

# HIGHER CHROMATIC THOM SPECTRA VIA UNSTABLE HOMOTOPY THEORY

SANATH DEVALAPURKAR

ABSTRACT. We investigate implications of an old conjecture in unstable homotopy theory related to the Cohen-Moore-Neisendorfer theorem and a conjecture about the  $E_2$ -topological Hochschild cohomology of certain Thom spectra (denoted  $A$ ,  $B$ , and  $T(n)$ ) related to Ravenel's  $X(p^n)$ . We show that these conjectures imply that the orientations  $M\mathrm{Spin} \rightarrow \mathrm{bo}$  and  $M\mathrm{String} \rightarrow \mathrm{tmf}$  admit spectrum-level splittings. This is shown by generalizing a theorem of Hopkins and Mahowald, which constructs  $\mathrm{HF}_p$  as a Thom spectrum, to construct  $\mathrm{BP}\langle n-1 \rangle$ ,  $\mathrm{bo}$ , and  $\mathrm{tmf}$  as Thom spectra (albeit over  $T(n)$ ,  $A$ , and  $B$  respectively, and not over the sphere). This interpretation of  $\mathrm{BP}\langle n-1 \rangle$ ,  $\mathrm{bo}$ , and  $\mathrm{tmf}$  offers a new perspective on Wood equivalences of the form  $\mathrm{bo} \wedge C\eta \simeq \mathrm{bu}$ : they are related to the existence of certain EHP sequences in unstable homotopy theory. This construction of  $\mathrm{BP}\langle n-1 \rangle$  also provides a different lens on the nilpotence theorem. Finally, we prove a  $C_2$ -equivariant analogue of our construction, describing  $\underline{\mathrm{HZ}}$  as a Thom spectrum.

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## 1. INTRODUCTION

**1.1. Statement of the main results.** One of the goals of this paper is to describe a program to prove the following old conjecture (studied, for instance, in [Lau04, LS19], and discussed informally in many places, such as [MR09, Section 7]):

**Conjecture 1.1.1.** *The Ando-Hopkins-Rezk orientation (see [AHR10])  $M\mathrm{String} \rightarrow \mathrm{tmf}$  admits a spectrum-level splitting.*

The key idea in our program is to provide a universal property for mapping out of the spectrum  $\mathrm{tmf}$ . We give a proof which is conditional on an old conjecture from unstable homotopy theory stemming from the Cohen-Moore-Neisendorfer theorem and a conjecture about the  $E_2$ -topological Hochschild cohomology of certain Thom spectra (the latter of which simplifies the proof of the nilpotence theorem from [DHS88]). This universal property exhibits  $\mathrm{tmf}$  as a certain Thom spectrum, similarly to the Hopkins-Mahowald construction of  $\mathrm{HZ}_p$  and  $\mathrm{HF}_p$  as Thom spectra.

To illustrate the gist of our argument in a simpler case, recall Thom's classical result from [Tho54]: the unoriented cobordism spectrum  $\mathrm{MO}$  is a wedge of suspensions of  $\mathrm{HF}_2$ . The simplest way to do so is

to show that  $\text{MO}$  is a  $\text{HF}_2$ -module, which in turn can be done by constructing an  $\text{E}_2$ -map  $\text{HF}_2 \rightarrow \text{MO}$ . The construction of such a map is supplied by the following theorem of Hopkins and Mahowald:

**Theorem** (Hopkins-Mahowald; see [Mah79] and [MRS01, Lemma 3.3]). *Let  $\mu : \Omega^2 S^3 \rightarrow \text{BO}$  denote the real vector bundle over  $\Omega^2 S^3$  induced by extending the map  $S^1 \rightarrow \text{BO}$  classifying the Möbius bundle. Then the Thom spectrum of  $\mu$  is equivalent to  $\text{HF}_2$  as an  $\text{E}_2$ -algebra.*

The Thomification of the  $\text{E}_2$ -map  $\mu : \Omega^2 S^3 \rightarrow \text{BO}$  produces the desired  $\text{E}_2$ -splitting  $\text{HF}_2 \rightarrow \text{MO}$ .

Our argument for Conjecture 1.1.1 takes this approach: we shall show that an old conjecture from unstable homotopy theory and a conjecture about the  $\text{E}_2$ -topological Hochschild homology of certain Thom spectra provide a construction of  $\text{tmf}$  (as well as  $\text{bo}$  and  $\text{BP}\langle n \rangle$ ) as a Thom spectrum, and utilize the resulting universal property of  $\text{tmf}$  to construct an (unstructured) map  $\text{tmf} \rightarrow \text{MString}$ .

Mahowald was the first to consider the question of constructing spectra like  $\text{bo}$  and  $\text{tmf}$  as Thom spectra (see [Mah87]). Later work by Rudyak in [Rud98] sharpened Mahowald's results to show that  $\text{bo}$  and  $\text{bu}$  cannot appear as the Thom spectrum of a  $p$ -complete spherical fibration. Recently, Chatham has shown in [Cha19] that  $\text{tmf}_2^\wedge$  cannot appear as the Thom spectrum of a structured 2-complete spherical fibration over a loop space. Our goal is to argue that these issues are alleviated if we replace “spherical fibrations” with “bundles of  $R$ -lines” for certain well-behaved spectra  $R$ .

The first hint of  $\text{tmf}$  being a generalized Thom spectrum comes from a conjecture of Hopkins and Hahn regarding a construction of the truncated Brown-Peterson spectra  $\text{BP}\langle n \rangle$  as Thom spectra. To state this conjecture, we need to recall some definitions. Recall (see [DHS88]) that  $X(n)$  denotes the Thom spectrum of the map  $\Omega \text{SU}(n) \rightarrow \Omega \text{SU} \simeq \text{BU}$ . Upon completion at a prime  $p$ , the spectra  $X(k)$  for  $p^n \leq k \leq p^{n+1} - 1$  split as a direct sum of suspensions of certain homotopy commutative ring spectra  $T(n)$ , which in turn filter the gap between the  $p$ -complete sphere spectrum and  $\text{BP}$  (in the sense that  $T(0) = \mathbb{S}$  and  $T(\infty) = \text{BP}$ ). Then:

**Conjecture 1.1.2** (Hahn, Hopkins). *There is a map  $f : \Omega^2 S^{|v_n|+3} \rightarrow \text{BGL}_1(T(n))$ , which detects an indecomposable element  $v_n \in \pi_{|v_n|} T(n)$  on the bottom cell of the source, whose Thom spectrum is a form of  $\text{BP}\langle n-1 \rangle$ .*

The primary obstruction to proving that a map  $f$  as in Conjecture 1.1.2 exists stems from the failure of  $T(n)$  to be an  $\text{E}_3$ -ring (due to Lawson; [Law19, Example 1.5.31]). Hahn suggested that one way to get past the failure of  $T(n)$  to be an  $\text{E}_3$ -ring would be via the following conjecture:

**Conjecture 1.1.3** (Hahn). *There is an indecomposable element  $v_n \in \pi_{|v_n|} T(n)$  which lifts to the  $\text{E}_2$ -topological Hochschild cohomology  $\mathfrak{Z}_3(X(p^n))$  of  $X(p^n)$ .*

We do not know how to prove this conjecture (and have no opinion on whether or not it is true). We shall instead show that Conjecture 1.1.2 is implied by the two conjectures alluded to above. We shall momentarily state these conjectures precisely as Conjectures **D** and **E**; let us first state our main results.

We need to introduce some notation. Let  $y(n)$  (resp.  $y_{\mathbb{Z}}(n)$ ) denote the Mahowald-Ravenel-Shick spectrum, constructed as a Thom spectrum over  $\Omega J_{p^{n-1}}(S^2)$  (resp.  $\Omega J_{p^{n-1}}(S^2)\langle 2 \rangle$ ) introduced in [MRS01] to study the telescope conjecture (resp. in [AQ19] as  $z(n)$ ). Let  $A$  denote the  $\text{E}_1$ -quotient  $\mathbb{S}/\nu$  of the sphere spectrum by  $\nu \in \pi_3(\mathbb{S})$ ; its mod 2 homology is  $H_*(A) \cong \mathbb{F}_2[\zeta_1^4]$ . Let  $B$  denote the  $\text{E}_1$ -ring introduced in [Dev19b, Construction 3.1]. It may be constructed as the Thom spectrum of a vector bundle over an  $\text{E}_1$ -space  $N$  which sits in a fiber sequence  $\Omega S^9 \rightarrow N \rightarrow \Omega S^{13}$ . The mod 2 homology of  $B$  is  $H_*(B) \cong \mathbb{F}_2[\zeta_1^8, \zeta_2^4]$ .

We also need to recall some unstable homotopy theory. In [CMN79a, CMN79b, Nei81], Cohen, Moore, and Neisendorfer constructed a map  $\phi_n : \Omega^2 S^{2n+1} \rightarrow S^{2n-1}$  whose composite with the double suspension  $S^{2n-1} \rightarrow \Omega^2 S^{2n+1}$  is the degree  $p$  map. Such a map was also constructed by Gray in [Gra89b, Gra88]. In Section 4.1, we introduce the related notion of a *charming map* (Definition 4.1.1), one example of which is the Cohen-Moore-Neisendorfer map.

Our main result is then:

**Theorem A.** *Suppose  $R$  is a base spectrum of height  $n$  as in the second line of Table 1. Let  $K_{n+1}$  denote the fiber of a charming map  $\Omega^2 S^{2p^{n+1}+1} \rightarrow S^{2p^{n+1}-1}$ . Then Conjectures **D** and **E** imply that there is a map  $K_{n+1} \rightarrow \mathrm{BGL}_1(R)$  such that the mod  $p$  homology of the Thom spectrum  $K_{n+1}^\mu$  is isomorphic to the mod  $p$  homology of the associated designer chromatic spectrum  $\Theta(R)$  as a Steenrod comodule<sup>1</sup>.*

*If  $R$  is any base spectrum other than  $B$ , the Thom spectrum  $K_{n+1}^\mu$  is equivalent to  $\Theta(R)$  upon  $p$ -completion for every prime  $p$ . If Conjecture **F** is true, then the same is true for  $B$ : the Thom spectrum  $K_{n+1}^\mu$  is equivalent to  $\Theta(B) = \mathrm{tmf}$  upon 2-completion.*

Height	0	1	2	$n$	$n$	$n$
Base spectrum $R$	$\mathbb{S}_p^\wedge$	$A$	$B$	$T(n)$	$\gamma(n)$	$\gamma_{\mathbb{Z}}(n)$
Designer chromatic spectrum $\Theta(R)$	$\mathrm{HZ}_p$	$\mathrm{bo}$	$\mathrm{tmf}$	$\mathrm{BP}\langle n \rangle$	$k(n)$	$k_{\mathbb{Z}}(n)$

TABLE 1. To go from a base spectrum “of height  $n$ ”, say  $R$ , in the second line to the third, one takes the Thom spectrum of a bundle of  $R$ -lines over  $K_{n+1}$ .

Naïvely making sense of Theorem **A** relies on knowing that  $T(n)$  admits the structure of an  $\mathbf{E}_1$ -ring; we shall interpret this phrase as in Warning 3.1.6. In Remark 5.4.7, we sketch how Theorem **A** relates to the proof of the nilpotence theorem.

Although the form of Theorem **A** does not resemble Conjecture 1.1.2, we show that Theorem **A** implies the following result.

**Corollary B.** *Conjectures **D** and **E** imply Conjecture 1.1.2.*

In the case  $n = 0$ , Corollary **B** recovers the Hopkins-Mahowald theorem constructing  $\mathrm{HF}_p$ . Moreover, Corollary **B** is true unconditionally when  $n = 0, 1$ .

Using the resulting universal property of  $\mathrm{tmf}$ , we obtain a result pertaining to Conjecture 1.1.1. If  $R$  is an  $\mathbf{E}_1$ - or  $\mathbf{E}_2$ -ring spectrum, let  $\mathfrak{Z}_3(R)$  denote the  $\mathbf{E}_2$ -topological Hochschild cohomology of  $R$ .

**Theorem C.** *Assume that the composite  $\mathfrak{Z}_3(B) \rightarrow B \rightarrow \mathrm{MString}$  is an  $\mathbf{E}_3$ -map. Then Conjectures **D**, **E**, and **F** imply that there is a spectrum-level unital splitting of the Ando-Hopkins-Rezk orientation  $\mathrm{MString}_{(2)} \rightarrow \mathrm{tmf}_{(2)}$ .*

In particular, Conjecture 1.1.1 follows (at least after localizing at  $p = 2$ ; a slight modification of our arguments should work at any prime). We believe that the assumption that the composite  $\mathfrak{Z}_3(B) \rightarrow B \rightarrow \mathrm{MString}$  is an  $\mathbf{E}_3$ -map is too strong: we believe that it can be removed using special properties of fibers of charming maps, and we will return to this in future work.

We stress that these splittings are unstructured; it seems unlikely that they can be refined to structured splittings. In [Dev19b], we showed (unconditionally) that the Ando-Hopkins-Rezk orientation  $\mathrm{MString} \rightarrow \mathrm{tmf}$  induces a surjection on homotopy, a result which is clearly implied by Theorem **C**.

We remark that the argument used to prove Theorem **C** shows that if the composite  $\mathfrak{Z}_3(A) \rightarrow A \rightarrow \mathrm{MSpin}$  is an  $\mathbf{E}_3$ -map, then Conjectures **D** and **E** imply that there is a spectrum-level unital splitting of the Atiyah-Bott-Shapiro orientation  $\mathrm{MSpin} \rightarrow \mathrm{bo}$ . This splitting was originally proved unconditionally (i.e., without assuming Conjecture **D** or Conjecture **E**) by Anderson-Brown-Peterson in [ABP67] via a calculation with the Adams spectral sequence.

**1.2. The statements of Conjectures **D**, **E**, and **F**.** We first state Conjecture **D**. The second part of this conjecture is a compilation of several old conjectures in unstable homotopy theory originally made by Cohen-Moore-Neisendorfer, Gray, and Selick in [CMN79a, CMN79b, Nei81, Gra89b, Gra88, Sel77]. The statement we shall give momentarily differs slightly from the statements made in the literature:

<sup>1</sup>We elected to use the symbol  $\Theta$  because the first two letters of the English spelling of  $\Theta$  and of Thom’s name agree.

for instance, in Conjecture [D\(b\)](#), we demand a  $\mathcal{Q}_1$ -space splitting (Notation [2.2.6](#)), rather than merely a H-space splitting.

**Conjecture D.** *The following statements are true:*

- (a) *The homotopy fiber of any charming map (Definition [4.1.1](#)) is equivalent as a loop space to the loop space on an Anick space.*
- (b) *There exists a  $p$ -local charming map  $f : \Omega^2 S^{2p^n+1} \rightarrow S^{2p^n-1}$  whose homotopy fiber admits a  $\mathcal{Q}_1$ -space retraction off of  $\Omega^2(S^{2p^n}/p)$ . There are also integrally defined maps  $\Omega^2 S^9 \rightarrow S^7$  and  $\Omega^2 S^{17} \rightarrow S^{15}$  whose composite with the double suspension on  $S^7$  and  $S^{15}$  respectively is the degree 2 map, whose homotopy fibers  $K_2$  and  $K_3$  (respectively) admit deloopings, and which admit  $\mathcal{Q}_1$ -space retractions off  $\Omega^2(S^8/2)$  and  $\Omega^2(S^{16}/2)$  (respectively).*

Next, we turn to Conjecture [E](#). This conjecture is concerned with the  $E_2$ -topological Hochschild cohomology of the Thom spectra  $X(p^n - 1)_{(p)}$ ,  $A$ , and  $B$  introduced above.

**Conjecture E.** *Let  $n \geq 0$  be an integer. Let  $R$  denote  $X(p^{n+1} - 1)_{(p)}$ ,  $A$  (in which case  $n = 1$ ), or  $B$  (in which case  $n = 2$ ). Then the element  $\sigma_n \in \pi_{|v_n|-1} R$  lifts to the  $E_2$ -topological Hochschild cohomology  $\mathfrak{H}_3(R)$  of  $R$ , and is  $p$ -torsion in  $\pi_* \mathfrak{H}_3(R)$  if  $R = X(p^{n+1} - 1)_{(p)}$ , and is 2-torsion in  $\pi_* \mathfrak{H}_3(R)$  if  $R = A$  or  $B$ .*

Finally, we state Conjecture [F](#). It is inspired by [[AP76](#), [AL17](#)]. We believe this conjecture is the most approachable of the conjectures stated here.

**Conjecture F.** *Suppose  $X$  is a spectrum which is bounded below and whose homotopy groups are finitely generated over  $\mathbf{Z}_p$ . If there is an isomorphism  $H_*(X; \mathbf{F}_p) \cong H_*(\mathrm{tmf}; \mathbf{F}_p)$  of Steenrod comodules, then there is a homotopy equivalence  $X_p^\wedge \rightarrow \mathrm{tmf}_p^\wedge$  of spectra.*

After proving Theorem [A](#) and Theorem [C](#), we explore relationships between the different spectra appearing on the second line of Table [1](#) in the remainder of the paper. In particular, we prove analogues of Wood's equivalence  $\mathrm{bo} \wedge C\eta \simeq \mathrm{bu}$  (see also [[Mat16](#)]) for these spectra. We argue that these are related to the existence of certain EHP sequences.

Finally, we describe a  $C_2$ -equivariant analogue of Corollary [B](#) at  $n = 1$  as Theorem [7.2.1](#), independently of a  $C_2$ -equivariant analogue of Conjecture [D](#) and Conjecture [E](#). This result constructs  $\mathrm{H}\underline{\mathbf{Z}}$  as a Thom spectrum of an equivariant bundle of invertible  $T(1)_{\mathbf{R}}$ -modules over  $\Omega^\rho S^{2\rho+1}$ , where  $T(1)_{\mathbf{R}}$  is the free  $E_\sigma$ -algebra with a nullhomotopy of the equivariant Hopf map  $\tilde{\eta} \in \pi_\sigma(\mathbb{S})$ , and  $\rho$  and  $\sigma$  are the regular and sign representations of  $C_2$ , respectively. We believe there is a similar result at odd primes, but we defer discussion of this. We discuss why our methods do not work to yield  $\mathrm{BP}\langle n \rangle_{\mathbf{R}}$  for  $n \geq 1$  as in Corollary [B](#).

**1.3. Outline.** Section [2](#) contains a review some of the theory of Thom spectra from the modern perspective, as well as the proof of the classical Hopkins-Mahowald theorem. The content reviewed in this section will appear in various guises throughout this project, hence its inclusion.

In Section [3](#), we study certain  $E_1$ -rings; most of them appear as Thom spectra over the sphere. For instance, we recall some facts about Ravenel's  $X(n)$  spectra, and then define and prove properties about the  $E_1$ -rings  $A$  and  $B$  used in the statement of Theorem [A](#). We state Conjecture [E](#), and discuss (Remark [5.4.7](#)) its relation to the nilpotence theorem, in this section.

In Section [4](#), we recall some unstable homotopy theory, such as the Cohen-Moore-Neisendorfer map and the fiber of the double suspension. These concepts do not show up often in stable homotopy theory, so we hope this section provides useful background to the reader. We state Conjecture [D](#), and then explore properties of Thom spectra of bundles defined over Anick spaces.

In Section [5](#), we state and prove Theorem [A](#) and Corollary [B](#), and state several easy consequences of Theorem [A](#).

In Section 6, we study some applications of Theorem A. For instance, we use it to prove Theorem C, which is concerned with the splitting of certain cobordism spectra. Then, we prove that the Thom spectra in the second line of Table 1 satisfy Wood-like equivalences, and argue that these equivalences are related to EHP sequences. We also study an approximation to the topological Hochschild homology of the chromatic Thom spectra of Table 1.

In Section 7, we prove an equivariant analogue of Corollary B at height 1. We construct equivariant analogues of  $X(n)$  and  $A$ , and describe why our methods fail to produce an equivariant analogue of Corollary B at all heights, even granting an analogue of Conjecture D and Conjecture E.

Finally, in Section 8, we suggest some directions for future research. There are also numerous interesting questions arising from our work, which we have indicated in the body of the paper.

**1.4. Conventions.** Unless indicated otherwise, or if it goes against conventional notational choices, a Latin letter with a numerical subscript (such as  $x_5$ ) denotes an element of degree given by its subscript. If  $X$  is a space and  $R$  is an  $E_1$ -ring spectrum, then  $X^\mu$  will denote the Thom spectrum of some bundle of invertible  $R$ -modules determined by a map  $\mu : X \rightarrow BGL_1(R)$ . We shall often quietly localize or complete at an implicit prime  $p$ . Although we have tried to be careful, *all limits and colimits will be homotopy limits and colimits*; we apologize for any inconvenience this might cause.

We shall denote by  $P^k(p)$  the mod  $p$  Moore space  $S^{k-1} \cup_p e^k$  with top cell in dimension  $k$ . The symbols  $\zeta_i$  and  $\tau_i$  will denote the *conjugates* of the Milnor generators (commonly written nowadays as  $\xi_i$  and  $\tau_i$ , although, as Haynes Miller pointed out to me, our notation for the conjugates was Milnor's original notation) in degrees  $2(p^i - 1)$  and  $2p^i - 1$  for  $p > 2$  and  $2^i - 1$  (for  $\zeta_i$ ) at  $p = 2$ . Unfortunately, we will use  $A$  to denote the  $E_1$ -ring in appearing in Table 1, and write  $A_*$  to denote the dual Steenrod algebra. We hope this does not cause any confusion, since we will always denote the homotopy groups of  $A$  by  $\pi_* A$  and not  $A_*$ .

If  $\mathcal{O}$  is an operad, we will simply write  $\mathcal{O}$ -ring to denote an  $\mathcal{O}$ -algebra object in spectra. A map of  $\mathcal{O}$ -rings respecting the  $\mathcal{O}$ -algebra structure will often simply be called a  $\mathcal{O}$ -map. Unless it is clear that we mean otherwise, all modules over non- $E_\infty$ -algebras will be left modules.

Hood Chatham pointed out to me that  $S^3\langle 4 \rangle$  would be the correct notation for what we denote by  $S^3\langle 3 \rangle = \text{fib}(S^3 \rightarrow K(\mathbb{Z}, 3))$ . Unfortunately, the literature seems to have chosen  $S^3\langle 3 \rangle$  as the preferred notation, so we stick to that in this project.

When we write that Theorem A, Corollary B, or Theorem C implies a statement P, we mean that Conjectures D and Conjecture E (and Conjecture F, if the intended application is to tmf) imply P via Theorem A, Corollary B, or Theorem C.

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## 2. BACKGROUND, AND SOME CLASSICAL POSITIVE AND NEGATIVE RESULTS

**2.1. Background on Thom spectra.** In this section, we will recall some facts about Thom spectra and their universal properties; the discussion is motivated by [ABG<sup>+</sup>14].

**Definition 2.1.1.** Let  $A$  be an  $E_1$ -ring, and let  $\mu : X \rightarrow \mathrm{BGL}_1(A)$  be a map of spaces. The Thom  $A$ -module  $X^\mu$  is defined as the homotopy pushout

$$\begin{array}{ccc} \Sigma_+^\infty \mathrm{GL}_1(A) & \longrightarrow & \Sigma_+^\infty \mathrm{fib}(\mu) \\ \downarrow & & \downarrow \\ A & \longrightarrow & X^\mu. \end{array}$$

**Remark 2.1.2.** Let  $A$  be an  $E_1$ -ring, and let  $\mu : X \rightarrow \mathrm{BGL}_1(A)$  be a map of spaces. The Thom  $A$ -module  $X^\mu$  is the homotopy colimit of the functor  $X \xrightarrow{\mu} \mathrm{BGL}_1(A) \rightarrow \mathrm{Mod}(A)$ , where we have abused notation by identifying  $X$  with its associated Kan complex. If  $A$  is an  $E_1$ - $R$ -algebra, then the  $R$ -module underlying  $X$  can be identified with the homotopy colimit of the composite functor

$$X \xrightarrow{\mu} \mathrm{BGL}_1(A) \rightarrow \mathrm{BAut}_R(A) \rightarrow \mathrm{Mod}(R),$$

where we have identified  $X$  with its associated Kan complex. The space  $\mathrm{BAut}_R(A)$  can be regarded as the maximal subgroupoid of  $\mathrm{Mod}(R)$  spanned by the object  $A$ .

The following is immediate from the description of the Thom spectrum as a Kan extension:

**Proposition 2.1.3.** Let  $R$  and  $R'$  be  $E_1$ -rings with an  $E_1$ -ring map  $R \rightarrow R'$  exhibiting  $R'$  as a right  $R$ -module. If  $f : X \rightarrow \mathrm{BGL}_1(R)$  is a map of spaces, then the Thom spectrum of the composite  $X \rightarrow \mathrm{BGL}_1(R) \rightarrow \mathrm{BGL}_1(R')$  is the base-change  $X^f \wedge_R R'$  of the (left)  $R$ -module Thom spectrum  $X^f$ .

**Corollary 2.1.4.** Let  $R$  and  $R'$  be  $E_1$ -rings with an  $E_1$ -ring map  $R \rightarrow R'$  exhibiting  $R'$  as a right  $R$ -module. If  $f : X \rightarrow \mathrm{BGL}_1(R)$  is a map of spaces such that the composite  $X \rightarrow \mathrm{BGL}_1(R) \rightarrow \mathrm{BGL}_1(R')$  is null, then there is an equivalence  $X^f \wedge_R R' \simeq R' \wedge \Sigma_+^\infty X$ .

Moreover (see e.g. [AB19, Corollary 3.2]):

**Proposition 2.1.5.** Let  $X$  be a  $k$ -fold loop space, and let  $R$  be an  $E_{k+1}$ -ring. Then the Thom spectrum of an  $E_k$ -map  $X \rightarrow \mathrm{BGL}_1(R)$  is an  $E_k$ - $R$ -algebra.

We will repeatedly use the following classical result, which is again a consequence of the observation that Thom spectra are colimits, as well as the fact that total spaces of fibrations may be expressed as colimits; see also [Bea17, Theorem 1].

**Proposition 2.1.6.** Let  $X \xrightarrow{i} Y \rightarrow Z$  be a fiber sequence of  $k$ -fold loop spaces (where  $k \geq 1$ ), and let  $R$  be an  $E_m$ -ring for  $m \geq k + 1$ . Suppose that  $\mu : Y \rightarrow \mathrm{BGL}_1(R)$  is a map of  $k$ -fold loop spaces. Then, there is a  $k$ -fold loop map  $\phi : Z \rightarrow \mathrm{BGL}_1(X^{\mu \circ i})$  whose Thom spectrum is equivalent to  $Y^\mu$  as  $E_{k-1}$ -rings. Concisely, if arrows are labeled by their associated Thom spectra, then there is a diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \longrightarrow & Z \\ & \searrow X^{\mu \circ i} & \downarrow \mu & & \downarrow \phi \\ & & \mathrm{BGL}_1(R) & \longrightarrow & \mathrm{BGL}_1(X^{\mu \circ i}). \end{array}$$

The argument to prove Proposition 2.1.6 also goes through with slight modifications when  $k = 0$ , and shows:

**Proposition.** Let  $X \xrightarrow{i} Y \rightarrow Z$  be a fiber sequence of spaces with  $Z$  connected, and let  $R$  be an  $E_m$ -ring for  $m \geq 1$ . Suppose that  $\mu : Y \rightarrow \mathrm{BGL}_1(R)$  is a map of Kan complexes. Then, there is a map  $\phi : Z \rightarrow \mathrm{BAut}_R(X^{\mu \circ i})$  whose Thom spectrum  $Z^\phi$  is equivalent to  $Y^\mu$  as an  $R$ -module.



We will abusively refer to this result in the sequel also as Proposition 2.1.6.

The following is a slight generalization of one of the main results of [AB19]:

**Theorem 2.1.7.** *Let  $R$  be an  $\mathbf{E}_{k+1}$ -ring for  $k \geq 0$ , and let  $\alpha : Y \rightarrow \mathrm{BGL}_1(R)$  be a map from a pointed space  $Y$ . For any  $0 \leq m \leq k$ , let  $\tilde{\alpha} : \Omega^m \Sigma^m Y \rightarrow \mathrm{BGL}_1(R)$  denote the extension of  $\alpha$ . Then the Thom spectrum  $(\Omega^m \Sigma^m Y)^{\tilde{\alpha}}$  is the free  $\mathbf{E}_m$ - $R$ -algebra  $A$  for which the composite  $Y \rightarrow \mathrm{BGL}_1(R) \rightarrow \mathrm{BGL}_1(A)$  is null. More precisely, if  $A$  is any  $\mathbf{E}_m$ - $R$ -algebra, then*

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbf{E}_m}^R}((\Omega^m \Sigma^m Y)^{\tilde{\alpha}}, A) \simeq \begin{cases} \mathrm{Map}_*(Y, \Omega^\infty A) & \text{if } \alpha : Y \rightarrow \mathrm{BGL}_1(R) \rightarrow \mathrm{BGL}_1(A) \text{ is null} \\ \emptyset & \text{else.} \end{cases}$$

**Remark 2.1.8.** Say  $Y = S^{n+1}$ , so  $\alpha$  detects an element  $\alpha \in \pi_n R$ . Corollary 2.1.7 suggests interpreting the Thom spectrum  $(\Omega^m \Sigma^{m+n+1})^{\tilde{\alpha}}$  as an  $\mathbf{E}_m$ -quotient; to signify this, we will denote it by  $R//_{\mathbf{E}_m} \alpha$ . If  $m = 1$ , then we will simply denote it by  $R//\alpha$ , while if  $m = 0$ , then the  $\mathbf{E}_m$ -quotient is simply the ordinary quotient  $R/\alpha$ .

**2.2. The Hopkins-Mahowald theorem.** The primary motivation for this project is the following miracle (see [Mah79] for  $p = 2$  and [MRS01, Lemma 3.3] for  $p > 2$ ):

**Theorem 2.2.1** (Hopkins-Mahowald). *Let  $\mathbb{S}_p^\wedge$  be the  $p$ -completion of the sphere at a prime  $p$ , and let  $f : S^1 \rightarrow \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$  detect the element  $1 - p \in \pi_1 \mathrm{BGL}_1(\mathbb{S}_p^\wedge) \simeq \mathbb{Z}_p^\times$ . Let  $\mu : \Omega^2 S^3 \rightarrow \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$  denote the  $\mathbf{E}_2$ -map extending  $f$ ; then there is a  $p$ -complete equivalence  $(\Omega^2 S^3)^\mu \rightarrow \mathrm{HF}_p$  of  $\mathbf{E}_2$ -ring spectra.*

It is not too hard to deduce the following result from Theorem 2.2.1:

**Corollary 2.2.2.** *Let  $S^3\langle 3 \rangle$  denote the 3-connected cover of  $S^3$ . Then the Thom spectrum of the composite  $\Omega^2 S^3\langle 3 \rangle \rightarrow \Omega^2 S^3 \xrightarrow{\mu} \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$  is equivalent to  $\mathrm{HZ}_p$  as an  $\mathbf{E}_2$ -ring.*

**Remark 2.2.3.** Theorem 2.2.1 implies a restrictive version of the nilpotence theorem: if  $R$  is an  $\mathbf{E}_2$ -ring spectrum, and  $x \in \pi_* R$  is a simple  $p$ -torsion element which has trivial  $\mathrm{HF}_p$ -Hurewicz image, then  $x$  is nilpotent. This is explained in [MNN15, Proposition 4.19]. Indeed, to show that  $x$  is nilpotent, it suffices to show that the localization  $R[1/x]$  is contractible. Since  $px = 0$ , the localization  $R[1/x]$  is an  $\mathbf{E}_2$ -ring in which  $p = 0$ , so the universal property of Theorem 2.1.7 implies that there is an  $\mathbf{E}_2$ -map  $\mathrm{HF}_p \rightarrow R[1/x]$ . It follows that the unit  $R \rightarrow R[1/x]$  factors through the Hurewicz map  $R \rightarrow R \wedge \mathrm{HF}_p$ . In particular, the multiplication by  $x$  map on  $R[1/x]$  factors as the indicated dotted map:

$$\begin{array}{ccccc} \Sigma^{|x|} R & \xrightarrow{x} & R & \longrightarrow & R[1/x]. \\ \downarrow & & \downarrow & \nearrow & \\ \mathrm{HF}_p \wedge \Sigma^{|x|} R & \xrightarrow{x} & R \wedge \mathrm{HF}_p & & \end{array}$$

However, the bottom map is null (because  $x$  has trivial  $\mathrm{HF}_p$ -Hurewicz image), so  $x$  must be null in  $\pi_* R[1/x]$ . This is possible if and only if  $R[1/x]$  is contractible, as desired. See Proposition 5.4.1 for the analogous connection between Corollary 2.2.2 and nilpotence.

Since an argument similar to the proof of Theorem 2.2.1 will be necessary later, we will recall a proof of this theorem. The key non-formal input is the following result of Steinberger's from [BMMS86]:

**Theorem 2.2.4** (Steinberger). *Let  $\zeta_i$  denote the conjugate to the Milnor generators  $\xi_i$  of the dual Steenrod algebra, and similarly for  $\tau_i$  at odd primes. Then*

$$Q^{p^i} \zeta_i = \zeta_{i+1}, \quad Q^{p^j} \tau_j = \tau_{j+1}$$

for  $i, j + 1 > 0$ .

*Proof of Theorem 2.2.1.* By Corollary 2.1.7, the Thom spectrum  $(\Omega^2 S^3)^\mu$  is the free  $E_2$ -ring with a nullhomotopy of  $p$ . Since  $HF_p$  is an  $E_2$ -ring with a nullhomotopy of  $p$ , we obtain an  $E_2$ -map  $(\Omega^2 S^3)^\mu \rightarrow HF_p$ . To prove that this map is a  $p$ -complete equivalence, it suffices to prove that it induces an isomorphism on mod  $p$  homology.

The mod  $p$  homology of  $(\Omega^2 S^3)^\mu$  can be calculated directly via the Thom isomorphism  $HF_p \wedge (\Omega^2 S^3)^\mu \simeq HF_p \wedge \Sigma_+^\infty \Omega^2 S^3$ . Note that this is *not* an equivalence as  $HF_p \wedge HF_p$ -comodules: the Thom twisting is highly nontrivial.

For simplicity, we will now specialize to the case  $p = 2$ , although the same proof works at odd primes. The homology of  $\Omega^2 S^3$  is classical: it is a polynomial ring generated by applying  $E_2$ -Dyer-Lashof operations to a single generator  $x_1$  in degree 1. Theorem 2.2.4 implies that the same is true for the mod 2 Steenrod algebra: it, too, is a polynomial ring generated by applying  $E_2$ -Dyer-Lashof operations to the single generator  $\zeta_1 = \xi_1$  in degree 1. Since the map  $(\Omega^2 S^3)^\mu \rightarrow HF_2$  is an  $E_2$ -ring map, it preserves  $E_2$ -Dyer-Lashof operations on mod  $p$  homology. By the above discussion, it suffices to show that the generator  $x_1 \in H_*(\Omega^2 S^3)^\mu \cong H_*(\Omega^2 S^3)$  in degree 1 is mapped to  $\zeta_1 \in H_* HF_2$ .

To prove this, note that  $x_1$  is the image of the generator in degree 1 in homology under the double suspension  $S^1 \rightarrow \Omega^2 S^3$ , and that  $\zeta_1$  is the image of the generator in degree 1 in homology under the canonical map  $\mathbb{S}/p \rightarrow HF_p$ . It therefore suffices to show that the Thom spectrum of the spherical fibration  $S^1 \rightarrow BGL_1(\mathbb{S}_p^\wedge)$  detecting  $1 - p$  is simply  $\mathbb{S}/p$ . This is an easy exercise.  $\square$

**Remark 2.2.5.** When  $p = 2$ , one does not need to  $p$ -complete in Theorem 2.2.1: the map  $S^1 \rightarrow BGL_1(\mathbb{S}_2^\wedge)$  factors as  $S^1 \rightarrow BO \rightarrow BGL_1(\mathbb{S})$ , where the first map detects the Möbius bundle over  $S^1$ , and the second map is the  $J$ -homomorphism.

**Notation 2.2.6.** Let  $\mathcal{Q}_1$  denote the (operadic nerve of the) *cup-1 operad* from [Law19, Example 1.3.6]: this is the operad whose  $n$ th space is empty unless  $n = 2$ , in which case it is  $S^1$  with the antipodal action of  $\Sigma_2$ . We will need to slightly modify the definition of  $\mathcal{Q}_1$  when localized at an odd prime  $p$ : in this case, it will denote the operad whose  $n$ th space is a point if  $n < p$ , empty if  $n > p$ , and when  $n = p$  is the ordered configuration space  $\text{Conf}_p(\mathbb{R}^2)$  with the permutation action of  $\Sigma_p$ . Any homotopy commutative ring admits the structure of a  $\mathcal{Q}_1$ -algebra at  $p = 2$ , but at other primes it is slightly stronger to be a  $\mathcal{Q}_1$ -algebra than to be a homotopy commutative ring. If  $k \geq 2$ , any  $E_k$ -algebra structure on a spectrum restricts to a  $\mathcal{Q}_1$ -algebra structure.

**Remark 2.2.7.** As stated in [Law19, Proposition 1.5.29], the operation  $Q_1$  already exists in the mod 2 homology of any  $\mathcal{Q}_1$ -ring  $R$ , where  $\mathcal{Q}_1$  is the cup-1 operad from Notation 2.2.6 — the entire  $E_2$ -structure is not necessary. With our modification of  $\mathcal{Q}_1$  at odd primes as in Remark 2.2.6, this is also true at odd primes.

**Remark 2.2.8.** We will again momentarily specialize to  $p = 2$  for convenience. Steinberger’s calculation in Theorem 2.2.4 can be rephrased as stating that  $Q_1 \zeta_i = \zeta_{i+1}$ , where  $Q_1$  is the *lower-indexed* Dyer-Lashof operation. As in Remark 2.2.7, the operation  $Q_1$  already exists in the mod  $p$  homology of any  $\mathcal{Q}_1$ -ring  $R$ . Since homotopy commutative rings are  $\mathcal{Q}_1$ -algebras in spectra, this observation can be used to prove results of Würdler [Wür86] and Pazhitnov-Rudyak [PR84].

**Remark 2.2.9.** The argument with Dyer-Lashof operations and Theorem 2.2.4 used in the proof of Theorem 2.2.1 will be referred to as the *Dyer-Lashof hopping argument*. It will be used again (in the same manner) in the proof of Theorem A.

**Remark 2.2.10.** Theorem 2.2.1 is *equivalent* to Steinberger’s calculation (Theorem 2.2.4), as well as to Bökstedt’s calculation of  $\text{THH}(\mathbb{F}_p)$  (as a ring spectrum, and not just the calculation of its homotopy).

**2.3. No-go theorems for higher chromatic heights.** In light of Theorem 2.2.1 and Corollary 2.2.2, it is natural to wonder if appropriate higher chromatic analogues of  $HF_p$  and  $H\mathbb{Z}$ , such as  $BP\langle n \rangle$ , bo,



or  $\mathrm{tmf}$  can be realized as Thom spectra of spherical fibrations. The answer is known to be negative (see [Mah87, Rud98, Cha19]) in many cases:

**Theorem 2.3.1** (Mahowald, Rudyak, Chatham). *There is no space  $X$  with a spherical fibration  $\mu : X \rightarrow \mathrm{BGL}_1(\mathbb{S})$  (even after completion) such that  $X^\mu$  is equivalent to  $\mathrm{BP}\langle 1 \rangle$  or  $\mathrm{bo}$ . Moreover, there is no 2-local loop space  $X'$  with a spherical fibration determined by a  $H$ -map  $\mu : X' \rightarrow \mathrm{BGL}_1(\mathbb{S}_2^\wedge)$  such that  $X'^\mu$  is equivalent to  $\mathrm{tmf}_2^\wedge$ .*

The proofs rely on calculations in the unstable homotopy groups of spheres.

**Remark 2.3.2.** Although not written down anywhere, a slight modification of the argument used by Mahowald to show that  $\mathrm{bu}$  is not the Thom spectrum of a spherical fibration over a loop space classified by a  $H$ -map can be used to show that  $\mathrm{BP}\langle 2 \rangle$  at  $p = 2$  (i.e.,  $\mathrm{tmf}_1(3)$ ) is not the Thom spectrum of a spherical fibration over a loop space classified by a  $H$ -map. We do not know a proof that  $\mathrm{BP}\langle n \rangle$  is not the Thom spectrum of a spherical fibration over a loop space classified by a  $H$ -map for all  $n \geq 1$  and all primes, but we strongly suspect this to be true.

**Remark 2.3.3.** A lesser-known no-go result due to Priddy appears in [Lew78, Chapter 2.11], where it is shown that  $\mathrm{BP}$  cannot appear as the Thom spectrum of a double loop map  $\Omega^2 X \rightarrow \mathrm{BGL}_1(\mathbb{S})$ . In fact, the argument shows that the same result is true with  $\mathrm{BP}$  replaced by  $\mathrm{BP}\langle n \rangle$  for  $n \geq 1$ ; we had independently come up with this argument for  $\mathrm{BP}\langle 1 \rangle$  before learning about Priddy's argument. Since Lewis' thesis is not particularly easy to acquire, we give a sketch of Priddy's argument. By the Thom isomorphism and the calculation (see [LN14, Theorem 4.3]):

$$H_*(\mathrm{BP}\langle n-1 \rangle; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[\zeta_1^2, \dots, \zeta_{n-1}^2, \zeta_n^2, \zeta_{n+1}, \dots] & p = 2 \\ \mathbb{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbb{F}_p}(\tau_n, \tau_{n+1}, \dots) & p > 2, \end{cases}$$

we find that the mod  $p$  homology of  $\Omega^2 X$  would be isomorphic as an algebra to a polynomial ring on infinitely many generators, possibly tensored with an exterior algebra on infinitely many generators. The Eilenberg-Moore spectral sequence then implies that the mod  $p$  cohomology of  $X$  is given by

$$H^*(X; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[b_1, \dots, b_n, c_{n+1}, \dots] & p = 2 \\ \mathbb{F}_p[b_1, b_2, \dots] \otimes \Lambda_{\mathbb{F}_p}(c_{n+1}, \dots) & p > 2, \end{cases}$$

where  $|b_i| = 2p^i$  and  $|c_i| = 2p^{i-1} + 1$ . If  $p$  is odd, then since  $|b_1| = 2p$ , we have  $P^1(b_1) = b_1^p$ . However, Liulevicius' formula for  $P^1$  in terms of secondary cohomology operations from [Liu62] involves cohomology operations of odd degrees (which kills  $b_1$  because everything is concentrated in even degrees in the relevant range) and a cohomology operation of degree  $4(p-1)$  (which also kills  $b_1$  since  $4(p-1) + 2p^i$  is never a sum of numbers of the form  $2p^k$  when  $p > 2$ ). It follows that  $b_1^p = 0$ , which is a contradiction. A similar calculation works at  $p = 2$ , using Adams' study of secondary mod 2 cohomology operations in [Ada60].

Our primary goal in this project is to argue that the issues in Theorem 2.3.1 are alleviated if we replace  $\mathrm{BGL}_1(\mathbb{S})$  with the delooping of the space of units of an appropriate replacement of  $\mathbb{S}$ . In the next section, we will construct these replacements of  $\mathbb{S}$ .

### 3. SOME THOM SPECTRA

In this section, we introduce certain  $\mathbf{E}_1$ -rings; most of them appear as Thom spectra over the sphere. The following table summarizes the spectra introduced in this section and gives references to their locations in the text. The spectra  $A$  and  $B$  were introduced in [Dev19b].

Thom spectrum	Definition	“Height”	BP-homology
$T(n)$	Theorem 3.1.5	$n$	Theorem 3.1.5
$y(n)$ and $y_Z(n)$	Definition 3.2.1	$n$	Proposition 3.2.2
$A$	Definition 3.2.7	1	Proposition 3.2.11
$B$	Definition 3.2.16	2	Proposition 3.2.18

TABLE 2. Certain Thom spectra and their homologies.

**3.1. Ravenel’s  $X(n)$  spectra.** The proof of the nilpotence theorem in [DHS88, HS98] crucially relied upon certain Thom spectra arising from Bott periodicity; these spectra first appeared in Ravenel’s seminal paper [Rav84].

**Definition 3.1.1.** Let  $X(n)$  denote the Thom spectrum of the  $E_2$ -map  $\Omega\mathrm{SU}(n) \subseteq \mathrm{BU} \xrightarrow{J} \mathrm{BGL}_1(\mathbb{S})$ , where the first map arises from Bott periodicity.

**Example 3.1.2.** The  $E_2$ -ring  $X(1)$  is the sphere spectrum, while  $X(\infty)$  is  $\mathrm{MU}$ . Since the map  $\Omega\mathrm{SU}(n) \rightarrow \mathrm{BU}$  is an equivalence in dimensions  $\leq 2n - 2$ , the same is true for the map  $X(n) \rightarrow \mathrm{MU}$ ; the first dimension in which  $X(n)$  has an element in its homotopy which is not detected by  $\mathrm{MU}$  is  $2n - 1$ .

**Remark 3.1.3.** The  $E_2$ -structure on  $X(n)$  does *not* extend to an  $E_3$ -structure (see [Law19, Example 1.5.31]). If  $X(n)$  admits such an  $E_3$ -structure, then the induced map  $H_*(X(n)) \rightarrow H_*(\mathrm{HF}_p)$  on mod  $p$  homology would commute with  $E_3$ -Dyer-Lashof operations. However, we know that the image of  $H_*(X(n))$  in  $H_*(\mathrm{HF}_p)$  is  $\mathbb{F}_p[\zeta_1^2, \dots, \zeta_n^2]$ ; since Steinberger’s calculation (Theorem 2.2.4) implies that  $Q_2(\zeta_i^2) = \zeta_{i+1}^2$  via the relation  $Q_2(x^2) = Q_1(x)^2$ , we find that the image of  $H_*(X(n))$  in  $H_*(\mathrm{HF}_p)$  cannot be closed under the  $E_3$ -Dyer-Lashof operation  $Q_2$ .

**Remark 3.1.4.** The proof of the nilpotence theorem shows that each of the  $X(n)$  detects nilpotence. However, it is known (see [Rav84]) that  $\langle X(n) \rangle > \langle X(n+1) \rangle$ .

After localizing at a prime  $p$ , the spectrum  $\mathrm{MU}$  splits as a wedge of suspensions of  $\mathrm{BP}$ ; this splitting comes from the Quillen idempotent on  $\mathrm{MU}$ . The same is true of the  $X(n)$  spectra, as explained in [Rav86, Section 6.5]: a multiplicative map  $X(n)_{(p)} \rightarrow X(n)_{(p)}$  is determined by a polynomial  $f(x) = \sum_{0 \leq i \leq n-1} a_i x^{i+1}$ , with  $a_0 = 1$  and  $a_i \in \pi_{2i}(X(n)_{(p)})$ . One can use this to define a truncated form of the Quillen idempotent  $\epsilon_n$  on  $X(n)_{(p)}$  (see [Hop84]), and thereby obtain a summand of  $X(n)_{(p)}$ . We summarize the necessary results in the following theorem.

**Theorem 3.1.5.** *Let  $n$  be such that  $p^n \leq k \leq p^{n+1} - 1$ . Then  $X(k)_{(p)}$  splits as a wedge of suspensions of the spectrum  $T(n) = \epsilon_{p^n} \cdot X(p^n)_{(p)}$ .*

- *The map  $T(n) \rightarrow \mathrm{BP}$  is an equivalence in dimensions  $\leq |v_{n+1}| - 2$ , so there is an indecomposable element  $v_i \in \pi_* T(n)$  which maps to an indecomposable element in  $\pi_* \mathrm{BP}$  for  $0 \leq i \leq n$ .*
- *This map induces the inclusion  $\mathrm{BP}_* T(n) = \mathrm{BP}_*[t_1, \dots, t_n] \subseteq \mathrm{BP}_*(\mathrm{BP})$  on BP-homology, and the inclusions  $\mathbb{F}_2[\zeta_1^2, \dots, \zeta_n^2] \subseteq \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \dots]$  and  $\mathbb{F}_p[\zeta_1, \dots, \zeta_n] \subseteq \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$  on mod 2 and mod  $p$  homology.*
- *$T(n)$  is a homotopy associative and  $\mathcal{Q}_1$ -algebra spectrum.*

**Warning 3.1.6.** We have not proved that  $T(n)$  admits the structure of an  $E_1$ -ring (even though it is true; this is unpublished work of Beardsley-Hahn-Lawson). We shall nonetheless abusively interpret the phrase “Thom spectrum  $X^\mu$  of a map  $\mu : X \rightarrow \mathrm{BGL}_1(T(n))$ ” where  $\mu$  arises via a map  $X \xrightarrow{\mu_0} \mathrm{BGL}_1(X(p^{n+1}-1))$  to mean the base-change  $X^{\mu_0} \wedge_{X(p^{n+1}-1)} T(n)$ . Note that the notation  $\mathrm{BGL}_1(T(n))$  is meaningless unless  $T(n)$  admits an  $E_1$ -structure.

It is believed that  $T(n)$  in fact admits more structure:

**Conjecture 3.1.7.** *The  $\mathcal{Q}_1$ -ring structure on  $T(n)$  extends to an  $E_2$ -ring structure.*

**Remark 3.1.8.** This is true at  $p = 2$  and  $n = 1$ . Indeed, in this case  $T(1) = X(2)$  is the Thom spectrum of the bundle given by the 2-fold loop map  $\Omega S^3 = \Omega^2 \text{BSU}(2) \rightarrow \text{BU}$ , induced by the inclusion  $\text{BSU}(2) \rightarrow \text{B}^3\text{U} = \text{BSU}$ .

**Remark 3.1.9.** Conjecture 3.1.7 is true at  $p = 2$  and  $n = 2$ . The Stiefel manifold  $V_2(\mathbf{H}^2)$  sits in a fiber sequence

$$S^3 \rightarrow V_2(\mathbf{H}^2) \rightarrow S^7.$$

There is an equivalence  $V_2(\mathbf{H}^2) \simeq \text{Sp}(2)$ , so  $\Omega V_2(\mathbf{H}^2)$  admits the structure of a double loop space. There is an  $\mathbf{E}_2$ -map  $\mu : \Omega V_2(\mathbf{H}^2) \rightarrow \text{BU}$ , given by taking double loops of the composite

$$\text{BSp}(2) \rightarrow \text{BSU}(4) \rightarrow \text{BSU} \simeq \text{B}^3\text{U}.$$

The map  $\mu$  admits a description as the left vertical map in the following map of fiber sequences:

$$\begin{array}{ccccc} \Omega V_2(\mathbf{H}^2) & \longrightarrow & \Omega S^7 & \longrightarrow & S^3 \\ \downarrow \mu & & \downarrow & & \downarrow \\ \text{BU} & \longrightarrow & * & \longrightarrow & \text{B}^2\text{U}. \end{array}$$

Here, the map  $S^3 \rightarrow \text{B}^2\text{U}$  detects the generator of  $\pi_2(\text{BU})$  (which maps to  $\eta \in \pi_2(\text{BGL}_1(\mathbb{S}))$ ) under the J-homomorphism). The Thom spectrum  $\Omega V_2(\mathbf{H}^2)^\mu$  is equivalent to  $T(2)$ , and it follows that  $T(2)$  admits the structure of an  $\mathbf{E}_2$ -ring. We do not know whether  $T(n)$  is the Thom spectrum of a  $p$ -complete spherical fibration over some space for  $n \geq 3$ .

It is possible to construct  $X(n+1)$  as an  $X(n)$ -algebra (see also [Bea18]):

**Construction 3.1.10.** There is a fiber sequence

$$\Omega \text{SU}(n) \rightarrow \Omega \text{SU}(n+1) \rightarrow \Omega S^{2n+1}.$$

According to Proposition 2.1.6, the spectrum  $X(n+1)$  is the Thom spectrum of an  $\mathbf{E}_1$ -map  $\Omega S^{2n+1} \rightarrow \text{BGL}_1(\Omega \text{SU}(n))^\mu = \text{BGL}_1(X(n))$ . This  $\mathbf{E}_1$ -map is the extension of a map  $S^{2n} \rightarrow \text{BGL}_1(X(n))$  which detects an element  $\chi_n \in \pi_{2n-1}X(n)$ . This element is equivalently determined by the map  $\Sigma_+^\infty \Omega^2 S^{2n+1} \rightarrow X(n)$  given by the Thomification of the nullhomotopic composite

$$\Omega^2 S^{2n+1} \rightarrow \Omega \text{SU}(n) \rightarrow \Omega \text{SU}(n+1) \rightarrow \Omega \text{SU} \simeq \text{BU},$$

where the first two maps form a fiber sequence. By Proposition 2.1.6,  $X(n+1)$  is the free  $\mathbf{E}_1$ - $X(n)$ -algebra with a nullhomotopy of  $\chi_n$ .

**Remark 3.1.11.** Another construction of the map  $\chi_n \in \pi_{2n-1}X(n)$  from Construction 3.1.10 is as follows. There is a map  $i : \mathbb{C}P^{n-1} \rightarrow \Omega \text{SU}(n)$  given by sending a line  $\ell \subseteq \mathbb{C}^n$  to the loop  $S^1 \rightarrow \text{SU}(n) = \text{Aut}(\mathbb{C}^n, \langle, \rangle)$  defined as follows:  $\theta \in S^1$  is sent to the (appropriate rescaling of the) unitary transformation of  $\mathbb{C}^n$  sending a vector to its rotation around the line  $\ell$  by the angle  $\theta$ . The map  $i$  Thomifies to a stable map  $\Sigma^{-2}\mathbb{C}P^n \rightarrow X(n)$ . The map  $\chi_n$  is then the composite

$$S^{2n-1} \rightarrow \Sigma^{-2}\mathbb{C}P^n \rightarrow X(n),$$

where the first map is the desuspension of the generalized Hopf map  $S^{2n+1} \rightarrow \mathbb{C}P^n$  which attaches the top cell of  $\mathbb{C}P^{n+1}$ . The fact that this map is indeed  $\chi_n$  follows immediately from the commutativity of the following diagram:

$$\begin{array}{ccccc} S^{2n-1} & \longrightarrow & \mathbb{C}P^{n-1} & \longrightarrow & \mathbb{C}P^n \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^2 S^{2n+1} & \longrightarrow & \Omega \text{SU}(n) & \longrightarrow & \Omega \text{SU}(n+1), \end{array}$$

where the top row is a cofiber sequence, and the bottom row is a fiber sequence.

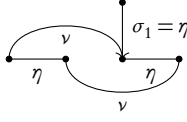


FIGURE 1.  $C\eta \wedge C\nu$  shown horizontally, with 0-cell on the left. The element  $\sigma_1$  is shown.

An easy consequence of the observation in Construction 3.1.10 is the following lemma.

**Lemma 3.1.12.** *Let  $\sigma_n \in \pi_{|v_{n+1}|-1}T(n)$  denote the element  $\chi_{p^{n+1}-1}$ . Then the Thom spectrum of the composite  $\Omega S^{|v_{n+1}|+1} \rightarrow \text{BGL}_1(X(p^{n+1}-1)) \rightarrow \text{BGL}_1(T(n))$  is equivalent to  $T(n+1)$ .*

**Example 3.1.13.** The element  $\sigma_0 \in \pi_{|v_1|-1}T(0) = \pi_{2p-3}\mathbb{S}_{(p)}$  is  $\alpha_1$ .

**Example 3.1.14.** Let us specialize to  $p = 2$ . Theorem 3.1.5 implies that  $H_*T(n) \cong \mathbb{F}_2[\zeta_1^2, \dots, \zeta_n^2]$ . Using this, one can observe that the 6-skeleton of  $T(1)$  is the smash product  $C\eta \wedge C\nu$ , and so  $\sigma_1 \in \pi_5(C\eta \wedge C\nu)$ . This element can be described very explicitly: the cell structure of  $C\eta \wedge C\nu$  is shown in Figure 1, and the element  $\sigma_1$  shown corresponds to the map defined by the relation  $\eta\nu = 0$ .

**Example 3.1.15.** The element  $\sigma_n$  in the Adams-Novikov spectral sequence for  $T(n)$  is represented by the element  $[t_{n+1}]$  in the cobar complex.

A calculation with the Adams-Novikov spectral sequence (as in [Rav02, Theorem 3.17], where  $\sigma_{n-1}$  is denoted by  $\alpha(\hat{v}_1/p)$ ) proves the following:

**Lemma 3.1.16.** *The group  $\pi_{|v_n|-1}X(p^n-1)$  is isomorphic to  $\mathbb{Z}/p$ .*

In particular, the element  $\sigma_{n-1} = \chi_{p^n-1} \in \pi_{|v_n|-1}X(p^n-1)$  is  $p$ -torsion, and the following is a consequence of Example 3.1.15:

**Proposition 3.1.17.** *The Toda bracket  $\langle p, \sigma_{n-1}, 1_{X(p^n)} \rangle$  in  $\pi_{|v_n|}X(p^n)$  contains an indecomposable  $v_n$ .*

**Corollary 3.1.18.** *The element  $\sigma_{n-1} \in \pi_{|v_n|-1}X(p^n-1)$  lifts to  $\pi_{|v_n|+1}(\mathbb{C}P^{|v_n|/2})$  along the map  $\Sigma^{-2}\mathbb{C}P^{|v_n|/2} \rightarrow X(p^n-1)$ .*

*Proof.* By Remark 3.1.11, the map  $\sigma_{n-1} : S^{|v_n|-1} \rightarrow X(p^n-1)$  is given by the composite of the generalized Hopf map  $S^{|v_n|-1} \rightarrow \Sigma^{-2}\mathbb{C}P^{p^n-1}$  with the map  $\Sigma^{-2}\mathbb{C}P^{p^n-1} \rightarrow X(p^n-1)$ . Moreover, this generalized Hopf map is the desuspension of the unstable generalized Hopf map  $S^{|v_n|+1} \rightarrow \mathbb{C}P^{p^n-1}$ , and so  $\sigma_{n-1}$  lifts to an element of the unstable homotopy group  $\pi_{|v_n|+1}(\mathbb{C}P^{|v_n|/2})$ .  $\square$

**3.2. Related Thom spectra.** We now introduce several Thom spectra related to the  $E_1$ -rings  $T(n)$  described in the previous section; some of these were introduced in [Dev19b]. (Relationships to  $T(n)$  will be further discussed in Section 6.2.) For the reader's convenience, we have included a table of the spectra introduced below with internal references to their definitions at the beginning of this section.

The following Thom spectrum was introduced in [MRS01].

**Definition 3.2.1.** Let  $y(n)$  denote the Thom spectrum of the composite

$$\Omega J_{p^n-1}(S^2) \rightarrow \Omega^2 S^3 \xrightarrow{1-p} \text{BGL}_1(\mathbb{S}_p^\wedge).$$

If  $J_{p^n-1}(S^2)\langle 2 \rangle$  denotes the 2-connected cover of  $J_{p^n-1}(S^2)$ , then let  $y_Z(n)$  denote the Thom spectrum of the composite

$$\Omega J_{p^n-1}(S^2)\langle 2 \rangle \rightarrow \Omega^2 S^3 \xrightarrow{1-p} \text{BGL}_1(\mathbb{S}_p^\wedge).$$

The following proposition is stated for  $y(n)$  in [MRS01, Equation 2.8].

**Proposition 3.2.2.** *As  $\mathrm{BP}_*$ -BP-comodules, we have*

$$\mathrm{BP}_*(y(n)) \cong \mathrm{BP}_*/I_n[t_1, \dots, t_n], \quad \mathrm{BP}_*(y_{\mathbf{Z}}(n)) \cong \mathrm{BP}_*/(v_1, \dots, v_{n-1})[t_1, \dots, t_n],$$

where  $I_n$  denotes the invariant ideal  $(p, v_1, \dots, v_{n-1})$ .

An immediate corollary is the following.

**Corollary 3.2.3.** *As  $A_*$ -comodules, we have*

$$H_*(y(n); \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_2[\zeta_1, \zeta_2, \dots, \zeta_n] & p = 2 \\ \mathbf{F}_p[\zeta_1, \zeta_2, \dots, \zeta_n] \otimes E(\tau_0, \dots, \tau_{n-1}) & p \geq 3, \end{cases}$$

and

$$H_*(y_{\mathbf{Z}}(n); \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_2[\zeta_1^2, \zeta_2, \dots, \zeta_n] & p = 2 \\ \mathbf{F}_p[\zeta_1, \zeta_2, \dots, \zeta_n] \otimes E(\tau_1, \dots, \tau_{n-1}) & p \geq 3. \end{cases}$$

We will now relate  $y(n)$  and  $y_{\mathbf{Z}}(n)$  to  $T(n)$ .

**Construction 3.2.4.** Let  $m \leq n$ , and let  $I_m$  be the ideal generated by  $p, v_1, \dots, v_{m-1}$ , where the  $v_i$  are some choices of indecomposables in  $\pi_{|v_i|}(T(n))$  which form a regular sequence. Inductively define  $T(n)/I_m$  as the cofiber of the map

$$T(n)/I_{m-1} \xrightarrow{v_m \wedge 1} T(n) \wedge T(n)/I_{m-1} \rightarrow T(n)/I_m.$$

The BP-homology of  $T(n)/I_m$  is  $\mathrm{BP}_*/I_m[t_1, \dots, t_n]$ . The spectrum  $T(n)/(v_1, \dots, v_{m-1})$  is defined similarly.

**Proposition 3.2.5.** *Let  $p > 2$ . There is an equivalence between  $T(n)/I_n$  (resp.  $T(n)/(v_1, \dots, v_{n-1})$ ) and the spectrum  $y(n)$  (resp.  $y_{\mathbf{Z}}(n)$ ) of Definition 3.2.1.*

*Proof.* We will prove the result for  $y(n)$ ; the analogous proof works for  $y_{\mathbf{Z}}(n)$ . By [Gra89a], the space  $\Omega J_{p^n-1}(S^2)$  is homotopy commutative (since  $p > 2$ ). Moreover, the map  $\Omega J_{p^n-1}(S^2) \rightarrow \Omega^2 S^3$  is a H-map, so  $y(n)$  is a homotopy commutative  $\mathbf{E}_1$ -ring spectrum. It is known (see [Rav86, Section 6.5]) that homotopy commutative maps  $T(n) \rightarrow y(n)$  are equivalent to partial complex orientations of  $y(n)$ , i.e., factorizations

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\quad} & \Sigma^{-2} \mathbf{C}P^{p^n-1} \\ & \searrow 1 & \downarrow \gamma_n \\ & & y(n). \end{array}$$

Such a  $\gamma_n$  indeed exists by obstruction theory: suppose  $k < p^n - 1$ , and we have a map  $\Sigma^{-2} \mathbf{C}P^k \rightarrow y(n)$ . Since there is a cofiber sequence

$$S^{2k-1} \rightarrow \Sigma^{-2} \mathbf{C}P^k \rightarrow \Sigma^{-2} \mathbf{C}P^{k+1}$$

of spectra, the obstruction to extending along  $\Sigma^{-2} \mathbf{C}P^{k+1}$  is an element of  $\pi_{2k-1}y(n)$ . However, the homotopy of  $y(n)$  is concentrated in even degrees in the appropriate range, so a choice of  $\gamma_n$  does indeed exist. Moreover, this choice can be made such that they fit into a compatible family in the sense that there is a commutative diagram

$$\begin{array}{ccc} \Sigma^{-2} \mathbf{C}P^{p^n-1} & \xrightarrow{\quad} & \Sigma^{-2} \mathbf{C}P^{p^{n+1}-1} \\ \gamma_n \downarrow & & \downarrow \gamma_{n+1} \\ y(n) & \xrightarrow{\quad} & y(n+1). \end{array}$$

The formal group law over  $\mathrm{HF}_p$  has infinite height; this forces the elements  $p, v_1, \dots, v_{n-1}$  (defined for the “ $(p^n-1)$ -bud” on  $\pi_* y(n)$ ) to vanish in the homotopy of  $y(n)$ . It follows that the orientation  $T(n) \rightarrow y(n)$  constructed above factors through the quotient  $T(n)/I_n$ . The induced map  $T(n)/I_n \rightarrow y(n)$  can be seen to be an isomorphism on homology (via, for instance, Definition 3.2.1 and Construction 3.2.4).  $\square$

**Remark 3.2.6.** Since  $y(n)$  has a  $v_n$ -self-map, we can form the spectrum  $y(n)/v_n$ ; its mod  $p$  homology is

$$H_*(y(n)/v_n; \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_2[\zeta_1, \dots, \zeta_n] \otimes \Lambda_{\mathbf{F}_2}(\zeta_{n+1}) & p = 2 \\ \mathbf{F}_p[\zeta_1, \dots, \zeta_n] \otimes \Lambda_{\mathbf{F}_p}(\tau_0, \dots, \tau_{n-1}, \tau_n) & p \geq 3. \end{cases}$$

It is in fact possible to give a construction of  $y(1)/v_1$  as a spherical Thom spectrum. We will work at  $p = 2$  for convenience. Define  $Q$  to be the fiber of the map  $2\eta : S^3 \rightarrow S^2$ . There is a map of fiber sequences

$$\begin{array}{ccccc} Q & \longrightarrow & S^3 & \xrightarrow{2\eta} & S^2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{BGL}_1(\mathbb{S}) & \longrightarrow & * & \longrightarrow & \mathrm{B}^2\mathrm{GL}_1(\mathbb{S}). \end{array}$$

By [DM81, Theorem 3.7], the Thom spectrum of the leftmost map is  $y(1)/v_1$ .

We end this section by recalling the definition of two Thom spectra which, unlike  $y(n)$  and  $y_{\mathbf{Z}}(n)$ , are not indexed by integers (we will see that they are only defined at “heights 1 and 2”). These were both introduced in [Dev19b].

**Definition 3.2.7.** Let  $S^4 \rightarrow \mathrm{BSpin}$  denote the generator of  $\pi_4 \mathrm{BSpin} \cong \mathbf{Z}$ , and let  $\Omega S^5 \rightarrow \mathrm{BSpin}$  denote the extension of this map, which classifies a real vector bundle of virtual dimension zero over  $\Omega S^5$ . Let  $A$  denote the Thom spectrum of this bundle.

**Remark 3.2.8.** As mentioned in the introduction, the spectrum  $A$  has been intensely studied by Mahowald and his coauthors in (for instance) [Mah79, DM81, Mah81b, Mah81a, Mah82, MU77], where it is often denoted  $X_5$ .

**Remark 3.2.9.** The map  $\Omega S^5 \rightarrow \mathrm{BSpin}$  is one of  $\mathbf{E}_1$ -spaces, so the Thom spectrum  $A$  admits the structure of an  $\mathbf{E}_1$ -ring with an  $\mathbf{E}_1$ -map  $A \rightarrow \mathrm{MSpin}$ .

**Remark 3.2.10.** There are multiple equivalent ways to characterize this Thom spectrum. For instance, the  $J$ -homomorphism  $\mathrm{BSpin} \rightarrow \mathrm{BGL}_1(\mathbb{S})$  sends the generator of  $\pi_4 \mathrm{BSpin}$  to  $\nu \in \pi_4 \mathrm{BGL}_1(\mathbb{S}) \cong \pi_3 \mathbb{S}$ . The universal property of Thom spectra in Theorem 2.1.7 shows that  $A$  is the free  $\mathbf{E}_1$ -ring  $\mathbb{S} // \nu$  with a nullhomotopy of  $\nu$ . Note that  $A$  is defined integrally, and not just  $p$ -locally for some prime  $p$ .

The following result is [Dev19b, Proposition 2.6]; it is proved there at  $p = 2$ , but the argument clearly works for  $p = 3$  too.

**Proposition 3.2.11.** *There is an isomorphism  $\mathrm{BP}_*(A) \cong \mathrm{BP}_*[y_2]$ , where  $|y_2| = 4$ . There is a map  $A_{(p)} \rightarrow \mathrm{BP}$ . Under the induced map on  $\mathrm{BP}$ -homology,  $y_2$  maps to  $t_1^2$  mod decomposables at  $p = 2$ , and to  $t_1$  mod decomposables at  $p = 3$ .*

**Remark 3.2.12.** For instance, when  $p = 2$ , we have  $\mathrm{BP}_*(A) \cong \mathrm{BP}_*[t_1^2 + v_1 t_1]$ .

An immediate corollary is the following.



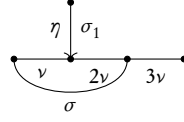


FIGURE 2. 15-skeleton of  $A$  at the prime 2 shown horizontally, with 0-cell on the left. The element  $\sigma_1$  is shown.

**Corollary 3.2.13.** *As  $A_*$ -comodules, we have*

$$H_*(A; \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_2[\zeta_1^4] & p = 2 \\ \mathbf{F}_3[\zeta_1] & p = 3 \\ \mathbf{F}_p[x_4] & p \geq 5, \end{cases}$$

where  $x_4$  is a polynomial generator in degree 4.

**Example 3.2.14.** Let us work at  $p = 2$  for convenience. Example 3.1.14 showed that  $\sigma_1$  is the element in  $\pi_5(C\eta \wedge C\nu)$  given by the lift of  $\eta$  to the 4-cell (which is attached to the bottom cell by  $\nu$ ) via a nullhomotopy of  $\eta\nu$ . In particular,  $\sigma_1$  already lives in  $\pi_5(C\nu)$ , and as such defines an element of  $\mathbb{S}/\nu = A$  (by viewing  $C\nu$  as the 4-skeleton of  $A$ ); note that, by construction, this element is 2-torsion. There is a canonical map  $A \rightarrow T(1)$ , and the image of  $\sigma_1 \in \pi_5(A)$  under this map is its namesake in  $\pi_5(T(1))$ . See Figure 2.

**Remark 3.2.15.** The element  $\sigma_1 \in \pi_5(A_{(2)})$  defined in Example 3.2.14 in fact lifts to an element of  $\pi_5(A)$ , because the relation  $\eta\nu = 0$  is true integrally, and not just 2-locally. An alternate construction of this map is the following. The Hopf map  $\eta_4 : S^5 \rightarrow S^4$  (which lives in the stable range) defines a map  $S^5 \rightarrow S^4 \rightarrow \Omega S^5$  whose composite to  $B\text{Spin}$  is null (since  $\pi_5(B\text{Spin}) = 0$ ). Upon Thomification of the composite  $S^5 \rightarrow \Omega S^5 \rightarrow B\text{Spin}$ , one therefore gets a map  $S^5 \rightarrow A$  whose composite with  $A \rightarrow M\text{Spin}$  is null. The map  $S^5 \rightarrow A$  is the element  $\sigma_1 \in \pi_5(A)$ .

Finally, we have:

**Definition 3.2.16.** Let  $BN$  be the space defined by the homotopy pullback

$$\begin{array}{ccc} BN & \longrightarrow & S^{13} \\ \downarrow & & \downarrow f \\ \text{BO}(9) & \longrightarrow & \text{BO}(10), \end{array}$$

where the map  $f : S^{13} \rightarrow \text{BO}(10)$  detects an element of  $\pi_{12}\text{O}(10) \cong \mathbf{Z}/12$ . There is a fiber sequence

$$S^9 \rightarrow \text{BO}(9) \rightarrow \text{BO}(10),$$

and the image of  $f$  under the boundary map in the long exact sequence of homotopy detects  $2\nu \in \pi_{12}(S^9) \cong \mathbf{Z}/24$ . In particular, there is a fiber sequence

$$S^9 \rightarrow BN \rightarrow S^{13}.$$

If  $N$  is defined to be  $\Omega BN$ , then there is a fiber sequence

$$N \rightarrow \Omega S^{13} \rightarrow S^9.$$

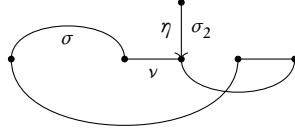


FIGURE 3. Steenrod module structure of the 20-skeleton of  $B$ ; the bottom cell (in dimension 0) is on the left; straight lines are  $Sq^4$ , and curved lines correspond to  $Sq^8$  and  $Sq^{16}$ , in order of increasing length. The bottom two attaching maps of  $B$  are labeled. The map  $\sigma_2$  is shown.

Define a map  $N \rightarrow BString$  via the map of fiber sequences

$$\begin{array}{ccccc} N & \longrightarrow & \Omega S^{13} & \longrightarrow & S^9 \\ \downarrow & & \downarrow & & \downarrow \\ BString & \longrightarrow & * & \longrightarrow & B^2String, \end{array}$$

where the map  $S^9 \rightarrow B^2String$  detects a generator of  $\pi_8 BString$ . The map  $N \rightarrow BString$  is in fact one of  $E_1$ -spaces. Let  $B$  denote the Thom spectrum of the induced bundle over  $N$ .

We introduced the spectrum  $B$  and studied its Adams-Novikov spectral sequence in [Dev19b]. The Steenrod module structure of the 20-skeleton of  $B$  is shown in [Dev19b, Figure 1], and is reproduced here as Figure 3. As mentioned in the introduction, the spectrum  $B$  has been briefly studied under the name  $\bar{X}$  in [HM02].

**Remark 3.2.17.** As with  $A$ , there are multiple different ways to characterize  $B$ . There is a fiber sequence

$$\Omega S^9 \rightarrow N \rightarrow \Omega S^{13},$$

and the map  $\Omega S^9 \rightarrow N \rightarrow BString$  is an extension of the map  $S^8 \rightarrow BString$  detecting a generator. Under the  $J$ -homomorphism  $BString \rightarrow BGL_1(\mathbb{S})$ , this generator maps to  $\sigma \in \pi_8 BGL_1(\mathbb{S}) \cong \pi_7 \mathbb{S}$ , so the Thom spectrum of the bundle over  $\Omega S^9$  determined by the map  $\Omega S^9 \rightarrow BString$  is the free  $E_1$ -ring  $\mathbb{S} // \sigma$  with a nullhomotopy of  $\nu$ . Proposition 2.1.6 now implies that  $N$  is the Thom spectrum of a map  $\Omega S^{13} \rightarrow BGL_1(\mathbb{S} // \sigma)$ . While a direct definition of this map is not obvious, we note that the restriction to the bottom cell  $S^{12}$  of the source detects an element  $\tilde{\nu}$  of  $\pi_{12} BGL_1(\mathbb{S} // \sigma) \cong \pi_{11} \mathbb{S} // \sigma$ . This in turn factors through the 11-skeleton of  $\mathbb{S} // \sigma$ , which is the same as the 8-skeleton of  $\mathbb{S} // \sigma$  (namely,  $C\sigma$ ). This element is precisely a lift of the map  $\nu : S^{11} \rightarrow S^8$  to  $C\sigma$  determined by a nullhomotopy of  $\sigma\nu$  in  $\pi_* \mathbb{S}$ . It is detected by  $h_{22}$  in the Adams spectral sequence.

The following result is [Dev19b, Proposition 3.2]; it is proved there at  $p = 2$ , but the argument clearly works for  $p \geq 3$  too.

**Proposition 3.2.18.** *The  $BP_*$ -algebra  $BP_*(B)$  is isomorphic to a polynomial ring  $BP_*[b_4, y_6]$ , where  $|b_4| = 8$  and  $|y_6| = 12$ . There is a map  $B_{(p)} \rightarrow BP$ . On  $BP_*$ -homology, the elements  $b_4$  and  $y_6$  map to  $t_1^4$  and  $t_2^2$  mod decomposables at  $p = 2$ , and  $y_6$  maps to  $t_1^3$  mod decomposables at  $p = 3$ .*

An immediate corollary is the following.

**Corollary 3.2.19.** *As  $A_*$ -comodules, we have*

$$H_*(B; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[\zeta_1^8, \zeta_2^4] & p = 2 \\ \mathbb{F}_3[\zeta_1^2, b_4] & p = 3 \\ \mathbb{F}_3[\zeta_1, x_{12}] & p = 5 \\ \mathbb{F}_p[x_8, x_{12}] & p \geq 7, \end{cases}$$

where  $x_8$  and  $x_{12}$  are polynomial generators in degree 8 and 12, and  $b_4$  is an element in degree 8.

**Example 3.2.20.** For simplicity, let us work at  $p = 2$ . There is a canonical ring map  $B \rightarrow T(2)$ , and the element  $\sigma_2 \in \pi_{13}T(2)$  lifts to  $B$ . We can be explicit about this: the 12-skeleton of  $B$  is shown in Figure 3, and  $\sigma_2$  is the element of  $\pi_{13}(B)$  existing thanks to the relation  $\eta\nu = 0$  and the fact that the Toda bracket  $\langle \eta, \nu, \sigma \rangle$  contains 0. This also shows that  $\sigma_2 \in \pi_{13}(B)$  is 2-torsion.

**Remark 3.2.21.** The element  $\sigma_2 \in \pi_{13}(B_{(2)})$  defined in Example 3.2.20 in fact lifts to an element of  $\pi_{13}(B)$ , because the relations  $\nu\sigma = 0$ ,  $\eta\nu = 0$ , and  $0 \in \langle \eta, \nu, \sigma \rangle$  are all true integrally, and not just 2-locally. An alternate construction of this map  $S^{13} \rightarrow B$  is the following. The Hopf map  $\eta_{12} : S^{13} \rightarrow S^{12}$  (which lives in the stable range) defines a map  $S^{13} \rightarrow S^{12} \rightarrow \Omega S^{13}$ . Moreover, the composite  $S^{13} \rightarrow \Omega S^{13} \rightarrow S^9$  is null, since it detects an element of  $\pi_{13}(S^9) = 0$ ; choosing a nullhomotopy of this composite defines a lift  $S^{13} \rightarrow N$ . (In fact, this comes from a map  $S^{14} \rightarrow BN$ .) The composite  $S^{13} \rightarrow N \rightarrow B\text{String}$  is null (since  $\pi_{13}(B\text{String}) = 0$ ). Upon Thomification, we obtain a map  $S^{13} \rightarrow B$  whose composite with  $B \rightarrow M\text{String}$  is null; the map  $S^{13} \rightarrow B$  is the element  $\sigma_2 \in \pi_{13}(B)$ .

The following theorem packages some information contained in this section.

**Theorem 3.2.22.** *Let  $R$  denote any of the spectra in Table 2, and let  $n$  denote its “height”. If  $R = T(n), \gamma(n)$ , or  $\gamma_{\mathbf{Z}}(n)$ , then there is a map  $T(n) \rightarrow R$ , and if  $R = A$  (resp.  $B$ ), then there is a map from  $T(1)$  (resp.  $T(2)$ ) to  $R$ . In the first three cases, there is an element  $\sigma_n \in \pi_{|v_{n+1}|-1}R$  coming from  $\sigma_n \in \pi_{|v_{n+1}|-1}T(n)$ , and in the cases  $R = A$  and  $B$ , there are elements  $\sigma_1 \in \pi_5(A)$  and  $\sigma_2 \in \pi_{13}(B)$  mapping to the corresponding elements in  $T(1)_{(2)}$  and  $T(2)_{(2)}$ , respectively. Moreover,  $\sigma_n$  is  $p$ -torsion in  $\pi_*R$ ; similarly,  $\sigma_1$  and  $\sigma_2$  are 2-torsion in  $\pi_*A_{(2)}$  and  $\pi_*B_{(2)}$ .*

*Proof.* The existence statement for  $T(n)$  is contained in Theorem 3.1.5, while the torsion statement is the content of Lemma 3.1.16. The claims for  $\gamma(n)$  and  $\gamma_{\mathbf{Z}}(n)$  now follow from Proposition 3.2.5. The existence and torsion statements for  $A$  and  $B$  are contained in Examples 3.2.14 and 3.2.20.  $\square$

The elements in Theorem 3.2.22 can in fact be extended to infinite families; this is discussed in Section 5.4.

**3.3. Centers of Thom spectra.** In this section, we review some of the theory of  $E_k$ -centers and state Conjecture E. We begin with the following important result, and refer to [Fra13] and [Lur16, Section 5.5.4] for proofs.

**Theorem 3.3.1.** *Let  $\mathcal{C}$  be a symmetric monoidal presentable  $\infty$ -category, and let  $A$  be an  $E_k$ -algebra in  $\mathcal{C}$ . Then the category of  $E_k$ - $A$ -modules is equivalent to the category of left modules over the factorization homology  $U(A) = \int_{S^{k-1} \times \mathbf{R}} A$  (known as the enveloping algebra of  $A$ ), which is an  $E_1$ -algebra in  $\mathcal{C}$ .*

**Definition 3.3.2.** The  $E_{k+1}$ -center  $\mathfrak{Z}(A)$  of an  $E_k$ -algebra  $A$  in  $\mathcal{C}$  is the  $(E_{k+1})$ -Hochschild cohomology  $\text{End}_{U(A)}(A)$ , where  $A$  is regarded as a left module over its enveloping algebra via Theorem 3.3.1.

**Remark 3.3.3.** We are using slightly different terminology than the one used in [Lur16, Section 5.3]: our  $E_{k+1}$ -center is his  $E_k$ -center. In other words, Lurie’s terminology expresses the structure on the input, while our terminology expresses the structure on the output.

The following proposition summarizes some results from [Fra13] and [Lur16, Section 5.3].

**Proposition 3.3.4.** *The  $E_{k+1}$ -center  $\mathfrak{Z}(A)$  of an  $E_k$ -algebra  $A$  in a symmetric monoidal presentable  $\infty$ -category  $\mathcal{C}$  exists, and satisfies the following properties:*

- (a)  $\mathfrak{Z}(A)$  is the universal  $\mathbf{E}_k$ -algebra of  $\mathcal{C}$  which fits into a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A \otimes \mathfrak{Z}(A) \\ & \searrow & \downarrow \\ & & A \end{array}$$

in  $\mathrm{Alg}_{\mathbf{E}_k}(\mathcal{C})$ .

- (b) The  $\mathbf{E}_k$ -algebra  $\mathfrak{Z}(A)$  of  $\mathcal{C}$  defined via this universal property in fact admits the structure of an  $\mathbf{E}_{k+1}$ -algebra in  $\mathcal{C}$ .  
(c) There is a fiber sequence

$$\mathrm{GL}_1(\mathfrak{Z}(A)) \rightarrow \mathrm{GL}_1(A) \rightarrow \Omega^{k-1} \mathrm{End}_{\mathrm{Alg}_{\mathbf{E}_k}(\mathrm{Sp})}(A)$$

of  $k$ -fold loop spaces.

In the sequel, we will need a more general notion:

**Definition 3.3.5.** Let  $m \geq 1$ . The  $\mathbf{E}_{k+m}$ -center  $\mathfrak{Z}_{k+m}(A)$  of an  $\mathbf{E}_k$ -algebra  $A$  in a presentable symmetric monoidal  $\infty$ -category  $\mathcal{C}$  with all limits is defined inductively as the  $\mathbf{E}_{k+m}$ -center of the  $\mathbf{E}_{k+m-1}$ -center  $\mathfrak{Z}_{k+m-1}(A)$ . In other words, it is the universal  $\mathbf{E}_{k+m}$ -algebra of  $\mathcal{C}$  which fits into a commutative diagram

$$\begin{array}{ccc} \mathfrak{Z}_{k+m-1}(A) & \longrightarrow & \mathfrak{Z}_{k+m-1}(A) \otimes \mathfrak{Z}_{k+m}(A) \\ & \searrow & \downarrow \\ & & \mathfrak{Z}_{k+m-1}(A) \end{array}$$

in  $\mathrm{Alg}_{\mathbf{E}_{k+m-1}}(\mathcal{C})$ .

Proposition 3.3.4 gives:

**Corollary 3.3.6.** Let  $m \geq 1$ . The  $\mathbf{E}_{k+m-1}$ -algebra  $\mathfrak{Z}_{k+m}(A)$  associated to an  $\mathbf{E}_k$ -algebra object  $A$  of  $\mathcal{C}$  exists, and in fact admits the structure of an  $\mathbf{E}_{k+m}$ -algebra in  $\mathcal{C}$ .

We can now finally state Conjecture E:

**Conjecture E.** Let  $n \geq 0$  be an integer. Let  $R$  denote  $X(p^{n+1}-1)_{(p)}$ ,  $A$  (in which case  $n = 1$ ), or  $B$  (in which case  $n = 2$ ). Then the element  $\sigma_n \in \pi_{|\sigma_n|} R$  lifts to the  $\mathbf{E}_3$ -center  $\mathfrak{Z}_3(R)$  of  $R$ , and is  $p$ -torsion in  $\pi_* \mathfrak{Z}_3(R)$  if  $R = X(p^{n+1}-1)_{(p)}$ , and is 2-torsion in  $\pi_* \mathfrak{Z}_3(R)$  if  $R = A$  or  $B$ .

**Remark 3.3.7.** If  $R$  is  $A$  or  $B$ , then  $\mathfrak{Z}_3(R)$  is the  $\mathbf{E}_3$ -center of the  $\mathbf{E}_2$ -center of  $R$ . This is a rather unwieldy object, so it would be quite useful to show that the  $\mathbf{E}_1$ -structure on  $A$  or  $B$  admits an extension to an  $\mathbf{E}_2$ -structure; we do not know if such extensions exist. Since neither  $\Omega S^5$  nor  $N$  admit the structure of a double loop space, such an  $\mathbf{E}_2$ -structure would not arise from their structure as Thom spectra. In any case, if such extensions do exist, then  $\mathfrak{Z}_3(R)$  in Conjecture E should be interpreted as the  $\mathbf{E}_3$ -center of the  $\mathbf{E}_2$ -ring  $R$ .

**Remark 3.3.8.** In the introduction, we stated Conjecture 1.1.3, which instead asked about whether  $v_n \in \pi_{|v_n|} X(p^n)$  lifts to  $\pi_* \mathfrak{Z}_3(X(p^n))$ . It is natural to ask about the connection between Conjecture E and Conjecture 1.1.3. Proposition 3.1.17 implies that if  $\mathfrak{Z}_3(X(p^n))$  admitted an  $X(p^n-1)$ -orientation factoring the canonical  $X(p^n-1)$ -orientation  $X(p^n-1) \rightarrow X(p^n)$ , and  $\sigma_{n-1} \in \pi_{|v_n|-1} X(p^n-1)$  was killed by the map  $X(p^n-1) \rightarrow \mathfrak{Z}_3(X(p^n))$ , then Conjecture E implies Conjecture 1.1.3. However, we do not believe that either of these statements are true.

**Remark 3.3.9.** One of the main results of [Kla18] implies that the  $\mathbf{E}_3$ -center of  $X(n)$  is  $X(n)^{b\mathrm{SU}(n)}$ , where  $\mathrm{SU}(n)$  acts on  $X(n)$  by a Thomification of the conjugation action on  $\Omega \mathrm{SU}(n)$ .

**Remark 3.3.10.** Note that the conjugation action of  $SU(n)$  on  $X(n)$  can be described very explicitly, via a concrete model for  $\Omega SU(n)$ . As explained in [PS86, Zhu17], if  $G$  is a reductive linear algebraic group over  $\mathbb{C}$ , the loop space  $\Omega G(\mathbb{C})$  of its complex points<sup>2</sup>, viewed as a complex Lie group, is equivalent to the homogeneous space  $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ ; this is also commonly studied as the complex points of the affine Grassmannian  $\mathrm{Gr}_G$  of  $G$ . The conjugation action of  $G(\mathbb{C})$  on  $\Omega G(\mathbb{C})$  arises by restricting the descent (to  $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ ) of the translation action by  $G(\mathbb{C}[[t]])$  on  $G(\mathbb{C}((t)))$  to the subgroup  $G(\mathbb{C}) \subseteq G(\mathbb{C}[[t]])$ . Setting  $G = \mathrm{SL}_n$  gives a description of the conjugation action of  $SU(n)$  on  $\Omega SU(n)$ . In light of its connections to geometric representation theory, we believe that there may be an algebro-geometric approach to proving that  $\chi_n$  is  $SU(n)$ -trivial in  $X(n)$  and in  $\Omega SU(n)$ .

**Example 3.3.11.** The element  $\chi_2 \in \pi_3 X(2)$  is central. To see this, note that  $\alpha \in \pi_* R$  (where  $R$  is an  $E_k$ -ring) is in the  $E_{k+1}$ -center of  $R$  if and only if  $\alpha$  is in the  $E_{k+1}$ -center of  $R_{(p)}$  for all primes  $p \geq 0$ . It therefore suffices to show that  $\chi_2$  is central after  $p$ -localizing for all  $p$ . First, note that  $\chi_2$  is torsion, so it is nullhomotopic (and therefore central) after rationalization. Next, if  $p > 2$ , then  $X(2)_{(p)}$  splits as a wedge of suspensions of spheres. If  $\chi_2$  is detected in  $\pi_3$  of a sphere living in dimension 3, then it could not be torsion, so it must be detected in  $\pi_3$  of a sphere living in dimension  $3 - k$  for some  $0 \leq k \leq 2$ . If  $k = 1$  or  $2$ , then  $\pi_3(S^{3-k})$  is either  $\pi_1(S^0)$  or  $\pi_2(S^0)$ , but both of these groups vanish for  $p > 2$ . Therefore,  $\chi_2$  must be detected in  $\pi_3$  of the sphere in dimension 0, i.e., in  $\pi_3 X(1)$ . This group vanishes for  $p > 3$ , and when  $p = 3$ , it is isomorphic to  $\mathbb{Z}/3$  (generated by  $\alpha_1$ ). Since  $X(1) = S^0$  is an  $E_\infty$ -ring, we conclude that  $\chi_2$  is central in  $X(2)_{(p)}$  for all  $p > 2$ .

At  $p = 2$ , we know the cell structure of  $X(2)$  in the bottom few dimensions (see Example 3.1.14; note that  $\sigma_1$  is *not*  $\chi_2$ ). In dimensions  $\leq 3$ , it is equivalent to  $C\eta$ , so  $\pi_3 X(2) \cong \pi_3 C\eta$ . However, it is easy to see that  $\pi_3 C\eta = 0$ , so  $\chi_2$  is nullhomotopic at  $p = 2$  (and is therefore vacuously central). We conclude from the above discussion that  $\chi_2$  is indeed central in  $X(2)$ .

#### 4. REVIEW OF SOME UNSTABLE HOMOTOPY THEORY

**4.1. Charming and Gray maps.** A major milestone in unstable homotopy theory was Cohen-Moore-Neisendorfer's result on the  $p$ -exponent of unstable homotopy groups of spheres from [CMN79a, CMN79b, Nei81]. They defined for all  $p > 2$  and  $k \geq 1$  a map  $\phi_n : \Omega^2 S^{2n+1} \rightarrow S^{2n-1}$  (the integer  $k$  is assumed implicit) such that the composite of  $\phi_n$  with the double suspension  $E^2 : S^{2n-1} \rightarrow \Omega^2 S^{2n+1}$  is homotopic to the  $p^k$ -th power map. By induction on  $n$ , they concluded via a result of Selick's (see [Sel77]) that  $p^n$  kills the  $p$ -primary component of the homotopy of  $S^{2n+1}$ . Such maps will be important in the rest of this paper, so we will isolate their desired properties in the definition of a *charming map*, inspired by [ST19]. (Our choice of terminology is non-standard, and admittedly horrible, but it does not seem like the literature has chosen any naming convention for the sort of maps we desire.)

**Definition 4.1.1.** A  $p$ -local map  $f : \Omega^2 S^{2np+1} \rightarrow S^{2np-1}$  is called a *Gray map* if the composite of  $f$  with the double suspension  $E^2$  is the degree  $p$  map, and the composite

$$\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$$

is nullhomotopic. Moreover, a  $p$ -local map  $f : \Omega^2 S^{2np+1} \rightarrow S^{2np-1}$  is called a *charming map* if the composite of  $f$  with the double suspension  $E^2$  is the degree  $p$  map, the fiber of  $f$  admits the structure of a  $\mathbb{Q}_1$ -space, and if there is a space  $BK$  which sits in a fiber sequence

$$S^{2np-1} \rightarrow BK \rightarrow \Omega S^{2np+1}$$

such that the boundary map  $\Omega^2 S^{2np+1} \rightarrow S^{2np-1}$  is homotopic to  $f$ .

**Remark 4.1.2.** If  $f$  is a charming map, then the fiber of  $f$  is a loop space. Indeed,  $\mathrm{fib}(f) \simeq \Omega BK$ .

<sup>2</sup>More precisely, the complex points of a maximal compact — but this distinction is irrelevant homotopically, because  $G(\mathbb{C})$  deformation retracts onto a maximal compact.

**Example 4.1.3.** Let  $f$  denote the Cohen-Moore-Neisendorfer map with  $k = 1$ . Anick proved (see [Ani93, GT10]) that the fiber of  $f$  admits a delooping, i.e., there is a space  $T^{2np+1}(p)$  (now known as an *Anick space*) which sits in a fiber sequence

$$S^{2np-1} \rightarrow T^{2np+1}(p) \rightarrow \Omega S^{2np+1}.$$

It follows that  $f$  is a charming map.

**Remark 4.1.4.** We claim that  $T^{2p+1}(p) = \Omega S^3\langle 3 \rangle$ , where  $S^3\langle 3 \rangle$  is the 3-connected cover of  $S^3$ . To prove this, we will construct a  $p$ -local fiber sequence

$$S^{2p-1} \rightarrow \Omega S^3\langle 3 \rangle \rightarrow \Omega S^{2p+1}.$$

This fiber sequence was originally constructed by Toda in [Tod62]. To construct this fiber sequence, we first note that there is a  $p$ -local fiber sequence

$$S^{2p-1} \rightarrow J_{p-1}(S^2) \rightarrow \mathbb{C}P^\infty,$$

where the first map is the factorization of  $\alpha_1 : S^{2p-1} \rightarrow \Omega S^3$  through the  $2(p-1)$ -skeleton of  $\Omega S^3$ , and the second map is the composite  $J_{p-1}(S^2) \rightarrow \Omega S^3 \rightarrow \mathbb{C}P^\infty$ . This fiber sequence is simply an odd-primary version of the Hopf fibration  $S^3 \rightarrow S^2 \rightarrow \mathbb{C}P^\infty$ ; the identification of the fiber of the map  $J_{p-1}(S^2) \rightarrow \mathbb{C}P^\infty$  is a simple exercise with the Serre spectral sequence. Next, we have the EHP sequence

$$J_{p-1}(S^2) \rightarrow \Omega S^3 \rightarrow \Omega S^{2p+1}.$$

Since  $\Omega S^3\langle 3 \rangle$  is the fiber of the map  $\Omega S^3 \rightarrow \mathbb{C}P^\infty$ , the desired fiber sequence is obtained by taking vertical fibers in the following map of fiber sequences:

$$\begin{array}{ccccc} J_{p-1}(S^2) & \longrightarrow & \Omega S^3 & \longrightarrow & \Omega S^{2p+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}P^\infty & \xlongequal{\quad} & \mathbb{C}P^\infty & \longrightarrow & * \end{array}$$

**Example 4.1.5.** Let  $W_n$  denote the fiber of the double suspension  $S^{2n-1} \rightarrow \Omega^2 S^{2n+1}$ . Gray proved in [Gra89b, Gra88] that  $W_n$  admits a delooping  $BW_n$ , and that after  $p$ -localization, there is a fiber sequence

$$BW_n \rightarrow \Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$$

for some map  $f$ . As suggested by the naming convention,  $f$  is a Gray map.

As proved in [ST19], Gray maps satisfy an important rigidity property:

**Proposition 4.1.6** (Selick-Theriault). *The fiber of any Gray map admits a H-space structure, and is H-equivalent to  $BW_n$ .*

**Remark 4.1.7.** It has been conjectured by Cohen-Moore-Neisendorfer and Gray in the papers cited above that there is an equivalence  $BW_n \simeq \Omega T^{2np+1}(p)$ , and that  $\Omega T^{2np+1}(p)$  retracts off of  $\Omega^2 p^{2np+1}(p)$  as a H-space, where  $p^k(p)$  is the mod  $p$  Moore space  $S^{k-1} \cup_p e^k$  with top cell in dimension  $k$ . For our purposes, we shall require something slightly stronger: namely, the retraction should be one of  $\mathbb{Q}_1$ -spaces. The first part of this conjecture would follow from Proposition 4.1.6 if the Cohen-Moore-Neisendorfer map were a Gray map. In [Ame19], it is shown that the existence of  $p$ -primary elements of Kervaire invariant one would imply equivalences of the form  $BW_{p^{n-1}} \simeq \Omega T^{2p^n+1}(p)$ .

Motivated by Remark 4.1.7 and Proposition 4.1.6, we state the following conjecture; it is slightly weaker than the conjecture mentioned in Remark 4.1.7, and is an amalgamation of slight modifications of conjectures of Cohen, Moore, Neisendorfer, Gray, and Mahowald in unstable homotopy theory, as well as an analogue of Proposition 4.1.6. (For instance, we strengthen having a H-space retraction to having a  $\mathbb{Q}_1$ -space retraction).



**Conjecture D.** *The following statements are true:*

- (a) *The homotopy fiber of any charming map is equivalent as a loop space to the loop space on an Anick space.*
- (b) *There exists a  $p$ -local charming map  $f : \Omega^2 S^{2p^n+1} \rightarrow S^{2p^n-1}$  whose homotopy fiber admits a  $\mathcal{Q}_1$ -space retraction off of  $\Omega^2 P^{2p^n+1}(p)$ . There are also integrally defined maps  $\Omega^2 S^9 \rightarrow S^7$  and  $\Omega^2 S^{17} \rightarrow S^{15}$  whose composite with the double suspension on  $S^7$  and  $S^{15}$  respectively is the degree 2 map, whose homotopy fibers  $K_2$  and  $K_3$  (respectively) admit deloopings, and which admits a  $\mathcal{Q}_1$ -space retraction off  $\Omega^2 P^9(2)$  and  $\Omega^2 P^{17}(2)$  (respectively).*

**Remark 4.1.8.** Conjecture D is already not known when  $n = 1$ . In this case, it asserts that  $\Omega^2 S^3\langle 3 \rangle$  retracts off of  $\Omega^2 P^{2p+1}(p)$ . A consequence of Selick's theorem that  $\Omega^2 S^3\langle 3 \rangle$  retracts off of  $\Omega^2 S^{2p+1}\{p\}$  for  $p$  odd is that  $\Omega^2 S^3\langle 3 \rangle$  retracts off of  $\Omega^2 P^{2p+2}(p)$ . In [Coh87, Observation 9.2], the question of whether  $\Omega^2 S^3\langle 3 \rangle$  retracts off of  $\Omega^2 P^{2p+1}(p)$  was shown to be equivalent to the question of whether there is a map  $\Sigma^2 \Omega^2 S^3\langle 3 \rangle \rightarrow P^{2p+1}(p)$  which is onto in homology. Some recent results regarding Conjecture D for  $n = 1$  can be found in [BT13].

It follows that a retraction of  $\Omega^2 S^3\langle 3 \rangle$  off  $\Omega^2 P^{2p+1}(p)$  will be compatible with the canonical map  $\Omega^2 S^3\langle 3 \rangle \rightarrow \Omega^2 S^3$  in the following manner. The  $p$ -torsion element  $\alpha_1 \in \pi_{2p}(\mathbb{S}^3)$  defines a map  $P^{2p-1}(p) \rightarrow \Omega^2 S^3$ , which extends to an  $E_2$ -map  $\Omega^2 P^{2p+1}(p) \rightarrow \Omega^2 S^3$ . We will abusively denote this extension by  $\alpha_1$ . The resulting composite

$$\Omega^2 S^3\langle 3 \rangle \rightarrow \Omega^2 P^{2p+1}(p) \xrightarrow{\alpha_1} \Omega^2 S^3$$

is homotopic to the canonical map  $\Omega^2 S^3\langle 3 \rangle \rightarrow \Omega^2 S^3$ .

The element  $\alpha_1 \in \pi_{2p-3}(\mathbb{S}_{(p)})$  defines a map  $S^{2p-2} \rightarrow \text{BGL}_1(\mathbb{S}_{(p)})$ , and since it is  $p$ -torsion, admits an extension to a map  $P^{2p-1}(p) \rightarrow \text{BGL}_1(\mathbb{S}_{(p)})$ . (This extension is in fact unique, because  $\pi_{2p-1}(\text{BGL}_1(\mathbb{S}_{(p)})) \cong \pi_{2p-2}(\mathbb{S}_{(p)})$  vanishes.) Since  $\text{BGL}_1(\mathbb{S}_{(p)})$  is an infinite loop space, this map further extends to a map  $\Omega^2 P^{2p+1}(p) \rightarrow \text{BGL}_1(\mathbb{S}_{(p)})$ . The discussion in the previous paragraph implies that if Conjecture D is true for  $n = 1$ , then the map  $\mu : \Omega^2 S^3\langle 3 \rangle \rightarrow \text{BGL}_1(\mathbb{S}_{(p)})$  from Corollary 2.2.2 is homotopic to the composite

$$\Omega^2 S^3\langle 3 \rangle \rightarrow \Omega^2 P^{2p+1}(p) \rightarrow \text{BGL}_1(\mathbb{S}_{(p)}).$$

**4.2. Fibers of charming maps.** We shall need the following proposition.

**Proposition 4.2.1.** *Let  $f : \Omega^2 S^{2p^n+1} \rightarrow S^{2p^n-1}$  be a charming map. Then there are isomorphisms of coalgebras:*

$$H_*(\text{fib}(f); \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p[x_{2^{n+1}-1}^2] \otimes \bigotimes_{k>1} \mathbb{F}_p[x_{2^{n+k}-1}] & p = 2 \\ \bigotimes_{k>0} \mathbb{F}_p[\gamma_{2(p^{n+k}-1)}] \otimes \bigotimes_{j>0} \Lambda_{\mathbb{F}_p}[x_{2^{p^n+j}-1}] & p > 2. \end{cases}$$

*Proof.* This is an easy consequence of the Serre spectral sequence coupled with the well-known coalgebra isomorphisms

$$H_*(\Omega^2 S^{2n+1}; \mathbb{F}_p) \cong \begin{cases} \bigotimes_{k>0} \mathbb{F}_p[x_{2^{k+1}-1}] & p = 2 \\ \bigotimes_{k>0} \mathbb{F}_p[\gamma_{2(n p^k-1)}] \otimes \bigotimes_{j>0} \Lambda_{\mathbb{F}_p}[x_{2^{n p^j}-1}] & p > 2, \end{cases}$$

where these classes are generated by the one in dimension  $2n - 1$  via the single Dyer-Lashof operation (coming already from the cup-1 operad; see Remark 2.2.8).  $\square$

**Remark 4.2.2.** The Anick spaces  $T^{2np+1}(p)$  from Example 4.1.3 sit in fiber sequences

$$S^{2np-1} \rightarrow T^{2np+1}(p) \rightarrow \Omega S^{2np+1},$$

and are homotopy commutative H-spaces. A Serre spectral sequence calculation gives an identification of coalgebras

$$H_*(T^{2np+1}(p); \mathbb{F}_p) \cong \mathbb{F}_p[a_{2np}] \otimes \Lambda_{\mathbb{F}_p}[b_{2np-1}],$$

with  $\beta(a_{2np}) = b_{2np-1}$ , where  $\beta$  is the Bockstein homomorphism. An argument with the bar spectral sequence recovers the result of Proposition 4.2.1 in this particular case.

**Remark 4.2.3.** The Serre spectral sequence calculating the integral homology of the delooping  $BK$  of the fiber of a charming map  $f : \Omega^2 S^{2np+1} \rightarrow S^{2np-1}$  only has a differential on the  $E_{2np-1}$ -page, and gives

$$H_*(BK; \mathbf{Z}) \cong \mathbf{Z}[x_{2(k+1)np-2k-1} | k \geq 2] / \langle x_i x_j, x_i^2, k x_{2(k+1)np-2k-1} \rangle,$$

where  $x_{2(k+1)np-2k-1}$  lives in degree  $2(k+1)np - 2k - 1$ . Explicitly:

$$H_i(BK; \mathbf{Z}) \cong \begin{cases} \mathbf{Z} & i = 0 \\ \mathbf{Z}/k & \text{if } i = 2(k+1)np - 2k - 1 \\ 0 & \text{else.} \end{cases}$$

We conclude this section by investigating Thom spectra of bundles defined over fibers of charming maps. Let  $R$  be a  $p$ -local  $E_1$ -ring, and let  $\mu : K \rightarrow BGL_1(R)$  denote a map from the fiber  $K$  of a charming map  $f : \Omega^2 S^{2np+1} \rightarrow S^{2np-1}$ . There is a fiber sequence  $\Omega S^{2np-1} \rightarrow K \rightarrow \Omega^2 S^{2np+1}$  of loop spaces, so we obtain a map  $\Omega S^{2np-1} \rightarrow BGL_1(R)$ . Such a map gives an element  $\alpha \in \pi_{2np-3} R$  via the effect on the bottom cell  $S^{2np-2}$ .

Theorem 2.1.7 implies that the Thom spectrum of the map  $\Omega S^{2np-1} \rightarrow BGL_1(R)$  should be thought of as the  $E_1$ -quotient  $R//\alpha$ , although this may not make sense if  $R$  is not at least  $E_2$ . However, in many cases (such as the ones we are considering here), the Thom  $R$ -module  $R//\alpha$  is in fact an  $E_1$ -ring such that the map  $R \rightarrow R//\alpha$  is an  $E_1$ -map. By Proposition 2.1.6, there is an induced map  $\phi : \Omega^2 S^{2np+1} \rightarrow BGL_1(R//\alpha)$  whose Thom spectrum is equivalent as an  $E_1$ -ring to  $K^\mu$ . We would like to determine the element<sup>3</sup> of  $\pi_* R//\alpha$  detected by the restriction to the bottom cell  $S^{2np-1}$  of the source of  $\phi$ . First, we note:

**Lemma 4.2.4.** *The element  $\alpha \in \pi_{2np-3} R$  is  $p$ -torsion.*

*Proof.* Since  $f$  is a charming map, the composite  $S^{2np-1} \rightarrow \Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$  is the degree  $p$  map. Therefore, the element  $p\alpha \in \pi_{2np-3} R$  is detected by the composite

$$S^{2np-2} \rightarrow \Omega S^{2np-1} \rightarrow \Omega^3 S^{2np-1} \xrightarrow{\Omega f} \Omega S^{2np-1} \rightarrow K \xrightarrow{\mu} BGL_1(R).$$

But there is a fiber sequence  $\Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1} \rightarrow BK$  by the definition of a charming map, so the composite detecting  $p\alpha$  is null, as desired.  $\square$

There is now a square

$$\begin{array}{ccc} S^{2np-2}/p & \longrightarrow & S^{2np-1} \\ \downarrow & & \downarrow \\ K & \longrightarrow & \Omega^2 S^{2np+1} \\ \alpha \downarrow & & \downarrow \\ BGL_1(R) & \longrightarrow & BGL_1(R//\alpha), \end{array}$$

and the following result is a consequence of the lemma and the definition of Toda brackets:

<sup>3</sup>Technically, this is bad terminology: there are multiple possibilities for the map  $\phi$ , and each gives rise to a map  $S^{2np-1} \rightarrow BGL_1(R//\alpha)$ . The elements in  $\pi_{2np-2}(R//\alpha)$  determined in this way need not agree, but they are the same modulo the indeterminacy of the Toda bracket  $\langle p, \alpha, 1_{R//\alpha} \rangle$ .

**Lemma 4.2.5.** *The element in  $\pi_{2np-2}(R//\alpha)$  detected by the vertical map  $S^{2np-1} \rightarrow \mathrm{BGL}_1(R//\alpha)$  lives in the Toda bracket  $\langle p, \alpha, 1_{R//\alpha} \rangle$ .*

The upshot of this discussion is the following:

**Proposition 4.2.6.** *Let  $R$  be a  $p$ -local  $\mathbf{E}_1$ -ring, and let  $\mu : K \rightarrow \mathrm{BGL}_1(R)$  denote a map from the fiber  $K$  of a charming map  $f : \Omega^2 S^{2np+1} \rightarrow S^{2np-1}$ , providing an element  $\alpha \in \pi_{2np-3}R$ . Assume that the Thom spectrum  $R//\alpha$  of the map  $\Omega S^{2np-1} \rightarrow \mathrm{BGL}_1(R)$  is an  $\mathbf{E}_1$ - $R$ -algebra. Then there is an element  $v \in \langle p, \alpha, 1_{R//\alpha} \rangle$  such that  $K^\mu$  is equivalent to the Thom spectrum of the map  $\Omega^2 S^{2np+1} \xrightarrow{v} \mathrm{BGL}_1(R//\alpha)$ .*

**Remark 4.2.7.** Let  $R$  be an  $\mathbf{E}_1$ -ring, and let  $\alpha \in \pi_d R$ . Then  $\alpha$  defines a map  $S^{d+1} \rightarrow \mathrm{BGL}_1(R)$ , and it is natural to ask when  $\alpha$  extends along  $S^{d+1} \rightarrow \Omega S^{d+2}$ , or at least along  $S^{d+1} \rightarrow J_k(S^{d+1})$  for some  $k$ . This is automatic if  $R$  is an  $\mathbf{E}_2$ -ring, but not necessarily so if  $R$  is only an  $\mathbf{E}_1$ -ring. Recall that there is a cofiber sequence

$$S^{(k+1)(d+1)-1} \rightarrow J_k(S^{d+1}) \rightarrow J_{k+1}(S^{d+1}),$$

where the first map is the  $(k+1)$ -fold iterated Whitehead product  $[\iota_{d+1}, [\dots, [\iota_{d+1}, \iota_{d+1}], \dots]]$ . In particular, the map  $S^{d+1} \rightarrow \mathrm{BGL}_1(R)$  extends along the map  $S^{d+1} \rightarrow J_k(S^{d+1})$  if and only if there are compatible nullhomotopies of the  $n$ -fold iterated Whitehead products  $[\alpha, [\dots, [\alpha, \alpha]], \dots] \in \pi_* \mathrm{BGL}_1(R)$  for  $n \leq k$ . These amount to properties of Toda brackets in the homotopy of  $R$ . We note, for instance, that the Whitehead bracket  $[\alpha, \alpha] \in \pi_{2d+1} \mathrm{BGL}_1(R) \cong \pi_{2d} R$  is the element  $2\alpha^2$ ; therefore, the map  $S^{d+1} \rightarrow \mathrm{BGL}_1(R)$  extends to  $J_2(S^{d+1})$  if and only if  $2\alpha^2 = 0$ .

**Remark 4.2.8.** Let  $R$  be a  $p$ -local  $\mathbf{E}_2$ -ring, and let  $\alpha \in \pi_d(R)$  with  $d$  even. Then  $\alpha$  defines an element  $\alpha \in \pi_{d+2} \mathrm{B}^2 \mathrm{GL}_1(R)$ . The  $p$ -fold iterated Whitehead product  $[\alpha, \dots, [\alpha, \alpha], \dots] \in \pi_{p(d+2)-(p-1)} \mathrm{B}^2 \mathrm{GL}_1(R) \cong \pi_{pd+(p-1)} R$  is given by  $p!Q_1(\alpha)$  modulo decomposables.

This is in fact true more generally. Let  $R$  be an  $\mathbf{E}_n$ -ring, and suppose  $\alpha \in \pi_d(R)$ . Let  $i < n$ , so  $\alpha$  defines an element  $\alpha \in \pi_{d+i} \mathrm{B}^i \mathrm{GL}_1(R)$ . The  $p$ -fold iterated Whitehead product  $[\alpha, \dots, [\alpha, \alpha], \dots] \in \pi_{p(d+i)-(p-1)} \mathrm{B}^i \mathrm{GL}_1(R) \cong \pi_{pd+(i-1)(p-1)} R$  is given by  $p!Q_{i-1}(\alpha)$  modulo decomposables. We shall expand on this in forthcoming work, but we note here that this is a consequence of the definition of Whitehead products and properties of  $(1\text{-fold})$  loop spaces of spheres.

## 5. CHROMATIC THOM SPECTRA

**5.1. Statement of the theorem.** To state the main theorem of this section, we set some notation. Fix an integer  $n \geq 1$ , and work in the  $p$ -complete stable category. For each Thom spectrum  $R$  of height  $n-1$  in Table 1 (reprinted below), let  $\sigma_{n-1} : S^{|\sigma_{n-1}|} \rightarrow \mathrm{BGL}_1(R)$  denote a map detecting  $\sigma_{n-1} \in \pi_{|\sigma_{n-1}|}(R)$  (which exists by Theorem 3.2.22). Let  $K_n$  denote the fiber of a  $p$ -local charming map  $\Omega^2 S^{2p^n+1} \rightarrow S^{2p^n-1}$  satisfying the hypotheses of Conjecture D, and let  $K_2$  (resp.  $K_3$ ) denote the fiber of an integrally defined charming map  $\Omega^2 S^9 \rightarrow S^7$  (resp.  $\Omega^2 S^{17} \rightarrow S^{15}$ ) satisfying the hypotheses of Conjecture D.

Then:

**Theorem A.** *Let  $R$  be a height  $n-1$  spectrum as in the second line of Table 1. Then Conjectures D and E imply that there is a map  $K_n \rightarrow \mathrm{BGL}_1(R)$  such that the mod  $p$  homology of the Thom spectrum  $K_n^\mu$  is isomorphic to the mod  $p$  homology of the associated designer chromatic spectrum  $\Theta(R)$  as a Steenrod comodule.*

*If  $R$  is any base spectrum other than  $B$ , the Thom spectrum  $K_n^\mu$  is equivalent to  $\Theta(R)$  upon  $p$ -completion for every prime  $p$ . If Conjecture F is true, then the same is true for  $B$ : the Thom spectrum  $K_n^\mu$  is equivalent to  $\Theta(B) = \mathrm{tmf}$  upon 2-completion.*

Height	0	1	2	$n$	$n$	$n$
Base spectrum	$\mathbb{S}_p^\wedge$	$A$	$B$	$T(n)$	$y(n)$	$y_Z(n)$
Designer chromatic Thom spectrum	$\mathrm{HZ}_p$	$\mathrm{bo}$	$\mathrm{tmf}$	$\mathrm{BP}\langle n \rangle$	$k(n)$	$k_Z(n)$

TABLE 3. A reprint of Table 1, used in the statement of Theorem A.

We emphasize again that naïvely making sense of Theorem A relies on knowing that  $T(n)$  admits the structure of an  $\mathbf{E}_1$ -ring; we shall interpret this phrase as in Warning 3.1.6.

**Remark 5.1.1.** Theorem A is proved independently of the nilpotence theorem. (In fact, it is even independent of Quillen’s identification of  $\pi_* \mathrm{MU}$  with the Lazard ring, provided one regards the existence of designer chromatic spectra as being independent of Quillen’s identification.) We shall elaborate on the connection between Theorem A and the nilpotence theorem in future work; a sketch is provided in Remark 5.4.7.

**Remark 5.1.2.** Theorem A is true unconditionally when  $n = 1$ , since that case is simply Corollary 2.2.2.

**Remark 5.1.3.** Note that Table 2 implies that the homology of each of the Thom spectra in Table 1 are given by the  $Q_0$ -Margolis homology of their associated designed chromatic spectra. In particular, the map  $R \rightarrow \Theta(R)$  is a rational equivalence.

Before we proceed with the proof of Theorem A, we observe some consequences.

**Corollary B.** *Conjecture D and Conjecture E imply Conjecture 1.1.2.*

*Proof.* This follows from Theorem A, Proposition 4.2.6, and Proposition 3.1.17.  $\square$

**Remark 5.1.4.** Corollary B is true *unconditionally* when  $n = 1$ , since Theorem A is true unconditionally in that case by Remark 5.1.2. See also Remark 4.1.4.

**Remark 5.1.5.** We can attempt to apply Theorem A for  $R = A$  in conjunction with Proposition 4.2.6. Theorem A states that Conjecture D and Conjecture E imply that there is a map  $K_2 \rightarrow \mathrm{BGL}_1(A)$  whose Thom spectrum is equivalent to  $\mathrm{bo}$ . There is a fiber sequence

$$\Omega S^7 \rightarrow K_2 \rightarrow \Omega^2 S^9,$$

so we obtain a map  $\mu : \Omega S^7 \rightarrow K_2 \rightarrow \mathrm{BGL}_1(A)$ . The proof of Theorem A shows that the bottom cell  $S^6$  of the source detects  $\sigma_1 \in \pi_5(A)$ . A slight variation of the argument used to establish Proposition 4.2.6 supplies a map  $\Omega^2 S^9 \rightarrow \mathrm{BAut}((\Omega S^7)^\mu)$  whose Thom spectrum is  $\mathrm{bo}$ . The spectrum  $(\Omega S^7)^\mu$  has mod 2 homology  $\mathbb{F}_2[\zeta_1^4, \zeta_2^2]$ . However, unlike  $A$ , it does not naturally arise as an  $\mathbf{E}_1$ -Thom spectrum over the sphere spectrum; this makes it unamenable to study via techniques of unstable homotopy.

More precisely,  $(\Omega S^7)^\mu$  is not the Thom spectrum of an  $\mathbf{E}_1$ -map  $X \rightarrow \mathrm{BGL}_1(\mathbb{S})$  from a loop space  $X$  which sits in a fiber sequence

$$\Omega S^5 \rightarrow X \rightarrow \Omega S^7$$

of loop spaces. Indeed,  $\mathrm{BX}$  would be a  $S^5$ -bundle over  $S^7$ , which by [Mah87, Lemma 4] implies that  $X$  is then equivalent as a loop space to  $\Omega S^5 \times \Omega S^7$ . The resulting  $\mathbf{E}_1$ -map  $\Omega S^7 \rightarrow \mathrm{BGL}_1(\mathbb{S})$  is specified by an element of  $\pi_5(\mathbb{S}) \cong 0$ , so  $(\Omega S^7)^\mu$  must then be equivalent as an  $\mathbf{E}_1$ -ring to  $A \wedge \Sigma_+^\infty \Omega S^7$ . In particular,  $\sigma_1 \in \pi_5(A)$  would map nontrivially to  $(\Omega S^7)^\mu$ , which is a contradiction.

The proof of Theorem A will also show:

**Corollary 5.1.6.** *Let  $R$  be a height  $n - 1$  spectrum as in the second line of Table 1, and assume Conjecture F if  $R = B$ . Let  $M$  be an  $\mathbf{E}_3$ - $R$ -algebra. Conjecture D and Conjecture E imply that if:*

- (a) *the composite  $\mathfrak{Z}_3(R) \rightarrow R \rightarrow M$  is an  $\mathbf{E}_3$ -algebra map,*
- (b) *the element  $\sigma_{n-1}$  in  $\pi_* M$  is nullhomotopic,*
- (c) *and the bracket  $\langle p, \sigma_{n-1}, 1_M \rangle$  contains zero,*

*then there is a unital map  $\Theta(R) \rightarrow M$ .*

**5.2. The proof of Theorem A.** This section is devoted to giving a proof of Theorem A, dependent on Conjecture D and Conjecture E. The proof of Theorem A will be broken down into multiple steps. The result for  $\gamma(n)$  and  $\gamma_Z(n)$  follow from the result for  $T(n)$  by Proposition 3.2.5, so we shall restrict ourselves to the cases of  $R$  being  $T(n)$ ,  $A$ , and  $B$ .

Fix  $n \geq 1$ . If  $R$  is  $A$  or  $B$ , we will restrict to  $p = 2$ , and let  $K_2$  and  $K_3$  denote the integrally defined spaces from Conjecture D. By Remarks 3.2.15 and 3.2.21, the elements  $\sigma_1 \in \pi_5(A)$  and  $\sigma_2 \in \pi_{13}(B)$  are defined integrally. We will write  $\sigma_{n-1}$  to generically denote this element, and will write it as living in degree  $|\sigma_{n-1}|$ . We shall also write  $R$  to denote  $X(p^n - 1)$  and *not*  $T(n)$ ; this will be so that we can apply Conjecture D. We apologize for the inconvenience, but hope that this is worth circumventing the task of having to read through essentially the same proofs for these slightly different cases.

*Step 1.* We begin by constructing a map  $\mu : K_n \rightarrow \text{BGL}_1(R)$  as required by the statement of Theorem A; the construction in the case  $n = 1$  follows Remark 4.1.8. By Conjecture D, the space  $K_n$  splits off  $\Omega^2 P^{|\sigma_{n-1}|+4}(p)$  (if  $R = T(n)$ , then  $|\sigma_{n-1}| + 4 = |v_n| + 3$ ). We are therefore reduced to constructing a map  $\Omega^2 P^{|\sigma_{n-1}|+4}(p) \rightarrow \text{BGL}_1(R)$ . Theorem 3.2.22 shows that the element  $\sigma_{n-1} \in \pi_* R$  is  $p$ -torsion, so the map  $S^{|\sigma_{n-1}|+1} \rightarrow \text{BGL}_1(R)$  detecting  $\sigma_{n-1}$  extends to a map

$$(5.1) \quad S^{|\sigma_{n-1}|+1}/p = P^{|\sigma_{n-1}|+2}(p) \rightarrow \text{BGL}_1(R).$$

Since  $\Omega^2 P^{|\sigma_{n-1}|+4}(p) \simeq \Omega^2 \Sigma^2 P^{|\sigma_{n-1}|+2}(p)$ , we would obtain an extension  $\tilde{\mu}$  of this map through  $\Omega^2 P^{|\sigma_{n-1}|+4}(p)$  if  $R$  admits an  $E_3$ -structure.

Unfortunately, this is not true; but this is where Conjecture E comes in: it says that the element  $\sigma_{n-1} \in \pi_{|\sigma_{n-1}|} R$  lifts to the  $E_3$ -center  $\mathfrak{Z}_3(R)$ , where it has the same torsion order as in  $R$ . (Here, we are abusively writing  $\mathfrak{Z}_3(T(n-1))$  to denote the  $E_3$ -center of  $X(p^n - 1)_{(p)}$ .) The lifting of  $\sigma_{n-1}$  to  $\pi_{|\sigma_{n-1}|} \mathfrak{Z}_3(R)$  provided by Conjecture E gives a factorization of the map from (5.1) as

$$S^{|\sigma_{n-1}|+1}/p = P^{|\sigma_{n-1}|+2}(p) \rightarrow \text{BGL}_1(\mathfrak{Z}_3(R)) \rightarrow \text{BGL}_1(R).$$

Since  $\mathfrak{Z}_3(R)$  is an  $E_3$ -ring,  $\text{BGL}_1(\mathfrak{Z}_3(R))$  admits the structure of an  $E_2$ -space. In particular, the map  $P^{|\sigma_{n-1}|+2}(p) \rightarrow \text{BGL}_1(\mathfrak{Z}_3(R))$  factors through  $\Omega^2 P^{|\sigma_{n-1}|+4}(p)$ , as desired. We let  $\tilde{\mu}$  denote the resulting composite

$$\tilde{\mu} : \Omega^2 P^{|\sigma_{n-1}|+4}(p) \rightarrow \text{BGL}_1(\mathfrak{Z}_3(R)) \rightarrow \text{BGL}_1(R).$$

*Step 2.* Theorem A asserts that there is an identification between the Thom spectrum of the induced map  $\mu : K_n \rightarrow \text{BGL}_1(R)$  and the associated designer chromatic spectrum  $\Theta(R)$  via Table 1. We shall identify the Steenrod comodule structure on the mod  $p$  homology of  $K_n^\mu$ , and show that it agrees with the mod  $p$  homology of  $\Theta(R)$ .

Designer chromatic spectrum	Mod $p$ homology	
$\text{BP}\langle n-1 \rangle$	$p = 2$	$\mathbf{F}_2[\zeta_1^2, \dots, \zeta_{n-1}^2, \zeta_n^2, \zeta_{n+1}, \dots]$
	$p > 2$	$\mathbf{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbf{F}_p}(\tau_n, \tau_{n+1}, \dots)$
$k(n-1)$	$p = 2$	$\mathbf{F}_2[\zeta_1, \dots, \zeta_{n-1}, \zeta_n^2, \zeta_{n+1}, \dots]$
	$p > 2$	$\mathbf{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbf{F}_p}(\tau_0, \dots, \tau_{n-2}, \tau_n, \tau_{n+1}, \dots)$
$k_Z(n-1)$	$p = 2$	$\mathbf{F}_2[\zeta_1^2, \zeta_2, \dots, \zeta_{n-1}, \zeta_n^2, \zeta_{n+1}, \dots]$
	$p > 2$	$\mathbf{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbf{F}_p}(\tau_1, \dots, \tau_{n-2}, \tau_n, \tau_{n+1}, \dots)$
bo	$p = 2$	$\mathbf{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \dots]$
	$p > 2$	$\mathbf{F}_p[x_4]/v_1 \otimes \mathbf{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbf{F}_p}(\tau_2, \tau_3, \dots)$
tmf	$p = 2$	$\mathbf{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \dots]$
	$p = 3$	$\Lambda_{\mathbf{F}_3}(b_4) \otimes \mathbf{F}_3[\zeta_1^3, \zeta_2, \dots] \otimes \Lambda_{\mathbf{F}_3}(\tau_3, \tau_4, \dots)$
	$p \geq 5$	$\mathbf{F}_p[c_4, c_6]/(v_1, v_2) \otimes \mathbf{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbf{F}_p}(\tau_3, \tau_4, \dots)$

TABLE 4. The mod  $p$  homology of designer chromatic spectra.

In Table 4, we have recorded the mod  $p$  homology of the designer chromatic spectra in Table 1 (see [LN14, Theorem 4.3] for  $\mathrm{BP}\langle n-1 \rangle$ ). It follows from Proposition 4.2.1 that there is an isomorphism

$$H_*(K_n^\mu) \cong \begin{cases} H_*(R) \otimes F_2[x_{2^{n+1}-1}^2] \otimes \bigotimes_{k>1} F_2[x_{2^{n+k}-1}] & p=2 \\ H_*(R) \otimes \bigotimes_{k>0} F_p[\gamma_{2(p^{n+k}-1)}] \otimes \bigotimes_{j>0} \Lambda_{F_p}[x_{2p^{n+j}-1}] & p>2. \end{cases}$$

Combining this isomorphism with Theorem 3.1.5, Proposition 3.2.3, Proposition 3.2.13, and Proposition 3.2.19, we find that there is an *abstract* equivalence between the mod  $p$  homology of  $K_n^\mu$  and the mod  $p$  homology of  $\Theta(R)$ .

We shall now work at  $p=2$  for the remainder of the proof; the same argument goes through with slight modifications at odd primes. We now identify the Steenrod comodule structure on  $H_*(K_n^\mu)$ . Recall that  $\tilde{\mu}$  is the map  $\Omega^2 P^{|\sigma_{n-1}|+4}(p) \rightarrow \mathrm{BGL}_1(R)$  from Step 1. By construction, there is a map  $K_n^\mu \rightarrow \Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}}$ . The map  $\Phi$  factors through a map  $\tilde{\Phi} : \Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}} \rightarrow \Theta(R)$ . The Thom spectrum  $\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}}$  admits the structure of a  $\mathcal{Q}_1$ -ring. Indeed, it is the smash product  $\Omega^2 P^{|\sigma_{n-1}|+4}(p)^\phi \wedge_{\mathfrak{Z}_3(R)} R$ , where  $\phi : \Omega^2 P^{|\sigma_{n-1}|+4}(p) \rightarrow \mathrm{BGL}_1(\mathfrak{Z}_3(R))$ ; it therefore suffices to observe that the Thom spectrum  $\Omega^2 P^{|\sigma_{n-1}|+4}(p)^\phi$  admits the structure of an  $E_1 \otimes \mathcal{Q}_1$ -ring. This is a consequence of the fact that  $\phi$  is a double loop, and hence an  $E_1 \otimes \mathcal{Q}_1$ -algebra, map. Moreover, the image of  $H_*(K_n^\mu)$  in  $H_*(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}})$  is generated under the single Dyer-Lashof operation (arising from the cup-1 operad; see Remark 2.2.8) by the indecomposables in the image of the map  $H_*(R) \rightarrow H_*(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}})$ .

The Postnikov truncation map  $\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}} \rightarrow H\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}})$  is one of  $\mathcal{Q}_1$ -rings. Since  $\Omega^2 P^{|\sigma_{n-1}|+4}(p)$  is highly connected,  $\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}}) \cong \pi_0(R)$ . In particular, there is an  $E_\infty$ -map  $H\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}}) \rightarrow \mathrm{HF}_p$ . The composite

$$\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}} \rightarrow H\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}}) \rightarrow \mathrm{HF}_p$$

is therefore a  $\mathcal{Q}_1$ -algebra map. Moreover, the composite

$$R \rightarrow \Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}} \rightarrow H\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}}) \rightarrow \mathrm{HF}_p$$

is simply the Postnikov truncation for  $R$ . It follows that the indecomposables in  $H_*(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}})$  which come from the indecomposables in  $H_*(R)$  are sent to the indecomposables in  $H_*(\mathrm{HF}_p)$ . Using the discussion in the previous paragraph, Steinberger's calculation (Theorem 2.2.4), and the Dyer-Lashof hopping argument of Remark 2.2.9, we may conclude that the Steenrod comodule structure on  $H_*(K_n^\mu)$  (which, recall, is abstractly isomorphic to  $H_*(\Theta(R))$ ) agrees with the Steenrod comodule structure on  $H_*(\Theta(R))$ .

*Step 3.* By Step 2, the mod  $p$  homology of the Thom spectrum  $K_n^\mu$  is isomorphic to the mod  $p$  homology of the associated designer chromatic spectrum  $\Theta(R)$  as a Steenrod comodule. The main result of [AL17] and [AP76, Theorem 1.1] now imply that unless  $R=B$ , the Thom spectrum  $K_n^\mu$  is equivalent to  $\Theta(R)$  upon  $p$ -completion for every prime  $p$ . Finally, if Conjecture F is true, then the same conclusion can be drawn for  $B$ : the Thom spectrum  $K_n^\mu$  is equivalent to  $\Theta(B) = \mathrm{tmf}$  upon  $p$ -completion for every prime  $p$ .

This concludes the proof of Theorem A.

**5.3. Remark on the proof.** Before proceeding, we note the following consequence of the proof of Theorem A.

**Proposition 5.3.1.** *Let  $p$  be an odd prime. Assume Conjecture D and Conjecture E. Then the composite*

$$g_2 : \Omega^2 S^{|\sigma_{n-1}|+3} \rightarrow \Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}} \rightarrow \mathrm{BGL}_1(X(p^n-1)) \rightarrow \mathrm{BGL}_1(\mathrm{BP}\langle n-1 \rangle)$$

*is null.*



*Proof.* Let  $R = X(p^n - 1)$ , and let  $\Theta(R) = \mathrm{BP}\langle n - 1 \rangle$ . The map  $g_2$  is the composite with  $\mathrm{BGL}_1(\mathfrak{Z}_3(R)) \rightarrow \mathrm{BGL}_1(\Theta(R))$  with the extension of the map

$$\sigma_{n-1} : S^{|\sigma_{n-1}|+1} \rightarrow \mathrm{BGL}_1(\mathfrak{Z}_3(R))$$

along the double suspension  $S^{|\sigma_{n-1}|+1} \rightarrow \Omega^2 S^{|\sigma_{n-1}|+3}$ . Since  $\sigma_{n-1}$  is null in  $\pi_* \Theta(R)$ , we would be done if  $g_2$  was homotopic to the dotted extension

$$\begin{array}{ccc} S^{|\sigma_{n-1}|+1} & \xrightarrow{\sigma_{n-1}} & \mathrm{BGL}_1(\Theta(R)) \\ \downarrow & \nearrow g'_2 & \\ \Omega^2 S^{|\sigma_{n-1}|+3} & & \end{array}$$

The potential failure of these maps to be homotopic stems from the fact that the composite  $\mathfrak{Z}_3(R) \rightarrow R \rightarrow \Theta(R)$  need not be a map of  $\mathbf{E}_3$ -rings. It is, however, a map of  $\mathbf{E}_2$ -rings; therefore, the maps

$$g_1 : \Omega S^{|\sigma_{n-1}|+2} \rightarrow \mathrm{BGL}_1(\mathfrak{Z}_3(R)) \rightarrow \mathrm{BGL}_1(\Theta(R))$$

and

$$g'_1 : \Omega S^{|\sigma_{n-1}|+2} \rightarrow \mathrm{BGL}_1(\Theta(R))$$

obtained by extending along the suspension  $S^{|\sigma_{n-1}|+1} \rightarrow \Omega S^{|\sigma_{n-1}|+2}$  are homotopic. We now utilize the following result of Serre's:

**Proposition 5.3.2** (Serre). *Let  $p$  be an odd prime. Then the suspension  $S^{2n-1} \rightarrow \Omega S^{2n}$  splits upon  $p$ -localization: there is a  $p$ -local equivalence*

$$E \times \Omega[\iota_{2n}, \iota_{2n}] : S^{2n-1} \times \Omega S^{4n-1} \rightarrow \Omega S^{2n}.$$

This implies that the suspension map  $\Omega S^{|\sigma_{n-1}|+2} \rightarrow \Omega^2 S^{|\sigma_{n-1}|+3}$  admits a splitting as loop spaces. In particular, this implies that the map  $g_2$  is homotopic to the composite

$$\Omega^2 S^{|\sigma_{n-1}|+3} \rightarrow \Omega S^{|\sigma_{n-1}|+2} \xrightarrow{g_1} \mathrm{BGL}_1(\mathfrak{Z}_3(R)) \rightarrow \mathrm{BGL}_1(\Theta(R)),$$

and similarly for  $g'_2$ . Since  $g_1$  and  $g'_1$  are homotopic, and  $g'_1$  (and hence  $g'_2$ ) is null, we find that  $g_2$  is also null, as desired.  $\square$

**5.4. Infinite families and the nilpotence theorem.** We now briefly discuss the relationship between Theorem A and the nilpotence theorem. We begin by describing a special case of this connection. Recall from Remark 2.2.3 that Theorem 2.2.1 implies that if  $R$  is an  $\mathbf{E}_2$ -ring spectrum, and  $x \in \pi_* R$  is a simple  $p$ -torsion element which has trivial MU-Hurewicz image, then  $x$  is nilpotent. A similar argument implies the following.

**Proposition 5.4.1.** *Assume Conjecture D when  $n = 1$ . Then Corollary 2.2.2 (i.e., Theorem A when  $n = 1$ ) implies that if  $R$  is a  $p$ -local  $\mathbf{E}_3$ -ring spectrum, and  $x \in \pi_* R$  is a class with trivial  $\mathrm{HZ}_p$ -Hurewicz image such that:*

- $\alpha_1 x = 0$  in  $\pi_* R$ ; and
- the Toda bracket  $\langle p, \alpha_1, x \rangle$  contains zero;

*then  $x$  is nilpotent.*

*Proof.* We claim that the composite

$$(5.2) \quad \Omega^2 S^3 \langle 3 \rangle \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)}) \rightarrow \mathrm{BGL}_1(R[1/x])$$

is null. Remark 4.1.8 implies that Conjecture D for  $n = 1$  reduces us to showing that the composite

$$\Omega^2 p^{2p+1}(p) \xrightarrow{\alpha_1} \mathrm{BGL}_1(\mathbb{S}_{(p)}) \rightarrow \mathrm{BGL}_1(R[1/x])$$

is null. Since this composite is one of double loop spaces, it further suffices to show that the composite

$$(5.3) \quad P^{2p-1}(p) \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)}) \rightarrow \mathrm{BGL}_1(R[1/x])$$

is null. The bottom cell  $S^{2p-2}$  of  $P^{2p-1}(p)$  maps trivially to  $\mathrm{BGL}_1(R[1/x])$ , because the bottom cell detects  $\alpha_1$  (by Remark 4.1.8), and  $\alpha_1$  is nullhomotopic in  $R[1/x]$ . Therefore, the map (5.3) factors through the top cell  $S^{2p-1}$  of  $P^{2p-1}(p)$ . The resulting map

$$S^{2p-1} \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)}) \rightarrow \mathrm{BGL}_1(R[1/x])$$

detects an element of the Toda bracket  $\langle p, \alpha_1, x \rangle$ , but this contains zero by hypothesis, so is nullhomotopic.

Since the map (5.2) is null, Corollary 2.2.2 and Theorem 2.1.7 implies that there is a ring map  $\mathrm{HZ}_p \rightarrow R[1/x]$ . In particular, the composite of the map  $x : \Sigma^{|x|}R \rightarrow R$  with the unit  $R \rightarrow R[1/x]$  factors as shown:

$$\begin{array}{ccccc} \Sigma^{|x|}R & \xrightarrow{x} & R & \longrightarrow & R[1/x]. \\ \downarrow & & \downarrow & \nearrow & \\ \mathrm{HZ}_p \wedge \Sigma^{|x|}R & \xrightarrow{x} & \mathrm{HZ}_p \wedge R & & \end{array}$$

The bottom map, however, is null, because  $x$  has zero  $\mathrm{HZ}_p$ -Hurewicz image. Therefore, the element  $x \in \pi_* R[1/x]$  is null, and hence  $R[1/x]$  is contractible.  $\square$

**Remark 5.4.2.** One can prove by a different argument that Proposition 5.4.1 is true without the assumption that Conjecture D holds when  $n = 1$ . At  $p = 2$ , this was shown by Astey in [Ast97, Theorem 1.1].

To discuss the relationship between Theorem A for general  $n$  and the nilpotence theorem (which we will expand upon in future work), we embark on a slight digression. The following proposition describes the construction of some infinite families.

**Proposition 5.4.3.** *Let  $R$  be a height  $n-1$  spectrum as in the second line of Table 2, and assume Conjecture E if  $R = A$  or  $B$ . Then there is an infinite family  $\sigma_{n-1,p^k} \in \pi_{p^k|v_n|-1}(R)$ . Conjecture E implies that  $\sigma_{n-1,p^k}$  lifts to  $\pi_{p^k|v_n|-1}(\mathfrak{Z}_3(R))$ , where  $\mathfrak{Z}_3(R)$  abusively denotes the  $\mathbf{E}_3$ -center of  $X(p^n - 1)$  if  $R = T(n-1)$ .*

*Proof.* We construct this family by induction on  $k$ . The element  $\sigma_{n-1,1}$  is just  $\sigma_{n-1}$ , so assume that we have defined  $\sigma_{n-1,p^k}$ . The element  $\sigma_{n-1,p^k} \in \pi_{p^k|v_n|-1}R$  defines a map  $\sigma_{n-1,p^k} : S^{p^k|v_n|} \rightarrow \mathrm{BGL}_1(R)$ . When  $R = T(n-1)$ , Lemma 3.1.12 (and the inductive hypothesis) implies that the map defined by  $\sigma_n$  factors through the map  $\mathrm{BGL}_1(X(p^n - 1)) \rightarrow \mathrm{BGL}_1(T(n-1))$ . When  $R = A$  or  $B$ , Conjecture E (and the inductive hypothesis) implies that the map defined by  $\sigma_n$  factors through the map  $\mathrm{BGL}_1(\mathfrak{Z}_3(R)) \rightarrow \mathrm{BGL}_1(R)$ . This implies that for all  $R$  as in the second line of Table 2, the map  $\sigma_{n-1,p^k} : S^{p^k|v_n|} \rightarrow \mathrm{BGL}_1(R)$  factors through an  $\mathbf{E}_1$ -space, which we shall just denote by  $\mathcal{Z}_R$  for the purpose of this proof. If we assume Conjecture E, then we may take  $\mathcal{Z}_R = \mathrm{BGL}_1(\mathfrak{Z}_3(R))$ .

Therefore, we get a map  $\sigma_{n-1,p^k} : \Omega S^{p^k|v_n|+1} \rightarrow \mathrm{BGL}_1(R)$  via the composite

$$\Omega S^{p^k|v_n|+1} \rightarrow \mathcal{Z}_R \rightarrow \mathrm{BGL}_1(R).$$

Since  $\mathcal{Z}_R$  is an  $\mathbf{E}_1$ -space, the map  $\Omega S^{p^k|v_n|+1} \rightarrow \mathcal{Z}_R$  is adjoint to a map

$$\bigvee_{j \geq 1} S^{j p^k|v_n|+1} \simeq \Sigma \Omega S^{|v_n|+1} \rightarrow \mathrm{B}\mathcal{Z}_R;$$

the source splits as indicated via the James splitting. These splittings are given by Whitehead products; in particular, the map  $S^{p^{k+1}|v_n|+1} = S^{p(p^k|v_n|+1)-(p-1)} \rightarrow \mathbf{B}\mathbb{Z}_R$  is given by the  $p$ -fold Whitehead product  $[\sigma_{n-1,p^k}, \dots, \sigma_{n-1,p^k}]$ . This is divisible by  $p$ , so it yields a map  $S^{p^{k+1}|v_n|} \rightarrow \mathbb{Z}_R$ , and hence a map  $S^{p^{k+1}|v_n|} \rightarrow \mathbf{BGL}_1(R)$  given by composing with the map  $\mathbb{Z}_R \rightarrow \mathbf{BGL}_1(R)$ . This defines the desired element  $\sigma_{n-1,p^{k+1}} \in \pi_{p^{k+1}|v_n|-1}(R)$ . As the construction makes clear, assuming Conjecture E and taking  $\mathbb{Z}_R = \mathbf{BGL}_1(\mathfrak{Z}_3(R))$  implies that  $\sigma_{n-1,p^k}$  lifts to  $\pi_{p^k|v_n|-1}(\mathfrak{Z}_3(R))$ .  $\square$

**Remark 5.4.4.** This infinite family is detected in the 1-line of the ANSS for  $R$  by  $\delta(v_n^k)$ , where  $\delta$  is the boundary map induced by the map  $\Sigma^{-1}R/p \rightarrow R$ .

**Remark 5.4.5.** The element  $\sigma_{n-1,1} \in \pi_{2p^n-3}(R)$  is precisely  $\sigma_{n-1}$ .

**Remark 5.4.6.** When  $n = 1$ , the ring  $R$  is the  $(p$ -local) sphere spectrum. The infinite family  $\sigma_{n-1,p^k}$  is the Adams-Toda  $\alpha$ -family.

We now briefly sketch an argument relating Theorem A to the proof of the nilpotence theorem; we shall elaborate on this discussion in forthcoming work.

**Remark 5.4.7.** The heart of the nilpotence theorem is what is called Step III in [DHS88]; this step amounts to showing that certain self-maps of  $T(n-1)$ -module skeleta (denoted  $G_k$  in [DHS88]) of  $T(n)$  are nilpotent. These self-maps are given by multiplication by  $b_{n,k} := \sigma_{n-1,p^k}^2 \in \pi_{p^k|v_n|-1}T(n-1)$  at  $p = 2$ , and by multiplication by the  $p$ -fold Toda bracket  $b_{n,k} = \langle \sigma_{n-1,p^k}, \dots, \sigma_{n-1,p^k} \rangle$  at an odd prime  $p$ . (When  $p = 2$ , the element  $\sigma_{n-1,p^k}$  is denoted by  $h$  in [Hop87, Theorem 3].) It therefore suffices to establish the nilpotency of the  $b_{n,k}$ .

This can be proven through Theorem A via induction on  $k$ ; we shall assume Conjecture D and Conjecture E for the remainder of this discussion. The motivation for this approach stems from the observation that if  $R$  is any  $\mathbf{E}_3\text{-F}_p$ -algebra and  $x \in \pi_*(R)$ , then there is a relation  $Q_1(x)^2 = Q_2(x^2)$  (at odd primes, one has a relation involving the  $p$ -fold Toda bracket  $\langle Q_1(x), \dots, Q_1(x) \rangle$ ). In our setting, Proposition 5.4.3 implies that the elements  $\sigma_{n-1,k}$  lift to  $\pi_*\mathfrak{Z}_3(X(p^n-1))$ . At  $p = 2$ , one can prove (in the same way that the Cartan relation  $Q_1(x)^2 = Q_2(x^2)$  is proven) that the construction of this infinite family implies that  $\sigma_{n-1,p^{k+1}}^2$  can be described in terms of  $Q_2(\sigma_{n-1,p^k}^2)$ . At odd primes, there is a similar relation involving the  $p$ -fold Toda bracket defining  $b_{n,k}$ . In particular, induction on  $k$  implies that the  $b_{n,k}$  are all nilpotent in  $\pi_*\mathfrak{Z}_3(X(p^n-1))$  if  $b_{n,0}$  is nilpotent.

To argue that  $b_{n,0}$  is nilpotent, one first observes that  $\sigma_{n-1}b_{n,0}^p = 0$  in  $\pi_*\mathfrak{Z}_3(X(p^n-1))$ ; when  $n = 0$ , this follows from the statement that  $\alpha_1\beta_1^p = 0$  in the sphere. To show that  $b_{n,0}$  is nilpotent, it suffices to establish that  $\mathfrak{Z}_3(X(p^n-1))[1/b_{n,0}^p]$  is contractible; when  $n = 0$ , this follows from Proposition 5.4.1. We give a very brief sketch of this nilpotence for general  $n$ , by arguing as in Proposition 5.4.1, and with a generous lack of precision which will be remedied in forthcoming work.

For notational convenience, we now denote  $d_{n,0} = b_{n,0}^p$ . It suffices to show that the multiplication-by- $d_{n,0}$  map

$$d_{n,0} : \Sigma^{|d_{n,0}|}\mathfrak{Z}_3(X(p^n-1)) \rightarrow \mathfrak{Z}_3(X(p^n-1))[1/d_{n,0}]$$

is nullhomotopic. Since  $\sigma_{n-1}$  kills  $d_{n,0}$ , we know that  $\sigma_{n-1}$  is nullhomotopic in  $\mathfrak{Z}_3(X(p^n-1))[1/d_{n,0}]$ . Moreover, the bracket  $\langle p, \sigma_{n-1}, 1_{\mathfrak{Z}_3(X(p^n-1))[1/d_{n,0}]} \rangle$  contains zero. By arguing as in Proposition 5.4.1, we can conclude that the composite

$$K_n \rightarrow \Omega^2 P^{|\sigma_{n-1}|+4} \xrightarrow{\phi} \mathbf{BGL}_1(\mathfrak{Z}_3(X(p^n-1))) \rightarrow \mathbf{BGL}_1(\mathfrak{Z}_3(X(p^n-1))[1/d_{n,0}])$$

is nullhomotopic, where the map  $\phi$  is constructed in Step 1 of the proof of Theorem A. (Recall that the proof of Theorem A shows that the Thom spectrum  $(\Omega^2 P^{|\sigma_{n-1}|+4})^\phi$  is an  $\mathbf{E}_1 \otimes \mathbb{Q}_1\text{-}\mathfrak{Z}_3(X(p^n-1))$ -algebra

such that  $\mathrm{BP}\langle n-1 \rangle$  splits off its base change along the map  $\mathfrak{Z}_3(X(p^n-1)) \rightarrow T(n-1)$ .) It follows from Theorem 2.1.7 that the multiplication-by- $d_{n,0}$  map factors as

$$\begin{array}{ccccc} \Sigma^{|d_{n,0}|} \mathfrak{Z}_3(X(p^n-1)) & \xrightarrow{d_{n,0}} & \mathfrak{Z}_3(X(p^n-1)) & \longrightarrow & \mathfrak{Z}_3(X(p^n-1))[1/d_{n,0}]. \\ \downarrow & & \downarrow & \nearrow \text{---} & \\ \Sigma^{|d_{n,0}|} (\Omega^2 P^{|\sigma_{n-1}|+4})^\phi & \xrightarrow{d_{n,0}} & (\Omega^2 P^{|\sigma_{n-1}|+4})^\phi & & \end{array}$$

To show that the top composite is null, it therefore suffices to show that the self map of  $K_n^\phi$  defined by  $d_{n,0}$  is nullhomotopic. This essentially follows from the fact that  $(\Omega^2 P^{|\sigma_{n-1}|+4})^\phi$  is an  $\mathbf{E}_1 \otimes \mathcal{Q}_1$ - $\mathfrak{Z}_3(X(p^n-1))$ -algebra: multiplication by  $d_{n,0}$  is therefore null on  $K_n^\phi$ , because  $d_{n,0}$  is built from  $\sigma_{n-1}$  (which is null in  $(\Omega^2 P^{|\sigma_{n-1}|+4})^\phi$ ) via  $\mathbf{E}_1$ -power operations.

## 6. APPLICATIONS

**6.1. Splittings of cobordism spectra.** The goal of this section is to prove the following.

**Theorem C.** *Assume that the composite  $\mathfrak{Z}_3(B) \rightarrow B \rightarrow \mathrm{MString}$  is an  $\mathbf{E}_3$ -map. Then Conjectures D, E, and F imply that there is a unital splitting of the Ando-Hopkins-Rezk orientation  $\mathrm{MString}_{(2)} \rightarrow \mathrm{tmf}_{(2)}$ .*

**Remark 6.1.1.** We believe that the assumption that the composite  $\mathfrak{Z}_3(B) \rightarrow B \rightarrow \mathrm{MString}$  is an  $\mathbf{E}_3$ -map is too strong: we believe that it can be removed using special properties of fibers of charming maps, and we will return to this in future work.

We only construct unstructured splittings; it seems unlikely that they can be refined to structured splittings. A slight modification of our arguments should work at any prime.

**Remark 6.1.2.** In fact, the same argument used to prove Theorem C shows that if the composite  $\mathfrak{Z}_3(A) \rightarrow A \rightarrow \mathrm{MSpin}$  is an  $\mathbf{E}_3$ -map, then Conjecture D and Conjecture E imply that there are unital splittings of the Atiyah-Bott-Shapiro orientation  $\mathrm{MSpin} \rightarrow \mathrm{bo}$ . This splitting was originally proved unconditionally (i.e., without assuming Conjecture D or Conjecture E) by Anderson-Brown-Peterson in [ABP67] via a calculation with the Adams spectral sequence.

**Remark 6.1.3.** The inclusion of the cusp on  $\overline{\mathcal{M}}_{\mathrm{ell}}$  defines an  $\mathbf{E}_\infty$ -map  $c : \mathrm{tmf} \rightarrow \mathrm{bo}$  as in [LN14, Theorem 1.2]. The resulting diagram

$$\begin{array}{ccc} \mathrm{MString} & \longrightarrow & \mathrm{MSpin} \\ \downarrow & & \downarrow \\ \mathrm{tmf} & \xrightarrow{c} & \mathrm{bo} \end{array}$$

commutes (see, e.g., [Dev19b, Lemma 6.4]). The splitting  $s : \mathrm{tmf} \rightarrow \mathrm{MString}$  of Theorem C defines a composite

$$\mathrm{tmf} \xrightarrow{s} \mathrm{MString} \rightarrow \mathrm{MSpin} \rightarrow \mathrm{bo}$$

which agrees with  $c$ .

**Remark 6.1.4.** The Anderson-Brown-Peterson splitting implies that if  $X$  is any compact space, then the Atiyah-Bott-Shapiro  $\hat{A}$ -genus (i.e., the index of the Dirac operator in families)  $\mathrm{MSpin}^*(X) \rightarrow \mathrm{bo}^*(X)$  is surjective. Similarly, if the composite  $\mathfrak{Z}_3(B) \rightarrow B \rightarrow \mathrm{MString}$  is an  $\mathbf{E}_3$ -map, then Conjectures D, E, and F imply that the Ando-Hopkins-Rezk orientation (i.e., the Witten genus in families)  $\mathrm{MString}^*(X) \rightarrow \mathrm{tmf}^*(X)$  is also surjective.

**Remark 6.1.5.** In [Dev19b], we proved (unconditionally) that the map  $\pi_* \text{MString} \rightarrow \pi_* \text{tmf}$  is surjective. Our proof proceeds by showing that the map  $\pi_* B \rightarrow \pi_* \text{tmf}$  is surjective via arguments with the Adams-Novikov spectral sequence and by exploiting the  $E_1$ -ring structure on  $B$  to lift the powers of  $\Delta$  living in  $\pi_* \text{tmf}$ .

The discussion preceding [MR99, Remark 7.3] implies that for a particular model of  $\text{tmf}_0(3)$ , we have:

**Corollary 6.1.6.** *Assume that the composite  $\mathfrak{Z}_3(B) \rightarrow B \rightarrow \text{MString}$  is an  $E_3$ -map. Then Conjectures D, E, and F imply that  $\Sigma^{16} \text{tmf}_0(3)$  is a summand of  $\text{MString}_2^\wedge$ .*

We now turn to the proof of Theorem C.

*Proof of Theorem C.* First, note that such a splitting exists after rationalization. Indeed, it suffices to check that this is true on rational homotopy; since the orientations under considerations are  $E_\infty$ -ring maps, the induced map on homotopy is one of rings. It therefore suffices to lift the generators.

We now show that the generators of  $\pi_* \text{tmf} \otimes \mathbb{Q} \cong \mathbb{Q}[c_4, c_6]$  lift to  $\pi_* \text{MString} \otimes \mathbb{Q}$ . Although one can argue this by explicitly constructing manifold representatives (as is done for  $c_4$  in [Dev19b, Corollary 6.3]), it is also possible to provide a more homotopy-theoretic proof: the elements  $c_4$  and  $c_6$  live in dimensions 8 and 12 respectively, and the map  $\text{MString} \rightarrow \text{tmf}$  is known to be an equivalence in dimensions  $\leq 15$ . It follows that the same is true rationally, so  $c_4$  and  $c_6$  indeed lift to  $\pi_* \text{MString} \otimes \mathbb{Q}$ , as desired.

We will now construct a splitting after  $p$ -completion (where  $p = 2$ ). By Corollary 5.1.6, we obtain a unital map  $\text{tmf} \simeq \Theta(B) \rightarrow \text{MString}$  upon  $p$ -completion which splits the orientation  $\text{MString} \rightarrow \Theta(B)$  because:

- (a) the map  $\mathfrak{Z}_3(B) \rightarrow B \rightarrow \text{MString}$  is an  $E_3$ -ring map (by assumption).
- (b) the element  $\sigma_2$  vanishes in  $\pi_{13} \text{MString}_{(2)}$  (because  $\pi_{13} \text{MString}_{(2)} \cong \pi_{13} \text{tmf}_{(2)} \cong 0$ ),
- (c) and the Toda bracket  $\langle 2, \sigma_2, 1_{\text{MString}_{(2)}} \rangle \subseteq \pi_* \text{MString}_{(2)}$  contains zero (because  $\pi_{14} \text{MString}_{(2)} \cong \pi_{14} \text{tmf}_{(2)}$ , and the corresponding bracket contains 0 in  $\pi_* \text{tmf}_{(2)}$ ).

To obtain a map  $\text{tmf}_{(p)} \rightarrow \text{MString}_{(p)}$ , we need to show that the induced map  $\text{tmf} \otimes \mathbb{Q} \rightarrow \text{tmf}_p^\wedge \otimes \mathbb{Q} \rightarrow \text{MString}_p^\wedge \otimes \mathbb{Q}$  agrees with the rational splitting constructed in the previous paragraph. However, this is immediate from the fact that the splittings  $\text{tmf}_p^\wedge \rightarrow \text{MString}_p^\wedge$  are constructed to be equivalences in dimensions  $\leq 15$ , and the fact that the map out of  $\text{tmf} \otimes \mathbb{Q}$  is determined by its effect on the generators  $c_4$  and  $c_6$ .  $\square$

**Remark 6.1.7.** The proof recalled in the introduction of Thom's splitting of MO proceeded essentially unstably: there is an  $E_2$ -map  $\Omega^2 S^3 \rightarrow \text{BO}$  of spaces over  $\text{BGL}_1(\mathbb{S})$ , whose Thomification yields the desired  $E_2$ -map  $\text{HF}_2 \rightarrow \text{MO}$ . This argument also works for  $\text{MSO}$ : there is an  $E_2$ -map  $\Omega^2 S^3 \langle 3 \rangle \rightarrow \text{BSO}$  of spaces over  $\text{BGL}_1(\mathbb{S})$ , whose Thomification yields the desired  $E_2$ -map  $\text{HZ} \rightarrow \text{MSO}$ . One might hope for the existence of a similar unstable map which would yield Theorem C. We do not know how to construct such a map. To illustrate the difficulty, let us examine how such a proof would work; we will specialize to the case of  $\text{MString}$ , but the discussion is the same for  $\text{MSpin}$ .

According to Theorem A, Conjecture D and Conjecture E imply that there is a map  $K_3 \rightarrow \text{BGL}_1(B)$  whose Thom spectrum is equivalent to  $\text{tmf}$ . There is a map  $BN \rightarrow \text{B}^2 \text{String}$ , whose fiber we will denote by  $Q$ . Then there is a fiber sequence

$$N \rightarrow \text{BString} \rightarrow Q,$$

and so Proposition 2.1.6 implies that there is a map  $Q \rightarrow \text{BGL}_1(B)$  whose Thom spectrum is  $\text{MString}$ . Theorem C would follow if there was a map  $f : K_3 \rightarrow Q$  of spaces over  $\text{BGL}_1(B)$ , since Thomification would produce a map  $\text{tmf} \rightarrow \text{MString}$ .

Conjecture [D](#) reduces the construction of  $f$  to the construction of a map  $\Omega^2 P^{17}(2) \rightarrow Q$ . This map would in particular imply the existence of a map  $P^{15}(2) \rightarrow Q$  (and would be equivalent to the existence of such a map if  $Q$  was a double loop space), which in turn stems from a 2-torsion element of  $\pi_{14}(Q)$ . The long exact sequence on homotopy runs

$$\cdots \rightarrow \pi_{14}(\text{BString}) \rightarrow \pi_{14}(Q) \rightarrow \pi_{13}(N) \rightarrow \pi_{13}(\text{BString}) \rightarrow \cdots$$

Bott periodicity states that  $\pi_{13}\text{BString} \cong \pi_{14}\text{BString} \cong 0$ , so we find that  $\pi_{14}(Q) \cong \pi_{13}(N)$ . The desired 2-torsion element of  $\pi_{14}(Q)$  is precisely the element of  $\pi_{13}(N)$  described in Remark [3.2.21](#). Choosing a particular nullhomotopy of twice this 2-torsion element of  $\pi_{14}(Q)$  produces a map  $g : P^{15}(2) \rightarrow Q$ . To extend this map over the double suspension  $P^{15}(2) \rightarrow \Omega^2 P^{17}(2)$ , it would suffice to show that there is a double loop space  $\tilde{Q}$  with a map  $\tilde{Q} \rightarrow Q$  such that  $g$  factors through  $\tilde{Q}$ .

Unfortunately, we do not know how to prove such a result; this is the unstable analogue of Conjecture [E](#). In fact, such an unstable statement would bypass the need for Conjecture [E](#) in Theorem [A](#). (One runs into the same obstruction for  $\text{MSpin}$ , except with the fiber of the map  $S^5 \rightarrow \text{B}^2\text{Spin}$ .) These statements are reminiscent of the conjecture (see Section [4.1](#)) that the fiber  $W_n = \text{fib}(S^{2n-1} \rightarrow \Omega^2 S^{2n+1})$  of the double suspension admits the structure of a double loop space.

**Remark 6.1.8.** The following application of Theorem [C](#) was suggested by Mike Hopkins. In [[HH92](#)], the Anderson-Brown-Peterson splitting is used to show that the Atiyah-Bott-Shapiro orientation  $\text{MSpin} \rightarrow \text{KO}$  induces an isomorphism

$$\text{MSpin}_*(X) \otimes_{\text{MSpin}_*} \text{KO}_* \xrightarrow{\cong} \text{KO}_*(X)$$

of  $\text{KO}_*$ -modules for all spectra  $X$ . In future work, we shall show that Theorem [C](#) can be used to prove the following height 2 analogue of this result: namely, Conjectures [D](#), [E](#), and [F](#) imply that the Ando-Hopkins-Rezk orientation  $\text{MString} \rightarrow \text{Tmf}$  induces an isomorphism

$$(6.1) \quad \text{MString}_*(X) \otimes_{\text{MString}_*} \text{Tmf}_* \xrightarrow{\cong} \text{Tmf}_*(X)$$

of  $\text{Tmf}_*$ -modules for all spectra  $X$ . The  $K(1)$ -analogue of this isomorphism was obtained by Laures in [[Lau04](#)].

**6.2. Wood equivalences.** The Wood equivalence states that  $\text{bo} \wedge C\eta \simeq \text{bu}$ . There are generalizations of this equivalence to  $\text{tmf}$  (see [[Mat16](#)]); for instance, there is a 2-local 8-cell complex  $DA_1$  whose cohomology is isomorphic to the double of  $A(1)$  as an  $A(2)$ -module such that  $\text{tmf}_{(2)} \wedge DA_1 \simeq \text{tmf}_1(3) \simeq \text{BP}\langle 2 \rangle$ . Similarly, if  $X_3$  denotes the 3-local 3-cell complex  $S^0 \cup_{\alpha_1} e^4 \cup_{2\alpha_1} e^8$ , then  $\text{tmf}_{(3)} \wedge X_3 \simeq \text{tmf}_1(2) \simeq \text{BP}\langle 2 \rangle \vee \Sigma^8 \text{BP}\langle 2 \rangle$ . We will use the umbrella term “Wood equivalence” to refer to equivalences of this kind.

Our goal in this section is to revisit these Wood equivalences using the point of view stemming from Theorem [A](#). In particular, we will show that these equivalences are suggested by the existence of certain EHP sequences; we find this to be a rather beautiful connection between stable and unstable homotopy theory. Each of the statements admit elementary proofs which only use the calculations in Section [3](#), but we will nonetheless spend a lot of time discussing their connections to EHP sequences.

The first Wood-style result was proved in Proposition [3.2.5](#). The next result, originally proved in [[Mah79](#), Section 2.5] and [[DM81](#), Theorem 3.7], is the simplest example of a Wood-style equivalence which is related to the existence of certain EHP sequences.

**Proposition 6.2.1.** *Let  $\mathbb{S} // \eta = X(2)$  (resp.  $\mathbb{S} // 2$ ) denote the  $\mathbf{E}_1$ -quotient of  $\mathbb{S}$  by  $\eta$  (resp. 2). If  $Y = C\eta \wedge \mathbb{S} / 2$  and  $A_1$  is a spectrum whose cohomology is isomorphic to  $A(1)$  as a module over the Steenrod algebra, then there are equivalences*

$$A \wedge C\eta \simeq \mathbb{S} // \eta, \quad A \wedge Y \simeq \mathbb{S} // 2, \quad A \wedge A_1 \simeq \gamma(1) / v_1$$

*of  $A$ -modules.*



**Remark 6.2.2.** Proposition 6.2.1 implies the Wood equivalence  $\mathrm{bo} \wedge C\eta \simeq \mathrm{bu}$ . Although this implication is already true before 2-completion, we will work in the 2-complete category for convenience. Recall that Theorem A states that Conjecture D and Conjecture E imply that there is a map  $\mu : K_2 \rightarrow \mathrm{BGL}_1(A)$  whose Thom spectrum is equivalent to  $\mathrm{bo}$  (as left  $A$ -modules). Moreover, the Thom spectrum of the composite  $K_2 \xrightarrow{\mu} \mathrm{BGL}_1(A) \rightarrow \mathrm{BGL}_1(T(1))$  is equivalent to  $\mathrm{BP}\langle 1 \rangle$ . Since this Thom spectrum is the base-change  $K_2^\mu \wedge_A T(1)$ , and Proposition 6.2.1 implies that  $T(1) = X(2) \simeq A \wedge C\eta$ , we find that

$$\mathrm{BP}\langle 1 \rangle \simeq K_2^\mu \wedge_A (A \wedge C\eta) \simeq K_2^\mu \wedge C\eta \simeq \mathrm{bo} \wedge C\eta,$$

as desired. Similarly, noting that  $\mathbb{S}/2 = \gamma(1)$ , we find that Proposition 6.2.1 also proves the equivalence  $\mathrm{bo} \wedge Y \simeq k(1)$ .

**Remark 6.2.3.** The argument of Remark 6.2.2 in fact proves that Theorem A for  $A$  implies Theorem A for  $T(1)$ ,  $\gamma_{\mathbb{Z}}(1)$ , and  $\gamma(1)$ .

**Remark 6.2.4.** There are EHP sequences

$$S^1 \rightarrow \Omega S^2 \rightarrow \Omega S^3, \quad S^2 \rightarrow \Omega S^3 \rightarrow \Omega S^5.$$

Recall that  $\mathbb{S}/2$ ,  $C\eta$ ,  $\mathbb{S}/2$ ,  $\mathbb{S}/\eta$  are Thom spectra over  $S^1$ ,  $S^2$ ,  $\Omega S^2$ ,  $\Omega S^3$ , and  $\Omega S^5$  respectively. Proposition 2.1.6 therefore implies that there are maps  $f : \Omega S^3 \rightarrow \mathrm{BAut}(\mathbb{S}/2)$  and  $g : \Omega S^5 \rightarrow \mathrm{BAut}(C\eta)$  whose Thom spectra are equivalent to  $\mathbb{S}/2$  and  $\mathbb{S}/\eta$ , respectively. The maps  $f$  and  $g$  define local systems of spectra over  $\Omega S^3$  and  $\Omega S^5$  whose fibers are equivalent to  $\mathbb{S}/2$  and  $C\eta$  (respectively), and one interpretation of Proposition 6.2.1 is that these local systems in fact factor as

$$\Omega S^3 \xrightarrow{\eta} \mathrm{BGL}_1(\mathbb{S}) \rightarrow \mathrm{BAut}(\mathbb{S}/2), \quad \Omega S^5 \xrightarrow{\nu} \mathrm{BGL}_1(\mathbb{S}) \rightarrow \mathrm{BAut}(C\eta).$$

If one could prove this directly, then Proposition 6.2.1 would be an immediate consequence. We argue this for the first case in Remark 6.2.5. We do not know how to argue this for the second case.

**Remark 6.2.5.** The first EHP sequence in Remark 6.2.4 splits via the Hopf map  $S^3 \rightarrow S^2$ . The map  $f : \Omega S^3 \rightarrow \mathrm{BAut}(\mathbb{S}/2)$  in fact factors through the dotted map in the following diagram:

$$\begin{array}{ccccc} S^2 & \longrightarrow & \Omega S^2 & \longrightarrow & \Omega S^3 \\ & & \downarrow & \swarrow \text{dotted} & \downarrow \\ & & \mathrm{BGL}_1(\mathbb{S}) & \longrightarrow & \mathrm{BAut}(\mathbb{S}/2). \end{array}$$

Indeed, the composite  $\Omega S^3 \rightarrow \Omega S^2 \rightarrow \mathrm{BGL}_1(\mathbb{S})$  is a loop map, and therefore is determined by the composite  $\phi : S^3 \rightarrow S^2 \rightarrow B^2\mathrm{GL}_1(\mathbb{S})$ . Since the map  $S^2 \rightarrow B^2\mathrm{GL}_1(\mathbb{S})$  detects the element  $-1 \in \pi_0(\mathbb{S})^\times$ , the map  $\phi$  does in fact determine a unit multiple of  $\eta$ . This implies the desired claim.

*Proof of Proposition 6.2.1.* For the first two equivalences, it suffices to show that  $A \wedge C\eta \simeq \mathbb{S}/\eta$  and that  $\mathbb{S}/\eta \wedge \mathbb{S}/2 \simeq \mathbb{S}/2$ . We will prove the first statement; the proof of the second statement is exactly the same. There is a map  $C\eta \rightarrow \mathbb{S}/\eta$  given by the inclusion of the 2-skeleton. There is also an  $E_1$ -ring map  $A \rightarrow \mathbb{S}/\eta$  given as follows. The multiplication on  $\mathbb{S}/\eta$  defines a unital map  $C\eta \wedge C\eta \rightarrow \mathbb{S}/\eta$ . But since the Toda bracket  $\langle \eta, 2, \eta \rangle$  contains  $\nu$ , there is a unital map  $C\nu \rightarrow C\eta \wedge C\eta$ . This supplies a unital map  $C\nu \rightarrow \mathbb{S}/\eta$ , which, by the universal property of  $A = \mathbb{S}/\nu$  (via Theorem 2.1.7), extends to an  $E_1$ -ring map  $A \rightarrow \mathbb{S}/\eta$ .

For the final equivalence, it suffices to construct a map  $A_1 \rightarrow \gamma(1)/v_1$  for which the induced map  $A \wedge A_1 \rightarrow \gamma(1)/v_1$  gives an isomorphism on mod 2 homology. Since  $A_1$  may be obtained as the cofiber of a  $v_1$ -self map  $\Sigma^2 Y \rightarrow Y$ , it suffices to observe that the following diagram commutes; our desired

map is the induced map on vertical cofibers:

$$\begin{array}{ccc} \Sigma^2 Y & \longrightarrow & \Sigma^2 y(1) \\ \downarrow v_1 & & \downarrow v_1 \\ Y & \longrightarrow & y(1). \end{array}$$

□

Next, we have the following result at height 2:

**Proposition 6.2.6.** *Let  $DA_1$  denote the double of  $A_1$  (see [Mat16]). There are 2-complete equivalences*

$$B \wedge DA_1 \simeq T(2), \quad B \wedge Z \simeq y(2), \quad B \wedge A_2 \simeq y(2)/v_2,$$

where  $Z$  is the spectrum “ $\frac{1}{2}A_2$ ” from [MT94, BE16]<sup>4</sup>, and  $A_2$  is a spectrum whose cohomology is isomorphic to  $A(2)$  as a module over the Steenrod algebra.

**Remark 6.2.7.** Arguing as in Remark 6.2.2 shows that Proposition 6.2.6 and Theorem A imply the Wood equivalences

$$\mathrm{tmf} \wedge DA_1 \simeq \mathrm{tmf}_1(3) = \mathrm{BP}\langle 2 \rangle, \quad \mathrm{tmf} \wedge Z \simeq k(2), \quad \mathrm{tmf} \wedge A_2 \simeq \mathrm{HF}_2.$$

**Remark 6.2.8.** Exactly as in Remark 6.2.3, the argument of Remark 6.2.7 in fact proves that Theorem A for  $B$  implies Theorem A for  $T(2)$ ,  $y_Z(2)$ , and  $y(2)$ .

**Remark 6.2.9.** The telescope conjecture, which we interpret as stating that  $L_n$ -localization is the same as  $L_n^f$ -localization, is known to be true at height 1. For odd primes, it was proved by Miller in [Mil81], and at  $p = 2$  it was proved by Mahowald in [Mah81a, Mah82]. Mahowald’s approach was to calculate the telescopic homotopy of the type 1 spectrum  $Y$ . In [MRS01], Mahowald-Ravenel-Shick proposed an approach to disproving the telescope conjecture at height 2: they suggest that for  $n \geq 2$ , the  $L_n$ -localization and the  $v_n$ -telescopic localization of  $y(n)$  have different homotopy groups. They show, however, that the  $L_1$ -localization and the  $v_1$ -telescopic localization of  $y(1)$  agree, so this approach (thankfully) does not give a counterexample to the telescope conjecture at height 1.

Motivated by Mahowald’s approach to the telescope conjecture, Behrens-Beaudry-Bhattacharya-Culver-Xu study the  $v_2$ -telescopic homotopy of  $Z$  in [BBB<sup>+</sup>19], with inspiration from the Mahowald-Ravenel-Shick approach. Propositions 6.2.1 and 6.2.6 can be used to relate these two (namely, the finite spectrum and the Thom spectrum) approaches to the telescope conjecture. As in Section 6.1, we will let  $R$  denote  $A$  or  $B$ . Moreover, let  $F$  denote  $Y$  or  $Z$  (depending on what  $R$  is), and let  $R'$  denote  $y(1)$  or  $y(2)$  (again depending on what  $R$  is), so that  $R \wedge F = R'$  by Propositions 6.2.1 and 6.2.6. Then:

**Corollary 6.2.10.** *If the telescope conjecture is true for  $F$  (and hence any type 1 or 2 spectrum) or  $R$ , then it is true for  $R'$ .*

*Proof.* Since  $L_n$ - and  $L_n^f$ -localizations are smashing, we find that if the telescope conjecture is true for  $F$  or  $R$ , then Propositions 6.2.1 and 6.2.6 yield equivalences

$$L_n^f R' \simeq R \wedge L_n^f F \simeq R \wedge L_n F \simeq L_n R'.$$

□

Finally, we prove Proposition 6.2.6.

---

<sup>4</sup>In the former source,  $Z$  is denoted by  $\overline{M}$ .

*Proof of Proposition 6.2.6.* We first construct maps  $B \rightarrow T(2)$  and  $DA_1 \rightarrow T(2)$ . The top cell of  $DA_1$  is in dimension 12, and the map  $T(2) \rightarrow BP$  is an equivalence in dimensions  $\leq 12$ . It follows that constructing a map  $DA_1 \rightarrow T(2)$  is equivalent to constructing a map  $DA_1 \rightarrow BP$ . However, both  $BP$  and  $DA_1$  are concentrated in even degrees, so the Atiyah-Hirzebruch spectral sequence collapses, and we find that  $BP^*(DA_1) \cong H^*(DA_1; BP_*)$ . The generator in bidegree  $(0, 0)$  produces a map  $DA_1 \rightarrow T(2)$ ; its effect on homology is the additive inclusion  $F_2[\zeta_1^2, \zeta_2^2]/(\zeta_1^8, \zeta_2^4) \rightarrow F_2[\zeta_1^2, \zeta_2^2]$ .

The map  $B \rightarrow T(2)$  may be defined via the universal property of Thom spectra from Section 2.1 and Remark 3.2.17. Its effect on homology is the inclusion  $F_2[\zeta_1^8, \zeta_2^4] \rightarrow F_2[\zeta_1^2, \zeta_2^2]$ . We obtain a map  $B \wedge DA_1 \rightarrow T(2)$  via the multiplication on  $T(2)$ , and this induces an isomorphism in mod 2 homology.

For the second equivalence, we argue similarly: the map  $B \rightarrow T(2)$  defines a map  $B \rightarrow T(2) \rightarrow \gamma(2)$ . Next, recall that  $Z$  is built through iterated cofiber sequences:

$$\Sigma^2 Y \xrightarrow{v_1} Y \rightarrow A_1, \quad \Sigma^5 A_1 \wedge C\nu \xrightarrow{\sigma_1} A_1 \wedge C\nu \rightarrow Z.$$

As an aside, we note that the element  $\sigma_1$  is intimately related to the element discussed in Example 3.1.14; namely, it is given by the self-map of  $A_1 \wedge C\nu$  given by smashing  $A_1$  with the following diagram:

$$\begin{array}{ccccc} \Sigma^5 C\nu & \xrightarrow{\quad} & \Sigma^5 A & \xrightarrow{\sigma_1 \wedge \text{id}} & A \wedge A & \longrightarrow & A \\ & \searrow & & & & & \uparrow \\ & & & \sigma_1 & & & C\nu. \end{array}$$

Using these cofiber sequences and Proposition 3.2.5, one obtains a map  $Z \rightarrow \gamma(2)$ , which induces the additive inclusion  $F_2[\zeta_1, \zeta_2]/(\zeta_1^8, \zeta_2^4) \rightarrow F_2[\zeta_1, \zeta_2]$  on mod 2 homology. The multiplication on  $\gamma(2)$  defines a map  $B \wedge Z \rightarrow \gamma(2)$ , which induces an isomorphism on mod 2 homology.

For the final equivalence, it suffices to construct a map  $A_2 \rightarrow \gamma(2)/v_2$  for which the induced map  $B \wedge A_2 \rightarrow \gamma(2)/v_2$  gives an isomorphism on mod 2 homology. Since  $A_2$  may be obtained as the cofiber of a  $v_2$ -self map  $\Sigma^6 Z \rightarrow Z$ , it suffices to observe that the following diagram commutes; our desired map is the induced map on vertical cofibers:

$$\begin{array}{ccc} \Sigma^6 Z & \longrightarrow & \Sigma^6 \gamma(2) \\ \downarrow v_2 & & \downarrow v_2 \\ Z & \longrightarrow & \gamma(2). \end{array}$$

□

Arguing exactly as in the proof of Proposition 6.2.6 shows the following result at the prime 3:

**Proposition 6.2.11.** *Let  $X_3$  denote the 8-skeleton of  $T(1) = \mathbb{S}/\alpha_1$ . There are 3-complete equivalences*

$$B \wedge X_3 \simeq T(2) \vee \Sigma^8 T(2), \quad B \wedge X_3 \wedge \mathbb{S}/(3, v_1) \simeq \gamma(2) \vee \Sigma^8 \gamma(2).$$

We now discuss the relation between Proposition 6.2.6 and EHP sequences, along the lines of Remark 6.2.4. We begin by studying the spectrum  $\gamma_Z(2)$ , which is a spectrum such that modding out by 2 yields  $\gamma(2)$ . The spectrum  $\gamma_Z(2)$  is the Thom spectrum of a map  $\Omega J_3(S^2)\langle 2 \rangle$ , but we will need a more explicit description of the space  $\Omega J_3(S^2)\langle 2 \rangle$ .

**Remark 6.2.12.** By [Ame18, Proposition 4.3], there is a fiber sequence

$$(6.2) \quad V_2(\mathbf{R}^5) \simeq J_3(S^2)\langle 2 \rangle \rightarrow J_3(S^2) \rightarrow \mathbf{C}P^\infty,$$

where  $V_k(\mathbf{R}^n)$  is the Stiefel manifold of orthonormal  $k$ -frames in  $\mathbf{R}^n$ . The fiber sequence (6.2) is split upon looping, so we obtain an equivalence  $\Omega J_3(S^2) \simeq \Omega V_2(\mathbf{R}^5) \times S^1$ . Recall that  $\gamma(2)$  arises as the Thom spectrum of a map  $\Omega J_3(S^2) \rightarrow \text{BGL}_1(\mathbb{S})$  such that the composite with  $S^1 \rightarrow \Omega J_3(S^2)$  detects

$-1 \in \pi_0(\mathbb{S})^\times$ . It follows from Proposition 2.1.6 that there is a map  $\phi_1 : \Omega V_2(\mathbf{R}^5) \rightarrow \mathrm{B} \mathrm{Aut}(\mathbb{S}/2)$  whose Thom spectrum is  $\gamma(2)$ . By arguing as in Remark 6.2.5, we find that  $\phi_1$  factors as

$$\Omega V_2(\mathbf{R}^5) \xrightarrow{\mu_1} \mathrm{BGL}_1(\mathbb{S}) \rightarrow \mathrm{B} \mathrm{Aut}(\mathbb{S}/2)$$

for some map  $\mu_1 : \Omega V_2(\mathbf{R}^5) \rightarrow \mathrm{BGL}_1(\mathbb{S})$ . It follows that

$$\gamma(2) \simeq \Omega V_2(\mathbf{R}^5)^{\mu_1} \wedge \mathbb{S}/2 \simeq \gamma_{\mathbf{Z}}(2) \wedge \mathbb{S}/2.$$

**Construction 6.2.13.** The map  $\mu_1$  admits the following explicit description. Recall that  $V_2(\mathbf{R}^5)$  sits in a fiber sequence

$$S^3 \rightarrow V_2(\mathbf{R}^5) \rightarrow S^4,$$

so taking loops defines a fiber sequence with total space  $\Omega V_2(\mathbf{R}^5)$ . We now define a real vector bundle over  $\Omega V_2(\mathbf{R}^5)$  via the map of fiber sequences

$$\begin{array}{ccccc} \Omega V_2(\mathbf{R}^5) & \longrightarrow & \Omega S^4 & \longrightarrow & S^3 \\ \downarrow \lambda_1 & & \downarrow & & \downarrow \\ \mathrm{BO} & \longrightarrow & * & \longrightarrow & \mathrm{B}^2\mathrm{O}, \end{array}$$

where the map  $S^3 \rightarrow \mathrm{B}^2\mathrm{O}$  detects the generator of  $\pi_2(\mathrm{BO}) \cong \mathbf{Z}/2$ . The map  $\mu_1$  is then the composite of  $\lambda_1$  with the J-homomorphism  $\mathrm{BO} \rightarrow \mathrm{BGL}_1(\mathbb{S})$ .

Next, we turn to the relationship between  $T(2)$  and  $\gamma_{\mathbf{Z}}(2)$ .

**Construction 6.2.14.** Remark 3.1.9 says that  $T(2)$  is the Thom spectrum of a bundle over  $\Omega V_2(\mathbf{H}^2)$ , and Remark 6.2.12 says that  $\gamma_{\mathbf{Z}}(2)$  is the Thom spectrum of a bundle over  $\Omega V_2(\mathbf{R}^5)$ . Just as in Remark 6.2.4, the equivalence  $T(2)/v_1 \simeq \gamma_{\mathbf{Z}}(2)$  of Proposition 3.2.5 is related to an “EHP” fiber sequence relating  $\Omega V_2(\mathbf{R}^5)$  and  $\Omega V_2(\mathbf{H}^2)$ . To describe this fiber sequence, recall that  $V_2(\mathbf{H}^2) \simeq \mathrm{Sp}(2)$ , which is isomorphic to  $\mathrm{Spin}(5)$ . There is therefore a map

$$V_2(\mathbf{H}^2) \simeq \mathrm{Sp}(2) \simeq \mathrm{Spin}(5) \rightarrow \mathrm{SO}(5) \rightarrow \mathrm{SO}(5)/\mathrm{SO}(3) \simeq V_2(\mathbf{R}^5).$$

There is a pullback square

$$\begin{array}{ccc} V_2(\mathbf{H}^2) & \longrightarrow & S^7 \\ \downarrow & & \downarrow \nu \\ V_2(\mathbf{R}^5) & \longrightarrow & S^4, \end{array}$$

where both the horizontal and vertical fibers are  $S^3$ . This results in a fiber sequence

$$(6.3) \quad \Omega V_2(\mathbf{H}^2) \rightarrow \Omega V_2(\mathbf{R}^5) \rightarrow S^3;$$

this fiber sequence is in fact split, because loops on the Hopf map  $\nu : S^7 \rightarrow S^4$  splits the Hopf map  $\Omega S^4 \rightarrow \Omega S^7$ . The composite

$$\Omega V_2(\mathbf{H}^2) \rightarrow \Omega V_2(\mathbf{R}^5) \xrightarrow{\lambda_1} \mathrm{BO}$$

agrees with the map  $\mu : \Omega V_2(\mathbf{H}^2) \rightarrow \mathrm{BO}$  from Remark 3.1.9, and the Thom spectrum of this composite is the canonical map  $T(2) \rightarrow \gamma_{\mathbf{Z}}(2)$ . The map  $S^3 \rightarrow \mathrm{BGL}_1(\Omega V_2(\mathbf{H}^2)^\mu) = \mathrm{BGL}_1(T(2))$  arising via Proposition 2.1.6 detects  $v_1 \in \pi_2 T(2)$ .

We now discuss the relationship between  $T(2)$  and  $B$ .

**Remark 6.2.15.** Recall that Remark 3.1.9 says that  $T(2)$  is the Thom spectrum of a bundle over  $\Omega V_2(\mathbf{H}^2)$ , and Definition 3.2.16 says that  $B$  is the Thom spectrum of a bundle over  $N$ . We may expect that there is a map of fiber sequences

$$(6.4) \quad \begin{array}{ccccc} \Omega S^3 & \longrightarrow & \Omega V_2(\mathbf{H}^2) & \longrightarrow & \Omega S^7 \\ \downarrow H^2 & & \downarrow & & \downarrow H \\ \Omega S^9 & \longrightarrow & N & \longrightarrow & \Omega S^{13}. \end{array}$$

The fiber of the leftmost map is  $J_3(S^2)$ , and the fiber of the rightmost map is  $S^6$ , so the fiber of the map  $\Omega V_2(\mathbf{H}^2) \rightarrow N$  would sit in a fiber sequence

$$J_3(S^2) \rightarrow \text{fib}(\Omega V_2(\mathbf{H}^2) \rightarrow N) \rightarrow S^6.$$

We do not know how to construct a map of fiber sequences as in (6.4), but one possible approach is as follows. In [Hop84, Section 3.2], Hopkins proves that the Thom space  $(\mathbf{C}P^{2n-2})^{\mathcal{L}^2}$  is a generating complex for  $\Omega \text{Sp}(n)$ , where  $\mathcal{L}$  is the tautological line bundle over  $\mathbf{C}P^{2n-2}$ . In particular, there is a map  $(\mathbf{C}P^2)^{\mathcal{L}^2} \rightarrow \Omega V_2(\mathbf{H}^2)$  whose image on homology generates  $H_*(\Omega V_2(\mathbf{H}^2))$ . (The homology of  $(\mathbf{C}P^2)^{\mathcal{L}^2}$  is free of rank one in degrees 2, 4, and 6.) One might therefore hope to construct a map  $\Omega V_2(\mathbf{H}^2) \rightarrow N$  via a map  $\Omega \Sigma(\mathbf{C}P^2)^{\mathcal{L}^2} \rightarrow N$  which factors through  $\Omega V_2(\mathbf{H}^2)$ . The map  $\Omega \Sigma(\mathbf{C}P^2)^{\mathcal{L}^2} \rightarrow N$  would be adjoint to a map

$$\bigvee_{k \geq 1} \Sigma((\mathbf{C}P^2)^{\mathcal{L}^2})^{\wedge k} \simeq \Sigma \Omega \Sigma(\mathbf{C}P^2)^{\mathcal{L}^2} \rightarrow BN,$$

where we have indicated the James splitting of the source. By connectivity, any map  $\Sigma(\mathbf{C}P^2)^{\mathcal{L}^2} \rightarrow BN$  is null, but there may be a map  $\Sigma(\mathbf{C}P^2)^{\mathcal{L}^2} \wedge (\mathbf{C}P^2)^{\mathcal{L}^2} \rightarrow BN$  producing the desired map  $\Sigma \Omega \Sigma(\mathbf{C}P^2)^{\mathcal{L}^2} \rightarrow BN$ . We leave this as an unresolved question.

**6.3. Topological Hochschild homology.** In this section, we calculate an approximation of the topological Hochschild homology of  $\text{BP}\langle n \rangle$ ,  $\text{bo}$ , and  $\text{tmf}$ .

**Definition 6.3.1.** The unstructured relative topological Hochschild homology of  $\text{BP}\langle n \rangle$ ,  $\text{bo}$ , and  $\text{tmf}$  over  $T(n)$ ,  $A$ , and  $B$  (respectively) is given by:

$$\widehat{\text{THH}}^{T(n)}(\text{BP}\langle n \rangle) \simeq \text{BP}\langle n \rangle \wedge \Sigma_+^\infty \text{BK}_{n+1}, \quad \widehat{\text{THH}}^A(\text{bo}) \simeq \text{bo} \wedge \Sigma_+^\infty \text{BK}_2, \quad \widehat{\text{THH}}^B(\text{tmf}) \simeq \text{tmf} \wedge \Sigma_+^\infty \text{BK}_3,$$

where, as usual,  $K_{n+1}$  denotes the fiber of a charming map  $\Omega^2 S^{2p^{n+1}+1} \rightarrow S^{2p^{n+1}-1}$  and  $K_2$  (resp.  $K_3$ ) denotes the fiber of a charming map  $\Omega^2 S^9 \rightarrow S^7$  (resp.  $\Omega^2 S^{17} \rightarrow S^{15}$ ).

**Remark 6.3.2.** Theorem A implies that there are unital maps  $\text{THH}(\text{BP}\langle n \rangle) \rightarrow \widehat{\text{THH}}^{T(n)}(\text{BP}\langle n \rangle)$ ,  $\text{THH}(\text{bo}) \rightarrow \widehat{\text{THH}}^A(\text{bo})$ , and  $\text{THH}(\text{tmf}) \rightarrow \widehat{\text{THH}}^B(\text{tmf})$ .

We then have the following calculation:

**Theorem 6.3.3.** *The homotopy of  $\widehat{\text{THH}}^{T(n)}(\text{BP}\langle n \rangle)$  is given by  $\pi_* \widehat{\text{THH}}^{T(n)}(\text{BP}\langle n \rangle) \cong \pi_* \text{BP}\langle n \rangle \oplus T_*$ , where  $T_*$  is a torsion graded abelian group concentrated in odd degrees, with a degreewise filtration whose associated graded in degree  $2d-1$  is*

$$\text{gr}(T_{2d-1}) \cong \bigoplus_{k \geq 0} \pi_{2(d-k)p^{n+1}-p^{n+1}+k} \text{BP}\langle n \rangle / p^{v_p(k)}.$$

**Remark 6.3.4.** It is probably also possible to compute  $\pi_* \widehat{\text{THH}}^A(\text{bo})$  and  $\pi_* \widehat{\text{THH}}^B(\text{tmf})$ , but, unlike Theorem 6.3.3, there are lots of possibilities for differentials in the Atiyah-Hirzebruch spectral

sequence. Since we do not have a good description at the moment, we leave this as an unresolved question that we hope to return to in future work. It would be very interesting to determine the Hurewicz image of  $\pi_* \widehat{\mathrm{THH}}^A(\mathrm{bo})$  in  $\pi_* \widehat{\mathrm{THH}}^B(\mathrm{bo})$  via the calculations in [AHL10, Section 7], and use it to make conjectures about the Hurewicz image of  $\pi_* \mathrm{THH}(\mathrm{tmf})$  in  $\pi_* \widehat{\mathrm{THH}}^B(\mathrm{tmf})$ .

**Remark 6.3.5.** The calculation of the additive extensions in the torsion subgroup of  $\pi_* \widehat{\mathrm{THH}}^{T(n)}(\mathrm{BP}\langle n \rangle)$  is quite complicated, so we have not attempted to give a complete description in the statement of Theorem 6.3.3. There may be a simpler combinatorial way to express the extensions.

**Remark 6.3.6.** The homotopy of  $\mathrm{THH}(\mathrm{BP}\langle 1 \rangle)$  was calculated in [AHL10, Theorem 2.6], where it was shown that

$$\pi_* \mathrm{THH}(\mathrm{BP}\langle 1 \rangle) \cong \mathrm{BP}\langle 1 \rangle_* \oplus \mathrm{BP}\langle 1 \rangle_* \left[ \frac{v_1^{p^k + \dots + p}}{p^k} \mid k \geq 1 \right] \oplus \bigoplus_{k \geq 0} \bigoplus_{m=1}^{p-1} \Sigma^{2mp^{k+2} + 2(p-1)} T_{k,m},$$

where  $T_{k,m}$  is an explicit torsion  $\pi_* \mathrm{BP}\langle 1 \rangle$ -module with generators living in degrees  $2 \sum_{i=1}^k m_i p^{n+2-i}$  where  $0 \leq m_i \leq p-1$ , and with numerous relations. The copy of  $\mathrm{BP}\langle 1 \rangle_*$  appearing as a summand in  $\pi_* \mathrm{THH}(\mathrm{BP}\langle 1 \rangle)$  is sent to the corresponding summand in  $\pi_* \widehat{\mathrm{THH}}^{T(1)}(\mathrm{BP}\langle 1 \rangle)$ . One can also attempt to compare the torsion summands. Theorem 6.3.3 gives a description of the torsion summand of  $\pi_* \widehat{\mathrm{THH}}^{T(1)}(\mathrm{BP}\langle 1 \rangle)$  up to additive extension problems. Although we have not computed these additive extensions even in this case, we believe that the torsion in  $\pi_* \mathrm{THH}(\mathrm{BP}\langle 1 \rangle)$  injects into the torsion in  $\pi_* \widehat{\mathrm{THH}}^{T(1)}(\mathrm{BP}\langle 1 \rangle)$ , and that this in fact true more generally at all heights (i.e., the torsion in  $\pi_* \mathrm{THH}(\mathrm{BP}\langle n \rangle)$  injects into the torsion in  $\pi_* \widehat{\mathrm{THH}}^{T(n)}(\mathrm{BP}\langle n \rangle)$ ).

*Proof of Theorem 6.3.3.* Definition 6.3.1 gives an Atiyah-Hirzebruch spectral sequence

$$E_{s,t}^2 = H_s(\mathrm{BK}_{n+1}; \pi_t \mathrm{BP}\langle n \rangle) \Rightarrow \pi_{s+t} \widehat{\mathrm{THH}}^{T(n)}(\mathrm{BP}\langle n \rangle),$$

where the differentials go  $d^r : E_{s,t}^r \rightarrow E_{s-r,t+r-1}^r$ . To calculate this spectral sequence, we use Remark 4.2.3 (after  $p$ -completion): this tells us that

$$E_{s,*}^2 \cong \begin{cases} \pi_* \mathrm{BP}\langle n \rangle & s = 0 \\ \pi_* \mathrm{BP}\langle n \rangle / p^{v_p(k)} & \text{if } s = 2(k+1)p^{n+1} - 2k - 1 \\ 0 & \text{else.} \end{cases}$$

A generic element of this  $E^2$ -page lives in bidegree  $(s, t) = (2(k+1)p^{n+1} - 2k - 1, 2m)$  for  $m = \sum_{i=1}^n m_i (p^i - 1)$  with  $m_i \in \mathbb{Z}_{\geq 0}$ .

We now need to compute the differentials in the spectral sequence. Notice that no differentials hit the  $s = 0$ -line. There are a few ways to see this: one way is to note that  $\mathrm{BP}\langle n \rangle$  splits off  $\widehat{\mathrm{THH}}^{T(n)}(\mathrm{BP}\langle n \rangle)$ ; another way is to note that any differential that hits the 0-line originates from a torsion group and lands in  $\pi_* \mathrm{BP}\langle n \rangle$ , and hence is automatically zero since the target is torsion-free.

We claim that there are no differentials between the torsion groups; this is essentially for degree reasons. Indeed, the  $d^r$ -differential changes bidegree by  $(-r, r-1)$ . Since all elements live in bidegrees of the form  $(s, t) = (2(k+1)p^{n+1} - 2k - 1, 2d)$  for  $k \geq 1$ , we find that any possibly nonzero differential  $d^r$  must have  $r = 2(k-\ell)(p^{n+1} - 1)$  for some  $k, \ell \geq 1$ . This differential would go from the  $(2(k+1)p^{n+1} - 2k - 1)$ -line to the  $(2(\ell+1)p^{n+1} - 2\ell - 1)$ -line. Since  $r$  must be even, the  $d^r$ -differential must increase the  $t$ -degree by an odd integer. However, it is clear from the above description of the  $E^2$ -page that all elements have even  $t$ -degree, and hence any differential necessarily vanishes. In particular, the Atiyah-Hirzebruch spectral sequence collapses at the  $E^2$ -page.  $\square$



## 7. $C_2$ -EQUIVARIANT ANALOGUE OF COROLLARY B

Our goal in this section is to study a  $C_2$ -equivariant analogue of Corollary B at height 1. The odd primary analogue of this result is deferred to the future; it is considerably more subtle.

**7.1.  $C_2$ -equivariant analogues of Ravenel's spectra.** In this section, we construct the  $C_2$ -equivariant analogue of  $T(n)$  for all  $n$ . We 2-localize everywhere until mentioned otherwise. There is a  $C_2$ -action on  $\Omega\mathrm{SU}(n)$  given by complex conjugation, and the resulting  $C_2$ -space is denoted  $\Omega\mathrm{SU}(n)_{\mathbf{R}}$ . Real Bott periodicity gives a  $C_2$ -equivariant map  $\Omega\mathrm{SU}(n)_{\mathbf{R}} \rightarrow \mathrm{BU}_{\mathbf{R}}$  whose Thom spectrum is the (genuine)  $C_2$ -spectrum  $X(n)_{\mathbf{R}}$ . This admits the structure of an  $E_{\rho}$ -ring. As in the nonequivariant case, the equivariant Quillen idempotent on  $\mathrm{MU}_{\mathbf{R}}$  restricts to one on  $X(m)_{\mathbf{R}}$ , and therefore defines a summand  $T(n)_{\mathbf{R}}$  of  $X(m)_{\mathbf{R}}$  for  $2^n \leq m \leq 2^{n+1} - 1$ . Again, this summand admits the structure of an  $E_1$ -ring.

**Construction 7.1.1.** There is an equivariant fiber sequence

$$\Omega\mathrm{SU}(n)_{\mathbf{R}} \rightarrow \Omega\mathrm{SU}(n+1)_{\mathbf{R}} \rightarrow \Omega S^{n\rho+1},$$

where  $\rho$  is the regular representation of  $C_2$ ; the equivariant analogue of Proposition 2.1.6 then shows that there is a map  $\Omega S^{n\rho+1} \rightarrow \mathrm{BGL}_1(X(n)_{\mathbf{R}})$  (detecting an element  $\chi_n \in \pi_{n\rho-1}X(n)_{\mathbf{R}}$ ) whose Thom spectrum is  $X(n+1)_{\mathbf{R}}$ .

If  $\tilde{\sigma}_n$  denotes the image of the element  $\chi_{2^{n+1}\rho-1}$  in  $\pi_{(2^{n+1}-1)\rho-1}T(n)_{\mathbf{R}}$ , then we have a  $C_2$ -equivariant analogue of Lemma 3.1.12:

**Lemma 7.1.2.** *The Thom spectrum of the map  $\Omega S^{(2^{n+1}-1)\rho+1} \rightarrow \mathrm{BGL}_1(X(2^{n+1}-1)_{\mathbf{R}}) \rightarrow \mathrm{BGL}_1(T(n)_{\mathbf{R}})$  detecting  $\tilde{\sigma}_n$  is  $T(n+1)_{\mathbf{R}}$ .*

**Example 7.1.3.** For instance,  $T(1)_{\mathbf{R}} = X(2)_{\mathbf{R}}$  is the Thom spectrum of the map  $\Omega S^{\rho+1} \rightarrow \mathrm{BU}_{\mathbf{R}}$ ; upon composing with the equivariant J-homomorphism  $\mathrm{BU}_{\mathbf{R}} \rightarrow \mathrm{BGL}_1(\mathbb{S})$ , this detects the element  $\tilde{\eta} \in \pi_{\sigma}\mathbb{S}$ , and the extension of the map  $S^{\rho} \rightarrow \mathrm{BGL}_1(\mathbb{S})$  to  $\Omega S^{\rho+1}$  uses the  $E_1$ -structure on  $\mathrm{BGL}_1(\mathbb{S})$ . The case of  $X(2)_{\mathbf{R}}$  exhibits a curious property:  $S^{\rho+1}$  is the loop space  $\Omega^{\sigma}\mathbf{HP}_{\mathbf{R}}^{\infty}$ , and there are equivalences (see [HW19])

$$\Omega S^{\rho+1} \simeq \Omega^{\sigma+1}\mathbf{HP}_{\mathbf{R}}^{\infty} \simeq \Omega^{\sigma}(\Omega\mathbf{HP}_{\mathbf{R}}^{\infty}).$$

However,  $\Omega\mathbf{HP}_{\mathbf{R}}^{\infty} \simeq S^{\rho+\sigma}$ , so  $\Omega S^{\rho+1} = \Omega^{\sigma}S^{\rho+\sigma}$ . The map  $\Omega^{\sigma}S^{\rho+\sigma} \rightarrow \mathrm{BGL}_1(\mathbb{S})$  still detects the element  $\tilde{\eta} \in \pi_{\sigma}\mathbb{S}$  on the bottom cell, but the extension of the map  $S^{\rho} \rightarrow \mathrm{BGL}_1(\mathbb{S})$  to  $\Omega^{\sigma}S^{\rho+\sigma}$  is now defined via the  $E_{\sigma}$ -structure on  $\mathrm{BGL}_1(\mathbb{S})$ . The upshot of this discussion is that  $X(2)_{\mathbf{R}}$  is not only the free  $E_1$ -ring with a nullhomotopy of  $\tilde{\eta}$ , but also the free  $E_{\sigma}$ -algebra with a nullhomotopy of  $\tilde{\eta}$ .

**Warning 7.1.4.** Unlike the nonequivariant setting, the element  $\tilde{\eta} \in \pi_{\sigma}\mathbb{S}$  is neither torsion nor nilpotent. This is because its geometric fixed points is  $\Phi^{C_2}\tilde{\eta} = 2 \in \pi_0\mathbb{S}$ .

**Example 7.1.5.** Consider the element  $\tilde{\sigma}_1 \in \pi_{3\rho-1}T(1)_{\mathbf{R}}$ . The underlying nonequivariant element of  $\pi_5T(1)_{\mathbf{R}}$  is simply  $\sigma_1$ . To determine  $\Phi^{C_2}\tilde{\sigma}_1 \in \pi_2\Phi^{C_2}T(1)_{\mathbf{R}}$ , we first note that  $\Phi^{C_2}T(1)_{\mathbf{R}}$  is the Thom spectrum of the map  $\Phi^{C_2}\tilde{\eta}: \Phi^{C_2}\Omega S^{\rho+1} \rightarrow \mathrm{BGL}_1(\mathbb{S})$ . Since  $\Phi^{C_2}\Omega S^{\rho+1} = \Omega S^2$  and  $\Phi^{C_2}\tilde{\eta} = 2$ , we find that  $\Phi^{C_2}T(1)_{\mathbf{R}}$  is the  $E_1$ -quotient  $\mathbb{S}/2$ . The element  $\Phi^{C_2}\tilde{\sigma}_1 \in \pi_2\mathbb{S}/2 \cong \pi_2\mathbb{S}/2$  is simply a map  $S^2 \rightarrow \mathbb{S}/2$  which is  $\eta$  on the top cell. Such a map exists because  $2\eta = 0$ .

As an aside, we mention that there is a  $C_2$ -equivariant lift of the spectrum  $A$ :

**Definition 7.1.6.** Let  $A_{C_2}$  denote the Thom spectrum of the Real bundle over  $\Omega S^{2\rho+1}$  defined by the extension of the map  $S^{2\rho} \rightarrow \mathrm{BSU}_{\mathbf{R}}$  detecting a generator of  $\pi_{2\rho}\mathrm{BSU}_{\mathbf{R}}$ .

**Remark 7.1.7.** The underlying spectrum of  $A_{C_2}$  is  $A$ . To determine the geometric fixed points of  $A_{C_2}$ , note that the equivariant J-homomorphism  $\mathrm{BU}_{\mathbf{R}} \rightarrow \mathrm{BGL}_1(\mathbb{S})$  sends the generator of  $\pi_{2\rho}\mathrm{BU}_{\mathbf{R}}$  to the equivariant Hopf map  $\tilde{v} \in \pi_{2\rho-1}\mathbb{S}$ . Therefore,  $\Phi^{C_2}A_{C_2}$  is the Thom spectrum of the map  $\Phi^{C_2}\tilde{v} : \Phi^{C_2}\Omega S^{2\rho+1} \rightarrow \mathrm{BGL}_1(\mathbb{S})$ . Since  $\tilde{\Phi}^{C_2}\tilde{v} = \eta$  and  $\Phi^{C_2}\Omega S^{2\rho+1} = \Omega S^3$ , we find that  $\Phi^{C_2}A_{C_2} = T(1)$ . In particular,  $A_{C_2}$  may be thought of as the free  $C_2$ -equivariant  $\mathbf{E}_1$ -ring with a nullhomotopy of  $\tilde{v}$ .

**Example 7.1.8.** The element  $\tilde{\sigma}_1$  lifts to  $\pi_{3\rho-1}A_{C_2}$ . Indeed, Remark 3.2.15 works equivariantly too: the equivariant Hopf map  $S^{3\rho-1} \rightarrow S^{2\rho}$  defines a composite  $S^{3\rho-1} \rightarrow S^{2\rho} \rightarrow \Omega S^{2\rho+1}$ . The composite  $S^{3\rho-1} \rightarrow \Omega S^{2\rho+1} \rightarrow \mathrm{BSU}_{\mathbf{R}}$  is null, since  $\pi_{3\rho-1}\mathrm{BSU}_{\mathbf{R}} = 0$ . It follows that upon Thomification, the map  $S^{3\rho-1} \rightarrow \Omega S^{2\rho+1}$  defines an element  $\tilde{\sigma}'_1$  of  $\pi_{3\rho-1}A_{C_2}$ . In order to show that this element indeed deserves to be called  $\tilde{\sigma}_1$ , we use Proposition 7.1.9. The map  $A_{C_2} \rightarrow T(1)_{\mathbf{R}}$  from the proposition induces a map  $\pi_{3\rho-1}A_{C_2} \rightarrow \pi_{3\rho-1}T(1)_{\mathbf{R}}$ , and we need to show that the image of  $\tilde{\sigma}'_1 \in \pi_{3\rho-1}A_{C_2}$  is in fact  $\tilde{\sigma}_1$ . By Example 7.1.5, it suffices to observe that the underlying nonequivariant map corresponding to  $\tilde{\sigma}'_1 \in \pi_{3\rho-1}T(1)_{\mathbf{R}}$  is  $\sigma_1$ , and that the geometric fixed points  $\Phi_{C_2}\tilde{\sigma}'_1 \in \pi_2\mathbb{S}/2$  is the lift of  $\eta$  appearing in Example 7.1.5.

We now prove the proposition used above.

**Proposition 7.1.9.** *There is a genuine  $C_2$ -equivariant  $\mathbf{E}_1$ -map  $A_{C_2} \rightarrow T(1)_{\mathbf{R}}$ .*

*Proof.* By Example 7.1.7, it suffices to show that  $\tilde{v} = 0 \in \pi_{3\rho-1}T(1)_{\mathbf{R}}$ . The underlying map is null, because  $v = 0 \in \pi_3T(1)$ . The geometric fixed points are also null, because  $\Phi^{C_2}\tilde{v} = \eta$  is null in  $\pi_2\Phi^{C_2}T(1)_{\mathbf{R}} = \pi_2\mathbb{S}/2$ . Therefore,  $\tilde{v}$  is null in  $\pi_{3\rho-1}T(1)_{\mathbf{R}}$ .  $\square$

In fact, it is easy to prove the following analogue of Proposition 6.2.1:

**Proposition 7.1.10.** *There is a  $C_2$ -equivariant equivalence  $A_{C_2} \wedge C\tilde{\eta} \simeq T(1)_{\mathbf{R}}$ .*

*Proof.* There are maps  $A_{C_2} \rightarrow T(1)_{\mathbf{R}}$  and  $C\tilde{\eta} \rightarrow T(1)_{\mathbf{R}}$ , which define a map  $A_{C_2} \wedge C\tilde{\eta} \rightarrow T(1)_{\mathbf{R}}$  via the multiplication on  $T(1)_{\mathbf{R}}$ . This map is an equivalence on underlying by Proposition 6.2.1, and on geometric fixed points induces the map  $T(1) \wedge \mathbb{S}/2 \rightarrow \mathbb{S}/2$ . This was also proved in the course of Proposition 6.2.1.  $\square$

**Remark 7.1.11.** As in Remark 6.2.2, one might hope that this implies the  $C_2$ -equivariant Wood equivalence  $\mathrm{bo}_{C_2} \wedge C\tilde{\eta} \simeq \mathrm{bu}_{\mathbf{R}}$  via some equivariant analogue of Theorem A.

**Remark 7.1.12.** The equivariant analogue of Remark 6.2.4 remains true: the equivariant Wood equivalence of Proposition 7.1.10 stems from the EHP sequence  $S^\rho \rightarrow \Omega S^{\rho+1} \rightarrow \Omega S^{2\rho+1}$ .

**7.2. The  $C_2$ -equivariant analogue of Corollary B at  $n = 1$ .** Recall (see [HK01]) that there are indecomposable classes  $\bar{v}_n \in \pi_{(2^n-1)\rho}\mathrm{BP}_{\mathbf{R}}$ ; as in Theorem 3.1.5, these lift to classes in  $\pi_*T(m)_{\mathbf{R}}$  if  $m \geq n$ . The main result of this section is the following:

**Theorem 7.2.1.** *There is a map  $\Omega^\rho S^{2\rho+1} \rightarrow \mathrm{BGL}_1(T(1)_{\mathbf{R}})$  detecting an indecomposable in  $\pi_\rho T(1)_{\mathbf{R}}$  on the bottom cell, whose Thom spectrum is  $\underline{\mathrm{HZ}}$ .*

Note that, as with Corollary B at  $n = 1$ , this result is *unconditional*. The argument is exactly as in the proof of Corollary B at  $n = 1$ , with practically no modifications. We need the following analogue of Theorem 2.2.1, originally proved in [BW18, HW19].

**Proposition 7.2.2** (Behrens-Wilson, Hahn-Wilson). *Let  $p$  be any prime, and let  $\lambda$  denote the 2-dimensional standard representation of  $C_p$  on  $\mathbf{C}$ . The Thom spectrum of the map  $\Omega^\lambda S^{\lambda+1} \rightarrow \mathrm{BGL}_1(S^0)$  extending the map  $1 - p : S^1 \rightarrow \mathrm{BGL}_1(S^0)$  is equivalent to  $\underline{\mathrm{HF}}_p$  as an  $\mathbf{E}_\lambda$ -ring. Moreover, if  $S^{\lambda+1}\langle\lambda+1\rangle$  denotes the*

$(\lambda + 1)$ -connected cover of  $S^{\lambda+1}$ , then the Thom spectrum of the induced map  $\Omega^\lambda S^{\lambda+1} \langle \lambda + 1 \rangle \rightarrow \mathrm{BGL}_1(S^0)$  is equivalent to  $\underline{\mathrm{HZ}}$  as an  $\mathrm{E}_\lambda$ -ring.

We can now prove Theorem 7.2.1.

*Proof of Theorem 7.2.1.* In [HW19], the authors prove that there is an equivalence of  $C_2$ -spaces between  $\Omega^\lambda S^{\lambda+1}$  and  $\Omega^\rho S^{\rho+1}$ , and that  $\underline{\mathrm{HF}}_2$  is in fact the Thom spectrum of the induced map  $\Omega^\rho S^{\rho+1} \rightarrow \mathrm{BGL}_1(S^0)$  detecting  $-1$ . Since both  $\Omega^\rho S^{\rho+1} \langle \rho + 1 \rangle$  and  $\Omega^\lambda S^{\lambda+1} \langle \lambda + 1 \rangle$  are defined as fibers of maps to  $S^1$  which are degree one on the bottom cell, Hahn and Wilson's equivalence lifts to a  $C_2$ -equivariant equivalence  $\Omega^\rho S^{\rho+1} \langle \rho + 1 \rangle \simeq \Omega^\lambda S^{\lambda+1} \langle \lambda + 1 \rangle$ , and we find that  $\underline{\mathrm{HZ}}$  is the Thom spectrum of the map  $\Omega^\rho S^{\rho+1} \langle \rho + 1 \rangle \rightarrow \mathrm{BGL}_1(S^0)$ .

Since  $T(1)_\mathbf{R}$  is the Thom spectrum of the composite map  $\Omega S^{\rho+1} \rightarrow \Omega^\rho S^{\rho+1} \langle \rho + 1 \rangle \rightarrow \mathrm{BGL}_1(S^0)$  detecting  $\tilde{\eta}$  on the bottom cell of the source, it follows from the  $C_2$ -equivariant analogue of Proposition 2.1.6 and the above discussion that it is sufficient to define a fiber sequence

$$\Omega S^{\rho+1} \rightarrow \Omega^\rho S^{\rho+1} \langle \rho + 1 \rangle \rightarrow \Omega S^{2\rho+1},$$

and check that the induced map  $\Omega^\rho S^{2\rho+1} \rightarrow \mathrm{BGL}_1(T(1)_\mathbf{R})$  detects an indecomposable element of  $\pi_\rho T(1)_\mathbf{R}$ . See Remark 4.1.4 for the nonequivariant analogue of this fiber sequence.

Since there is an equivalence  $\Omega S^{\rho+1} \simeq \Omega^\sigma S^{\rho+\sigma}$ , it suffices to prove that there is a fiber sequence

$$(7.1) \quad S^{\rho+\sigma} \rightarrow \Omega S^{\rho+1} \langle \rho + 1 \rangle \rightarrow \Omega S^{2\rho+1};$$

taking  $\sigma$ -loops produces the desired fiber sequence. The fiber sequence (7.1) can be obtained by taking vertical fibers in the following map of fiber sequences

$$\begin{array}{ccccc} S^\rho & \longrightarrow & \Omega S^{\rho+1} & \longrightarrow & \Omega S^{2\rho+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{CP}_\mathbf{R}^\infty & \xlongequal{\quad} & \mathrm{CP}_\mathbf{R}^\infty & \longrightarrow & *. \end{array}$$

Here, the top horizontal fiber sequence is the EHP fiber sequence

$$S^\rho \rightarrow \Omega S^{\rho+1} \rightarrow \Omega S^{2\rho+1}.$$

To identify the fibers, note that there is the Hopf fiber sequence

$$S^{\rho+\sigma} \xrightarrow{\tilde{\eta}} S^\rho \rightarrow \mathrm{CP}_\mathbf{R}^\infty.$$

The fiber of the middle vertical map is  $\Omega S^{\rho+1} \langle \rho + 1 \rangle$  via the definition of  $S^{\rho+1} \langle \rho + 1 \rangle$  as the homotopy fiber of the map  $S^{\rho+1} \rightarrow \mathrm{BCP}_\mathbf{R}^\infty$ .

It remains to show that the map  $\Omega^\rho S^{2\rho+1} \rightarrow \mathrm{BGL}_1(T(1)_\mathbf{R})$  detects an indecomposable element of  $\pi_\rho T(1)_\mathbf{R}$ . Indecomposability in  $\pi_\rho T(1)_\mathbf{R} \cong \pi_\rho \mathrm{BP}_\mathbf{R}$  is the same as not being divisible by 2, so we just need to show that the dotted map in the following diagram does not exist:

$$\begin{array}{ccc} S^{\rho+1} & & \\ \downarrow E^2 & \searrow 2 & \\ \Omega^\rho S^{2\rho+1} & \longrightarrow & S^{\rho+1} \\ \downarrow & \swarrow \text{dotted} & \\ \mathrm{BGL}_1(T(1)_\mathbf{R}) & & \end{array}$$

If this factorization existed, there would be an orientation  $\underline{\mathrm{HZ}} \rightarrow T(1)_\mathbf{R}$ , which is absurd.  $\square$

We now explain why we do not know how to prove the equivariant analogue of Corollary B at higher heights. One could propose an equivariant analogue of Conjecture D, and such a conjecture would obviously be closely tied with the existence of some equivariant analogue of the work of Cohen-Moore-Neisendorfer. We do not know if any such result exists, but it would certainly be extremely interesting.

Suppose that one wanted to prove a result like Corollary B, stating that the equivariant analogues of Conjecture D and Conjecture E imply that there is a map  $\Omega^\rho S^{2^n \rho + 1} \rightarrow \mathrm{BGL}_1(T(n)_\mathbb{R})$  detecting an indecomposable in  $\pi_{(2^n - 1)\rho} T(n)_\mathbb{R}$  on the bottom cell, whose Thom spectrum is  $\mathrm{BP}\langle n - 1 \rangle_\mathbb{R}$ . One could then try to run the same proof as in the nonequivariant case by constructing a map from the fiber of a charming map  $\Omega^\rho S^{2^n \rho + 1} \rightarrow S^{(2^n - 1)\rho + 1}$  to  $\mathrm{BGL}_1(T(n - 1)_\mathbb{R})$ , but the issue comes in replicating Step 1 of Section 5.2: there is *no* analogue of Lemma 3.1.16, since the equivariant element  $\tilde{\sigma}_n \in \pi_* T(n)$  is neither torsion nor nilpotent. See Warning 7.1.4. This is intimately tied with the failure of an analogue of the nilpotence theorem in the equivariant setting. In future work, we shall describe a related project connecting the  $T(n)$  spectra to the Andrews-Gheorghe-Miller  $w_n$ -periodicity in C-motivic homotopy theory.

However, since there is a map  $\Omega^\lambda S^{\lambda + 1} \langle \lambda + 1 \rangle \rightarrow \mathrm{BGL}_1(\mathbb{S})$  as in Proposition 7.2.2, there may nevertheless be a way to construct a suitable map from the fiber of a charming map  $\Omega^\rho S^{2^n \rho + 1} \rightarrow S^{(2^n - 1)\rho + 1}$  to  $\mathrm{BGL}_1(T(n - 1)_\mathbb{R})$ . Such a construction would presumably provide a more elegant construction of the nonequivariant map used in the proof of Theorem A.

## 8. FUTURE DIRECTIONS

In this section, we suggest some directions for future investigation. This is certainly not an exhaustive list; there are numerous questions we do not know how to address that are spattered all over this document, but we have tried to condense some of them into the list below. We have tried to order the questions in order of our interest in them. We have partial progress on many of these questions.

- (a) Some obvious avenues for future work are the conjectures studied in this paper: Conjectures D, E, F, and 3.1.7. Can the  $E_3$ -assumption in the statement of Theorem C be removed? Prove the existence of a map of fiber sequences as in Remark 6.2.15.
- (b) One of the Main Goals<sup>TM</sup> of this project is to rephrase the proof of the nilpotence theorem from [DHS88, HS98]. As mentioned in Remark 2.2.3, the Hopkins-Mahowald theorem for  $\mathrm{HF}_p$  immediately implies the nilpotence theorem for simple  $p$ -torsion classes in the homotopy of a homotopy commutative ring spectrum (see also [Hop84]). We will expand on the relation between the results of this paper and the nilpotence theorem in forthcoming work; see Remark 5.4.7 for a sketch.

From this point of view, Theorem A is very interesting: it connects torsion in the unstable homotopy groups of spheres (via Cohen-Moore-Neisendorfer) to nilpotence in the stable homotopy groups of spheres. We are not sure how to do so, but could the Cohen-Moore-Neisendorfer bound for the exponents of unstable homotopy groups of spheres be used to obtain bounds for the nilpotence exponent of the stable homotopy groups of spheres?

- (c) It is extremely interesting to contemplate the interaction between unstable homotopy theory and chromatic homotopy theory apparent in this paper. Connections between unstable homotopy theory and the chromatic picture have appeared elsewhere in the literature (e.g., in [AM99, Aro98, Mah82, MT94]), but their relationship to the content of this project is not clear to me. It would be interesting to have this clarified. One naïve hope is that such connection could stem from a construction of a charming map (such as the Cohen-Moore-Neisendorfer map) via Weiss calculus.
- (d) Let  $R$  denote  $\mathbb{S}$  or  $A$ . The map  $R \rightarrow \Theta(R)$  is an equivalence in dimensions  $< |\sigma_n|$ . Moreover, the  $\Theta(R)$ -based Adams-Novikov spectral sequence has a vanishing line of slope  $1/|\sigma_n|$  (see [Mah81a])

for the case  $R = A$ ). Can another proof of this vanishing line be given using general arguments involving Thom spectra? We have some results in this direction which we shall address in future work.

- (e) The unit maps from each of the Thom spectra on the second line of Table 1 to the corresponding designer spectrum on the third line are surjective on homotopy. In the case of  $\mathrm{tmf}$ , this requires some computational effort to prove, and has been completed in [Dev19b]. This behavior is rather unexpected: in general, the unit map from a structured ring to some structured quotient will not be surjective on homotopy. Is there a conceptual reason for this surjectivity?
- (f) In [BBB<sup>+</sup>19], the  $\mathrm{tmf}$ -resolution of a certain type 2 spectrum  $Z$  is studied. Mahowald uses the Thom spectrum  $A$  to study the  $\mathrm{bo}$ -resolution of the sphere in [Mah81a], so perhaps the spectrum  $B$  could be used to study the  $\mathrm{tmf}$ -resolution of  $Z$ . This is work in progress. See also Corollary 6.2.10 and the discussion preceding it.
- (g) Is there an equivariant analogue of Theorem A at higher heights and other primes? Currently, we have such an analogue at height 1 and at  $p = 2$ ; see Section 7.
- (h) The Hopkins-Mahowald theorem may be used to define Brown-Gitler spectra. Theorem A produces “relative” Brown-Gitler spectra for  $\mathrm{BP}\langle n \rangle$ ,  $\mathrm{bo}$ , and  $\mathrm{tmf}$ . In future work, we will study these spectra and show how they relate to the Davis-Mahowald *non*-splitting of  $\mathrm{tmf} \wedge \mathrm{tmf}$  as a wedge of shifts of  $\mathrm{bo}$ -Brown-Gitler spectra smashed with  $\mathrm{tmf}$  from [DM10].
- (i) The story outlined in the introduction above could fit into a general framework of “fp-Mahowald spectra” (for “finitely presented Mahowald spectrum”, inspired by [MR99]), of which  $A$ ,  $B$ ,  $T(n)$ , and  $\gamma(n)$  would be examples. One might then hope for a generalization of Theorem A which relates fp-Mahowald spectra to fp-spectra. It would also be interesting to prove an analogue of Mahowald-Rezk duality for fp-Mahowald spectra which recovers their duality for fp-spectra upon taking Thom spectra as above.
- (j) One potential approach to the question about surjectivity raised above is as follows. The surjectivity claim at height 0 is the (trivial) statement that the unit map  $\mathbb{S} \rightarrow \mathrm{HZ}$  is surjective on homotopy. The Kahn-Priddy theorem, stating that the transfer  $\lambda : \Sigma^\infty \mathbf{R}P^\infty \rightarrow \mathbb{S}$  is surjective on  $\pi_{* \geq 1}$ , can be interpreted as stating that  $\pi_* \Sigma^\infty \mathbf{R}P^\infty$  contains those elements of  $\pi_* \mathbb{S}$  which are *not* detected by  $\mathrm{HZ}$ . One is then led to wonder: for each of the Thom spectra  $R$  on the second line of Table 1, is there a spectrum  $P$  along with a map  $\lambda_R : P \rightarrow R$  such that each  $x \in \pi_* R$  in the kernel of the map  $R \rightarrow \Theta(R)$  lifts along  $\lambda_R$  to  $\pi_* P$ ? (The map  $R \rightarrow \Theta(R)$  is an equivalence in dimensions  $< |\sigma_n|$  (if  $R$  is of height  $n$ ), so  $P$  would have bottom cell in dimension  $|\sigma_n|$ .)  
Since  $\Sigma^\infty \mathbf{R}P^\infty \simeq \Sigma^{-1} \mathrm{Sym}^2(\mathbb{S})/\mathbb{S}$ , the existence of such a result is very closely tied to an analogue of the Whitehead conjecture (see [Kuh82]; the Whitehead conjecture implies the Kahn-Priddy theorem). In particular, one might expect the answer to the question posed above to admit some interaction with Goodwillie calculus.
- (k) Let  $p \geq 5$ . Is there a  $p$ -primary analogue of  $B$  which would provide a Thom spectrum construction (via Table 1) of the conjectural spectrum  $\mathrm{eo}_{p-1}$ ? Such a spectrum would be the Thom spectrum of a  $p$ -complete spherical fibration over a  $p$ -local space built via  $p-1$  fiber sequences from the loop spaces  $\Omega S^{2k(p-1)+1}$  for  $2 \leq k \leq p$ .
- (l) The spectra  $T(n)$  and  $\gamma(n)$  have algebro-geometric interpretations: the stack  $\mathcal{M}_{T(n)}$  associated (see [DFHH14, Chapter 9]; this stack is well-defined since  $T(n)$  is homotopy commutative) to  $T(n)$  classifies  $p$ -typical formal groups with a coordinate up to degree  $p^{n+1}-1$ , while  $\gamma(n)$  is the closed substack of  $\mathcal{M}_{T(n)}$  defined by the vanishing locus of  $p, v_1, \dots, v_{n-1}$ . What are the moduli problems classified by  $A$  and  $B$ ? We do not know if this question even makes sense at  $p = 2$ , since  $A$  and  $B$  are *a priori* only  $E_1$ -rings. Nonetheless, in [Dev19a], we provide a description of  $\mathrm{tmf} \wedge A$  in terms of the Hodge filtration of the universal elliptic curve (even at  $p = 2$ ).

- (m) Theorem A shows that the Hopkins-Mahowald theorem for  $H\mathbb{Z}_p$  can be generalized to describe forms of  $BP\langle n \rangle$ ; at least for small  $n$ , these spectra have associated algebro-geometric interpretations (see [DFHH14, Chapter 9]). What is the algebro-geometric interpretation of Theorem A?

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Email address: [sanathd@mit.edu](mailto:sanathd@mit.edu)