YOU CAN CONSTRUCT G-COMMUTATIVE ALGEBRAS ONE NORM AT A TIME

NATALIE STEWART

Abstract. We define the category of G-operads and the hierarchy of *generalized $\mathcal{N}_\infty$-operads*, which are G-suboperads of $\mathrm{Comm}_G^{\otimes}$. We exhibit an isomorphism between the category of subterminal operads and the self-join poset

$$\mathrm{Op}_G^{\mathcal{N}_\infty} \simeq \mathrm{Ind}-\mathrm{Sys}_G \star \mathrm{Ind}-\mathrm{Sys}_G,$$

where $\mathrm{Ind}-\mathrm{Sys}_G$ is the poset of *indexing systems* in G. This generalized $\mathcal{N}_\infty$-operads as parameterizing *some commutative multiplicative transfers and possibly a commutative multiplication*. Indeed, their algerbas in semiadditive Cartesian categories are incomplete Mackey functors and their algebras in Mackey functors recover incomplete Tambara functors when they are N_∞ *operads*, i.e. when they contain $\mathbb{E}_\infty$.

After this, we discuss some in-progress research. Namely, we construct a *Boardman-Vogt tensor product* of G-operads, and tensor products of genereralized $\mathcal{N}_\infty$ operads correspond with joins in $\mathrm{Ind}-\mathrm{Sys}_G \star \mathrm{Ind}-\mathrm{Sys}_G$ i.e. there is an $\mathcal{N}_{(I\vee J)\infty}$-monoidal equivalence

$$\mathbf{Alg}^{\mathcal{N}_{I\infty}} \mathbf{Alg}^{\mathcal{N}_{J\infty}} \mathcal{C} \simeq \mathbf{Alg}^{\mathcal{N}_{(I\vee J)\infty}} \mathcal{C}$$

for all $\mathcal{N}_{(I\vee J)\infty}$-monoidal categories $\mathcal{C}$, allowing G-commutative structures to be constructed "one norm at a time."

Foreword. The following are notes prepared for a casual talk in the zygotop seminar concerning research which is currently in-progress cite . Though I will attempt to confine these notes to their own proofs, citations to the literature, and well-marked conjecture, the reader should read with the understanding that they are particularly error-prone.

1. Introduction

In [Dre71], the concept of a *Mackey functor* was introduced; this structure was described as consisting of functors $M_I : \mathcal{O}_G \to \mathbf{Mod}_R$ and $M_R : \mathcal{O}_G^{\mathrm{op}} \to \mathbf{Mod}_R$ which agree on $\mathcal{O}_G^{\simeq}$ and satisfying the *double coset formula*

$$R^H_J I^H_K = \prod_{x \in [J\backslash H/K]} I^J_{J\cap xKx^{-1}} \cdot \mathrm{conj}_X R_{x^{-1}Jx\cap K}$$

for all $J, K \subset H$, where $R^K_J := M_R(G/J \to G/K)$ and similar for I. The ur-example of this is the assignment $H \mapsto \mathbf{Rep}_H(R)$ with covariant functoriality Ind and contravariant functoriality Res. This was repackaged and generalized into the modern definition of the *category of $\mathcal{C}$-valued G-Mackey functors*

$$\mathcal{M}_G(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_G), \mathcal{C}),$$

where $\mathbb{F}_G$ denotes the category of finite G-sets.

In parallel, the concept of *transfer maps in group cohomology* was being developed in [**Evens**], later lifted to genuine equivariant cohomology in [**Greenlees**], and finally developed as a functor

$$N_H^G : \mathrm{Sp}_H \to \mathrm{Sp}_G$$

in [HHR16], which played a crucial role in the solution to the Kervaire invariant one problem. These were noted in [HH16] to satisfy the conditions of a *Symmetric monoidal Mackey functor*, a notion they distinguished from their notion of *G-symmetric monoidal categories* due to coherence issues.

In the broad program announced in [Bar+16], the correct notion of *G-symmetric monoidal G-∞-categories* (henceforth G-symmetric monoidal categories) was introduced:

Definition 1.1. Let $\mathcal{C}$ have finite products. Then, the category of G-commutative monoids in $\mathcal{C}$ is

$$\mathrm{CMon}_G(\mathcal{C}) := \mathcal{M}_G(\mathcal{C}).$$

The category of G-symmetric monoidal categories is $\mathrm{CMon}_G(\mathbf{Cat})$.

We similarly define the *category of small G-categories* as

$$\mathbf{Cat}_G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Cat}) \simeq \mathbf{Cat}^{\mathrm{cocart}}_{/\mathcal{O}_G^{\mathrm{op}}},$$

where the equivalence is the *straightening-unstraightening construction* of [HTT]. We may summarize the structure $\mathcal{C}^\otimes \in \mathrm{CMon}_G(\mathbf{Cat})$ a G-symmetric monoidal category, as consisting of, for every conjugacy class (H) of G, a category with Weyl group action $\mathcal{C}_H \in \mathbf{Cat}^{BW_G H}$, as well as functors

$$\begin{aligned} \otimes_H^2 &: \mathcal{C}_H^2 \to \mathcal{C}_H, \\ N_H^K &: \mathcal{C}_H \to \mathcal{C}_K, \\ \mathrm{Res}_H^K &: \mathcal{C}_K \to \mathcal{C}_H \end{aligned}$$

which are associative, commutative, unital, and compatible with each other and the Weyl group action, together with coherence. The maps Res encode an underlying G-category $\mathcal{C}$ of $\mathcal{C}^\otimes$, and N_H^K is pronounced "the norm from H to K."

Given $\mathcal{C}$ a G-symmetric monoidal category, we may informally define a *G-commutative monoid* to be a tuple of objects $(X_H)_{H \in \mathcal{O}_G}$ satisfying

$$X_H \simeq \mathrm{Res}_H^G X_G$$

together with structure maps

$$\begin{aligned} \otimes_H^2 &: X_H^{\otimes 2} \to X_H \\ \mathrm{tr}_H^K &: N_H^K X_H \to X_K, \end{aligned}$$

for all $H \subset K$. The map tr_H^K is pronounced "the transfer from H to K."

This talk concerns various relaxations of the notion of G-commutative algebras. Namely, we will define a symmetric monoidal closed category Op_G of *(colored) G-operads*, whose internal hom $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})^\otimes$ is called the *operad of algebras under pointwise tensors*, and whose tensor product is called the *Boardman-Vogt tensor product*.

We will define $\mathcal{N}_\infty$ *operads*, which interpolate between $\mathbb{E}_\infty$ and the G-operad Comm_G which encodes G-commutative algebras by adding a subset of the transfers parameterized by Comm_G:

Definition 1.2. A *G-transfer system* is a core-preserving wide subcategory $\mathcal{O}_G^{\simeq} \subset T \subset \mathcal{O}_G$ which is closed under base change, i.e. for any diagram in $\mathcal{O}_G$

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\alpha'"] & V \arrow[d, "\alpha"] \\ U' \arrow[r] & V' \end{tikzcd}$$

with $U \hookrightarrow V' \times_{U'} V$ a summand inclusion (pullback taken in $\mathbb{F}_G$) and $\alpha \in T$, we have $\alpha' \in T$.

An *indexing system* is a subcategory $I \subset \underline{\mathbb{F}}_G$ induced by a transfer system under taking coproducts. A *generalized indexing system* is a core-preserving subcategory $I \subset \underline{\mathbb{F}}_G$ which is either an indexing system or is attained by one by removing all non-isomorphisms of trivial G-sets. The poset of indexing systems under inclusion is denoted $\mathrm{Ind}-\mathrm{Sys}_G$, and the poset of generalized indexing systems is denoted .

It is not hard to see that there is an equivalence of posets

$$\widehat{\mathrm{Ind}-\mathrm{Sys}}_G \simeq \mathrm{Ind}-\mathrm{Sys}_G \star \mathrm{Ind}-\mathrm{Sys}_G,$$

and in particular, generalized indexing systems decompose into two different join-stable copies of indexing systems, depending on whether maps of trivial G-sets are included.

The main theorem of this talk follows:

Theorem A. *There is a fully faithful and symmetric monoidal inclusion*

$$\mathcal{N}_{(-)\infty} : \widehat{\mathrm{Ind}-\mathrm{Sys}}_G^{\amalg} \hookrightarrow \mathrm{Op}_G^{\otimes}$$

whose image consists of the suboperads of Comm_G, *and when restricted to the indexing systems has image consisting of operads* $\mathcal{O}$ *possessing diagrams* $\mathbb{E}_\infty \subset \mathcal{O} \subset \mathrm{Comm}_G$. *In particular, for* $\mathcal{C}$ *an* $\mathcal{N}_{(I\vee J)\infty}$*-monoidal category, there is a canonical* $\mathcal{N}_{(I\vee J)\infty}$*-monoidal equivalence*

$$\mathbf{Alg}^{\mathcal{N}_{I\infty}} \mathbf{Alg}^{\mathcal{N}_{J\infty}} \mathcal{C} \simeq \mathbf{Alg}^{\mathcal{N}_{(I\vee J)\infty}} \mathcal{C}.$$

We say an inclusion of subgroup $H \subset K$ is *atomic* if it is proper and there exist no chains of proper subgroup inclusions $H \subset J \subset K$. More generally, we say that a conjugacy class $(H) \in \mathrm{Conj}(G)$ is an *atomic subclass* of (K) if there exists an atomic inclusion $\tilde{H} \subset \tilde{K}$ with $\tilde{H} \in (H)$ and $\tilde{K} \in (K)$, and we say that (K) is atomic if the canonical inclusion $1 \hookrightarrow K$ is atomic.

Given $(H) \subset (K)$ an atomic subclass, we refer to the $\mathcal{N}^\infty$-operad corresponding to the minimal index system containing the inclusion $H \hookrightarrow K$ as $\mathcal{N}^\infty(H, K)$. When $(H) = (1)$, we instead simply write $\mathcal{N}^\infty(K)$.

Corollary B. *Let* $1 = G_n \subset G_{n-1} \subset \cdots \subset G_0 = G$ *be a maximal subgroup series of a finite group, and let* $\mathcal{C}$ *be a* G*-symmetric monoidal category. Then, there exists a canonical* G*-symmetric monoidal equivalence*

$$\mathbf{Alg}^{\mathcal{N}^\infty(G_1,G_0)} \cdots \mathbf{Alg}^{\mathcal{N}^\infty(G_n,G_{n-1})} \mathcal{C} \simeq \mathrm{CAlg}_G \mathcal{C}.$$

Furthermore, if $G \simeq H \times J$, *then*

$$\mathrm{CAlg}_H \mathrm{CAlg}_J \mathcal{C} \simeq \mathrm{CAlg}_G \mathcal{C}.$$

Remark. One may worry about the comparison between models for G-operads, as our notion of N^∞-operads is ostensibly embedded deep within the world of G-∞-operads, which are not known to be equivalent to the ∞-category presented by the graph model structure or by genuine G operads. However, in addition to being too complicated to work with, the model of ref, all notions of N^∞ operads coincide .

2. The ideas

2.1. Fibrous patterns.

Definition 2.1. An *algebraic pattern* is an ∞-category $\mathcal{O}$, together with a factorization system $(\mathcal{O}^{\mathrm{int}}, \mathcal{O}^{\mathrm{act}})$ of $\mathcal{O}$ and a full subcategory $\mathcal{O}^{\mathrm{el}} \subset \mathcal{O}^{\mathrm{int}}$. The *category of algebraic patterns* is the full subcategory

$$\mathrm{AlgPatt} \subset \mathrm{Fun}(D, \mathbf{Cat})$$

spanned by algebraic patterns, where $D := \bullet \to \bullet \to \bullet \leftarrow \bullet$.

Maps in $\mathcal{O}^{\mathrm{int}}$ and $\mathcal{O}^{\mathrm{act}}$ are pronounced *inert and active maps*, and objects of $\mathcal{O}^{\mathrm{el}}$ are pronounced *elementary objects*. For instance, $\mathbb{F}_*$, together with its inert and active maps as defined in [HA, § 2] and elementary objects $\{\langle 1 \rangle\}$ determines an algebraic pattern. In analogy with [HA, § 2], we will use these to develop a notion of operads, called *fibrous patterns*.

Definition 2.2. Let $\mathcal{O}$ be an algebraic pattern. A *fibrous* $\mathcal{O}$*-pattern* is a map of algebraic patterns $\pi : \mathcal{P} \to \mathcal{O}$ such that

(1) $\mathcal{P}$ has π-cocartesian lifts for inert morphisms of $\mathcal{O}$,
(2) (Segal condition for colors) For every active morphism $\omega : O_0 \to O_1$ in $\mathcal{O}$, the functor

$$\mathcal{P}^{\simeq}_{O_0} \to \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O_1/}} \mathcal{P}^{\simeq}_{\omega_{\alpha,!}O_1}$$

induced by cocartesian transport along ω_α is an equivalence, where $\omega_{(-)} : \mathcal{O}^{\mathrm{el}}_{Y/} \to \mathcal{O}^{\int}_{X/}$ is the inert morphism appearing in the inert-active factorization of $\alpha \circ \omega$, and
(3) (Segal condition for multimorphism) for every active morphism $\omega : O_1 \to O_2$ in $\mathcal{O}$ and all objects $X_i \in \mathcal{P}_{O_i}$, the commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{P}}(X_0, X_1) \arrow[r] \arrow[d] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O_1/}} \mathrm{Map}_{\mathcal{P}}(X_0, \omega_{\alpha,!}X_1) \arrow[d] \\
\mathrm{Map}_{\mathcal{O}}(O_0, O_1) \arrow[r] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O_1/}} \mathrm{Map}_{\mathcal{O}}(O_0, \omega_{\alpha,!}O_1)
\end{tikzcd}$$

is cartesian.

A fibrous $\mathcal{O}$-pattern $\pi : \mathcal{P} \to \mathcal{O}$ is a *Segal $\mathcal{O}$-category* if π is a cocartesian fibration. The category of fibrous $\mathcal{O}$-patterns is the full subcategory

$$\mathrm{Fbrs}(\mathcal{O}) \subset \mathrm{AlgPatt}_{/\mathcal{O}}$$

spanned by fibrous patterns, and the category of Segal $\mathcal{O}$-∞-category is the full subcategory of

$$\mathrm{Seg}_{\mathcal{O}}(\mathbf{Cat}) \subset \mathrm{Fbrs}(\mathcal{O}) \times_{\mathbf{Cat}_{/\mathcal{O}}} \mathbf{Cat}^{\mathrm{cocart}}_{/\mathcal{O}}$$

spanned by Segal $\mathcal{O}$-categories.

We state one technical lemma:

Lemma 2.3. *All of the inclusions*

$$\mathrm{Seg}(\mathcal{O}) \to \mathrm{Fbrs}(\mathcal{O}) \hookrightarrow \mathrm{AlgPatt}_{/\mathcal{O}} \to \mathbf{Cat}_{/\mathcal{O}} \to \mathbf{Cat}$$

have left adjoints; in particular, the full subcategory $\mathrm{Fbrs}(\mathcal{O}) \subset \mathrm{AlgPatt}_{/\mathcal{O}}$ *is localizing.*

We refer to the left adjoint $\mathrm{Env} : \mathrm{Fbrs}(\mathcal{O}) \to \mathrm{Seg}(\mathcal{O})$ as the *Segal envelope*, and we use it analogously to the *symmetric monoidal envelope*, reducing the question of characterizing maps of fibrous patterns into Segal $\mathcal{O}$-categories into simply a question of characterizing maps of Segal $\mathcal{O}$-categories, which is much simpler.

Example 2.4:

Definition 2.5. Given the data of $\mathcal{X}$ a category, $\mathcal{X}_b, \mathcal{X}_f$ wide subcategories, and $\mathcal{X}_0 \subset \mathcal{X}_b$ a full subcategory, we define the *span pattern* $\mathrm{Span}_{b,f}(\mathcal{X}; \mathcal{X}_0)$ to have:

- underlying category $\mathrm{Span}_{b,f}(\mathcal{X})$ whose objects are objects in $\mathcal{X}$ and whose morphisms $X \to Z$ are spans
$$X \xleftarrow{B} Y \xrightarrow{F} Z$$
with $B \in \mathcal{X}_b$ and $F \in \mathcal{X}_f$.
- inert morphisms $\mathcal{X}_b^{\mathrm{op}} \subset \mathrm{Span}(\mathcal{X})$.
- active morphisms $\mathcal{X}_f \subset \mathrm{Span}(\mathcal{X})$.
- Elementary objects $\mathcal{X}_0^{\mathrm{el}} \subset \mathcal{X}_b^{\mathrm{op}}$.

Then, for instance we have the following:

Theorem 2.6 ([BHS22])**.** *Pullback along the inclusion* $\mathbb{F}_* \hookrightarrow \mathrm{Span}(\mathbb{F})$ *induces an equivalence on the categories of fibrous patterns and Segal categories.*

2.2. **G-operads and $\mathcal{I}$-operads.** There is an adjunction

$$\mathrm{Tot} : \mathbf{Cat}_G \rightleftarrows \mathbf{Cat} : \mathrm{CoFr}^G$$

where Tot takes the total category of a cocartesian fibration and $\mathrm{CoFr}^G(\mathcal{C})$ is classified by functor categories

$$\mathrm{CoFr}^G(\mathcal{C})_H := \mathrm{Fun}(\mathcal{O}_H^{\mathrm{op}}, \mathcal{C})$$

with functoriality dictated by pullback. In particular, the *G-category of small G-categories* $\underline{\mathbf{Cat}}_G := \mathrm{CoFr}^G(\mathcal{C})$ has G-fixed points given by $\mathbf{Cat}$.

Remark. Elmendorf's theorem may be reinterpreted in this language as the statement that the *G-category of G-spaces* $\mathcal{S}_G$ is cofreely generated by $\mathcal{S}$.

Let $\underline{\mathbb{F}}_G := \mathrm{CoFr}^G(\mathbb{F})$ and let $\underline{\mathbb{F}}_{G,*} := \mathrm{CoFr}^G(\mathbb{F}_*)$. Then, there is an equivariant lift of ref :

Theorem 2.7 ([BHS22])**.** *Pullback along the composition* $\underline{\mathbb{F}}_{G,*} \hookrightarrow \mathrm{Span}(\mathrm{Tot}\underline{\mathbb{F}}_G) \xrightarrow{U} \mathrm{Span}(\mathbb{F}_G)$ *induces an equivalence on the categories of fibrous patterns and Segal categories, where* $\mathbb{F}_G$ *is the category of G-sets.*

Definition 2.8. The *category of G-operads* is the category of fibrous patterns

$$\mathrm{Op}_G := \mathrm{Fbrs}(\mathrm{Span}(\mathbb{F}_G)).$$

A good sanity check is to verify that the category of G-symmetric monoidal categories agrees with the category of Segal $\mathrm{Span}(\mathbb{F}_G)$-categories; after some argumentation, one finds that the Segal conditions associated with the unstraightening of a cocartesian fibration over $\mathrm{Span}(\mathbb{F}_G)$ are precisely the condition that the unstraightened functor preserves products in $\mathrm{Span}(\mathbb{F}_G)$.

This is a straightforward, but heavy, generalization of the ∞-operads of [HA] to the equivariant world, and we suggest the interested reader consult [BHS22] for a less heavy variant or [NS22] for the original. In particular, postcomposition along the inclusion functor $\mathbb{F} \hookrightarrow \mathrm{Tot}\underline{\mathbb{F}}_{G,*}$ induces a fully faithful functor $\mathrm{Op} \hookrightarrow \mathrm{Op}_G$.

An early observation about genuine equivariant homotopy coherent algebraic structures is that the structure of transfers *does not come canonically* from an $\mathbb{E}_\infty$-structure; that is, $\mathbb{E}_\infty \in \mathrm{Op}_G$ is not terminal. The failure of $\mathbb{E}_\infty$ to be terminal is parameterized by the category of *generalized N^∞-operads*:

Definition 2.9. Write $\mathrm{Comm}_G^{\otimes} := (\mathrm{Span}(\mathbb{F}_G) = \mathrm{Span}(\mathbb{F}_G))$ for the terminal G-operad. A G-operad $\mathcal{O}^{\otimes}$ is a *generalized $\mathcal{N}^\infty$-operad* if the unique morphism $\mathcal{O}^{\otimes} \to \mathrm{Comm}_G^{\otimes}$ is a monomorphism, i.e. $\mathcal{O}_U^{\otimes} \simeq *$ for all U and $\mathrm{Map}_{\mathcal{O}}^{\psi}(x, y) \in \{*, \varnothing\}$ for all $\psi : \pi(x) \to \pi(y)$.

A generalized $\mathcal{N}^\infty$ operad $\mathcal{N}_{\infty I}$ is an N^∞ *operad* if it admits a map

$$\mathbb{E}_\infty \to \mathcal{O}^{\otimes}.$$

Write $\mathrm{Op}_G^{GN\infty}$ for the full subcategory consisting of generalized $\mathcal{N}_\infty$-operads. The following proposition is an exercise in category theory, and establishes that a map to an $\mathcal{N}_\infty$ operad is a *property*, not a structure.

Proposition 2.10. *Given $\mathcal{N}_{I\infty} \in \mathrm{Op}_G^{GN\infty}$ a generalized $\mathcal{N}_\infty$ operad, the forgetful functor*

$$\mathrm{Op}_{G,/\mathcal{N}_{I\infty}} \to \mathrm{Op}_G$$

is fully faithful.

Proof idea. It is equivalent to prove that $\mathrm{Map}(\mathcal{O}, \mathcal{N}_{I\infty}) \in \{*, \varnothing\}$ for all $\mathcal{O} \in \mathrm{Op}_G$ In fact, there is a localizing (1-) subcategory $N : \mathrm{Op}_{1,G} \hookrightarrow \mathrm{Op}_G$ consisting of operads whose structure spaces are discrete, and whose localization functor $h : \mathrm{Op}_G \to \mathrm{Op}_{1,G}$ takes π_0 of the structure spaces. $\mathcal{N}_{I\infty}$ evidently lies in $\mathrm{Op}_{1,G}$, so we have

$$\mathrm{Map}_{\mathrm{Op}_G}(\mathcal{O}, \mathcal{N}_{I\infty}) \simeq \mathrm{Hom}_{\mathrm{Op}_{1,G}}(h\mathcal{O}, \mathcal{N}_{I\infty}).$$

Hence it suffices to check that the latter set is empty or contractible. This is easy to see in $\mathrm{Op}_{1,G}$, since $\mathrm{Hom}(-, *)$ and $\mathrm{Hom}(-, \varnothing)$ are always either empty or contractible. $\square$

In particular, this implies that $\mathrm{Op}_G^{GN\infty}$ is a poset, so we'd like to identify this poset. There is a functor

$$A : \mathrm{Op}_G \to \widehat{\mathrm{Ind} - \mathrm{Sys}}_G$$

called the *admissible sets* with value over G/H given by

$$A(\mathcal{O})_{/(G/H)} := \left\{S \to G/H \mid \pi_{\mathcal{O}}^{-1}(S \to G/H) \neq \varnothing\right\}$$

and extended to general G-sets by coproducts. The following proposition is an exercise in category theory (see [NS22, Ex 2.4.7], or the original references [BH15; GW18; Rub21]):

Proposition 2.11. *The restricted functor*

$$A : \mathrm{Op}_G^{GN\infty} \to \widehat{\mathrm{Ind} - \mathrm{Sys}}_G$$

is an equivalence of categories.

We denote by $\mathcal{N}_{(-)\infty}$ the composite functor

$$\mathcal{N}_{(-)\infty} : \widehat{\mathrm{Ind} - \mathrm{Sys}}_G \xrightarrow{A^{-1}} \mathrm{Op}_G^{GN\infty} \hookrightarrow \mathrm{Op}_G$$

Using this, we finally define *I-operads*.

Definition 2.12. Let I be a generalized indexing system. Then, the *category of I-operads* is the slice category

$$\mathrm{Op}_I := \mathrm{Op}_{G,/\mathcal{N}_{\infty I}^{\otimes}}.$$

Given $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes} \in \mathrm{Op}_I$, the *category of $\mathcal{O}$-algebras in $\mathcal{P}$* is the full subcategory

$$\mathbf{Alg}_{c\mathcal{O}}(\mathcal{C}) \subset \mathrm{Fun}_{/\mathcal{N}_{\infty I}^{\otimes}}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$$

spanned by maps of $\mathcal{I}$-operads.

2.3. The BV tensor product.

2.4. The internal hom on G-operads.

Proposition 2.13. *The following are equivalent:*

1. *The forgetful functor $\mathrm{CAlg}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*
2. *For all one-object I-operads $\mathcal{O}$, the forgetful functor $\mathbf{Alg}^{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*
3. *The I-restricted operad is cocartesian*

Corollary 2.14. *If $\mathcal{O}$ is a one-object G-operad, then the map $\mathcal{N}^\infty(I) \to \mathcal{N}^\infty(I) \otimes \mathcal{O}$ is an I-equivalence; in particular, $\mathcal{N}^\infty(I)$ is $\otimes$-idempotent.*

Proof. Cf [NS22, Cor 5.3.9] □

3. Technical nonsense

Let I, J be indexing systems. Then, we have a unique diagram

$$\begin{tikzcd}
\mathcal{N}^\infty(I) \arrow[dr] \arrow[drr] & & \\
& \mathcal{N}^\infty(I)\otimes\mathcal{N}^\infty(J) \arrow[r, "\varphi"] & \mathcal{N}^\infty(I\cup J)\otimes\mathcal{N}^\infty(I\cup J) = \mathcal{N}^\infty(I\cup J) \\
\mathcal{N}^\infty(J) \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

We are tasked with proving that φ is an equivalence, for which we have several tricks:

1. By checking on after carefully chosen colocalization functors, we reduce to the case $I \subset J$.
2. By examining the monad on $\mathcal{S}_G$ associated with a G-operad, we reduce to verifying that the forgetful forgetful functor $U : \mathbf{Alg}^{\mathcal{N}_{\infty I}}\mathbf{Alg}^{\mathcal{N}_{\infty J}}(\mathcal{S}_J) \to \mathbf{Alg}^{\mathcal{N}_{\infty J}}(\mathcal{S}_J)$, where $\mathcal{S}_J$ is a J-*colocalization* of the Cartesian G-symmetric monoidal category of spaces.
3. We verify that U is an equivalence by showing that the pointwise tensor product on $\mathbf{Alg}^{\mathcal{N}_{\infty J}}(\mathcal{S}_J)$ is *cocartesian*.

3.1. $\mathcal{I}$-operads.

Definition 3.1. Let $\mathcal{I}$ be a generalized indexing system. Then, the *category of $\mathcal{I}$-operads* is the slice category

$$\mathrm{Op}_{\mathcal{I}} \simeq \mathrm{Op}_{G,/\mathcal{N}_{\infty\mathcal{I}}}.$$

Let $\iota : \mathcal{N}_{\infty I} \to \mathcal{N}_{\infty J}$ be a map of $\mathcal{N}_\infty$ operads. By ref , the pushforward $\iota_! : \mathrm{Op}_{\mathcal{I}} \to \mathrm{Op}_{\mathcal{J}}$ is fully faithful for all $\mathcal{I} \subset \mathcal{J}$; by elementary category theory, it also attains a left adjoint $R = i^* : \mathrm{Op}_{\mathcal{J}} \to \mathrm{Op}_{\mathcal{I}}$, making the $\mathcal{I}$ operads a *colocalizing subcategory* of the $\mathcal{J}$ operads. In particular, choosing $\mathcal{N}_{\infty J} = \mathrm{Comm}_G$, the $\mathcal{I}$-operads are a colocalizing subcategory of the $\mathcal{J}$ operads.

In fact, rune cite yields an immediate intrinsic characterization of $\mathcal{I}$-operads:

Proposition 3.2. *The category of $\mathcal{I}$-operads has objects consisting of functors*

$$\mathcal{O}^{\otimes} \to \mathcal{N}_{\infty I}$$

with cocartesian lifts for inert morphisms of $\mathcal{N}_{\infty I}$ satisfying the analogs of ref for all $(S \to U)$ and ψ in $\mathcal{I}$.

The main result of this subsection is the following:

Proposition 3.3. *The product $\iota_!^{\mathcal{I}} \times \iota_!^{\mathcal{J}} : \mathrm{Op}_{\mathcal{I}\cup\mathcal{J}} \to \mathrm{Op}_{\mathcal{I}} \times \mathrm{Op}_{\mathcal{J}}$ is conservative.*

Proof. Let $C : \mathrm{Op}_{\mathcal{I}} \to \mathbf{Cat}_G$ be the *G-category of colors* functor and let $S : \mathrm{Op}_{\mathcal{I}} \to \mathcal{S}^{\mathrm{Mor}(\mathcal{I})_{/\mathrm{orb}}}$ be the functor taking O to the fibers over morphisms in $\mathcal{I}$ over orbits. Then, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Op}_{\mathcal{I} \cup \mathcal{J}} \arrow[r] \arrow[d] & \mathbf{Cat}_G \times \mathcal{S}^{\mathrm{Mor}(\mathcal{I} \cup \mathcal{J})_{/\mathrm{orb}}} \arrow[d, "\Delta \times F"] \\
\mathrm{Op}_{\mathcal{I}} \times \mathrm{Op}_{\mathcal{J}} \arrow[r] & \mathbf{Cat}_G \times \mathbf{Cat}_G \times \mathcal{S}^{\mathrm{Mor}(\mathcal{I})_{/\mathrm{orb}}} \times \mathcal{S}^{\mathrm{Mor}(\mathcal{J})_{/\mathrm{orb}}}
\end{tikzcd}$$

where $F : \mathcal{S}^{\mathrm{Mor}(I \cup J)} \to \mathcal{S}^{\mathrm{Mor}(I)_{/mathrmorb}} \times \mathcal{S}^{\mathrm{Mor}(J)_{\mathrm{orb}}}$ is the uhhhhhhhh □

3.2. **Tensor products of subterminal operads.**

Proposition 3.4. *Let $O^{\otimes}$ be a G-operad. The following are equivalent:*

(1) *$O^{\otimes}$ is $\mathcal{N}^{\infty}$.*
(2) *the functor $\mathrm{Env}O \to \underline{\mathbb{F}}_G^{\amalg}$ is an equifibrous core-preserving G-symmetric monoidal subcategory.*
(3) *the functor $\mathrm{Env}O \to \underline{\mathbb{F}}_G^{\amalg}$ is a wide subcategory.*

Proof. Obvious. □

Endow $\mathrm{CMon}_G(\mathbf{Cat})_{/\underline{\mathbb{F}}_G^{\amalg}}$ with the sliced Day convolution symmetric monoidal structure. Let $\mathrm{CMon}_G(\mathbf{Cat})^{\mathrm{ecs}}_{\underline{\mathbb{F}}_G^{\amalg}} \subset \mathrm{CMon}_G(\mathbf{Cat})_{/\underline{\mathbb{F}}_G^{\amalg}}$ be the full subcategory spanned by equifibrous core-preserving G-symmetric monoidal subcategories of $\underline{\mathbb{F}}_G^{\amalg}$

Proposition 3.5. $\mathrm{CMon}_G(\mathbf{Cat})^{\mathrm{eqw}}_{\underline{\mathbb{F}}_G^{\amalg}}$ *is closed under binary tensor products.*

Proof. Note that $\underline{\mathbb{F}}_G^{\amalg}$ is idempotent under the Day convolution tensor product, so it suffices to prove that the *external* tensor product $\mathcal{C} \times \mathcal{D} \to \underline{\mathbb{F}}_G^{\amalg} \times \underline{\mathbb{F}}_G^{\amalg}$ is an equifibrous core-preserving subcategory.[1] In fact, by the pointwise formula for left Kan extensions, it suffices to prove that products and colimits in preserve equifibrous core-preserving subcategories of $\underline{\mathbb{F}}_G^{\amalg}$.

It is easy to verify this for products and coproducts, so we must verify this for geometric realizations . By the argument due to Thomas Blom in the AT discord , the core-preserving property is preserved under sifted colimits, and the hom-spaces are sifted colimits of the hom spaces. Hence it suffices to prove that the geometric realization of a summand inclusion in sSet is a summand inclusion in $\mathcal{S}$. But this is just simple homotopy theory. Geometric realizations should just always preserve equifibrous summand inclusions. Need to check the equifibrous conditions! In general, BV tensor products may have to pass through a localization to be equifibrous. This can probably be done by classifying the core-preserving symmetric monoidal wide subcategories of $\underline{\mathbb{F}}_G^{\amalg}$ and computing directly that the result is equifibrous. A cheap trick–we can verify that $\mathcal{N}^{\infty}(I) \otimes \mathcal{N}^{\infty}(J) \xrightarrow{\sim} \mathcal{N}^{\infty}(I \cup J) \otimes \mathcal{N}^{\infty}(I \cup J)$ by day conv before taking L_{eqfbrs}! □

Remark. The equifibrous wide condition is necessary, but i doubt the 1-category condition is necessary.

3.3. **Synthesis.**

Proof of theorem A. By groth , image of N infty , and closure of N infty , the sub-poset $\widehat{\mathcal{N}}_G^{\infty} \subset \mathrm{Op}_G$ is closed under tensor products, so by HA citation , it is endowed with a canonical symmetric monoidal structure. It remains to prove that this structure is cocartesian. Clearly O_{triv} is both the initial object and the unit, so it suffices to prove that binary tensor products are computed by the join of indexing systems.

[1] Argue this using G-symmetric monoidality of the grothendieck construction. Alternatively, are cocartesian symmetric monoidal categories idempotent?

In fact, the universal property for the BV tensor product constructs a diagram in $\mathcal{N}_G^\infty$

$$\begin{tikzcd}
\mathcal{N}^\infty(I) \arrow[dr] \arrow[drr] & & \\
& \mathcal{N}^\infty(I) \otimes \mathcal{N}^\infty(J) \arrow[r] & \mathcal{N}^\infty(I \cup J) = \mathcal{N}^\infty(I) \cup \mathcal{N}^\infty(J) \\
\mathcal{N}^\infty(J) \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

so that $\mathcal{N}^\infty(I), \mathcal{N}^\infty(J) \subset \mathcal{N}^\infty(I) \otimes \mathcal{N}^\infty(J) \subset \mathcal{N}^\infty(I) \cup \mathcal{N}^\infty(J)$. By the universal property for the join, this guarantees that the map $\mathcal{N}^\infty(I) \otimes \mathcal{N}^\infty(J) \to \mathcal{N}^\infty(I \cup J)$ is an equivalence. $\square$

Proof of corollary B. The first statement is immediate. For the second statement, by the structure theorem for finitely generated Abelian groups, it suffices to prove this for C_{p^n}, i.e. to prove that there are exactly two indexing systems for C_{p^n}. $\square$

References

[BHS22] Shaul Barkan, Rune Haugseng, and Jan Steinebrunner. *Envelopes for Algebraic Patterns*. 2022. arXiv: 2208.07183 [math.CT].

[Bar+16] Clark Barwick et al. *Parametrized higher category theory and higher algebra: A general introduction*. 2016. arXiv: 1608.03654 [math.AT].

[BH15] Andrew J. Blumberg and Michael A. Hill. “Operadic multiplications in equivariant spectra, norms, and transfers”. In: *Adv. Math.* 285 (2015), pp. 658–708. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2015.07.013. URL: https://doi.org/10.1016/j.aim.2015.07.013.

[Dre71] Andreas W. M. Dress. *Notes on the theory of representations of finite groups. Part I: The Burnside ring of a finite group and some AGN-applications*. With the aid of lecture notes, taken by Manfred Küchler. Universität Bielefeld, Fakultät für Mathematik, Bielefeld, 1971, iv+158+A28+B31 pp. (loose errata).

[GW18] Javier J. Gutiérrez and David White. “Encoding equivariant commutativity via operads”. In: *Algebr. Geom. Topol.* 18.5 (2018), pp. 2919–2962. ISSN: 1472-2747,1472-2739. DOI: 10.2140/agt.2018.18.2919. URL: https://doi.org/10.2140/agt.2018.18.2919.

[HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. “On the nonexistence of elements of Kervaire invariant one”. In: *Ann. of Math. (2)* 184.1 (2016), pp. 1–262. ISSN: 0003-486X. DOI: 10.4007/annals.2016.184.1.1. URL: https://people.math.rochester.edu/faculty/doug/mypapers/Hill_Hopkins_Ravenel.pdf.

[HH16] Michael A. Hill and Michael J. Hopkins. *Equivariant symmetric monoidal structures*. 2016. arXiv: 1610.03114 [math.AT].

[HTT] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.

[HA] Jacob Lurie. *Higher Algebra*. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[NS22] Denis Nardin and Jay Shah. *Parametrized and equivariant higher algebra*. 2022. arXiv: 2203.00072 [math.AT].

[Rub21] Jonathan Rubin. “Combinatorial N_∞ operads”. In: *Algebr. Geom. Topol.* 21.7 (2021), pp. 3513–3568. ISSN: 1472-2747,1472-2739. DOI: 10.2140/agt.2021.21.3513. URL: https://doi.org/10.2140/agt.2021.21.3513.