrm{id}\circledast\eta"']
\end{tikzcd}$$

In particular, by inverting both $\mathrm{Env}(\mathrm{id}\otimes\eta)$ and $\mathrm{id}\circledast\eta$, we construct an equivalence

$$\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}\left(\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes\right) \simeq \mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}\left(\mathcal{O}^\otimes\right) \circledast \mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}\left(\mathcal{P}^\otimes\right)$$

over $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$, as desired. □

Corollary 4.3. *When $\mathcal{T} = *$, there is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Op}_*^{\otimes} \simeq \mathrm{Op}^{\otimes},$$

where the latter is the Boardman-Vogt symmetric monoidal ∞-category of [BS24a]. In particular, this takes $\overset{BV}{\otimes}$ to the Boardman-Vogt tensor product of [HM23; HA].

Proof. In [Ste24a] we supplied an equivalence $\mathrm{Op}_* \simeq \mathrm{Op}$, so it suffices to upgrade this to a symmetric monoidal equivalence. In fact, the forgetful functor $\mathrm{Cat}^{\otimes}_{\infty,\mathbb{F}^{\sqcup}} \to \mathrm{Cat}^{\otimes}_{\infty}$ is symmetric monoidal (as all "unslicing" forgetful functors are), so Corollary 4.2 constructs a symmetric monoidal structure on the composite induced $\mathrm{Op}_*^{\otimes} \to \mathrm{Cat}_{\infty}^{\otimes}$, the latter having the mode symmetric monoidal structure. Thus [BS24a, Thm E] constructs a symmetric monoidal equivalence extending the equivalence $\mathrm{Op}_* \simeq \mathrm{Op}$ and shows that $\overset{\mathrm{BV}}{\otimes}$ is the tensor product of [HA]. □

Corollary 4.4. *Let I be a one color weak indexing system. Then, $\mathrm{Op}_I \subset \mathrm{Op}_{\mathcal{T}}$ is a symmetric monoidal subcategory.*

Proof. Since $\mathrm{Atriv}_{\mathcal{T}} \simeq \mathbb{F}_{\mathcal{T}}^{\simeq} \subset I$, the $\overset{\mathrm{BV}}{\otimes}$-unit $\mathrm{triv}_{\mathcal{T}}^{\otimes}$ is an I-operad. Corollary 2.20 implies that $\mathrm{Op}_I \subset \mathrm{Op}_{\mathcal{T}}$ is closed under binary tensor products, so it is a symmetric monoidal subcategory. □

We will write $\mathrm{Op}_I^{\mathrm{uni}} \coloneqq \mathrm{Op}_{\mathcal{T}}^{\mathrm{uni}} \cap \mathrm{Op}_I$, and similar for various other conditions.

Corollary 4.5. *Let I be a unital weak indexing system. Then, $\mathrm{Op}_I^{\mathrm{uni}} \subset \mathrm{Op}_I$ is a smashing localization, and in particular, it possesses a canonical symmetric monoidal structure such that $\mathbb{E}_0^{\otimes} \overset{BV}{\otimes} (-) \colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T}}^{\mathrm{uni}}$ is symmetric monoidal.*

Proof. This is a consequence of Proposition 3.13. □

In particular, the $\overset{\mathrm{BV}}{\otimes}$-unit in $\mathrm{Op}_I^{\mathrm{uni}}$ is $\mathbb{E}_0^{\otimes}$.

Corollary 4.6. *Let I be a one color weak indexing system and $n \in \mathbb{N} \cup \{\infty\}$. Then, the following are symmetric monoidal subcategory inclusions:*

$$\begin{array}{ccccccc} \mathrm{Op}_{I,\geq n}^{aE\,\mathrm{red}} & \subset & \mathrm{Op}_I^{aE\,\mathrm{red}} & \subset & \mathrm{Op}_I^{aE\mathrm{uni}} & \subset & \mathrm{Op}_I \\ \mathrm{Op}_{I,\geq n}^{\mathrm{red}} & \subset & \mathrm{Op}_I^{\mathrm{red}} & \subset & \mathrm{Op}_I^{\mathrm{uni}} & & \end{array}$$

Proof. $\mathrm{triv}_{\mathcal{T}}\otimes$ and $\mathbb{E}_0^{\otimes}$ are ∞-connected; in particular, the symmetric monoidal unit is contained in each of these subcategories. Thus we're left with verifying that each subcategory is closed under tensor products. The lefthand inclusions both follow from Corollary 3.21; the middle inclusions follow from Corollary 2.16; the righthand inclusion is Corollary 3.19. □

4.2. **Equivariant infinitary Dunn additivity.** An inclusion $V \subset W$ yields a map of graph G-operads $D_V \subset D_W$, and hence a map of G-operads $\mathbb{E}_V^{\otimes} \to \mathbb{E}_W^{\otimes}$. This yields a map of weak indexing systems $\underline{\mathbb{F}}^V \to \underline{\mathbb{F}}^W$; in [Ste24b] we showed that this is additive, i.e.

$$\underline{\mathbb{F}}^V \vee \underline{\mathbb{F}}^W = \underline{\mathbb{F}}^{V \oplus W}. \tag{20}$$

Corollary G (Equivariant infinitary Dunn additivity)**.** *Let G be a finite group and V, W real orthogonal G-representations satisfying at least one of the following conditions:*

(a) V, W are weak G-universes, or
(b) the canonical map $\mathbb{E}_V^{\otimes} \simeq \mathbb{E}_{V\oplus W}^{\otimes}$ is an equivalence.

Then the canonical map

$$\mathbb{E}_V^{\otimes} \overset{BV}{\otimes} \mathbb{E}_W^{\otimes} \to \mathbb{E}_{V\oplus W}^{\otimes}$$

is an equivalence; equivalently, for any G-symmetric monoidal category $\mathcal{C}$, the pullback functors

$$\mathrm{Alg}_{\mathbb{E}_V} \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_W}(\mathcal{C}) \leftarrow \mathrm{Alg}_{\mathbb{E}_{V\oplus W}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_W} \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C})$$

are equivalences.

Proof. Given Corollary 2.30, case (a) follows from Theorem F and Eq. (20) and case (b) follows from Corollary D. □

4.3. **Iterated Real topological Hochschild homology.** Let $\mathcal{O}^\otimes \in \mathrm{Op}_{\mathcal{T}}$ be an *arbitrary* $\mathcal{T}$-operad and $\mathcal{C}^\otimes$ a $\mathcal{T}$-symmetric monoidal ∞-category. Recall that the *pointwise* $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})$ has algebras characterized by the mapping property

$$\mathrm{Alg}_{\mathcal{P}} \mathrm{Fun}_{\mathcal{T}}(\mathrm{Env}(\mathcal{O}),\mathcal{C})^{\otimes-\mathrm{ptws}} \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathrm{Env}(\mathcal{O}), \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C})\right).$$

In particular, we may construct a natural transformation

$$\begin{aligned}
\mathrm{Alg}_{\mathcal{P}}\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) &\simeq \mathrm{Fun}^{\otimes}_{\mathcal{T}}\left(\mathrm{Env}(\mathcal{P}), \underline{\mathrm{Fun}}^{\otimes}_{\mathcal{T}}\left(\mathrm{Env}(\mathcal{O}), \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})\right)\right)\\
&\simeq \mathrm{Fun}^{\otimes}_{\mathcal{T}}\left(\mathrm{Env}(\mathcal{O}), \underline{\mathrm{Fun}}^{\otimes}_{\mathcal{T}}\left(\mathrm{Env}(\mathcal{P}), \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})\right)\right)\\
&\simeq \mathrm{Fun}^{\otimes}_{\mathcal{T}}\left(\mathrm{Env}(\mathcal{O}), \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C})\right)\\
&\to \mathrm{Fun}_{\mathcal{T}}\left(\mathrm{Env}(\mathcal{O}), \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C})\right)\\
&\simeq \mathrm{Alg}_{\mathcal{P}} \mathrm{Fun}_{\mathcal{T}}(\mathrm{Env}(\mathcal{O}),\mathcal{C})^{\otimes-\mathrm{ptws}};
\end{aligned}$$

Yoneda's lemma applied in $\mathrm{Op}_{\mathcal{T}}$ implies that this is implemented by a unique lax $\mathcal{T}$-symmetric monoidal functor $F\colon \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{T}}(\mathrm{Env}(\mathcal{O}),\mathcal{C})^{\otimes-\mathrm{ptws}}$.

Proposition 4.7. *$F\colon \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{T}}(\mathrm{Env}(\mathcal{O}),\mathcal{C})^{\otimes-\mathrm{ptws}}$ is $\mathcal{T}$-symmetric monoidal.*

Proof. Fix an orbit $V \in \mathcal{T}$, a finite V-set $S \in \mathbb{F}_V$, and an S-tuple of $\mathcal{O}$-algebras $(X_U) \in \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})_V$. We're tasked with proving that, for all orbits $U \to V$ and finite U-sets $T \in \mathbb{F}_U$, the canonical assembly map

$$h_S : {}^{\mathcal{C}}\bigotimes{}_U^S F(X_U)_T \to {}^{\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})}\left(\bigotimes_U^S F\right)(X_U)_T$$

is an equivalence. In fact, this follows from the fact that $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ is symmetric monoidal. $\square$

In particular, this constructs a G-symmetric monoidal lift for *genuine equivariant factorization homology.*

Corollary 4.8. *Given M a V-framed smooth G-manifold, M-factorization homology lifts to a G-symmetric monoidal functor*

$$\int_M : \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C}) \to \mathcal{C}^{\otimes};$$

in particular, it further lifts to a G-symmetric monoidal functor

$$\int_M : \underline{\mathrm{CAlg}}^{\otimes}_{AV}(\mathcal{C}) \to \underline{\mathrm{CAlg}}^{\otimes}_{AV}(\mathcal{C}).$$

Proof. In the notation of [Hor19], let $\iota^{\otimes} : \underline{\mathrm{Disk}}^{G,V-fr,\sqcup} \to \underline{\mathrm{Mfld}}^{G,V-fr,\sqcup}$ be the symmetric monoidal inclusion of V-framed G-disks into V-framed G-manifolds. By [Hor19, Horev 4.1.4], $\int_M$ may be presented as the G-value of a composition

$$\int_M : \underline{\mathrm{Alg}}_{\mathbb{E}_V}(\mathcal{C}) \simeq \underline{\mathrm{Fun}}^{\otimes}_G\left(\underline{\mathrm{Disk}}^{G,V-fr},\mathcal{C}\right) \xrightarrow{U} \underline{\mathrm{Fun}}_G\left(\underline{\mathrm{Disk}}^{G,V-fr},\mathcal{C}\right) \xrightarrow{\iota_!} \underline{\mathrm{Fun}}_G\left(\underline{\mathrm{Mfld}}^{G,V-fr},\mathcal{C}\right) \xrightarrow{\mathrm{ev}_M} \mathcal{C}.$$

To construct the lift of $\int_M$, we may compose G-symmetric monoidal lifts of U, $\iota_!$, and ev_M; these are given by Proposition 4.7 and Observation 1.26. $\square$

Corollary 4.9. *Real topological Hochschild homology lifts to a C_2-symmetric monoidal functor*

$$\mathrm{THR} : \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_\sigma}(\mathrm{Sp}) \to \underline{\mathrm{Sp}}_{C_2};$$

in particular, if V contains infinitely many copies of σ, then THR *lifts to a C_2-symmetric monoidal functor*

$$\mathrm{THR} : \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C}).$$

Furthermore, given $A \in \mathrm{CAlg}_{C_2}(\mathcal{C})$, there is an equivalence

$$\mathrm{THR}(A) \simeq \mathrm{colim}_{S^\sigma} A,$$

with colimit taken in $\mathrm{CAlg}_{C_2}(\mathcal{C})$.

Proof. The last sentence is the only part which does not follow immediately from combining Horev's facotization homology formula [Hor19, Rmk 7.1.2] with Corollary 4.8 in view of the equivariant infinitary Dunn additivity of Corollary G. In fact, the collar decomposition formula of [Hor19, Prop 7.1.1] yields a coequalizer diagram

$$\begin{tikzcd} N_e^{C_2} A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, phantom, "\wr"] & A \otimes A \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{THR}(A) \arrow[d, equal] \\ \mathrm{CoInd}_e^{C_2} \mathrm{Res}_e^{C_2} A \arrow[r, shift left] \arrow[r, shift right] & A \oplus A \arrow[r] & \mathrm{THR}(A) \end{tikzcd}$$

Pulling A out of the bottom expression, we find that $\mathrm{THR}(A) \simeq \mathrm{colim}_X A$, where X is the C_2-space $\mathrm{CoEq}([C_2/e] \rightrightarrows 2*_{C_2}) \xrightarrow{\sim} X$; this is just the standard C_2-cell presentation of $X = S^\sigma$. □

Remark 4.10. The computation $\mathrm{THR}(A) = \mathrm{colim}_{S^\sigma} A$ when A is pulled back from a C_2-commutative algebra is not new; indeed, it appears as [QS19, Rmk 5.4]. In fact, the ambiguity induced by the potential discrepancy between our construction $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ and that of [NS22, Thm 5.3.4] vanishes for the I-symmetric monoidal structure on $\mathrm{CAlg}_I(\mathcal{C})$ by applying Theorem B' in view of the fact that each are cocartesian [NS22, Thm 5.3.9]. The new element of this identification is that the operation on C_2-commutative algebras is induced canonically from the operation on $\mathbb{E}_\sigma$-algebras. ◁

4.4. **Some conjectures.**

4.4.1. *Closing the gap between models.* Furthermore, several papers such as [BH15; Rub21b; Szc23] have characterized the behaviour of various "Boardman-Vogt" tensor products on examples in various models. We propose means to close the loop.

Conjecture 4.11. *The Boardman-Vogt tensor products of [BH15; Rub21b; Szc23] lift to a common symmetric monoidal ∞-category* $g\mathrm{Op}_G^{\otimes}$ *possessing a G-symmetric monoidal equivalence*

$$g\mathrm{Op}_G^{\otimes} \simeq \mathrm{Op}_G^{\otimes}.$$

We are interested in this conjecture for two reasons; on one hand, some tensor products of G-operads have been computed in models, such as tensor products of models for N_∞-operads in [Rub21b] and tensor products of models for $\mathbb{E}_V$ operads in [Szc23]. On the other, the model categories are hard to work with, and to the author's knowledge, no BV tensor product on models has been lifted to a homotopical symmetric monoidal closed structure, so these results are difficult to apply to constructions of algebras.

We suggest two possible lines of argumentation for the equivalence of ∞-categories. First, note that $N^\otimes$ is a conservative functor between two ∞-categories who are each monadic over $\mathrm{Fun}(\underline{\Sigma}_G, \mathcal{S})$; To compare our notions, it suffices to characterize the *free G-operad on a G-symmetric sequence* and provide an explicit comparison between it and the genuine equivariant operad monad of [BP21, § 4.2]. If these monads are shown to be equivalent via $N^\otimes$, then $N^\otimes$ itself with be an equivalence.

Another line of argumentation is to generalize the non-equivariant case; for instance, we conjecture that [Bar18, § 10] applied to the perfect operator category $\underline{\mathbb{F}}_G$ will provide an equivalence between G-operads and $\mathrm{Seg}_{\Delta^{\mathrm{op}}_{\underline{\mathbb{F}}_G}}(\mathcal{S})$, the latter being comparable to the equivariant dendroidal Segal spaces of [BP20; Per18] by an equivariant lift of the argument of [CHH18] in the language of algebraic patterns and using the recognition principle for Morita equivalences of patterns due to [Bar23, Thm 2.63].

The underlying tensor products and norms seem amenable to argumentation once pushed to structures on a common ∞-category; for instance, the universal property of BV tensor products in [Szc23, Def 1.7.2] bears resemblance to the fact that our BV tensor product corepresents bifunctors of G-operads.

4.4.2. *The equivariant homotopical Eckmann-Hilton argument.* We conjecture a strengthening of Corollary D.

Conjecture 4.12. *Suppose I is an aE-unital weak indexing system and* $\mathcal{O}, \mathcal{P}$ *are* d_1, d_2*-connected reduced I-operads with* $A\mathcal{O} = A\mathcal{P}$*. Then,* $\mathcal{O} \otimes \mathcal{P}$ *is* $(d_1 + d_2 - 2)$*-connected.*

Note that this immediately implies a weak form of *infinite loop space theory*, i.e. the map

$$\mathrm{colim}_n \left(\mathcal{O}^{\otimes}\right)^{\otimes n} \to \mathcal{N}_{A\mathcal{O}\infty}$$

is an equivalence for all aE-reduced $\mathcal{O}$, or equivalently, letting $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O},n}(\mathcal{C}) := \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O},n-1}(\mathcal{C})$ with $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O},0}(\mathcal{C}) = \mathcal{C}$,

$$\lim_n \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O},n}(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}^{\otimes}_{A\mathcal{O}}(\mathcal{C}).$$

The author hopes to fulfill this in upcoming work bearing similarity to [SY19]. In view of Proposition 2.4, we will acquire an inductive strategy to construct algebras over *any* aE-unital weak N_∞ operad, using at each step e.g. the associative or free I-operads of [Rub21a].

We also would immediately acquire an intrinsic characterization of almost-unital weak N_∞-operads, and hence of A; since infinite tensor products of almost-reduced $\mathcal{T}$ operads are weak $\mathcal{N}_\infty$-operads, and weak $\mathcal{N}_\infty$-operads are idempotent by Theorem F, the argument of Remark 3.33 will immediately show that the $\overset{\mathrm{BV}}{\otimes}$-idempotent algebras in $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}$ are precisely the almost-unital weak $\mathcal{N}_\infty$-operads.

4.4.3. *Equivariant Dunn additivity.* In the thesis [Szc23], the non-homotopical graph-operad equivalent to the following conjecture was proved.

Conjecture 4.13. *The map $\mu : \mathbb{E}_V^{\otimes} \otimes \mathbb{E}_W^{\otimes} \to \mathbb{E}_{V\oplus W}^{\otimes}$ is an equivalence of G-operads.*

In forthcoming work, the author plans to prove this theorem after stabilizing to spectral G-operads.

4.4.4. *Discrete models for G-operads.* Much of the strategy employed in sources such as [HA] which characterize $\mathbb{E}_n$-algebras consists of reduction to the $\mathbb{E}_1$-case via Dunn's additivity theorem; $\mathbb{E}_1$ is a discrete operad, and hence it is amenable to combinatorial study. Unfortunately, Conjecture 4.13 does not predict such a luxury in the equivariant setting; for instance, if $|G|$ is odd, then G admits no nontrivial 1-dimensional real orthogonal G-representation. Given V of finite dimension at least 2, $\mathbb{E}_V(2*_e) \simeq \mathrm{Conf}^e_{[2]}(V) \simeq S(V)^e$, which is *not* discrete, as it has nonvanishing $\dim V$th homotopy group. Thus we are inspired to ask the following difficult question.

Question 4.14. Does there exist a family of G-operads $\mathbb{O}$ such that $\mathbb{E}_V \in \mathbb{O}$ for all V and such that $\mathbb{O}$ is generated under $\overset{\mathrm{BV}}{\otimes}$ by discrete G-operads? ◁

One potentially fruitful source of examples is the subject of the next set of questions.

4.4.5. *Coinduced V-operads and free equivariant symmetric sequences.*

Question 4.15. Let $\mathcal{O}$ be a $\underline{V}$ operad and $U \to W$ a map. What structure does a $\mathrm{CoInd}_U^V \mathcal{O}$-algebra have? ◁

This is nontrivial, as coinduced operads are characterized by *mapping-in properties*, but their algebras are maps *out*. It is useful, as Construction 2.23 uses this mapping-in property to argue that $\mathrm{CoInd}_U^V \mathcal{O}^{\otimes}$ is the universal structure borne by V-norms of $\mathcal{O}^{\otimes}$-algebras. It is old, as coinduced operads appear in the graph model structure as early as [BH15, § 6.2.1]

For instance, Proposition 2.21 leads to the following perplexing observations:

Observation 4.16. $\mathrm{CoInd}_e^G \mathbb{E}_1$ is a discrete G-operad whose underlying weak indexing system is complete; $\mathrm{CoInd}_e^G \mathbb{E}_2$ is a 1-truncated G-operad whose underlying weak indexing system is complete. ◁

The author is frustrated to report that she has guesses as to what $\mathrm{CoInd}_e^G \mathbb{E}_n$ is when $1 < n < \infty$ despite its structure being borne by HHR norms of all $\mathbb{E}_n$-rings.

Observation 4.17. Let $X_\bullet$ be a $\underline{V}$-symmetric sequence. Then,

$$\begin{aligned} \mathrm{Map}_{\mathrm{sseq}}(X_\bullet, \mathrm{sseq}\,\mathrm{CoInd}_U^V \mathcal{O}) &\simeq \mathrm{Map}(\mathrm{Fr}(X_\bullet)^{\otimes}, \mathrm{CoInd}_U^V \mathcal{O}^{\otimes}) \\ &\simeq \mathrm{Map}(\mathrm{Res}_U^V \mathrm{Fr}(X_\bullet)^{\otimes}, \mathcal{O}^{\otimes}) \\ &\simeq \mathrm{Map}(\mathrm{Fr}(\mathrm{Res}_U^V X_\bullet)^{\otimes}, \mathcal{O}^{\otimes}) \\ &\simeq \mathrm{Map}_{\mathrm{sseq}}(\mathrm{Res}_U^V X_\bullet, \mathrm{sseq}\,\mathcal{O}). \end{aligned}$$

In particular, if $\mathrm{Fr}(S)$ is the free $\underline{V}$-symmetric sequence on $S \in \underline{\mathbb{F}}_V$, this demonstrates that

$$\mathrm{CoInd}_U^V \mathcal{O}(S) \simeq \mathrm{Map}_{\mathrm{sseq}}(\mathrm{Res}_U^V \mathrm{Fr}(S), \mathrm{sseq}\,\mathcal{O});$$

thus, combinatorial control of free $\underline{V}$-symmetric sequences is likely to yield information about the equivariant symmetric sequence underlying coinduced V-operads; in particular, since the underlying V-symmetric sequence functor is conservative, this is a potential avenue by which to "guess and check" the identity of coinduced V-operads, giving intrinsic characterization of the structure of HHR norms of $\mathcal{O}$-algebras. ◁

4.4.6. *On developing global operads.*

Definition 4.18. Let $\mathcal{T}$ be an ∞-category. Then, a *weak indexing datum of* $\mathcal{T}$ is a pair (P, I_P), where P is an atomic orbital subcategory and I_P is a P-weak indexing category. ◁

There is a cartesian symmetric monoidal subcategory $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \subset \mathrm{Span}_{\mathcal{P}}(\mathbb{F}_{\mathcal{T}})$, yielding on this category the structure of a symmetric monoidal algebraic pattern, allowing one to define the Boardman-Vogt tensor product.

Definition 4.19. Let $\mathcal{T}$ be an ∞-category. Then, the $\otimes$-*category of* $\mathcal{T}$-I-*operads* is

$$\mathrm{Op}_{\mathcal{T},I} := \left(\mathrm{Fbrs}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})), \overset{\mathrm{BV}}{\otimes}\right). \qquad ◁$$

Question 4.20. Does the work of this paper and [NS22; Ste24b] extend to $\mathrm{Op}_{\mathcal{T},I}$? ◁

Recollection 4.21. In [CLL23, § 4.7], the free $\mathcal{T}$-∞-category $\underline{\mathbb{F}}_{P,*} := \underline{\mathbb{F}}^P_{\mathcal{T},*}$ admitting P-coproducts on a point was constructed; in particular, since $\mathrm{Span}_P(\underline{\mathbb{F}}_{\mathcal{T}})$ admits finite P-products and is P-semiadditive, it admits finite P-coproducts, and hence admits a unique P-coproduct preserving $\mathcal{T}$-functor

$$\iota : \underline{\mathbb{F}}_{P,*} \to \mathrm{Span}_P(\underline{\mathbb{F}}_{\mathcal{T}})$$

sending $*_+ \mapsto *$. ◁

If one would like to repeat arguments from Appendix A and [NS22; Ste24b] verbatim, one needs a [HA]-style pattern modelling $\mathrm{Op}_{\mathcal{T},I}$; this is especially important for Proposition 1.18, whose conclusion can't easily be formulated over effective Burnside patterns in the first place. Thus we formulate the following conjecture:

Conjecture 4.22. $\underline{\mathbb{F}}_{P,*}$ *admits a structure as a sound algebraic pattern such that the composite functor*

$$\underline{\mathbb{F}}_{P,*} \to \mathrm{Span}_P(\underline{\mathbb{F}}_{\mathcal{T}}) \to \mathrm{Span}_P(\mathbb{F}_{\mathcal{T}})$$

is a Morita equivalence.

Appendix A. Cartesian and cocartesian I-symmetric monoidal ∞-categories

Fix I a unital weak indexing category. This appendix can be understood as a lift of [HA, § 2.4.1-2.4.3] to the setting of (co)cartesian I-symmetric monoidal ∞-categories; we proceed by an essentially similar strategy, complicated only by less convenient combinatorics. In particular, we use the combinatorics of $\underline{\mathbb{F}}_{I,*}$-fibrous patterns throughout, so we will freely synonymize $\mathrm{Op}_{\mathcal{T}}$ and $\mathrm{Fbrs}(\underline{\mathbb{F}}_{I,*})$ throughout.

Define the $\mathcal{T}$-1-category $\underline{\Gamma}^*_I$ to have V-values

$$\Gamma^*_{I,V} := \left\{ U_+ \xrightarrow{s.i.} S_+ \mid U \in \underline{V} \right\} \subset \mathrm{Ar}(\underline{\mathbb{F}}_{I,*})_V;$$

that is, the objects of $\Gamma_{I,V}$ are pointed I-admissible V-sets with a distinguished orbit, and the morphisms of $\Gamma_{I,V}$ preserve distinguished orbits. This possesses a tautological forgetful functor $\underline{\Gamma}^*_I \to \underline{\mathbb{F}}_{I,*}$. We use this to construct an ∞-category $\mathcal{C}$ over $\underline{\mathbb{F}}_{I,*}$ in Appendix A.2 satisfying the following universal property.

Proposition A.1. *Given* $\mathcal{C}$ *a* $\mathcal{T}$-∞-*category, there exists an* ∞-*category* $\mathcal{C}^{I-\sqcup}$ *over* $\underline{\mathbb{F}}_{I,*}$ *satisfying the universal property that there is a natural equivalence*

$$\mathrm{Fun}_{\underline{\mathbb{F}}_{I,*}}(\mathcal{D}, \mathcal{C}^{I-\sqcup}) \simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{D} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I, \mathcal{C});$$

that is, the functor $(-) \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I : \mathrm{Cat}_{\infty,/\underline{\mathbb{F}}_{I,*}} \to \mathrm{Cat}_{\mathcal{T}}$ *possesses a right adjoint* $(-)^{I-\sqcup}$.

An object of $\mathcal{C}^{I-\sqcup}$ may be viewed as S_+ a pointed V-set and $\mathbf{C} = (C_W) \in \mathcal{C}_S$ an S-tuple of elements of $\mathcal{C}$; a morphism $f : \mathbf{C} \to \mathbf{D}$ may be viewed as a $\underline{\mathbb{F}}_{I,*}$-map $(S_+ \to V_{S,+}) \xrightarrow{f} (T_+ \to V_{T,+})$ together with a collection of maps

$$\left\{ f_W : N^U_W C_W \to D_U \mid W \in f^{-1}(U) \right\}$$

for all $U \in \mathrm{Orb}(T)$. In particular, we have the following:

Lemma A.2. $\mathcal{C}^{I-\sqcup}$ *satisfies the Segal conditions (b) and (c) of [NS22, Def 2.1.7].*

Furthermore, unwinding definitions and applying [HTT, Cor 3.2.2.13], we find the following.

Proposition A.3. *A morphism $f : (\mathbf{C}, S) \to (\mathbf{D}, T)$ is π-cocartesian if and only if $\{f_W\}$ witnesses D_U as the coproduct*

$$\coprod_{W \in f^{-1}(U)} N_W^U C_W \simeq D_U$$

for all $U \in \mathrm{Orb}(T)$. In particular, f is inert if and only if the following conditions are satisfied:

(a) The projected morphism $\pi(f) : S \to T$ is inert.

(b) The associated map $C_{f^{-1}(U)} \to D_U$ is an equivalence for all $U \in \mathrm{Orb}(T)$.

Hence $\mathcal{C}^{I-\sqcup}$ is an I-operad, which is an I-symmetric monoidal ∞-category if and only if $\mathcal{C}$ admits I-indexed coproducts.

Thus, when $\mathcal{C}^{\otimes}$ admits I-indexed products, we've constructed an I-symmetric monoidal ∞-category whose indexed tensor products are coproducts; we will now compute its algebras, eventually forcing all other such I-symmetric monoidal structures to be equivalent to this one.

A.1. **$\mathcal{O}$-comonoids and (co)cartesian rigidity.** Define a diagram of Cartesian squares.

$$\begin{tikzcd} \mathcal{O} \arrow[r, hook, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}_\Gamma^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes} \arrow[d] \\ \mathcal{T}^{\mathrm{op}} \arrow[r, hook] & \underline{\Gamma}_I^* \arrow[r] & \underline{\mathbb{F}}_{I,*} \end{tikzcd}$$

Note that the objects of $\mathcal{O}_\Gamma^{\otimes}$ consist of triples $(S_+ \to V_+, U, X)$ where $U \in \mathrm{Orb}(S)$ and $X \in \mathcal{O}_S$, and the image of ι is equivalent to the triples where $S \in \underline{V}$, hence $U = S$.

Further tote that cocartesian transport along inert morphism $U_+ \hookrightarrow S_+$ induces an equivalence

$$\mathrm{Map}_{\mathcal{O}_\Gamma^{\otimes}}(Y, (S_+ \to V_+, U, X))) \simeq \mathrm{Map}_{\mathcal{O}_\Gamma^{\otimes}}(Y, (U_+ \to V_+, U, X_U)))$$

for all $Y \in \mathcal{O}$. In particular, ι witnesses $\mathcal{O}$ as a *colocalizing subcategory*, with colocalization functor

$$R(S_+ \to V_+, U, X) \simeq (U_+ \to V_+, U, X_U).$$

Lemma A.4. *Fix a functor $A : \mathcal{O}_\Gamma^{\otimes} \to \mathcal{C}$. Then, the following are equivalent*

(a) The corresponding map $\mathcal{O}^{\otimes} \to \mathcal{C}^{I-\sqcup}$ is a functor of I-operads.

(b) For all morphisms α in $\mathcal{O}_\Gamma^{\otimes}$ whose image in $\mathcal{O}^{\otimes}$ is inert, $A(\alpha)$ is an equivalence in $\mathcal{C}$.

(c) If $f : (S_+ \to V_+, U, X) \to (U_+ \to V_+, U, X_U)$ is a cocartesian lift of the corresponding inert morphism, then $A(f)$ is an equivalence.

(d) A is left Kan extended from $\mathcal{O}$.

Furthermore, every functor $F : \mathcal{O} \to \mathcal{C}$ admits a left Kan extension along $\mathcal{O} \hookrightarrow \mathcal{O}_\Gamma^{\otimes}$; in particular, the forgetful functor $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_G(\mathcal{O}, \mathcal{C})$ is an equivalence.

Proof. (a) $\iff$ (b) follows immediately from Proposition A.3. (b) $\iff$ (c) is immediate by definition. (c) $\iff$ (d) and the remining statement both follow by the more general observation that the left Kan extension of $F : \mathcal{C} \to \mathcal{D}$ along a functor $L : \mathcal{C} \to \mathcal{E}$ with right adjoint R is given by the composite $FR : \mathcal{E} \to \mathcal{C} \to \mathcal{D}$. ☐

We would additionally like to characterize I-symmetric monoidal functors into $\mathcal{C}^{I-\sqcup}$. The following lemma follows immediately from Proposition A.3.

Lemma A.5. *Assume $\mathcal{C}$ has I-indexed coproducts and $\mathcal{D}^{\otimes}$ is an I-symmetric monoidal ∞-category. Then, TFAE for a lax I-symmetric monoidal functor $\varphi : \mathcal{D}^{\otimes} \to \mathcal{C}^{I-\sqcup}$:*

(1) φ is a map of I-symmetric monoidal categories.

(2) The corresponding $\mathcal{T}$-functor $F : \mathcal{D}^{\otimes} \to \mathcal{C}$ satisfies the property that, for all $(X_U) \in \mathcal{D}_S$, the canonical maps $\mathrm{Ind}_U^V F(X_U) \to F(X)$ exhibit $F(X)$ as the indexed coproduct

$$\coprod_U^S F(X_U) \simeq F(X).$$

We use this for the following fundamental proposition underlying (co)cartesian rigidity.

Proposition A.6. *Suppose $\mathcal{D}^\otimes$ is an I-symmetric monoidal category satisfying the condition that its action maps $f_\otimes : \mathcal{D}_S \to \mathcal{D}_V$ are left adjoint to the restriction map $f^* : \mathcal{D}_V \to \mathcal{D}_S$. Then, the forgetful functor*

$$U : \mathrm{Fun}_I^\otimes(\mathcal{D}^\otimes, \mathcal{C}^{I-\sqcup}) \to \mathrm{Fun}_\mathcal{T}(\mathcal{D}, \mathcal{C})$$

is fully faithful with image spanned by the I-coproduct preserving functors; dually, if $\mathcal{E}^\otimes$ is an I-symmetric monoidal category satisfying the condition that its action maps $f_\otimes : \mathcal{E}_S \to \mathcal{E}_V$ are right *adjoint to the restriction map $f^* : \mathcal{E}_V \to \mathcal{E}_S$, then the forgetful functor*

$$U : \mathrm{Fun}_I^\otimes(\mathcal{E}^\otimes, (\mathcal{C}^{I-\times})^{v\,\mathrm{op}}) \to \mathrm{Fun}_\mathcal{T}(\mathcal{E}, \mathcal{C})$$

is fully faithful with image spanned by the I-product preserving functors, $(-)^{v\,\mathrm{op}}$ denoting the fiberwise opposite over $\underline{\mathbb{F}}_{I,*}$.

Proof. The first statement follows by noting that those $\mathcal{T}$-functors $\mathcal{D}^\otimes \to \mathcal{C}$ satisfying the conditions of Lemma A.5 are precisely those which are left Kan extended along the (fully faithful) $\mathcal{T}$-functor $\mathcal{D} \hookrightarrow \mathcal{D}^\otimes$ from I-coproduct preserving functors. The second follows by taking fiberwise opposites. □

We are now ready to prove our main generalization for Theorem B' (see p. 16).

Proof of Theorem B'. The two cases are dual, so we prove it for $(-)^{I-\sqcup}$. To see that it's fully faithful, it suffices to note that the action maps in $\mathcal{C}^{I-\sqcup}$ are left adjoint to restriction and apply Proposition A.6. The compatibility with U is obvious, and the description of the image follows immediately from Proposition A.6. □

A.2. **A quasicategory modeling $\mathcal{C}^{I-\sqcup}$.** Let $\mathcal{T}$ be a quasicategory and $\mathcal{C} \in \mathrm{sSet}^{\mathrm{cocart}}_{/\mathcal{T}}$ a cocartesian fibration to $\mathcal{T}$. There exists a simplicial set $\mathcal{C}^{I-\sqcup}$ satisfying the universal property

$$\mathrm{Hom}_{\underline{\mathbb{F}}_{I,*}}(K, \mathcal{C}^{I-\sqcup}) \simeq \mathrm{Hom}_\mathcal{T}(K \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I, \mathcal{C}). \tag{21}$$

Lemma A.7. *The map of simplicial sets $\mathcal{C}^{I-\sqcup} \to \underline{\mathbb{F}}_{I,*}$ is an inner fibration; hence $\mathcal{C}^{I-\sqcup}$ is a quasicategory.*

Proof. The proof is exactly analogous to the analogous part of [HA, Prop 2.4.3.3]; that is, we may apply the universal property

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "f_0"] \arrow[d] & \mathcal{C}^{I-\sqcup} \arrow[d] \\
\Delta^n \arrow[r, "(S_{0,+} \to \cdots \to S_{n,+})"'] \arrow[ur, dashed] & \underline{\mathbb{F}}_{I,*}
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
\Lambda^n_i \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I \arrow[d] & \simeq & \coprod\limits_{\substack{U \in \mathrm{Orb}(S) \\ f(U) \in S^\circ_{n,+}}} \Lambda^n_i \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\Delta^n \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I & \simeq & \coprod\limits_{\substack{U \in \mathrm{Orb}(S) \\ f(U) \in S^\circ_{n,+}}} \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{T}^{\mathrm{op}}
\end{tikzcd}$$

after which the lifting problem on the RHS has solutions in bijection with the tuples of solutions to the lifting problems made up of the summands, which exist by assumption that the functor $\mathcal{C} \to \mathcal{T}$ is a cocartesian fibration (hence an inner fibration).

The remaining claim follows by noting that $\underline{\mathbb{F}}_{I,*}$ is a quasicategory, so the composite map of simplicial sets $\mathcal{C}^{I-\sqcup} \to \underline{\mathbb{F}}_{I,*} \to *$ is an inner fibration. □

Proof of Proposition A.1. Unwinding the above work, we've verified that $\mathcal{C}^{I-\sqcup}$ is a quasicategory over $\underline{\mathbb{F}}_{I,*}$. Fixing some quasicategory $\mathcal{D}$ over $\underline{\mathbb{F}}_{I,*}$ and applying Eq. (21) for $K := \mathcal{D} \times \Delta^n$, we find that $\mathrm{Fun}(K, \mathcal{C}^{I-\sqcup}) \simeq \mathrm{Fun}_\mathcal{T}(K \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I, \mathcal{C})$. The result then follows by replacing "quasicategory" with "∞-category." □

A.3. **$\mathcal{O}$-monoids.** Recall that an *$\mathcal{O}$-monoid in $\mathcal{C}$* is a functor $\mathcal{O}^\otimes \to \mathcal{C}$ satisfying the condition that for all $X = (X_U) \in \mathcal{C}_S$, the canonical maps $F(X) \to F(X_U)$ witness $F(X)$ as the indexed product

$$F(X) \simeq \prod_U^S F(X_U).$$

We are tasked with proving the following.

Proposition 1.18. *Fix $\mathcal{C}$ a $\mathcal{T}$-category. Then, the postcomposition functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\times}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}^{\otimes}, \mathcal{C})$ is fully faithful with image spanned by the $\mathcal{O}$-monoids.*

In order to do so, we introduce a construction.

Construction A.8. The (non-full) $\mathcal{T}$-subcategory $\underline{\Gamma}_I^{\times} \subset \mathrm{Ar}(\underline{\mathbb{F}}_{I,*})$ has V-objects given by summand inclusions of pointed V-sets $\overline{S} \hookrightarrow S$ and morphisms of V-objects given by maps $\alpha : S \to T$ with the property that $\alpha^{-1}(\overline{T}) \subset \overline{S}$. ◁

Recollection A.9 ([NS22, Def 2.1.2])**.** A morphism f in $\underline{\mathbb{F}}_{I,*}$ from $S_+ \in \mathbb{F}_{I,*,U}$ to $T_+ \in \mathbb{F}_{I,*,V}$ may be modelled as a morphism of spans

$$\begin{tikzcd}
S \arrow[dd] & & f^{-1}(T) \arrow[ll] \arrow[r, "f^\circ"] \arrow[dd] \arrow[dl, "\iota_f"'] & T \arrow[dd] \\
& \mathrm{Res}_U^V S \arrow[ul] \arrow[dr] & & \\
U & & V \arrow[ll] \arrow[r, equal] & V
\end{tikzcd}$$

such that $f^\circ \in I$. Such a morphism is $\pi_{\underline{\mathbb{F}}_{I,*}}$-cocartesian if f° and ι_f are both equivalences, i.e. it witnesses an equivalence $\mathrm{Res}_U^V S_+ \xrightarrow{\sim} T_+$. ◁

Let $T_+ \to S_+$ be a map in $\underline{\mathbb{F}}_{I,*}$ lying over an orbit map $U \to V$ and let $\overline{S} \subset S$ be an element of $\underline{\Gamma}_I^{\times}$ lying over S_+. We would like to construct a Cartesian edge landing on $\overline{S} \subset S$; we do so by setting $\overline{T} := f^{-1}(\mathrm{Res}_U^V \overline{S}) \subset f^{-1}(S) \subset T$, and letting the associated map $t : \left(f^{-1}(\mathrm{Res}_U^V \overline{S}) \subset T\right) \to \left(\overline{S} \subset S\right)$ be the canonical one. The following lemma then follows by unwinding definitions, where $U : \underline{\Gamma}_I^{\times} \to \underline{\mathbb{F}}_{I,*}$ denotes the forgetful functor.

Lemma A.10. *t is a U-cartesian arrow; in particular, U is a cartesian fibration.*

Given $\mathcal{C}$ a $\mathcal{T}$-∞-category, modelled as a quasicategory cocartesian fibered over a fixed model for $\mathcal{T}^{\mathrm{op}}$, we may define a simplicial set $\tilde{\mathcal{C}}^{I-\times}$ over $\underline{\mathbb{F}}_{I,*}$ by the universal property

$$\mathrm{Hom}_{/\underline{\mathbb{F}}_{I,*}}(K, \tilde{\mathcal{C}}^{I-\times}) \simeq \mathrm{Hom}_{/\mathcal{T}^{\mathrm{op}}}(K \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_I^{\times}, \mathcal{C}).$$

For $S_+ \in \underline{\mathbb{F}}_{I,*}$, we view objects in $\tilde{\mathcal{C}}^{I-\times}_{S_+}$ over V as V-functors $\mathscr{P}_V(S)^{\mathrm{op}} \to \mathcal{C}_V$, where $\mathscr{P}_V(S)$ is the poset of V-subsets of S.

The following lemma is then immediately implied by [HTT, Cor 3.2.2.13].

Lemma A.11. *Let $\tilde{p} : \tilde{\mathcal{C}}^{I-\times} \to \underline{\mathbb{F}}_{I,*}$ be the projection, and let $\tilde{\alpha} : F \to G$ be a morphism lying over a morphism $\alpha : T \to S$ lying over an orbit map $U \to V$. Then, $\tilde{\alpha}$ is $\tilde{p}$-cocartesian in the sense of [HTT] if and only if, for all $T' \subset T$, the induced map $F(\alpha^{-1}(\mathrm{Res}_U^V T')) \to \mathrm{Res}_U^V G(T')$ is an equivalence; in particular, $\tilde{p}$ is a cocartesian fibration of simplicial sets*

Since $\tilde{\mathcal{C}}^{I-\times} \to \underline{\mathbb{F}}_{I,*}$ is a cocartesian fibration of simplicial sets, it is an inner fibration, so $\tilde{\mathcal{C}}^{I-\times}$ is a quasicategory. Using this, we henceforth treat $\tilde{\mathcal{C}}^{I-\times} \to \underline{\mathbb{F}}_{I,*}$ as a cocartesian fibration of ∞-categories. Let $\mathcal{C}^{I-\times} \subset \tilde{\mathcal{C}}^{I-\times}$ be the full subcategory spanned by those functors $\mathscr{P}(S)^{\mathrm{op}} \to \mathcal{C}_{\underline{V}}$ satisfying the property that, for all $T \subset S$, the maps

$$F(T) \to \mathrm{CoInd}_U^V \mathrm{Res}_U^V F(U)$$

exhibit $F(T)$ as the T-indexed product $F(T) \simeq \prod_U^T F(U)$ in $\mathcal{C}$. Once again, the following follows by definition.

Proposition A.12. *A morphism in $\mathcal{C}^{I-\times}$ is p-cocartesian if and only if it lifts to a $\tilde{p}$-cocartesian morphism of $\mathcal{C}^{I-\times}$. In particular, the projection $p : \mathcal{C}^{I-\times} \to \underline{\mathbb{F}}_{I,*}$ is an I-symmetric monoidal category if and only if $\mathcal{C}$ admits I-indexed products.*

Observation A.13. $\mathcal{C}^{I-\times}$ is a cartesian I-symmetric monoidal ∞ category with underlying $\mathcal{T}$-∞-category $\mathcal{C}$, so we have not created a clash in notation. ◁

Observation A.14. The structure map $\mathcal{O}^{\otimes} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_{I,*} \to \mathcal{O}^{\otimes}$ admits a left adjoint L sending $X \in \mathcal{O}^{\otimes}_{S_+}$ to $(X, S \subset S)$; the unit map of this adjunction is evidently an equivalence, so $L : \mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_{I,*}$ is fully faithful. ◁

Fix a $\mathcal{T}$ functor $A : \mathcal{O}^{\otimes} \times_{\underline{\mathbb{F}}_{I,*}} \Gamma^{\times} \to \mathcal{C}$ with corresponding functor $\varphi : \mathcal{O}^{\otimes} \to \tilde{\mathcal{C}}^{I-\times}$ and restricted functor $A' : \mathcal{O}^{\otimes} \to \mathcal{C}$. Lemma A.11 immediately implies the following.

Lemma A.15. *Suppose A' is a $\mathcal{T}$-functor. Then, the following conditions are equivalent:*

(a) The map φ is a functor of I-operads.
(b) For all morphisms α in $\mathcal{O}^{\otimes} \times_{\mathbb{F}_{I,*}} \underline{\Gamma}_I^{\times}$ whose image in $\mathcal{O}^{\otimes}$ is inert $A(\alpha)$ is an equivalence in $\mathcal{C}$.
(c) If $f : (\overline{S}_+ \to V_+, \overline{S}, F, X) \to (S_+ \to V_+, \overline{S}, F, X)$ is a cocartesian lift of the corresponding inert morphism, then $A(f)$ is an equivalence.
(d) A is right Kan extended from A' along L.

In this case, the composite map $\mathcal{O}^{\otimes} \to \tilde{\mathcal{C}}^{I-\times} \to \mathcal{C}$ is homotopic to A'.

We use this to finally identify Cartesian algebras in the following lemma, which also follows imeediately from Lemma A.11.

Lemma A.16. *Suppose φ is a functor of I-operads. Then, the following conditions are equivalent:*

(a) φ factors through the inclusion $\mathcal{C}^{I-\times} \subset \tilde{\mathcal{C}}^{I-\times}$.
(b) A' is an $\mathcal{O}$-monoid.

Proof of Proposition 1.18. $\mathcal{C}^{I-\times} \hookrightarrow \tilde{\mathcal{C}}^{I-\times}$ is fully faithful, and hence it is a monomorphism in **Cat**. This implies that the associated functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\times}) \hookrightarrow \mathrm{Fun}^{int-cocart}_{/\mathbb{F}_{I,*}}\left(\mathcal{O}^{\otimes}, \tilde{\mathcal{C}}^{I-\times}\right) \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{O}^{\otimes}, \mathcal{C}\right)$$

is fully faithful. By Lemma A.16, its image is the $\mathcal{O}$-monoids. □

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