ORBITAL CATEGORIES AND WEAK INDEXING SYSTEMS

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Abstract. Fix $\mathcal{T}$ an atomic orbital ∞-category. In this exposé, we initiate the combinatorial study of the poset $\mathrm{wIndex}_{\mathcal{T}}$ of *weak $\mathcal{T}$-indexing systems*, which yields arities for equivariant algebraic structures which are closed under their own operations. Within this sits a natural $\mathcal{T}$-analog $\mathrm{Index}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}$ of Blumberg-Hill's *indexing systems*, consisting of weak indexing systems which have all binary and nullary operations. For instance, we conclude from results of Balchin-Barnes-Roitzheim that the lattice of $C_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}_p$-indexing systems is equivalent to the infinite associahedron.

Along the way, we characterize the relationship between the posets of *unital weak indexing systems* and *indexing systems*, the latter remaining isomorphic to *transfer systems* on this level of generality, and we with a particular closed form expression in the C_{p^N}-equivariant case.

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1. Introduction

Fix G a finite group. In [BH15], the notion of *$\mathcal{N}_\infty$-operads for G* was introduced, encapsulating a collection of *blueprints* for G-equivariantly commutative multiplicative structures on Mackey functors which possess underlying Green functors. They demonstrated that the ∞-category of $\mathcal{N}_\infty$-operads for G is an embedded sub-poset of the category of *indexing systems* Index_G.

Subsequently, the embedding $\mathcal{N}_\infty-\mathrm{Op}_G \subset \mathrm{Index}_G$ was shown to be an equivalence in several independent works [BP21; GW18; Rub21]; of particular interest is the equivalent redefinition of indexing systems as a poset of subcategories $\mathrm{Index}_G \subset \mathrm{Sub}(\mathbb{F}_G)$ (referred to as *indexing categories*) and the observation of Rubin that indexing categories only depend on their intersections with the orbit category $\mathcal{O}_G = \{G/H\} \subset \mathbb{F}_G$, the resulting embedded subposet

$$\begin{tikzcd} \mathrm{Index}_G \arrow[d, hook] & \mathrm{IndexCat}_G \arrow[l, "\sim"'] \arrow[rr, "\sim"] \arrow[d, hook] & & \mathrm{Transf}_G \arrow[d, hook] \\ \mathrm{FullSub}_G(\underline{\mathbb{F}}_G) & \mathrm{Sub}(\mathbb{F}_G) \arrow[l, "\mathbb{F}_{(-)}"] \arrow[r, "(-)\cap\mathcal{O}_G"'] & \mathrm{Sub}(\mathcal{O}_G) \arrow[r, "\mathrm{p.b.}"'] & \mathrm{Sub}_{\mathrm{Poset}}\mathrm{Sub}_{\mathrm{Grp}}(G) \end{tikzcd}$$

being referred to as *transfer systems*.

Furthermore, Transf_G was remarked in ref to depend only on its image on the poset completion $[\mathcal{O}_G]$ of $\mathcal{O}_G$, instantiating the study of *transfer systems in a poset*. It is in this form that the burgeoning subfield of *homotopical combinatorics* (coined in [Bal+23], where it is related to finite model category theory) has attacked enumerative problems concerning $\mathcal{N}_\infty$-algebras.

Using the synonymous language of *norm maps* and noting that $\left[\mathcal{O}_{C_{p^n}}\right] = [n+1]$, this approach was used in [BBR21] to prove that $\mathrm{Transf}_{C_{p^n}}$ is equivalent to the $(n+1)$st associahedron K_{n+1}. Furthermore, this has powered a large amount of further work on the topic; for instance, $\mathrm{Transf}_{C_{pqr}}$ is enumerated for p, q, r distinct primes in [Bal+20], with some indications on how to generalize this to arbitrary squarefree integers.

In this paper, we aim to demonstrate how one may extend these enumerative efforts to *arbitrary* blueprints for unital G-equivariantly commutative multiplicative structures, called *unital weak $\mathcal{N}_\infty$-operads for G*, crucially using knowledge of Transf_G and the poset Fam_G of subconjugacy-closed families of subgroups of G. Indeed, in [St] we establish a symmetric monoidal equivalence between the notions of unital weak $\mathcal{N}_\infty$-operads and *unital weak indexing systems* (the latter monoidal under joins), which we define below.

Before doing so, we introduce the scope of our objects; this is that of *atomic orbital categories*, an axiomatic replacement for $\mathcal{O}_G$ which is wide-reaching enough to simultaneously generalize many examples of interest.

1.1. **Orbital categories.**
We briefly review the setting introduced in [Bar+16].

Construction 1.1 (c.f. [Gla17])**.** Given $\mathcal{T}$ an ∞-category[1], its *finite coproduct completion* is the full subcategory $\mathbb{F}_{\mathcal{T}} \subset \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S})$ spanned by coproducts of representables. ◁

Example 1.2. If G is a finite group, then $\mathbb{F}_{\mathcal{O}_G}$ is equivalent to the category of finite G-sets; more generally, if $\mathcal{F} \subset \mathcal{O}_G$ is a subconjugacy-closed family of subgroups, then $\mathbb{F}_{\mathcal{F}} \subset \mathbb{F}_{\mathcal{O}_G}$ is equivalent to the subcategory of finite G-sets whose stabilizers lie in $\mathcal{F}$. ◁

Inspired by the above example, given $S \in \mathbb{F}_{\mathcal{T}}$, there is a canonical expression $S \simeq \bigoplus_I V$ for some elements $(V) \subset \mathcal{T}$. We refer to these (V) as *orbits*, and refer to the set of orbits of S as $\mathrm{Orb}(S)$. An important property of the finite coproduct completion is existence of equivalences

$$\mathbb{F}_{\mathcal{T},/S} \simeq \prod_{V \in \mathrm{Orb}(S)} \mathbb{F}_{\mathcal{T},/V}; \qquad\qquad \mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_{\mathcal{T}_{/V}}.$$

We henceforth refer to $\mathcal{T}_{/V}$ simply as $\underline{V}$, and $\mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_{\underline{V}}$ as $\mathbb{F}_V$. Note that, in the case $\mathcal{T} = \mathcal{O}_G$, induction furnishes an equivalence $\mathcal{O}_{G,/[G/H]} \simeq \mathcal{O}_H$, so $\mathbb{F}_{[G/H]} \simeq \mathbb{F}_H$.

Fundamental to representation theory is the *effective Burnside category*, $\mathrm{Span}(\mathbb{F}_G)$; for instance, G-Mackey functors may be presented as product-preserving functors $\mathrm{Span}(\mathbb{F}_G) \to \mathbf{Ab}$. In fact, the spectral Mackey functor theorem of [GM17] presents G-spectra as product-preserving functors of ∞-categories $\mathrm{Span}(\mathbb{F}_G) \to \mathrm{Sp}$, a perspective which has been greatly exploited e.g. in [Bar14; BGS20].

In $\mathrm{Span}(\mathbb{F}_G)$, composition of morphisms is accomplished via the pullback

$$\begin{tikzcd} & & R_{fg} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] & & \\ & R_g \arrow[dl] \arrow[dr] & & R_f \arrow[dl] \arrow[dr] & \\ S & & T & & Q \end{tikzcd}$$

Indeed, given $\mathcal{T}$ an arbitrary ∞-category, the triple $(\mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}})$ is *adequate* in the sense of [Bar14] if and only if $\mathbb{F}_{\mathcal{T}}$ has pullbacks, in which case the triple is *disjunctive*. Thus, Barwick's construction [Bar14, Def 5.5]

[1] 1-categories embed fully faithfully into ∞-categories, and the reader is free to safely assume all categorical terminology refer to 1-categories (and spaces as sets) if they so choose, at the expense of some examples regarding parameterization over spaces or non-discrete groups.

defines a $\mathcal{T}$-effective Burnside ∞-category $\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) = A^{eff}(\mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}})$ precisely if $\mathcal{T}$ is *orbital* in the sense of the following definition.

Definition 1.3 ([Nar16, Def 4.1])**.** An ∞-category is *orbital* if $\mathbb{F}_{\mathcal{T}}$ has pullbacks; an orbital ∞-category is *atomic* if all retracts in $\mathcal{T}$ are equivalences. $\triangleleft$

We will not discuss the Burnside ∞-category in the remainder of this paper, as it is not crucial to our current combinatorics.

Remark 1.4. We show in Section 2.1 that, if $\mathcal{T}$ is an atomic orbital ∞-category, then $\mathrm{ho}(\mathcal{T})$ is as well, and the main combinatorial objects of this paper are the same between $\mathcal{T}$ and $\mathrm{ho}(\mathcal{T})$; hence the reader may uniformly assume that $\mathcal{T}$ is a 1-category, at the loss of essentially none of the combinatorics. $\triangleleft$

Example 1.5. Given X a space considered as an ∞-category, X is atomic orbital; by [Gla18, Thm 2.13], the associated stable category is the Ando-Hopkins-Rezk category of parameterized spectra over X (c.f. [And+14]). $\triangleleft$

Example 1.6. Given P a meet semilattice, P is atomic orbital; the associated stable category contains that of parameterized spectra over P. $\triangleleft$

Given G a Lie group, let $\mathcal{S}_G$ denote the ∞-category presented by orthogonal G-spaces, and let $\mathcal{O}_G \subset \mathcal{S}_G$ denote the full subcategory spanned by the homogeneous G-spaces G/H for $H \subset G$ a closed subgroup. A famous issue with equivariant homotopy theory over positive-dimensional Lie groups is that $\mathcal{O}_G$ is not *orbital*; the G-Burnside category does not exist, as $\mathbb{F}_G$ does not have pullbacks with which to define composition of spans.

Nevertheless, this has been rectified in various contexts. One particularly lucid treatment due to [CLL23] uses the slightly more general setting of *global homotopy theory.*

Definition 1.7 ([CLL23, Def 4.2.2, 4.3.2])**.** If $\mathcal{T}$ is an ∞-category, an *atomic orbital subcategory* of $\mathcal{T}$ is a wide subcategory $\mathcal{P} \subset \mathcal{T}$ satisfying the following conditions:

(1) Denote by $\mathbb{F}_{\mathcal{T}}^{\mathcal{P}} \subset \mathbb{F}_{\mathcal{T}}$ the wide subcategory consisting of morphisms which are disjoint unions of morphisms in $\mathcal{P}$. Then, $\mathbb{F}_{\mathcal{T}}^{\mathcal{P}}$ is stable under pullbacks along arbitrary maps in $\mathbb{F}_{\mathcal{T}}$, and all such pullbacks exist.
(2) Any morphism $A \to B$ in $\mathcal{P}$ admitting a section in $\mathcal{T}$ is an equivalence. $\triangleleft$

An ∞-category is atomic orbital if and only if it's an atomic orbital subcategory of itself. We have a partial converse:

Lemma 1.8. *Suppose $\mathcal{P} \subset \mathcal{T}$ is an atomic orbital subcategory. Then, $\mathcal{P}$ is atomic orbital as an ∞-category.*

Proof. First, assume we have a square in $\mathbb{F}_{\mathcal{P}}$, which is canonically extended to be the outer square of the following $\mathcal{T}$-diagram

$$\begin{tikzcd}
T' \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g'"] \arrow[ddr, bend right, "f'"'] & & \\
& T \times_S S' \arrow[r, dashed, "\pi_T"'] \arrow[d, dashed, "\pi_{S'}"] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "f"] \\
& S' \arrow[r, "g"'] & S
\end{tikzcd}$$

To prove that $\mathcal{P}$ is orbital, it suffices to verify that the inner square is a pullback, for which it suffices to check that all of the involved maps are in $\mathcal{P}$. First note that, $\pi_{S'}$ and π_T are in $\mathcal{P}$ since $\mathcal{P} \subset \mathcal{T}$ is orbital; h is then in $\mathcal{P}$ since atomic orbital subcategories are left cancellable by [CLL23, Lem 4.3.5], so we've proved that $\mathcal{P}$ is orbital. To see that $\mathcal{P}$ is atomic, note that this immediately follows from the second condition of Definition 1.7. $\square$

Definition 1.9. Given $\mathcal{T}$ an ∞-category, a $\mathcal{T}$*-family* is a full subcategory $\mathcal{F} \subset \mathcal{T}$ satisfying the condition that, given $F : V \to W$ a morphism with $W \in \mathcal{F}$, we have $V \in \mathcal{T}$. A $\mathcal{T}$*-cofamily* is a full subcategory $\mathcal{F}^{\perp} \subset \mathcal{T}$ such that $\mathcal{F}^{\perp,\mathrm{op}} \subset \mathcal{T}$ is a $\mathcal{T}^{\mathrm{op}}$-family.

Given $\mathcal{T}$ an ∞-category, an *interval family* of $\mathcal{T}$ is an intersection of a family and a cofamily; equivalently, it is a full subcategory $\mathcal{F}$ with the property that whenever $U, W \in \mathcal{F}$ and there is a path $U \to V \to W$, we have $V \in \mathcal{F}$. $\triangleleft$

Observation 1.10. If $\mathcal{F} \subset \mathcal{T}$ is an interval family in an atomic orbital ∞-category satisfying the condition that, for all cospans $U \to W \leftarrow V \in \mathcal{T}$ with $U, W \in \mathcal{F}$, there is a span $U \leftarrow W' \to V$ with $W \in \mathcal{F}$, then the inclusion $\mathbb{F}_{\mathcal{F}} \subset \mathbb{F}_{\mathcal{T}}$ creates pullbacks. In particular, $\mathcal{F}$ is an atomic orbital ∞-category. $\triangleleft$

Example 1.11. Let G be a Lie group and $\mathcal{O}_G^{f.i.} \subset \mathcal{O}_G$ the wide subcategory of the orbit ∞-category spanned by projections $G/K \to G/H$ corresponding with finite-index closed subgroup inclusions $K \subset H$. Then, by [CLL23, Ex 4.2.6], $\mathcal{O}_G^{f.i.} \subset \mathcal{O}_G$ is an orbital subcategory. In fact, it follows quickly from definition that it is atomic as well; hence $\mathcal{O}_G^{f.i.}$ is an atomic orbital ∞-category. The pullbacks in $\mathbb{F}_G^{f.i.}$ are computed by a double coset formula.

In fact, by Observation 1.10, the $\mathcal{O}_G$ interval families consisting of *finite subgroups* and of *finite-index closed subgroups* are atomic orbital ∞-categories as well. The former in the case $G = \mathbb{T}$ yields the *cyclonic orbit category* of [BG16]. $\triangleleft$

Example 1.12. Given $H \subset G$ a closed subgroup, the cofamily $\mathcal{O}_{G,\geq[G/H]}^{f.i.}$ spanned by homogeneous G-spaces G/J admitting a quotient map from G/H satisfies the assumptions of Observation 1.10, so it is atomic orbital; in the case $H = N \subset G$ is normal, it is equivalent to $\mathcal{O}_{G/N}^{f.i.}$. In any case, the associated stable homotopy theory is the value category of H*-geometric fixed points* with residual genuine G/H-structure (c.f. [Gla17]). $\triangleleft$

1.2. **Weak indexing systems and weak indexing categories.** Throughout the remainder of this introduction, we fix $\mathcal{T}$ an atomic orbital ∞-category. In the case $\mathcal{T} = \mathcal{O}_G$ is the orbit category of a compact Lie group G, Elmendorf's theorem [DK84; Elm83] implies that the ∞-category of G-spaces is equivalent to the functor ∞-category

$$\mathcal{S}_G \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}),$$

i.e. they are (homotopy coherent) *indexing systems of spaces*. It has become traditional to allow G to act on the *category theory* surrounding equivariant homotopy theory, culminating in the following definition.

Definition 1.13. The *2-category of $\mathcal{T}$-1-categories* is the functor 2-category[2]

$$\mathbf{Cat}_{\mathcal{T},1} \coloneqq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Cat}_1) \simeq \mathrm{Fun}(h_2\mathcal{T}^{\mathrm{op}}, \mathbf{Cat}_1),$$

where $\mathbf{Cat}_1$ is the 2-category of 1-categories. $\triangleleft$

We refer to the morphisms in $\mathbf{Cat}_{\mathcal{T},1}$ as $\mathcal{T}$*-functors*. Given a $\mathcal{T}$-1-category $\mathcal{C}$ and an object $V \in \mathcal{T}$, there is a V*-value* 1-category $\mathcal{C}_V \coloneqq \mathcal{C}(V)$, and given a map $V \to W$ in $\mathcal{T}$, there is an associated *restriction functor* $\mathcal{C}_W \to \mathcal{C}_V$.

Example 1.14. By [NS22, Prop 2.5.1], the ∞-category $\underline{V}$ is a 1-category, so $\mathbb{F}_V \simeq \mathbb{F}_{\underline{V}} \simeq \mathbb{F}_{\mathcal{T},/V}$ is a 1-category. Hence the functor $\mathcal{T}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ sending $V \mapsto \mathbb{F}_{\mathcal{T},/V}$ is a $\mathcal{T}$-1-category, which we call $\underline{\mathbb{F}}_{\mathcal{T}}$. $\triangleleft$

Evaluation is functorial in the $\mathcal{T}$-category; given a $\mathcal{T}$-functor $\mathcal{C} \to \mathcal{D}$, there is a canonical functor

$$\mathrm{Res}_V^W : \mathcal{C}_V \to \mathcal{D}_V.$$

We refer to a $\mathcal{T}$-functor whose V-values are fully faithful as a *fully faithful $\mathcal{T}$-functor*; if $\iota : \mathcal{C} \to \mathcal{D}$ is a fully faithful $\mathcal{T}$-functor, we say that $\mathcal{C}$ is a *full $\mathcal{T}$-subcategory of* $\mathcal{D}$. A full $\mathcal{T}$-subcategory of $\mathcal{D}$ is uniquely determined by an equivalence-closed and restriction-stable class of objects in $\mathcal{D}$; see [Sha23] for details.

Definition 1.15 (c.f. [HHR16, § 2.2.3])**.** Fix $\mathcal{C}$ a $\mathcal{T}$-1-category. The functor $\mathrm{Ind}_U^V : \mathcal{C}_U \to \mathcal{C}_V$, if it exists, is the left adjoint to Res_U^V. Furthermore, given a V-set S and a tuple $(T_U)_{U \in \mathrm{Orb}(S)}$, the *$S$-indexed coproduct of* T_U is, if it exists, the element

$$\coprod_U^S T_U \coloneqq \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^V T_U \in \mathcal{C}_W.$$

Dually, $\mathrm{CoInd}_U^V : \mathcal{C}_U \to \mathcal{C}_V$ denote the right adjoint to Res_U^V (if it exists), and the S-indexed product is (if it exists), the element

$$\prod_U^S T_U \coloneqq \prod_{U \in \mathrm{Orb}(S)} \mathrm{CoInd}_U^V T_U \in \mathcal{C}_U. \quad \triangleleft$$

[2] Throughout this paper, *n-category* will mean $(n,1)$-category, i.e. ∞-category whose mapping spaces are $(n-1)$-truncated.

Example 1.16. Given a subgroup inclusion $K \subset H \subset G$, the associated functor $\mathbb{F}_H \to \mathbb{F}_K$ is restriction, and hence its left adjoint $\mathbb{F}_K \to \mathbb{F}_H$ is *G-set induction*, matching the *indexed coprodudcts* of [HHR16, § 2.2.3]. $\triangleleft$

Given $S \in \mathbb{F}_V$, we write

$$\mathcal{C}_S \coloneqq \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_V;$$

we say that $\mathcal{C}$ *strongly admits finite coproducts* if $\coprod_U^S T_U$ always exists, in which case it amounts to a functor

$$\coprod_{-}^{S}(-) : \mathcal{C}_S \to \mathcal{C}_V.$$

It follows from construction that $\underline{\mathbb{F}}_{\mathcal{T}}$ strongly admits finite coproducts.

Definition 1.17. Given a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ and a full $\mathcal{T}$-subcategory $\mathcal{E} \subset \mathcal{D}$, we say that $\mathcal{E}$ is *closed under $\mathcal{C}$-indexed coproducts* if, for all $S \in \mathcal{C}_V$ and $(T_U) \in \mathcal{E}_S$, we have $\coprod_U^S T_U \in \mathcal{E}_V$. $\triangleleft$

Definition 1.18. We say that a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is *closed under self-indexed coproducts* if it is closed under $\mathcal{C}$-indexed coproducts. $\triangleleft$

Definition 1.19. Given $\mathcal{T}$ an orbital category, a *$\mathcal{T}$-weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ satisfying the following conditions:

(IS-a) Whenever $\underline{\mathbb{F}}_{I,V} \neq \varnothing$, we have $*_V \in \underline{\mathbb{F}}_{I,V}$.
(IS-b) $\underline{\mathbb{F}}_I$ is closed under self-indexed coproducts.

We denote by $\mathrm{wIndex}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathbf{Cat}_{\mathcal{T}}}(\mathbb{F}_{\mathcal{T}})$ the embedded sub-poset spanned by $\mathcal{T}$-weak indexing systems. Moreover, we say that a $\mathcal{T}$-weak indexing system *has one color* if it satisfies the following condition

(IS-i) For all $V \in \mathcal{T}$, we have $\underline{\mathbb{F}}_{I,V} \neq \varnothing$;

these span an embedded subposet $\mathrm{wIndex}_{\mathcal{T}}^{oc} \subset \mathrm{wIndex}_{\mathcal{T}}$. We say that a $\mathcal{T}$-weak indexing system is almost E-unital if it satisfies the condition

(IS-ii) For all noncontractible V-sets $S \sqcup S' \in \mathbb{F}_{I,V}$, we have $S, S' \in \mathbb{F}_{I,V}$.

An almost E-unital $\mathcal{T}$-weak indexing system is *almost unital* if it has one color. These are denoted $\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}$. We say that a $\mathcal{T}$-weak indexing system is *E-unital* if it satisfies the condition

(IS-iii) For all $S \sqcup S' \in \mathbb{F}_{I,V}$, we have $S, S' \in \mathbb{F}_{I,V}$.

and an E-unital $\mathcal{T}$-weak indexing system is *unital* if it has one color. We write $\mathrm{wIndex}_{\mathcal{T}}^{E\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}$. Lastly, a $\mathcal{T}$-weak indexing system is an *indexing system* if it satisfies the following condition.

(IS-iv) The subcategory $\mathbb{F}_{I,V} \subset \mathbb{F}_V$ is closed under finite coproducts for all $V \in \mathcal{T}$.

We denote the resulting poset by $\mathrm{Index}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}}$. $\triangleleft$

In practice, we will find that non-almost E-unital weak indexing systems are not well behaved, and questions involving almost E-unital weak indexing systems are usually quickly reducible to the unital case; the non-combinatorial user is encouraged to focus primarily on unital weak indexing systems for this reason.

Example 1.20. The terminal $\mathcal{T}$-weak indexing system is $\underline{\mathbb{F}}_{\mathcal{T}}$; the initial one-color $\mathcal{T}$-weak indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}}$ is defined by

$$\mathbb{F}_{\mathcal{T},V}^{\mathrm{triv}} \coloneqq \mathbb{F}_V^{\simeq}. \quad \triangleleft$$

Remark 1.21. In ninfty cite we define the *underlying $\mathcal{T}$-symmetric sequence* $\mathcal{O}(-)$ of a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$; $\mathcal{O}^{\otimes}$ parameterizes a type of equivariant multiplicative structures, and the space $\mathcal{O}(S)$ parameterizes the *S-ary operations* endowed on an $\mathcal{O}$-algebra. There we define the *arity support*

$$\mathbb{F}_{A\mathcal{O},V} \coloneqq \{S \in \mathbb{F}_V \mid \mathcal{O}(S) \neq \varnothing\};$$

in cite, we show that this possesses a fully faithful right adjoint, making $\mathcal{T}$-weak indexing systems equivalent to *weak $\mathcal{N}_\infty$-$\mathcal{T}$-operads*, i.e. subterminal objects in the ∞-category of $\mathcal{T}$-operads.

This inspires our naming; cites establishes that $\underline{\mathbb{F}}_{A\mathrm{triv}_{\mathcal{T}}} = \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}}$ and $\underline{\mathbb{F}}_{A\mathrm{Comm}_{\mathcal{T}}} = \underline{\mathbb{F}}_{\mathcal{T}}$. $\triangleleft$

Proposition 1.22. *Given $\underline{\mathbb{F}}_I$ a $\mathcal{T}$-weak indexing system, the following are $\mathcal{T}$-families:*

$$c(I) := \{V \in \mathcal{T} \mid *_V \in \mathbb{F}_{I,V}\}$$
$$\upsilon(I) := \{V \in \mathcal{T} \mid \varnothing_V \in \mathbb{F}_{I,V}\}$$
$$\nabla(I) := \{V \in \mathcal{T} \mid 2*_V \in \mathbb{F}_{I,V}\}$$

Note that $c(I) \leq \upsilon(I) \cap \nabla(I)$. In particular, we find that the one-color $\mathcal{T}$-weak indexing systems are $c^{-1}(\mathcal{T})$, the unital $\mathcal{T}$-weak indexing systems are $\upsilon^{-1}(\mathcal{T})$, and the $\mathcal{T}$-indexing systems are $\upsilon^{-1}(\mathcal{T}) \cap \nabla^{-1}(\mathcal{T})$.

Construction 1.23. Given $\mathcal{F}$ a $\mathcal{T}$-family and $\underline{\mathbb{F}}_I$ an $\mathcal{F}$-weak indexing system, we may define the $\mathcal{T}$-weak indexing system $E_{\mathcal{F}}^{\mathcal{T}}\underline{\mathbb{F}}_I$ by

$$\left(E_{\mathcal{F}}^{\mathcal{T}}\underline{\mathbb{F}}_I\right)_V := \begin{cases} \mathbb{F}_{I,V} & V \in \mathcal{F}; \\ \varnothing & \text{otherwise.} \end{cases}$$

this is an injective monotone map $\mathrm{wIndex}_{\mathcal{F}} \to \mathrm{wIndex}_{\mathcal{T}}$. ◁

Proposition 1.24. *The fiber of $c : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ is the image of $E_{\mathcal{F}}^{\mathcal{T}}|_{oc} : \mathrm{wIndex}_{\mathcal{F}}^{oc} \to \mathrm{wIndex}_{\mathcal{T}}$.*

In particular, we find that $E_{\mathcal{F}}^{\mathcal{T}}\underline{\mathbb{F}}_{\mathcal{F}}$ and $E_{\mathcal{F}}^{\mathcal{T}}\underline{\mathbb{F}}_{\mathcal{F}}^{\simeq}$ are terminal and initial among $c^{-1}(\mathcal{F})$.

Example 1.25. The intial unital $\mathcal{T}$-weak indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^0$ is defined by

$$\mathbb{F}_{\mathcal{T},V}^0 := \{\varnothing_V, *_V\};$$

we see in ninfty that this is equal to $\underline{\mathbb{F}}_{A\mathbb{E}_0}$. ◁

Example 1.26. The initial $\mathcal{T}$-indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^{\infty}$ is defined by

$$\mathbb{F}_V^{\infty} := \{n \cdot *_V \mid n \in \mathbb{N}\};$$

we see in ninfty that this is equal to $\underline{\mathbb{F}}_{A\mathbb{E}_\infty}$. ◁

Example 1.27. Choosing $\mathcal{T} = \mathcal{O}_{C_p}$ with standard representation λ, we show that in cite that the *little $\infty\lambda$-disks C_p-operad* has arity support

$$\mathbb{F}_{A\mathbb{E}_{\infty\lambda},e} = \mathbb{F}_e, \qquad \mathbb{F}_{A\mathbb{E}_{\infty\lambda},C_p} = \{n \cdot [C_p/e] \mid n \in \mathbb{N}\} \sqcup \left\{*_{C_p} + n \cdot [C_p/e] \mid n \in \mathbb{N}\right\};$$

in particular, this unital weak indexing system corresponds with an interesting algebraic theory and it is *not* an indexing system. ◁

With a wealth of examples under our belt, we begin on the road towards other perspectives on weak indexing systems.

Observation 1.28. Denote by $\mathrm{Ind}_V^{\mathcal{T}} S \to V$ the map corresponding a V-set S under the equivalence $\mathbb{F}_V \simeq \mathbb{F}_{\mathcal{T},/V}$. This equivalence implies a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is determined by its subgraph

$$I(\mathcal{C}) := \left\{ \coprod_i \mathrm{Ind}_{V_i}^{\mathcal{T}} S_i \to V_i \;\middle|\; \forall i, \quad S \in \mathcal{C}_{V_i} \right\} \subset \mathbb{F}_{\mathcal{T}}.$$

In other words, the construction I yields an embedding of posets

$$I(-) : \mathrm{wIndex}_{\mathcal{T}} \hookrightarrow \mathrm{Sub}_{\mathrm{graph}}(\mathbb{F}_{\mathcal{T}}).$$ ◁

Theorem A. *The image of $I(-)$ consists of the subcategories $I \subset \mathcal{C}$ satisfying the following conditions*

(IC-a) (restrictions) I is stable under arbitrary pullbacks in $\mathbb{F}_{\mathcal{T}}$;
(IC-b) (segal condition) the pair $T \to S$ and $T' \to S'$ are in I if and only if $T \coprod T' \to S \coprod S'$ is in I; and
(IC-c) ($\Sigma_{\mathcal{T}}$-action) if $S \in I$, then all automorphisms of S are in I.

moreover, for all numbers n, condition (IS-n) of Definition 1.19 is equivalent to condition (IC-n) below:

(IC-i) (one color) I is wide; equivalently, I contains $\mathbb{F}_{\mathcal{T}}^{\simeq}$.
(IC-ii) (aE-unital) if $S \coprod S' \to T$ is a non-isomorphism identity in I, then $S \to T$ and $S' \to T$ are in I.
(IC-iii) (E-unital) if $S \coprod S' \to T$ is in I, then $S \to T$ and $S' \to T$ are in I.
(IC-iv) (indexing category) the fold maps $n \cdot V \to V$ are in I for all $n \in \mathbb{N}$ and $V \in \mathcal{T}$.

We refer to the image of $I(-)$ as the *weak indexing categories* $\mathrm{wIndexCat}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_{\mathcal{T}})$. In general, we will refer to a generic weak indexing category as I and its corresponding weak indexing system as $\underline{\mathbb{F}}_I$.

The following observations form the basis for the proof of Theorem A.

Observation 1.29. By a basic inductive argument, Condition (IC-b) is equivalent to the following condition:

(IC-b') $S \to T$ is in I if and only if $S_U = S \times_T U \to U$ is in I for all $U \in \mathrm{Orb}(T)$.

in particular, I is uniquely determined by the maps to orbits. ◁

Observation 1.30. By Observation 1.29, in the presence of Condition (IC-b), Condition (IC-a) is equivalent to the following condition:

(IC-a') for all Cartesian diagrams in $\mathbb{F}_{\mathcal{T}}$

$$\begin{tikzcd} T \times_V U \arrow[r] \arrow[d, "\alpha'"] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "\alpha"] \\ U \arrow[r] & V \end{tikzcd} \tag{1}$$

with $U, V \in \mathcal{T}$ and $\alpha \in I$, we have $\alpha' \in I$. ◁

One of the major reasons for this formalism is the technology of *equivariant algebra*. If $\iota : I \subset \mathbb{F}_{\mathcal{T}}$ is a pullback-stable subcategory write $\mathbb{F}_{c(I)}$ for the coproduct closure of the essential image of ι. Then $(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I)$ is an adequate triple, so we may form the span category

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \coloneqq A^{eff}(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I),$$

whose forward maps are I and backwards maps are arbitrary. If $\mathcal{C}$ is an ∞-category, the category of *I-commutative monoids* is the product preserving functor category

$$\mathrm{CMon}_I(\mathcal{C}) \coloneqq \mathrm{Fun}^\times(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C});$$

the *I-symmetric monoidal 1-categories* are

$$\mathbf{Cat}_{I,1}^{\otimes} \coloneqq \mathrm{CMon}_I(\mathbf{Cat}_1),$$

where $\mathbf{Cat}_1$ denotes the 2-category of 1-categories. These are a form of *I-symmetric monoidal Mackey functors.*

$\mathcal{T}$-commutative monoids yields I-commutative monoids by neglect of structure. By ninfty cite, a full $\mathcal{T}$-subcategory of a cocartesian I-symmetric monoidal category $\mathcal{C} \subset \mathcal{D}^{I-\sqcup}$ is I-symmetric monoidal if and only if it's closed under I-indexed coproducts. Hence we have the following.

Corollary B. *Fix a collection of objects $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ containing the contractible $c(I)$-sets and $I \subset \mathbb{F}_{\mathcal{T}}$ the corresponding collection of maps satisfying Condition (IC-b). Then, the following conditions are equivalent:*

(1) I is a weak indexing category;
(2) $\underline{\mathbb{F}}_I$ is a weak indexing system;
(3) $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}^{I-\sqcup}$ is an I-symmetric monoidal subcategory.

We explore this further in ninfty cite.

1.3. Weak indexing categories and transfer systems.

Definition 1.31. Given $\mathcal{T}$ an orbital category, an *orbital transfer system in $\mathcal{T}$* is a core-containing subcategory $\mathcal{T}^{\simeq} \subset R \subset \mathcal{T}$ which is stable under *base change* in the sense that for all $\mathcal{T}$ digarams

$$\begin{tikzcd} V' \arrow[r] \arrow[d, "\alpha'"] & V \arrow[d, "\alpha"] \\ U' \arrow[r] & U \end{tikzcd}$$

whose associated $\mathbb{F}_{\mathcal{T}}$ map $V' \to V \times_U U$ is a summand inclusion, if $\alpha \in R$, we have $\alpha' \in R$. The associated embedded sub-poset is

$$\mathrm{Transf}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_{\mathcal{T}}).$$ ◁

Observation 1.32. If I is a unital weak indexing category, the intersection $\mathfrak{R}(I) \coloneqq I \cap \mathcal{T}$ is an orbital transfer system; hence it yields a monotone map

$$\mathfrak{R}(-) : \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}}.$$ ◁

Proposition 1.33 ([NS22, Rmk 2.4.9])**.** *$\mathfrak{R}(-)$ restricts to an equivalence*

$$\mathfrak{R}(-) : \mathrm{Index}_{\mathcal{T}} \xrightarrow{\sim} \mathrm{Transf}_{\mathcal{T}}.$$

Remark 1.34. In the case $\mathcal{T} = \mathcal{O}_G$, it was shown independently in [Rub19, Thm 3.7] and [BBR21, Cor 8] that the composite inclusion $\mathrm{Sub}_{\mathbf{Grp}}(G) \hookrightarrow \mathcal{O}_G \hookrightarrow \mathbb{F}_G$ induces an embedding $\mathrm{Index}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathbf{Poset}}(\mathrm{Sub}_{\mathbf{Grp}}(G))$ whose image is identified by those subposets which are closed under restriction and conjugation, which were called *G-transfer systems*; this and Proposition 1.33, together imply that pullback along the *homogeneous G-set* functor $\mathrm{Sub}_{\mathbf{Grp}}(G) \to \mathcal{O}_G$ induces an equivalence between the poset of G-transfer systems of [BBR21; Rub19] and the orbital $\mathcal{O}_G$-transfer systems of Definition 1.31. ◁

In view of Remark 1.34, we henceforth in this paper refer to orbital transfer systems simply as *transfer systems*, never referring to the other notion.

In Corollary 2.23, we fact show that the composite

$$\mathrm{Transf}_{\mathcal{T}} \simeq \mathrm{Index}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$$

is a fully faithful right adjoint to $\mathfrak{R}$, i.e. the poset of unital weak indexing systems possessing a given transfer system has a terminal object, given by the unique such indexing system. However, we show that the fibers can be quite large; for instance, in 2.24, we will see that $\mathfrak{R}$ also attains a fully faithful left adjoint, which is distinct from the right adjoint over all transfer systems when $\mathcal{T}$ has a terminal object (e.g. when $\mathcal{T} = \mathcal{O}_G$).

The upshot is that unital weak indexing systems are not determined by their transitive V-sets. Nevertheless, they are defined by their transitive V-sets of *at most 2 orbits with any particular stabilizer.* To this end, denote by $\underline{\mathbb{F}}_{\mathcal{T}}^{\leq n} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ the collection of objects with at most n summands of any particular orbit.

Given $\mathcal{C}^{\leq 2} \subset \underline{\mathbb{F}}_{\mathcal{T}}^{\leq 2}$, we may form the full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ generated by $\mathcal{C}^{\leq 2}$ under $\mathcal{C}^{\leq 2}$-indexed colimits. We say that $\mathcal{C}^{\leq 2}$ is *closed under applicable self-indexed coproducts* if $\mathcal{C}^{\leq 2} = \mathcal{C} \cap \underline{\mathbb{F}}_{\mathcal{F}}^{\leq 2}$.

Theorem C. *Restriction along the inclusion $\underline{\mathbb{F}}_{\mathcal{T}}^{\leq 2} \hookrightarrow \underline{\mathbb{F}}_{\mathcal{T}}$ yields an embedding of posets $\mathrm{wIndex}_{\mathcal{T}} \subset \mathrm{Coll}(\underline{\mathbb{F}}_{\mathcal{T}}^{\leq 2})$ whose image is spanned by those collections which are closed under applicable self-indexed coproducts.*

Corollary 1.35. *If $\mathcal{T}$ is an orbital ∞-category such that $\pi_0(\mathcal{T})$ is finite and $\mathcal{T}_{/V}$ is finite as a 1-category for all $V \in \pi_0(\mathcal{T})$, then there exist finitely many $\otimes$-idempotent weak $\mathcal{N}_\infty$-$\mathcal{T}$-operads.*

Remark 1.36. Let $\mathcal{T} = \mathcal{O}_G$. By Theorem C, one may devise an inefficient algorithm to compute $\mathrm{wIndex}_G^{\mathrm{uni}}$. Namely, given a collections of objects $\mathcal{C}^{\leq 2} \subset \underline{\mathbb{F}}_G^{\leq 2}$, one may compute all of its self-indexed coproducts in finite time using the double coset formula in order to determine whether $\mathcal{C}^{\leq 2}$ is closed under applicable self-indexed coproducts. One may simply iterate over the finite poset $\mathrm{Coll}(\underline{\mathbb{F}}_G^{\leq 2})$, performing the above computation at each step to determine which collections correspond with unital weak indexing systems. ◁

The above algorithm is quite inefficient; in practice, we instead prefer to divide and conquer, first computing Fam_G and Transf_G, then computing the fibers under $\mathfrak{R}$ and ∇. We will do this for $G = C_{p^N}$, but first we need notation. Given $R \in \mathrm{Transf}_G$, we define the families

$$\mathrm{Dom}(R) := \left\{ U \in \mathcal{O}_G \mid \quad \exists U \to V \xrightarrow{f} W \ \text{ s.t. } \ f \in R \right\};$$
$$\mathrm{Cod}(R) := \left\{ U \in \mathcal{O}_G \mid \quad \exists V \xrightarrow{f} W \leftarrow U \ \text{ s.t. } \ f \in R \right\}.$$

Given a full subcategory $\mathcal{F} \subset \mathcal{O}_G$ and a G-transfer system T, we denote by $\mathrm{Sieve}_T(\mathcal{F})$ the poset of precomposition-closed wide subcategories of $T \cap \mathcal{F}$.

Theorem D. *Fix $N \in \mathbb{N} \cup \{\infty\}$. Then, there is a cocartesian fibration*

$$(\mathfrak{R}, \nabla) : \mathrm{wIndex}_{C_{p^N}}^{\mathrm{uni}} \to K_N \times [N]$$

with fibers satisfying

$$\mathfrak{R}^{-1}(R) \cap \nabla^{-1}(\mathcal{F}) = \begin{cases} \varnothing & \mathrm{Dom}(R) \not\leq \mathcal{F}; \\ * & \mathrm{Cod}(R) \leq \mathcal{F}; \\ \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) & \text{otherwise.} \end{cases}$$

Moreover, cocartesian transport is computed along $R \leq R'$ by the inclusion

$$\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \hookrightarrow \mathrm{Sieve}_{R'}(\mathrm{Cod}(R') - \mathcal{F})$$

and computed along $\mathcal{F} \leq \mathcal{F}'$ by the restriction

$$\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \twoheadrightarrow \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}')$$