Kan Seminar Notes

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This will be a rough collection of live-LaTeXed notes covering the Kan seminar talks given in Fall 2021. I'll make no promises that the contents of this are readable, or without significant clerical error. Last update: September 22, 2021.

Contents

1 Gabrielle Li: Serre, Cohomologie modulo 2 des complexes d'Eilenberg Mac Lane (i)

This talk was delivered September 15, 2021 by Gabrielle Li. Throughout, $H^*(-) := H^*(-;\mathbb{F}_2)$.

1.1 Steenrod operations

The *Steenrod operations* are a family of cohomology operations $\mathrm{Sq}^n : H^*(X) \to H^{*+n}(X)$ such that:

(1) Each Sq^n is natural in X.

(2) Each Sq^n is stable: $\mathrm{Sq}^n(\Sigma X) = \Sigma\,\mathrm{Sq}^n(X)$.

(3) When $|x| = n$, $\mathrm{Sq}^n(x) = x \cup x$.

(4) $\mathrm{Sq}^0 = \mathrm{id}$.

We give a basis for these:

Definition 1.1. A sequence $I = (i_1) \subset \mathbb{Z}_{>0}$ is *admissable* if $i_k \geq 2i_{k+1}$ for each k. We define the *degree* $n(I) := \sum i_k$ and the *excess* $e(I) = \sum(i_k - i_{k+1}) = 2i_i - n(I)$ (padding with zeros).

1.2 Borel's theorem

Let $F \hookrightarrow E \to B$ be a Serre fibration. Recall that, in the cohomological Serre spectral sequence, we have transgression morphisms $\tau : E_r^{0,r-1} \to E_r^{r,0}$, whose domain is a subset of $H^{r-1}(F)$ and whose codomain is a quotient of $H^r(B)$. This is an additive relation between $H^{r-1}(F)$ and $H^r(B)$. We say that $x \in H^{r-1}(F)$ is *transgressive* if it survives to the r page.

We hold off on proving the following proposition until the next talk:

Proposition 1.2. *τ commutes with Steenrod operations.*

We need a bit more language to use this:

Definition 1.3. For a space X, an ordered family of elements $(x_i) \subset H^*(X)$ is a *simple system of generators* if:

(1) Each x_i is homogeneous.

(2) The increasing products $x_{i_1} \cdots x_{i_j}$ (for $i_k < i_{k+1}$) form a $\mathbb{F}_2$-basis of $H^*(X)$.

The following examples are important:

Example 1.4:
$\mathbb{F}_2[x_1, x_2, \ldots]$ has simple system of generators $(x_j^{2^i})$. Similar systems apply to the exterior algebra $E[x]$ and the truncated poltnomial algebra $\mathbb{F}_2[x]/(x^{2^i})$.

We're finally ready to state our theorem:

Theorem 1.5 (Borel)**.** *Given a fibration $F \hookrightarrow E \to B$ satisfying the following properties:*

(1) $E_2^{s,t} = H^s(B) \otimes H^t(F)$ (for instance, when B is 1-connected and $H^*(B), H^*(F)$ are f.g.).

(2) $H^i(E) = 0$ for $i > 0$.

(3) $H^*(F)$ have a simple system of transgressive generators (x_i).

Then, $H^*(B)$ is a polynomial algebra generated (independently) by the any choice of representatives $y_i \in H^*(B)$ which map to $\tau(x_i)$ in $E_*^{*,0}$.

Note that, whenever $H^*(F)$ is a polynomial algebra generated by z_i, we know that $H^*(F)$ has a simple system of generators $z_i^{2^r}$. In order to use this, we introduce a bit of notation:

Notation. $L(a, r) := \{2^{r-1}a, 2^{r-2}a, \cdots, 2a, a\}$.

Note that $z_i^{2^r} = \mathrm{Sq}^{L(n_i,r)}(z_i)$. Hence

$$\tau\left(z_i^{2^r}\right) = \mathrm{Sq}^{L(n_i,r)}\, t_i$$

where $t_i := \tau(z_i)$. Hence $H^*(B)$ is a polynomial algebra generated by $\mathrm{Sq}^{L(n_i,r)}(z_i)$.

1.3 Performing the calculation

We will use Borel's theorem soon, but first, a lemma:

Lemma 1.6. *An admissible sequence $J = \{j_1, \ldots, j_k\}$ with $e(J) < q - 1$. Then, we may define a sequence*

$$J' := \left\{2^{r-1}s_J, 2^{r-2}s_J, \ldots, s_J, j_1, j_2, \ldots, j_k\right\},$$

where $s_J = q - 1 + n(J)$. Then, J' is admissible, with $e(J') < q$; furthermore, all admissible sequences of excess $< q$ arise this way.

The reversal is surprisingly easy; simply take the longest prefix satisfying $j_1 = 2j_2 = \cdots = 2^i j_i$.

We will need a few more constructions to prepare for the calculation:

(1) There is a fibration $K(\mathbb{F}_2, q - 1) \hookrightarrow E \to K(\mathbb{F}_2, q)$ where E is contractible.

(2) By Hurewicz, $H^q(K(\mathbb{F}_2, q)) = \mathbb{F}_2$, with a generator that we call u_q.

Theorem 1.7. *$H^*(K(\mathbb{Z}/2, q), \mathbb{Z}/2)$ is a polynomial algebra (independently) generated by $\mathrm{Sq}^I(u_q)$ where I runs over the admissible sequences of excess $e(I) < q$.*

Proof. We prove this via induction. The $q = 1$ case is easy, as we have $K(\mathbb{F}_2, 1) = \mathbb{RP}^\infty$, and $H^*(\mathbb{RP}^\infty) = \mathbb{F}_2[u_q]$ via the usual computation.

For the inductive step, assume we've proven the theorem for $q - 1$. We use the fibration from (1). For an admissible sequence J, let

$$S_J := \left|\mathrm{Sq}^J(u_{q-1})\right| = q - 1 + n(J).$$

We have transgression additive relation $H^{q-1}(K(\mathbb{F}_2, q - 1)) \rightsquigarrow H^q(K(\mathbb{F}_2, q))$. Note that the transgression sends $\tau(u_{q-1}) = u_q$ (this will be justified later). Using our trick,

$$\tau(\mathrm{Sq}^J(u_{q-1})) = \mathrm{Sq}^J u_q.$$

By Borel, the $H^*(K(\mathbb{F}_2, q))$ is generated by $\mathrm{Sq}^{L(s_J,r)}\, \mathrm{Sq}^J u_q = \mathrm{Sq}^{L(s_j,r)J}\, u_q = \mathrm{Sq}^I u_q$, where I is an admissible sequence with $e(I) < q$, and all such I are generated this way. ☐

The other computations are routine and similar.

2 Weixiao Lu: Serre, Cohomologie modulo 2 des complexes d'Eilenberg Mac Lane (ii)

This talk was delivered September 17, 2021 by Weixiao Lu. We'll first cover some preliminaries.

2.1 Preliminaries

Theorem 2.1 (Serre spectral sequence)**.** *Let $F \hookrightarrow E \xrightarrow{p} B$ be a Serre fibration. Then, there is a spectral sequence*

$$E_2^{s,t} = H^s(B; H^t(p^{-1}(-); G)) \implies H^{s+t}(E; G).$$

If $\pi_1(B)$ acts trivially on $H^n(p^{-1}(-))$, then

$$E_2^{2,t} = H^s(B; H^t(F; G)).$$

Proof sketch. If F^*C^* is a filtered cochain complex, we have an SS,

$$E_0^{s,t} = \mathrm{gr}^s(C^{s+t}) \implies H^{s+t}(C^*).$$

Assume B is a CW complex with n-skeleton B^n. Then, $E_n := p^{-1}(B^n)$. We have $F^s S^*(E) = S^*(E, E_{s-1}) = \ker(S^*(E) \to S^*(E_{s-1}))$, which gives the right E_0 page. □

In any upper-right quadrant SS, we have a transgression morphism $d^n : E^{0,n-1} \to E_n^{n,0}$. Note that $E_n^{0,n-1} \subset E_{n-1}^{0,n-1} \subset \cdots \subset H^{n-1}(F)$. The transgressive elements of $H^{n-1}(F)$ map to some quotient of $H^n(B)$.

We can create a diagram

$$\begin{tikzcd}
H^n(B,b) \arrow[r, "p^*"] \arrow[d, "\sim"] & H^n(E,F) & \\
H^n(B) \arrow[ur] & & H^{n-1}(F) \arrow[ul, "\partial"']
\end{tikzcd}$$

Theorem 2.2 (Transgression theorem)**.** *The transgression relation coincides with this diagram.*

This comes down to how the Serre SS was constructed.

Proposition 2.3. *The Steenrod square Sq_i "commutes" with transgression in the sense that any $x \in H^{n-1}(F; \mathbb{Z}/2)$ transgressive has $\mathrm{Sq}^i x$ transgressive, and $\tau(\mathrm{Sq}^i x) = \mathrm{Sq}^i(\tau x)$.*

Proof. Recall that a functor is stable iff it commutes with coboundary operators, so Sq_1 commutes with coboundary operators. Further, recall that it's natural. Hence the following diagram commutes, so Sq^i "commutes with the transgression relation" (is a morphism of cospans):

$$\begin{tikzcd}
& H^{n+i}(E,F) & \\
H^{n+i}(B) \arrow[ur, "p^*"] & & H^{n+i-1}(F) \arrow[ul, "\partial"'] \\
& H^n(E,F) \arrow[uu, "\mathrm{Sq}^i"'] & \\
H^n(B \arrow[uu, "\mathrm{Sq}^i"] \arrow[ur, "p^*"] & & H^{n-1}(F) \arrow[ul, "\partial"'] \arrow[uu, "\mathrm{Sq}^i"]
\end{tikzcd}$$

□

Reclal that for G a f.g. Abelian group,

1. $H^*(K(G \times H; q)) = H^*(K(G; q)) \otimes H^*(K(H; q))$.
2. $H^*(K(\mathbb{F}_2; q)) = \mathbb{F}_2[\mathrm{Sq}^I u_q \mid I \text{ admissible, s.t. } e(I) < q]$.
3. $H^*(K(\mathbb{F}_2; q)) = \mathbb{F}_2[\mathrm{Sq}^I u_q \mid I \text{ admissible, s.t. } e(I) < q, 1 \text{ does not appear in } i]$.
4. $H^*(K(\mathbb{F}_2^h; q)) = \mathbb{F}_2[\mathrm{Sq}^I u_q, \mathrm{Sq}^J k_{q+1}]$ where $k_{q+1} \in H^{q+1}(K(\mathbb{F}_{2^h}, q))$ for admissibles $e(I) < q$, $e(J) \leq q$ where no Sq^1 term appears in both Sq^I and Sq^J. This comes from a fibration fill in from notes later .
5. $H^*(K(\mathbb{F}_{p^h}; q)) = \mathbb{Z}/2$ for p odd with $q > 0$.

Remark. We have a different choice of generators related to universal classes, but as graded $\mathbb{F}_2$-algebras,

$$H^*(K(\mathbb{F}_{2^h}; q)) \simeq H^*(K(\mathbb{F}_2; q)).$$

We will aim towards the following theorem:

Theorem 2.4. *For all $n > 1$, there are infinitely many indices i at which $\pi_i(S^n)$ has nonzero 2-torsion.*

Our tool will be Poincaré series. The accents in Poincaré's name are to be understood from here on out.

2.2 Poincaré series

For L_* a finite type graded k-vector space, define the series

$$L(t) = \sum_{n \in \mathbb{N}} \dim L^n t^n \in \mathbb{Z}[[t]].$$

This is called the *Poincare series*, called $\theta(G; q; t)$ in the case of $H^*(K(G; q))$.

Example 2.5:
For $L^* = \mathbb{Z}/2[u]$, we have

$$L(t) = \frac{1}{1 - t^m}.$$

Note that $(N^* \otimes M^*)(t) = L(t)M(t)$. Hence $L^{'*} = k[u_1, \dots]$ with finite type has

$$L(t) = \prod_{n \geq 1} \frac{1}{1 - t^{\deg u_i}}$$

which converges t-adically.

Hence

$$\theta(\mathbb{F}_2, q, t) = \prod_{e(I) < q} \frac{1}{1 - t^{\deg(\mathrm{Sq}^I u_q)}} = \prod_{e(I) < q} \frac{1}{1 + t^{q + n(I)}}.$$

We can give this another combinatorial description:

Proposition 2.6.

$$\theta(\mathbb{F}_2, q, t) = \prod_{n_1 \geq n_2 \geq \cdots \geq n_{q-1} \geq 0} \frac{1}{1 - t^{2^{n_1} + \cdots 2^{n_{q-1}} + 1}}.$$

The radius of convergence of this is 1 considered as a complex power series. We can continue to analyze this series along these lines:

Theorem 2.7.

$$\lim_{x \to \infty} \frac{\log_2 \theta\left(\mathbb{F}_2, q, 1 - 2^{-x}\right)}{x^q / q!} = 1.$$

In general there is an essential singularity at 1. Serre used this replacement to reign it in, but we won't work with it very explicitly.

2.3 Applications

Theorem 2.8. *Suppose X is a 1-connected space satisfying the following conditions:*

1. *$H_*(X;\mathbb{Z})$ is of finite type.*
2. *$H_i(X;\mathbb{F}_2) = 0$ for $i \gg 0$.*

Then, for infinitely many indices i, $\pi_i(X)$ has a subspace isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/2$.

This directly implies Theorem 2.4 once you know that only finitely many homotopy groups of spheres are infinite.

We see this using a whitehead tower

$$\begin{tikzcd}
\cdots \arrow[d] & \\
X_{n+1} \arrow[d] \arrow[dr] & \\
X_n \arrow[r] \arrow[d] & X \\
X_{n-1} \arrow[d] \arrow[ur] & \\
\cdots &
\end{tikzcd}$$

where X_n is n-connected, and a π_i iso to X and X_{n-1} for $i > n$. We'll use another piece of machinery, seen by the Serre SS directly.

Lemma 2.9. *For $F \hookrightarrow E \to B$ a Serre fibration with B simply connected, $B(t)F(t) \geq E(t)$.*

Proof of Theorem 2.8. Otherwise, there is some largest q with $\pi_q(X) \otimes \mathbb{Z}/2 \neq 0$. Then, there is some j smallest such that $H_j(X;\mathbb{Z}/2) \neq 0$. Then, $\pi_j(X) \otimes \mathbb{Z}/2 \neq 0$.

In the whitehead tower, $X_q \to X_{q-1}$ is trivial on $\pi_*(-) \otimes \mathbb{Z}/2$, so $H^*(X_q, \mathbb{Z}/2)$ is trivial. Using the fibration $X_q \hookrightarrow X_{q-1} \to K(\pi_q(X), q)$ from the whitehead tower, we must have $H^*(X_{q-1}) = H^*(K(\pi_q(X), q))$. Then,

$$X_{q-1}(t) = \theta(\pi_q(x), q, t).$$

Further, the fibbrations in the whitehead series imply that

$$X_{i+1}(t) \leq X_i(t)\theta(\pi_{i+1}(X), i, t)$$

for each i, Chaining these together forever, what we get is

$$\theta(\pi_q(X), q, t) \leq X_1(t)\theta(\pi_2(X), 1, t) \cdots \theta(\pi_{q-1}(x), q-2, t).$$

Note that $X_1(t)$ is a polynomial, so bounded on $[0, 1]$. Applying our asymptotic bound on θ yields a contradiction. $\square$

3 Zihong Chen: Moore, Semi-simplicial complexes and Postnikov systems

This talk was delivered September 20, 2021 by Zihong (Peter) Chen.

3.1 Review of simplicial sets

The talk began with a very brief review of simplicial sets: let Δ be the category of finite ordered sets and order preserving maps. Recall that such maps are generated by distinguished maps $\delta_i : [n] \to [n+1]$ and $s_i : [n+1] \to [n]$, called the *face and degeneracy maps.*

Definition 3.1. A *simplicial set* is a functor $X : \Delta^n \to \mathbf{Set}$.

The morphism set is completely characterised by their images on face and degeneracy maps, which must satisfy a collection of combinatorial relations, which I won't write down here.

> **Example 3.2:**
> The *standard n-simplex* is given by the representable functor $\Delta[n] := \mathrm{Hom}(-, [n])$.

By Yoneda's lemma, $X_n = \mathrm{Hom}(\Delta[n], X)$, where $X_n = X([n])$.

> **Example 3.3:**
> If $X \in \mathbf{Top}$, the singular simplicial set $\mathrm{Sing}(X)$ is familiar. It participates in an adjunction, with left adjoint $|\cdot|$ the *Geometric realization.*

> **Example 3.4:**
> Define the *ith face* $\delta_i : \Delta[n-1] \to \Delta[n]$. The *ith horm* is $\bigvee_i^n := \cup_{k \neq i} \delta_i$. The *boundary* is $\partial\Delta[n] = \bigcup_i \delta_i$.

This allows us to define the combinatorial equivalent of a topological space:

Definition 3.5. A simplicial set X is a *Kan complex* if every morphism $\bigvee_k^n \to X$ factors through $\Delta[n] \to X$; you can *fill any horn* (not necessarily uniquely).

A morphism $p : E \to B$ is a *Kan fibration* if it has the right lifting property against horn inclusions:

$$\begin{tikzcd}
\bigvee_k^n \arrow[r] \arrow[d] & E \arrow[d] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

Examples of this include $\mathrm{Sing}(X)$, and any simplicial group (which we won't prove).

Definition 3.6. For X a Kan complex, define the *path components* $\pi_0(X) = X_0/\sim$ where $x \sim y$ if there exists some p with $d_1 p = x$ and $d_0 p = y$.

This is in fact an equivalence relation: you can do this via horn filling, which was drawn on the board, but which I will not spell out. We can define higher homotopy groups after defining the internal hom:

Definition 3.7. For $A \subset X$ and $B \subset Y$, define the *mapping object*

$$\mathrm{Map}((X, A), (Y, B)) = \mathrm{Hom}(\Delta[n] \times (X, A), (Y, B))$$

i.e. the maps $\Delta[n] \times X \to Y$ restricting to a map $\Delta[n] \times A \to B$. The maps $\Delta[i] \to \mathbf{Set}$ form a covariant functor, so this is a contravariant functor, i.e. a simplicial set.

We use the following Theorem of Kan:

Theorem 3.8 (Kan)**.** *If Y, B are Kan complexes, then so is* $\mathrm{Map}((X, A), (Y, B))$.

We finally define homotopy groups.

Definition 3.9. If X is a Kan complex, define $\pi_n(X,x) := \pi_0(\mathrm{Map}((\Delta[n], \partial\Delta[n]), (X,x)))$. A Kan complex is $K(\Pi, n)$ if $\pi_q(X,x) = \Pi$ when $q = n$ and 0 otherwise.[1]

We will use these to decompose Kan complexes.

3.2 Postnikov systems

Let $\Delta[q]_n$ be the n-skeleton of $\Delta[q]$. For X a Kan complex, define the complex $X^{(n)}$ via

$$X_q^{(n)} = X_q/\sim \qquad\qquad x \sim y \iff x|_{\Delta[q]_n} = y|_{\Delta[q]_n}.$$

The maps are induced by X. We have the following properties:

1. $X^{(n)}$ is a Kan complex.
2. There is a quotient Kan fibration $X^{(n)} \xrightarrow{p} X^{(k)}$ if $n > k$.
3. $\pi_q(X^{(n)}, x) = 0$ if $q > n$.
4. $p_* : \pi_q(X^{(n)}, x) \xrightarrow{\sim} \pi_q(X^{(k)}, x)$ is an iso if $n \geq k \geq q$.

As in topology, Kan fibrations induce LES of homotopy groups; hence the fiber $F^{(n+1)} \hookrightarrow X^{(n+1)} \xrightarrow{p} X^{(n)}$ is a $K(\pi_{n+1}(X), x+1)$. We finally give this a name:

Definition 3.10. $(X^0, X^{(1)}, \dots)$ is called the *natural Postnikov system* of X.

This motivates a question: How far is X from $\prod_n K(\pi_n, n)$? It's always a colimit, but we'll measure how complex it is in the following section.

The idea is that $K(\pi_{n+1}, n+1) \hookrightarrow X^{(n+1)} \to X^{(n+1)}$ will be seen as something like a "principal $K(\pi_{n+1}, n+1)$-bundle." We will construct something like a "classifying space" $\overline{W}(K(\pi_{n+1}, n+1)) = K(\pi_{n+1}, n+2)$, and derive algebraic invariants from this. Let's actually do this now:

3.3 Principally twisted cartesian products

Definition 3.11. A *principally twisted Cartesian product* (PTCP) with simplicial group G and base G is written

$$E(T) = G \times_T B$$

where $E(T)_n = G_n \times B_n$ with degeneracy maps all the same, except that

$$\partial_0(g, b) = (T(b) \cdot d_0 g, d_0 b)$$

and T is a *twisting function* $B_q \to G_{q-1}$ for $q \geq 1$.

This is a combinatorial version of *holonomy*, as per a comment from Prof. Miller.

Definition 3.12. A PTCP is *of type (W)* if $B_0 = \{b_0\}$ and

$$\partial_0|_{\{e_q\}\times B_q} : [e_q] \times B_q \xrightarrow{\sim} E(T)_{q-1}$$

is an iso. Let $\int$ be its inverse.

Theorem 3.13. *If $G \times_T B$, $G' \times_{T'} B'$, and $\gamma : G \to G'$ is a morphism of simplicial group, then there exists a unique γ-equivariant map $\theta : G \times_T B \to G' \times_{T'} B'$ and Some condition holds of θ–fill in later .*

I couldn't follow this part; use $\int$ to construct this "upwards" from b_0, or something like that.

Corollary 3.14. *A PTCP of type (W) with group G is unique, if it exists.*

[1] This *actually* has a requirement of minimality, but we handwave this away.

Theorem 3.15. *If $E(T)$ is PTCP of type (W), it is contractible.*

They do exist! We can construct them by $B := \overline{W}(G)$, $W(G) = G \times_{T(G)} \overline{W(G)}$, where $\overline{W}_n(G) = G_{n-1} \times \cdots \times G_0$ for $n \geq 1$, and terminal for $n = 0$. put face and degen maps here . It has twisting function

$$T(G)[g_n, \dots, g_0] = g_n.$$

It can be checked explicitly that this is type (W).[2]

Corollary 3.16. *Every PTCP with group G is by*

$$B \xrightarrow{\pi} \overline{W}(G)$$

with $\pi(b) = \left[T(b), T(\partial_0 b), \dots, T(\partial_0^{n-1} b)\right]$.

This is a simplicial version of the bar construction??

This allows us to explicitly construct $K(\pi, n)$! Define $K(\pi, 0)$ to be π in each degree and $\partial_i s_i$ all identity. Define $K(\pi, n) = \overline{W}(K(\pi, n-1))$ inductively. We can see this is in fact a $K(\pi_1)$ via a fibration

$$K(\pi, n) \to W(K(\pi, n)) \to \overline{W}(K(\pi, n)),$$

where we know $W(*)$ to be contractible.

The main technical result follows:

Lemma 3.17. *Suppose there is no nontrivial morphism $\pi_1 \to \mathrm{Aut}(\pi_n)$. Then, $X^{(n)}$ is a PTCP with group $K(\pi_{n+1}, n+1)$.*[3]

To handwave, the idea for this is that minimal Kan fibrations are fiber bundles. Given the π_1 assumption, the structure group is $K(\pi_{n+1}, n+1)$. Then, a "principal G-bundle" is the same thing as a PTCP, in some intuitive way.

We can define the k-invariants via the fibrations $K(\pi_{n+1}, n+1) \hookrightarrow X^{(n+1)} \to X^{(n)}$: there is a universal class

$$u \in H^{n+2}(K(\pi_{n+1}, n+2))$$

and via the map $X^{(n+1)} \xrightarrow{f^{n+2}} \overline{W}(K(\pi_{n+1}, n+1)) = K(\pi_{n+1,n+2})$, we can define k-invariants as $\left(f^{n+2}\right)^* u = k^{n+2}$.

[2] This was written down in class.

[3] Per a comment of Prof. Miller, we only need simplicity, not total nontriviality of morphisms $\pi_1 \to \mathrm{Aut}(\pi_n)$.

4 Dylan Pentland: Borel, La cohomologie modulo 2 de certains espaces homogenes

This talk was delivered September 22, 2021 by Dylan Pentland.

4.1 Motivation and preprequisites

Characteristic classes We have a functor

$$\mathrm{Bun}_{O(n)} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$$

sending X to the isomorphism classes of principal $O(n)$ bundles mod isomorphisms. We know that this is representable, i.e. expressable as $\mathrm{Bun}_{O(n)}(-) = \mathrm{Hom}(-, \mathrm{BO}(n))$ (in the homotopy category).

Definition 4.1. A *characteristic class* is a natural transformation $\mathrm{Bun}_{O(n)} \implies H^i(-)$, where coefficients are understood mod 2. By the Yoneda lemma, this is the same thing as an element of $H^i(\mathrm{BO}(n))$.

We're going to characterize these via a cohomology computation. The main theorem is as follows: let $Q(n) \subset O(n)$ be the diagonal matrices. From this inclusion, we get a projection $\mathrm{BQ}(n) \xrightarrow{\rho} \mathrm{BO}(n)$, wich yields an induced map

$$\rho^* : H^*(\mathrm{BO}(n)) \to H^*(\mathrm{BQ}(n)) \simeq \mathbb{F}_2[x_1, \ldots, x_n].$$

Theorem 4.2. *The map ρ^* satisfies the following properties:*

- *ρ^* is injective.*
- *the image of ρ^* is $\mathbb{F}_2[x_1, ;x_n]^{\Sigma_{in}}$.*
- *$p^*(w_i) = e_i$.*

The splitting principle We give a modern POV on this:

Theorem 4.3. *Let X be paracompact and $E \to X$ a bundle. There is an undiced bundle $f : \mathrm{Fl}(E) \to X$ so that*

$$f^* : H^*(X) \to H^*(\mathrm{Fl}(E))$$

is injective, and f^*E splits into a direct sum of line bundles.

This winds up telling you the injectivity of Theorem 4.2, but not the image statement (only a containment). Either way, the proof is not much easier.

Spectral sequences breaking down We'll keep some assumptions about finite type cohomology. Dylan stated the requirement of simply connected spaces or principal G-bundle.[4]

Theorem 4.4. *Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration. The associated Serre spectral sequence (SSS) is trivial if and only if $H^*(E) \to H^*(F)$ is surjective. In this case, we say that F is* totally non-homologous to zero*, and we have the following properties:*

- *p^* is injective.*
- *$P(E) = P(B) \cdot P(F)$.*

The condition is called totally non-homologous to zero because the dual condition $H_*(F) \hookrightarrow H_*(E)$ makes sense for this name.

[4] Haynes had some comments about this; principality is not enough in general. There's secretly some connectedness condition.

4.2 Cohomology of $\mathrm{BO}(n)$

Outline of the proof of Theorem 4.2 Let $F_n = O(N)/Q(N)$. We will use the fibration

$$F_n \hookrightarrow \mathrm{BQ}(n) \xrightarrow{\rho} \mathrm{BO}(n).$$

We call this fibration $(\star)$. We follow the following steps:

(1) $H^*(F_n) = \langle H^1(F_n)\rangle$ so that $P(F_n) = (1-t)\cdots(1-t)^n \cdot (1-t)^{-n}$.

(2) The SSS for $(\star)$ is trivial, so $P(\mathrm{BQ}(n)) = P(\mathrm{BO}(n))P(F_n)$, giving injectivity.

(3) $\operatorname{im}\rho^* \subset \mathbb{F}_2[x_1,\dots,x_n]^{\Sigma_{in}}$, and dimensions yield that this is an equality.

General rule of theorem: every spectral sequence written today will be trivial.

Step 1: cohomology of F_n. We use induction via the fibration

$$F_{n-1} \hookrightarrow F_n \to \mathbb{P}^{n-1}$$

Lemma 4.5. $\dim H^1(F_n) \geq n-1$.

Write ${}^nE_r = \bigoplus_{s+t=n} E_r^{s,t}$.

Proof. Recall the fibration $F_n \hookrightarrow \mathrm{BSQ}(n) \to \mathrm{BSO}(n)$. The base space is simply connected, so

$$E_2^{1,0} = H^1(\mathrm{BSO}(n), H^0(F_n)) = 0.$$

Hence

$$\dim{}^1 E_2 \geq \dim{}^1 E_\infty = \dim H^1(\mathrm{BSQ}(n)) = n-1.$$

This implies that $E_2^{0,1} = H^1(F_n)$. □

Proposition 4.6. *$P(F_n) = (1-t)\cdots(1-t^n)(1-t)^{-n}$ and $H^*(F_n) = \langle H^1(F_n)\rangle$.*

Proof. Return to the fibration from the beginning of the fibration. We know the Poincaré polynomial for projective space, and we just have to prove that the SSS is trivial.[5] Write $H^*(F_n) \xrightarrow{i^*} H^*(F_{n-1})$. Assume both claims for $n-1$, so $\dim H^1(F_{n-1}) = n-2$. Note the following:

- $\dim E_2^{1,0} = \dim H^1(\mathbb{P}^{n-1}) = 1$.
- $\dim E_2^{0,1} = \dim H^0(\mathbb{P}^{n-1}, H^1(F_n)) \leq n-2$.

Look at $\dim H^1(F_n) = {}^1E_\infty \leq n-1$; combined with our previous bound, we have $\dim H^1(F_n) = n-1$. This implies that ${}^1E_n ={}^1 E_\infty$ since they have equal dimensions. This implies that $E_2^{0,1}$ are cocycles for differentials.

Further, note that $\operatorname{im} i^*|_{\deg 1} = E_\infty^{0,1} = H^1(F_{n-1})$. Since cohomology of the codomain is generated in degree 1, this implies that i^* is surjective, so the SSS is trivial. This implies the Poincaré polynomial is as we said it is, by a familiar technique. □

Step 2: triviality of the SSS of $(\star)$.

Proposition 4.7. *The SSS for $(\star)$ is trivial.*

Proof. Note that $\dim{}^1 E_2 = \dim H^1(\mathrm{BO}(n), H^0(F_n)) + \dim H^0(\mathrm{BO}(n), H^1(F_n))$. The first is equal to 1, and the second is $\leq n+1$, and the second is $\leq n-1$, so the total is $\leq n$.

Now look at $\dim{}^1E_\infty \leq \dim{}^1 E_2$, which is an equality for dimension reasons. We have $\dim{}^1 E_2 \geq \dim{}^1 E_\infty$, and hence $H^0(\mathrm{BO}(n), H^1(F_n)) = H^1(F_n)$. For reasons relating to generation t degree 1, we also have $H^0(\mathrm{BO}(n), H^k(F_n)) = H^k(F_n)$. Hence $H^*(\mathrm{BQ}(n)) \twoheadrightarrow H^*(F_n)$. Hence the SSS is trivial. □

This allows us to immediately compute the Poincaré polynomial

$$P(\mathrm{BO}(n)) = \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)}.$$

[5] In particular, $P(\mathbb{P}^{n-1}) = \frac{1-t^n}{1-t}$, so since passing to the associated graded preserves graded dimension, triviality implies that the Poincaré series are multiplicative, and we can prove the Poincaré series computation inductively. I'll skip this.

Step 3: containment of the image of ρ^* in the symmetric polynomials. Combinatorics exists:

Lemma 4.8. $P(\mathbb{F}_2[x_1, \ldots, x_n]^{\Sigma_{in}}) = P(\mathrm{BO}(n))$.

Use Schur polynomials. Back to the topology.

Proposition 4.9. in $\rho^* \subset \mathbb{F}_2[x_1, \ldots, x_n]^{\Sigma_{in}}$.

Proof. $\Sigma_{in} = N_n/Q(n)$. Write down the classifying space fibration:

$$Q(n) \hookrightarrow EQ(n) \to BQ(n)$$

N_n acts on this, and acts on polynomials by permuting the generators in $\mathbb{F}_2[x_1, \ldots, x_n]$. The normalizer N_n also acts on

$$O(n) \to EO(n) \to \mathrm{BO}(n).$$

the action on $\mathrm{BO}(n)$ is homotopically trivial[6], which we could use... Instead, we know the groups, so we can check concretely that this acts trivially on the cohomology, which gives the image containment. ☐

Hidden step 4: talking about $p^*(w_i) = e_i$. Once we know the $p^*(w_i) = e_i$ statement, universal relations on Steifel-Whitney classes come down to relations on $H^*(\mathrm{BO}(n))$.

[6] This is a general fact stated by Haynes, which comes down to some categories thing I didn't catch...