Homotopy-coherent interchange and equivariant little disk operads

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INI Equivariant Homotopy Theory in Context
Queen's University Belfast, April 9 2025

Interchange and little *V*-disks

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Historical context

G-∞-categorical operads
G-(pre)operads, algebras
BV-tensor products
The Dunn map
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A question you might happen upon

In trace methods for Real *K*-theory, THH has a Real analogue:[1]

$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) & & \mathrm{Alg}_{\mathbb{E}_\sigma}(\mathrm{Sp}_{C_2}) \\
\downarrow^{\mathrm{THH}} & \xleftrightarrow{\text{"Reality"}} & \downarrow^{\mathrm{THR}} \\
\mathrm{Sp} & & \mathrm{Sp}_{C_2}
\end{array}
$$

In this, σ is the sign representation and $\mathbb{E}_\sigma$-algebras are a genuine-equivariant version of rings with anti-involution.

Question (c.f. Dotto-Moi-Patchkoria-Reeh[2] '17)

What algebraic structure does THR of highly structured C_2-ring spectra have?

---

[1] Emanuele Dotto. *Stable real K-theory and real topological Hochschild homology*. Thesis (Ph.D.)–Uniiversity of Copenhagen. 2012. arXiv: 1212.4310 [math.AT].

[2] Emanuele Dotto, Kristian Moi, Irakli Patchkoria, and Sune Precht Reeh. "Real topological Hochschild homology". In: *J. Eur. Math. Soc. (JEMS)* 23.1 (2021), pp. 63–152.

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How to construct structure on THH

Observation

THH can be given a symmetric monoidal structure, so we may lift

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathrm{Sp}) \quad \simeq \quad \mathrm{Alg}_{\mathcal{O}}\mathrm{Alg}_{\mathbb{E}_1}^{\otimes}(\mathrm{Sp}) \dashrightarrow \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$$

$$U \downarrow \qquad\qquad\qquad U \downarrow$$

$$\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) \xrightarrow{\quad\mathrm{THH}\quad} \mathrm{Sp}$$

Theorem (Dunn[3] '88, Lurie[4] '09)

$\mathbb{E}_n \simeq \mathbb{E}_{n-1} \otimes \mathbb{E}_1$; hence THH takes $\mathbb{E}_n$-rings to $\mathbb{E}_{n-1}$-rings.

---

[3] Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258.

[4] Jacob Lurie. *Derived Algebraic Geometry VI: Ek Algebras*. 2009. arXiv: 0911.0018 [math.AT].

How to construct structure on TH**R**

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Observation (S.[5])

TH**R** can be given a C_2-symmetric monoidal structure, so we may lift

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_\sigma}(\mathrm{Sp}_{C_2}) \quad\simeq\quad \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_\sigma}(\mathrm{Sp}_{C_2}) \dashrightarrow \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}_{C_2})$$

$$\mathrm{Alg}_{\mathbb{E}_\sigma}(\mathrm{Sp}_{C_2}) \xrightarrow{\quad\mathrm{TH}\mathbf{R}\quad} \mathrm{Sp}_{C_2}$$

with vertical map U.

Conjecture

In the 1-categorical world, there is an equivalence $\mathbb{E}_V \otimes \mathbb{E}_W \simeq \mathbb{E}_{V\oplus W}$.

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[5] Natalie Stewart. *On tensor products of equivariant commutative operads (draft)*. 2025.

Statement of the additivity theorem

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For this talk, all terms are defined ∞-categorically.

Theorem (S.[6])

Given V, W orthogonal G-representations, we have

$$\mathbb{E}_V \otimes \mathbb{E}_W \simeq \mathbb{E}_{V \oplus W};$$

hence there is an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_V} \underline{\mathrm{Alg}}_{\mathbb{E}_W}^{\otimes} \left(\mathrm{Sp}_G\right) \simeq \mathrm{Alg}_{\mathbb{E}_{V \oplus W}} \left(\mathrm{Sp}_G\right).$$

Corollary

THR of $\mathbb{E}_{V \oplus \sigma}$-rings has a natural $\mathbb{E}_V$-ring structure, and no more.

---

[6] Natalie Stewart. *On homotopical additivity of equivariant little disks operads (forthcoming)*. 2025.

Myriad versions of this theorem

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1. *Dunn '88:*[7] $C_1^{\otimes n} \simeq C_n$ w.r.t. a **point-set** tensor product.

2. *Brinkmeier '00:*[8] $C_n \otimes C_m \simeq C_{n+m}$ w.r.t. a **point-set** tensor product.

3. *Lurie '09:*[9] $\mathbb{E}_n^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{E}_m^\otimes \simeq \mathbb{E}_{n+m}^\otimes$ with respect to a **homotopical** tensor product. Also, tangential structures.

4. *Fiedorowicz-Vogt '15:*[10] Dunn-Brinkmeier's result extends to **arbitrary cofibrant** $\mathbb{E}_n$**-operads**.

5. *Szczesny '24:*[11] $D_V \otimes D_W \simeq D_{V \oplus W}$ w.r.t. a **point-set** tensor product.

6. *S. '25:*[12] $\mathbb{E}_V^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^\otimes \simeq \mathbb{E}_{V \oplus W}^\otimes$ with respect to a **homotopical** tensor product. Also, stalkwise-linearizable tangential structures.

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[7] Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258.

[8] Michael Brinkmeier. *On Operads*. Thesis (Ph.D.)–Universität Osnabrück. 2000.

[9] Jacob Lurie. *Derived Algebraic Geometry VI: Ek Algebras*. 2009. arXiv: 0911.0018 [math.AT].

[10] Z. Fiedorowicz and R. M. Vogt. "An additivity theorem for the interchange of E_n structures". In: *Adv. Math.* 273 (2015), pp. 421–484.

[11] Ben Szczesny. *Equivariant Framed Little Disk Operads are Additive*. 2024. arXiv: 2410.20235 [math.AT].

[12] Natalie Stewart. *On homotopical additivity of equivariant little disks operads (forthcoming)*. 2025.

A heavily abridged history of G-∞-categorical operads

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0 *Hill-Hopkins '16:*[13] G-commutative monoids are **semi-Mackey functors**.

1 *Nardin-Shah '22:*[14] G-operads are a type of fibration over a genuine equivariant version $\underline{\mathbb{F}}_{G,*}$ of Segal's category Γ^{op}.

2 *Barkan-Haugseng-Steinebrunner '22:*[15] G-operads are also **fibrations over the effective Burnside 2-category** $\mathrm{Span}(\mathbb{F}_G)$.

3 *S. '25:*[16] homotopy-coherent interchange is corepresented by **BV tensor products** and G-operads are monadic over G-symmetric sequences.

4 *S. '25:*[17] Algebras in **(co)cartesian G-symmetric monoidal structures** have concrete descriptions and $\mathcal{N}_{I\infty} \otimes \mathcal{N}_{J\infty} \simeq \mathcal{N}_{I\vee J\infty}$

---

[13] Michael A. Hill and Michael J. Hopkins. *Equivariant symmetric monoidal structures*. 2016. arXiv: 1610.03114 [math.AT].

[14] Denis Nardin and Jay Shah. *Parameterized and equivariant higher algebra*. 2022. arXiv: 2203.00072 [math.AT].

[15] Shaul Barkan, Rune Haugseng, and Jan Steinebrunner. *Envelopes for Algebraic Patterns*. 2022. arXiv: 2208.07183 [math.CT].

[16] Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT].

[17] Natalie Stewart. *On tensor products of equivariant commutative operads (draft)*. 2025.

A quasi-definition of G-operads

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Definition

- A *G-preoperad* is a functor $\mathcal{O}^{\otimes} \to \mathrm{Span}(\mathbb{F}_G)$ with cocartesian lifts over backwards maps.
- A *G-operad* is required to satisfy "Segal conditions."
- An $\mathcal{O}$-**algebra in** $\mathcal{C}^{\otimes}$ is a functor preserving cocartesian arrows :

$$
\begin{array}{ccc}
\mathcal{O}^{\otimes} & \xrightarrow{\quad \varphi \quad} & \mathcal{C}^{\otimes} \\
& \searrow \quad \swarrow & \\
& \mathrm{Span}(\mathbb{F}_G) &
\end{array}
$$

$$
\mathrm{Op}_G \underset{\xrightarrow{\quad}}{\overset{L_{\mathrm{Op}_G}}{\underset{\perp}{\xleftarrow{\quad}}}} \mathrm{POp}_G \;=\!=\!=\; \mathrm{Cat}_{\infty,/\,\mathrm{Span}(\mathbb{F}_G)}^{\mathrm{Backwards-cocart}}
$$

The underlying G-symmetric sequence

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Construction

Given $\mathcal{O} \in \mathrm{Op}_G$, $H \subset G$, and $S \in \mathbb{F}_G$, define the **structure space**

$$
\begin{array}{ccc}
\mathcal{O}(S) & \longrightarrow & \mathrm{Mor}\,(\mathcal{O}^{\otimes}) \\
\downarrow & & \downarrow \\
\left\{\mathrm{Ind}_H^G S \to [G/H]\right\} & \longrightarrow & \mathrm{Mor}\,(\mathrm{Span}(\mathbb{F}_G))
\end{array}
$$

Proposition (S.[18])

If $\mathcal{O}^{\otimes}$ has "one color" then is conservatively identified by $(\mathcal{O}(S))_{\substack{H \subset G \\ S \in \mathbb{F}_H}}$.

---

[18] Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT].

Ur-examples

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Example

The **little *V*-disks *G*-operad** has *S*-ary structure space given by *H*-**equivariant configurations of *S* in *V*:**

$$\mathbb{E}_V(S) := \mathrm{Emb}^{H,\mathrm{Affine}}\left(S \cdot D(V), D(V)\right) \simeq \mathrm{Conf}_S^H(V).$$

Example

Given a *G*-symmetric monoidal category $\mathcal{C}^{\otimes} : \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$, its unstraightening is a *G*-operad. Given $X \in \mathcal{C}(G/G)$, there is an **endomorphism *G*-operad** with

$$\mathrm{End}_X(S) \simeq \mathrm{Map}_{\mathcal{C}_H}\left(\left(\mathrm{Res}_H^G X\right)^{\otimes S}, \mathrm{Res}_H^G X\right)$$

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In particular, an $\mathbb{E}_V$-algebra in $\mathcal{C}$ consists of an object $X \in \mathcal{C}_G$ and homotopy-coherently compatible maps

$$\mathrm{Conf}_S^H(V) \to \mathrm{Map}_{\mathcal{C}_H}\left(\left(\mathrm{Res}_H^G X\right)^{\otimes S}, \mathrm{Res}_H^G X \right).$$

Example (Horev-Klang-Zou[19] '20)

Let $\underline{\mathcal{S}}_G^{G-\times}$ be the *cartesian structure* on *G*-spaces. Then, for all $X \in \mathcal{S}_G$, we have $\Omega^V X \in \mathrm{Alg}_{\mathbb{E}_V}(\mathcal{S}_G)$.

Example (Horev-Klang-Zou '20, loc. cit.)

Let $\underline{\mathrm{Sp}}_G^{\otimes}$ be the HHR *G*-symmetric monoidal structure. If $f \colon \Omega^V X \to \underline{\mathrm{Pic}}(\underline{\mathrm{Sp}}_G)$ is a *V*-**loop map**, then $\mathrm{Th}(f) \in \mathrm{Alg}_{\mathbb{E}_V}(\mathrm{Sp}_G)$.

---

[19] Jeremy Hahn, Asaf Horev, Inbar Klang, Dylan Wilson, and Foling Zou. *Equivariant nonabelian Poincaré duality and equivariant factorization homology of Thom spectra*. 2024. arXiv:
2006.13348 [math.AT].

Modelling *G*-operadic constructions, two ways

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Model categorist: **(co)fibrantly replace**, then apply a construction to monoids in *G*-symmetric sequences.

∞-*categorist*: apply a *G*-preoperadic construction, then L_{Op_G}-**localize**.

Shared goal: model corepresenting object for **pairings** (aka interchanging algebras, bifunctors, etc.) akin to May.[20]

$$
\begin{array}{ccc}
\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} & \xrightarrow{\text{"pairing"}} & \mathcal{Q}^{\otimes} \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
\mathrm{Span}(\mathbb{F}_G) \times \mathrm{Span}(\mathbb{F}_G) & \xrightarrow{\wedge} & \mathrm{Span}(\mathbb{F}_G)
\end{array}
$$

Today, we are ∞-categorists.

[20] J. P. May. "Pairings of categories and spectra". In: *J. Pure Appl. Algebra* 19 (1980), pp. 299–346.

The Boardman-Vogt tensor product

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Definition

$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} := L_{\mathrm{Op}_G}\left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \longrightarrow \mathrm{Span}(\mathbb{F}_G)^2 \overset{\wedge}{\longrightarrow} \mathrm{Span}(\mathbb{F}_G)\right)$$

$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ lifts to a "pointwise" *G*-symmetric monoidal category $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$.

Proposition (S.[21])

$(-) \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes}$ is left adjoint to $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(-)$, so

$$\mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\mathcal{C}).$$

---

[21] Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT].

The Dunn map

$$\mathbb{E}_V^{\otimes} \wr \mathbb{E}_W^{\otimes} \xrightarrow{\varphi_{\mathbb{E}}} \mathbb{E}_{V \oplus W}^{\otimes}$$

$\mathbb{E}_W(*_H)$

$\mathbb{E}_W(*_G \sqcup [G/K])$

$\mathbb{E}_W(2 \cdot *_H)$

$\mathbb{E}_{V \oplus W}\left((*_G \sqcup [G/K]) \sqcup \mathrm{Ind}_H^G *_H \sqcup \mathrm{Ind}_H^G *_H\right)$

$V \oplus W$

W $\mathrm{Res}_H^G W$

$\mathbb{E}_V(*_G \sqcup [G/H] \sqcup [G/H])$ V

$$\mathrm{Ar}^{\mathrm{act}}\left(\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}\right) \times_{\mathrm{Ar}(\mathbb{F}_G)} \left\{\mathrm{Ind}_H^G S \to [G/H]\right\}$$

$$\| \wr$$

$$\coprod_{T \to S} \mathcal{O}(S) \times \prod_{[H/K] \subset S} \mathcal{P}\left(T \times_S [H/K]\right)$$

Preoperad models for 1-colored G-operads

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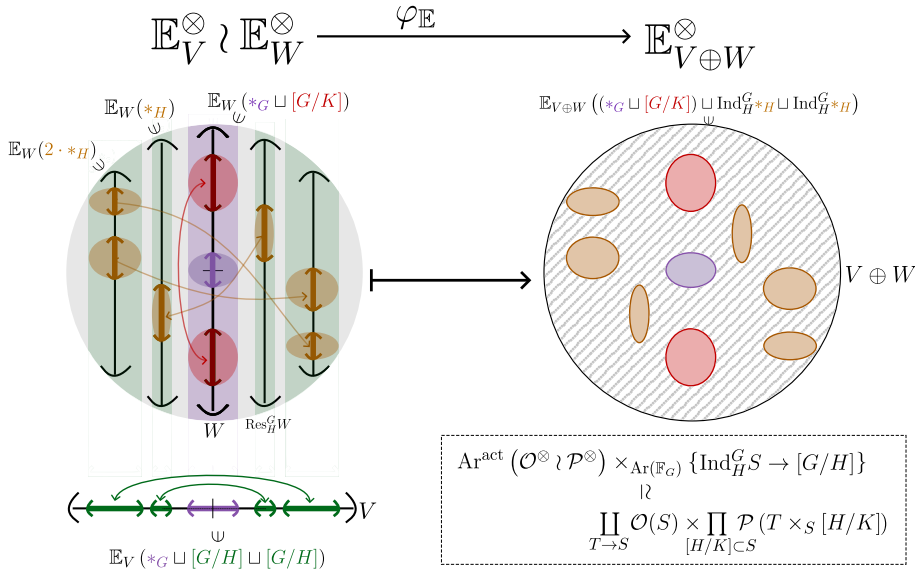BV-tensor products
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Given $\mathcal{P}^{\otimes} \in \mathrm{POp}_G$, We let $\mathcal{P}^{\mathrm{act}}_{/P}$ be the ∞-category of arrows $P' \to P$ projecting to a forward map $T = T \to S$.

Definition

A map of G-preoperads $\alpha \colon \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ with $\mathcal{O}^{\otimes}$ a "one color" G-operad is a **weak approximation** if

1 The G-∞-category of colors $U\mathcal{P}$ **has a terminal G-object**, and

2 For all $P \in \mathcal{P}^{\otimes}$ and $T \to \pi(P)$, **the map of spaces**

$$B\left(\mathcal{P}^{\mathrm{act}}_{/P} \times_{\mathbb{F}_{G,/\pi(P)}} \{T \to \pi(P)\}\right) \to \prod_{[H/K] \subset \pi(P)} \mathcal{O}\left(T \times_{\pi(P)} [H/K]\right)$$

is a weak equivalence.

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Proposition (Harpaz[22] '19 + reinterpretation)

If $\alpha \colon \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a weak approximation, pullback is fully faithful

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{S}_G) \subset \mathrm{Alg}_{\mathcal{P}}(\mathcal{S}_G)$$

with image the $\mathcal{P}$-monoids whose "color" G-functors $U\mathcal{P} \to \underline{\mathcal{S}}_G$ are constant.

Weak approximations can be made to have many colors; a weak approximation α is a **strong approximation** if $U\mathcal{P} \to U\mathcal{O}$ is an equivalence.

Proposition (S.[23])

$\mathrm{Alg}_{(-)}(\mathcal{S}_G)$ detects L_{Op_G}-equivalences when $U\mathcal{P} \to U\mathcal{O}$ is an equivalence; in particular, **strong approximations are L_{Op_G}-equivalences**.

---

[22] Yonatan Harpaz. *Little cubes algebras and factorization homology (course notes)*.

[23] Natalie Stewart. *On tensor products of equivariant commutative operads (draft)*. 2025.

The L_{Op_G}-conservativity argument, in short

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Proof sketch.

Examine the free $\mathcal{O}$-G-space monad:[24]

$$(T_{\mathcal{O}}X)^H \simeq \coprod_{S \in \mathbb{F}_H} \left(\mathcal{O}(S) \times \left(X^S \right)^H \right)_{h \operatorname{Aut}_H S}.$$

The inclusion $\operatorname{Aut}_H(S) \subset \operatorname{End}_H(S) = \left(S^S \right)^H$ yields natural splitting

$$(T_{\mathcal{O}}S)^H \simeq \mathcal{O}(S) \oplus \mathbf{Junk}.$$

Use monadicity of $\operatorname{Alg}_{\mathcal{O}}(\mathcal{S}_G) \to \mathcal{S}_G$ and conservativity of $(\mathcal{O}(S))$. $\qquad\square$

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[24] Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT].

Approximating manifolds

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Proposition (Dugger-Isaksen[25] '01)

If X is a topological space and $\mathfrak{O} \subset \mathscr{P}(X)$ a basis of contractible open subsets, then we get a weak equivalence

$$B\mathfrak{O} \overset{\sim}{\longrightarrow} X$$

Corollary

Let $\mathfrak{O}_S^H(V) \subset \mathrm{Conf}_S^H(V)$ be the basis of configurations in affinely $\coprod_S D(V)$-shaped invariant subspaces of $D(V)$. We get a weak equivalence

$$B\mathfrak{O}_S^H(V) \overset{\sim}{\longrightarrow} \mathrm{Conf}_S^H(V) \simeq \mathbb{E}_V(S).$$

---

[25] Daniel Dugger and Daniel C. Isaksen. "Topological hypercovers and A^1-realizations". In: *Math. Z.* 246.4 (2004), pp. 667–689.

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$$
\begin{array}{ccccccccc}
\mathbb{P}_V^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{P}_W^\otimes & \xleftarrow{\ L_\mathbb{P}\ } & \mathbb{P}_V^\otimes \times \mathbb{P}_W^\otimes & \xrightarrow{\ M_\mathbb{P}\ } & \mathbb{P}_V^\otimes \wr \mathbb{P}_W^\otimes & \xrightarrow{\ \widetilde{\varphi}_\mathbb{P}\ } & \mathbb{P}_{V|W}^\otimes & \longleftarrow & \mathbb{P}_{V\oplus W}^\otimes \\
{\scriptstyle \alpha_V \otimes \alpha_W}\downarrow & & {\scriptstyle \alpha_V \times \alpha_W}\downarrow & & {\scriptstyle \alpha_V \wr \alpha_W}\downarrow & & {\scriptstyle \alpha_{V|W}}\downarrow & & \downarrow{\scriptstyle \alpha_{V\oplus W}} \\
\mathbb{E}_V^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^\otimes & \xleftarrow{\ L_\mathbb{E}\ } & \mathbb{E}_V^\otimes \times \mathbb{E}_W^\otimes & \xrightarrow{\ M_\mathbb{E}\ } & \mathbb{E}_V^\otimes \wr \mathbb{E}_W^\otimes & \xrightarrow{\ \varphi_\mathbb{E}\ } & \mathbb{E}_{V|W}^\otimes & = = & \mathbb{E}_{V\oplus W}^\otimes
\end{array}
$$

We define a *G*-preoperad $\mathbb{P}_V^\otimes$ such that $\mathbb{P}_{V,/P}^{\mathrm{act}} \simeq \mathfrak{O}_S^H(V)$, yielding a weak approximation $\alpha_V \colon \mathbb{P}_V^\otimes \to \mathbb{E}_V^\otimes$. Then, we define a $\mathbb{P}$-Dunn map fitting into the above diagram.

Here, $\mathbb{P}_{V|W}^\otimes$ is the "*G*-preoperadic image, i.e. "decomposed little disks."

We're tasked with verifying that $\varphi_\mathbb{E} \circ M_\mathbb{E}$ induces an equivalence

$$
\mathrm{Alg}_{\mathbb{E}_{V\oplus W}}(\mathcal{S}_G) \xrightarrow{\ \sim\ } \mathrm{Alg}_{\mathbb{E}_V \times \mathrm{Alg}_{\mathbb{E}_W}}(\mathcal{S}_G) \xleftarrow{\ \sim\ } \mathrm{Alg}_{\mathbb{E}_V \otimes \mathbb{E}_W}(\mathcal{S}_G)
$$

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$$\mathbb{P}_V^\otimes \overset{\mathbf{BV}}{\otimes} \mathbb{P}_W^\otimes \xleftarrow{\ L_\mathbb{P}\ } \mathbb{P}_V^\otimes \times \mathbb{P}_W^\otimes \xrightarrow{\ M_\mathbb{P}\ } \mathbb{P}_V^\otimes \wr \mathbb{P}_W^\otimes \xrightarrow{\ \widetilde{\varphi}_\mathbb{P}\ } \mathbb{P}_{V|W}^\otimes \longleftrightarrow \mathbb{P}_{V\oplus W}^\otimes$$

$$\downarrow{\alpha_V \otimes \alpha_W} \qquad \downarrow{\alpha_V \times \alpha_W} \qquad \downarrow{\alpha_V \wr \alpha_W} \qquad \downarrow{\alpha_{V|W}} \qquad \downarrow{\alpha_{V\oplus W}}$$

$$\mathbb{E}_V^\otimes \overset{\mathbf{BV}}{\otimes} \mathbb{E}_W^\otimes \xleftarrow{\ L_\mathbb{E}\ } \mathbb{E}_V^\otimes \times \mathbb{E}_W^\otimes \xrightarrow{\ M_\mathbb{E}\ } \mathbb{E}_V^\otimes \wr \mathbb{E}_W^\otimes \xrightarrow{\ \varphi_\mathbb{E}\ } \mathbb{E}_{V\oplus W}^\otimes \ =\!=\ \mathbb{E}_{V\oplus W}^\otimes$$

1 $L_\mathbb{P}$ and $L_\mathbb{E}$ are L_{Op_G}-equivalences by fiat.

2   A variant of Harpaz's strategy[26] shows that $M_\mathbb{P}$ and $M_\mathbb{E}$ are L_{Op_G}-equivalences.

3   Simple ∞-category theory shows that $\alpha_V \times \alpha_W$ is a weak approximation.

4   A variant of Dunn's strategy[27] shows that $\alpha_{V|W}$ is a weak approximation.

5   Explicit 1-category theory shows that $\widetilde{\varphi}_\mathbb{P}$ is a strong approximation.

6   Routine bookkeeping then shows that $\mathbf{Alg}_{\mathbb{E}_{V\oplus W}}(\mathcal{S}_G) \xrightarrow{\ \sim\ } \mathbf{Alg}_{\mathbb{E}_V \times \mathbb{E}_W}(\mathcal{S}_G)$.

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[26] Yonatan Harpaz. *Little cubes algebras and factorization homology (course notes)*.

[27] Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258.

Interchange and
little *V*-disks

Natalie Stewart

Cold open

Historical context

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G-(pre)operads, algebras
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Stalkwise-linearizable tangential structures

Define $\mathbb{E}^{\otimes}_{B^{\mathrm{lin}}_G \mathrm{Top}(n)}$ to have colors the linearizable G-actions on $\mathbb{R}^n$ and operations the *topological* embeddings.

Given a G-space X with stalkwise-linearizable equivariant $\mathbb{R}^n$-bundle $T_\bullet \colon X \to B^{\mathrm{lin}}_G \mathrm{Top}(n)$, we define the assembly

$$\mathbb{E}^{\otimes}_X := L_{\mathrm{Op}_G} \left(\mathbb{E}^{\otimes}_{B^{\mathrm{lin}}_G \mathrm{Top}(n)} \times_{B^{\mathrm{linTop}(n)^{G-\sqcup}}_G} X^{G-\sqcup} \right).$$

Theorem (S.[28])

There is a natural equivariant colimit expression of operads

$$\underline{\mathrm{colim}}_{x \in X} \mathbb{E}^{\otimes}_{T_x} \overset{\sim}{\longrightarrow} \mathbb{E}^{\otimes}_X$$

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[28] Natalie Stewart. *On homotopical additivity of equivariant little disks operads (forthcoming)*. 2025.

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Corollary (S.)

Given stalkwise-linearizable equivariant $\mathbb{R}^k$-bundles $X \to B_G^{\mathrm{lin}}\mathrm{Top}(n)$ and
$Y \to B_G^{\mathrm{lin}}\mathrm{Top}(m)$, fixing the direct sum structure

$$X \times Y \longrightarrow B_G^{\mathrm{lin}}\mathrm{Top}(n) \times B_G^{\mathrm{lin}}\mathrm{Top}(m) \xrightarrow{\ \oplus\ } B_G^{\mathrm{lin}}\mathrm{Top}(n+m),$$

there is an equivalence.

$$\mathbb{E}_X^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_Y^{\otimes} \xrightarrow{\ \sim\ } \mathbb{E}_{X \times Y}^{\otimes}.$$

Other cases that are covered by the original argument:

- (Equivariant) swiss cheese
- Stratified little disks

A curious corollary about geometric topology

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Let $\underline{\mathrm{Disk}}_G^{X-\mathrm{fr},\sqcup} \subset \underline{\mathrm{Mfld}}_G^{X-\mathrm{fr},\sqcup}$ be the *G*-symmetric monoidal full category whose objects are disjoint unions of *X*-framed *G*-disks and whose mapping spaces are *X*-framed disk embeddings.

Corollary

There is a *G*-symmetric monoidal equivalence

$$\underline{\mathrm{Disk}}_G^{X-\mathrm{fr},\sqcup} \square \underline{\mathrm{Disk}}_G^{Y-\mathrm{fr},\sqcup} \simeq \underline{\mathrm{Disk}}_G^{X\times Y-\mathrm{fr},\sqcup}.$$

Here, $\square$ is the box product of semi-Mackey functors.

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This presentation was made in Beamer, with figures via tikz-cd and Inkscape, presented via Impressive. The tex is on my website. The title slide is $H^*_{C_2}(*_{C_2}; \underline{\mathbb{Z}})$ [Dug05].