AN ECKMANN-HILTON ARGUMENT IN EQUIVARIANT HIGHER ALGEBRA

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ABSTRACT. Let $\mathcal{O}^\otimes$ and $\mathcal{P}^\otimes$ be k- and ℓ-connected unital G-operads subject to the condition for all S that $\mathcal{O}(S) = \varnothing$ if and only if $\mathcal{P}(S) = \varnothing$. We show that the Boardman-Vogt tensor product $\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes$ is $(k + \ell + 2)$-connected; equivalently, $\mathcal{O} \otimes \mathcal{P}$-monoids in any $(k + \ell + 3)$-category lift uniquely to incomplete semi-Mackey functors. As a consequence, we show that the smashing localizations on unital G-operads correspond precisely with unital $\mathcal{N}_\infty$-operads, and hence the (finite) poset of unital weak indexing systems. Along the way we characterize ℓ-connectivity of a unital G-operad $\mathcal{O}^\otimes$ equivalently as ℓ-connectivity of $\mathcal{O}$-admissible Wirthmüller maps of $\mathcal{O}$-monoid spaces.

In the discrete case, under no connectivity assumptions, $\mathcal{O} \otimes \mathcal{P}$-monoids lift uniquely to incomplete semi-Mackey functors, recovering an Eckmann-Hilton argument for "C_p-unital magmas." In the limiting case of infinite tensor powers, we *take the loops out of equivariant infinite loop space theory,* constructing algebraic approximations to incompletely stable G-spectra over arbitrary transfer systems.

INTRODUCTION

The classical *Eckmann-Hilton argument* shows that, given a set with two unital multiplications $(M, *, \cdot)$ satisfying the interchange law

$$(a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d),$$

the unital magmas $(M, *)$ and $(M, \cdot)$ are isomorphic to each other and are commutative monoids. This result is foundational to algebraic topology; for instance, it is used to show that $\pi_n(X, x)$ is Abelian for all $n \geq 2$, and for $n = 1$ if X admits an H-space structure. We will study equivariant variations of this result, beginning with a weakening of Dress' Mackey functors [Dre71].

Definition 1. Let $\mathcal{C}$ be a 1-category with finite products and C_p the cyclic group of prime order p. A C_p-*unital magma in* $\mathcal{C}$ is a unital magma M^e with a C_p action by unital magma homomorphisms, a unital magma M^{C_p} (with trivial C_p-action), and C_p-equivariant *restriction and transfer* homomorphisms

$$r \colon M^{C_p} \to M^e, \qquad\qquad t \colon M^e \to M^{C_p}$$

subject to the condition that $r \circ t$ is multiplication by p. A *homomorphism* $M \to N$ is a pair of unital magma homomorphisms $F^e \colon M^e \to N^e$ and $F^{C_p} \colon M^{C_p} \to M^e$ such that $F^{C_p} \circ t = t \circ F^e$ and $F^e \circ r = r \circ F^{C_p}$. ◄

Example 2. The $(\lambda + 1)$st homotopy coefficient system of a C_p-space attains a natural C_p-unital magma structure under the evident analog of Lewis' *unstable Mackey structure* [Lew92]. A similar structure holds for ρth homotopy when $p = 2$.[1] ◄

In this article, we prove and vastly generalize the following theorem.

Theorem A. *Suppose (M, M') is a pair of C_p-unital magma structures on the same coefficient system satisfying suitable interchange relations. Then, $M \simeq M'$ and each underlie a semi-Mackey functor; in particular, if the multiplications on M^e and M^{C_p} are invertible, then M and M' are isomorphic Mackey functors.*

For instance, given X a C_2-unital magma valued in spaces, the C_2-space UX defined by $UX^H = X(H)$ has an induced C_2-unital magma structure on $\underline{\pi}_\rho(X)$ which interchanges with that of Example 2. Theorem A implies that these two structures agree and lift to a Mackey functor.

Example 3. The above argument confirms that the Mackey structure from Real Bott periodicity [Ati66] and the additive Mackey structure on Real vector bundles induce the same structure on $\underline{\pi}_\rho \mathrm{BU}_\mathbb{R} = \underline{\pi}_0 \mathrm{BU}_\mathbb{R}$. ◄

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[1] Explicitly, by V-Mackey functor, we mean a functor $\mathscr{B}_G(V) \to \mathbf{Ab}$ sending disjoint unions to direct sums, where $\mathscr{B}_G(V)$ is Lewis' V-Burnside category; the transfer map $\Sigma_+^{\lambda+1} *_{C_p} \to \Sigma_+^{\lambda+1}[C_p/e]$ is constructed by the usual $\mathbb{S}_G$-duality construction along an embedding $[C_p/e] \hookrightarrow \lambda$ (see [Wir75]). λ refers to any nontrivial 2-dimensional C_p-representation.

To prove Theorem A, we embed C_p-unital magmas in the theory of *algebras over G-operads* in the sense of [NS22]; in particular, we show in Section 3 that C_p-unital magmas are algebras over a particular C_p-operad $\mathbb{A}_{2,C_p}^{\otimes}$ in C_p-coefficient systems valued in $\mathcal{C}$, and spell out the correct interchange relations there. We recommend that the reader familiarizes themself with the language of equivariant higher algebra via the introductions to [Ste25a; Ste25b].

Crucially, the *Boardman-Vogt tensor product* of [Ste25a] corepresents interchanging G-operad algebras:

$$\mathrm{Alg}_{\mathcal{O} \otimes \mathcal{P}}(\mathcal{D}) \simeq \mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{D}).$$

In particular, pairs of interchanging C_p-unital magma structures correspond with $\mathbb{A}_{2,C_p}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{A}_{2,C_p}^{\otimes}$-algebras.

Now, G-operads are manifestly homotopy-theoretic gadgets; indeed, their algebras subsume the homotopy-coherent incomplete (semi-) Mackey functors of [BH18; Gla17] by [Ste25b], the homotopy-coherent bi-incomplete Tambara functors of [BH22; EH23] by [Cha24; CHLL24], and the algebraic structure on iterated loop G-spaces and their Thom spectra by [GM11; HHKWZ24]. In particular, the first and second examples are incarnated by the *weak $\mathcal{N}_\infty$-operads* of [Ste25a], which are characterized together by the fact that their nonempty structure spaces are contractible, and classified by their "arity support" weak indexing category

$$A\mathcal{O} := \{T \to S \ | \ \forall [G/H] \subset S, \ \mathcal{O}(T \times_S [G/H]) \neq \varnothing\} \subset \mathbb{F}_G;$$

here, $\mathcal{O}(S)$ is the "S-ary structure space," such as the S-ary structure space $\mathbb{E}_V(S) = \mathrm{Conf}_S^H(V)$ for the little V-disks G-operad incarnating the third example.

Thankfully, $\mathcal{O}$-algebras in a G-symmetric monoidal n-category are canonically equivalent to algebras over the *homotopy n-operad $h_n \mathcal{O}^{\otimes}$*, whose structure spaces are the $(n-1)$-truncations of the structure spaces of $\mathcal{O}^{\otimes}$ [Ste25a].[2] In particular, if the structure spaces of $\mathcal{O}^{\otimes}$ are n-connected, then $h_n \mathcal{O}^{\otimes}$ possesses a (unique) equivalence with a weak $\mathcal{N}_\infty$-operad; for instance, its algebras in coefficient systems $\underline{\mathrm{CoFr}}^G(\mathcal{D})^{G-\times}$ valued in an n-category $\mathcal{D}$ are (homotopy-coherent) incomplete semi-Mackey functors. In this situation, we say that $\mathcal{O}^{\otimes}$ is *n-connected*.

From this, we identify Theorem A with the statement that $\mathbb{A}_{2,C_p}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{A}_{2,C_p}^{\otimes}$ is connected, together with the easy observation that $A\mathbb{A}_{2,C_p} = \mathbb{F}_{C_p}$, so the corresponding incomplete semi-Mackey functors have all transfers. Our main lift of Theorem A is the following equivariant generalization of [SY19, Thm 1.0.1].

Theorem B. *If $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ are k and ℓ-connected almost essentially unital G-operads with $A\mathcal{O} = A\mathcal{P}$, then $\mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes}$ is $(k+\ell+2)$-connected.*

Corollary 4. *If $\mathcal{O}^{\otimes}$ is a nonempty almost essentially unital G-operad, then $\mathcal{O}^{\otimes n}$ is $(n-1)$-connected.*

For instance, Corollary 4, lax G-symmetric monoidality of $\underline{\pi}_0 \colon \underline{\mathrm{Sp}}_G^{\otimes} \to \underline{\mathrm{Mack}}_G^{\square}(\mathbf{Ab})$, and the results of [Cha24] together construct a natural $A\mathcal{O}$-Tambara structure on the 0th homotopy groups of $\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{O}$-ring G-spectrum;[3] this and a forthcoming equivariant Dunn additivity result will construct a natural AV-Tambara structure on the 0th homotopy Mackey functors of $\mathbb{E}_{2V}$-ring G-spectra.

We may remove the assumption $A\mathcal{O} = A\mathcal{P}$ in Theorem B, but we will need a more refined notion of connectivity. In general, given a weak indexing category I, we say that $\mathcal{O}^{\otimes}$ is *k-connected at I* if, for all elements of the corresponding weak indexing system

$$T \in \mathbb{F}_{I,H} := \left\{ S \in \mathbb{F}_H \ \middle| \ \mathrm{Ind}_H^G S \to [G/H] \in I \right\},$$

the structure space $\mathcal{O}(T)$ is k-connected. We define the *connectivity function*

$$\mathrm{Conn}_{\mathcal{O}} \colon \mathrm{wIndexCat}_G^{a\mathrm{uni}} \longrightarrow \mathbb{Z} \cup \{\infty\}$$

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[2] Throughout this article, *n-category* will be used to refer to $(n,1)$-*categories*, i.e. ∞-categories whose mapping spaces are all $(n-1)$-truncated. The reader should feel free to think mostly in terms of familiar examples, such as the n-categoy of $(n-1)$-truncated spaces, of $(n-1)$-truncated connective spectra, of small $(n-1)$-categories, or of the hammock localization of chain complexes with homology concentrated in degrees $[d, d+n-1]$ for some uniform d.

[3] To construct this lax symmetric monoidality, first note that $\underline{\mathrm{Sp}}_{G,\geq 0}^{\otimes} \subset \underline{\mathrm{Sp}}_G^{\otimes}$ is closed under tensor products, so the localization G-functor $\underline{\mathrm{Sp}}_G \to \underline{\mathrm{Sp}}_{G,\geq 0}^{\otimes}$ is given a lax G-symmetric monoidal structure by Proposition 40. Moreover, to construct a lax G-symmetric monoidal structure on $\tau_{\leq 0} = \pi_0 \colon \underline{\mathrm{Sp}}_{G,\geq 0} \to \underline{\mathrm{Sp}}_G$, in light of [NS22] we need only note that $\otimes$ takes π_0-equivalences to π_0-equivalences and that the resulting structure agrees with the usual one on Mackey functors; the former follows by the same fact applied to geometric fixed points combined with induction up the poset of families using the isotropy separation sequence.

by the formula $\mathrm{Conn}_{\mathcal{O}}(S) := \min\{k \mid \mathcal{O}^{\otimes} \text{ is } k\text{-connected at } I\}$. This is a G-operadic version of the *connectivity dimension-function* of a G-space.

Now, $(\mathbb{Z} \cup \{\infty\})^{\mathrm{wIndexCat}_G^{\mathrm{Auni}}}$ forms a *pointwise* commutative monoidal poset, i.e.

$$f \leq g \iff \forall S, \, f(S) \leq g(S).$$

A cleaner statement of Theorem B is the following.

Theorem C. *Given $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ a pair of almost-unital G-operads, the following inequality holds:*

$$\mathrm{Conn}_{\mathcal{O}} + \mathrm{Conn}_{\mathcal{P}} + 2 \leq \mathrm{Conn}_{\mathcal{O} \otimes \mathcal{P}}.$$

To put Theorems B and C into context, note that a G-operad $\mathcal{O}^{\otimes}$ is a weak $\mathcal{N}_{\infty}$-operad if and only if $\mathrm{Conn}_{\mathcal{O}}$ has all values -2 or ∞; in this case, Theorem C says that weak $\mathcal{N}_{\infty}$-operads are closed under tensor products and $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes}$ is classified by a weak indexing category contained in the join $I \vee J$.

The above specialization of Theorem C is the difficult part of the main theorem of [Ste25b]. To explain how it is proved, we must introduce a definition; for $S \in \mathbb{F}_H$, the *S-indexed Wirthmüller map* in a (suitably pointed) G-∞-category is defined to be the S-indexed semiadditive norm map as in [CLL24; Nar16]; that is, the $[H/K]$-indexed Wirthmüller map $W_{[H/K],X}: \mathrm{Ind}_K^H X \to \mathrm{CoInd}_K^H X$ is adjunct to the map

$$X \longrightarrow \mathrm{Res}_K^H \mathrm{CoInd}_K^H X \simeq \prod_{g \in [K \backslash H/K]} \mathrm{CoInd}_{H \cap gKg^{-1}}^H \mathrm{Res}_{H \cap gKg^{-1}}^H X$$

whose projection onto the factor indexed by the identity double coset is the identity and whose other projections are zero. More generally, the $\coprod_i [H/K_i]$-indexed Wirthmüller map

$$W_{\coprod_i [H/K_i],(X_i)}: \coprod_{K_i}^H X_i \simeq \coprod_i \mathrm{Ind}_{K_i}^H X_i \longrightarrow \prod_i \mathrm{CoInd}_{K_i}^H X_i \simeq \prod_{K_i}^H X_i$$

is classified by the diagonal matrix whose ith entry is $W_{[H/K_i],X_i}$.

The key strategy of [Ste25b] was to recognize that a G-operad $\mathcal{O}^{\otimes}$ has mWirthmüller isomorphisms indexed by I if and only if, for all $\mathrm{Ind}_H^G S \to [G/H]$ lying in I, $\mathcal{O}(S)$ is contractible. In particular, this identified $\mathcal{N}_{I\infty}^{\otimes}$ as the *unique* G-operad $\mathcal{O}^{\otimes}$ with $A\mathcal{O} \leq I$ such that $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{S}_G)$ has I-indexed Wirthmüller isomorphisms.

In this article, our key strategy will be to identify ℓ-connectivity of $\mathcal{O}^{\otimes}$ via the category theory of $\mathcal{O}$-algebra G-spaces. To that end, we will prove the following central theorem.

Theorem D. *Let $\mathcal{P}^{\otimes}$ be a G-operad, I an almost unital weak indexing category, and ℓ a natural number. Then, the following conditions are equivalent:*

(a) *$\mathcal{P}^{\otimes}$ is ℓ-connected at I.*

(b) *For all n-toposes $\mathcal{C}$ (with $n \leq \infty$), I-admissible H-sets $S \in \mathbb{F}_{I,H}$, and S-indexed tuples of $\mathcal{P}$-monoids $(X_K) \in \prod_{[H/K] \in \mathrm{Orb}(S)} \mathrm{Mon}_{\mathrm{Res}_K^G \mathcal{P}}(\mathcal{C})$, the S-indexed $\mathcal{P}$-monoid Wirthmüller map*

$$W_{S,(X_K)}: \coprod_K^S X_K \longrightarrow \prod_K^S X_K$$

is ℓ-connected.

(c) *For all I-admissible H-sets $S \in \mathbb{F}_{I,H}$ and S-indexed tuples of $\mathcal{P}$-monoid G-spaces $(X_K) \in \prod_{[H/K] \in \mathrm{Orb}(S)} \mathrm{Mon}_{\mathrm{Res}_K^G \mathcal{P}}(\mathcal{S})$, the S-indexed $\mathcal{P}$-G-space Wirthmüller map*

$$W_{S,(X_K)}: \coprod_K^S X_K \longrightarrow \prod_K^S X_K$$

is ℓ-connected.

For Theorem D, a morphism $g: X \to Y$ in an ∞-category $\mathcal{C}$ is *ℓ-truncated* if, for all $Z \in \mathcal{C}$, the map of spaces $\mathrm{Map}(Z,X) \to \mathrm{Map}(Z,Y)$ is ℓ-truncated, and $f: A \to B$ is *ℓ-connected* if, for all diagrams

$$\begin{array}{ccc} A & \longrightarrow & X \\ f \downarrow & \nearrow h & \downarrow g \\ B & \longrightarrow & Y \end{array}$$

such that g is ℓ-truncated, the space of lifts h is contractible.

Remark 5. In the case that $\mathcal{C}$ is an n-topos for some $0 \leq n \leq \infty$, the above definitions are equivalent to ℓ-*truncatedness* and $(\ell-1)$-*connectiveness* in the sense of [HTT, Def 6.5.1.10] by [SY19, Lem 4.2.6] and [HTT, Prop 6.5.1.12, Prop 6.5.1.19]. ◄

Remark 6. In the course of proving Theorem D, we will verify that Condition (b) is further equivalent to the condition that the $\mathrm{CoFr}^H \mathcal{C}$-map underlying $W_{S,(X_K)}$ is pointwise ℓ-connected; moreover, Condition (c) is equivalent to the condition that the underlying H-space map is ℓ-connected, i.e. its associated maps on J-fixed point spaces are surjective on path components with ℓ-connected fiber for each $J \subset H$. ◄

The rest of this paper replaces the orbit category $\mathcal{O}_G$ with an arbitrary atomic orbital ∞-category $\mathcal{T}$; we will prove Theorems B to D in that level of generality. We encourage the reader to either globally specialize to $\mathcal{T} = \mathcal{O}_G$ or familiarize themself with the atomic orbital setting via [Ste25a] .

Consequences in higher algebra. The specialization of Theorem B to infinite tensor powers is the following.

Corollary 7. *Suppose* $\mathcal{O}^\otimes$ *is an almost-reduced* $\mathcal{T}$-*operad. Then, the following conditions are equivalent.*

(a) $\mathcal{O}^\otimes$ *is an almost-unital weak* $\mathcal{N}_\infty$-*operad.*
(b) *($\mathcal{O}^\otimes$-EHA) the unique map* $\mathrm{triv}_{\mathcal{T}}^\otimes \to \mathcal{O}^\otimes$ *yields an equivalence*

$$\mathcal{O}^\otimes \simeq \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{\mathcal{T}}^\otimes \xrightarrow{\mathrm{id} \otimes \mathrm{can}} \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{O}^\otimes.$$

(c) *(abstract* $\otimes$-*idempotence) there exists an equivalence* $\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{O}^\otimes \simeq \mathcal{O}^\otimes.$

Proof. The implication (a) $\implies$ (b) is one of the main results of [Ste25b], and is also implied by Theorem B. The implication (b) $\implies$ (c) is obvious. To see the implication (c) $\implies$ (a), note that Theorem B implies that $\mathcal{O}^\otimes$ is ∞-connected, i.e. all of its nonempty structure spaces are contractible. The result follows by the identification of such almost-reduced $\mathcal{T}$-operads with almost-unital weak $\mathcal{N}_\infty$-operads [Ste25a]. □

To see why we may view Condition (b) as an *Eckmann-Hilton argument*, note that it is equivalent to the condition that $\mathcal{O}^\otimes$ possesses a unital magma structure in $\mathrm{Op}_{\mathcal{T}}^\otimes$ whose multiplication map $\mu \colon \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{O}^\otimes \to \mathcal{O}^\otimes$ is an equivalence; unitality of μ is precisely the condition that the associated diagonal natural transformation

$$\delta \colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\mathcal{C})$$

is split by restriction to either $\mathcal{O}$-algebra structure, and the fact that μ is an equivalence is precisely the condition that δ is a natural equivalence, i.e. pairs of interchanging $\mathcal{O}$-algebra structures agree, and there is one such pair for all $\mathcal{O}$-algebra structures.

On the other hand, Condition (b) is equivalent to the assertion that $\mathcal{O}^\otimes$ admits a (unique) structure as an *idempotent algebra* in $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni},\otimes}$; taking modules yields a bijective monotone correspondence between these and the smashing localizations on $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni},\otimes}$ (see [GGN15, § 3] and [CSY20, § 5.1]).

Corollary 7 classifies smashing localizations on $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}$; define the full subcategory

$$\mathrm{Op}_{\mathcal{T}}^{I-\mathrm{Wirth}} := \left\{ \mathcal{O}^\otimes \; \middle| \; \forall\, S \in \underline{\mathbb{F}}_I, \mathcal{C}^\otimes \in \mathrm{Cat}_{\mathcal{T}}^\otimes, \; \overset{S}{\bigotimes} \simeq \overset{S}{\coprod} \text{ in } \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \right\} \subset \mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}.$$

In [Ste25b] we showed that this is the smashing localization for $\mathcal{N}_{I\infty}^\otimes$ in order to compute tensor products of $\mathcal{N}_\infty$-operads. We also showed that idempotent algebras in $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}$ are almost-reduced, yielding the following.

Corollary E. *The construction* $I \mapsto \mathrm{Op}_{\mathcal{T}}^{I-\mathrm{Wirth}}$ *yields an isomorphism of posets*

$$\mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \xrightarrow{\;\sim\;} \left\{ \text{Smashing localizations of } \mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}} \text{ under reverse inclusion} \right\}$$

A striking corollary of this is that there are finitely many smashing localizations on $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}$ whenever $\mathcal{T}$ is essentially finite [Ste24]; moreover, they have rich combinatorial structure, as they are naturally cocartesian-fibered over $\mathcal{T}$-transfer systems, giving e.g. a cocartesian fibration from smashing localizations of $\mathrm{Op}_{C_{p^n}}^{a\mathrm{uni}}$ to the $(n+2)$nd associahedron, whose fibers can be explicitly described [BBR21; Ste24].

Consequences in algebraic topology. Let I be an indexing category and Sp_I the ∞-category presented by Blumberg-Hill's stable model category of I-spectra [BH21]. We say that an I-spectrum E is *connected* if $\underline{\pi}_n(E) \simeq 0$ for all $n \leq 0$, i.e. it is the suspension of a connective I-spectrum. We see that *any* loop space theory with arity support I reaches connected I-spectra after infinite iteration.

Corollary 8. *If $\mathcal{O}^{\otimes}$ is a reduced G-operad with $\mathcal{O}(2 \cdot *_G) \neq 0$ and X is a connected G-space with infinitely many interchanging $\mathcal{O}$-algebra structures, then X is the 0th G-space of an essentially unique connected $A\mathcal{O}$-spectrum compatibly with its $\mathcal{O}^{\otimes \infty}$-structure.*

Proof. Note that $\mathcal{O}^{\otimes \infty} := \mathrm{colim}_{n \to \infty} \mathcal{O}^{\otimes n}$ is abstractly $\overset{\mathrm{BV}}{\otimes}$-idempotent, so $\mathcal{O}^{\otimes \infty} \simeq \mathcal{N}_{A\mathcal{O}}^{\otimes}$ by Corollary 7, i.e.

$$\tag{1} \underline{\mathrm{CAlg}}_{A\mathcal{O}}^{\otimes}(\mathcal{C}) \xrightarrow{\sim} \lim_{n \to \infty} \overbrace{\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes} \cdots \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}}^{n\text{-fold}}(\mathcal{C}).$$

Moreover, given a model $\mathcal{O}^t \in \mathrm{Op}(\mathrm{sSet}_G)$ for $\mathcal{N}_{A\mathcal{O}}^{\otimes}$, [Ste25b] and [Mar24] yield equivalences

$$\mathrm{CAlg}_{A\mathcal{O}}\left(\underline{\mathcal{S}}_{G,\geq 1}^{G-\times}\right) \simeq \mathrm{CMon}_{A\mathcal{O}}(\mathcal{S}_{\geq 1}) \simeq \mathrm{Alg}_{\mathcal{O}^t}\left(\mathrm{Top}_{G,\geq 1}\right)\left[\mathrm{WEQ}^{-1}\right]$$

over $\mathcal{S}_{G,\geq 1}$, the right hand side denoting the Hammock localization inverting the class of (point-set) $\mathcal{P}$-algebra morphisms whose underlying function of topological G-spaces is a G-weak equivalence.[4] The defining equivalence $\mathrm{Sp}_{A\mathcal{O},\geq 0} \simeq \mathrm{Alg}_{\mathcal{O}^t}^{\mathrm{grplike}}\left(\mathrm{Top}_G\right)\left[\mathrm{WEQ}^{-1}\right]$ then embeds $\mathrm{Alg}_{\mathcal{O}^t}\left(\mathrm{Top}_{G,\geq 1}\right)\left[\mathrm{WEQ}^{-1}\right]$ as those $A\mathcal{O}$-spectra whose 0th G-space is connected; it follows by unwinding definitions that this is precisely $\mathrm{Sp}_{A\mathcal{O},\geq 1}$, so Eq. (1) restricts to an equivalence, lying over, $\mathcal{S}_{G,\geq 1}$ of the form

$$\mathrm{Sp}_{I,\geq 1} \simeq \lim_{n \to \infty} \overbrace{\underline{\mathrm{Alg}}_{\mathcal{O}} \cdots \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}}^{n\text{-fold}}(\mathcal{S}_{G,\geq 1}). \qquad \square$$

To construct an infinite loop space theory for I-spectra, one is left with the following question.

Question 9. Given an indexing category I, does there exist a reduced G-operad $\mathcal{O}^{\otimes}$ with $A\mathcal{O} = I$ and a space S^I such that $\mathcal{O}$-monoid structures on a connected G-space X are equivalent to S^I-loop space structures? ◁

Remark 10. Corollary 8 has a strong philosophical implication running transverse to Question 9: regardless of the topology, it constructs a flexible machine which inputs algebraic theories and outputs towers of ∞-categories converging to equivariant stable homotopy theory. For instance, iterating algebras over Rubin's free or associative N-opeards [Rub21] yields such a tower converging to arbitrary Sp_I.

In essence, Corollary 8 *takes the loops out of equivariant infinite loop space theory*, extending algebraic versions of the theory to arbitrary incompletely stable categories regardless of the answer to Question 9. ◁

Remark 11. We chose to specialize to the connected setting for convenience; one could instead assume that there exists some $\mu \in \mathcal{O}(2 \cdot *_G)$ whose action on one of the $\mathcal{O}$-structures on X induces an *invertible* magma structure on the coefficient system $\underline{\pi}_0 X$, in which case the corresponding $A\mathcal{O}$-commutative algebra has an underlying grouplike commutative monoid structure; the variation of Corollary 8 follows *mutatis mutandis*. ◁

Additionally, we acquire Ω^V-spectrum structures in a similar circumstance.

Corollary 12. *Fix V an orthogonal G-representation, $n \in [1, \infty]$, and $\mathcal{O}^{\otimes}$ an almost-reduced G-operad with $\mathcal{O}(S) \neq \varnothing$ whenever there exists an embedding $S \hookrightarrow \mathrm{Res}_H^G V$. If X is a connected and n-truncated G-space admitting $(n+1)$-many interchanging $\mathcal{O}$-algebra structures, then X underlies a V-infinite loop space.*

Proof. Note that the G-∞-category of n-truncated connected G-spaces is a G-n-category; indeed, Elmendorf's theorem yields an equivalence

$$\left(\underline{\mathcal{S}}_{G,[1,n]}\right)_H \simeq \mathrm{Fun}\left(\mathcal{O}_H^{\mathrm{op}}, \mathcal{S}_{[1,n]}\right),$$

and $\mathcal{S}_{[1,n]}$ is an n-category as whenever X, Y are n-truncated connected spaces, $\mathrm{Map}(\Sigma^{n+1} X, Y) \simeq *$; hence [HTT, Cor 2.3.4.8] implies that each value $\left(\underline{\mathcal{S}}_{G,[1,n]}\right)_H$ is an n-category.

The desired V-infinite loop space structure corresponds under the recognition principle of [GM17; RS00] with the $\mathbb{E}_{\infty V}$-structure pulled back along the unique map $\mathbb{E}_{\infty V}^{\otimes} \simeq \mathcal{N}_{AV}^{\otimes} \to \mathcal{N}_{A\mathcal{O}}^{\otimes} \simeq h_n \mathcal{O}^{\otimes(n+1)}$ of Corollary 4. $\square$

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[4] Here, $\mathrm{sSet}_G := \mathrm{sSet}^{BG}$ and $\mathrm{Top}_G := \mathrm{Top}^{BG}$ are the 1-categories of *simplicial sets* and *topological spaces* with G-action.

Philosophy. The following significantly motivated this article and its prequels [Ste24; Ste25a; Ste25b].

Question 13. For what higher-algebraic, universal, and operadic reasons do $\mathcal{N}_\infty$ operads arise? ◁

Of course, there are preexisting higher-algebraic reasons: the several incomplete variants of the *spectral Mackey functor theorem* verify that $\mathcal{N}_{I\infty}$-monoids are intimately connected with I-admissible Wirthmüller isomorphisms, which are close to the heart of equivariant stable homotopy theory [BH18; CLL24; GM11; Mar24; Nar16]. Since the value $\underline{\mathrm{Alg}}_{\mathcal{N}_{I\infty}}(\underline{\mathcal{S}}_G)$ uniquely pins $\mathcal{N}_{I\infty}^\otimes$ as a G-operad [Ste25b], this characterizes $\mathcal{N}_\infty$-operads. This is not a complete answer, as it requires us to care about I-indexed Wirthmüller isomorphisms a-priori; that is, while our reason is higher-algebraic and universal, it is not quite operadic in its philosophy.

Now, if we admit *weak $\mathcal{N}_\infty$-operads*, a universal operadic characterization is easy to come by: in [Ste25a] we confirmed that weak $\mathcal{N}_\infty$-operads are precisely the subterminal objects of Op_G. Unfortunately, *algebra* lives in the mapping spaces from one-object G-operads to G-symmetric monoidal ∞-categories, and no nontrivial $\mathcal{N}_\infty$-operads are G-symmetric monoidal ∞-categories, so the mapping-in property identifying subterminal objects is not higher-algebraic in nature.

In the almost-unital locus (or, for that matter, the unital locus), Corollary 7 gives a characterization with all three properties: almost-unital weak $\mathcal{N}_\infty$-operads are characterized universally as the corepresenting G-operads at the limit of of (infinitary) Eckmann-Hilton arguments in equivariant higher algebra.

Sharpness. Theorems B and C are not sharp for all examples. One reason is the discrepancy between unions and joins of weak indexing systems.

Example 14. It follows by definition that

$$\mathrm{Conn}_{\mathcal{N}_{I\infty}}(S) = \begin{cases} \infty & S \in \underline{\mathbb{F}}_I, \text{ and} \\ -2 & \text{otherwise;} \end{cases}$$

we also found in [Ste25b] that $\mathcal{N}_{I\infty}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^\otimes \simeq \mathcal{N}_{I\vee J\infty}^\otimes$. Generically, this defeats sharpness of Theorem C, as

$$\left(\mathrm{Conn}_{\mathcal{N}_{I\infty}} + \mathrm{Conn}_{\mathcal{N}_{J\infty}} + 2\right)^{-1}(\infty) = \underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J \subsetneq \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = \mathrm{Conn}_{\mathcal{N}_{I\infty}\otimes\mathcal{N}_{J\infty}}^{-1}(\infty). \quad ◁$$

Another issue is topological; in forthcoming work, given V an orthogonal G-representation, we will show that the little V-disks G-operad $\mathbb{E}_V^\otimes$ is ℓ-connected at the minimal unital weak indexing category $I_S \vee I^0$ containing S if and only if the following conditions are satisfied:

(a) For all orbits $[H/K] \subset S$ and intermediate inclusions $K \subset J \subset H$, we have $\dim V^J \geq \dim V^K + \ell + 2$, and
(b) if $\left|S^H\right| \geq 2$, then $\dim V^H \geq \ell + 2$.

Moreover, we will show that $\mathbb{E}_V$ is additive under tensor products, i.e. $\mathbb{E}_V^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^\otimes \simeq \mathbb{E}_{V\oplus W}^\otimes$.

Example 15. Let $G := C_2$, with sign representation σ. Then, we have fixed point dimensions

$$\dim(a + b\sigma)^e = a + b; \qquad \dim(a + b\sigma)^{C_2} = a.$$

In particular, the connectivity function has

$$\mathrm{Conn}_{\mathbb{E}_{a+b\sigma}}(k*_e) = a + b - 2$$

$$\mathrm{Conn}_{\mathbb{E}_{a+b\sigma}}(c*_{C_2} + d[C_2/e]) = \begin{cases} a - 2 & d = 0 \\ b - 2 & c < 2 \\ \min(a,b) - 2 & \text{otherwise.} \end{cases}$$

where $\mathrm{Conn}(S) := \mathrm{Conn}(I_S \vee I^0)$. Note that $\mathrm{Conn}_{\mathbb{E}_{a+b\sigma}}(c*_{C_2} + d[C_2/e])$ is as non-additive as is possible in the last case; indeed, the examples $1 + b\sigma$ and $a' + \sigma$ have the same arity-support, but when $a', b > 1$, we have

$$\mathrm{Conn}_{1+b\sigma}(2*_{C_2} + [C_2/e]) + \mathrm{Conn}_{a'+\sigma}(2*_{C_2} + [C_2/e]) - 2 = 0$$
$$< \min(a', b) - 1$$
$$= \mathrm{Conn}_{a'+1+(b+1)\sigma}(2*_{C_2} + [C_2/e]). \quad ◁$$

Nevertheless, equality is sometimes attained.

Example 16. For all orthogonal G-representations V, it follows from the above description that

$$\mathrm{Conn}_{\mathbb{E}_V\otimes\mathbb{E}_V} = \mathrm{Conn}_{\mathbb{E}_{2V}} = 2\mathrm{Conn}_{\mathbb{E}_V} - 2. \quad ◁$$

The strategy. We begin with a convenient simplification.

Lemma 17. *The following theorems imply each other:*

(a) *Theorem B in all cases.*
(b) *Theorem B in the case that $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ are unital and $\mathcal{T}$ has a terminal object.*
(c) *Theorem C.*

Proof. The implication (a) $\implies$ (b) is obvious. The implication (c) $\implies$ (a) follows by monotonicity the function $\min_{S \in \underline{\mathbb{F}}_{A\mathcal{O}}} f(S) \colon (\mathbb{Z} \cup \{\infty\})^{\mathrm{wIndex}^{a\mathrm{uni}}} \to \mathbb{Z} \cup \{\infty\}$.

To prove the remaining implication, recall that restriction assembles to a (tautological) symmetric monoidal equivalence $\mathrm{Op}_{\mathcal{T}}^{\otimes} \simeq \lim_{V \in \mathcal{T}} \mathrm{Op}_{V}^{\otimes}$ such that, given a morphism $V \to W$ in $\mathcal{T}$ and S a finite V-set, $\mathrm{Res}_V^{\mathcal{T}} \mathcal{O}(S) \simeq \mathcal{O}(S)$. In particular, Condition (a) is equivalent to Theorem B for all restrictions, in which case $\mathcal{T}$ has a terminal object.

Moreover, each $\mathcal{O}(S), \mathcal{P}(S)$ is easily determined by arity support except in the case $V \in \nu(\mathcal{O})$, and arity support is additive in the predicted way by [Ste25b]; Condition (a) is equivalent to Theorem B for all restrictions for which V satisfies the above condition, in which case $\mathrm{Res}_V^{\mathcal{T}} \mathcal{O}^{\otimes}$ and $\mathrm{Res}_V^{\mathcal{T}} \mathcal{P}^{\otimes}$ are unital. $\qquad\square$

We're left with proving Theorem B in the unital case where $\mathcal{T}$ has a terminal object. We will perform a similar reduction to [SY19]; namely, by examining the free $\mathcal{O}$-algebra monad, we reduce this to $(k + 1)$-*connectivity of the reduced endomorphism $A\mathcal{O}$-operad in $\underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{C})^{I \text{-} \times}$ in the case $\mathcal{C}$ is the $\mathcal{T}$-∞-category of coefficient systems in a presheaf ∞-topos.

We express the structure space $\mathrm{End}_X\left(\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C})^{I \text{-} \times}\right)(S)$ as the spaces of lifts of $\Delta \colon X^{\sqcup S} \to X$ along the S-indexed Wirthmüller map $W_{X,S} \colon X^{\sqcup S} \to X^{\times S}$, which is directly related to truncatedness of X and connectedness of $W_{X,S}$ [SY19]; hence it suffices to prove Theorem D in the unital case.

We finish by directly relating ℓ-connectivity of $W_{X,S}$ in $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ and $\mathrm{Mon}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C})$, reducing Theorem D to the fact that $\mathrm{Mon}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C})$ is I-semiadditive when $\mathcal{O}$ is ℓ-connected at I, which we verified in [Ste25b].

1. I-OPERADS

Throughout this article, we fix $\mathcal{T}$ an atomic orbital ∞-category in the sense of [NS22]; that is, we assume that all retracts in $\mathcal{T}$ are equivalences and that the finite coproduct completion $\mathbb{F}_{\mathcal{T}} \coloneqq \mathcal{T}^{\sqcup}$ has pullbacks.

We begin in Section 1.1 by recalling the simultaneous generalization and weakening of Blumberg-Hill's G-indexing systems and I-Mackey functors to $\mathcal{T}$-*weak* indexing systems and I-commutative monoids. We go on to Section 1.3 where we recall the relevant background from [NS22; Ste25a; Ste25b] on $\mathcal{T}$-operads, as well as establishing a few foundational results concerning the *doctrinal adjunction* and *reduced endomorphism I-operads*.

1.1. **Preliminaries on $\mathcal{T}$-∞-categories and weak indexing systems.** Recall that a $\mathcal{T}$-*coefficient system* is a functor out of $\mathcal{T}^{\mathrm{op}}$:

$$\mathrm{CoFr}^{\mathcal{T}}(\mathcal{C}) \coloneqq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C}).$$

Generalizing Elmendorf's theorem, we define d-truncated $\mathcal{T}$-spaces and $\mathcal{T}$-d-categories as coefficient systems:

$$\mathcal{S}_{\mathcal{T}, \leq d} \coloneqq \mathrm{CoFr}^{\mathcal{T}}(\mathcal{S}_{\leq d}); \qquad \mathrm{Cat}_{\mathcal{T}, d} \coloneqq \mathrm{CoFr}^{\mathcal{T}}(\mathrm{Cat}_d).$$

We write $\mathrm{Cat}_{\mathcal{T}} \coloneqq \mathrm{Cat}_{\mathcal{T}, \infty}$ and $\mathcal{S}_{\mathcal{T}} \coloneqq \mathcal{S}_{\mathcal{T}, \leq \infty}$. Given a $\mathcal{T}$-∞-category $\mathcal{C}$, we write $\mathcal{C}_V$ for the value $\mathcal{C}(V)$ and $\mathrm{Res}_V^W \colon \mathcal{C}_W \to \mathcal{C}_V$ for the functoriality under a map $V \to W$. The ∞-category of $\mathcal{T}$-coefficient systems lifts to a $\mathcal{T}$-∞-category with V-value the $\mathcal{T}_{/V}$-coefficient systems

$$\underline{\mathrm{CoFr}}^{\mathcal{T}}(\mathcal{C})_V \coloneqq \mathrm{CoFr}^{\mathcal{T}_{/V}}(\mathcal{C});$$

the functoriality is given by restriction. We acquire $\mathcal{T}$-∞-categories $\underline{\mathcal{S}}_{\mathcal{T}, \leq d}$ and $\underline{\mathrm{Cat}}_{\mathcal{T}, d}$ similarly.

Example 18. We may define a $\mathcal{T}$-∞-category by $\underline{\mathbb{F}}_{\mathcal{T}}$ by values

$$(\underline{\mathbb{F}}_{\mathcal{T}})_V := \mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_{\mathcal{T}_{/V}}$$

with functoriality given by pullback. We write $\mathbb{F}_V := \mathbb{F}_{\mathcal{T},/V}$. Note that this is a $\mathcal{T}$-1-category since $\mathcal{T}_{/V}$ is a 1-category [NS22, Prop 2.5.1]. ◄

Example 19. Given $\mathcal{C}$ an arbitrary n-category, $\underline{\mathrm{CoFr}}^{\mathcal{T}}(\mathcal{C})$ is a $\mathcal{T}$-n-category [HTT, Cor 2.3.4.8]. In particular, if $\mathcal{C}$ is an ∞-topos and $\tau_{\leq n-1}\mathcal{C}$ its n-topos of $(n-1)$-truncated objects, then $\underline{\mathrm{CoFr}}^{\mathcal{T}}(\tau_{\leq n-1}\mathcal{C})$ is a $\mathcal{T}$-n-category. ◄

Example 20. The ∞-category of $\mathcal{T}$-∞-categories is Cartesian closed with internal hom characterized by values

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})_V \simeq \mathrm{Fun}_{\mathcal{T}_{/V}}(\mathrm{Res}_V^{\mathcal{T}}\mathcal{C},\mathrm{Res}_V^{\mathcal{T}}\mathcal{D}),$$

where $\mathrm{Res}_V^{\mathcal{T}}\colon \mathrm{Cat}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}_{/V}}$ is pullback and $\mathrm{Fun}_{\mathcal{T}}(-,-)$ denotes the evident ∞-category of natural transformations [BDGNS16]. By unwinding definitions and applying [HTT, Cor 2.3.4.8], we find that whenever $\mathcal{D}$ is a $\mathcal{T}$-n-category, $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$ is a $\mathcal{T}$-n-category. ◄

Example 21. We refer to the adjunction between limits and constant diagrams as the *inflation and fixed point* adjunction

$$\mathrm{Cat} \underset{\Gamma^{\mathcal{T}}}{\overset{\mathrm{Infl}_e^{\mathcal{T}}}{\rightleftarrows}} \mathrm{Cat}_{\mathcal{T}}$$

In the case that $\mathcal{T}$ has a terminal object V, the image of $\mathrm{Infl}_e^{\mathcal{T}}$ consists of the $\mathcal{T}$-∞-categories whose restriction functors Res_W^V are all equivalences. In any case, we may string together natural equivalences

$$\begin{aligned}
\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}} K, \underline{\mathrm{CoFr}}^{\mathcal{T}}\mathcal{C}\right)_V &\simeq \mathrm{Fun}_V\left(\mathrm{Infl}_e^{\mathcal{T}_{/V}} K, \underline{\mathrm{CoFr}}^{\mathcal{T}_{/V}}\mathcal{C}\right) \\
&\simeq \mathrm{Fun}\left(K, \mathrm{Fun}\left((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{C}\right)\right) \\
&\simeq \mathrm{Fun}\left((\mathcal{T}_{/V})^{\mathrm{op}}, \mathrm{Fun}(K,\mathcal{C})\right) \\
&\simeq \underline{\mathrm{CoFr}}^{\mathcal{T}}\left(\mathcal{C}^K\right)_V
\end{aligned}$$

to construct a $\mathcal{T}$-equivalence $\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}} K, \underline{\mathrm{CoFr}}^{\mathcal{T}}\mathcal{C}\right) \simeq \underline{\mathrm{CoFr}}^{\mathcal{T}}\left(\mathcal{C}^K\right)$; in particular, choosing $\mathcal{C} = \mathcal{K}$, $\mathcal{T}$-coefficient systems in presheaves of spaces on K can equivalently be realized as $\mathcal{T}$-equivariant presheaves of $\mathcal{T}$-spaces on K with trivial $\mathcal{T}$-equivariant structure. We henceforth write

$$\underline{\mathcal{S}}_{\mathcal{T},\leq n}^K := \underline{\mathrm{CoFr}}^{\mathcal{T}}\left(\mathcal{S}_{\leq n}^K\right); \qquad\qquad \underline{\mathcal{S}}_{\mathcal{T}}^K := \underline{\mathrm{CoFr}}^{\mathcal{T}}\left(\mathcal{S}^K\right). \qquad\qquad ◄$$

Given $V \in \mathcal{T}$ an orbit and $S \in \mathbb{F}_V$ a finite V-set, we write $\varphi_{SV}\colon \mathrm{Ind}_V^{\mathcal{T}}S \to V$ for the corresponding map in $\mathbb{F}_{\mathcal{T}}$, and we write

$$\mathcal{C}_S := \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_U \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathrm{Ind}_V^{\mathcal{T}}S, \mathcal{C}\right).$$

Pullback along the structure map φ_{SV} yields an *indexed diagonal* functor

$$\Delta^S\colon \mathcal{C}_V \to \mathcal{C}_S;$$

its values are $\Delta^S X = (\mathrm{Res}_U^V X)_{U \in \mathrm{Orb}(S)}$. The *$S$-indexed coproduct* (if it exists) is the left adjoint $\coprod^S\colon \mathcal{C}_S \to \mathcal{C}_V$ to Δ^S, and the *S-indexed product* $\prod^S\colon \mathcal{C}_S \to \mathcal{C}_V$ is the right adjoint. These are the ür-examples of *equivariantly indexed operations*, whose combinatorics we control using *weak indexing systems*.

Definition 22. A *one-color weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ which is closed under $\underline{\mathbb{F}}_I$-indexed coproducts and contains $*_V$ for all $V \in \mathcal{T}$. A *one-color weak indexing category* is a pullback-stable wide subcategory $I \subset \mathbb{F}_{\mathcal{T}}$ subject to the condition that $\coprod_i (T_i \to S_i)$ lies in I if and only if each map $T_i \to S_i$ lies in I. ◄

Given I a one-color weak indexing category, we define the *I-admissible V-sets* as

$$\underline{\mathbb{F}}_I := \left\{S \mid \mathrm{Ind}_V^{\mathcal{T}}S \to V \in I\right\} \subset \underline{\mathbb{F}}_{\mathcal{T}};$$

we verified in [Ste24] that $\mathbb{F}_{(-)}$ furnishes an equivalence between one-color weak indexing systems and one-color weak indexing categories, so we safely conflate these notions. For the following example, a full subcategory $\mathcal{F} \subset \mathcal{T}$ is called a $\mathcal{T}$-*family* if, whenever there exists a morphism $V \to W$ with $W \in \mathcal{F}$, we have $V \in \mathcal{F}$.

Example 23. The terminal one-color weak indexing system is $\mathbb{F}_{\mathcal{T}}$. We define the following other examples, where $\mathcal{F} \subset \mathcal{T}$ is a fixed $\mathcal{T}$-family:

$$(\mathbb{F}_{\text{triv}})_V := \{*_V\}$$

$$\left(\mathbb{F}_{0,\mathcal{F}}\right)_V := \begin{cases} \{\varnothing_V, *_V\} & V \in \mathcal{F} \\ \{*_V\} & \text{otherwise.} \end{cases}$$

$$(\mathbb{F}_\infty)_V := \{n \cdot *_V \mid n \in \mathbb{N}\}.$$

The corresponding one-color weak indexing categories are denoted $I_{\text{triv}}, I_{0,\mathcal{F}}, I_\infty$. ◄

Construction 24. We write

$$v(I) := \left\{ V \in \mathcal{T} \mid \varnothing_V \in (\mathbb{F}_I)_V \right\} \subset \mathcal{T}.$$

This is a $\mathcal{T}$-family, called the *unit family* of I [Ste24]. ◄

We say that $\mathbb{F}_I$ is *almost-unital* if, whenever $\{*_V\} \subsetneq \mathbb{F}_{I,V}$, we have $\varnothing_V \in \mathbb{F}_{I,V}$; that is, $\mathbb{F}_I$ is unital over all orbits for which $\mathbb{F}_I$ has nontrivial arities. We say $\mathbb{F}_I$ is *unital* if $\varnothing_V \in \mathbb{F}_{I,V}$ for all V.

1.2. Preliminaries on I-commutative monoids and I-symmetric monoidal ∞-categories.

Let I be a one-color weak indexing category. The pair $(\mathbb{F}_{\mathcal{T}}, I)$ is a *span pair* in the sense of [EH23] (i.e. $(\mathbb{F}_{\mathcal{T}}, I, I)$ is an *adequate triple* in the sense of [Bar14]), so it yields a wide subcategory

$$\text{Span}_I(\mathbb{F}_{\mathcal{T}}) \hookrightarrow \text{Span}(\mathbb{F}_{\mathcal{T}})$$

of the effective Burnside ∞-category whose morphisms are given by spans $X \leftarrow R \xrightarrow{f} Y$ with $f \in I$. Given I a one-color weak indexing category and $\mathcal{C}$ an ∞-category, we define the ∞-category of I-*commutative monoids* in $\mathcal{C}$ as

$$\text{CMon}_I(\mathcal{C}) := \text{Fun}^\times(\text{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C}).$$

We define the ∞-category of *small I-symmetric monoidal ∞-categories* as

$$\text{Cat}_I^\otimes := \text{CMon}_I(\text{Cat}).$$

We henceforth ignore size issues and omit the adjective "small." Given an I-symmetric monoidal ∞-category $\mathcal{C}$ and $S \in \mathbb{F}_{I,V}$ an I-admissible V-set, we denote the functoriality of $\mathcal{C}^\otimes$ under the structure map $\text{Ind}_S^{\mathcal{T}} S = \text{Ind}_S^{\mathcal{T}} S \to V$ by

$$\bigotimes^S : \mathcal{C}_S \to \mathcal{C}_V.$$

If I is almost-unital, $S \in \mathbb{F}_{I,V}$ is I-admissible, and $1_U \in \mathcal{C}_U$ is initial whenever it exists, then given an S-indexed tuple $(X_U) \in \mathcal{C}_S$ in an I-symmetric monoidal ∞-category with S-indexed coproducts, we define an S-*indexed tensor Wirthmüller map*

$$W_{S,(X_U)} : \coprod_U^S X_U \longrightarrow \bigotimes_U^S X_U$$

by defining its composite map $\text{Ind}_W^V X_W \hookrightarrow \coprod_U^S X_U \to \bigotimes_U^S X_U$ to be adjunct to the map

$$X_W \simeq X_W \otimes \bigotimes_W^{\text{Res}_W^V S - *_W} 1_U \xrightarrow{(\text{id}, \eta)} X_W \otimes \bigotimes^{\text{Res}_W^V S - *_W} X_U \simeq \text{Res}_U^V \bigotimes_U^S X_U;$$

intuitively, on the W'th factor, $W_{S,(X_U)}$ takes x to the simple tensor with x in the W'th place and units elsewhere. Given $J \subset I$, we say that $\mathcal{C}$ is J-*cocartesian* if $W_{S,(X_U)}$ is an equivalence for all $S \in \mathbb{F}_I$ and $(X_U) \in \mathcal{C}_S$, and we say that $\mathcal{C}$ is J-*cartesian* if its "vertical opposite"

$$\text{Span}_I(\mathbb{F}_{\mathcal{T}}) \xrightarrow{\mathcal{C}^\otimes} \text{Cat} \xrightarrow{\text{op}} \text{Cat}$$

is a J-cocartesian I-symmetric monoidal ∞-category..

In [Ste25b], given $\mathcal{C}$ a $\mathcal{T}$-∞-category with I-indexed (co)products, we constructed essentially unique *(co)cartesian* I-symmetric monoidal structures on $\mathcal{C}$ and verified that $\mathcal{C}$ is I-*semiadditive* in the sense of [CLL24] if and only if there exists an equivalence $\mathcal{C}^{I-\sqcup} \simeq \mathcal{C}^{I-\times}$, which can be chosen (uniquely) to lie over the identity endofunctor.

1.3. **Preliminaries on I-operads.** In [NS22], an ∞-category $\mathrm{Op}_{\mathcal{T}}$ of $\mathcal{T}$-*operads* was introduced, and in [Ste25a; Ste25b] it was given a symmetric monoidal closed $\mathcal{T}$-∞-category structure $\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes}$. We review the relevant formal properties here; in particular, outside of a small part of the verification of another formal property in Proposition 40, we will only use formal properties of $\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes}$, instead probing its objects via the various functors

$$\mathrm{Cat}_{\mathcal{T}}^{\otimes} \longhookrightarrow \mathrm{Op}_{\mathcal{T}} \xrightarrow{\ \mathrm{sseq}\ } \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$$

$$\mathrm{Cat}_{\mathcal{T}} \qquad\qquad \mathrm{Cat}_{\mathcal{T}} \qquad\qquad \mathrm{Cat}_{\mathcal{T}}$$

with maps U, $\underline{\mathrm{Alg}}_{(-)}(\mathcal{C})$, $\underline{\mathrm{Alg}}_{\mathcal{P}}(-)$.

In this way, this paper can be considered agnostic to the presentation of $\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes}$ and the above functors.

1.3.1. $\mathcal{T}$-*symmetric sequences and I-operads.* Writing $\underline{\Sigma}_{\mathcal{T}}$ for the composite $\mathcal{T}$-∞-category

$$\mathcal{T}^{\mathrm{op}} \xrightarrow{\ \mathbb{F}_{\mathcal{T}}\ } \mathrm{Cat} \xrightarrow{\ (-)^{\simeq}\ } \mathcal{S} \hookrightarrow \mathrm{Cat}$$

and writing $\mathrm{Tot} \colon \mathrm{Cat}_{\mathcal{T}} \simeq \mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}}^{\mathrm{cocart}} \to \mathrm{Cat}$ for the total category functor, in [Ste25a] we defined a *underlying $\mathcal{T}$-symmetric sequence* functor

$$\mathcal{O}(-) \colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S}).$$

To characterize this, we need a definition.

Definition 25. We say that an I-operad $\mathcal{O}^{\otimes}$ *has at least one color* if $\mathcal{O}(*_V) \neq \varnothing$ for all $V \in \mathcal{T}$ and *has one color* if $\mathcal{O}(*_V) \simeq *$ for all $V \in \mathcal{T}$, ◁

Proposition 26 ([Ste25a])**.** *The functor* $\mathcal{O}(-) \colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ *has a left adjoint* Fr*; in particular, letting* $\mathrm{Fr}_{\mathrm{Op}}(S)$ *be the free $\mathcal{T}$-operad on the left Kan extended $\mathcal{T}$-symmetric sequence*

$$\{S\} \xrightarrow{\quad * \quad} \mathcal{S}$$
$$\downarrow \qquad \nearrow_{\mathrm{Fr}_{\Sigma,S}(*)}$$
$$\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}},$$

the adjunctions construct a natural equivalence

$$\mathrm{Alg}_{\mathrm{Fr}_{\mathrm{Op}}(S)}(\mathcal{O}) \simeq \mathcal{O}(S).$$

Moreover, the restricted functor $\mathcal{O}(-) \colon \mathrm{Op}_{\mathcal{T}}^{\mathrm{oc}} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ *is monadic.*

In particular, identifying an object of $\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}$ with a pair (V, S) where $V \in \mathcal{T}$ and $S \in \mathbb{F}_V$, $\mathcal{T}$-operads are identified conservatively by the functor

$$\mathcal{O} \mapsto \prod_{V,S} \mathcal{O}(S).$$

Intuitively, we view $\mathcal{O}(S)$ as the space of S-*ary operations* $\left(\mathrm{Res}_V^{\mathcal{T}} X\right)^{\otimes S} \to \mathrm{Res}_V^{\mathcal{T}} X$ borne by an $\mathcal{O}$-algebra X. This technology allowed us to define the *arity support* functor

$$A\mathcal{O} := \left\{ T \to S \ \middle| \ \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T \times_S U) \neq \varnothing \right\} \subset \mathbb{F}_{\mathcal{T}};$$

which we verified in [Ste25a] to be a weak indexing category. In fact, we verified that the essential surjection associated with A possesses a fully faithful right adjoint

$$(2) \qquad\qquad\qquad \mathrm{Op}_{\mathcal{T}} \underset{\mathcal{N}_{(-)\infty}^{\otimes}}{\overset{A}{\rightleftarrows}} \perp \ \mathrm{wIndexCat}_{\mathcal{T}};$$

we refer to the $\mathcal{T}$-operad $\mathcal{N}_{I\infty}^{\otimes}$ as the *weak $\mathcal{N}_{\infty}$-operad associated with I*. Now, we further verified in [Ste25a] that, given a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$, the unique map $\mathcal{O}^{\otimes} \to \mathrm{Comm}_{\mathcal{T}}^{\otimes}$ is a monomorphism if and only if the counit map $\mathcal{O}^{\otimes} \to \mathcal{N}_{A\mathcal{O}}^{\otimes}$ is an equivalence; in particular, we acquire an equality of full subcategories

$$\mathrm{Op}_{\mathcal{T},/\mathcal{N}_{I\infty}^{\otimes}} = A^{-1}(\mathrm{wIndexCat}_{\mathcal{T},\leq I}) \ \subset \ \mathrm{Op}_{\mathcal{T}},$$

and a full subcategory of $\mathrm{Op}_{\mathcal{T}}$ has a terminal object if and only if it is of this form. We refer to $\mathrm{Op}_I \coloneqq \mathrm{Op}_{\mathcal{T},/\mathcal{N}_{I\infty}^{\otimes}}$ as the ∞-category of *I-operads*; see [Ste25a] for an intrinsic characterization of Op_I.

Monomorphisms are right-cancellable, so all inclusions $I \subset J$ induce monomorphisms $\iota_I^J \colon \mathcal{N}_{I\infty}^{\otimes} \to \mathcal{N}_{J\infty}^{\otimes}$; in other words, the push-pull adjunction

$$\mathrm{Op}_I \underset{\mathrm{Bor}_I^J = \iota_I^{J*}}{\overset{E_I^J = \iota_{I!}^J}{\rightleftarrows}} \mathrm{Op}_J$$

witnesses $\mathrm{Op}_I \subset \mathrm{Op}_J$ as a colocalizing subcategory. Moreover, it behaves well with $\overset{\mathrm{BV}}{\otimes}$.

Proposition 27 ([Ste25a]). *Suppose $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ have at least one color. Then, there is an equality*

$$A(\mathcal{O} \otimes \mathcal{P}) \simeq A\mathcal{O} \vee A\mathcal{P}.$$

In particular, $\mathrm{Op}_I \subset \mathrm{Op}_{\mathcal{T}}$ is a symmetric monoidal full subcategory.

1.3.2. *I-symmetric monoidal categories and $\mathcal{O}$-algebras.* [NS22] constructed a (non-full) subcategory inclusion

$$\iota \colon \mathrm{Cat}_I^{\otimes} \to \mathrm{Op}_{\mathcal{T}};$$

$\mathcal{T}$-operad maps between I-symmetric monoidal categories are called *lax I-symmetric monoidal functors*, and morphisms in the image of ι are called *I-symmetric monoidal functors*.

Moreover, given $\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$, we define *$\mathcal{O}$-algebras in $\mathcal{C}^{\otimes}$* to be $\mathcal{T}$-operad maps $\mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$, which naturally fit into an ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. These have a *pointwise $\mathcal{T}$-operad structure* $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ given by the internal hom in a presentably symmetric monoidal structure on $\mathrm{Op}_{\mathcal{T}}$, whose tensor product we write as $\overset{\mathrm{BV}}{\otimes}$ [Ste25a; Ste25b]. The unit for this symmetric monoidal structure is the $\mathcal{T}$-operad $\mathrm{triv}_{\mathcal{T}}^{\otimes} \coloneqq \mathcal{N}_{I^{\mathrm{triv}}\infty}^{\otimes}$ [Ste25a], i.e. there is a canonical equivalence

$$(3) \qquad \underline{\mathrm{Alg}}_{\mathrm{triv}_{\mathcal{T}}}^{\otimes}(\mathcal{O}) \simeq \mathcal{O}^{\otimes}$$

Moreover, we verified in [Ste25a] that whenever $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal ∞-category, $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is as well, and given a $\mathcal{T}$-operad map $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ and an I-symmetric monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$, the induced lax I-symmetric monoidal functors

$$\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}); \qquad\qquad \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{D})$$

are I-symmetric monoidal. In particular, when $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal ∞-category and $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are I-operads, there are natural I-symmetric monoidal equivalence

$$(4) \qquad \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes} \underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \simeq \underline{\mathrm{Alg}}_{\mathcal{O}\otimes\mathcal{P}}^{\otimes}(\mathcal{C}) \simeq \underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes} \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$$

1.3.3. *The underlying $\mathcal{T}$-∞-category.* An I-operad $\mathcal{O}^{\otimes}$ has an underlying $\mathcal{T}$-∞-category $U\mathcal{O}$ [NS22]; indeed, $\mathcal{T}$-operads are equivariantizations of the classical notions of *colored operads*, and $U\mathcal{O}$ the *∞-category of colors*. Moreover, the composite functor $\mathrm{Cat}_I^{\otimes} \to \mathrm{Op}_I \xrightarrow{U} \mathrm{Cat}_{\mathcal{T}}$ is the usual *underlying $\mathcal{T}$-∞-category* functor.

U behaves well with respect to $\underline{\mathrm{Alg}}^{\otimes}$; indeed, we verified in [Ste25a] that the underlying $\mathcal{T}$-∞-category has values

$$U\left(\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})\right)_V \simeq \mathrm{Alg}_{\mathrm{Res}_V^{\mathcal{T}}\mathcal{O}}\left(\mathrm{Res}_V^{\mathcal{T}}\mathcal{C}\right),$$

where $\mathrm{Res}_V^{\mathcal{T}} \colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Op}_V$ is a restriction functor, and furthermore

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \simeq \Gamma^{\mathcal{T}} U \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}).$$

It was observed in [NS22] that the composite functor $\mathrm{Op}_{I^{\mathrm{triv}}} \subset \mathrm{Op}_{\mathcal{T}} \xrightarrow{U} \mathrm{Cat}_{\mathcal{T}}$ is an equivalence, and that U factors as $\mathrm{Op}_{\mathcal{T}} \xrightarrow{\mathrm{Bor}_{I^\infty}^{\mathrm{triv}}} \mathrm{Op}_{I^{\mathrm{triv}}} \simeq \mathrm{Cat}_{\mathcal{T}}$. We write $\mathrm{triv}(-)^\otimes$ for the composite functor

$$\mathrm{triv}(-)^\otimes \colon \mathrm{Cat}_{\mathcal{T}} \xrightarrow{U^{-1}} \mathrm{Op}_{I^\infty} \hookrightarrow \mathrm{Op}_{\mathcal{T}};$$

unwinding definitions, we find that there is a natural equivalence

$$\underline{\mathrm{Alg}}_{\mathrm{triv}(\mathcal{C})}(\mathcal{O}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, U\mathcal{O});$$

that is, $\mathrm{triv}(\mathcal{C})$ algebras are simply $\mathcal{C}$-indexed diagrams of objects.

1.3.4. *$\mathcal{T}$-operadic inflation and fixed points.* In [Ste25a] we constructed an equivalence

$$\varphi \colon \mathrm{Op}_{I_\infty} \xrightarrow{\sim} \mathrm{CoFr}^{\mathcal{T}} \mathrm{Op}$$

exhibiting natural equivalences $\varphi\mathcal{O}_V(n) \simeq \mathcal{O}(n \cdot *_V)$. Limits and constant diagrams yields an *inflation and fixed point* adjunction

$$\mathrm{Op} \underset{\Gamma^{\mathcal{T}}}{\overset{\mathrm{Infl}_e^{\mathcal{T}}}{\underset{\perp}{\rightleftarrows}}} \mathrm{Op}_{I_\infty} \underset{\mathrm{Bor}_{I_\infty}^{\mathcal{T}}}{\overset{E_{I_\infty}^{\mathcal{T}}}{\underset{\perp}{\rightleftarrows}}} \mathrm{Op}_{\mathcal{T}};$$

we refer to the composite adjunction $\mathrm{Op} \rightleftarrows \mathrm{Op}_{\mathcal{T}}$ also as $\mathrm{Infl}_e^{\mathcal{T}} \dashv \Gamma^{\mathcal{T}}$. For instance we have

$$(5) \qquad\qquad \mathrm{Alg}_{\mathrm{Infl}(\mathcal{O})}(\mathcal{P}) \simeq \mathrm{Alg}_{\mathcal{O}}\left(\Gamma^{\mathcal{T}}\mathcal{P}\right);$$

moreover, we can identify the image of $\mathrm{Infl}_e^{\mathcal{T}}$ easily: they are the I_∞-operads $\mathcal{O}^\otimes$ whose underlying $\mathcal{T}$-∞-category is inflated and whose restriction maps

$$\mathcal{O}(C; D) \to \mathcal{O}(\mathrm{Res}_U^V C; \mathrm{Res}_U^V D)$$

are all equivalences.

Example 28. The above description yields a natural equivalence $\mathrm{Infl}_e^{\mathcal{T}}(\mathrm{triv}(\mathcal{C})^\otimes) \simeq \mathrm{triv}(\mathrm{Infl}_e^{\mathcal{T}}\mathcal{C})^\otimes$. ◁

Example 29. The $\mathcal{T}$-operads $\mathbb{E}_0^\otimes := \mathcal{N}_{I_{0,\mathcal{T}}}^\otimes$ and $\mathbb{E}_\infty^\otimes := \mathcal{N}_{I_\infty}^\otimes$ are inflated from operads of the same names; in particular, unwinding definitions, we may identify $\mathbb{E}_0$-algebras by the formula

$$\underline{\mathrm{Alg}}_{\mathbb{E}_0}(\mathcal{C})_V \simeq \mathcal{C}_{V, 1_V/}.$$

If 1_V is terminal for all $V \in \mathcal{T}$, then this is the $\mathcal{T}$-category of pointed objects $\mathcal{C}_*$. ◁

1.3.5. *Unital I-operads.* Assume that I is an almost unital weak indexing category. In [Ste25b] we introduced the following gamut of definitions, each of which will be useful.

Definition 30. We say that an I-operad $\mathcal{O}^\otimes$
- is *almost unital* if it has at least one color and whenever there exists some $S \in \mathbb{F}_V$ such that $\mathcal{O}(S) \neq \varnothing$, we have $\mathcal{O}(\varnothing_V) \simeq *$,
- is *unital* if it has at least one color and $\mathcal{O}(\varnothing_V) \simeq \mathcal{N}_{I_\infty}(\varnothing_V)$ for all $V \in \mathcal{T}$, and
- is *almost reduced* if it is almost unital and has one color, and
- is *reduced* if it is unital and has one color. ◁

A $\mathcal{T}$-operad is almost unital if and only if it's a unital I-operad for *some* almost-unital weak indexing category I. For this reason, we'll usually focus on either unital I-operads or almost-unital $\mathcal{T}$-operads. It will be important to keep the I-symmetric monoidal case in mind.

Example 31. We verified in [Ste25b] that an I-symmetric monoidal ∞-category $\mathcal{C}^\otimes$ is a unital I-operad if and only if, for all $V \in v(I)$, the unit object $1_V \in \mathcal{C}_V$ is initial. ◁

Write $\mathbb{E}_{0,v(I)}^\otimes := \mathcal{N}_{I_{0,v(I)}}^\otimes$. We will largely use the following result of [Ste25b] to access unital I-operads.

Proposition 32 ([Ste25b])**.** *The full subcategory* $\mathrm{Op}_I^{\mathrm{uni}} \subset \mathrm{Op}_I$ *of unital I-operads is both a localizing and colocalizing subcategory, i.e. the inclusion participates in a double adjunction*

$$\mathrm{Op}_I \underset{\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{0,v(I)}}(-)}{\overset{(-)\overset{\mathrm{BV}}{\otimes}\mathbb{E}^{\otimes}_{0,v(I)}}{\underset{\perp}{\overset{\perp}{\rightleftarrows}}}} \mathrm{Op}_I^{\mathrm{uni}}.$$

In particular, if $\mathcal{O}^{\otimes}$ and $\mathcal{C}^{\otimes}$ are unital, then there are natural equivalences

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) \simeq \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}\otimes\mathbb{E}_{0,v(I)}}(\mathcal{C});$$
$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{D}) \simeq \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{0,v(I)}}(\mathcal{D}).$$

We accomplished this in part by recognizing an equality of full subcategories $\mathrm{Op}_I^{\mathrm{uni}} = \mathrm{Op}_I^{I_{0,v(I)}-\mathrm{Wirth}}$; that is, an I-operad is unital if and only if its I-symmetric monoidal ∞-categories of algebras have V-units which are initial for each $V \in v(I)$, which is true if and only if they are unital by Example 31. Moreover, since the $\overset{\mathrm{BV}}{\otimes}$-unit $\mathrm{triv}_{\mathcal{T}}^{\otimes}$ is initial among one color I-operads, this yields the following easy corollary.

Corollary 33. $\mathbb{E}^{\otimes}_{0,v(I)}$ *is initial among reduced I-operads.*

$\mathrm{Op}_I^{\mathrm{red}}$ has initial unit object; interestingly, it has *absorptive* terminal object.

Proposition 34 ([Ste25b])**.** *If $\mathcal{O}^{\otimes}$ is a unital I-operad, then the map $\mathbb{E}^{\otimes}_{0,v(I)} \to \mathcal{O}^{\otimes}$ induces a (unique) equivalence*

$$\mathcal{N}^{\otimes}_{I\infty} \simeq \mathcal{N}^{\otimes}_{I\infty}\overset{\mathrm{BV}}{\otimes}\mathbb{E}^{\otimes}_{0,v(I)} \overset{\sim}{\longrightarrow} \mathcal{N}^{\otimes}_{I\infty}\overset{\mathrm{BV}}{\otimes}\mathcal{O}^{\otimes}.$$

1.3.6. *Cartesian and cocartesian I-symmetric monoidal ∞-categories.* In [Ste25b], given $\mathcal{C}$ a $\mathcal{T}$-∞-category with I-indexed (co)products, we defined *cocartesian* and *cartesian* I-symmetric monoidal ∞-categories $\mathcal{C}^{I-\sqcup}$ and $\mathcal{C}^{I-\times}$, which are determined by the properties that their I-indexed tensor products are canonically equivalent to indexed (co)products. We gave algebras in cartesian I-symmetric monoidal ∞-categories an explicit presentation generalizing the $\mathcal{O}$-*monoids* of [HA] (as $\mathcal{T}$-functors satisfying "Segal conditions") which we will not mention explicitly here; as a relic of this, we will simply use the notation

$$(6) \qquad \underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{D}) := \underline{\mathrm{Alg}}_{\mathcal{O}}\big(\mathcal{D}^{I-\times}\big); \qquad \mathrm{Mon}_{\mathcal{O}}(\mathcal{D}) := \mathrm{Alg}_{\mathcal{O}}\big(\mathcal{D}^{I-\times}\big).$$

The associated I-symmetric monoidal structure is cartesian [Ste25b]. When $\mathcal{C}$ is an ∞-category, we will write

$$(7) \qquad \underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}) := \underline{\mathrm{Mon}}_{\mathcal{O}}\big(\underline{\mathrm{CoFr}}^{\mathcal{T}}\mathcal{C}\big); \qquad \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) := \mathrm{Mon}_{\mathcal{O}}\big(\underline{\mathrm{CoFr}}^{\mathcal{T}}\mathcal{C}\big).$$

instead we will use their monadic presentation, which goes as follows.

Proposition 35 ([Ste25a])**.** *Suppose $\mathcal{C}$ is a presentable and cartesian closed ∞-category. Then, the monad $T_{\mathcal{O}}$ associated with the monadic functor $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{CoFr}^{\mathcal{T}}\mathcal{C}$ has fixed points*

$$(T_{\mathcal{O}}X)^W \simeq \coprod_{S \in \mathbb{F}_{I,W}} \left(\mathrm{Fr}_{\mathcal{C}}\,\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S)} X^U \right)_{h\,\mathrm{Aut}_W(S)},$$

where $\mathrm{Fr}_{\mathcal{C}} \colon \mathcal{S} \to \mathcal{C}$ is the unique left adjoint sending $*$ to the terminal object of $\mathcal{C}$.

Moreover, in the case that $\mathcal{O}^{\otimes}$ is unital, we characterized cocartesian algebras simply as diagrams

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}\big(\mathcal{C}^{I-\sqcup}\big) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{O},\mathcal{C})^{I-\sqcup};$$

in fact, $\mathcal{C}^{I-\sqcup}$ still exists as an I-operad with the above algebras in when $\mathcal{C}$ is not assumed to have I-indexed coproducts. In particular, in the unital case, we acquire a double adjunction

$$(8) \qquad \mathrm{Cat}_{\mathcal{T}} \underset{(-)^{I-\sqcup}}{\overset{\mathrm{triv}(-)^{\otimes}\overset{\mathrm{BV}}{\otimes}\mathbb{E}_{0,v(I)}}{\underset{\perp}{\overset{\perp}{\rightleftarrows}}}} \mathrm{Op}_I^{\mathrm{uni}}.$$

Example 36. In [Ste25b] we gave a general formula for $\mathcal{C}^{I-\sqcup}$, but the mapping-in property makes it easy enough to determine this in the case that $\mathcal{C}$: there is an equivalence

$$\mathrm{Alg}_{\mathcal{O}}\left(*_{\mathcal{T}}^{I-\sqcup}\right) \simeq * \simeq \mathrm{Alg}_{\mathcal{O}}\left(\mathcal{N}_{I\infty}^{\otimes}\right),$$

natural in the unital I-operad $\mathcal{O}^{\otimes}$, constructing an equivalence $\mathcal{N}_{I\infty}^{\otimes} \simeq *_{\mathcal{T}}^{I-\sqcup}$ by Yoneda's lemma.   ◄

1.3.7. *I-d-operads.* In [Ste25a], we defined the full subcategory $\mathrm{Op}_{\mathcal{T},d} \subset \mathrm{Op}_{\mathcal{T}}$ of $\mathcal{T}$-d-operads to be those such that $\mathcal{O}(S)$ is a $(d-1)$-truncated space for all $S \in \underline{\mathbb{F}}_{A\mathcal{O}}$, and verified the following.

Proposition 37 ([Ste25a]). *Fix $d \geq -1$ and $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$.*

  (1) The inclusion $\mathrm{Op}_{\mathcal{T},d} \subset \mathrm{Op}_{\mathcal{T}}$ has a left adjoint $h_d \colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T},d}$ satisfying

$$h_d\mathcal{O}(S) \simeq \tau_{\leq d-1}\mathcal{O}(S).$$

  (2) The unit of the h_0-localization adjunction is the map $\mathcal{O}^{\otimes} \to \mathcal{N}_{A\mathcal{O}}^{\otimes}$; in particular, $\mathcal{N}_{(-)\infty}^{\otimes}$ factors through an equivalence

$$\mathrm{wIndexCat}_{\mathcal{T}} \simeq \mathrm{Op}_{\mathcal{T},0}.$$

  (3) When $\mathcal{P}^{\otimes}$ is a $\mathcal{T}$-d-operad, there is a natural equivalence

$$\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{P}) \simeq \underline{\mathrm{Alg}}_{h_d\mathcal{O}}^{\otimes}(\mathcal{P}),$$

  and each are $\mathcal{T}$-d-operads.
  (4) An I-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is a $\mathcal{T}$-d-operad if and only if $U\mathcal{C}$ is a $\mathcal{T}$-d-category.

  We call $h_d\mathcal{O}^{\otimes}$ the *homotopy d-operad* of $\mathcal{O}^{\otimes}$.

1.3.8. *$\mathcal{O}$-algebras in I-symmetric monoidal 1-categories.* Fix $\mathcal{C}^{\otimes}$ an I-symmetric monoidal 1-category; in light of Proposition 37, to characterize $\mathcal{O}$-algebras in $\mathcal{C}^{\otimes}$, we may equivalently characterise $h_1\mathcal{O}$-algebras in $\mathcal{C}$, so assume $\mathcal{O}^{\otimes}$ is an I-1-operad, i.e. its structure spaces are sets.

    We gave a simple combinatorial model for I-1-operads in [Ste25a], which we will not relitigate here, instead focusing only on algebras. Given a $\mathcal{T}$-object $X \in \Gamma^{\mathcal{T}}\mathcal{C}$, we defined the *unreduced endomorphism I-operad* $\mathrm{End}_X(\mathcal{C})$ as a one-colored I-1-operad with structure sets

$$\mathrm{End}_X(\mathcal{C})(S) \simeq \mathrm{Hom}_{\mathcal{C}_V}\left(X_{\underline{V}}^{\otimes S}, X_{\underline{V}}\right),$$

where $X_{\underline{V}} \in \mathcal{C}_V$ is the V-object underlying X. 1-categorical algebras take a familiar form.

Proposition 38 ([Ste25a]). *Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_{I,1}^{\mathrm{oc}}$, $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is a 1-category whose objects are pairs $(X \in \Gamma^{\mathcal{T}}\mathcal{C}, \varphi \colon \mathcal{O} \to \mathrm{End}_X(\mathcal{C}))$ and whose morphisms are $\Gamma^{\mathcal{T}}\mathcal{C}$-maps $f \colon X \to Y$ such that the corresponding diagram commutes*

$$\begin{array}{ccc} & & \mathrm{End}_X(\mathcal{C}) \\ & \nearrow & \\ \mathcal{O}^{\otimes} & & \downarrow{\scriptstyle \mathrm{End}_f} \\ & \searrow & \\ & & \mathrm{End}_Y(\mathcal{C}) \end{array}$$

    Moreover, we may exploit this to explicitly describe interchange.

Corollary 39 ([Ste25a]). *Given $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes} \in \mathrm{op}_{I,1}^{\mathrm{oc}}$, an $\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{P}$-algebra structure on X is precisely a pair of $\mathcal{O}$-algebra and $\mathcal{P}$-algebra structures such that, for all $\mu \in \mathcal{O}(S)$, the corresponding $\mathcal{C}$-map $X_{\underline{V}}^{\otimes S} \to X_{\underline{V}}$ is a morphism of $\mathcal{P}$-algebras; a morphism of $\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{P}$-algebras is a $\Gamma^{\mathcal{T}}\mathcal{C}$-map which is separately an $\mathcal{O}$-algebra and $\mathcal{P}$-algebra morphism.*

1.4. **The doctrinal adjunction.** The following proposition will play a crucial role in constructing I-symmetric monoidal left adjoints. We temporarily assume that the reader is familiar with [Ste25a, § 2].

Proposition 40 (Doctrinal adjunction). *Suppose $L^\otimes \colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is an I-symmetric monoidal functor whose underlying $\mathcal{T}$-functor L admits a right adjoint R. Then, R lifts to a canonical lax I-symmetric monoidal right adjoint $R^\otimes \vdash L^\otimes$. Moreover, for any $\mathcal{T}$-operad $\mathcal{O}^\otimes$ the postcomposition lax I-symmetric monoidal functors partake in a lax I-symmetric monoidal adjunction*

$$L_*^\otimes \colon \underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\mathcal{C}) \rightleftarrows \underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\mathcal{D}) \colon R_*^\otimes$$

such that $L^{\otimes}{}_*$ is I-symmetric monoidal. If $R^\otimes$ is symmetric monoidal then $R_*^\otimes$ is symmetric monoidal; if R is also fully faithful, then $R_*^\otimes$ is fully faithful.

Proof. Applying [HA, Prop 7.3.2.6] to the fibrations on opposite categories, we acquire a right adjoint $R^\otimes \vdash L^\otimes$ relative to $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$. Moreover, an identical argument to [HA, Cor 7.3.2.7] shows that $R^\otimes$ preserves cocartesian lifts for inert morphisms. The lax I-symmetric monoidal functors $L_*^\otimes$ and $R_*^\otimes$ are then constructed in [Ste25a], where postcomposition along an I-symmetric monoidal functor is verified to be I-symmetric monoidal; in particular, $L_*^\otimes$ is always I-symmetric monoidal and $R_*^\otimes$ is I-symmetric monoidal whenever $R^\otimes$ is.

Note that postcomposition along the unit and counit data for $L^\otimes \dashv R^\otimes$ yield unit and counit data for $L_*^\otimes$ and $R_*^\otimes$ in any case. When $R^\otimes, L^\otimes$ are symmetric monoidal and R is fully faithful, the counit $\varepsilon \colon L^\otimes R^\otimes \mathcal{C}^\otimes \to \mathcal{C}^\otimes$ is an I-symmetric monoidal functor whose underlying $\mathcal{T}$-functor is an equivalence, so ε is an I-symmetric monoidal equivalence; in particular, this implies that the counit of $L_*^\otimes \dashv R_*^\otimes$ is an equivalence, so $R_*^\otimes$ is fully faithful. $\square$

1.5. **Recognizing h_{n+1}-equivalences.** Theorem D recognizes morphisms of $\mathcal{T}$-operads which become equivalences after applying h_{n+1}, so we now spell out some of its antecedents.

Proposition 41. *Let $\varphi \colon \mathcal{O}^\otimes \to \mathcal{P}^\otimes$ be a morphism of $\mathcal{T}$-operads. The following are equivalent:*

(a) *for all $S \in \underline{\mathbb{F}}_{A\mathcal{O}} \cup \underline{\mathbb{F}}_{A\mathcal{P}}$, the map of spaces*

$$\varphi(S) \colon \mathcal{O}(S) \to \mathcal{P}(S)$$

is an n-equivalence;

(b) *φ is an h_{n+1}-equivalence;*

(c) *for all $\mathcal{T}$-symmetric monoidal $(n+1)$-categories $\mathcal{C}$, the pullback $\mathcal{T}$-symmetric monoidal functor*

$$\underline{\mathrm{Alg}}_{\mathcal{P}}^\otimes(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\mathcal{C})$$

is an equivalence;

(d) *the pullback functor*

$$\mathrm{Mon}_{\mathcal{P}}(\mathcal{S}_{\leq n}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{S}_{\leq n})$$

is an equivalence; and

(e) *for all ∞-categories K, the pullback map of spaces*

$$\mathrm{Mon}_{\mathcal{P}}\left(\mathcal{S}_{\leq n}^K\right)^{\simeq} \to \mathrm{Mon}_{\mathcal{O}}\left(\mathcal{S}_{\leq n}^K\right)^{\simeq}$$

is an equivalence.

To prove this, we apply the following lemma.

Lemma 42. *Given a $\mathcal{T}$-operad $\mathcal{P}^\otimes$ and a pair of ∞-categories $\mathcal{D}, K$ such that $\mathcal{D}$ admits finite products, there is an equivalence*

$$\underline{\mathrm{Mon}}_{\mathcal{P}}\left(\mathcal{D}^K\right) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}} K, \underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{D})\right),$$

natural in functors of K, product-preserving functors of $\mathcal{D}$, and $\mathcal{T}$-operad maps of $\mathcal{P}$; in particular, taking $\mathcal{T}$-fixed points yields a natural equivalence of categories

$$\mathrm{Mon}_{\mathcal{P}}\left(\mathcal{D}^K\right) \simeq \mathrm{Mon}_{\mathcal{P}}(\mathcal{D})^K.$$

Proof. We construct a chain of equivalences

$$\underline{\mathrm{Mon}}_{\mathcal{P}}\left(\mathcal{D}^K\right) \simeq \underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathrm{CoFr}}^{\mathcal{T}}(\mathcal{D}^K)^{\mathcal{T}-\times}) \hfill \text{Eqs. (6) and (7)}$$

$$\simeq \underline{\mathrm{Alg}}_{\mathcal{P}}\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}}K, \underline{\mathrm{CoFr}}^{\mathcal{T}}\mathcal{D}\right)^{\mathcal{T}-\times} \hfill \text{Example 21}$$

$$\simeq \underline{\mathrm{Alg}}_{\mathcal{P}}\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}(\mathrm{Infl}_e^{\mathcal{T}}K)}\left(\underline{\mathrm{CoFr}}^{\mathcal{T}}\mathcal{D}^{\mathcal{T}-\times}\right) \hfill \text{Eq. (3)}$$

$$\simeq \underline{\mathrm{Alg}}_{\mathcal{P}}\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{Infl}_e^{\mathcal{T}}\mathrm{triv}(K)}\left(\underline{\mathrm{CoFr}}^{\mathcal{T}}\mathcal{D}^{\mathcal{T}-\times}\right) \hfill \text{Example 28}$$

$$\simeq \underline{\mathrm{Alg}}_{\mathrm{Infl}_e^{\mathcal{T}}\mathrm{triv}(K)}\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}\left(\underline{\mathrm{CoFr}}^{\mathcal{T}}\mathcal{D}^{\mathcal{T}-\times}\right) \hfill \text{Eq. (4)}$$

$$\simeq \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}}K, \underline{\mathrm{Alg}}_{\mathcal{P}}\left(\underline{\mathrm{CoFr}}^{\mathcal{T}}, \mathcal{D}^{\mathcal{T}-\times}\right)\right) \hfill \text{Eq. (5)}$$

$$\simeq \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}}K, \underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{D})\right) \hfill \text{Eqs. (6) and (7)}$$

The remaining equivalence follows by noting that $\Gamma^{\mathcal{T}}\mathrm{Infl}_e^{\mathcal{T}}\mathcal{C} \simeq \mathcal{C}$, naturally in $\mathcal{C}$. $\hfill\square$

Proof of Proposition 41. A generalization of the equivalence between Conditions (a) to (d) was proved in [Ste25a], and Condition (c) clearly implies Condition (e). Moreover, fixing $\mathcal{D} = \mathcal{S}_{\leq n}$ and taking cores of Lemma 42 yields a natural equivalence

$$\mathrm{Mon}_{\mathcal{P}}\left(\mathcal{S}_{\leq n}^K\right)^{\simeq} \simeq \mathrm{Map}_{\mathrm{Cat}}\left(K, \mathrm{Mon}_{\mathcal{P}}(\mathcal{S}_{\leq n})\right)$$

so Condition (e) and Yoneda's lemma together imply Condition (d). $\hfill\square$

We say that $\mathcal{O}^{\otimes}$ is *n-connected* if the unique map $\mathcal{O}^{\otimes} \to \mathcal{N}_{A\mathcal{P}}^{\otimes}$ is an h_{n+1}-equivalence. In [Ste25b] we acquired the following additional characterizations for n-connected $\mathcal{T}$-operads:

Proposition 43. *Suppose $\mathcal{O}^{\otimes}$ is an almost-unital $\mathcal{T}$-operad. Then, the following conditions are equivalent:*

- *(b') $\mathcal{O}^{\otimes}$ is n-connected.*
- *(f') For all $A\mathcal{O}$-symmetric monoidal $(n+1)$-categories $\mathcal{C}^{\otimes}$, the $A\mathcal{O}$-symmetric monoidal $(n+1)$-category $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is cocartesian.*
- *(g') The $\mathcal{T}$-$(n+1)$-category $\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{S}_{\leq n})$ is $A\mathcal{O}$-semiadditive.*

1.6. **The reduced endomorphism I-operad as a right adjoint.** In [Ste25b], we introduced the *reduced endomorphism I-operad* of a $\mathcal{T}$-operad for the purpose of lifting the disintegration and assembly process of [HA]. In this section, we gain explicit computational control over reduced endomorphism I-operads of unital I-symmetric monoidal ∞-categories.

Proposition 44. *The inclusion $\mathrm{Op}_I^{\mathrm{red}} \simeq \mathrm{Op}_{I,\mathbb{E}_{0,v(I)}/}^{\mathrm{red}} \hookrightarrow \mathrm{Op}_{I,\mathbb{E}_{0,v(I)}/}^{\mathrm{uni}}$ has a right adjoint computed by the pullback*

(9)
$$\begin{array}{ccc}
\mathrm{End}_X^{I,\mathrm{red}} & \longrightarrow & \mathcal{O}^{\otimes} \\
\downarrow & \lrcorner & \downarrow{\scriptstyle\eta} \\
\mathcal{N}_{I\infty}^{\otimes} & \xrightarrow{\{X\}} & \mathcal{O}^{\mathcal{T}-\sqcup}
\end{array}$$

In the case that $\mathcal{C}^{\otimes}$ is a unital I-symmetric monoidal ∞-category and $X \in \mathcal{C}_V$ is a V-object, mapping in from the free unital I-operad $\mathrm{Fr}_{\mathrm{Op}}(S)\overset{\mathrm{BV}}{\otimes}\mathbb{E}_{0,v(I)}$ on an operation in arity $S \in \mathbb{F}_{I,V}$ yields a pullback

$$\begin{array}{ccc}
\mathrm{End}_X^{I,\mathrm{red}}(S) & \longrightarrow & \mathrm{Map}_{\mathcal{C}_V}\left(X^{\otimes S}, X\right) \\
\downarrow & \lrcorner & \downarrow{\scriptstyle W_{S,X}^*} \\
\{\nabla\} & \longrightarrow & \mathrm{Map}_{\mathcal{C}_V}\left(X^{\sqcup S}, X\right)
\end{array}$$

i.e. $\mathrm{End}_X^{I,\mathrm{red}}(S)$ *is equivalent to the space of lifts along the following dashed arrow in* $\mathcal{C}_V$

$$
\begin{array}{ccc}
X^{\sqcup S} & \xrightarrow{\nabla} & X \\
{\scriptstyle W_{S,X}}\downarrow & \nearrow & \downarrow{\scriptstyle !} \\
X^{\otimes S} & \xrightarrow{!} & *
\end{array}
$$

Proof. We will apply the general reduction procedure of [SY19, Prop 2.1.5], applied to the *sliced* adjunction

$$U_* \colon \mathrm{Op}^{\mathrm{uni}}_{I,\mathbb{E}^{\otimes}_{0,v(I)}/} \rightleftarrows \mathrm{Cat}_{\mathcal{T},*} \colon \eta^*(-)^{I-\sqcup},$$

whose right adjoint is $(-)^{I-\sqcup}$ together with the precomposed structure map

$$\mathbb{E}^{\otimes}_{0,v(I)} \xrightarrow{\eta} \mathcal{N}^{\otimes}_{I\infty} \simeq *^{I-\sqcup}_{\mathcal{T}} \to \mathcal{C}^{I-\sqcup}.$$

Indeed, $\mathrm{Cat}_{\mathcal{T},*}$ admits an initial object $*_{\mathcal{T}} \simeq U\mathbb{E}_{0,v(I)}$, and $\mathrm{Op}^{\otimes}_{I,\mathbb{E}^{\otimes}_{0,v(I)}/}$ admits all limits, which are preserved by U since it is a right adjoint by Eq. (8). Moreover, $\mathbb{E}_{0,v(I)} \in \mathrm{Op}^{\mathrm{red}}_I$ is initial by Corollary 33, there is a unique equivalence $\mathcal{N}^{\otimes}_{I\infty} \simeq *^{I-\sqcup}_{\mathcal{T}}$ by Eq. (2) and Example 36, and $\mathcal{O}^{\otimes} \in \mathrm{Op}^{\mathrm{uni}}_{I,\mathbb{E}_{0,v(I)}/}$ corresponds with a reduced I-operad if and only if $U\mathcal{O}^{\otimes} \in \mathrm{Cat}_{\mathcal{T},*}$ is initial, so the first claim follows by [SY19, Prop 2.1.5].

To acquire the second pullback square, one need only note that the natural equivalences

$$\mathrm{Map}_{\mathrm{Op}_{\mathcal{T}}}\left(\mathrm{Fr}_{\mathrm{Op}}(S) \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{0,v(I)}, \mathcal{C}^{\otimes}\right) \simeq \mathrm{Map}_{\mathcal{C}_V}\left(X^{\otimes S}, X\right),$$

$$\mathrm{Map}_{\mathrm{Op}_{\mathcal{T}}}\left(\mathrm{Fr}_{\mathrm{Op}}(S) \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{0,v(I)}, \mathcal{N}^{\otimes}_{I\infty}\right) \simeq *$$

follow by Propositions 26 and 32. What remains is to verify that the right vertical arrow is $W^*_{S,X}$ and the bottom arrow includes the fold map ∇; both facts were verified in [Ste25b]. $\square$

In fact, [SY19, Prop 4.2.8] introduced a result on connectivity of such spaces of lifts, immediately yielding the following corollary.

Corollary 45. *If* $X \in \mathcal{C}_V$ *is a* $(k+\ell+2)$*-truncated object and the Wirthmüller map* $W_{S,X} \colon X^{\sqcup S} \to X^{\otimes S}$ *is* ℓ*-connected, then the space* $\mathrm{End}_X^{I,\mathrm{red}}(\mathcal{C})(S)$ *is* k*-truncated.*

In general, reduction is an incarnation of the *disintegration and assembly* procedure of [HA; Ste25b]; given a reduced I-operad $\mathcal{P}^{\otimes}$ and a V-object $X \in \mathcal{O}_V$, applying $\mathcal{P}$-algebras to Eq. (9) yields a pullback

(10)
$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{P}} \mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O}) & \longrightarrow & \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{O})_V \\
\downarrow & \lrcorner & \downarrow{\scriptstyle U} \\
\{X\} & \lhook\joinrel\longrightarrow & U\mathcal{O}_V
\end{array}
$$

In the case that $U\mathcal{O}$ is a $\mathcal{T}$-space, U is a automatically cocartesian fibration, so $\mathcal{O}$-algebras are $U\mathcal{O}$-indexed diagrams of $\mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O})$-algebras. Unfortunately, this is far from our case; the best we can do is take cores of the above pullback square, resulting in the following proposition.

Proposition 46. *Suppose* $\mathcal{P}^{\otimes} \to \mathcal{Q}^{\otimes}$ *is a morphism of* I*-operads inducing an equivalence of spaces*

$$\varphi^{*,\simeq}_X \colon \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{Q}} \mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O})^{\simeq} \longrightarrow \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{P}} \mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O})^{\simeq}$$

for all $V \in \mathcal{T}$ *and* $X \in U\mathcal{O}_V$*. Then, the induced map of* $\mathcal{T}$*-spaces*

$$\underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{O})^{\simeq} \to \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{O})^{\simeq}$$

is an equivalence; in particular, passing to $\mathcal{T}$*-fixed points, the induced map of spaces*

$$\mathrm{Alg}_{\mathcal{Q}}(\mathcal{O})^{\simeq} \to \mathrm{Alg}_{\mathcal{P}}(\mathcal{O})^{\simeq}$$

is an equivalence.

Proof. Taking cores of Eq. (10), we find that that $\varphi_X^{*,\simeq}$ is the induced map on the homotopy fiber over X of the following map of $\mathcal{T}$-spaces over $U\mathcal{O}$:

$$\underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{O})^{\simeq} \xrightarrow{\quad\varphi^{*,\simeq}\quad} \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{O})^{\simeq}$$
$$U\mathcal{O}$$

$\varphi^{*,\simeq}$ is an equivalence if and only if its V-fixed points are an equivalence for all $V \in \mathcal{T}$, and the homotopy fibers of $\varphi^{*,\simeq,V}$ are contractible by the above argument, so $\varphi^{*,\simeq,V}$ is an equivalence for all V. Hence $\varphi^{*,\simeq}$ is an equivalence, proving the proposition. $\qquad\square$

2. Connectivity and Wirthmüller maps

In this section, we verify Theorem D; in particular, we will acquire the following technical corollary.

Corollary 47. *If $\mathcal{P}^{\otimes}$ is ℓ-connected at I, then for all $(k + \ell + 2)$-toposes $\mathcal{C}$, the reduced endomorphism I-operad $\mathrm{End}_X\left(\underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{C})^{I-\times}\right)$ is an I-$(k+1)$-operad.*

Proof. Since $\mathcal{C}$ is a $(k + \ell + 2)$-category, X is $(k + \ell + 2)$-truncated, and Theorem D implies that $W_{X,S}$ is ℓ-connected, so the result follows from Corollary 45. $\qquad\square$

Before moving on, we show how this yields the atomic orbital generalization of Theorem B.

Proof of Theorem B. By passing to restrictions and Borelifications, we assume that $\mathcal{O}, \mathcal{P}$ are almost reduced. By Proposition 41, we're tasked with verifying that, for all presheaf $(k + \ell + 2)$-toposes $\mathcal{C}$, the map of spaces

$$\mathrm{Mon}_{\mathcal{O}}\mathrm{Mon}_{\mathcal{P}}(\mathcal{C})^{\simeq} \to \mathrm{CMon}_{A\mathcal{O}}(\mathcal{C})^{\simeq}.$$

is an equivalence; since $\mathcal{N}_{A\mathcal{O}\infty}^{\otimes} \simeq \mathcal{P}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{A\mathcal{O}\infty}^{\otimes}$ by Proposition 34, we may equivalently require that the map

$$\mathrm{Mon}_{\mathcal{O}}\mathrm{Mon}_{\mathcal{P}}(\mathcal{C})^{\simeq} \to \mathrm{CMon}_{A\mathcal{O}}\mathrm{Mon}_{\mathcal{P}}(\mathcal{C})^{\simeq}$$

is an equivalence. In particular, by Propositions 41 and 46, it suffices to prove that $\mathrm{End}_X\left(\underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{C})^{A\mathcal{O}-\times}\right)$ is an $A\mathcal{O}$-$(k+1)$-operad, which is Corollary 47. $\qquad\square$

2.1. Connectivity of algebras can be detected in the value topos.

Fix $\mathcal{C}$ an n-topos for some $n \leq \infty$.

Lemma 48. *A map $f\colon C \to D$ in $\mathrm{CoFr}^{\mathcal{T}}\mathcal{C}$ is ℓ-connected if and only if, for all $V \in \mathcal{T}^{\mathrm{op}}$, the fixed point map $C^V \to D^V$ is ℓ-connected.*

Proof. Per Remark 5, it is equivalent to prove that ℓ-connectiveness of a morphism in $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C})$ is measured elementwise. Indeed, since (co)limits in $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C})$ are computed elementwise, effective epimorphisms and diagonals are as well. The former proves the statement for (-2)-connectiveness, and the latter together with the diagonal presentation of [HTT, Prop 6.5.1.18] shows that the statement for $(\ell - 1)$-connectiveness implies the statement for ℓ-connectiveness, so the lemma follows by induction. $\qquad\square$

Proposition 49. *Given a map $f\colon X \to Y$ in $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$, if the underlying map Uf in $\mathrm{CoFr}^{\mathcal{T}}\mathcal{C}$ is ℓ-connected, then f is ℓ-connected.*

Proof. In view of [SY19, Lem 4.4.1], it suffices to verify that the monad $T_{\mathcal{O}}\colon \mathrm{CoFr}^{\mathcal{T}}\mathcal{C} \to \mathrm{CoFr}^{\mathcal{T}}\mathcal{C}$ preserves ℓ-connected morphisms; by Lemma 48, it suffices to verify that whenever each $\mathcal{C}$-diagram $X^V \to Y^V$ is ℓ-connected, each induced map $T_{\mathcal{O}}X^W \to T_{\mathcal{O}}X^W$ is ℓ-connected. But by Proposition 35, it suffices to note that ℓ-connected morphisms in an ∞-topos are closed under cartesian products and colimits [HTT, Cor 6.5.1.13, Prop 5.2.8.6]. $\qquad\square$

For instance, U preserves the terminal object and is conservative, so it also reflects the property of being terminal; applying Proposition 49 in the case $Y = *$ shows that U reflects n-connectivity of objects.

Remark 50. Since U is a right adjoint, it preserves n-truncatedness and n-truncated objects. $\qquad\triangleleft$

Warning 51. Proposition 49 is delicate for a few reasons.

(1) 1f $\mathcal{O}$ is not n-connected, then the free $\mathcal{O}$-algebra monad $T_{\mathcal{O}}\colon \mathcal{C}_V \to \mathcal{C}_V$ may itself may fail to preserve n-connected objects; indeed, we have $T_{\mathcal{O}}*_V \simeq \coprod_{S \in \mathbb{F}_V} \mathrm{Fr}_{\mathcal{C}} \mathcal{O}(S)_{h\mathrm{Aut}_V S}$, which is often not much more highly connected than the individual spaces $\mathcal{O}(S)_{h\mathrm{Aut}_V S}$.

(2) U does not generally *preserve* ℓ-connectivity of objects or morphisms for instance, given an $\ell \geq (k+1)$-connected space X, the equivalence $\Omega^k \colon \mathcal{S}_{*, \geq k+1} \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S}_{\geq 1})$ exhibits Ω^k as an ℓ-connected $\mathbb{E}_k$-algebra such that $U\Omega^n$ is only in general $(\ell - k)$-connected.

(3) For a similar reason, U does not usually reflect ℓ-truncatedness of morphisms or objects. ◄

2.2. **The proof of Theorem D.** We now begin to reduce Theorem D to the case $n \leq \ell + 1$ with the following.

Lemma 52. *The truncation functor $\tau_{\leq \ell} \colon \mathcal{C} \to \tau_{\leq \ell}\mathcal{C}$ extends to a $\mathcal{T}$-functor*

$$\tau_{\mathcal{O}} \colon \underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Mon}}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C})$$

satisfying $\tau_{\mathcal{O}}W_{S,X} = W_{S,\tau_{\mathcal{O}}X}$. Moreover, the inclusion $\iota \colon \tau_{\leq \ell}\mathcal{C} \to \mathcal{C}$ extends to a fully faithful $\mathcal{T}$-functor

$$\iota_{\mathcal{O}} \colon \underline{\mathrm{Mon}}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C}) \hookrightarrow \underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C})$$

such that $\tau_{\mathcal{O}}W_{S,\iota_{\mathcal{O}}X} = W_{S,X}$.

Proof. Since $\tau_{\leq \ell}$ is product-preserving [HTT, Lem 6.5.1.2], $\tau_{\leq \ell} \colon \underline{\mathrm{CoFr}}^{\mathcal{T}}\mathcal{C} \to \underline{\mathrm{CoFr}}^{\mathcal{T}}\tau_{\leq \ell \mathcal{C}}$ is a $\mathcal{T}$-symmetric monoidal left adjoint for the cartesian structure [Ste25b]; everything other than the equalities involving $W_{S,X}$ then follows straightforwardly from Proposition 40.
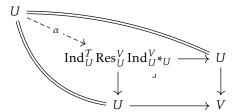
In particular, $\tau_{\mathcal{O}}$ is a $\mathcal{T}$-functor which preserves indexed products and coproducts; this implies that $\tau_{\mathcal{O}}W_{S,X} = W_{S,\tau_{\mathcal{O}}X}$. The remaining equality follows from fully faithfulness by noting that

$$\tau_{\mathcal{O}}W_{S,\iota_{\mathcal{O}}X} = W_{S,\tau_{\mathcal{O}}\iota_{\mathcal{O}}X} = W_{S,X}. \qquad \square$$

We say that a map $f \colon X \to Y$ in an n-topos is an *ℓ-equivalence* if it is a $\tau_{\leq \ell}$-equivalence; if f admits a section, this is equivalent to f being ℓ-connected (see [SY19, Prop 4.3.5] or note that this follows by splitting the long exact sequence in homotopy). We apply this by equivariantizing [SY19, Lem 5.1.1].

Lemma 53. *If $\mathcal{C}^{I-\times}$ is a Cartesian I-symmetric monoidal ∞-category and $S \in \underline{\mathbb{F}}_I$, then the image of the $\mathcal{O}$-algebra Wirthmüller map $W_{X,S} \colon \coprod_U^S X_U \to \prod_U^S X_U$ under $U \colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_V \to \mathcal{C}_V$ admits a section.*

Given (Y_U) an S-tuple and $U \in \mathrm{Orb}(S)$ a distinguished orbit, choose the distinguished fixed point α whose induction is the following

(see [Ste24] for the fact that this is indeed a summand inclusion). Let

$$\beta \colon Y_U \to \mathrm{Res}_U^V \mathrm{CoInd}_U^V Y_U \simeq \prod_W^{\mathrm{Res}_U^V \mathrm{Ind}_{U*U}^V} \mathrm{CoInd}_W^U \mathrm{Res}_W^U Y_U$$

be the map whose corresponding map $\mathrm{Res}_W^U Y_U \to \mathrm{Res}_W^U Y_U$ is 0 when $W \neq \alpha$ and the identity otherwise. Let

$$\iota_U \colon Y_U \to \mathrm{Res}_U^V \prod_W^S Y_U \simeq \mathrm{Res}_U^V \mathrm{CoInd}_U^V Y_U \times \mathrm{Res}_U^V \prod_W^{S-U} Y_W$$

be the map corresponding with β on the first factor and 0 on the other. Let $i_U \colon Y_U \to \mathrm{Res}_U^V \coprod_{U'}^S Y_{U'}$ be adjunct to the inclusion $\mathrm{Ind}_U^V Y_U \hookrightarrow \coprod_{U'}^S Y_{U'}$.

Proof of Lemma 53. Fix some operation $\mu \in \mathcal{O}(S)$. We will verify that the following diagram commutes. Then, $\mu\sigma_1 f$ will be the desired section for $W_{X,S}$.

$$
\begin{array}{ccccc}
\prod\limits_U^S\left(\mathrm{Res}_U^V \coprod\limits_U^V X_U\right) & \xrightarrow{\ \sim\ } & \left(\coprod\limits_U^S X_U\right)^{\times S} & \xrightarrow{\ \mu\ } & \coprod\limits_U^S X_U \\[2em]
\Big\uparrow{\scriptstyle f=(i_U)_{U\in\mathrm{Orb}(S)}} & & \Big\downarrow{\scriptstyle h=\left(W_{\mathrm{Res}_U^V X,\mathrm{Res}_U^V S}\right)_{U\in\mathrm{Orb}(S)}} & & \Big\downarrow{\scriptstyle W_{X,S}} \\[2em]
& & \left(\prod\limits_U^S X_U\right)^{\times S} & \xrightarrow{\ \mu\ } & \prod\limits_U^S X_U \\[2em]
& & \Big\downarrow{\scriptstyle \sigma_2} & & \Big\| \\[2em]
\prod\limits_U^S X_U & \xrightarrow{g=(\iota_U)_{U\in\mathrm{Orb}(S)}} & \prod\limits_U^S X_U^{\times\mathrm{Res}_U^V S} & \xrightarrow{\ \mu\ } & \prod\limits_U^S X_U
\end{array}
$$

Note that the top right square is commutative by the fact that $W_{S,X}$ is an $\mathcal{O}$-algebra morphism and the bottom right follows by unwinding the definition of μ.

Now, note that $\mu \circ g$ is the external product of a collection of endomorphisms $X_U \xrightarrow{\iota_U} X_U^{\times\mathrm{Res}_U^V S} \xrightarrow{\mu} X_U$; unwinding definitions, ι_U is the inclusion of a unit on all but one factor:

$$
\begin{array}{ccccc}
X_U & \xrightarrow{\ \iota_U\ } & X_U^{\times\mathrm{Res}_U^V S} & \xrightarrow{\ \mu\ } & X_U \\[1em]
\Big\| & & \Big\| & & \nearrow \\[1em]
X_U \times \prod\limits_W^{\mathrm{Res}_U^V S-\{\alpha\}} 1_W & \xrightarrow{(\mathrm{id},\eta)} & X_U \times \prod\limits_W^{\mathrm{Res}_U^V S-\{\alpha\}} X_W & &
\end{array}
$$

in particular, $\mu \circ \iota_U$ is homotopic to the identity, so $\mu \circ g$ is homotopic to the identity, and the bottom triangle commutes.

To characterize the composite morphism of the left rectangle, we may equivalently characterize the composite map $\pi_U\sigma_2 h\sigma_1 f\colon \prod\limits_U^S X_U \to \mathrm{CoInd}_U^V X_U^{\times\mathrm{Res}_U^V S}$; in fact, under the expression $X_U^{\times\mathrm{Res}_U^V S} \simeq \prod\limits_W^{\mathrm{Res}_U^V S} \mathrm{Res}_W^U X_U$, it suffices to characterize the composite morphism $\prod\limits_U^S X_U \to \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U$ and verify that it is homotopic to the relevant projection of g for each W, U.

In particular, relevant projection of g is the composite morphism

$$
\prod\limits_U^S X_U \twoheadrightarrow \mathrm{CoInd}_U^V X_U \xrightarrow{\ \delta_{U,W}\ } \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U
$$

where $\delta_{U,W}$ is a Kronecker delta

$$
\delta_{U,W} = \begin{cases} \mathrm{id} & U = W; \\ 0 & \text{otherwise.} \end{cases}
$$

Moreover, note that the projection $\pi_U\sigma_2 h\sigma_1\colon \prod\limits_U^S X_U \to X_U^{\times\mathrm{Res}_U^V S}$ itself factors as

$$
\prod\limits_U^S\left(\mathrm{Res}_U^V \coprod\limits_U^V X_U\right) \twoheadrightarrow \mathrm{CoInd}_U^V X_U \xrightarrow{\ \widetilde{f_U}\ } \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U,
$$

so we're tasked with verifying that $\widetilde{f}_U$ is homotopic to $\delta_{U,W}$. Indeed, this follows by examining the following diagram:

$$\prod_U^S X_U \xrightarrow{f} \prod_U^S \left(\mathrm{Res}_U^V \coprod_U^V X_U\right) \quad \simeq \quad \left(\coprod_U^S X_U\right)^{\times S} \xrightarrow{\quad h \quad} \left(\prod_U^S X_U\right)^{\times S} \quad \simeq \quad \prod_U^S X_U^{\times \mathrm{Res}_U^V S}$$

$$\mathrm{CoInd}_U^V X_U \xrightarrow{\mathrm{CoInd}_U^V i_U} \mathrm{CoInd}_U^V \mathrm{Res}_U^V \coprod_U^V X_U \xrightarrow{\mathrm{CoInd}_U^V W} \mathrm{CoInd}_U^V \mathrm{Res}_U^V \prod_U^V X_U \quad \simeq \quad X_U^{\times \mathrm{Res}_U^V S}$$

$$\xrightarrow{\delta_{U,W}} \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U$$

$\square$

Proof of Theorem D. Assume $\mathcal{O}^\otimes$ is ℓ-connected at I, i.e. Condition (a). We study the behavior of $W_{S,X}$ under the following diagram:

$$\begin{array}{ccc}
\underline{\mathrm{Mon}}_{\mathcal{O}}(\tau_{\leq\ell}\mathcal{C}) \xrightarrow{\iota_{\mathcal{O}}} \underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{L_{\mathcal{O}}} \underline{\mathrm{Mon}}_{\mathcal{O}}(\tau_{\leq\ell}\mathcal{C}) \\
\downarrow U_{\leq\ell} \qquad\qquad \downarrow U \qquad\qquad \downarrow U_{\leq\ell} \\
\underline{\mathrm{CoFr}}^{\mathcal{T}} \tau_{\leq\ell}\mathcal{C} \xrightarrow{\iota} \underline{\mathrm{CoFr}}^{\mathcal{T}} \mathcal{C} \xrightarrow{L} \underline{\mathrm{CoFr}}^{\mathcal{T}} \tau_{\leq\ell}\mathcal{C}
\end{array}$$

In particular, by Proposition 41 and Lemma 52, $L_{\mathcal{O}} W_{S,X} = W_{S,L_{\mathcal{O}}X}$ is an equivalence, so $U_{\leq\ell} L_{\mathcal{O}} W_{S,X} = LU W_{S,X}$ is an equivalence, i.e. $UW_{S,X}$ is an ℓ-equivalence. In turn, by Lemma 53 this implies that $UW_{S,X}$ is ℓ-connected, so Proposition 49 implies that $W_{S,X}$ is ℓ-connected, i.e. Condition (b).

The implication Condition (b) $\implies$ Condition (c) is immediate, so assume Condition (c), i.e. fix the case $\mathcal{C} := \mathcal{S}$ and assume that that $W_{S,X}$ is ℓ-connected for all $X \in \mathrm{Alg}_{\mathcal{O}}\mathcal{S}$ and $S \in \underline{\mathbb{F}}_I$. We may invert the above argument: this time, we find that $UW_{S,\iota_{\mathcal{O}}Y}$ is an ℓ-equivalence for all $Y \in \mathrm{Alg}_{\mathcal{O}}\mathcal{S}_{\leq\ell}$, so $LU W_{S,Y} = U_{\leq\ell} L_{\mathcal{O}} W_{S,\iota_{\mathcal{O}}Y} = U_{\leq\ell} W_{S,Y}$ is an equivalence. By conservativity of $U_{\leq\ell}$, this implies that $W_{S,Y}$ is an equivalence, so $\mathcal{O}^\otimes$ is ℓ-connected at I by Proposition 41, proving Condition (a). $\square$

3. The C_p-operads $\mathbb{A}_{2,C_p}^\otimes$ and $\mathbb{A}_{2,C_p}^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{A}_{2,C_p}^\otimes$

For the rest of this article, we specialize to $\mathcal{T} = \mathcal{O}_{C_p}$, where C_p is the group of prime order p, and $\mathcal{C}$ is a 1-category. As in Proposition 26, let $\mathrm{Fr}_\Sigma(S)$ denote the free C_p-symmetric sequence on an operation in arity S. Now, the pointwise formula for left Kan extensions yields equivalences

$$\mathrm{Fr}_{\Sigma, p \cdot *_{C_p}}(*)(p \cdot *_e) \simeq \Sigma_p;$$

$$\mathrm{Fr}_{\Sigma, [C_p/e]}(*)(p \cdot *_e) \simeq \Sigma_p.$$

We define the C_p-symmetric sequence of sets F_{2,C_p} as the coequalizer

$$F_{2,C_p} := \mathrm{CoEq}\left(\Sigma_p[p \cdot *_e] \rightrightarrows \left(\mathrm{Fr}_{\Sigma,[C_p/e]}(*) \sqcup \mathrm{Fr}_{\Sigma, p \cdot *_{C_p}}(*)\right)\right),$$

where $\Sigma_p[p \cdot *_e]$ is the C_p-symmetric sequence defined by

$$\Sigma_p[p \cdot *_e](S) := \begin{cases} \Sigma_p & S = p \cdot *_e; \\ \varnothing & \text{otherwise.} \end{cases}$$

and the two arrows are the inclusions of $\Sigma_p[p \cdot *_e]$. We define the unital C_p-operad $\mathbb{A}_{2,C_p}^\otimes$ by the Boardman-Vogt tensor product

$$\mathbb{A}_{2,C_p}^\otimes := \mathbb{E}_0^\otimes \overset{\mathrm{BV}}{\otimes} \mathrm{Fr}_{\mathrm{Op}}\left(F_{2,C_p}\right).$$

As promised, we verify that $\mathbb{A}_{2,C_p}$-monoids are the same as C_p-unital magmas.

Proposition 54. *There is an equivalence between $\mathrm{Mon}_{\mathbb{A}_{2,C_p}}(\mathcal{C})$ and C_p-unital magmas in $\mathcal{C}$.*

Proof. By Example 29 and Proposition 32 we have

$$\mathrm{Mon}_{\mathbb{A}_{2,C_p}}(\mathcal{C}) \simeq \mathrm{Mon}_{\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})}\underline{\mathrm{Mon}}^{\otimes}_{\mathbb{E}_0}(\mathcal{C}) \simeq \mathrm{Mon}_{\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})}\mathcal{C}_*.$$

Moreover, by Proposition 38, the data of an $\mathbb{A}_{2,C_p}$-monoid structure on $X \in \mathrm{CoFr}^{C_p}\mathcal{C}$ is equivalently viewed as a map $\eta\colon *_{C_p} \to X$ (which we identify with an element $\widetilde{X} \in \mathrm{CoFr}^{C_p}\mathcal{C}_*$) and an element of

$$\mathrm{Mon}_{\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})}(\mathrm{End}_{\widetilde{X}}(\mathcal{C}_*))^{\simeq} \simeq \mathrm{Hom}_{\mathrm{Fun}\left(\mathrm{Tot}\underline{\Sigma}_{C_p},\mathcal{S}\right)}\left(F_{2,C_p},\mathrm{End}_{\widetilde{X}}(\mathcal{C}_*)\right)$$

$$\simeq \mathrm{Hom}_{\mathrm{CoFr}^{C_p}\mathcal{C}_*}\left(\widetilde{X}^p, \widetilde{X}\right) \times_{\mathrm{Hom}_{\mathcal{C}_*}\left((\widetilde{X}^e)^p, \widetilde{X}^e\right)} \mathrm{Hom}_{\mathrm{CoFr}^{C_p}\mathcal{C}_*}\left(\mathrm{CoInd}_e^{C_p}\widetilde{X}^e, \widetilde{X}\right).$$

We're left with interpreting this concretely: by a standard argument, $\mathrm{Hom}_{\mathrm{CoFr}^{C_p}\mathcal{C}_*}(\widetilde{X}^p, \widetilde{X})$ corresponds bijectively with the set of unital magma structures on X with unit η, and this corresponds bijectively with the pairs of unital magma structures on X^{C_p} and X^e with unit maps η^{C_p} and η^e such that the restriction map is a homomorphism. Under this bijection, the forgetful map $\mathrm{Hom}_{\mathrm{CoFr}^{C_p}\mathcal{C}_*}\left(\widetilde{X}^p, \widetilde{X}\right) \to \mathrm{Hom}_{\mathcal{C}_*}\left(\left(\widetilde{X}^e\right)^p, \widetilde{X}\right)$ simply forgets the data of X^{C_p} and the restriction.

Similarly, since C_p-coefficient coinduction is presented by the coefficient system $X^p \xleftarrow{\Delta} X$ with permutation action, $\mathrm{Hom}_{\mathrm{CoFr}^{C_p}\mathcal{C}_*}\left(\mathrm{CoInd}_e^{C_p}\widetilde{X}^e, \widetilde{X}\right)$ corresponds bijectively with the set of unital C_p-equivariant transfers $t\colon X^e \to X^{C_p}$ and unital magma structures on X^e with unit η^e satisfying the condition that the following diagram commutes.

$$\begin{array}{ccc} X^e & \xrightarrow{t} & X^{C_p} \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle r} \\ (X^e)^p & \xrightarrow{*} & X^e \end{array}$$

Once again, the forgetful map restricts to the unital magama structure on η^e; thus the fiber product corresponds exactly with G-unital magma structures on X with units η^e and η^{C_p}.

Now, what we've described is a bijective assignment of *sets* $\mathrm{Ob}\,\mathrm{Mon}_{\mathbb{A}_{2,C_p}}(\mathcal{C}) \to \mathrm{Ob}\,\mathrm{Magma}^{\mathrm{uni}}_{C_p}(\mathcal{C})$ over $\mathrm{Ob}\,\mathcal{C}$. To conclude, it suffices to prove that a $\mathrm{CoFr}^{C_p}\mathcal{C}$ morphism between a pair of C_p-unital magmas is a C_p-unital magma homomorphism if and only if it's an $\mathbb{A}_{2,C_p}$-algebra homomorphism.

To prove this, note that an $\mathbb{A}_{2,C_p}$-monoid morphism is equivalently a $\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})$-monoid morphism of pointed objects, i.e. a pair of maps $F^e\colon M^e \to N^e$ and $F^{C_p}\colon M^{C_p} \to N^{C_p}$ which are compatible with units, satisfying $F^{C_p} \circ t = t \circ F^e$ and $F^e \circ r = r \circ F^{C_p}$ together with p-degree additivity

$$\begin{array}{ccc} \left(M^{C_p}\right)^p & \longrightarrow & \left(N^{C_p}\right)^p \\ \downarrow & & \downarrow \\ M^{C_p} & \longrightarrow & N^{C_p} \end{array} \qquad \begin{array}{ccc} (M^e)^p & \longrightarrow & (M^e)^p \\ \downarrow & & \downarrow \\ M^e & \longrightarrow & N^e \end{array}$$

It suffices to note that a map between the pointed sets underlying unital magmas is a homomorphism if and only if it intertwines with $n$ary addition for *some* $n \geq 2$; indeed, one can simply identify binary addition with n-ary addition whose first $(n-2)$-factors are the unit. $\qquad\square$

We now spell out the interchange relations explicitly.

Proposition 55. *There is an equivalence between* $\mathrm{Mon}_{\mathbb{A}_{2,C_p} \otimes \mathbb{A}_{2,C_p}}(\mathcal{C})$ *and pairs of G-unital magma structures* $(M, *, \bullet, t_*, t_\bullet)$ *in $\mathcal{C}$ satisfying the interchange relations* $1_* = 1_\bullet$ *and*

$$\begin{array}{ccc} (X^p)^p & \xrightarrow{(\bullet)} & X^p \\ \downarrow{\scriptstyle(*)} & & \downarrow{\scriptstyle *} \\ X^p & \xrightarrow{\bullet} & X \end{array} \qquad \begin{array}{ccccc} X^{C_p} & \xleftarrow{t_\bullet} & X^e & \xrightarrow{t_*} & X^{C_p} \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle r} \\ X^e & \xleftarrow{*} & (X^e)^p & \xrightarrow{\bullet} & X^e \end{array} \qquad \begin{array}{ccc} (X^e)^p & \xrightarrow{(t_\bullet)} & \left(X^{C_p}\right)^p \\ \downarrow{\scriptstyle *} & & \downarrow{\scriptstyle *} \\ X^e & \xrightarrow{t_\bullet} & X^{C_p} \end{array} \qquad \begin{array}{ccc} (X^e)^p & \xrightarrow{(t_*)} & \left(X^{C_p}\right)^p \\ \downarrow{\scriptstyle\bullet} & & \downarrow{\scriptstyle\bullet} \\ X^e & \xrightarrow{t_*} & X^{C_p} \end{array}$$

Proof. Example 29 and Proposition 32 yields an equivalence.

$$\mathrm{Mon}_{\mathbb{A}_{2,C_p}^{\otimes 2}}(\mathcal{C}) \simeq \mathrm{Mon}_{\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})^{\otimes 2}}(\mathcal{C}_*).$$

This is characterized explicitly by Corollary 39 and Proposition 54; it suffices to note that the specified interchange relations correspond precisely with the conditions that $t_\bullet$ and $\bullet$ are C_p-unital magma homomorphisms. $\square$

We conclude the following form of Theorem A.

Corollary 56. *Given $\mathcal{C}$ a 1-category, the forgetful functor*

$$\mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_{C_p}),\mathcal{C}) \longrightarrow \mathrm{Mon}_{\mathbb{A}_{2,C_p}\otimes\mathbb{A}_{2,C_p}}(\mathcal{C})$$

$$\simeq \Big\{\textit{Interchanging pairs of } C_p\textit{-unital magmas in } \mathcal{C}\Big\}$$

is an equivalence of categories.

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