ORBITAL CATEGORIES AND WEAK INDEXING SYSTEMS

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Abstract. We initiate the combinatorial study of the poset $\mathrm{wIndex}_{\mathcal{T}}$ of *weak $\mathcal{T}$-indexing systems*, consisting of composable collections of arities for $\mathcal{T}$-equivariant algebraic structures, where $\mathcal{T}$ is an orbital ∞-category, such as the orbit category of a finite group. In particular, we show that these are equivalent to *weak $\mathcal{T}$-indexing categories* and characterize various unitality conditions.

Within this sits a natural generalization $\mathrm{Index}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}$ of Blumberg-Hill's *indexing systems*, consisting of arities for structures possessing binary operations and unit elements. For instance, in this setting, results of Balchin-Barnes-Roitzheim quickly imply that the lattice of $C_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}_p$-indexing systems is equivalent to the infinite associahedron.

We characterize the relationship between the posets of *unital weak indexing systems* and *indexing systems*, the latter remaining isomorphic to *transfer systems* on this level of generality. We use this to compute the poset of unital C_{p^N}-weak indexing systems for $N \in \mathbb{N} \cup \{\infty\}$.

Contents

1. Introduction

Fix G a finite group. In [BH15], the notion of $\mathcal{N}_\infty$*-operads for* G was introduced, encapsulating a collection of *blueprints* for G-equivariantly commutative multiplicative structures on Mackey functors which possess underlying Green functors. They demonstrated that the ∞-category of $\mathcal{N}_\infty$-operads for G is an embedded sub-poset of the lattice of *indexing systems* Index_G.

Subsequently, the embedding $\mathcal{N}_\infty-\mathrm{Op}_G \subset \mathrm{Index}_G$ was shown to be an equivalence in several independent works [BP21; GW18; Rub21]; of particular note is the equivalent characterization of indexing systems as a poset of wide subcategories $\mathrm{IndexCat}_G \subset \mathrm{Sub}(\mathbb{F}_G)$ (referred to as *indexing categories*) [BH18, § 3.2] and the

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observation that indexing categories only depend on their pullbacks to the subgroup lattice $\mathrm{Sub}_{\mathrm{Grp}}(G)$, the resulting embedded subposet

$$\begin{tikzcd}
\mathrm{Index}_G \arrow[d, hook] & \mathrm{IndexCat}_G \arrow[l, "\sim"'] \arrow[rr, "\sim"] \arrow[d, hook] & & \mathrm{Transf}_G \arrow[d, hook] \\
\mathrm{FullSub}_G(\underline{\mathbb{F}}_G) & \mathrm{Sub}(\mathbb{F}_G) \arrow[l, "\mathbb{F}_{(-)}"] \arrow[r, "(-)\cap\mathcal{O}_G"'] & \mathrm{Sub}(\mathcal{O}_G) \arrow[r, "\mathrm{p.b.}"'] & \mathrm{Sub}_{\mathrm{Poset}}\mathrm{Sub}_{\mathrm{Grp}}(G)
\end{tikzcd}$$

being referred to as *transfer systems* [BBR21; Rub19]. It is in this language that enumerative problems concerning $\mathcal{N}_\infty$-operads are often solved.

Noting that $\mathrm{Sub}_{\mathrm{Grp}}(\mathcal{O}_{C_{p^n}}) = [n+1]$, the transfer system approach was used in [BBR21] to prove that $\mathrm{Transf}_{C_{p^n}}$ is equivalent to the $(n+2)$nd associahedron K_{n+2}, where C_m is the cyclic group of order m.[1] Furthermore, transfer systems have powered a large amount of further work on the topic; for instance, $\mathrm{Transf}_{C_{pqr}}$ is enumerated for p, q, r distinct primes in [BBPR20], with some indications on how to generalize this to arbitrary squarefree integers.

In this paper, we aim to demonstrate how one may extend this work in two ways:

(1) we will remove the assumption on indexing systems that they are closed under coproducts; on the side of algebra, we will see in [Ste24] this corresponds with removing the assumption that algebras over the corresponding G-operad $\mathcal{N}_{I\infty}^{\otimes}$ in Mackey functors possess underlying green functors.
(2) we will replace the orbit category $\mathcal{O}_G$ with an axiomatic version, called an *atomic orbital ∞-category*; this allows us to fluently describe equivariance under families and cofamilies, as well as extending to more general orbit categories, such as the finite-index orbit categories of a compact Lie group or profinite group.

For the former, we find in Example 1.32 that the poset of *weak* indexing systems is always infinite; nevertheless, when we assert a unitality assumption, we find that $\mathrm{wIndex}_G^{\mathrm{uni}}$ is finite when G is finite, and it can usually be explicitly described in terms of transfer systems and G-families (c.f. Theorem C and Corollary D). Moreover, unitality is compatible with joins (c.f. Proposition 2.57), and in [Ste24] we will establish that joins compute tensor products of unital weak $\mathcal{N}_\infty$-operads.

We assure the skeptical reader that they may freely assume $\mathcal{T}$ is (the orbit category of) a G-family and replace all instances of orbits $V \in \mathcal{T}$ with homogeneous G-spaces $[G/H]$ for $H \in \mathcal{F}$ (or with the subgroup $H \subset G$ itself, depending on which is contextually appropriate);[2] then, our results will only be novel in way (1). Regardless, we will now review the axiomatic setting of *(atomic) orbital ∞-categories.*

1.1. Orbital ∞-categories.

We briefly review the setting introduced in [BDGNS16] generalizing the orbit category $\mathcal{O}_G$; we assume basic intuition for $\mathcal{O}_G$, consistent e.g. with the characterization in [Die09, § 1.2-1.3].

Construction 1.1 (c.f. [Gla17])**.** Given $\mathcal{T}$ an ∞-category[3], its *finite coproduct completion* is the full subcategory $\mathbb{F}_{\mathcal{T}} \subset \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S})$ spanned by finite coproducts of representable presheaves, where $\mathcal{S}$ denotes the ∞-category of spaces. ◁

Example 1.2. If G is a finite group, then $\mathbb{F}_{\mathcal{O}_G}$ is equivalent to the 1-category of finite G-sets; more generally, if $\mathcal{F} \subset \mathcal{O}_G$ is (the orbit category of) a G-family, then $\mathbb{F}_{\mathcal{F}} \subset \mathbb{F}_{\mathcal{O}_G}$ is the full subcategory spanned by finite G-sets S such that the stabilizer $\mathrm{stab}_G(x)$ lies in $\mathcal{F}$ for all $x \in S$. ◁

$\underline{\mathbb{F}}_{\mathcal{T}}$ is *freely* generated by $\mathcal{T}$ under finite coproducts; in particular, given $S \in \mathbb{F}_{\mathcal{T}}$, there is a unique expression $S \simeq \bigoplus_{V \in \mathrm{Orb}(S)} V$ for some set of S-*orbits* $\mathrm{Orb}(S) \to \mathrm{Ob}\mathcal{T}$. Another important property of the finite

[1] This is off by one from the indexing used in [BBR21]; following the combinatorial literature, we use K_n for the associahedron parameterizing parenthesizations of n-letters, so that e.g. K_3 has 2 elements.

[2] Throughohut this paper, a G-*family* will always refer to a subconjugacy closed collection of subgroups of G, That the reader understands weak indexing systems over G-families will become non-negotiable over the course of this paper, as we critically employ change of universe functors throughout the text, such as *Borelification.*

[3] 1-categories embed fully faithfully into ∞-categories, and the reader is free to safely assume all categorical terminology refer to 1-categories (and spaces as sets) *except* for the 2-category Cat_1 of 1-categories, which must be a 2-category in order for the definition of I-symmetric monoidal 1-categories to have coherences compatible with the ∞-categorical case.

coproduct completion is existence of equivalences

$$\mathbb{F}_{\mathcal{T},/S} \simeq \prod_{V \in \mathrm{Orb}(S)} \mathbb{F}_{\mathcal{T},/V}; \qquad \mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_{\mathcal{T}_{/V}}.$$

We henceforth refer to $\mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_{\mathcal{T}_{/V}}$ as $\mathbb{F}_V$. Note that, in the case $\mathcal{T} = \mathcal{O}_G$, induction furnishes an equivalence $\mathcal{O}_{G,/[G/H]} \simeq \mathcal{O}_H$, so $\mathbb{F}_{[G/H]} \simeq \mathbb{F}_H$.

Fundamental to genuine-equivariant mathematics is the *effective Burnside category* $\mathsf{Span}(\mathbb{F}_G)$; for instance, the G-Mackey functors of [Dre71] may be presented as product-preserving functors $\mathsf{Span}(\mathbb{F}_G) \to \mathbf{Ab}$. In fact, the spectral Mackey functor theorem of [GM17] presents G-spectra as product-preserving functors of ∞-categories $\mathsf{Span}(\mathbb{F}_G) \to \mathsf{Sp}$, a perspective which has been greatly exploited e.g. in [Bar14; BGS20].

In $\mathsf{Span}(\mathbb{F}_G)$, composition of morphisms is accomplished via the pullback

$$\begin{tikzcd}[column sep=small] & & R_{fg} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & R_g \arrow[dl] \arrow[dr] & & R_f \arrow[dl] \arrow[dr] & \\ S & & T & & Q \end{tikzcd} \tag{1}$$

Indeed, given $\mathcal{T}$ an arbitrary ∞-category, the triple $(\mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}})$ is *adequate* in the sense of [Bar14] if and only if $\mathbb{F}_{\mathcal{T}}$ has pullbacks, in which case the triple is *disjunctive*. Thus, Barwick's construction [Bar14, Def 5.5] defines an effective Burnside ∞-category $\mathsf{Span}(\mathbb{F}_{\mathcal{T}}) = A^{eff}(\mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}})$ precisely if $\mathcal{T}$ is *orbital* in the sense of the following definition.

Definition 1.3 ([Nar16, Def 4.1])**.** A (small) ∞-category $\mathcal{T}$ is *orbital* if $\mathbb{F}_{\mathcal{T}}$ has pullbacks; an orbital ∞-category $\mathcal{T}$ is *atomic* if all retracts in $\mathcal{T}$ are equivalences. ◃

If $\mathcal{T}$ is an orbital 1-category, then the effective Burnside ∞-category $\mathsf{Span}(\mathbb{F}_{\mathcal{T}})$ is a 2-category with objects the finite $\mathcal{T}$-sets, morphisms the spans of finite $\mathcal{T}$-sets, 2-cells the isomorphisms of spans, and composition defined by Eq. (1). We will not discuss the Burnside ∞-category for the main combinatorial results of this paper, but it factors greatly into the parallel study of *genuine equivariant algebra*, and hence in the parallel article [Ste24].

Remark 1.4. We show in Section 2.1 that, if $\mathcal{T}$ is an orbital ∞-category, then $\mathsf{ho}(\mathcal{T})$ is as well; furthermore, the main combinatorial objects of this paper are the same between $\mathcal{T}$ and $\mathsf{ho}(\mathcal{T})$. Hence the reader may uniformly assume that $\mathcal{T}$ is a 1-category, at the loss of essentially none of the combinatorics. ◃

Example 1.5. Given X a space considered as an ∞-category, X is atomic orbital; by [Gla18, Thm 2.13], the associated stable ∞-category is the Ando-Hopkins-Rezk ∞-category of parameterized spectra over X (c.f. [ABGHR14]). In particular, for $X = BG$, this recovers *spectra with G-action*. ◃

Example 1.6. Given P a meet semilattice, P is atomic orbital, as the meets in $\mathbb{F}_P$ are easily computed in terms of meets in P. ◃

Given G a topological group, let $\mathcal{S}_G$ denote the ∞-category of G-spaces, presented for instance by the simplicial localization of topological spaces with G-action at the maps inducing weak equivalences on point-set fixed points for each closed subgroup. Let $\mathcal{O}_G \subset \mathcal{S}_G$ denote the full subcategory spanned by the homogeneous G-spaces $[G/H]$ for $H \subset G$ a closed subgroup. We call this the *orbit ∞-category*.

A famous issue with equivariant homotopy theory over an infinite group G is that the orbit ∞-category $\mathcal{O}_G$ is not *orbital*; the G-Burnside category does not exist, as $\mathbb{F}_G$ does not have pullbacks with which to define composition of spans (the double coset formula constructs infinitely many elements in many such pullbacks). Nevertheless, this has been rectified in various homotopical contexts. One particularly lucid treatment due to Cnossen-Lenz-Linskens uses the slightly more general setting of *global homotopy theory*.

Definition 1.7 ([CLL23, Def 4.2.2, 4.3.2])**.** Given $\mathcal{P} \subset \mathcal{T}$ a wide subcategory of an ∞-category, we denote by $\mathbb{F}_{\mathcal{T}}^{\mathcal{P}} \subset \mathbb{F}_{\mathcal{T}}$ the wide subcategory consisting of morphisms which are disjoint unions of morphisms in $\mathcal{P}$. $\mathcal{P} \subset \mathcal{T}$ is an *orbital subcategory* if $\mathbb{F}_{\mathcal{T}}^{\mathcal{P}} \subset \mathbb{F}_{\mathcal{T}}$ is stable under pullbacks along arbitrary maps in $\mathbb{F}_{\mathcal{T}}$, and all such pullbacks exist. An orbital subcateogry is *atomic* if any morphism in $\mathcal{P}$ which admits a section in $\mathcal{T}$ is an equivalence. ◃

Note that an ∞-category is atomic orbital if and only if it's an atomic orbital subcategory of itself, so the orbital setting specializes the global setting. On the other hand, many global examples can be pulled back to the orbital setting.

Lemma 1.8. *Suppose $\mathcal{P} \subset \mathcal{T}$ is an atomic orbital subcategory. Then, $\mathcal{P}$ is atomic orbital as an ∞-category.*

Proof. First, assume we have a square in $\mathbb{F}_{\mathcal{P}}$; taking a pullback in $\mathbb{F}_{\mathcal{T}}$, we extended it to be the outer square of the following $\mathcal{T}$-diagram

$$\begin{tikzcd} T' \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g'"] \arrow[ddr, bend right, "f'"'] & & \\ & T \times_S S' \arrow[r, dashed, "\pi_T"'] \arrow[d, dashed, "\pi_{S'}"] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "f"] \\ & S' \arrow[r, "g"'] & S \end{tikzcd}$$

To prove that $\mathcal{P}$ is orbital, it suffices to verify that the inner square is a pullback, for which it suffices to check that all of the involved maps are in $\mathcal{P}$. First note that, $\pi_{S'}$ and π_T are in $\mathcal{P}$ since $\mathcal{P} \subset \mathcal{T}$ is orbital subcategory; h is then in $\mathcal{P}$ since atomic orbital subcategories are left cancellable by [CLL23, Lem 4.3.5], so we've proved that $\mathcal{P}$ is orbital. To see that $\mathcal{P}$ is atomic, note that this immediately follows from the second condition of Definition 1.7. □

To use this for equivariance over infinite groups, we make the following definition.

Definition 1.9. Given $\mathcal{T}$ an ∞-category, a *$\mathcal{T}$-family* is a full subcategory $\mathcal{F} \subset \mathcal{T}$ satisfying the condition that, given $F : V \to W$ a morphism with $W \in \mathcal{F}$, we have $V \in \mathcal{T}$. A *$\mathcal{T}$-cofamily* is a full subcategory $\mathcal{F}^{\perp} \subset \mathcal{T}$ such that $\mathcal{F}^{\perp,\mathrm{op}} \subset \mathcal{T}^{\mathrm{op}}$ is a $\mathcal{T}^{\mathrm{op}}$-family.

Given $\mathcal{T}$ an ∞-category, an *interval family* of $\mathcal{T}$ is an intersection of a family and a cofamily; equivalently, it is a full subcategory $\mathcal{F}$ with the property that whenever $U, W \in \mathcal{F}$ and there is a path $U \to V \to W$, we have $V \in \mathcal{F}$. ◁

Observation 1.10. If $\mathcal{F} \subset \mathcal{T}$ is an interval family in an atomic orbital ∞-category satisfying the condition that, for all cospans $U \to W \leftarrow V \in \mathcal{T}$ with $U, W \in \mathcal{F}$, there is a span $U \leftarrow W' \to V$ with $W \in \mathcal{F}$, then the inclusion $\mathbb{F}_{\mathcal{F}} \subset \mathbb{F}_{\mathcal{T}}$ creates pullbacks. In particular, $\mathcal{F}$ is an atomic orbital ∞-category. ◁

Example 1.11. Let G be a Lie group and $\mathcal{O}_G^{f.i.} \subset \mathcal{O}_G$ the wide subcategory of the orbit ∞-category spanned by projections $G/K \to G/H$ corresponding with finite-index closed subgroup inclusions $K \subset H$. Then, by [CLL23, Ex 4.2.6], $\mathcal{O}_G^{f.i.} \subset \mathcal{O}_G$ is an orbital subcategory. In fact, it follows quickly from definition that it is atomic as well; hence $\mathcal{O}_G^{f.i.}$ is an atomic orbital ∞-category and pullbacks in $\mathbb{F}_G^{f.i.}$ are computed by a double coset formula.

In fact, by Observation 1.10, the $\mathcal{O}_G^{f.i.}$-interval families consisting of *finite subgroups* and of *finite-index closed subgroups* are atomic orbital ∞-categories as well. The former in the case $G = \mathbb{T}$ yields the *cyclonic orbit category*, so its stable homotopy theory is that of *cyclonic spectra*, i.e. *finitely genuine S^1-spectra* (c.f. [BG16, Thm 2.8]). ◁

Example 1.12. Given $H \subset G$ a closed subgroup, the cofamily $\mathcal{O}_{G,\geq[G/H]}^{f.i.}$ spanned by homogeneous G-spaces G/J admitting a quotient map from G/H satisfies the assumption of Observation 1.10, so it is atomic orbital; in the case $H = N \subset G$ is normal, it is equivalent to $\mathcal{O}_{G/N}^{f.i.}$. In any case, the associated stable homotopy theory is the value category of *H-geometric fixed points* with residual genuine G/H-structure (c.f. [Gla17]). ◁

1.2. **Weak indexing systems and weak indexing categories.** Throughout the remainder of this introduction, we fix $\mathcal{T}$ an orbital ∞-category.

1.2.1. *Weak indexing systems.* In the case $\mathcal{T} = \mathcal{O}_G$ is the orbit category of a compact Lie group G, Elmendorf's theorem [DK84; Elm83] implies that the ∞-category of G-spaces is equivalent to the functor ∞-category

$$\mathcal{S}_G \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}),$$

i.e. they are (homotopy coherent) *coefficient systems of spaces*. It is becoming traditional to allow G to act on the *category theory* surrounding genuine equivariant mathematics, culminating in the following definition.

Definition 1.13. The *2-category of $\mathcal{T}$-1-categories* is the functor 2-category[4]

$$\mathrm{Cat}_{\mathcal{T},1} := \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}_1) \simeq \mathrm{Fun}(h_2\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}_1),$$

where Cat_1 is the 2-category of 1-categories and $h_2(-)$ denotes the homotopy 2-category. ◁

We refer to the morphisms in $\mathrm{Cat}_{\mathcal{T},1}$ as *$\mathcal{T}$-functors*. Given a $\mathcal{T}$-1-category $\mathcal{C}$ and an object $V \in \mathcal{T}$, $\mathcal{C}$ has a *V-value* 1-category $\mathcal{C}_V := \mathcal{C}(V)$, and given a map $V \to W$ in $\mathcal{T}$, $\mathcal{C}$ has an associated *restriction functor* $\mathrm{Res}_V^W \colon \mathcal{C}_W \to \mathcal{C}_V$.

Example 1.14. By [NS22, Prop 2.5.1], the ∞-category $\mathcal{T}_{/V}$ is a 1-category, so $\mathbb{F}_V := \mathbb{F}_{\mathcal{T}_{/V}} \simeq \mathbb{F}_{\mathcal{T},/V}$ is a 1-category. Hence the functor $\mathcal{T}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sending $V \mapsto \mathbb{F}_{\mathcal{T},/V}$ is a $\mathcal{T}$-1-category, which we call *the $\mathcal{T}$-1-category of finite $\mathcal{T}$-sets* and denote as $\underline{\mathbb{F}}_{\mathcal{T}}$. ◁

Notation 1.15. We refer to the terminal object $(V = V) \in \mathbb{F}_V$ as $*_V$ and call it the *contractible V-set*. We refer to the initial object $(\varnothing \to V) \in \mathbb{F}_V$ as $\varnothing_V$ and call it the *empty V-set*. ◁

Evaluation is functorial in the $\mathcal{T}$-1-category; indeed, a $\mathcal{T}$-functor $F \colon \mathcal{C} \to \mathcal{D}$ is just a collection of functors

$$F_V \colon \mathcal{C}_V \to \mathcal{D}_V$$

intertwining with restriction. We refer to a $\mathcal{T}$-functor whose V-values are fully faithful as a *fully faithful $\mathcal{T}$-functor*; if $\iota \colon \mathcal{C} \to \mathcal{D}$ is a fully faithful $\mathcal{T}$-functor, we say that $\mathcal{C}$ is a *full $\mathcal{T}$-subcategory of $\mathcal{D}$*. A full $\mathcal{T}$-subcategory of $\mathcal{D}$ is uniquely determined by an equivalence-closed and restriction-stable class of objects in $\mathcal{D}$; see [Sha23] for details.

Definition 1.16 (c.f. [HHR16, § 2.2.3])**.** Fix $\mathcal{C}$ a $\mathcal{T}$-1-category. The *induced V-set functor* $\mathrm{Ind}_U^V \colon \mathcal{C}_U \to \mathcal{C}_V$, if it exists, is the left adjoint to Res_U^V. Furthermore, given a V-set S and a tuple $(T_U)_{U \in \mathrm{Orb}(S)}$, the *$S$-indexed coproduct of T_U* is, if it exists, the element

$$\coprod_U^S T_U := \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^V T_U \in \mathcal{C}_V.$$

Dually, the *coinduced V-set* $\mathrm{CoInd}_U^V \colon \mathcal{C}_U \to \mathcal{C}_V$ is the right adjoint to Res_U^V (if it exists), and the S-indexed product is (if it exists), the element

$$\prod_U^S T_U := \prod_{U \in \mathrm{Orb}(S)} \mathrm{CoInd}_U^V T_U \in \mathcal{C}_V. \qquad ◁$$

Example 1.17. Given a subgroup inclusion $K \subset H \subset G$, the associated functor $\mathbb{F}_H \to \mathbb{F}_K$ is restriction, and hence its left adjoint $\mathbb{F}_K \to \mathbb{F}_H$ is *G-set induction*, matching the *indexed coproducts* of [HHR16, § 2.2.3]. ◁

Given $S \in \mathbb{F}_V$, we write

$$\mathcal{C}_S := \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_V;$$

we say that $\mathcal{C}$ *strongly admits finite indexed coproducts* if $\coprod_U^S T_U$ always exists, in which case it is a functor

$$\coprod_U^S (-) \colon \mathcal{C}_S \to \mathcal{C}_V.$$

Remark 1.18. Given $S \in \mathbb{F}_V$, we may define the functor $\Delta^S \colon \mathcal{C}_V \to \mathcal{C}_S$ so that for each $U \in \mathrm{Orb}(S)$, the associated functor $\mathcal{C}_V \to \mathcal{C}_U$ is restriction along the composite map $U \to S \to V$. This is the rightwards horizontal composition in the following:

$$\begin{tikzcd}
\mathcal{C}_V \arrow[r, "\Delta"] & \prod\limits_{U \in \mathrm{Orb}(S)} \mathcal{C}_V \arrow[l, bend right, "\coprod_{U \in \mathrm{Orb}(S)}(-)"'] \arrow[l, bend left, "\prod_{U \in \mathrm{Orb}(S)}(-)"] \arrow[r, "(\mathrm{Res}_U^V)"] & \prod\limits_{U \in \mathrm{Orb}(S)} \mathcal{C}_U \arrow[l, bend right, "\mathrm{Ind}_U^V"'] \arrow[l, bend left, "\mathrm{CoInd}_U^V"]
\end{tikzcd}$$

[4] Throughout this paper, *n-category* will mean $(n,1)$-category, i.e. ∞-category whose mapping spaces are $(n-1)$-truncated.

In particular, by composing adjoints, we acquire adjunctions $\coprod_U^S(-) \dashv \Delta^S \dashv \prod_U^S(-)$, i.e. we've constructed indexed (co)limits in the sense of [Sha22]. ◁

It follows from construction that $\underline{\mathbb{F}}_{\mathcal{T}}$ strongly admits finite indexed coproducts; indeed, $\mathbb{F}_{\mathcal{T},/V} = \mathbb{F}_{\mathcal{T}_{/V}}$ admits finite coproducts by definition, and $\mathcal{T}$-set induction along a map $f : V \to W$ is implemented by the postcomposition $f_! : \mathbb{F}_{\mathcal{T},/V} \to \mathbb{F}_{\mathcal{T},/W}$, as it participates in the categorical push-pull adjunction $f_! \dashv f^*$. Similarly, $\underline{\mathbb{F}}_{\mathcal{T}}$ strongly admits finite indexed products, so in particular, Res_U^V preserves coproducts.

Definition 1.19. Given a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ and a full $\mathcal{T}$-subcategory $\mathcal{E} \subset \mathcal{D}$, we say that $\mathcal{E}$ is *closed under $\mathcal{C}$-indexed coproducts* if, for all $S \in \mathcal{C}_V$ and $(T_U) \in \mathcal{E}_S$, we have $\coprod\limits_U^S T_U \in \mathcal{E}_V$. ◁

Definition 1.20. We say that a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is *closed under self-indexed coproducts* if it is closed under $\mathcal{C}$-indexed coproducts. ◁

Definition 1.21. Given $\mathcal{T}$ an orbital ∞-category, a *$\mathcal{T}$-weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ with V-values $\mathbb{F}_{I,V} := (\underline{\mathbb{F}}_I)_V$ satisfying the following conditions:

(IS-a) whenever $\mathbb{F}_{I,V} \neq \varnothing$, we have $*_V \in \mathbb{F}_{I,V}$; and
(IS-b) $\underline{\mathbb{F}}_I$ is closed under self-indexed coproducts.

We denote by $\mathsf{wIndex}_{\mathcal{T}} \subset \mathsf{FullSub}_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$ the embedded sub-poset spanned by $\mathcal{T}$-weak indexing systems. Moreover, we say that a $\mathcal{T}$-weak indexing system *has one color* if it satisfies the following condition:

(IS-i) for all $V \in \mathcal{T}$, we have $\mathbb{F}_{I,V} \neq \varnothing$;

these span an embedded subposet $\mathsf{wIndex}_{\mathcal{T}}^{oc} \subset \mathsf{wIndex}_{\mathcal{T}}$. We say that a $\mathcal{T}$-weak indexing system is almost essentially unital or (aE-unital) if it satisfies the following condition:

(IS-ii) for all noncontractible V-sets $S \sqcup S' \in \mathbb{F}_{I,V}$, we have $S, S' \in \mathbb{F}_{I,V}$.

An almost essentially unital $\mathcal{T}$-weak indexing system is *almost unital* if it has one color. These are denoted $\mathsf{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \subset \mathsf{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \subset \mathsf{wIndex}_{\mathcal{T}}$. We say that a $\mathcal{T}$-weak indexing system is *essentially unital* (or E-unital) if it satisfies the following condition:

(IS-iii) for all V-sets $S \sqcup S' \in \mathbb{F}_{I,V}$, we have $S, S' \in \mathbb{F}_{I,V}$.

We say that an essentially unital $\mathcal{T}$-weak indexing system is *unital* if it has one color. We write $\mathsf{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \subset \mathsf{wIndex}_{\mathcal{T}}^{E\mathrm{uni}} \subset \mathsf{wIndex}_{\mathcal{T}}$. Lastly, a $\mathcal{T}$-weak indexing system is an *indexing system* if it satisfies the following condition:

(IS-iv) the subcategory $\mathbb{F}_{I,V} \subset \mathbb{F}_V$ is closed under finite coproducts for all $V \in \mathcal{T}$.

We denote the resulting poset by $\mathsf{Index}_{\mathcal{T}} \subset \mathsf{wIndex}_{\mathcal{T}}^{\mathrm{uni}}$. ◁

Remark 1.22. The indexing systems of [BH15] are seen to be equivalent to ours when $\mathcal{T} = \mathcal{O}_G$ by unwinding definitions. The weak indexing systems of [BP21; Per18] are equivalent to our *unital* weak indexing systems when $\mathcal{T} = \mathcal{O}_G$ by [Per18, Rem 9.7] and [BP21, Rem 4.60]. ◁

In practice, we will find that non-aE-unital weak indexing systems are not well behaved, and questions involving aE-unital weak indexing systems are usually quickly reducible to the unital case; the non-combinatorial reader is encouraged to focus primarily on unital weak indexing systems for this reason.

1.2.2. *Some examples.* We begin with some universal examples.

Example 1.23. The terminal $\mathcal{T}$-weak indexing system is $\underline{\mathbb{F}}_{\mathcal{T}}$; the initial $\mathcal{T}$-weak indexing system is the empty subcategory; the initial one-color $\mathcal{T}$-weak indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}}$ is defined by

$$\mathbb{F}_{\mathcal{T},V}^{\mathrm{triv}} := \{*_V\}.$$ ◁

To understand the conditions of Definition 1.21, we introduce some associated invariants.

Proposition 1.24. *Given $\underline{\mathbb{F}}_I$ a $\mathcal{T}$-weak indexing system, the following are $\mathcal{T}$-families:*

$$c(I) := \{V \in \mathcal{T} \mid *_V \in \mathbb{F}_{I,V}\}$$
$$\upsilon(I) := \{V \in \mathcal{T} \mid \varnothing_V \in \mathbb{F}_{I,V}\}$$
$$\nabla(I) := \{V \in \mathcal{T} \mid 2 \cdot *_V \in \mathbb{F}_{I,V}\}$$

Proof. This follows by noting that $\mathrm{Res}_U^V n \cdot *_V = n \cdot *_U$, where we write

$$n \cdot S := \overbrace{S \sqcup \cdots \sqcup S}^{n\text{-fold}}. \qquad \square$$

We call $c(I)$ the *color family of* I, $v(I)$ the *unit family*, and $\nabla(I)$ the *fold map family*. Note that $c(I) \leq v(I) \cap \nabla(I)$; that is, Condition (IS-a) implies that whenever $\underline{\mathbb{F}}_I$ prescribes a unit or a fold map over V, it possesses a color over V. We will use the following lemma ubiquitously.

Lemma 1.25. *Let $\underline{\mathbb{F}}_I$ be a $\mathcal{T}$-weak indexing system.*

1. *$\underline{\mathbb{F}}_I$ has one color if and only if $c(I) = \mathcal{T}$.*
2. *$\underline{\mathbb{F}}_I$ is E-unital if and only if $v(I) = c(I)$.*
3. *$\underline{\mathbb{F}}_I$ is unital if and only if $v(I) = \mathcal{T}$.*
4. *$\underline{\mathbb{F}}_I$ is an indexing system if and only if $v(I) \cap \nabla(I) = \mathcal{T}$.*

Proof. (1) follows immediately by unwinding definitions. For (2), if $\underline{\mathbb{F}}_I$ is E-unital and $V \in c(I)$, then choosing $\varnothing_V \sqcup *_V \in \mathbb{F}_{I,V}$ yields $\varnothing_V \in \mathbb{F}_{I,V}$, i.e. $V \in v(I)$. Conversely, if $v(I) = c(I)$ and $S \sqcup S' \in \mathbb{F}_{i,V}$, then

$$S = \coprod_U^{S \sqcup S'} \chi_S(U), \qquad \text{where } \chi_S(U) := \begin{cases} *_U & U \in S \\ \varnothing_U & U \notin S \end{cases}$$

so $S \in \underline{\mathbb{F}}_I$, i.e. $\underline{\mathbb{F}}_I$ is E-unital. (3) follows by combining (1) and (2).

For (4), note that $\underline{\mathbb{F}}_I$ an indexing system implies that $v(I) \cap \nabla(I) = \mathcal{T}$ by taking nullary and binary copowers of $*_V \in \mathbb{F}_{I,V}$. Conversely, if $v(I) \cap \nabla(I) = \mathcal{T}$, then by iterating binary coproducts $(n-1)$-times, we find that $n \cdot *_V = (*_V \sqcup (n-1) \cdot *_V) \in \mathbb{F}_{I,V}$ for all V and $n \in \mathbb{N}$. Applying Condition (IS-b), we find that $\mathbb{F}_{I,V}$ is closed under n-ary coproudcts for all $n \in \mathbb{N}$, i.e. $\underline{\mathbb{F}}_I$ is an indexing system. $\square$

In fact, the proof of (2) shows more; we may use the same argument to show the following.

Lemma 1.26. *$\underline{\mathbb{F}}_I$ is aE-unital if and only if, whenever $S \in \mathbb{F}_{I,V}$ is noncontractible, $V \in v(I)$.*

We may use c to reduce study of weak indexing systems to the one-color case via the following.

Construction 1.27. Given $\mathcal{F}$ a $\mathcal{T}$-family and $\underline{\mathbb{F}}_I$ an $\mathcal{F}$-weak indexing system, we may define the $\mathcal{T}$-weak indexing system $E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_I$ by

$$\left(E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_I\right)_V := \begin{cases} \mathbb{F}_{I,V} & V \in \mathcal{F}; \\ \varnothing & \text{otherwise.} \end{cases}$$

◁

This yields an embedding of posets $\mathrm{wIndex}_{\mathcal{F}} \to \mathrm{wIndex}_{\mathcal{T}}$. In Proposition 2.29, we prove the following.

Proposition 1.28. *The fiber of $c \colon \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ is the image of $E_{\mathcal{F}}^{\mathcal{T}}|_{oc} \colon \mathrm{wIndex}_{\mathcal{F}}^{oc} \to \mathrm{wIndex}_{\mathcal{T}}$.*

In particular, we find that $E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}$ and $E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^{\mathrm{triv}}$ are terminal and initial among $c^{-1}(\mathcal{F})$.

Example 1.29. In [Ste24] we define the *underlying $\mathcal{T}$-symmetric sequence* $\mathcal{O}(-)$ of a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$; the space $\mathcal{O}(S)$ parameterizes the *S-ary operations* endowed on an $\mathcal{O}$-algebra. We define the *arity support*

$$\mathbb{F}_{A\mathcal{O},V} := \{S \in \mathbb{F}_V \mid \mathcal{O}(S) \neq \varnothing\};$$

in [Ste24], we show that this possesses a fully faithful right adjoint, making $\mathcal{T}$-weak indexing systems equivalent to *weak $\mathcal{N}_\infty$-$\mathcal{T}$-operads*, i.e. subterminal objects in the ∞-category of $\mathcal{T}$-operads. This inspires our naming; [Ste24] establishes that $\underline{\mathbb{F}}_{A\mathrm{triv}_{\mathcal{T}}} = \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}}$ and $\underline{\mathbb{F}}_{A\mathrm{Comm}_{\mathcal{T}}} = \underline{\mathbb{F}}_{\mathcal{T}}$.

In general, we may choose $\mathcal{T} = \mathcal{O}_G$ and R a real orthogonal G-representation. Then, the arity support of the little R-disks operad is defined by

$$\mathbb{F}_H^R := \mathbb{F}_{A\mathbb{E}_R,H} = \{S \in \mathbb{F}_H \mid \exists \text{ H-equivariant embedding } S \hookrightarrow R\}$$

(c.f. [Hor19] as recalled in [Ste24]). The unital weak indexing system $\underline{\mathbb{F}}_R$ is not always an indexing system; for instance, choosing $G = C_p$ and λ a 2-dimensional irreducible real orthogonal C_p-representation, we see by unwinding definitions that

$$\mathbb{F}_e^\lambda = \mathbb{F}_e, \qquad \mathbb{F}_{C_p}^\lambda = \left\{n \cdot [C_p/e] \mid n \in \mathbb{N}\right\} \sqcup \left\{*_{C_p} + n \cdot [C_p/e] \mid n \in \mathbb{N}\right\}.$$

In fact, a unital G-weak indexing system $\underline{\mathbb{F}}_I$ is an indexing system if and only if it contains $2 \cdot *_G$ (in which case, it must contain its restrictions $2 \cdot *_H$ for all $H \subset G$), and R admits a G-equivariant embedding of $2 \cdot *_G$ if and only if the inclusion $\{0\} \subset R^G$ is proper, i.e. R has positive-dimensional fixed points. Thus $\underline{\mathbb{F}}^R$ is not an indexing system when R has 0-dimensional fixed points. ◁

We will see in Section 2.3 that the construction $R \mapsto \underline{\mathbb{F}}^R$ is monotone and compatible with direct sums.

Example 1.30. The intial unital $\mathcal{T}$-weak indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^0$ is defined by

$$\mathbb{F}_{\mathcal{T},V}^0 \coloneqq \{\varnothing_V, *_V\};$$

we will see in [Ste24] that this is equal to $\underline{\mathbb{F}}_{A\mathbb{E}_0}$. ◁

Example 1.31. The initial $\mathcal{T}$-indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^\infty$ is defined by

$$\mathbb{F}_V^\infty \coloneqq \{n \cdot *_V \mid n \in \mathbb{N}\};$$

we will see in [Ste24] that this is equal to $\underline{\mathbb{F}}_{A\mathbb{E}_\infty}$. ◁

Example 1.32. Let $\mathcal{T} = *$ be the terminal category. Then, a full subcategory $\underline{\mathbb{F}}_I \subset \mathbb{F}$ can be identified with a subset $n(I) \subset \mathbb{N}$, Condition (IS-a) with the condition that $n(I)$ is empty or contains 1, and Condition (IS-b) with the condition that $n(I)$ is closed under k-fold sums for all $k \in n(I)$. There are many such things; for instance, for each $n \in \mathbb{N}$, the set $\{1\} \cup n\mathbb{N} \subset \mathbb{N}$ gives a nonunital $*$-weak indexing system.

Nevertheless, if we assert that $\varnothing \in n(I)$ (i.e. $\underline{\mathbb{F}}_I$ is unital), then $n(I)$ is closed under summands, i.e. it is lower-closed in $\mathbb{N}$. Thus we have the following computations for $\mathcal{T} = *$:

condition	poset
indexing system	$\mathbb{F}$
unital	$\mathbb{F}^0 \longrightarrow \mathbb{F}$
almost unital	$\mathbb{F}^{\mathrm{triv}} \longrightarrow \mathbb{F}^0 \longrightarrow \mathbb{F}$
essentially unital	$\varnothing \longrightarrow \mathbb{F}^0 \longrightarrow \mathbb{F}$
almost essentially unital	$\varnothing \longrightarrow \mathbb{F}^{\mathrm{triv}} \longrightarrow \mathbb{F}^0 \longrightarrow \mathbb{F}$

◁

Example 1.33. We will see in Corollary 2.4 that when X is a space, there is a canonical equivalence $\mathrm{wIndex}_X \simeq \mathrm{wIndex}_*$ respecting our various conditions. In particular, the computations for *Borel* equivariant weak indexing systems mirror those of Example 1.32. ◁

1.2.3. *Weak indexing categories.* With a wealth of examples under our belt, we now simplify the combinatorics.

Observation 1.34. Denote by $\mathrm{Ind}_V^{\mathcal{T}} S \to V$ the map corresponding with a finite V-set S under the equivalence $\mathbb{F}_V \simeq \mathbb{F}_{\mathcal{T},/V}$. This equivalence implies a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is determined by the subgraph

$$I(\mathcal{C}) \coloneqq \left\{ \coprod_i \mathrm{Ind}_{V_i}^{\mathcal{T}} S_i \to V_i \;\middle|\; \forall i, \quad S \in \mathcal{C}_{V_i} \right\} \subset \mathbb{F}_{\mathcal{T}}.$$

In other words, the construction I yields an embedding of posets

$$I(-) : \mathrm{wIndex}_{\mathcal{T}} \hookrightarrow \mathrm{Sub}_{\mathrm{graph}}(\mathbb{F}_{\mathcal{T}}).$$ ◁

We will prove the following in Section 2.2.

Theorem A. *Fix $\mathcal{T}$ an orbital ∞-category. Then, the image of the map $I(-)$ consists of the subcategories $I \subset \mathbb{F}_{\mathcal{T}}$ satisfying the following conditions*

(IC-a) (restrictions) I is stable under arbitrary pullbacks in $\mathbb{F}_{\mathcal{T}}$;
(IC-b) (segal condition) the pair $T \to S$ and $T' \to S'$ are in I if and only if $T \sqcup T' \to S \sqcup S'$ is in I; and
(IC-c) ($\Sigma_{\mathcal{T}}$-action) if S is an object of I, then all automorphisms of S are in I.

Moreover, for all numbers n, condition (IS-n) of Definition 1.21 is equivalent to condition (IC-n) below:

(IC-i) (one color) I is wide; equivalently, I contains $\mathbb{F}_{\mathcal{T}}^{\simeq}$.
(IC-ii) (aE-unital) if $S \sqcup S' \to T$ is a non-isomorphism map in I, then $S \to T$ and $S' \to T$ are in I.
(IC-iii) (E-unital) if $S \sqcup S' \to T$ is a map in I, then $S \to T$ and $S' \to T$ are in I.
(IC-iv) (indexing category) the fold maps $n \cdot V \to V$ are in I for all $n \in \mathbb{N}$ and $V \in \mathcal{T}$.

We refer to the image of $I(-)$ as the *weak indexing categories* $\mathrm{wIndexCat}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathrm{Cat}}(\mathbb{F}_{\mathcal{T}})$. In general, we will refer to a generic weak indexing category as I and its corresponding weak indexing system as $\underline{\mathbb{F}}_I$. The following observations form the basis for the proof of Theorem A.

Observation 1.35. By a basic inductive argument, Condition (IC-b) is equivalent to the following condition:

(IC-b') $T \to S$ is in I if and only if $T_U = T \times_S U \to U$ is in I for all $U \in \mathrm{Orb}(S)$.

in particular, I is uniquely determined by the maps to orbits. ◁

Observation 1.36. By Observation 1.35, in the presence of Condition (IC-b), Condition (IC-a) is equivalent to the following condition:

(IC-a') for all Cartesian diagrams in $\mathbb{F}_{\mathcal{T}}$

$$\begin{tikzcd} T \times_V U \arrow[r] \arrow[d, "\alpha'"] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "\alpha"] \\ U \arrow[r] & V \end{tikzcd} \tag{2}$$

with $U, V \in \mathcal{T}$ and $\alpha \in I$, we have $\alpha' \in I$. ◁

One of the major reasons for this formalism is the technology of *equivariant algebra*. If $\iota\colon I \subset \mathbb{F}_{\mathcal{T}}$ is a pullback-stable subcategory, write $\mathbb{F}_{c(I)}$ for the coproduct closure of the essential image of ι. Then $(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I)$ is an adequate triple in the sense of [Bar14], so we may form the span ∞-category

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) := A^{eff}(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I),$$

whose forward maps are I and backwards maps are arbitrary. If $\mathcal{C}$ is an ∞-category, the ∞-category of *I-commutative monoids in* $\mathcal{C}$ is the product preserving functor ∞-category

$$\mathrm{CMon}_I(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C});$$

the *I-symmetric monoidal* 1*-categories* are

$$\mathrm{Cat}^{\otimes}_{I,1} := \mathrm{CMon}_I(\mathrm{Cat}_1),$$

where Cat_1 denotes the 2-category of 1-categories. These are a form of *I-symmetric monoidal Mackey functors* in the sense of [HH16].

$\mathcal{T}$-commutative monoids yields I-commutative monoids by neglect of structure.[5] By [Ste24], a $\mathcal{T}$-1-category $\mathcal{D}$ with I-indexed coproducts possesses an essentially unique *cocartesian I-symmetric structure* $\mathcal{D}^{I-\sqcup}$ satisfying the property that its I-indexed tensor products implement I-indexed coproducts; a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \mathcal{D}$ is I-symmetric monoidal under this structure if and only if it's closed under I-indexed coproducts. Hence we have the following.

Corollary B. *Fix a collection of objects $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ containing the contractible $c(I)$-sets and $I \subset \mathbb{F}_{\mathcal{T}}$ the corresponding collection of maps satisfying Condition (IC-b). Then, the following conditions are equivalent:*

(1) I is a weak indexing category;
(2) $\underline{\mathbb{F}}_I$ is a weak indexing system;
(3) $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is an I-symmetric monoidal subcategory under indexed coproducts.

Remark 1.37. If $\mathcal{C}$ is an I-symmetric monoidal category, $V \to W$ a map in I, and $U \to W$ a map in $\mathcal{T}$, then there is an associated commutative diagram

$$\begin{tikzcd} & U \times_V W \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ U \arrow[dr] & & W \arrow[dl] \\ & V & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & \mathcal{C}_{U \times_V W} \arrow[dr] \arrow[d, phantom, "\wr"] & \\ \mathcal{C}_U \arrow[ur] \arrow[r, "\Delta^S"] \arrow[dr, "N_U^V"'] & \prod\limits_{X \in \mathrm{Orb}(U \times_V W)} \mathcal{C}_X \arrow[r, "\otimes^{U \times_V W}"] & \mathcal{C}_W \\ & \mathcal{C}_V \arrow[ur, "\mathrm{Res}_V^W"'] & \end{tikzcd}$$

[5] In particular, this is modeled by pullback along the product-preserving inclusion $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ induced by the inclusion of adequate triples $(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I) \hookrightarrow (\mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}})$.

In particular, this encodes the double coset formula $\mathrm{Res}_W^V N_U^V R_U = \bigotimes_X^{U \times_V W} \mathrm{Res}_X^U R_U$.

In the case that $\mathcal{C}$ has I-indexed coproducts, it possesses a *cocartesian I-symmetric monoidal structure* (see [Ste24]), so this recovers a more traditional double coset formula. Replacing U with some V-set S, we get the formula

$$\mathrm{Res}_V^W \coprod_U^S Z_U \simeq \coprod_X^{\mathrm{Res}_V^W S} \mathrm{Res}_X^{o(X)} Z_{o(X)},$$

where $o(X)$ is the orbit of S satisfying $X \subset \mathrm{Res}_V^W o(X) \subset \mathrm{Res}_V^W S$. ◁

We explore this further in [Ste24], wherein we frequently use that indexed coproducts of arities compute the arities of composite operations in the theory of equivariant operads.

1.3. **Unital weak indexing categories and transfer systems.** We now turn to transfer systems.

Definition 1.38. Given $\mathcal{T}$ an orbital ∞-category, an *orbital transfer system in $\mathcal{T}$* is a core-containing wide subcategory $\mathcal{T}^{\simeq} \subset R \subset \mathcal{T}$ satisfying the "base change" condition that for all $\mathcal{T}$ digarams

$$\begin{tikzcd} V' \arrow[r] \arrow[d, "\alpha'"] & V \arrow[d, "\alpha"] \\ U' \arrow[r] & U \end{tikzcd}$$

whose associated $\mathbb{F}_{\mathcal{T}}$ map $V' \to V \times_U U'$ is a summand inclusion, if $\alpha \in R$, we have $\alpha' \in R$. The associated embedded sub-poset is denoted $\mathrm{Transf}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathrm{Cat}}(\mathbb{F}_{\mathcal{T}})$. ◁

Observation 1.39. If I is a unital weak indexing category, the intersection $\mathfrak{R}(I) := I \cap \mathcal{T}$ is an orbital transfer system; hence it yields a monotone map

$$\mathfrak{R}(-) \colon \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}}. \qquad ◁$$

Transfer systems were first defined because of the following phenomenon.

Proposition 1.40 ([NS22, Rmk 2.4.9])**.** $\mathfrak{R}(-)$ *restricts to an equivalence*

$$\mathfrak{R}(-) \colon \mathrm{Index}_{\mathcal{T}} \xrightarrow{\sim} \mathrm{Transf}_{\mathcal{T}}.$$

Remark 1.41. In the case $\mathcal{T} = \mathcal{O}_G$, before Nardin-Shah's result, it was shown independently in [Rub19, Thm 3.7] and [BBR21, Cor 8] that the composite inclusion $\mathrm{Sub}_{\mathrm{Grp}}(G) \hookrightarrow \mathcal{O}_G \hookrightarrow \mathbb{F}_G$ induces an embedding $\mathrm{Index}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathbf{Poset}}\left(\mathrm{Sub}_{\mathrm{Grp}}(G)\right)$ whose image is identified by those subposets which are closed under restriction and conjugation, which were called *G-transfer systems*; this and Proposition 1.40, together imply that pullback along the *homogeneous G-set* functor $\mathrm{Sub}_{\mathrm{Grp}}(G) \to \mathcal{O}_G$ induces an equivalence between the poset of G-transfer systems of [BBR21; Rub19] and the orbital $\mathcal{O}_G$-transfer systems of Definition 1.38. ◁

In view of Remark 1.41, we henceforth in this paper refer to orbital transfer systems simply as *transfer systems*, never referring to the other notion. Proposition 1.40 additionally allows for a reformulation of transfer systems which may be familiar to global equivariant homotopy theorists.

Observation 1.42. Let $\mathcal{T}$ be an orbital ∞-category. Then, a wide subcategory $R \subset \mathcal{T}$ is a transfer system if and only if it is an orbital subcategory in the sense of Definition 1.7; indeed, the axioms for an orbital subcategory encapsulate that of a transfer system, and give a transfer system, [NS22, Rmk 2.4.9] argues that $\mathbb{F}_{\mathcal{T}}^R$ is indexing category, so in particular it is pullback-stable.[6] Furthermore, if $\mathcal{T}$ is atomic orbital, then all of its orbital subcategories are atomic orbital. ◁

In Proposition 2.40, we will show that the composite

$$\mathrm{Transf}_{\mathcal{T}} \simeq \mathrm{Index}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$$

is a fully faithful right adjoint to $\mathfrak{R}$, i.e. the poset of unital weak indexing systems possessing a given transfer system has a terminal object, given by the unique such indexing system. However, the fibers can be quite

[6] In essence, the foundational difference between the orbital and global settings is that the orbital setting develops stable homotopy theory over a transfer system by specialization from the complete transfer system, whereas the global setting characterizes this directly; the latter strategy is more complicated, but allows for base categories which are not themselves orbital, such as the global indexing category.

large; for instance, in Remark 2.45, we will see that $\mathfrak{R}$ also attains a fully faithful left adjoint, which is distinct from the right adjoint over all transfer systems when $\mathcal{T}$ has a terminal object (e.g. when $\mathcal{T} = \mathcal{O}_G$).

The upshot is that unital weak indexing systems are not determined by their transitive V-sets. Nevertheless, we can specify them by a small collection of data, for which we need the following definition.

Definition 1.43. Denote by $\pi_0\mathcal{T}$ the set of isomorphism classes of $\mathcal{T}$. Given $\mathcal{C}$ a $\mathcal{T}$-1-category, there is an underlying diagram $\mathrm{Ob}'\mathcal{C}\colon \pi_0\mathcal{T} \to \mathsf{Set}$; We refer to a $\pi_0\mathcal{T}$-graded subset of $\mathrm{Ob}'\mathcal{C}$ as a *$\mathcal{C}$-collection.* We will generally refer to $\underline{\mathbb{F}}_{\mathcal{T}}$-collections simply as *collections.* ◁

Construction 1.44. If $\mathcal{T}$ is an orbital ∞-category, then we define the collection of *sparse objects* $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ to have V-value spanned by the V-sets

$$\varepsilon \cdot *_V \sqcup W_1 \sqcup \cdots \sqcup W_n,$$

for $\varepsilon \in \{0,1\}$ and $W_1,\dots,W_n \in \mathcal{T}_{/V}$ subject to the condition that there exist no maps $W_i \to W_j$ for $i \neq j$. ◁

Example 1.45. Let G be a finite group. Then, for (H) a conjugacy class of G, the *sparse H-sets* are precisely the H-sets

$$\varepsilon \cdot *_H \sqcup [H/K_1] \sqcup \cdots \sqcup [H/K_n],$$

where none of the conjugacy classes $(K_1),\dots,(K_n)$ include into each other. ◁

Given $\mathcal{C}^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}$, we may form the full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ generated by $\mathcal{C}^{\mathrm{sprs}}$ under iterated $\mathcal{C}^{\mathrm{sprs}}$-indexed coproducts We say that $\mathcal{C}^{\mathrm{sprs}}$ is *closed under applicable self-indexed coproducts* if $\mathcal{C}^{\mathrm{sprs}} = \mathcal{C} \cap \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}$. We prove the following in Section 3.1.

Theorem C. *Suppose $\mathcal{T}$ is an atomic orbital ∞-category. Then restriction along the inclusion $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}} \hookrightarrow \underline{\mathbb{F}}_{\mathcal{T}}$ yields an embedding of posets*

$$\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \subset \mathrm{Coll}(\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}})$$

whose image is spanned by the aE-unital collections which are closed under applicable self-indexed coproducts.

In Remark 3.5, we will see that Theorem C is compatible with the conditions of Definition 1.21; namely, the conditions of almost unitality, essential unitality, unitality, and being an indexing system correspond with the same conditions on the sparse collection.

We will prove in [Ste24] that the aE-unital weak indexing systems are isomorphic to the poset of $\otimes$-idempotent weak $\mathcal{N}_\infty$-operads. Thus we may conclude the following.

Corollary 1.46. *If $\mathcal{T}$ is an atomic orbital ∞-category such that $\pi_0(\mathcal{T})$ is finite and $\mathcal{T}_{/V}$ is finite as a 1-category for all $V \in \pi_0(\mathcal{T})$, then there exist finitely many $\otimes$-idempotent weak $\mathcal{N}_\infty$-$\mathcal{T}$-operads.*

Proof. By [Ste24], we're tasked with provng that $\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}}$ is finite. Theorem C yields an injective map

$$\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \hookrightarrow \prod_{V \in \pi_0\mathcal{T}} \mathscr{P}(\mathrm{Ob}\,\underline{\mathbb{F}}_{\mathcal{T}_{/V}}^{\mathrm{sprs}}),$$

where $\mathscr{P}(-)$ denotes the power set. By assumption, for all V, $\mathcal{T}_{/V}$ is finite, so $\underline{\mathbb{F}}_{\mathcal{T}_{/V}}^{\mathrm{sprs}}$ is finite, and hence $\mathscr{P}(\mathrm{Ob}\,\underline{\mathbb{F}}_{\mathcal{T}_{/V}}^{\mathrm{sprs}})$ is finite. Since $\pi_0\mathcal{T}$ is finite, this implies that the $\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}}$ injects into a finite poset, so it is finite. □

For instance, if G is finite, then there are finitely many subgroups of G, and hence finitely many transitive G-sets; this implies that $\pi_0\mathcal{O}_G$ is finite, and more generally, $\mathcal{O}_H$ is finite for all $H \subset G$. Hence Corollary 1.46 implies that there are finitely may $\otimes$-idempotent weak $\mathcal{N}_\infty$-G-operads.

Remark 1.47. Let $\mathcal{T} = \mathcal{O}_G$ for G a finite group. By Theorem C, one may devise an inefficient algorithm to compute $\mathrm{wIndex}_G^{\mathrm{uni}}$. Namely, given a sparse collection $\mathcal{C}^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_G^{\mathrm{sprs}}$, one may compute all of its self-indexed coproducts in finite time using the double coset formula in order to determine whether $\mathcal{C}^{\mathrm{sprs}}$ is closed under applicable self-indexed coproducts. One may simply iterate over the finite poset $\mathrm{Coll}(\underline{\mathbb{F}}_G^{\mathrm{sprs}})$, performing the above computation at each step to determine the unital weak indexing systems. ◁

The above algorithm is quite inefficient; in practice, we instead prefer to divide and conquer, first computing Fam_G and Transf_G, then computing the fibers under $\mathfrak{R}$ and ∇. When $N \in \mathbb{N} \cup \{\infty\}$, we will state the result of this for $G = C_{p^N} = \mathrm{colim}_{n\le N}\mathbb{Z}/p^n\mathbb{Z}$, but first we need notation. Given $R \in \mathrm{Transf}_G$, we define the families

$$\mathrm{Dom}(R) := \left\{ U \in \mathcal{O}_G \mid \ \exists U \to V \xrightarrow{f} W \text{ s.t. } f \in R - R^{\simeq} \right\};$$
$$\mathrm{Cod}(R) := \left\{ U \in \mathcal{O}_G \mid \ \exists V \xrightarrow{f} W \leftarrow U \text{ s.t. } f \in R - R^{\simeq} \right\}.$$

Given a full subcategory $\mathcal{F} \subset \mathcal{O}_G$ and a G-transfer system R, we denote by $\mathrm{Sieve}_R(\mathcal{F})$ the poset of precomposition-closed wide subcategories of $R \cap \mathcal{F}$. We let K_N be the Nth associahedron.

Corollary D. *Fix $N \in \mathbb{N} \cup \{\infty\}$. Then, there is a map of posets*

$$(\mathfrak{K}, \nabla) : \mathrm{wIndex}^{\mathrm{uni}}_{C_{p^N}} \to K_{N+2} \times [N+1]$$

with fibers satisfying

$$\mathfrak{K}^{-1}(R) \cap \nabla^{-1}(\mathcal{F}) = \begin{cases} \varnothing & \mathrm{Dom}(R) \not\leq \mathcal{F}; \\ * & \mathrm{Cod}(R) \leq \mathcal{F}; \\ \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) & \text{otherwise.} \end{cases}$$

Moreover, the associated surjection onto its image is a cocartesian fibration, with cocartesian transport computed along $R \le R'$ given by the map

$$\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \to \mathrm{Sieve}_{R'}(\mathrm{Cod}(R') - \mathcal{F})$$

sending $\mathfrak{S} \mapsto R^{\simeq} \cup \{J \subset K \subsetneq H \mid J \subset K \in R',\ K \subsetneq H \in \mathfrak{S}\}$ and cocartesian transport computed along $\mathcal{F} \le \mathcal{F}'$ by the restriction

$$\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \twoheadrightarrow \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}').$$

This completely determines $\mathrm{wIndex}^{\mathrm{uni}}_{C_{p^N}}$. Nevertheless, we draw this explicitly for $N \le 2$ in Section 3.

1.4. **Why (unital) weak indexing systems?** The author finds weak indexing systems compelling for two reasons:

(1) once the algebraist is convinced that they want finite H-sets to index their G-equivariant algebraic structures, weak indexing systems are forced upon them, and our various support properties classify useful properties of algebraic theories;
(2) $\mathbb{E}_V$-spaces and $\mathbb{E}_V$-ring spectra frequently appear in algebraic topology, sometimes for V a representation which has *zero-dimensional fixed points*, and hence the associated G-operad $\mathbb{E}_V$ has arities supported only on a (unital) *weak* indexing system.

Hopefully this paper and [Ste24] will demonstrate the first point handily; indeed, we will see in [Ste24] $\mathrm{wIndexCat}_{\mathcal{T}}$ occurs "in nature" as the poset of sub-terminal objects in the ∞-category $\mathrm{Op}_{\mathcal{T}}$ of $\mathcal{T}$-operads, and aE-unitality of I classifies the property of the weak *Eckmann-Hilton argument* that

$$\mathrm{CAlg}_I \underline{\mathrm{CAlg}}^{\otimes}_I(\mathcal{C}) \xrightarrow{U} \mathrm{CAlg}_I(\mathcal{C})$$

is an equivalence.

The author's favorite example behind the second point is the sign C_2-representation σ; as explained above, its arity-support (which is shared with $\infty\sigma$) is *not* an indexing system. Nevertheless, the evident conjectural extension of Dunn's additivity theorem [Dun88] in the equivariant setting would imply that $\mathbb{E}_\sigma^{\otimes\infty} \simeq \mathbb{E}_{\infty\sigma}$, so one should expect this structure to arise around constructions using $\mathbb{E}_\sigma$ structures (such as Real topological Hochschild homology [AGH21, § 3]).

Indeed, we will crucially utilize weak indexing systems in [Ste24] to show that whenever V is a real orthogonal C_2-representation containing an $\infty\sigma$-summand, there is an equivalence $\mathbb{E}_V \otimes \mathbb{E}_\sigma \simeq \mathbb{E}_V$; hence the forgetful functors are equivalences of ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_V} \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_\sigma}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{E}_V}(\mathcal{C}) \xleftarrow{\sim} \mathrm{Alg}^{\otimes}_{\mathbb{E}_\sigma} \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C}).$$

This allows one to take arbitrary *iterated THR* of $\mathbb{E}_V$ algebras without assuming V is a complete C_2-universe.

1.5. **Notation and conventions.** There is an equivalence of categories between that of posets and that of categories whose hom sets have at most one point; we safely conflate these notions. In doing so, we use categorical terminology to describe posets.

A *sub-poset* of a poset P is an injective monotone map $P' \hookrightarrow P$, i.e. a relation on a subset of the elements of P refining the relation on P. A *embedded sub-poset* (or *full sub-poset*) is a sub-poset $P' \hookrightarrow P$ such that $x \leq_{P'} y$ if and only if $x \leq_P y$ for all $x, y \in P'$.

An *adjunction of posets* (or *monotone Galois connection*) is a pair of opposing monotone maps $L : P \rightleftarrows Q : R$ satisfying the condition that

$$Lx \leq_Q y \qquad \iff \qquad x \leq_P Ry \quad \forall\, x \in P,\ y \in Q.$$

In this case, we refer to L as the *left adjoint* and R as the *right adjoint*, as L is uniquely determined by R and vice versa.

A *cocartesian fibration of posets* is a monotone map $\pi : P \to Q$ satisfying the condition that, for all pairs $q \leq q'$ and $p \in \pi^{-1}(q)$, there exists an element $t_q^{q'} p \in \pi^{-1}(q')$ characterized by the property

$$p \leq p' \qquad \iff \qquad t_q^{q'} p \leq p' \quad \forall\, p' \in \pi^{-1}(q');$$

in this case, we note that $t_q^{q'} : \pi^{-1}(q) \to \pi^{-1}(q')$ is a monotone map, and we may express P as the set $\coprod_{q \in Q} \pi^{-1}(q)$ with relation determined entirely by the above formula.

Acknowledgements. I would like to thank Clark Barkwick for numerous helpful conversations on this topic; for instance, his skepticism at an early (erroneous) sketch of the classification of weak $\mathcal{N}_\infty$-operads motivated me to take a careful look at the combinatorics of weak indexing systems, which grew into this work. I would be remiss to fail to mention that this project is closely linked with [Ste24], about which many illuminating conversations were had with Clark Barwick, Dhilan Lahoti, Mike Hopkins, Piotr Pstrągowski, Maxime Ramzi, and Andy Senger.

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2. Weak indexing systems

This section concerns non-enumerative aspects of the study of weak indexing systems and weak indexing categories. We begin in Section 2.1 by recognizing weak indexing categories as indexed collections of weak indexing categories of the slice categories of $\mathbb{F}_{\mathcal{T}}$ over orbits, allowing us to universally reduce structural statements about $\mathsf{wIndexCat}_{\mathcal{T}}$ to the case that $\mathcal{T}$ possesses a terminal object, so it is a 1-category. Using this, in Section 2.2, we prove Theorem A.

Following this, we dedicate some study to structural statements about $\mathsf{wIndex}_{\mathcal{T}}$, developing a litany of adjunctions and cocartesian fibrations involving it and its variants. We begin in Section 2.3 by developing the technology of *weak indexing system closures*, and using it to combinatorially characterize joins in the poset $\mathsf{wIndex}_{\mathcal{T}}$; as examples, we compute joins of the arity support $\underline{\mathbb{F}}^R$ of the little R-disks G-operad and characterize weak indexing system coinduction.

Next, in Section 2.4, we characterize the families c and v; the former is a fully faithful left and right adjoint (so we may reduce to the one-object case), and the latter has a fully faithful left adjoint, but interacts with joins in a complicated way. Following this, in Section 2.5, we characterize the map $\mathfrak{R} \colon \mathsf{wIndexCat}_{\mathcal{T}}^{\mathrm{uni}} \to \mathsf{Transf}_{\mathcal{T}}$ of Observation 1.39, showing it possesses fully faithful left and right adjoints, which seldom agree; we then characterize ∇, showing that it has fully faithful left and right adjoints. We additionally develop another family ϵ, and use it to characterize adjoins and join-compatibility of the various conditions of Definition 1.21.

Lastly, in Section 2.6, we take a detour and generalize the theory of *compatible pairs of indexing systems* to the setting of weak indexing systems, showing that the multiplicative hull of a weak indexing system exists and is an indexing system.

2.1. **Recovering weak indexing categories from their slice categories.** Recall that the poset of weak indexing categories $\mathsf{wIndexCat} \subset \mathsf{Sub}_{\mathsf{Cat}}(\mathbb{F}_{\mathcal{T}})$ is the embedded subposet spanned by those subcategories satisfying Conditions (IC-a) to (IC-c) of Theorem A; that is, they are pullback stable subcategories which are extended by coproducts from their maps to orbits and are full on cores.

We refer to $\mathcal{T}$-1-categories $\mathcal{C}$ whose V-values $\mathcal{C}_V$ are posets for all $V \in \mathcal{T}$ as $\mathcal{T}$*-posets*.

Construction 2.1. Given $\mathcal{C} \subset \mathbb{F}_{\mathcal{T}}$ a subcategory and $V \in \mathcal{T}$ an object, we write

$$\mathcal{C}_V := \left\{ f : \begin{tikzcd} S \arrow[rr, "\widetilde{f}"] \arrow[dr] & & T \arrow[dl] \\ & V & \end{tikzcd} \;\middle|\; \widetilde{f} \in \mathcal{C} \right\};$$

that is, maps in $\mathcal{C}_V$ are maps over V whose underlying map in $\mathbb{F}_{\mathcal{T}}$ lies in $\mathcal{C}$. For every map $V \to W$, this yields a map $(-)_V : \mathrm{Sub}_{\mathrm{Cat}_W}(\underline{\mathbb{F}}_W) \to \mathrm{Sub}_{\mathrm{Cat}_V}(\underline{\mathbb{F}}_V)$, compatibly with composition. We let $\underline{\mathrm{Sub}}_{\underline{\mathrm{Cat}}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}})$ be the resulting $\mathcal{T}$-poset ◁

Proposition 2.2. *If $I \subset \mathbb{F}_{\mathcal{T}}$ is a $\mathcal{T}$-weak indexing category, then $I_V \subset \mathbb{F}_V$ is a $\mathcal{T}_{/V}$-weak indexing category.*

Proof. Condition (IC-c) for I_V follows quickly by noting that automorphisms in I_V have underlying automorphisms in I, and Condition (IC-b) for I_V follows by unwinding definitions, noting that $\mathrm{Ind}_V^{\mathcal{T}} : \mathbb{F}_V \to \mathbb{F}_{\mathcal{T}}$ is coproduct-preserving. Lastly, Condition (IC-a) follows by unwinding definitions, noting that the pullback functor $\mathbb{F}_V \to \mathbb{F}_W$ is pullback-preserving for each $W \to V$. □

Thus the subcategories $\mathrm{wIndexCat}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}})$ yield an embedded $\mathcal{T}$-subposet

$$\underline{\mathrm{wIndexCat}}_{\mathcal{T}} \subset \underline{\mathrm{Sub}}_{\underline{\mathrm{Cat}}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}}),$$

Given a $\mathcal{T}$-poset $P : \mathcal{T}^{\mathrm{op}} \to \mathrm{Poset}$, we denote by $\Gamma^{\mathcal{T}} P$ the associated limit. There is a monotone map

$$\widetilde{\gamma} : \mathrm{Sub}_{\mathrm{Cat}}(\mathbb{F}_{\mathcal{T}}) \to \Gamma \underline{\mathrm{Sub}}_{\underline{\mathrm{Cat}}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}})$$

defined by $\widetilde{\gamma}(\mathcal{C})_V := \mathcal{C}_V$. We may use γ to recover $\mathrm{wIndexCat}_{\mathcal{T}}$ from $\underline{\mathrm{wIndexCat}}_{\mathcal{T}}$.

Proposition 2.3. *$\widetilde{\gamma}$ restricts to an equivalence*

$$\gamma : \mathrm{wIndexCat}_{\mathcal{T}} \xrightarrow{\sim} \Gamma \underline{\mathrm{wIndexCat}}_{\mathcal{T}}$$

Proof. Proposition 2.2 implies that $\widetilde{\gamma}$ restricts to a monotone map of posets $\gamma : \mathrm{wIndexCat}_{\mathcal{T}} \to \Gamma^{\mathcal{T}} \underline{\mathrm{wIndexCat}}_{\mathcal{T}}$, so it suffices to prove that this is bijective. If $\gamma I = \gamma J$, then for a map $f : T \to V$, the canonical $\mathcal{T}_{/V}$-map $! : T \to *_V$ lies I_V if and only if it lies in J_V, so f lies in I if and only if it lies in J; thus Condition (IC-b') implies that $I = J$, so γ is injective.

It remains to prove that γ is surjective, so we fix $I_\bullet \in \Gamma^{\mathcal{T}} \underline{\mathrm{wIndexCat}}_{\mathcal{T}}$. Define the subcategory

$$I := \{ T \to S \mid \forall U \in \mathrm{Orb}(S), \quad T \times_S U \to U \in I_U \} \subset \mathbb{F}_{\mathcal{T}}.$$

By definition, $\gamma I = I_\bullet$, so it suffices to verify that I is a weak indexing category. First note that I satisfies Condition (IC-b') by definition. Furthermore, since any automorphism of V is isomorphic to $*_V \in \mathbb{F}_V$, the subcategory I satisfies Condition (IC-c). Lastly, Condition (IC-a') is precisely the condition that $I_{(-)}$ is an element of $\underline{\mathrm{wIndexCat}}_{\mathcal{T}}$. Hence I is a $\mathcal{T}$-weak indexing system, proving that γ_W is an isomorphism of posets. □

Noting that spaces (as ∞-categories) have *contractible* slice categories, this implies the following.

Corollary 2.4. *If X is a space, then the forgetful map $\mathrm{wIndex}_X \to \mathrm{wIndex}_*$ is an equivalence.*

We would like to use this to uniformly replace $\mathcal{T}$ with a 1-category, for which we need the following.

Example 2.5. The atomic orbital ∞-category $\mathcal{T}_{/V}$ has a terminal object; by [NS22, Prop 2.5.1], this implies that $\mathcal{T}_{/V}$ is a 1-category. In general for $F : J \to \mathcal{T}$ a diagram in an atomic orbital ∞-category indexed by a finite 1-category, $\mathcal{T}_{/J}$ is also a 1-category; in particular, the top arrow

$$\begin{tikzcd} \mathcal{T}_{/J} \arrow[r] \arrow[dr] & \mathrm{ho}(\mathcal{T})_{/J} \arrow[d, phantom, "\wr"] \\ & \mathrm{ho}(\mathcal{T}_{/J}) \end{tikzcd}$$

is an equivalence. This implies that $\mathbb{F}_{\mathrm{ho}\mathcal{T}}$ has pullbacks, i.e. $\mathrm{ho}(\mathcal{T})$ is orbital; because $\mathcal{T}$ is atomic, retracts in $\mathrm{ho}(\mathcal{T})$ are isomorphisms, i.e. $\mathrm{ho}(\mathcal{T})$ is atomic orbital. ◁

Using this and fact that the 1-category of posets is a 1-category, we an equivalence

$$\begin{tikzcd} & \mathrm{Sub}(\mathbb{F}_{\mathcal{T}}) \arrow[r, "\mathrm{ho}"] & \mathrm{Sub}(\mathbb{F}_{\mathrm{ho}(\mathcal{T})}) \\ & \mathrm{wIndexCat}_{\mathcal{T}} \arrow[u, hook] \arrow[r, "\sim"] & \mathrm{wIndexCat}_{\mathrm{ho}(\mathcal{T})} \arrow[u, hook] \\ & \lim_{V \in \mathcal{T}^{\mathrm{op}}} \mathrm{wIndexCat}_{\mathcal{T}_{/V}} \arrow[u, phantom, "\wr"] \arrow[r, "\sim"] & \lim_{V \in \mathrm{ho}\mathcal{T}^{\mathrm{op}}} \mathrm{wIndexCat}_{\mathrm{ho}(\mathcal{T})_{/V}} \arrow[u, phantom, "\wr"] \end{tikzcd}$$

In other words, we've observed the following.

Corollary 2.6. *The homotopy category construction restricts to an equivalence* $\mathrm{wIndexCat}_{\mathcal{T}} \simeq \mathrm{wIndexCat}_{\mathrm{ho}(\mathcal{T})}$.

Using this, for the rest of the paper, we will assume that $\mathcal{T}$ is a 1-category.

2.2. Weak indexing categories vs weak indexing systems.

Construction 2.7. Given $I \subset \mathbb{F}_{\mathcal{T}}$ a subgraph, define the class of *I-admissible V-sets*

$$\mathbb{F}_{V,I} := \left\{ S \mid \mathrm{Ind}_V^{\mathcal{T}} S \to V \in I \right\} \subset \mathbb{F}_V.$$

Taken altogether, we refer to this as $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$. ◁

Recall the notation $I(-)$ used in Observation 1.34.

Observation 2.8. Given $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ a collection of objects, we have $\mathbb{F}_{V,I(\mathcal{C})} \simeq \mathcal{C}$; conversely, if $I \subset \mathbb{F}_{\mathcal{T}}$ satisfies Condition (IC-b), then $I(\underline{\mathbb{F}}_I) = I$. ◁

These are candidates for inverse maps $\mathrm{wIndex}_{\mathcal{T}} \rightleftarrows \mathrm{wIndexCat}_{\mathcal{T}}$, and they are well behaved:

Observation 2.9. If $S \simeq S'$ as V-sets, then there exists an equivalence $\mathrm{Ind}_V^{\mathcal{T}} S \simeq \mathrm{Ind}_V^{\mathcal{T}} S'$ over V. Hence whenever $I \subset \mathbb{F}_{\mathcal{T}}$ is a subcategory satisfying Condition (IC-c) and $S \in \underline{\mathbb{F}}_I$, the map $\mathrm{Ind}_V^{\mathcal{T}} S' \to V$ is in I, i.e. $\mathbb{F}_{V,I} \subset \mathbb{F}_V$ is closed under equivalence; these objects determine a unique full subcategory, which we henceforth refer to by the same name.

Conversely, if $\underline{\mathbb{F}}_I$ is a $\mathcal{T}$-weak indexing system and $\mathcal{T}$ has a terminal object $*_{\mathcal{T}}$, then the fact that $\mathbb{F}_{I,*_{\mathcal{T}}}$ contains all automorphisms immediately implies that $I(\underline{\mathbb{F}}_I)$ contains all automorphisms. ◁

Observation 2.10. By definition, the restriction functor $\mathrm{Res}_V^W : \mathbb{F}_W \to \mathbb{F}_V$ is implemented by the pullback

$$\begin{tikzcd} \mathrm{Ind}_V^{\mathcal{T}} \mathrm{Res}_V^W S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ind}_W^{\mathcal{T}} S \arrow[d] \\ V \arrow[r] & W \end{tikzcd}$$

thus I satisfies Condition (IC-a') if and only if $\mathrm{Res}_V^W \mathbb{F}_{W,I} \subset \mathbb{F}_{V,I}$ for all maps $V \to W$; in particular, in this case, $\{\mathbb{F}_{V,I}\}_{V \in \mathcal{T}}$ corresponds with a unique full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$. ◁

We are now ready to verify that $I(-)$ and $\underline{\mathbb{F}}_{(-)}$ restrict to maps between $\mathrm{wIndex}_{\mathcal{T}}$ and $\mathrm{wIndexCat}_{\mathcal{T}}$.

Proposition 2.11. *If $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a weak indexing system, then $I(\mathcal{C})$ is a weak indexing category.*

Proof. By Proposition 2.3, we may assume that $\mathcal{T}$ has a terminal object. By Observations 1.35 and 1.36, it suffices to verify Conditions (IC-a'), (IC-b') and (IC-c). Condition (IC-a') is verified by Observation 2.10; Condition (IC-b') follows immediately from construction; Condition (IC-c) is verified in Observation 2.9. □

Proposition 2.12. *If $I \subset \mathbb{F}_{\mathcal{T}}$ is a weak indexing category, then $\underline{\mathbb{F}}_I$ is a weak indexing system.*

Proof. Observations 2.9 and 2.10 verify that $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a full $\mathcal{T}$-subcategory, and the fact that the identity arrow on V corresponds with the contractible V-set implies that whenever $\underline{\mathbb{F}}_{I,V} \neq \varnothing$ (i.e. $V \in I$), $*_V \in \underline{\mathbb{F}}_{I,V}$. Thus it suffices to verify that $\underline{\mathbb{F}}_I$ is closed under self-indexed coproducts.

Let $(T_U) \in \mathbb{F}_{I,S}$ be an S-tuple of elements of $\underline{\mathbb{F}}_I$ for some $S \in \mathbb{F}_{I,V}$. Then, the indexed coproduct of (T_U) corresponds with the composite arrow

$$\mathrm{Ind}_V^{\mathcal{T}} \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^V T_U = \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_V^{\mathcal{T}} \mathrm{Ind}_U^V T_U = \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^{\mathcal{T}} T_U \to S \to V;$$

the left arrow is in I by Condition (IC-b) applied to the structure maps for each T_U and the right arrow is in I by assumption. Thus the composite is in I, i.e. $\coprod_U^S T_U \in \underline{\mathbb{F}}_I$, as desired. □

Having done this, we're poised to conclude that $I(-)$ and $\underline{\mathbb{F}}_{-}$ are inverse equivalences.

Proof of Theorem A. By Propositions 2.11 and 2.12, $I\colon \mathrm{wIndex}_{\mathcal{T}} \rightleftarrows \mathrm{wIndexCat}_{\mathcal{T}} \colon \underline{\mathbb{F}}_{(-)}$ are well defined monotone maps; by Observation 2.8, they are inverse to each other, so they are equivalences.

What remains is to verify that (IC-n) is equivalent to (IS-n) in Definition 1.21 and Theorem A. For $n = i$, this follows immediately by noting that $V \in I \iff \mathrm{id}_V \in I \iff *_V \in \mathbb{F}_{I,V} \iff \mathbb{F}_{I,V} \neq \varnothing$. For $n = ii$ and $n = iii$, this follows by unwinding definitions using Condition (IC-b'). For $n = iv$, this follows by noting that the fold map $n \cdot V \to V$ corresponds with the element $n \cdot *_V \in \mathbb{F}_V$. □

2.3. **Joins and coinduction.** We move on to intrinsic statements concerning $\mathrm{wIndex}_{\mathcal{T}}$.

2.3.1. *Prerequisites on adjunctions and cocartesian fibrations.* Recall that a monotone map $\pi\colon \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration (i.e. a Grothendieck opfibration) if and only if, for all related pairs $D \le D'$ in $\mathcal{D}$ and elements $C \in \pi^{-1}(D)$, there is an element $t_D^{D'} C \in \pi^{-1}(D')$ satisfying the property

$$C \le C' \qquad \iff \qquad t_D^{D'} C \le C' \quad \forall\, C' \text{ s.t. } D' \le \pi(C')$$

In this section, we relate these to adjunctions of posets (i.e. monotone Galois connections). We critically use the following.

Lemma 2.13. *Let $\pi : \mathcal{C} \to \mathcal{D}$ be a monotone map. The following are equivalent.*

(a) π possesses a fully faithful left adjoint L.
(b) For all $D \in \mathcal{D}$, the preimage $\pi^{-1}(\mathcal{D}_{\ge D})$ possesses an initial object $L(D)$ with $\pi L(D) = D$.
(c) For all $D \in \mathcal{D}$, the fiber $\pi^{-1}(D)$ has an initial object $L(D)$, and $D \le D'$ implies $L(D) \le L(D')$.

Furthermore, the element $L(D)$ agrees between these three constructions.

Proof. By definition, π has a left adjoint L if and only if there are initial objects in $\pi^{-1}(\mathcal{D}_{\le D})$, which are $L(D)$. By the usual category theoretic nonsense, L is fully faithful if and only if the unit relation $D \le \pi L(D)$ is an equality, i.e. $L(D) \in \pi^{-1}(D)$; hence (a) $\iff$ (b).

To see (b) $\iff$ (c), first note that

$$L(D) \le C' \quad \iff \quad D \le \pi(C') \quad \iff \quad L(D) \le L\pi(C');$$

if (b), then when $D = L(D) \le L\pi L(D') = D'$, we have $L(D) \le L(D')$, so (c). Conversely, if (c) and $L(D) \le C'$, then we have $D \le \pi(C')$, so D is initial in $\pi^{-1}(\mathcal{D}_{\le D})$, so (b). □

Proposition 2.14. *Suppose $\mathcal{C}$ has binary joins and $\pi\colon \mathcal{C} \to \mathcal{D}$ is a monotone map which is compatible with binary joins and possesses a fully faithful left adjoint L. Then, π is a cocartesian fibration with*

$$t_D^{D'} C = L(D') \vee C.$$

Proof. First note that

$$\pi(L(D') \vee C) = \pi L(D') \vee \pi(C) = D' \vee \pi(C) = D'.$$

Thus the property for cocartesian transport is given by

$$L(D') \vee C \le C' \iff L(D') \le C' \text{ and } C \le C';$$

indeed, when we restrict to the case $L(D') \le C'$ (i.e. $D' \le \pi(C')$), we then have $C \le C'$ if and only if $L(D') \vee C \le C'$, as desired. □

Remark 2.15. If π possesses a *right* adjoint R, then it is compatible with joins, as left adjoint functors are compatible with colimits.[7] The adjoint functor theorem for posets states the converse; indeed, if π is compatible with joins, then its right adjoint is computed by

$$R(Z) = \bigvee_{\pi(Y) \le Z} Y.$$

[7] We may see this directly in the binary case by noting that, for $X, Y \in \mathcal{C}$, the universal property for joins is satisfied by

$$\pi(X \vee Y) \le Z \iff X \vee Y \le R(Z) \iff X \le R(Z) \text{ and } Y \le R(Z) \iff \pi(X) \le Z \text{ and } \pi(Y) \le Z.$$

Thus Proposition 2.14 may be weakened to state that whenever π has a left and right adjoint and the left is fully faithful, π is a cocartesian fibration with transport computed as stated. In fact, the left adjoint is fully faithful if and only if the right adjoint is fully faithful [DT87, Lem 1.3], so we may stipulate that either (or both) are fully faithful.

This is manifestly self-dual; in this setting, the dual of Proposition 2.14 implies that π is a cartesian fibration with cartesian transport given by $t_D^{D'} C = R(D) \wedge C$. We will not use this explicitly in this text, but the author suggests that homotopical combinatorialists keep this trick in mind. ◃

2.3.2. *Closures and joins of weak indexing systems.* The following construction will be used often.

Construction 2.16. Given collections $\mathcal{D}, \mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$, inductively define $\mathrm{Cl}_{\mathcal{D},0}(\mathcal{C}) \coloneqq \mathcal{C}$ and

$$\mathrm{Cl}_{\mathcal{D},n}(\mathcal{C})_V = \left\{ \coprod_U^S T_U \mid (T_U) \in \mathrm{Cl}_{n-1}(\mathcal{C})_S,\ \ S \in \mathcal{D} \right\},$$

with $\mathrm{Cl}_{\mathcal{D},\infty}(\mathcal{C}) \coloneqq \bigcup_n \mathrm{Cl}_{\mathcal{D},n}(\mathcal{C})$. and $\mathrm{Cl}_n(\mathcal{C}) \coloneqq \mathrm{Cl}_{\mathcal{C},n}(\mathcal{C})$. We call this the *$n$-step closure of $\mathcal{C}$ under $\mathcal{D}$-indexed coproducts* or just the *closure of $\mathcal{C}$ under $\mathcal{D}$-indexed coproducts* when $n = \infty$. ◃

Observation 2.17. If $\mathcal{D}$ is a weak indexing system, then the canonical inclusion

$$\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C}) \subset \mathrm{Cl}_{\mathcal{D}}(\mathcal{C})$$

is an equality for all $\mathcal{C}$; indeed, as in the proof of Proposition 2.12, given $S \in \mathcal{D}_V$ and $(T_U) \in \mathcal{D}_S$, S-indexed coproducts of T_U-indexed coproducts are $\coprod_U^S T_U$-indexed coproducts, so this follows from Condition (IS-b). ◃

Observation 2.18. If $\mathcal{D}$ satisfies Condition (IS-a) and $c(\mathcal{D}) \supset c(\mathcal{C})$, then by taking $*_V$-indexed coproducts for all $V \in c(\mathcal{C})$, we find that $\mathcal{C} \subset \mathrm{Cl}_{\mathcal{D},1}(\mathcal{C})$. Similarly, if $\mathcal{C}$ satisfies Condition (IS-a) and $c(\mathcal{D}) \subset c(\mathcal{C})$, by taking indexed coproducts of $(*_U)$, we find that $\mathcal{C} \subset \mathrm{Cl}_{\mathcal{C},1}(\mathcal{D})$. Combining these, if $\mathcal{C}$ and $\mathcal{D}$ satisfy Condition (IS-a) and $c(\mathcal{C}) = c(\mathcal{D})$ (e.g. they each have one color), then we have

$$\mathcal{C}, \mathcal{D} \subset \mathrm{Cl}_{\mathcal{D},1}(\mathcal{C}).$$

Furthermore, note that $c(\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C})) = c(\mathcal{C})$ in this situation, so $\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C})$ satisfies Condition (IS-a). ◃

Let $\mathrm{FullSub}^*_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}}) \subset \mathrm{FullSub}_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$ denote the full subposet of elements satisfying Condition (IS-a).

Lemma 2.19. *The fully faithful map* $\iota\colon \mathrm{wIndex}_{\mathcal{T}} \hookrightarrow \mathrm{FullSub}^*_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$ *is right adjoint to* Cl_∞.

Proof. If $\mathrm{Cl}_\infty(\mathcal{C})$ is a weak indexing system, then it is clearly minimal among those containing $\mathcal{C}$, so it suffices to prove that it's a weak indexing system. By Observation 2.18, $\mathrm{Cl}_\infty(\mathcal{C})$ satisfies Condition (IS-a), so it suffices to verify Condition (IS-b).

In fact, by the argument used in Proposition 2.12, we find that $\mathrm{Cl}_i(\mathcal{C})$-indexed coproducts of elements of $\mathrm{Cl}_j(\mathcal{C})$ are $\mathrm{Cl}_{i+1}(\mathcal{C})$-indexed coproducts of elements of $\mathrm{Cl}_{j-1}(\mathcal{C})$; applying this j-many times, we find that $\mathrm{Cl}_i(\mathcal{C})$-indexed coproducts of elements in $\mathrm{Cl}_j(\mathcal{C})$ are in $\mathrm{Cl}_\infty(\mathcal{C})$, so taking a union, we find that $\mathrm{Cl}_\infty(\mathcal{C})$ satisfies Condition (IS-b). ☐

Define the rectified closure

$$\widehat{\mathrm{Cl}}_{\mathcal{C},1}(\mathcal{D}) = \mathrm{Cl}_{\mathcal{C} \cup \mathbb{F}^{\mathrm{triv}}_{c(\mathcal{D})},1}(\mathcal{D}) = \mathrm{Cl}_{\mathcal{C}}(\mathcal{D}) \cup \mathcal{D};$$

the equalities follow from Observation 2.18, so in particular, when $c(\mathcal{C}) \supset c(\mathcal{D})$, this agrees with $\mathrm{Cl}_{\mathcal{C},1}(\mathcal{D})$. We write the notoation $\widehat{\mathrm{Cl}}_I \coloneqq \widehat{\mathrm{Cl}}_{\underline{\mathbb{F}}_I}$, and we use this to characterize binary joins in $\mathrm{wIndex}_{\mathcal{T}}$.

Proposition 2.20. $\mathrm{wIndex}_{\mathcal{T}}$ *is a lattice; the meets in* $\mathrm{wIndex}_{\mathcal{T}}$ *are intersections, and the joins are*

$$\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = \bigcup_{n \in \mathbb{N}} \overbrace{\widehat{\mathrm{Cl}}_I \widehat{\mathrm{Cl}}_J \cdots \widehat{\mathrm{Cl}}_I \widehat{\mathrm{Cl}}_J}^{2n} (\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J).$$

Proof. By Lemma 2.19, $\mathrm{wIndex}_{\mathcal{T}}$ has meets computed in $\mathrm{FullSub}^*_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$, which are clearly given by intersections. Furthermore, Lemma 2.19 implies that $\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J)$. Thus is suffices to note that, for arbitrary $\mathcal{C},\mathcal{D},\mathcal{E}$, we have

$$\mathrm{Cl}_{\mathcal{C}\cup\mathcal{D},\infty}(\mathcal{E}) = \bigcup_{n\in\mathbb{N}} \overbrace{\mathrm{Cl}_{\mathcal{C}\cup\mathbb{F}^{\mathrm{triv}}_{c(\mathcal{D})}} \mathrm{Cl}_{\mathcal{D}\cup\mathbb{F}^{\mathrm{triv}}_{c(\mathcal{C})}} \cdots \mathrm{Cl}_{\mathcal{C}\cup\mathbb{F}^{\mathrm{triv}}_{c(\mathcal{D})}} \mathrm{Cl}_{\mathcal{D}\cup\mathbb{F}^{\mathrm{triv}}_{c(\mathcal{C})}}}^{2n}(\mathcal{E}),$$

and set $\mathcal{C} = \underline{\mathbb{F}}_I$, $\mathcal{D} = \underline{\mathbb{F}}_J$, and $\mathcal{E} = \underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J$. □

Remark 2.21. In fact, Lemma 2.19 constructs arbitrary meets in $\mathrm{wIndex}_{\mathcal{T}}$. Furthermore, chains in $\mathrm{wIndex}_{\mathcal{T}}$ have joins computed by unions; hence $\mathrm{wIndex}_{\mathcal{T}}$ is a complete lattice. ◁

Observation 2.22. Similarly, if $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a collection, then the full $\mathcal{T}$-subcategory $\widehat{\mathcal{C}}$ defined by

$$\widehat{\mathcal{C}}_V = \begin{cases} \{*_V\} \cup \bigcup_{V\to W} \mathrm{Res}_V^W \mathcal{C}_W & \mathcal{C}_V \neq \varnothing, \\ \varnothing & \mathcal{C}_V = \varnothing \end{cases}$$

is initial among full $\mathcal{T}$-subcategories containing $\mathcal{C}$ and satisfying Condition (IS-a) Combining adjunctions, we find that the fully faithful map $\iota : \mathrm{wIndex}_{\mathcal{T}} \hookrightarrow \mathrm{Coll}(\underline{\mathbb{F}}_{\mathcal{T}})$ possesses a left adjoint $\mathrm{Cl}_\infty(\widehat{-})$, which we write simply as $\mathrm{Cl}_\infty(-)$ for brevity. ◁

Given $S \in \mathbb{F}_V$, let $\mathbb{F}_{I_S,V}$ be the closure of $\{*_V\}$ under S-indexed coproducts; more generally, let $\mathbb{F}_{I_S,W} := \bigcup_{f\colon W\to V} \mathrm{Res}_W^V \mathbb{F}_{I_S,V}$, and let $\left(\underline{\mathbb{F}}_{I_S}\right)_W := \mathbb{F}_{I_S,W}$.

Proposition 2.23. *Given $S \in \mathbb{F}_V$, we have $\mathrm{Cl}_\infty(\{S\}) = \underline{\mathbb{F}}_{I_S}$.*

Proof. First, note that $\underline{\mathbb{F}}_{I_S} \subset \mathrm{Cl}_\infty(\{S\})$. By Lemma 2.19, it suffices to prove that $\underline{\mathbb{F}}_{I_S}$ is weak indexing system containing S. By construction, $\underline{\mathbb{F}}_{I_S} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a full $\mathcal{T}$-subcategory satisfying the property that

$$*_W \in \mathbb{F}_{I_S,W} \qquad \Longleftrightarrow \qquad \exists f\colon W \to V \qquad \Longleftrightarrow \qquad \varnothing \neq \mathbb{F}_{I_S,W}.$$

i.e. it satisfies Condition (IS-a). Hence it suffices to prove that $\underline{\mathbb{F}}_{I_S}$ is closed under self-indexed coproducts.

First, note that if $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is closed under T-indexed coproducts and X_U-indexed coproducts for $(X_U) \in \mathbb{F}_T$, then $\mathcal{C}$ is closed under $\coprod_U^T X_U$-indexed coproducts, as they are T-indexed coproducts of X_U-indexed coproducts; hence $\underline{\mathbb{F}}_{I_S}$ is closed under $\mathbb{F}_{I_S,V}$-indexed coproducts.

Second, note that if $\mathcal{C}_W$ is generated under restrictions by $\mathcal{C}_U$ and $\mathcal{C}_U$ is closed under T-indexed coproducts, then $\mathcal{C}_W$ is closed under $\mathrm{Res}_W^U T$-indexed coproducts, as they are restrictions of T-indexed coproducts; hence $\underline{\mathbb{F}}_{I_S}$ is closed under self-indexed coproducts, as desired. □

2.3.3. *Joins and $\mathbb{F}^R$.* Let G be a finite group and R a real orthogonal G-representation. Recall from Example 1.29 that there is a weak indexing system $\underline{\mathbb{F}}^R$ satisfying

$$\mathbb{F}_H^R = \{S \in \mathbb{F}_H \mid \exists\, H\text{-equivariant embedding } S \hookrightarrow R\}.$$

Observation 2.24. If $S \in \mathbb{F}_V^R$ and R is a subrepresentation of R', then the composite embedding $S \hookrightarrow R \hookrightarrow R'$ witnesses the membership $S \in \mathbb{F}_V^{R'}$; that is, $\underline{\mathbb{F}}^{(-)}$ is *monotone* under inclusions of subrepresentations. ◁

In particular, monotonicity yields relations $\underline{\mathbb{F}}^R, \underline{\mathbb{F}}^{R'} \subset \underline{\mathbb{F}}^{R\oplus R'}$, and hence a relation $\underline{\mathbb{F}}^R \vee \underline{\mathbb{F}}^{R'} \subset \underline{\mathbb{F}}^{R\oplus R'}$. We verify that this relation is an equality in the following argument; throughout the argument when $x \in T$ is an element of an H-set, we will write $[x]_T$ for its orbit under the H-action.

Proposition 2.25. *For R, R' real orthogonal G-representations, we have $\underline{\mathbb{F}}^R \vee \underline{\mathbb{F}}^{R'} = \underline{\mathbb{F}}^{R\oplus R'}$.*

Proof. By the above argument, it suffices to verify the relation $\underline{\mathbb{F}}^{R\oplus R'} \subset \underline{\mathbb{F}}^R \vee \underline{\mathbb{F}}^R$. Let $S \in \mathbb{F}_H^{R\oplus R'}$ be a finite H-set embedding into $R \oplus R'$. The associated map $S \to R \oplus R' \to R$ possesses an image factorization

$$\begin{tikzcd} S \arrow[r, hook, "\iota"] \arrow[d, two heads, "\psi"] & R \oplus R' \arrow[d, two heads, "\pi"] \\ S_R \arrow[r, hook, "\mathrm{im}(\pi\iota)"'] & R \end{tikzcd}$$

Given $x \in S_R$, note that there is an isomorphism

$$\psi^{-1}[x]_{S_R} \simeq \mathrm{Ind}_{\mathrm{stab}_H(x)}^H \psi^{-1}(x),$$

where the $\mathrm{stab}_H(x)$ action on $\psi^{-1}(x)$ is restricted from the H-action on S. Furthermore, note that the fiber of $R \oplus R'$ over $\iota(x)$ is invariant under the $\mathrm{stab}_H(x)$ action, and the resulting $\mathrm{stab}_H(x)$-space taken isomorphically onto $R' \simeq R \oplus \{0\}$ by $(-) + \iota(x)$; thus $\psi^{-1}(x)$ admits a $\mathrm{stab}_H(x)$-equivariant embedding into R'.

To summarize, we may make a choice of an element x_{K_i} in each orbit $[H/K_i] \subset S_R$ and apply the above argument to conclude that $S_R \in \mathbb{F}_H^R$, that $\psi^{-1}(x_{K_i}) \in \mathbb{F}_{K_i}^{R'}$, and that

$$S = \coprod_{[H/K_i] \in \mathrm{Orb}(S_R)} \psi^{-1}([H/K_i]) = \coprod_{[H/K_i] \in \mathrm{Orb}(S_R)} \mathrm{Ind}_{\mathrm{stab}_H(x)}^H \psi^{-1}(x) = \coprod_{K_i}^{S_R} \psi^{-1}(x_{K_i}).$$

In particular, this shows that

$$\underline{\mathbb{F}}^{R \oplus R'} \subset \mathrm{Cl}_{\underline{\mathbb{F}}_R}(\underline{\mathbb{F}}_{R'}) \subset \underline{\mathbb{F}}^R \vee \underline{\mathbb{F}}^{R'},$$

proving the proposition. □

2.3.4. *Coinduction.* If it exists, the right adjoint to $\mathrm{Res}_V^W : \mathrm{wIndex}_W \to \mathrm{wIndex}_V$ is denoted CoInd_V^W.

Proposition 2.26. *Let $\underline{\mathbb{F}}_I$ be a weak indexing system. Then, $\mathrm{CoInd}_V^W \underline{\mathbb{F}}_I$ exists and is computed by*

$$\left(\mathrm{CoInd}_V^W \underline{\mathbb{F}}_I\right)_U = \left\{ S \in \mathbb{F}_U \;\middle|\; \forall\ W \leftarrow U \leftarrow U' \to V,\ \ \mathrm{Res}_{U'}^U S \in \mathbb{F}_{I,U'} \right\}$$

Proof. Denote by $\mathcal{C}$ the right hand side of the above equation. Note that $\mathcal{C} \subset \underline{\mathbb{F}}_W$ is the maximum full $\mathcal{T}$-subcategory such that $\mathrm{Res}_V^W \mathcal{C} \le \underline{\mathbb{F}}_I$. Indeed, if $S \in \mathbb{F}_U - \mathcal{C}_U$, then for some $U' \to V$, we have $\mathrm{Res}_{U'}^U S \notin \mathbb{F}_{I,U'}$; thus whenever $\underline{\mathbb{F}}_J \not\le \mathrm{Res}_V^W \mathcal{C}$, we have $\underline{\mathbb{F}}_J \not\le \underline{\mathbb{F}}_I$. Hence it suffices to prove that $\mathcal{C}$ is a weak indexing system.

First, suppose that $S \in \mathcal{C}_U$; then, $\mathrm{Res}_{U'}^U S \in \mathbb{F}_{I,U}$ for all $U' \to V$, so $*_{U'} = \mathrm{Res}_{U'}^U *_U \in \mathbb{F}_{I_U}$ for all $U' \to V$. Hence $*_U \in \mathcal{C}_U$, i.e. $\mathcal{C}$ satisfies Condition (IS-a). Now, fix $(T_X) \in \mathcal{C}_S$ an S-tuple. What remains is to verify that for all $U' \to V$,

$$\mathrm{Res}_{U'}^U \coprod_X^S T_X \simeq \coprod_{X'}^{\mathrm{Res}_{U'}^U S} \mathrm{Res}_{X'}^{o(X')} T_{o(X')} \in \mathbb{F}_{I,U'},$$

the equivalence coming from Remark 1.37. But by assumption, we have $\mathrm{Res}_{U'}^U S$, $\mathrm{Res}_{X'}^{o(X')} T_{o(X')} \in \underline{\mathbb{F}}_I$, so this is in $\underline{\mathbb{F}}_I$ by Condition (IS-b), as desired. □

We will use this in [Ste24] to see that $\mathrm{CoInd}_V^W A\mathcal{O} = A\mathrm{CoInd}_V^W \mathcal{O}$ for all $\mathcal{T}$-operads $\mathcal{O}^{\otimes}$.

2.4. **The color and unit fibrations.** Recall the maps c, υ, and ∇ of Proposition 1.24 and $\aleph$ of Observation 1.39. In this subsection, we study c and υ, for which we start at the following observation.

Observation 2.27. By definition, we find that c, υ, ∇, and $\aleph$ are compatible with joins, in the sense that for each $F \in \{c, \upsilon, \nabla, \aleph\}$, and set of collections $(C_\alpha)_{\alpha \in A}$ we have an equality

$$\bigcup_{\alpha \in A} F(C_\alpha) = F\left(\bigcup_{\alpha \in A} C_\alpha\right). \qquad \triangleleft$$

Much of the following work concerns *joins* and these maps, beginning with c.

2.4.1. *The color-support fibration.* We will reduce the analysis of $\mathrm{wIndex}_{\mathcal{T}}$ to the one-color case.

Proposition 2.28. *The monotone map $c\colon \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ has a fully faithful left adjoint $\underline{\mathbb{F}}_{(-)}^{\mathrm{triv}}$ and a fully faithful right adjoint $\underline{\mathbb{F}}_{(-)}$.*

Proof. By Lemma 2.13 it suffices to note that $\underline{\mathbb{F}}_{c(\underline{\mathbb{F}}_I)}^{\mathrm{triv}} \le \underline{\mathbb{F}}_I \le \underline{\mathbb{F}}_{c(\underline{\mathbb{F}}_I)}$ for all $\mathcal{F}$, and that $\underline{\mathbb{F}}_{\mathcal{F}}^{\mathrm{triv}} \le \underline{\mathbb{F}}_{\mathcal{F}'}^{\mathrm{triv}}$ and $\underline{\mathbb{F}}_{\mathcal{F}} \le \underline{\mathbb{F}}_{\mathcal{F}'}$ whenever $\mathcal{F} \le \mathcal{F}'$. □

The following proposition additionally follows by unwinding definitions.

Proposition 2.29. *The fiber $c^{-1}(\mathrm{Fam}_{\mathcal{T}, \le \mathcal{F}})$ is equivalent to $\mathrm{wIndex}_{\mathcal{F}}$, and the associated fully faithful functor $E_{\mathcal{F}}^{\mathcal{T}}\colon \mathrm{wIndex}_{\mathcal{F}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ is left adjoint to $\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}}(-) \coloneqq (-) \cap \underline{\mathbb{F}}_{\mathcal{F}}$ and has values given by*

$$E_{\mathcal{F}}^{\mathcal{T}} \mathcal{C}_V = \begin{cases} \mathcal{C}_V & V \in \mathcal{F}; \\ \varnothing & \text{otherwise}. \end{cases}$$

In particular, the fiber $c^{-1}(\{\mathcal{F}\}))$ *is the image of* $E_{\mathcal{F}}^{\mathcal{T}}\colon \mathrm{wIndex}_{\mathcal{F}}^{\mathrm{oc}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$.

Finally, in order to understand cocartesian transport, we make the following observation.

Observation 2.30. Since $\mathbb{F}_{\mathcal{F},V}^{\mathrm{triv}}$ is $*_V$ when $V \in \mathcal{F}$ and empty otherwise, a finite V-set X is a $\underline{\mathbb{F}}_{\mathcal{F}}^{\mathrm{triv}}$-indexed coproduct of elements in $\mathcal{C}$ if and only if $V \in \mathcal{F}$ and $X \in \mathcal{C}_V$. In other words, we have

$$\mathrm{Cl}_{I_{\mathcal{F}}^{\mathrm{triv}}}(\mathcal{C}) = \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}}(\mathcal{C}).$$

In fact, extending this logic, if $\mathrm{Bor}_{c(I)}^{\mathcal{T}}\mathcal{C}$ is closed under I-indexed coprocts, then we have $\mathrm{Cl}_I(\mathcal{C}) = \mathrm{Bor}_{c(I)}^{\mathcal{T}}\mathcal{C}$; hence $\widehat{\mathrm{Cl}}_I(\mathcal{C}) = \mathcal{C}$. In particular, applying Proposition 2.20, we find that

$$\underline{\mathbb{F}}_{\mathcal{F}}^{\mathrm{triv}} \vee \underline{\mathbb{F}}_I = \underline{\mathbb{F}}_{\mathcal{F}}^{\mathrm{triv}} \cup \underline{\mathbb{F}}_I. \quad \triangleleft$$

Thus, applying Remark 2.15, Propositions 2.28 and 2.29, and Observation 2.30, we arrive at the following.

Corollary 2.31. *Let* $\mathcal{T}$ *be an orbital* ∞*-category.*

1. *The map* $c\colon \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration with fiber* $c^{-1}(\mathcal{F}) = \mathrm{wIndex}_{\mathcal{F}}^{oc}$ *and with cocartesian transport along* $\mathcal{F} \le \mathcal{F}'$ *sending* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_{\mathcal{F}'}^{\mathrm{triv}} \vee E_{\mathcal{F}}^{\mathcal{F}'}\underline{\mathbb{F}}_I$.
2. *The map* $c\colon \mathrm{wIndex}_{\mathcal{T}}^{E\mathrm{uni}} \to \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration with fiber* $c^{-1}(\mathcal{F}) = \mathrm{wIndex}_{\mathcal{F}}^{\mathrm{uni}}$ *and cocartesian transport along* $\mathcal{F} \le \mathcal{F}'$ *sending* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_{\mathcal{F}'}^{\mathrm{triv}} \vee E_{\mathcal{F}}^{\mathcal{F}'}\underline{\mathbb{F}}_I$.
3. *The map* $c\colon \mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \to \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration with fiber* $c^{-1}(\mathcal{F}) = \mathrm{wIndex}_{\mathcal{F}}^{a\mathrm{uni}}$ *and cocartesian transport along* $\mathcal{F} \le \mathcal{F}'$ *sending* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_{\mathcal{F}'}^{\mathrm{triv}} \vee E_{\mathcal{F}}^{\mathcal{F}'}\underline{\mathbb{F}}_I$.

Remark 2.32. Entailed in this corollary are the statements that $\underline{\mathbb{F}}_I$ is E-unital if and only if $\underline{\mathbb{F}}_I = E_{c(I)}^{\mathcal{T}}\mathrm{Bor}_{c(I)}^{\mathcal{T}}\underline{\mathbb{F}}_I$ and $\mathrm{Bor}_{c(I)}^{\mathcal{T}}\underline{\mathbb{F}}_I$ is unital; in particular, we find that the E-unital weak indexing systems are those which come about by applying $E_{(-)}^{\mathcal{T}}$ to unital weak indexing systems. $\triangleleft$

2.4.2. *The unit fibration.* We study the map v using the following.

Proposition 2.33. *The map* $v : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ *has fully faithful left adjoint given by* $E_{-}^{\mathcal{T}}\underline{\mathbb{F}}_{(-)}^0$.

Proof. In view of Lemma 2.13, we're tasked with proving that $E_{\mathcal{F}}^{\mathcal{T}}\underline{\mathbb{F}}_{\mathcal{F}}^0 \in v^{-1}(\mathrm{Fam}_{\mathcal{T},\ge\mathcal{F}})$ is initial, and $v\left(\underline{\mathbb{F}}_{\mathcal{F}}^0\right) = \mathcal{F}$, both of which follow by unwinding definitions. □

Once again, we would like to simplify our expression for cocartesian transport.

Observation 2.34. Let $V \in \mathcal{F}$. Note that a V-set is an S-indexed coproduct of elements of $E_{\mathcal{F}}^{\mathcal{T}}\underline{\mathbb{F}}_{\mathcal{F}}^0$ if and only if it is a summand of S; in particular, if $\underline{\mathbb{F}}_I$ is closed under *nonempty* summands, then $\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_{c(I)}^0 = \mathrm{Cl}_I(\underline{\mathbb{F}}_{c(I)}^0)$. Thus, in this case we have

$$\underline{\mathbb{F}}_I \vee E_{\mathcal{F}}^{\mathcal{T}}\underline{\mathbb{F}}_{cF}^0 = \cdots \widehat{\mathrm{Cl}}_{\underline{\mathbb{F}}_I}\widehat{\mathrm{Cl}}_{E_{\mathcal{F}}^{\mathcal{T}}\underline{\mathbb{F}}_{\mathcal{F}}^0}(\underline{\mathbb{F}}_I \cup E_{\mathcal{F}}^{\mathcal{T}}\underline{\mathbb{F}}_{\mathcal{F}}^0) = \underline{\mathbb{F}}_I \cup E_{\mathcal{F}}^{\mathcal{T}}\underline{\mathbb{F}}_{\mathcal{F}}^0.$$

In particular, if $\underline{\mathbb{F}}_I$ is aE-unital, then it is closed under nonempty summands, so this applies. $\triangleleft$

We may use this to reduce enumerative problems from the almost unital setting (or the aE-unital setting in view of Corollary 2.31) to the unital setting.

Proposition 2.35. *The restricted map* $v_a\colon \mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \to \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration with fiber* $v_a^{-1}(\mathcal{F}) = \mathrm{wIndex}_{\mathcal{F}}^{\mathrm{uni}}$ *embedded along* $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}} \cup E_{\mathcal{F}}^{\mathcal{T}}(-)$. *Moreover, the cocartesian transport map* $t_{\mathcal{F}}^{\mathcal{F}'}\colon \mathrm{wIndex}_{\mathcal{F}}^{\mathrm{uni}} \to \mathrm{wIndex}_{\mathcal{F}'}^{\mathrm{uni}}$ *is implemented by*

$$t_{\mathcal{F}}^{\mathcal{F}'}\underline{\mathbb{F}}_I = \underline{\mathbb{F}}_{\mathcal{F}'}^0 \cup E_{\mathcal{F}}^{\mathcal{F}'}\underline{\mathbb{F}}_I$$

Proof. The property $v_a^{-1}(\mathcal{F}) = \mathrm{wIndex}_{\mathcal{F}}^{\mathrm{uni}}$ follows by unwinding definitions using Lemma 1.25. For the remaining property, we're tasked with proving that $\underline{\mathbb{F}}_{\mathcal{F}'}^0 \cup E_{\mathcal{F}}^{\mathcal{F}'}\underline{\mathbb{F}}_I \in \mathrm{wIndex}_{\mathcal{F}'}^{\mathrm{uni}}$ is the initial unital $\mathcal{F}'$-weak indexing system which embeds $\underline{\mathbb{F}}_I$ after each are embedded into $\mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}}$ along $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}} \cup E_{-}^{\mathcal{T}}(-)$. Unwinding definitions, this universal property is satisfied of $\underline{\mathbb{F}}_{\mathcal{F}'}^0 \vee E_{\mathcal{F}}^{\mathcal{F}'}\underline{\mathbb{F}}_I$; thus the proposition follows from Observation 2.34. □

The fibers of the unrestricted map υ have terminal objects, which are sometimes useful counterexamples.

Proposition 2.36. *Given $\mathcal{F} \in \mathrm{Fam}_{\mathcal{T}}$, the fiber $\upsilon^{-1}(\mathcal{F})$ has a terminal object computed by*

$$\mathbb{F}_{\mathcal{F}^{\perp}-nu,V} = \begin{cases} \mathbb{F}_V & V \in \mathcal{F}; \\ \mathbb{F}_V - \{S \mid \forall\, U \in \mathrm{Orb}(S), U \in \mathcal{F}\} & V \notin \mathcal{F} \end{cases}$$

Proof. We begin by noting that $\underline{\mathbb{F}}_{\mathcal{F}^{\perp}-nu}$ contains all $\mathcal{T}$-weak indexing systems with unit family $\mathcal{F}$; indeed for contradiction, if $\underline{\mathbb{F}}_J$ satisfies $\upsilon(J) = \mathcal{F}$ and there is some $S \in \mathbb{F}_{J,V} - \mathbb{F}_{\mathcal{F}^{\perp}-nu,V}$, then we must have $U \in \mathcal{F} \subset \upsilon(J)$ for all $U \in \mathrm{Orb}(S)$ and $V \notin \mathcal{F}$, so

$$\coprod_U^S \varnothing_U = \varnothing_V \in \mathbb{F}_{J,V},$$

implying that $\upsilon(J) < \mathcal{F}$ (which contradicts our assumption). Thus it suffices to verify that $\underline{\mathbb{F}}_{\mathcal{F}^{\perp}-nu}$ is a $\mathcal{T}$-weak indexing system. Since it contains all contractible V-sets, it suffices to prove that it's closed under self-indexed coproducts.

Fix some $S \in \mathbb{F}_{\mathcal{F}^{\perp}-nu,V}$ and $(T_U) \in \mathbb{F}_{\mathcal{F}^{\perp}-nu,S}$. If $V \in \mathcal{F}$, then there is nothing to prove, so suppose $V \notin \mathcal{F}$. Then, note that

$$\mathrm{Orb}\left(\coprod_U^S T_U\right) = \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Orb}(T_U).$$

S must contain some orbit U outside of $\mathcal{F}$, and by assumption, T_U contains an orbit outside of $\mathcal{F}$; thus $\coprod_U^S T_U$ contains an orbit outside of $\mathcal{F}$, i.e. $\coprod_U^S T_U \in \underline{\mathbb{F}}_{\mathcal{F}^{\perp}-nu}$, as desired. □

Warning 2.37. υ does not admit a right adjoint, as it is not even compatible with binary joins; for instance, if $\mathcal{T} = \mathcal{O}_G$, then note that the weak indexing system $\underline{\mathbb{F}}_{\varnothing^{\perp}-nu}$ consists of all nonempty H-sets, and $E_{BG}^G \underline{\mathbb{F}}_{BG}^0$ contains only the e-sets $\{\varnothing_e, *_e\}$. Nevertheless, the join $\underline{\mathbb{F}}_{\varnothing^{\perp}-nu,V} \vee E_{BG}^G \underline{\mathbb{F}}_{BG}^0$ contains the inductions $\mathrm{Ind}_e^H \varnothing_e = \varnothing_H$, so it is equal to the complete indexing system $\underline{\mathbb{F}}_G$. Thus when G is nontrivial, we have a proper family inclusion

$$\upsilon(\underline{\mathbb{F}}_{\varnothing^{\perp}-nu}) \cup \upsilon(E_{BG}^G \underline{\mathbb{F}}_{BG}^0) = BG \subsetneq \mathcal{O}_G = \upsilon(\underline{\mathbb{F}}_{\varnothing^{\perp}-nu} \vee E_{BG}^G \underline{\mathbb{F}}_{BG}^0). \quad \triangleleft$$

Remark 2.38. Despite Warning 2.37, υ is *lax*-compatible with joins, in the sense that there is a relation

$$\upsilon(I) \cup \upsilon(J) \le \upsilon(I \vee J);$$

this follows by simply noting that $I \vee J$ contains I and J. In particular, by Proposition 1.24, we find that joins of unital weak indexing systems are unital. ◁

Observation 2.39. Despite Warning 2.37, υ is compatible with joins *on aE-unital weak indexing systems*; indeed, if $\underline{\mathbb{F}}_I$ is aE-unital, then we have

$$\underline{\mathbb{F}}_I = E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(I)}^{\mathrm{triv}} \cup E_{\upsilon(I)}^{\mathcal{T}} \mathrm{Bor}_{\upsilon(I)}^{\mathcal{T}} \underline{\mathbb{F}}_I,$$

so that

$$\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(I)}^{triv} \cup E_{c(J)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(J)}^{\mathrm{triv}} \cup E_{\upsilon(I)\cup\upsilon(J)}^{\mathcal{T}} \mathrm{Bor}_{\upsilon(I)\cup\upsilon(J)}^{\mathcal{T}} \left(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J\right).$$

Thus we have

$$\upsilon(I) \cup \upsilon(J) \le \upsilon(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J) = \upsilon\left(\mathrm{Bor}_{\upsilon(I)\cup\upsilon(J)}^{\mathcal{T}}\left(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J\right)\right) \le \upsilon(I) \cup \upsilon(J). \quad \triangleleft$$

2.5. **The transfer system and fold map fibrations.** We further reduce our classification using $\aleph$ and ∇.

2.5.1. *The transfer system fibration.* Recall that the monotone map $\mathfrak{R}\colon \mathrm{wIndexCat}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}}$ is defined by $\mathfrak{R}(I) = I \cap \mathcal{T}$; we denote the composite $\mathrm{wIndex}_{\mathcal{T}} \simeq \mathrm{wIndexCat}_{\mathcal{T}} \to \mathrm{Transf}_{\mathcal{T}}$ as $\mathfrak{R}$ as well. Given R a transfer system, define the weak indexing system

$$\overline{\underline{\mathbb{F}}}_R \coloneqq \underline{\mathbb{F}}_{\mathcal{T}}^0 \vee \mathrm{Cl}_\infty\left(\left\{\mathrm{Res}_V^W U \mid U \to W \in R,\ \ V \to W \in \mathcal{T}\right\}\right)$$

Our main statements about $\mathfrak{R}$ will be the following proposition and its immediate corollary

Proposition 2.40. *The map of posets $\mathfrak{R}\colon \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}}$ has fully faithful right adjoint given by the composite $\mathrm{Transf}_{\mathcal{T}} \simeq \mathrm{Index}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ and fully faithful left adjoint given by $\overline{\underline{\mathbb{F}}}_{(-)}$.*

Corollary 2.41. *If I, J are unital weak indexing categories, then*

$$\mathfrak{R}(I) \vee \mathfrak{R}(J) = \mathfrak{R}(I \vee J) \qquad \text{and} \qquad \mathfrak{R}(I) \cap \mathfrak{R}(J) = \mathfrak{R}(I \cap J).$$

We begin with an easy technical lemma concerning closures and transfer systems.

Lemma 2.42. $\mathfrak{R}\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C}) = \mathfrak{R}\mathrm{Cl}_{\mathfrak{R}(\mathcal{D}),1}(\mathfrak{R}\mathcal{C})$.

Proof. Since $\mathfrak{R}\mathrm{Cl}_{\mathfrak{R}(\mathcal{D}),1}(\mathfrak{R}\mathcal{C}) \subset \mathfrak{R}\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C})$, it suffices to prove the opposite inclusion; indeed, whenever $\coprod_U^S T_U \in \mathrm{Cl}_{\mathcal{D},1}(\mathcal{C})$ is an orbit, there is exactly one T_U which is nonempty, in which case $\mathrm{Ind}_U^V T_U = \coprod_U^S T_U$, implying that T_U is an orbit, so that $\coprod_U^S T_U \in \mathfrak{R}\mathrm{Cl}_{\mathfrak{R}(\mathcal{D}),1}(\mathfrak{R}\mathcal{C})$. □

We use this to give compatibility of $\mathfrak{R}$ with joins in a restricted setting.

Lemma 2.43. *If I, J unital satisfy $\mathfrak{R}(I) \le \mathfrak{R}(J)$, then $\mathfrak{R}(I \vee J) = \mathfrak{R}(J)$.*

Proof. Note that $\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J$ is closed under I-indexed induction, so we have

$$\mathfrak{R}\mathrm{Cl}_{\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J,1}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) = \mathfrak{R}\mathrm{Cl}_{\mathfrak{R}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J),1}(\mathfrak{R}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J)) = \mathfrak{R}\mathrm{Cl}_{\mathfrak{R}(J),1}(\mathfrak{R}(J)) = \mathfrak{R}(J).$$

Iterating this and taking a union, we find that

$$\mathfrak{R}(I \vee J) = \mathfrak{R}\mathrm{Cl}_{\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J,\infty}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) = \mathfrak{R}(J). \qquad □$$

We additionally note the following.

Lemma 2.44. *$\overline{\underline{\mathbb{F}}}_R$ is initial in $\mathfrak{R}^{-1}(\mathrm{Transf}_{\mathcal{T},\ge R})$ and $\mathfrak{R}\overline{\underline{\mathbb{F}}}_R = R$.*

Proof. The only nontrivial part is showing that $\mathfrak{R}\overline{\underline{\mathbb{F}}}_R = R$; in fact, this follows by unwinding definitions and applying Lemma 2.42. □

Proof of Proposition 2.40. The left adjoint is Lemma 2.44, so we're left with proving that we've constructed the right adjoint. By Lemma 2.43, the indexing category $I_{\mathcal{T}}^\infty \vee I$ satisfies $\mathfrak{R}(I_{\mathcal{T}}^\infty \vee I) = \mathfrak{R}(I)$ and is an upper bound for I. In fact, by Proposition 1.40, $I_{\mathcal{F}}^\infty \vee I$ is the *unique* indexing system with $\mathfrak{R}(I \vee I_{\mathcal{F}}^\infty) = I$, and so it is an upper bound for all J with $\mathfrak{R}(I) = \mathfrak{R}(J)$. In fact, if $\mathfrak{R}(I) \ge \mathfrak{R}(J)$, then $J \le J \vee I \le I_{\mathcal{F}}^\infty \vee I$ by the same argument, so $I_{\mathcal{F}}^\infty \vee I$ satisfies the conditions of Lemma 2.13, as desired. □

Remark 2.45. If $\mathcal{T}$ is an atomic orbital ∞-category with a terminal object V, then $2 \cdot *_V$ is not in $\overline{\underline{\mathbb{F}}}_R$ for any R, since $2 \cdot *_V$ is not a summand in the restriction of any orbital W-sets for any $W \in \mathcal{T}$; indeed, since $\mathcal{T}$ is atomic, there are no non-isomorphisms $V \to W$, so this would require that $2 \cdot *_V$ is an orbit, but it is not. Hence $\overline{\underline{\mathbb{F}}}_R$ is not an indexing system, or equivalently, $\mathfrak{R}^{-1}(R)$ has multiple elements. We may interpret this as saying that unital weak indexing systems are seldom determined by their transitive V-sets. ◁

2.5.2. *The fold map fibration.* Our first statement about ∇ will be the following.

Proposition 2.46. *For all unital weak indexing systems $\underline{\mathbb{F}}_I$ and $\underline{\mathbb{F}}_J$, we have $\nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J) = \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J)$.*

To prove this, we work through the formula in Proposition 2.20 one step at a time.

Lemma 2.47. *Suppose $\underline{\mathbb{F}}_I$ is unital. If $\nabla(\underline{\mathbb{F}}_I), \nabla(\mathcal{C}) \le \mathcal{F}'$, then $\nabla(\mathrm{Cl}_{\underline{\mathbb{F}}_I,1}(\mathcal{C})) \le \mathcal{F}'$.*

Proof. Suppose $V \in \nabla(\mathrm{Cl}_{\underline{\mathbb{F}}_I,1}(\mathcal{C}))$, i.e. there exists some $S \in \mathbb{F}_{I,V}$, some $(X_U) \in \mathcal{C}_S$, and some $n \geq 2$ such that $\coprod_U^S X_U = n \cdot *_V$. We would like to prove that $V \in \mathcal{F}'$. Since $\underline{\mathbb{F}}_I$ is unital, writing $S = S_{ne} \sqcup S_{\varnothing}$ for $S_{\varnothing}$ the disjoint union of S-orbits over which X_U is empty, we have $S_{ne} \in \mathbb{F}_{I,V}$ and

$$\coprod_U^S X_U = \coprod_U^{S_{ne}} X_U;$$

hence we may replace S with S_{ne} and assume that X_U is nonempty for all U.

Note that, for all $U \in \mathrm{Orb}(S)$, we have $\mathrm{Ind}_U^V X_U = m \cdot *_V$ for some $m \geq 1$; in particular, this implies $U = V$. Hence $S = k \cdot *_V$ for some $k \geq 1$. Writing our decomposition as $S = \{1, \dots, k\}$ and $X_i = m_i *_V$, we find that $n = \sum_{i=1}^k m_i \geq 2$, so either $m_i \geq 2$ for some i or $k \geq 2$. In either case, we find $V \in \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\mathcal{C}) \subset \mathcal{F}'$, as desired. □

Observation 2.48. For any nonempty set of collections $(\mathcal{C}_i)_{i \in I}$, it follows by unwinding definitions that we have $\nabla\left(\bigcup_{i \in I} \mathcal{C}_i\right) = \bigcup_{i \in I} \nabla(\mathcal{C}_i)$. ◁

We use this to compute the fold family of a join of unital weak indexing systems.

Proof of Proposition 2.46. By Observation 2.48, we have $\nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J) = \nabla(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) \leq \nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J)$, so we are tasked with proving the opposite inclusion. By Lemma 2.47, we find inductively that $\nabla \mathrm{Cl}_{\underline{\mathbb{F}}_I,1} \mathrm{Cl}_{\underline{\mathbb{F}}_J,1} \cdots \mathrm{Cl}_{\underline{\mathbb{F}}_J,1}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) \leq \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J)$; applying Observation 2.48 to take a union, we find that $\nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J) \leq \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J)$, as desired. □

Now we're ready to use this to show that ∇ is a cocartesian fibration.

Proposition 2.49. *The restricted map $\nabla_u \colon \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Fam}_{\mathcal{T}}$ has fully faithful left adjoint given by $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{-}^{\mathcal{T}} \underline{\mathbb{F}}_{(-)}^{\infty}$ and a fully faithful right adjoint; hence it is a cocartesian fibration, and the cocartesian transport map $t_{\mathcal{F}}^{\mathcal{F}'}$ is implemented by*

$$t_{\mathcal{F}}^{\mathcal{F}'} \underline{\mathbb{F}}_I \simeq \underline{\mathbb{F}}_I \vee E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^{\infty}$$

Proof. First note that Observation 2.48 and Proposition 2.49 together imply that $\nabla(-)$ is compatible with *arbitrary* joins; since $\mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}}$ has arbitrary joins, the adjoint functor theorem recalled in Remark 2.15 implies that $\nabla(-)$ has a right adjoint. In light of Remark 2.15, it thus suffices to prove that the monotone map $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{(-)}^{\mathcal{T}} \underline{\mathbb{F}}_{(-)}^{\infty}$ is a fully faithful left adjoint to ∇_u, or equivalently by Lemma 2.13, that $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^{\infty}$ is an initial element of $\nabla_u^{-1}(\mathcal{F})$.

First note that it follows from Lemma 1.25 and Observation 2.34 that $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^{\infty}$ is a weak indexing system; additionally, it follows from Proposition 2.46 that $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cap E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^{\infty} \in \nabla_u^{-1}(\mathcal{F})$, i.e. it's unital and has fold family $\mathcal{F}$. Lastly, it follows from Lemma 1.25 that every unital $\mathcal{T}$-weak indexing system with fold family $\mathcal{F}$ contains $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^{\infty}$, as desired. □

Remark 2.50. The author is not aware of an informative formula for the right adjoint to ∇_u, but there are interesting examples; for instance, if λ is a nontrivial irreducible real orthogonal C_p-representation, then we show in Section 3.2 that $\underline{\mathbb{F}}^{\lambda}$ is terminal among the C_p-weak indexing systems with fold maps over the trivial subgroup. In algebra, this may be interpreted as saying that $\mathbb{E}_{\lambda\infty}$ presents the terminal sub-C_p-commutative algebraic theory prescribing a multiplication on the underlying Borel type of a genuine C_p-object, but not on genuine C_p-fixed points. ◁

We would like to compute examples with many transfers and few folds.

Observation 2.51. Given $V \to W$ a map in $\mathcal{T}$, write $\underline{\mathbb{F}}_{I_{\mathrm{Ind}_V^W *_V}}$ for the weak indexing system of Proposition 2.23. In view of Observation 2.34, we may compute the associated fold family as

$$\nabla\left(\mathbb{F}_{\mathcal{T}}^0 \vee \underline{\mathbb{F}}_{I_U}\right) = \left\{U \in \mathcal{T} \;\middle|\; \exists U \to W \text{ s.t. } 2 \cdot *_U \subset \mathrm{Res}_U^W \mathrm{Ind}_V^W *_V\right\},$$

Furthermore, if R is a transfer system, then Propositions 2.23 and 2.40 yield an equality

$$\overline{\underline{\mathbb{F}}}_R = \underline{\mathbb{F}}_{\mathcal{T}}^0 \vee \bigvee_{V \to W \in R} \underline{\mathbb{F}}_{I_{\mathrm{Ind}_V^W *_V}} = \bigvee_{V \to W \in R} \underline{\mathbb{F}}_{\mathcal{T}}^0 \vee \underline{\mathbb{F}}_{I_{\mathrm{Ind}_V^W *_V}};$$

thus Proposition 2.46 yields

$$\begin{aligned}\nabla\underline{\mathbb{F}}_R &= \bigcup_{V\to W\in R} \nabla\left(\underline{\mathbb{F}}^0_{\mathcal{T}} \vee \underline{\mathbb{F}}_{I_{\mathrm{Ind}^{\mathcal{T}}_W V}}\right)\\ &= \left\{U \in \mathcal{T} \mid\ \exists U \to W \xleftarrow{f} V \text{ s.t. } f\in R \text{ and } 2\cdot *_U \subset \mathrm{Res}^W_U \mathrm{Ind}^W_V *_V\right\}.\end{aligned}$$

We write $\mathrm{Dom}(R) := \nabla\underline{\mathbb{F}}_R$ for the above expression. ◁

We may simplify this in a number of equivariant examples.

Remark 2.52. If $\mathcal{T} = \mathcal{F} \subset \mathcal{O}_G$ is a family of normal subgroups of a finite group (e.g. any family of subgroups of a finite Dedekind group), then for every pair of proper subgroup inclusions $H,K \subset J$, the double coset formula implies that $\mathrm{Res}^J_K \mathrm{Ind}^J_H *_H = |K\backslash J/H| \cdot [H/H\cap K]$. In particular, $2*_H \subset \mathrm{Res}^J_K \mathrm{Ind}^J_H *_H$ if and only if $H \subset K$.

Unwinding definitions, we find in this case that $\mathrm{Dom}(R)$ is the family

$$\mathrm{Dom}(R) = \left\{K \in \mathcal{F} \;\middle|\; \exists K \to H \xrightarrow{f} G,\ f\in R,\ H\neq G\right\},$$

where we conflate $[G/K]$ with K; that is, it is the family generated by domains of nontrivial transfers in R. ◁

2.5.3. *The essence fibration.* Given $\underline{\mathbb{F}}_I$ a weak indexing system, define the *essence family*

$$\epsilon(I) := \{V \in \mathcal{T} \mid \mathbb{F}_{I,V} - \{*_V\} \neq \varnothing\}$$

so that $\underline{\mathbb{F}}_I$ is aE-unital if and only if $\epsilon(I) = \upsilon(I)$. This behaves similarly to c and ∇.

Lemma 2.53. *$\epsilon(I)$ is a $\mathcal{T}$-family.*

Proof. This follows by noting that the restriction of a non-contractible $\mathcal{T}$-sets remain non-contractible. □

Lemma 2.54. *If $\epsilon(\mathcal{C}) \subset \epsilon(\mathcal{D})$, then*

$$\epsilon\left(\widehat{\mathrm{Cl}}_{\mathcal{C},1}(\mathcal{D})\right) = \epsilon(D).$$

Proof. Fix some non-contractible V-set $T \in \mathrm{Cl}_{\mathcal{C},1}(\mathcal{D})$, and express it as an S-indexed colimit

$$T = \coprod_U^S T_U$$

For $S \in \mathcal{C}_V$ and $(T_U) \in \mathcal{D}_S$. Since T is non-contractible, either S is non-contractible or T_U is non-contractible; either way, this implies that $V \in \epsilon(D)$, so $\epsilon\left(\widehat{\mathrm{Cl}}_{\mathcal{C},1}(\mathcal{D})\right) \subset \epsilon(D)$. The opposite inclusion follows by the fact $D \subset \widehat{\mathrm{Cl}}_{\mathcal{C},1}(\mathcal{D})$. □

Observation 2.55. For all A-indexed diagrams in $\mathrm{wIndex}_{\mathcal{T}}$, we have $\epsilon\left(\bigcup_{\alpha\in A}\underline{\mathbb{F}}_{I_\alpha}\right) = \bigcup_{\alpha\in A}\epsilon\left(\underline{\mathbb{F}}_{I_\alpha}\right)$. ◁

Proposition 2.56. *ϵ is compatible with arbitrary joins.*

Proof. ϵ is clearly compatible with unions; hence it suffices to prove that it's compatible with binary joins. In fact, we may inductively prove using Lemma 2.54 that

$$\epsilon(\overbrace{\widehat{\mathrm{Cl}}_I\widehat{\mathrm{Cl}}_J\cdots\widehat{\mathrm{Cl}}_I\widehat{\mathrm{Cl}}_J}^{2n}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J)) = \epsilon(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) = \epsilon(\underline{\mathbb{F}}_I) \cup \epsilon(\underline{\mathbb{F}}_J);$$

taking a union as $n \to \infty$ yields the desired statement. □

We're finally ready to round up localizations to our various conditions.

Proposition 2.57. *Let $\mathcal{T}$ be an orbital ∞-category.*

(1) The inclusion $\mathrm{wIndex}^{a\mathrm{Euni}}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ is right adjoint to $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}^0_{\epsilon(I)}$.
(2) The inclusion $\mathrm{wIndex}^{E\mathrm{uni}}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ is right adjoint to $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}^0_{c(I)}$.
(3) The inclusion $\mathrm{wIndex}^{\mathrm{oc}}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ is right adjoint to $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{T}}$.
(4) The inclusion $\mathrm{wIndex}^{a\mathrm{uni}}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ is right adjoint to $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^0_{\epsilon(\underline{\mathbb{F}}_I)}$.

(5) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ *is right adjoint to* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_{\mathcal{T}}^0$.
(6) The inclusion $\mathrm{Index}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ *is right adjoint to* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_{\mathcal{T}}^{\infty}$.

Furthermore, each inclusion is additionally compatible with joins.

Proof. We begin with compatibility of each condition with joins. First, note by Propositions 2.28 and 2.56 that the maps $c, \epsilon : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ are compatible with joins, by Remark 2.38 the map υ is lax-compatible with joins, and by Proposition 2.46, ∇ is compatible with joins of unital weak indexing systems. This implies that the conditions that $c(I) = \mathcal{T}$, that $\upsilon(I) = c(I)$, that $\upsilon(I) = \mathcal{T}$, and that $\nabla(I) \cap \upsilon(I) = \mathcal{T}$ are all compatible with joins, so we are left with proving that aE-unital weak indexing systems are closed under joins. But this follows by noting whenever I, J are aE-unital that

$$\epsilon(I \vee J) = \epsilon(I) \cup \epsilon(U) = \upsilon(I) \cup \upsilon(J) = \upsilon(I \vee J)$$

in view of Observation 2.39. Thus we are left with constructing left adjoints.

We begin by proving (1). By Lemma 2.13, we are tasked with verifying that $\underline{\mathbb{F}}_I \vee E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{\epsilon(I)}^0$ is initial among aE-unital weak indexing systems $\mathcal{C}$ satisfying the property that $\underline{\mathbb{F}}_I \leq \mathcal{C}$. In fact, if $\underline{\mathbb{F}}_I \leq \underline{\mathbb{F}}_J$ and $\underline{\mathbb{F}}_J$ is aE-unital, then $\epsilon(I) \leq \epsilon(J) = \upsilon(J)$ and $c(I) \leq c(J)$, so we have $E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(I)}^{\mathrm{triv}}, E_{\epsilon(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{\epsilon(I)}^0 \leq \underline{\mathbb{F}}_J$. Taking a join, this implies that

$$\underline{\mathbb{F}}_I \vee E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{\epsilon(I)}^0 = \underline{\mathbb{F}}_I \vee E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(I)}^{\mathrm{triv}} \vee E_{\epsilon(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{\epsilon(I)}^0 \leq \underline{\mathbb{F}}_J.$$

Thus we're left with verifying that $\underline{\mathbb{F}}_I \vee E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{\upsilon(I)}^0$ is aE-unital; in fact, we have

$$\upsilon(\underline{\mathbb{F}}_I \vee E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{\epsilon(I)}^0) \geq \upsilon\left(E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{\epsilon(I)}^0\right) = \epsilon(I),$$

and by Proposition 2.56 we have

$$\epsilon(\underline{\mathbb{F}}_I \vee E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{\epsilon(I)^0}) = \epsilon(I).$$

Together these imply that $\epsilon(\underline{\mathbb{F}}_I \vee E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{\epsilon(I)}^0) \geq \upsilon(\underline{\mathbb{F}}_I \vee E_{c(I)}^{\mathcal{F}} \underline{\mathbb{F}}_{\epsilon(I)}^0)$, so it is aE-unital, proving (1).

The proof of (2) is analogous, instead concluding the relation $\upsilon(\underline{\mathbb{F}}_I \vee E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(I)}^0) = c(\underline{\mathbb{F}}_I \vee E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(I)}^0)$ by the same argument, replacing Proposition 2.56 with Proposition 2.28. The proof of (3) is easier, as we only need to use Proposition 2.28 to verify that $c(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}}) = \mathcal{T}$ Similarly, the proof of (6) uses Proposition 2.46 and Remark 2.38 to verify that $\mathcal{T} \geq \nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_{\mathcal{T}}^{\infty}) \cap \upsilon(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_{\mathcal{T}}^{\infty}) \geq \mathcal{T}$. (4) follows by combining (1) and (3), and (5) follows by combining (1) and (2). $\square$

2.5.4. *The combined transfer-fold fibration.* We now combine ∇ and $\mathfrak{R}$.

Observation 2.58. By Lemma 2.44 and Observation 2.51, if $\mathrm{Dom}(R) \not\subset \mathcal{F}$, then $\mathfrak{R}^{-1}(R) \cap \nabla^{-1}(\mathcal{F})$ is empty. In fact, by Proposition 2.46 and Observation 2.51 we find that $\overline{\underline{\mathbb{F}}}_R \vee \underline{\mathbb{F}}_{\mathcal{F}}^{\infty} \in \mathcal{F}^{-1}(R) \cap \nabla^{-1}(\mathcal{F} \cup \mathrm{Dom}(R))$ is *initial*; in particular the condition $\mathrm{Dom}(R) \subset \mathcal{F}$ is necessary and sufficient for $\mathfrak{R}^{-1}(R) \cap \nabla^{-1}(\mathcal{F})$ to be nonempty. Furthermore, this is functorial in R and $\mathcal{F}$, since $\overline{\underline{\mathbb{F}}}_R \leq \overline{\underline{\mathbb{F}}}_{R'}$ and $\underline{\mathbb{F}}_{\mathcal{F}}^{\infty} \leq \underline{\mathbb{F}}_{\mathcal{F}'}^{\infty}$ whenever $R \leq R'$ and $\mathcal{F} \leq \mathcal{F}'$. $\triangleleft$

Define the embedded subposet $(\mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}})^{\mathrm{admsbl}} \subset \mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}}$ spanned by the pairs $(R, \mathcal{F})$ such that $\mathrm{Dom}(R) \subset \mathcal{F}$. Note that $(\mathfrak{R}, \nabla)$ is compatible with joins by Propositions 2.40 and 2.46, and joins of admissible pairs are admissible; in light of Lemma 2.13, we may rephrase this together with Observation 2.58 as follows.

Proposition 2.59. *The map* $(\mathfrak{R}, \nabla) : \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}}$ *has image* $(\mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}})^{\mathrm{admsbl}}$ *and factors as the following diagram of join-preserving maps*

$$\begin{tikzcd} \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \arrow[d, "(\mathfrak{R},\nabla)"'] \arrow[dr, "(\mathfrak{R},\nabla)"] & \\ (\mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}})^{\mathrm{admsbl}} \arrow[r, hook] & \mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}} \end{tikzcd}$$

where the lefthand vertical map admits a fully faithful left adjoint computed by $(R, \mathcal{F}) \mapsto \overline{\underline{\mathbb{F}}}_R \vee \underline{\mathbb{F}}_{\mathcal{F}}^{\infty}$. *Thus the left vertical map is a cocartesian fibration with cocartesian transport computed by*

$$t_{(R,\mathcal{F})}^{(R',\mathcal{F}')} \underline{\mathbb{F}}_I = \underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'} \vee \underline{\mathbb{F}}_{\mathcal{F}'}^{\infty}.$$

2.6. **Compatible pairs of weak indexing systems.** We finish the section with a discussion of *compatible pairs of weak indexing systems*, generalizing the setting of [BH22].

Definition 2.60. A pair of one-object weak indexing categories (I_a, I_m) is *compatible* if $\underline{\mathbb{F}}_{I_a} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is closed under I_m-indexed products, i.e. $\underline{\mathbb{F}}_{I_a} \subset \underline{\mathbb{F}}_{\mathcal{T}}^{I_m-\times}$ is an I_m-symmetric monoidal full subcategory. ◁

We'd like to compare these with the notions from [CHLL24], beginning with the following.

Observation 2.61. $\mathbb{F}_{\mathcal{T}}$ is *extensive* in the sense of [CHLL24, Def 2.2.1]. Furthermore, a subcategory $I \subset \mathbb{F}_{\mathcal{T}}$ furnishes a *span pair* $(\mathbb{F}_{c(I)}, I)$ if and only if it satisfies Conditions (IC-a) and (IC-c); thus a span pair $(\mathbb{F}_{c(I)}, I)$ is *weakly extensive* in the sense of [CHLL24, Def 2.2.1] if and only if I is a weak indexing category. Furthermore, by Lemma 1.25, a weakly extensive pair $(\mathbb{F}_{c(I)}, I)$ is *extensive* if and only if I is an indexing category. ◁

They have their own notion of compatibility, which generalizes ours.

Observation 2.62. A bispan triple $(\mathbb{F}_{\mathcal{T}}, I_m, I_a)$ whose span pairs are weakly extensive is called a *semiring context* in [CHLL24, Def 4.1.1] when the right adjoint $f_*\colon \mathbb{F}_{\mathcal{T},/X} \to \mathbb{F}_{\mathcal{T},/Y}$ to pullback along a map $f : X \to Y$ in I_m preserves morphisms whose image in $\mathbb{F}_{\mathcal{T}}$ lies in I_a; unwinding definitions, this is precisely the condition that (I_a, I_m) is a compatible pair of one-object weak indexing systems. ◁

In particular, when (I_a, I_m) is a compatible pair of indexing systems in the sense of [BH22], the triple $(\mathbb{F}_{\mathcal{T}}, I_m, I_a)$ is a semiring context in the sense of [CHLL24]. This is useful, as [CHLL24, Thm 4.2.4] yields an operadic presentation for the associated theory of *bi-incomplete Tambara functors* valued in cocomplete cartesian closed ∞-categories.

Our main contribution to this is to concretely characterize the terminal (weak) indexing system $m(I)$ such that $(I, m(I))$ is a compatible pair, generalizing [BH22, Cor 6.19].

Proposition 2.63 (Multiplicative hull)**.** *Given $\underline{\mathbb{F}}_I$ a one-object weak indexing system, the subcategories*

$$\mathbb{F}_{m(I),V} := \{S \in \mathbb{F}_V \mid \underline{\mathbb{F}}_I \text{ closed under } S\text{-indexed products}\}$$

form an indexing system characterized by the property that, for all $I_m \in \mathrm{wIndex}_{\mathcal{T}}$, the pair (I, I_m) is compatible if and only if $I_m \leq m(I)$.

Proof. It follows directly from construction that $I_m \leq m(I)$ if and only if (I, I_m) is compatible. Furthermore, the $*_V$-indexed product functor is the identity, so $*_V \in \mathbb{F}_{m(I),V}$ for all V. Hence it suffices to prove that $\varnothing_V \in \mathbb{F}_{m(I),V}$ for all $V \in \mathcal{T}$ and that $\underline{\mathbb{F}}_{m(I)}$ is closed under binary coproducts and self-induction.

For the first statement, empty products are terminal objects (i.e. $*_V$), so $\varnothing_V \in \mathbb{F}_{m(I),V}$ for all $V \in \mathcal{T}$. For binary coproduts, note that $T \sqcup T'$-indexed products simply binary products of T- and T'-indexed products, so it suffices to prove that $\mathbb{F}_{I,V}$ is closed under binary products. Indeed, by distributivity of finite products and coproducts, we have

$$S \times S' = \coprod_{U \in \mathrm{Orb}(S)} U \times S' = \coprod_U^S \mathrm{Res}_U^V S',$$

which is in $\mathbb{F}_{I,V}$ by closure under restrictions and self-indexed coproducts. For self-induction, note that

$$\begin{aligned}
\prod_U^{\mathrm{Ind}_W^V S} T_U &= \prod_{U \in \mathrm{Orb}(\mathrm{Ind}_W^V S)} \mathrm{CoInd}_U^V T_U \\
&= \prod_{U \in \mathrm{Orb}(S)} \mathrm{CoInd}_W^V \mathrm{CoInd}_U^W T_U \\
&= \mathrm{CoInd}_W^V \prod_{U \in \mathrm{Orb}(S)} \mathrm{CoInd}_U^W T_U \\
&= \mathrm{CoInd}_W^V \prod_U^S T_U;
\end{aligned}$$

if S and $\mathrm{Ind}_W^V *_W$ are in $\underline{\mathbb{F}}_{m(I)}$, then this implies that $\prod_U^{\mathrm{Ind}_W^V S} T_U \in \mathbb{F}_{I,V}$ whenever $(T_U) \in \mathbb{F}_{I,\mathrm{Ind}_W^V S}$, so $\mathrm{Ind}_W^V S \in \mathbb{F}_{m(I),V}$. In other words, $\underline{\mathbb{F}}_{m(I)}$ is closed under self-indexed coproducts, as desired. □

3. Enumerative results

Having developed the main beats of the theory of (unital) weak indexing systems in Section 2, we now turn to enumerating weak indexing systems under a number of unitality assumptions. In Section 3.1, we prove Theorem C; we use this in Section 3.2 to draw a Hasse diagram for $\mathrm{wIndex}_{C_p}^{a\mathcal{E}\mathrm{uni}}$. Finally, in Section 3.3, we prove Corollary D and draw a Hasse diagram for $\mathrm{wIndex}_{C_{p^2}}^{\mathrm{uni}}$.

3.1. Sparsely indexed coproducts.

The following is the heart of our enumerative efforts.

Proposition 3.1. *If $\mathcal{T}$ is an atomic orbital ∞-category and $\underline{\mathbb{F}}_I$ is an aE-unital $\mathcal{T}$-weak indexing system, then*

$$\underline{\mathbb{F}}_I = \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\mathrm{sprs}})$$

In order to show this, given S a V-set, we let $\mathrm{Istrp}(S) \coloneqq \{U \in \mathcal{T}_{/V} \mid \exists \text{ summand inclusion } U \hookrightarrow S\} \subset \mathcal{T}_{/V}$ be the *isotropy of* S. Let $\overline{\mathrm{Istrp}}(S) \subset \mathrm{Istrp}(S)$ be the full subcategory spanned by orbits which are either V or which are *maximal* among non-V elements. We may make a (noncanonical) choice $f_{(-)} \colon \mathrm{Orb}(S) \to \mathrm{Ar}(\mathrm{Istrp})$ such that for all $U \in \mathrm{Orb}(S)$, the codomain of $f_U \colon U \to e(U)$ is in $\overline{\mathrm{Istrp}}(S)$ is not V unless $U = V$. Then, for all $W \in \overline{\mathrm{Istrp}}(S)$, define

$$\overline{S} \coloneqq \coprod_{W \in \overline{\mathrm{Istrp}}(S)} \mathrm{Ind}_W^V *_W; \qquad\qquad S_{(\overline{W})} \coloneqq \coprod_{e(U)=W} \mathrm{Ind}_U^W *_U.$$

We think of $S_{(\overline{W})}$ as the locus of points in S of orbit type distinguished by W.

Observation 3.2. For all $S \in \mathbb{F}_{I,V}$, $\overline{S}$ is a summand in S; if $\overline{S} = S$, the obviously we have $\overline{S} \in \mathbb{F}_{I,V}$. Furthermore, if $\overline{S} \subsetneq S$, S must have at least two orbits, so it is non-contractible. This implies that $V \in v(S)$, so $\overline{S} \in \mathbb{F}_{I,V}$. ◁

In particular, we find that

$$S = \coprod_W^{\overline{S}} S_{(\overline{W})}, \tag{3}$$

so that S is a sparsely I-indexed coproduct of $\left(S_{(\overline{W})}\right)$. We'd like to identify each $S_{(\overline{W})}$ as an element of $\mathbb{F}_{I,W}$, for which we write $S^V \coloneqq S_{(\overline{V})}$ for the set of *V-fixed points of* S. Our trick is to use the following lemma.

Lemma 3.3. *If $\mathcal{T}$ is an atomic orbital ∞-category, the U-set $\mathrm{Res}_U^V \mathrm{Ind}_U^V *_U$ has a fixed point.*

Proof. We have a diagram

$$\begin{tikzcd}
U \arrow[dr, dashed] \arrow[drr, bend left, equals] \arrow[ddr, bend right, equals] & & \\
& \mathrm{Ind}_U^{\mathcal{T}} \mathrm{Res}_U^V \mathrm{Ind}_U^V *_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d] \\
& U \arrow[r] & V
\end{tikzcd}$$

Taking slices over U, the lefthand triangle establishes $*_U$ as a retract of $\mathrm{Res}_U^V \mathrm{Ind}_U^V *_U$, i.e. it is a retract of an orbital summand $*_U \rightleftarrows S \subset \mathrm{Res}_U^V \mathrm{Ind}_U^V *_U$. By the atomic assumption, this establishes $*_U = S$, as desired. □

Observation 3.4. By Lemma 3.3, there is a summand inclusion

$$\begin{tikzcd}
S_{(\overline{W})} \arrow[rr, hook] \arrow[d, phantom, "\wr"] & & \mathrm{Res}_W^V S \arrow[d, phantom, "\wr"] \\
\coprod_{e(U)=W} \mathrm{Ind}_U^W *_U \arrow[r, hook] & \coprod_{e(U)=W} \mathrm{Res}_W^V \mathrm{Ind}_W^V \mathrm{Ind}_U^W *_U & \sqcup \coprod_{W' \in \overline{\mathrm{Istrp}}(S) - \{W\}} \mathrm{Ind}_{W'}^V S_{(\overline{W})}
\end{tikzcd}$$

so $S_{(\overline{W})} \in \mathbb{F}_{I,W}$. In particular, we find that whenever $\left|\overline{\mathrm{Istrp}(S)}\right| = 1$, the I-admissible V-set S is an I-indexed induction of elements of $\underline{\mathbb{F}}_I \cap \underline{\mathbb{F}}_{\mathcal{T}}^\infty$. ◁

We're now ready to prove that aE-unital weak indexing systems are generated by their sparse collections.

Proof of Proposition 3.1. First note that, since $n \cdot *_V \simeq *_V \sqcup (n-1) \cdot *_V$, the usual an inductive argument shows that $\underline{\mathbb{F}}_I \cap \underline{\mathbb{F}}_{\mathcal{T}}^{\infty}$ is generated under sparsely I-indexed coproducts by $\underline{\mathbb{F}}_I^{\mathrm{sprs}}$. Hence it suffices to prove that $\underline{\mathbb{F}}_I$ is generated under sparsely I-indexed coproducts by $\underline{\mathbb{F}}_I^{\mathrm{sprs}} \cup \left(\underline{\mathbb{F}}_{\mathcal{T}}^{\infty} \cap \underline{\mathbb{F}}_I\right)$.

Fix $S \in \underline{\mathbb{F}}_I$. We will prove that $S \in \mathrm{Cl}_{\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}},\infty}\left(\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}} \cup \left(\underline{\mathbb{F}}_I \cap \underline{\mathbb{F}}_{\mathcal{T}}^{\infty}\right)\right)$ inductively on $|\mathrm{Orb}(S)|$. In fact, Observation 3.4 implies that this is true whenever $\left|\overline{\mathrm{Istrp}}(S)\right| = 1$ (including the base case $|\mathrm{Orb}(S)| = 1$), so it suffices to prove this in the case that $\left|\overline{\mathrm{Istrp}}(S)\right| \geq 2$ under the inductive assumption that the statement is true for all $T \in \underline{\mathbb{F}}_{\mathcal{T}}$ with $|\mathrm{Orb}(T)| < |\mathrm{Orb}(S)|$.

In this case, by Observation 3.4, S is a sparsely I-indexed coproduct of $\left(S_{(\overline{W})}\right)_{W \in \overline{\mathrm{Istrp}}(T)}$; by the assumption $\overline{\mathrm{Istrp}}(S) \geq 2$, $\mathrm{Ind}_W^V S_{(\overline{W})} \subsetneq S$, so in particular, we have $\left|\mathrm{Orb}\left(S_{(\overline{W})}\right)\right| < |\mathrm{Orb}(S)|$. By the inductive hypothesis, we $S_{(\overline{W})} \in \mathrm{Cl}_{\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}}\left(\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}} \cup \left(\underline{\mathbb{F}}_I \cap \underline{\mathbb{F}}_{\mathcal{T}}^{\infty}\right)\right)$ for each W; in view of Observation 3.2, Eq. (3) thus expresses S as an iterated sparsely I-indexed coproduct of elements of $\underline{\mathbb{F}}_I^{\mathrm{sprs}} \cup \left(\underline{\mathbb{F}}_I \cap \underline{\mathbb{F}}_{\mathcal{T}}^{\infty}\right)$, which is what we set out to do. □

Proof of Theorem C. By Proposition 3.1, $(-)^{\mathrm{sprs}}$ is a section of $\mathrm{Cl}_\infty(-)$ and a right adjoint; this implies that $(-)^{\mathrm{sprs}}$ is an embedding by Lemma 2.13, with image spanned by those collections $\mathcal{C}$ satisfying $\mathcal{C} \simeq \mathrm{Cl}_\infty(\mathcal{C})^{\mathrm{sprs}}$. Unwinding definitions, this is what we set out to prove. □

Remark 3.5. Note that the maps $\upsilon, c, \nabla, \aleph$ all factor as

$$\begin{tikzcd}
\mathrm{wIndex}_{\mathcal{T}} \arrow[rr, "{\upsilon,c,\nabla,\aleph}"] \arrow[d, "-\cap\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}"'] & & \mathcal{C} \\
\mathrm{Coll}(\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}) \arrow[r, hook] & \mathrm{Coll}(\underline{\mathbb{F}}_{\mathcal{T}}) \arrow[ur, "{\upsilon,c,\nabla,\aleph}"'] &
\end{tikzcd}$$

where $\mathcal{C} = \mathrm{Transf}_{\mathcal{T}}$ for $\aleph$ and $\mathrm{Fam}_{\mathcal{T}}$ otherwise. By using Lemma 1.25, we find that:

(1) $\aleph(\underline{\mathbb{F}}_I) = \aleph(\underline{\mathbb{F}}_I^{\mathrm{sprs}})$.
(2) $\underline{\mathbb{F}}_I$ has one-color if and only if $\underline{\mathbb{F}}_I^{\mathrm{sprs}}$ has one color.
(3) $\underline{\mathbb{F}}_I$ is E-unital if and only if $\underline{\mathbb{F}}_I^{\mathrm{sprs}}$ is E-unital.
(4) $\underline{\mathbb{F}}_I$ is unital if and only if $\underline{\mathbb{F}}_I^{\mathrm{sprs}}$ is unital.
(5) $\underline{\mathbb{F}}_I$ is an indexing system if and only if $\upsilon(\underline{\mathbb{F}}_I^{\mathrm{sprs}}) \cap \nabla(\underline{\mathbb{F}}_I^{\mathrm{sprs}}) = \mathcal{T}$.

In particular, we may enumerate the associated posets using Theorem C. ◁

In fact, our description in terms of sparse V-sets is not as compact as it could be.

Observation 3.6. If $\underline{\mathbb{F}}_I$ is aE-unital and contains the sparse V-set $S = \varepsilon \cdot *_V \sqcup V_1 \sqcup \cdots \sqcup V_n$ and the transfer $U \to V_1$, then $\underline{\mathbb{F}}_I$ contains the sparse V-set $\varepsilon \cdot *_V \sqcup U \sqcup V_2 \sqcup \cdots \sqcup V_n$, as it's an S-indexed coproduct of elements of $\underline{\mathbb{F}}_I$. ◁

3.2. **Warmup: the (aE-)unital C_p-weak indexing systems.** The orbit category of the prime-order cyclic group C_p may be presented as follows:

$$\left\langle \begin{tikzcd} {[C_p]} \arrow[loop left, "\tau"] \arrow[r, "r_{e,C_p}"] & *_{C_p} \end{tikzcd} \;\middle|\; \tau^p = \mathrm{id}_{[C_p]}, \quad r_{e,C_p} = r_{e,C_p}\tau \right\rangle$$

It is easy to see that there are precisely two C_p-transfer systems: R_0 contains no transfers, and R_1 contains the transfer $e \to C_p$. Thus the poset Transf_{C_p} is $(R_0 \to R_1)$. Furthermore, there are exactly three C_p families, and the poset is $\left(\varnothing \to \{e\} \to \left\{e, C_p\right\}\right)$. We will use this to perform the following computation.

Theorem 3.7. *The poset* $\mathrm{wIndex}_{C_p}^{aE\mathrm{uni}}$ *is presented by the following*

$$\begin{tikzcd}
\varnothing \arrow[d] & & & & \\
E_e^{C_p}\mathbb{F}^{\mathrm{triv}} \arrow[r] \arrow[d] & E_e^{C_p}\mathbb{F}^{0} \arrow[r] \arrow[d] \arrow[ddr, bend left] & E_e^{C_p}\mathbb{F}^{\infty} \arrow[d] \arrow[dd, bend left] & & \\
\underline{\mathbb{F}}_{C_p}^{\mathrm{triv}} \arrow[r] & \underline{\mathbb{F}}_{e}^{0} \arrow[r] \arrow[d] & \underline{\mathbb{F}}_{C_p}^{\mathrm{triv}} \vee E_e^{C_p}\mathbb{F}^{\infty} \arrow[d] & & \\
 & \underline{\mathbb{F}}_{C_p}^{0} \arrow[r] & \underline{\mathbb{F}}_{e}^{\infty} \arrow[rr] \arrow[d] & & \underline{\mathbb{F}}_{C_p}^{\infty} \arrow[d] \\
 & & \overline{\underline{\mathbb{F}}}_{C_p} \arrow[r] & \underline{\mathbb{F}}^{\lambda} \arrow[r] & \underline{\mathbb{F}}_{C_p}
\end{tikzcd}$$

where $\left\{\underline{\mathbb{F}}_{C_p}^{\infty}, \underline{\mathbb{F}}_{C_p}\right\}$ *are the indexing systems,* $\left\{\underline{\mathbb{F}}_{C_p}^{0}, \underline{\mathbb{F}}_{e}^{\infty}, \overline{\underline{\mathbb{F}}}_{C_p}, \underline{\mathbb{F}}^{\lambda}\right\}$ *are the otherwise-unital weak indexing systems,* $\left\{\underline{\mathbb{F}}_{e}^{0}, \underline{\mathbb{F}}_{C_p}^{\mathrm{triv}} \vee E_e^{C_p}\mathbb{F}^{\infty}\right\}$ *are the otherwise-almost unital weak indexing systems, and* $\left\{E_e^{C_p}\mathbb{F}^{0}, E_e^{C_p}\mathbb{F}^{\infty}\right\}$ *are the otherwise-essentially unital weak indexing systems.*

Remark 3.8. Already, we see that none of $\mathrm{wIndex}_{C_p}^{\mathrm{uni}}$, $\mathrm{wIndex}_{C_p}^{a\mathrm{uni}}$, $\mathrm{wIndex}_{C_p}^{E\mathrm{uni}}$, or $\mathrm{wIndex}_{C_p}^{aE\mathrm{uni}}$ are self-dual, since each embed the poset $\bullet \to \bullet \to \bullet \leftarrow \bullet$ as a cofamily, but none embed its dual as a family. This heavily contrasts the cases of $\mathrm{Index}_G = \mathrm{Transf}_G$ and of Fam_G, which are known to be self-dual for arbitrary abelian G by [FOOQW22]. ◃

Note that $\underline{\mathbb{F}}_{C_p}^{\infty} \subset \underline{\mathbb{F}}_{C_p}$ are C_p-indexing systems; Proposition 1.40 shows that this is the poset of indexing systems. This completely characterizes $\nabla^{-1}(\mathcal{T}) \cap \mathfrak{R}^{-1}(-)$, and we will extend this to arbitrary fibers. First, those with no transfers:

Observation 3.9. For any atomic orbital ∞-category $\mathcal{T}$, the map $\nabla\colon \mathfrak{R}^{-1}(\mathcal{T}^{\simeq}) \to \mathrm{Fam}_{\mathcal{T}}$ is an equivalence by Proposition 3.1; the fibers of this are

$$\nabla^{-1}(\mathcal{F}) \cap \mathfrak{R}^{-1}(\mathcal{T}^{\simeq}) = \left\{\underline{\mathbb{F}}_{\mathcal{F}}^{\infty}\right\}. \quad ◃$$

The only remaining case is $\nabla^{-1}(\{e\}) \cap \mathfrak{R}^{-1}(R_1)$. Unwinding definitions, we find that there are two options for unital sparse collections closed under applicable self-indexed coproducts with the specified transfers and fold maps; they each must have e-values given by $\{\varnothing_e, *_e, 2 \cdot *_e\}$, and the two options for C_p-values are

$$\overline{\mathbb{F}}_{C_p}^{\mathrm{sprs}} = \left\{\varnothing_{C_p}, *_{C_p}, [C_p/e]\right\}, \qquad \mathbb{F}_{C_p}^{\lambda,\mathrm{sprs}} = \left\{\varnothing_{C_p}, *_{C_p}, [C_p/e], *_{C_p} + [C_p/e]\right\}.$$

Furthermore, in view of Corollary 2.4, we have $\mathrm{wIndex}_{BC_p}^{\mathrm{uni}} \simeq \mathrm{wIndex}_{*}^{\mathrm{uni}}$. Applying Example 1.32, we've arrived at the following computations:

$$\mathrm{wIndex}_{BC_p}^{\mathrm{uni}}: \qquad \underline{\mathbb{F}}^{0} \longrightarrow \underline{\mathbb{F}}^{\infty}$$

$$\mathrm{wIndex}_{C_p}^{\mathrm{uni}}: \qquad \begin{tikzcd}
\underline{\mathbb{F}}_{C_p}^{0} \arrow[r] & \underline{\mathbb{F}}_{e}^{\infty} \arrow[rr] \arrow[d] & & \underline{\mathbb{F}}_{C_p}^{\infty} \arrow[d] \\
 & \overline{\underline{\mathbb{F}}}_{C_p} \arrow[r] & \underline{\mathbb{F}}^{\lambda} \arrow[r] & \underline{\mathbb{F}}_{C_p}
\end{tikzcd}$$

Theorem 3.7 then follows by applying Corollary 2.31 and Proposition 2.35.

3.3. **The fibers of the C_{p^N}-transfer-fold fibration.** Now, fix $\mathcal{T} = \mathcal{O}_{C_{p^N}}$ for $N \in \mathbb{N} \cup \{\infty\}$. Recall that when $\mathcal{F} \subset \mathcal{O}_{C_{p^N}}$ is a collection of objects and R a C_{p^N}-transfer system, we refer to precomposition-closed wide subcategories of $R \cap \mathcal{F}$ as *R-sieves on $\mathcal{F}$*, and write the resulting poset as $\mathrm{Sieve}_R(\mathcal{F}) \subset \mathrm{Sub}_{\mathrm{Cat}}(R \cap \mathcal{F})$. Given

$\underline{\mathbb{F}}_I^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_{C_{p^N}}$ a sparse collection which is closed under applicable self-indexed coproducts, let $\mathscr{S}(\underline{\mathbb{F}}_I^{\mathrm{sprs}}) \subset \mathrm{Cod}(\aleph(\underline{\mathbb{F}}_I^{\mathrm{sprs}})) - \nabla(\underline{\mathbb{F}}_I)$ be the wide subcategory consisting of maps $U \to V$ such that $*_V + U \in \mathbb{F}_{I,V}^{\mathrm{sprs}}$.

Proposition 3.10. *The restricted map* $\mathscr{S}\colon \aleph^{-1}(R) \cap \nabla^{-1}(\mathcal{F}) \to \mathrm{Sub}_{\mathrm{Cat}}(\mathrm{Cod}(\aleph(R)) - \mathcal{F})$ *is an embedding with image spanned by the* R*-sieves on* $\mathcal{F}$*.*

Proof. In view of Theorem C, a unital $\mathcal{T}$-weak indexing system lying over $(R, \mathcal{F})$ is determined by its nontrivial V-sets S such that:

(a) $S^V = *_V$; and
(b) $S - S^V = U_1 \sqcup \cdots \sqcup U_n \neq \varnothing$ and there exist no maps $U_i \to U_j$ for $i \neq j$; and
(c) $V \in \mathrm{Cod}(\aleph(\underline{\mathbb{F}}_I)) - \mathcal{F}$.

Thus we may restrict fully faithfully to just these sparse V-sets.

In fact, since the subconjugacy lattice $\left[\mathcal{O}_{C_{p^N}}\right]$ is a total order, such a sparse H-set is exactly an H-set of the form $*_H + [H/J]$ for some $J \subsetneq H$. Thus $\mathscr{S}$ is an embedding, so it suffices to characterize its image.

On one hand, the precomposition in $\mathscr{S}(I)$ is corresponds with the assignment $*_H \sqcup [H/J] \mapsto *_H \sqcup [H/K]$ for $K \subset J$ a transfer in I, which is implemented by I-indexed coproducts; hence closure under self-indexed coproducts implies that $\mathscr{S}(I)$ is an R-Sieve on $\mathrm{Cod}(\aleph(\underline{\mathbb{F}}_I^{\mathrm{sprs}})) - \mathcal{F}$.

Conversely, if $\mathfrak{S}$ is an R-sieve on $\mathrm{Cod}(\aleph(\underline{\mathbb{F}}_I^{\mathrm{sprs}})) - \mathcal{F}$, we define the unital weak indexing system $\underline{\mathbb{F}}_{\mathfrak{S}}$ by

$$\mathbb{F}_{\mathfrak{S},H} = \{S \mid \forall [H/K] \in \mathrm{Orb}(S),\ K \to H \in R\}$$

when $H \in \mathcal{F}$, and

$$\mathbb{F}_{\mathfrak{S},H} = \left\{ \coprod_i n_i \cdot [H/K_i] \;\middle|\; \forall i,\quad n_i \in \mathbb{N}, \text{ and } K_i \subsetneq H \in R \right\} \cup \left\{ *_H \sqcup \coprod_i n_i \cdot [H/K_i] \;\middle|\; \forall i,\quad n_i \in \mathbb{N}, \text{ and } K_i \subsetneq H \in \mathfrak{S} \right\}$$

when $H \notin \mathcal{F}$. It follows immediately by definition that $\nabla(\underline{\mathbb{F}}_{\mathfrak{S}}) = \mathcal{F}$, that $\aleph(\underline{\mathbb{F}}_{\mathfrak{S}}) = R$, that $v(\underline{\mathbb{F}}_{\mathfrak{S}}) = \mathcal{O}_{C_{p^N}} = c(\underline{\mathbb{F}}_{\mathfrak{S}})$, and that $\mathscr{S}(\underline{\mathbb{F}}_{\mathfrak{S}}) = \mathfrak{S}$, so to conclude that $\underline{\mathbb{F}}_{\mathfrak{S}} \in \mathscr{S}^{-1}(\mathfrak{S})$ (and hence the proposition), it remains to show that $\underline{\mathbb{F}}_{\mathfrak{S}}$ is closed under self-indexed coproducts. To that end, we fix $S \in \mathbb{F}_{\mathfrak{S},H}$ and $(T_{K_i}) \in \mathbb{F}_{\mathfrak{S},S}$ and break into cases.

The case $|\mathrm{Orb}(S)| = 1$. If $|\mathrm{Orb}(S)| = 1$, i.e. $S = \mathrm{Ind}_K^H *_U$ for some U, then we're tasked with proving that self-induction preserves elements of $\underline{\mathbb{F}}_{\mathfrak{S}}$. If $K = H$ then there is noting to prove, so assume $K \subsetneq H$; note that S is fixed-point free, so we're left with verifying that all orbital summands of S lie in $R_{/H}$. In fact, their structure maps to H are composites of maps in R, so these are in R, as desired.

The cases $H \in \mathcal{F}$ *or* $S^H = \varnothing$. In either of these cases, we're tasked with proving that the orbital summands of S lie in $R_{/H}$. In any case, all orbital summands of T_{K_i} lie in $R_{/K_i}$ by construction; if $H \in \mathcal{F}$ or $S^H = \varnothing$, then all orbital summands of S are R-indexed inductions of orbital summands of T_{K_i}, so their structure maps are composites of maps in R. Unwinding definitions, we see that orbital summands of S lie in $R_{/H}$, as desired.

The case $H \notin \mathcal{F}$ *and* $S^H \neq \varnothing$. Write $T = \coprod_{K_i}^S T_{K_i}$. Then, the decomposition $S = S^H \sqcup S'$ yields a decomposition $T = T_H \sqcup T'$ where $T_H \in \underline{\mathbb{F}}_{\mathfrak{S}}$ and T' is a coproduct of nontrivial R-indexed inductions; in particular, this implies that $T^H = T_H^H \sqcup (T')^H = T_H^G = S^H = *_H$. Thus we're left with proving that the nontrivial orbital summands of T lie in $\mathfrak{S}_{/H}$.

Indeed, a nontrivial orbital summand $[H/K] \subset T$ lies in either T_H or T'; if $[H/K] \subset T_H$ then $[H/K] \in \mathfrak{S}_{/H}$ since $T \in \mathbb{F}_{\mathfrak{S},H}$ with $T^H = *_H$. We're left with the case $[H/K] \subset T'$, in which case we have $[H/K] \subset \mathrm{Ind}_J^H T_J$ for some $J \subsetneq H \in \mathfrak{S}$. This implies that the structure map of K factors as $K \subset J \subsetneq H$ with the left inclusion lying in R and the right inclusion lying in $\mathfrak{S}$; since $\mathfrak{S}$ is closed under precomposition by maps in R, we have $[H/K] \in \mathfrak{S}_{/H}$, as desired. □

In order to prove Corollary D, we need to identify $\mathrm{Transf}_{C_{p^N}}$; this was already done in [BBR21] when N is finite, and the infinite case follows immediately from Proposition 2.3.

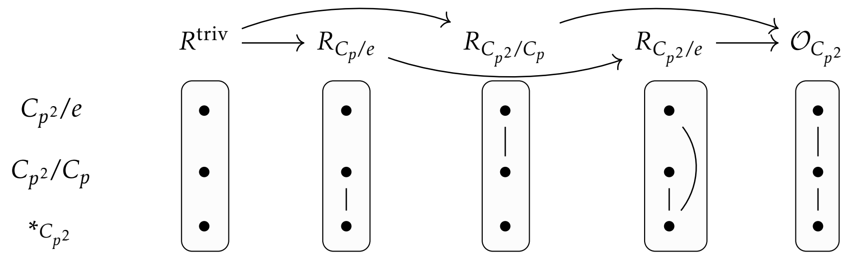

Figure 1. Pictured is the result of Rubin and Balchin-Barns-Rotzheim's computation of $\mathrm{Transf}_{C_{p^2}}$.

Proposition 3.11 ([BBR21, Thm 25]). *For $N \in \mathbb{N} \cup \{\infty\}$, there is an equivalence of posets*

$$K_{N+2} \simeq \mathrm{Transf}_{C_{p^N}},$$

the left side denoting the Nth associahedron.

Proof of Corollary D . In view of Proposition 3.11, the combined transfer-fold fibration has signature $(\aleph, \nabla)\colon \mathrm{wIndex}^{\mathrm{uni}}_{C_{p^N}} \to K_{N+2} \times [N+1]$. After Propositions 2.59, 3.10 and 3.11, we've identified the fibers and proved that the restricted map is a cocartesian fibration. Thus it suffices to understand cocartesian transport, which is implemented by

$$t_{(R,\mathcal{F})}^{(R',\mathcal{F}')} \underline{\mathbb{F}}_I = \underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'} \vee \underline{\mathbb{F}}_{\mathcal{F}'}^{\infty},$$

by Proposition 2.14, in terms of R-sieves. When $R = R'$, it is clear that this is given by the restriction $\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \twoheadrightarrow \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}')$, so it suffices to characterize this in the case $\mathcal{F} = \mathcal{F}'$. Unwinding definitions, we're tasked with characterizing for which $K \hookrightarrow H$, we have

$$*_H + [H/K] \in \underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'}.$$

By Theorem C, it suffices to characterise which of these are presented as sparse indexed coproducts of elements of $\underline{\mathbb{F}}_I$ and $\overline{\underline{\mathbb{F}}}_{R'}$.

Let $t_R^{R'}\colon \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \hookrightarrow \mathrm{Sieve}_{R'}(\mathrm{Cod}(R') - \mathcal{F})$ be the map sending and R-sieve $\mathfrak{S}$ to the R'-sieve whose non-isomorphisms are the composites $J \subset K \subsetneq H$ with $K \subset H \in \mathfrak{S} - \mathfrak{S}^{\simeq}$ and $J \subset K \in R'$. On one hand, note that, for all $J \subset K \subsetneq H$ in $\iota(\mathfrak{S})$, we have

$$*_H \sqcup [H/J] = *_H \sqcup \mathrm{Ind}_K^H [K/J],$$

i.e. $*_H \sqcup [H/J]$ is a $*_H \sqcup [H/K]$-indexed coproducts of elements of $\overline{\underline{\mathbb{F}}}_{R'}$; unwinding definitions, this implies that $\mathscr{S}\left(\underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'}\right) \geq t_R^{R'} \mathscr{S}(\underline{\mathbb{F}}_I)$.

On the other hand, note that $\underline{\mathbb{F}}_{t_R^{R'} \mathscr{S}(\underline{\mathbb{F}}_I)}$ is a unital weak indexing system containing both $\underline{\mathbb{F}}_I$ and $\overline{\underline{\mathbb{F}}}_{R'}$; this implies that $\underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'} \leq \underline{\mathbb{F}}_{t_R^{R'} \mathscr{S}(\underline{\mathbb{F}}_I)}$, so applying $\mathscr{S}$ together with the above inequality yields $\mathscr{S}\left(\underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'}\right) = t_R^{R'} \mathscr{S}(\underline{\mathbb{F}}_I)$, which is what we set out to prove. □

We finish by drawing this out for $N = 2$. We may illustrate $\mathcal{O}_{C_{p^2}}$ as follows

$$\begin{tikzcd} \left[C_{p^2}/e\right] \arrow[r] \arrow[loop below, "C_{p^2}"] & \left[C_{p^2}/C_p\right] \arrow[r] \arrow[loop below, "C_p"] & *_{C_p^2} \end{tikzcd}$$

with $\mathrm{Map}([C_{p^2}/e], [C_{p^2}/C_p])$ a C_p-torsor and $\mathrm{Map}([C_{p^2}/C_p], *_{C_{p^2}}) = *$. The independent computations of [BBR21; Rub21] verify the that $\mathrm{Transf}_{C_{p^2}}$ agrees with Fig. 1.

Given $R \in \mathrm{Transf}_{C_{p^2}}$, we let $\underline{\mathbb{F}}_R$ be the corresponding indexing system. Corollary D implies the following.

Corollary E. *Let $\lambda_{C_{p^N}}$ denote a nontrivial irreducible real orthogonal C_{p^N}-representation. Then, the poset of unital C_{p^2}-weak indexing systems is that of Fig. 2*

To be explicit, we use the following examples, where $x \in C_{p^2}$ is a distinguished generator.

Example 3.12. Let $\lambda_{C_{p^2}}$ be the 2-dimensional real orthogonal C_{p^2}-representation wherein x acts by a rotation of order p^2. Then, both $\lambda_{C_{p^2}}$ and $\mathrm{Res}_{C_p}^{C_{p^2}} \lambda_{C_{p^2}}$ have 0-dimensional fixed points, so they do not possess fold maps; hence $\nabla\left(\underline{\mathbb{F}}^{\lambda_{C_{p^2}}}\right) = \{e\}$.

The non-fixed points of $\lambda_{C_{p^2}}$ have orbit type $[C_{p^2}/e]$ and the non-fixed points of $\mathrm{Res}_{C_p}^{C_{p^2}} \lambda^{C_{p^2}}$ have orbit type $[C_p/e]$; together these imply that $\Re\left(\underline{\mathbb{F}}^{\lambda_{C_{p^2}}}\right) = R_{C_{p^2}/e}$ as in Fig. 1. Furthermore, $\mathrm{Conf}^H_{*_H+S}(V) = \mathrm{Conf}^H_S(V - \{0\})$ for all H, so $\mathscr{S}\left(\underline{\mathbb{F}}^V\right)$ is the maximal $\Re\left(\underline{\mathbb{F}}^V\right)$-Sieve on $\mathrm{Cod}\left(\Re\left(\underline{\mathbb{F}}^V\right)\right) - \nabla\left(\underline{\mathbb{F}}^V\right)$ for all V. Thus we've completely determined the position of $\underline{\mathbb{F}}^{\lambda_{C_{p^2}}}$ in the classification of Corollary D. ◃

Example 3.13. Similarly to Example 3.12, let λ_{C_p} be the irreducible C_{p^2}-representation wherein x acts by a rotation of order p; this is 1-dimensional (and the sign representation) if $p = 2$, and 2-dimensional if $p > 2$. Note that λ_{C_p} has 0-dimensional fixed points, but $\mathrm{Res}_{C_p}^{C_{p^2}} \lambda_{C_p}$ is trivial; hence $\nabla\left(\underline{\mathbb{F}}^{\lambda_{C_p}}\right) = \left\{e, C_p\right\}$.

Furthermore, the orbit type of non-fixed points in λ_{C_p} is $[C_{p^2}/C_p]$; this implies that $\Re\left(\underline{\mathbb{F}}^{\lambda_{C_{p^2}}}\right) = R_{C_{p^2}/C_p}$ as in Fig. 1. Using the sieve maximality of Example 3.12, we've completely determined the position of $\underline{\mathbb{F}}^{\lambda_{C_p}}$ in the classification of Corollary D. ◃

Note that $\underline{\mathbb{F}}_R$ corresponds with the minimal R-sieve on $\mathrm{Cod}(R) - \mathrm{Dom}(R)$. Together with Examples 1.29, 3.12 and 3.13, this completely characterizes the image of the join generators of Fig. 2 under $(\Re, \nabla, \mathscr{S})$; since these are compatible with joins, this completely characterizes the image of the entirety of Fig. 2 under $(\Re, \nabla, \mathscr{S})$. What remains is to verify that Fig. 2 bijects onto the Sieve posets of Corollary D and that cocartesian transport as described by Corollary D is implemented by horizontal arrows; this follows straightforwardly by unwinding definitions.

3.4. **Questions and future directions.** To stimulate further development in this area, we now pose a litany of questions concerning the structure and tabulation of weak indexing systems. The first arose to the author out of consternation concerning the apparent lack of structure arising in Corollary E.

Question 3.14. Is there a closed form expression for $\mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{O}_{C_{p^N}}}$ or $\left|\mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{O}_{C_{p^N}}}\right|$? ◃

The author believes that, akin to the strategy employed in [BBR21], this may be solved by characterizing change-of-group functors such as restriction, Borelificaiton, and inflation. In particular, given $H \subset G$ a subgroup, the cofamily $\mathcal{O}_{G/H}$ consisting of transitive G-sets on which H acts trivially is an atomic orbital ∞-category, so it possesses a well-defined theory of weak indexing systems, which should participate in an adjunction

$$\mathrm{Infl}_H^G \colon \mathrm{wIndex}_{G/H} \rightleftarrows \mathrm{wIndex}_G \colon F_H^G,$$

where F_H^G metaphorically represents "fixed points with residual genuine $W_G(H)$-action," and literally sends $\underline{\mathbb{F}}_I$ to a $\mathcal{O}_{G/H}$-weak indexing system satisfying $F_H^G \mathbb{F}_{I,V} = \mathbb{F}_{I,V}$ for all $V \in \mathcal{O}_{G/H} \subset \mathcal{O}_G$. In the setting where $N \subset G$ is normal, $\mathcal{O}_{G/N}$ is canonically equivalent to the orbit category for the group G/N, so given a choice of a *normal* subgroup, this produces an inductive procedure: characterize $\mathcal{O}_G$ weak indexing systems by picking a normal subgroup and inductively characterizing weak indexing systems for $\mathcal{O}_{G,\leq N}$ (related to $\mathcal{O}_N$ by Proposition 2.3), weak indexing systems for $\mathcal{O}_{G/N}$, and the possible transfers from outside $\mathcal{O}_{G/N}$ to inside (as well as the possible additional data of H-sets S for which N acts trivially on G/H but not on $G/\mathrm{stab}_H(x)$ for all $x \in S$).

Outside of closed form expressions, the following question is evident as an extension of Corollary D.

Question 3.15. Is there a good combinatorial expression of $\nabla^{-1}(\mathcal{F}) \cap \Re^{-1}(R)$ over an arbitrary dedekind, nilpotent, or general finite group? ◃

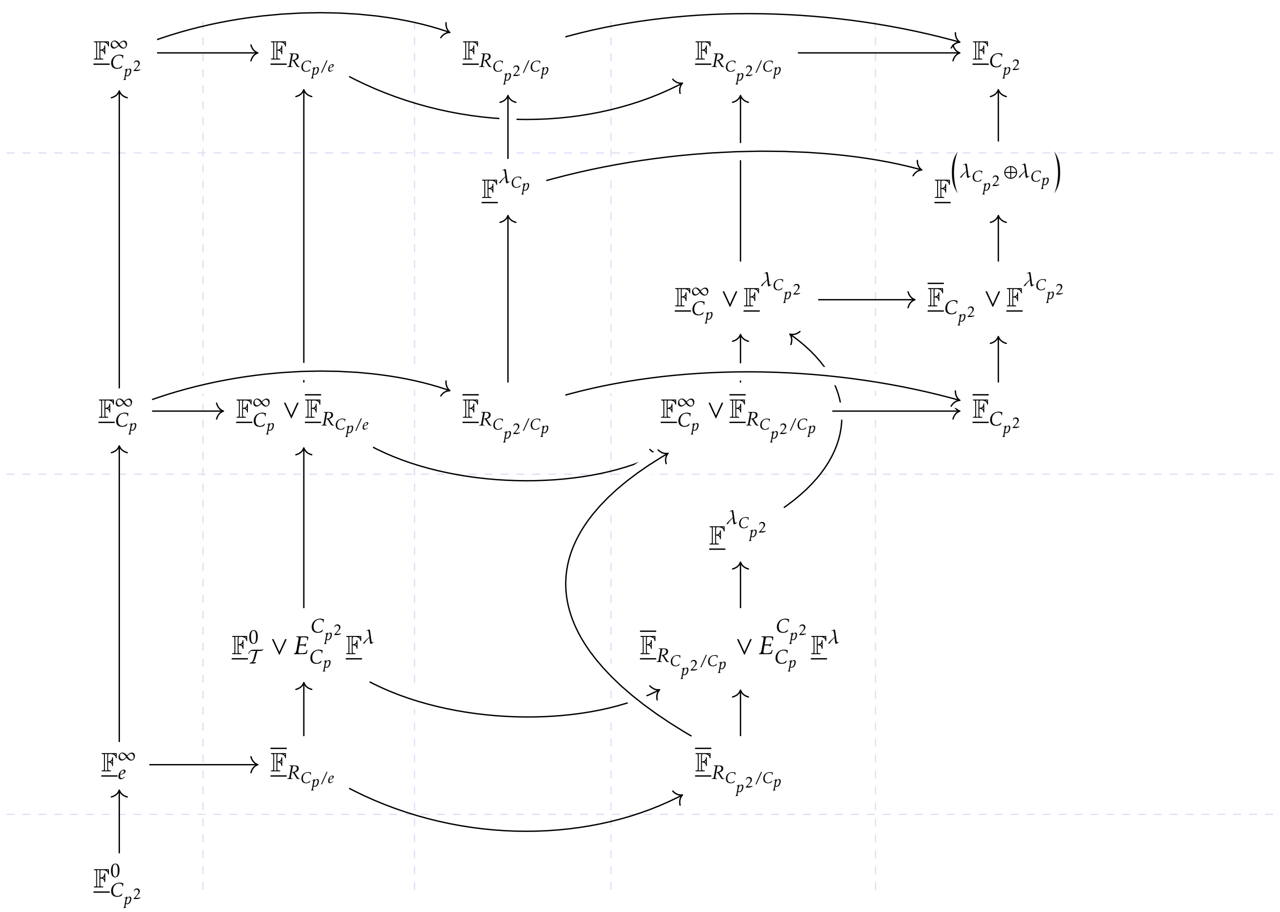

Figure 2. Pictured is a Hasse diagram for the poset of unital C_{p^2}-weak indexing systems. Dashed lines separate the fibers of the cocartesian fibration $(\mathfrak{R}, \nabla)$.

The author expects that our techniques may be extended to a similar sieve-based presentation for $\nabla^{-1}(\mathcal{F}) \cap fR^{-1}(R)$ over more general families of groups.

Another question arises by looking closely at Corollary E; we were able to tabulate all 20 unital C_{p^2}-weak indexing systems using only the families $\underline{\mathbb{F}}_R$, $\overline{\underline{\mathbb{F}}}_R$, and $\underline{\mathbb{F}}^V$ together with joins and the functors $E^{C_{p^2}}_{(-)}$.[8] Thus we ask the following.

Question 3.16. Which unital weak indexing systems are realizable via tensor products of $\left\{\mathbb{F}^V\right\}$ under various change of group functors? ◁

In particular, all instances of the right adjoint to ∇ occur as the arity support $\underline{\mathbb{F}}^V$ of an $\mathbb{E}_V$-G-operad, so we ask the following.

Question 3.17. What is the right adjoint to ∇? Is it related to $\mathbb{E}_V$? ◁

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[8] To see this, note that $\underline{\mathbb{F}}^0_G$ is the arity support of the 0 G-representation and $\underline{\mathbb{F}}^{\infty}_G$ is the arity support of any positive-dimensional trivial G-representation.

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