ON INDEXED TENSOR PRODUCTS OF EQUIVARIANT COMMUTATIVE OPERADS

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Abstract. We lift the Boardman-Vogt tensor product to a G-symmetric monoidal closed ∞-category $\underline{\mathrm{Op}}_G^{\otimes}$ of $\mathcal{O}_G$-∞-operads. Using this, we construct a G-symmetric monoidal colocalizing subcategory

$$\mathcal{N}_{(-)\infty} : \underline{\mathrm{wIndex}}_G^{\otimes} \to \underline{\mathrm{Op}}_G^{\otimes}$$

called the *poset of weak $\mathcal{N}_\infty$-G-operads*, whose colocalization functor constructs the weak indexing system of admissible H-sets. Additionally, we combinatorially characterize $\underline{\mathrm{wIndex}}_G^{\otimes}$, finding that the G-subcategory $\underline{\mathrm{wIndex}}_G^{\mathrm{uni},\vee} = \underline{\mathrm{Op}}_G^{\mathrm{uni},\otimes} \cap \underline{\mathrm{wIndex}}_G^{\otimes}$ of *unital weak $\mathcal{N}_\infty$-G-operads* is *cocartesian G-symmetric monoidal*, i.e. its indexed tensor products are indexed joins.

As a special case, we recognize Blumberg-Hill's $\mathcal{N}_\infty$-operads as a symmetric monoidal sub-poset $\mathrm{Index}_G^{\vee} \subset \mathrm{wIndex}_G^{\mathrm{uni},\vee}$ confirming a conjecture of Blumberg-Hill. In particular, for I, J unital weak indexing systems and $\mathcal{C}$ an $I \vee J$-symmetric monoidal ∞-category, we construct a canonical $I \vee J$-symmetric monoidal equivalence

$$\underline{\mathrm{CAlg}}_I^{\otimes} \underline{\mathrm{CAlg}}_J^{\otimes} \mathcal{C} \simeq \underline{\mathrm{CAlg}}_{I \vee J}^{\otimes} \mathcal{C}.$$

Contents

1. Introduction

1.1. **Summary of main results.** Homotopy-coherent algebraic structures in genuine-equivariant mathematics are naturally founded in the notion of *G-commutative monoids*. In the context of this paper, the *∞-category of G-commutative monoids* in an ∞-category $\mathcal{D}$ is the ∞-category of product-preserving functors

$$\mathrm{CMon}_G(\mathcal{D}) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_G), \mathcal{D}),$$

where $\mathbb{F}_G$ denotes the category of finite G-sets.[1] The ∞-category of *small G-symmetric monoidal ∞-categories* is $\mathrm{CMon}_G(\mathbf{Cat})$, where $\mathbf{Cat}$ denotes the ∞-category of small ∞-categories.

Give $\mathcal{C}^{\otimes} \in \mathrm{CMon}_G(\mathbf{Cat})$ a G-symmetric monoidal ∞-category, the product-preserving functor

$$\iota_H : \mathrm{Span}(\mathbb{F}) \xrightarrow{* \mapsto G/H} \mathrm{Span}(\mathbb{F}_G)$$

constructs a symmetric monoidal ∞-category $\mathcal{C}_H^{\otimes} := \iota_H^* \mathcal{C}^{\otimes}$ whose underlying ∞-category $\mathcal{C}_H$ is the *value* of $\mathcal{C}^{\otimes}$ on the orbit G/H.[2] For all subgroups $K \subset H \subset G$, the covariant and contravariant functoriality of $\mathcal{C}^{\otimes}$ then yield symmetric monoidal *restriction and norm* functors

$$\mathrm{Res}_K^H : \mathcal{C}_H^{\otimes} \to \mathcal{C}_K^{\otimes},$$
$$N_K^H : \mathcal{C}_K^{\otimes} \to \mathcal{C}_H^{\otimes}.$$

We use this structure to encode *algebras* in $\mathcal{C}^{\otimes}$, for which we need *operads*.

Various notions of G-operad have been introduced for this. In Section 4.1 we introduce an ∞-category Op_G of *$\mathcal{O}_G$-∞-operads* (henceforth G-operads) equivalent to that of [NS22]. Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_G$ a G-operad and $S \in \mathbb{F}_H$ an H-set for some $H \subset G$, we construct a *space of S-ary operations* $\mathcal{O}(S)$, together with *operadic composition maps*

$$\mathcal{O}(S) \otimes \bigotimes_{H/K_i \in \mathrm{Orb}(S)} \mathcal{O}(T_i) \to \mathcal{O}\left(\coprod_{H/K_i \in \mathrm{Orb}(S)} \mathrm{Ind}_{K_i}^H T_i \right), \tag{1}$$

operadic restriction maps

$$\mathcal{O}(S) \to \mathcal{O}(\mathrm{Res}_K^H S), \tag{2}$$

and *equivariant symmetric group action*

$$\mathrm{Aut}_H(S) \times \mathcal{O}(S) \to \mathcal{O}(S), \tag{3}$$

the with Eqs. (2) and (3) ascending to a structure of a *G-symmetric sequence*; we go on to show in Corollary 4.23 that this structure is *monadic* under a reducedness assumption.

Definition 1.1. We say that $\mathcal{O}^{\otimes}$ *has at least one color* when $\mathcal{O}(*_H)$ is nonempty for all subgroups $H \subset G$, and we say $\mathcal{O}^{\otimes}$ *has at most one color* if $\mathcal{O}(*_H) \in \{*, \emptyset\}$ for all $H \subset G$. We say that $\mathcal{O}^{\otimes}$ *has one color* if it has at least one color and at most one color. ◃

[1] In this paper we will call ∞-categories *∞-categories* and 0-truncated ∞-categories *1-categories*. We hope this prevents avoidable confusion in older readers.

[2] In this paper, "orbits" refer to transitive G-sets, i.e. objects of the orbit category $\mathcal{O}_G$.

When $\mathcal{O}^{\otimes}$ has one color, an $\mathcal{O}$-algebra in the G-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ can intuitively be viewed as a tuple $(X_H \in \mathcal{C}_H)_{G/H \in \mathcal{O}_G}$ satisfying $X_K \simeq \mathrm{Res}^H_K X_H$, together with $\mathcal{O}(S)$-actions

$$\mu_S : \mathcal{O}(S) \otimes X_H^{\otimes S} \to X_H \tag{4}$$

for all $S \in \mathbb{F}_H$ and $H \subset G$, homotopy-coherently compatible with the maps Eqs. (1) to (3), where we write

$$X_H^{\otimes S} := \bigotimes_{H/K \in \mathrm{Orb}(S)} N_K^H \mathrm{Res}^H_K X_H.$$

for the *indexed tensor products* in $\mathcal{C}^{\otimes}$. In this paper, we are concerned with defining *indexed tensor products* of $\mathcal{O}$-algebras, as well as $\mathcal{P}$-algebras in the resulting G-symmetric monoidal ∞-category. Mirroring the nonequivariant case, we will accomplish this by realizing the operad of $\mathcal{O}$-alebras in $\mathcal{P}$ as the *internal hom* with respect to a symmetric monoidal structure on the ∞-category of G-operads.

Theorem A. *There exists a G-symmetric monoidal ∞-category $\underline{\mathrm{Op}}_G^{\otimes}$ with the following properties:*

(1) The H-value ∞-category of $\underline{\mathrm{Op}}_G^{\otimes}$ is Op_H as in [NS22].

(2) In the case $G = e$, there exists an equivalence of symmetric monoidal ∞-categories

$$\mathrm{Op}_e^{\otimes} \simeq \mathrm{Op}_{\infty}^{\otimes}$$

where $\mathrm{Op}_{\infty}^{\otimes}$ is the Boardman-Vogt symmetric monoidal ∞-category of ∞-operads of [BS24]; in particular, the tensor product of $\mathrm{Op}_e^{\otimes}$ agrees under this equivalence with the Boardman-Vogt tensor product *of [BV73; HM23; HA].*[3]

(3) The underlying norm functor $N_H^G : \mathrm{Op}_H \to \mathrm{Op}_G$ satisfies $N_H^G \mathcal{O}^{\otimes} \simeq \mathrm{Ind}_H^G \mathcal{O}^{\otimes}$.

(4) The underlying tensor functor $- \otimes \mathcal{O} : \mathrm{Op}_G \to \mathrm{Op}_G$ has is right adjoint to $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(-)$ as in [NS22].

(5) The underlying G-∞-category of $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{P})$ is the G-∞-category of algebras $\underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{P})$; the associated ∞-category is the ∞-category of algebras $\mathbf{Alg}_{\mathcal{O}}(\mathcal{P})$.

(6) When $\mathcal{C}$ is G-symmetric monoidal, $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is G-symmteric monoidal.

(7) When $\mathcal{O}$ is reduced and $\mathcal{C}$ is G-symmetric monoidal, the forgetful G-functor $\underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ lifts to an G-symmetric monoidal functor

$$\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathcal{C}^{\otimes}.$$

Remark 1.2. After this introduction, we replace $\mathcal{O}_G$ with an atomic orbital ∞-category $\mathcal{T}$ for the remainder of the paper; we prove Theorem A as well as other theorems in this introduction in this setting, greatly generalizing the stated results, at the cost of intuition. ◁

References. $\underline{\mathrm{Op}}_G^{\otimes}$ is constructed in Definition 4.1, and statements (1) and (3) are Remark 4.2. The remining statements are Corollaries 4.13 and 4.71. □

Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_G^{\mathrm{oc}}$ a G-operad with one color and $\psi : T \to S$ a map of finite H-sets, we also define the *space of multimorphisms*[4]

$$\mathrm{Mul}_{\mathcal{O}}^{\psi}(T; S) := \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T \times_S U).$$

We then define the subcategory[5] $A\mathcal{O} \subset \mathbb{F}_G$ of *$\mathcal{O}$-admissible maps* by

$$A\mathcal{O} := \left\{ \psi : T \to S \mid \mathrm{Mul}_{\mathcal{O}}^{\psi}(T; S) \neq \emptyset \right\} \subset \mathbb{F}_G.$$

[3] The symmetric monoidal structure of [HA] is derived from a model-categorical structure, and as such, only its underlying tensor functor is known to be well-behaved. Our result mimics that of [BS24], as each symmetric monodidal structure is canonically induced from the Day convolution structure on G-symmetric monoidal ∞-category by the G-symmetric monoidal envelope (see Corollary 4.17), hence its coherences additionally satisfy a useful universal property.

[4] We only make the assumption that $\mathcal{O}^{\otimes}$ has one color for ease of exposition; throughout the remainder of text following the introduction, we will not make this assumption.

[5] Throughout this paper, we say *subobject* to mean monomorphism in the sense of [HTT, § 5.5.6]; in the case the ambient ∞-category is a 1-category, this agrees with the traditional notion.

In the case our objects are in the ∞-category **Cat** of small ∞-categories, we call this a *subcategory*; in the case that the containing ∞-category is a 1-category, this is canonically expressed as a *wide subcategory of a full subcategory*, and it is uniquely determined by its morphisms, so we will express subcategories of $\mathcal{C}$ a 1-category as a sub-poset of $2^{\mathrm{Mor}(\mathcal{C})}$.

In essence, taking tensor products of Eq. (4) yields an action

$$\mathrm{Mul}_{\mathcal{O}}^{\psi}(T;S) \otimes X_H^{\otimes T} \to X_H^{\otimes S},$$

and $A\mathcal{O}$ consists of the *pairs of arities* over which this produces structure on X.

The fact that $\varnothing$ accepts no maps from nonempty sets potentially obstructs construction of maps as in Eqs. (1) and (2), so $A\mathcal{O}$ can't be an *arbitrary* subcategory. In Sections 4.3 and 4.5, we combinatorially characterize the image of A in $\mathrm{Sub}(\mathbb{F}_G)$ as the poset wIndex_G of *weak indexing systems*, a weakened variant of the notion introduced in [BP21].

We say that a G-operad $\mathcal{O}^{\otimes}$ is *weakly unital* if

$$\mathcal{O}(\varnothing_V) \in \begin{cases} * & \mathcal{O}(*_V) \neq \varnothing; \\ \varnothing & \mathcal{O}(*_V) = \varnothing. \end{cases}$$

We say that $\mathcal{O}^{\otimes}$ is *unital* if it is weakly unital and has at least one color. we denote the full subcategory spanned by unital G-operads by $\mathrm{Op}_G^{\mathrm{uni}} \subset \mathrm{Op}_G$.

Theorem B. *The following posets are each equivalent:*

1. *The poset* $\mathrm{Sub}_{\mathbf{Cat}_G^{\otimes}}(\mathbb{F}_G^{\amalg})$ *of G-symmetric monoidal subcategories of* $\mathbb{F}_G^{\amalg}$.
2. *The poset* $\mathrm{Sub}_{\mathrm{Op}_G}(\mathrm{Comm}_G)$ *of* sub-commutative G-operads
3. *The category* $\mathrm{Op}_{G,\leq -1}$ *of* G-(-1)-operads.
4. *The image* $A(\mathrm{Op}_G) \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_G)$
5. *The sub-poset* $\mathrm{wIndex}_G \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_G)$ *spanned by subcategories $I \subset \mathbb{F}_{\mathcal{T}}$ which are closed under base change and automorphisms and satisfy the Segal condition that*
$$T \to S \in I \qquad \iff \qquad \forall U \in \mathrm{Orb}(S), \quad T \times_S U \to U \in I$$
6. *The sub-poset* $\mathrm{Sub}^{\mathrm{full}}_{\mathbf{Cat}_G}(\underline{\mathbb{F}}_G)$ *spanned by full G-subcategories $\mathcal{C} \subset \underline{\mathbb{F}}_G$ which are closed under self-indexed coproducts and have $*_H \in \mathcal{C}_H$ whenever $\mathcal{C}_H \neq \varnothing$.*

Furthermore, setting $\mathrm{wIndex}_G^{\mathrm{uni}} = A\left(\mathrm{Op}_G^{\mathrm{uni}}\right)$, *there is an equivalence*

$$\mathrm{wIndex}_G^{\mathrm{uni}} \simeq \mathrm{Transf}_G \times \mathrm{Fam}_G$$

whose image $\mathrm{Transf}_G \times \{\mathcal{O}_G\} \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_G)$ *under (5) is the* indexing systems *of [BP21; GW18; Rub21a].*

References. The *sliced G-symmetric monoidal envelope* is shown to implement (1) $\iff$ (2) in Corollary 4.17. Then in Remark 4.66, we show that (2) and (3) are equivalent as subcategories. We combinatorially characterize $A(\mathrm{Op}_G)$ in Section 4.3, constructing equivalences (4) $\iff$ (5) $\iff$ (6) as Proposition 4.26 and Proposition 4.38. We construct the equivalence (3) $\iff$ (4) in Corollary 4.67 by constructing a fully faithful right adjoint to

$$A : \mathrm{Op}_{\mathcal{T}} \rightleftarrows \mathrm{wIndex}_G : \mathcal{N}_{(-)\infty}. \tag{5}$$

□

The remaining equivalence is Theorem 4.52.

Using this, under the assumption that $\mathcal{O}^{\otimes}$ is *unital*, the information of $A\mathcal{O}$ may be understood as simply specifying the colors over which $\mathcal{O}^{\otimes}$ prescribes a binary multiplication

$$X_H^{\otimes 2} \to X_H$$

and the maps $K \to H$ over which $\mathcal{O}^{\otimes}$ prescribes a transfer

$$X_K \to X_H.$$

In general, we find that a full subcategory $\mathcal{C} \subset \mathrm{Op}_G$ has a terminal object if and only if it there exists a weak indexing system I such that $\mathcal{C} = \mathrm{Op}_I \subset \mathrm{Op}_G$ is spanned by the G-operads satisfying $A\mathcal{O} \leq I$, in which case the terminal object is the G-operad $\mathcal{N}_{I\infty}^{\otimes}$ constructed in Eq. (5). These G-operads are called *weak $\mathcal{N}_\infty$-G-operads.*

We may understand $\mathcal{N}_{I\infty}^{\otimes}$ in a hands-on manner in many ways; for instance, it is constructed explicitly in Proposition 4.4. On the other hand, the equivalence (2) $\iff$ (5) Theorem B shows that $\mathcal{N}_{I\infty}^{\otimes}$ is uniquely identified by the property

$$\mathcal{N}_{I\infty}(S) = \begin{cases} * & \mathrm{Ind}_H^G S \to G/H \quad \text{is in } I; \\ \emptyset & \text{otherwise.} \end{cases} \tag{6}$$

Alternatively, we may see this indirectly using the existence of free G-operads on symmetric sequences (see Corollary 4.23) and the equivalence $\mathrm{Op}_I \simeq \mathrm{Op}_{\mathcal{T},/\mathcal{N}_{I\infty}^{\otimes}}$.

In fact, there are many weak $\mathcal{N}_\infty$-G-operads of interest outside of the world of $\mathcal{N}_\infty$-G-operads:

Example 1.3. Given $\mathcal{F} \subset \mathcal{O}_G^{\mathrm{op}}$ a G-family, the operad $\mathbf{triv}_{\mathcal{F}}^{\otimes} := \mathcal{N}_{\mathbb{F}_{\mathcal{F}}^{\simeq}}^{\otimes}$ is characterized by a natural equivalence

$$\underline{\mathbf{Alg}}_{\mathrm{triv}_{\mathcal{F}}}^{\otimes}(\mathcal{C}) = \mathrm{Bor}_{\mathcal{F}}^G(\mathcal{C}^{\otimes}),$$

in Proposition 4.15, where $\mathrm{Bor}_{\mathcal{F}}^G$ is the $\mathcal{F}$*-Borelification* discussed in Section 4.6. ◁

Example 1.4. Additionally, given R a transfer system, Theorem B constructs an associated unital weak $\mathcal{N}_\infty$-G-operad $\mathrm{Norm}_R^{\otimes}$ with no binary multiplications; we see that algebras over this G-opeard are *G-objects with R-indexed unital norms* in ref . ◁

In general, in Corollary 5.10, we characterize the *∞-category of I-commutative monoids* in $\mathcal{C}$ a complete ∞-category as

$$\mathrm{CMon}_I\mathcal{C} := \mathbf{Alg}_{\mathcal{N}_{I\infty}}(\mathcal{C}^{\times}) \simeq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_G), \mathcal{C}),$$

where $\mathrm{Span}_I(\mathbb{F}_G) \subset \mathrm{Span}(\mathbb{F}_G)$ is the subcategory whose forward maps are in I; we define the *∞-category of I-symmetric monoidal ∞-categories* as $\mathrm{CMon}_I\mathbf{Cat}$. We also show in Proposition 4.8 that I-symmetric monoidal ∞-categories have underlying I-operads; for $\mathcal{C} \in \mathrm{CMon}_I\mathbf{Cat}$, we define the ∞-category of *I-commutative algebras* in $\mathcal{C}$ as

$$\mathrm{CAlg}_I\mathcal{C} := \mathbf{Alg}_{\mathcal{N}_{I\infty}}(\mathcal{C}).$$

We show in Corollary 4.71, that analogs of Theorem A (5) and (6) hold for I-commutative algebra objects in I-symmetric monoidal categories.

We go on to compute the I-indexed tensor products in $\underline{\mathrm{CAlg}}_I^{\otimes}\mathcal{C}$ under a distributivity assumption; they are *I-cocartesian*, in the sense that their I-indexed tensor products are indexed coproducts (c.f. Section 5.2).

Theorem C. *Let $\mathcal{O}^{\otimes}$ be a G-operad. Then, the following properties are equivalenent.*

(a) The $A\mathcal{O}$-symmetric monoidal ∞-category $\underline{\mathbf{Alg}}_{A\mathcal{O}}^{\otimes}\underline{\mathcal{S}}_G$ is $A\mathcal{O}$-cocartesian.

(b) The unique map $\mathcal{O}^{\otimes} \to \mathcal{N}_{A\mathcal{O}\infty}^{\otimes}$ is an equivalence.

Furthermore, $\underline{\mathrm{CAlg}}_I^{\otimes}\mathcal{C}$ is I-cocartesian for any distributive I-symmetric monoidal ∞-category $\mathcal{C}$ and weak indexing system I.

In Corollary 5.10, we use this to prove that $\mathrm{CMon}_I(\mathcal{C})$ is an *I-semiadditive*, I-symmetric monoidal category, in the sense that it is simultaneously cartesian and cocartesian. We use this to recognize I-commutative monoids within the world of [CLL24], hinting at an operadic presentation for equivariant lifts of [GGN15].

We say that an I-operad $\mathcal{O}^{\otimes}$ is *reduced* if the (unique) map $\mathcal{O}^{\otimes} \to \mathcal{N}_{I\infty}$ induces equivalences

$$\mathcal{O}(S) \simeq \mathcal{N}_{I\infty}(S) \qquad \forall \;\; S \in \mathbb{F}_H \quad \text{empty or transitive}$$

(c.f. Eq. (6)). We characterize algebras in cocartesian I-symmetric monoidal categories in Theorem 5.5, and from this Theorem C entirely characterizes the tensor products of reduced I-operads with $\mathcal{N}_{I\infty}^{\otimes}$.

Corollary D. *If $\mathcal{O}^{\otimes}$ is an reduced I-operad, then the unique map $\mathcal{O}^{\otimes} \otimes \mathcal{N}_{I\infty}^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$ is an equivalence.*

This immediately characterizes many tensor products of weak $\mathcal{N}_\infty$-operads, since $\mathcal{N}_{I\infty}$ is a J-operad whenever $I \leq J$. We go on to completely characterize indexed tensor products of weak $\mathcal{N}_\infty$-operads, confirming Conjecture 6.27 of [BH15].

Theorem E. *The functor* $\mathcal{N}^{\otimes}_{(-)\infty} : \mathrm{wIndex}_G \to \mathrm{Op}_G$ *lifts to a G-symmetric monoidal colocalizing subcategory inclusion*

$$\begin{tikzcd} \underline{\mathrm{wIndex}}^{\otimes}_G \arrow[rr, bend left, "\mathcal{N}^{\otimes}_{(-)\infty}"] \arrow[rr, phantom, "\perp"] & & \underline{\mathrm{Op}}^{\otimes}_G \arrow[ll, bend left, "A"] \end{tikzcd}$$

and the resulting G-symmetric monoidal structure on $\underline{\mathrm{wIndex}}^{\otimes}_G$ is uniquely determined by the following conditions:

(1) *the H-value category is $\left(\underline{\mathrm{wIndex}}^{\otimes}_G\right)_H = \mathrm{wIndex}_H$, with tensor product*
$$I \otimes J = \mathrm{Bor}^G_{\mathrm{cSupp}(I \cap J)}(I \vee J)$$
where $I \vee J$ is the join in wIndex_G;
(2) *the restriction functors are $\mathrm{Res}^G_H I = I \cap \mathbb{F}_H$; and*
(3) *the norm functors are the left adjoint $\mathrm{Ind}^G_H I = \iota^G_{H!} I \subset \mathbb{F}_G$ to Res^G_H.*

Corollary F. *When I, J are weak indexing systems, we have*

$$\begin{aligned} \mathcal{N}^{\otimes}_{I\infty} \otimes \mathcal{N}^{\otimes}_{J\infty} &\simeq \mathcal{N}^{\otimes}_{(I\otimes J)\infty} \\ \mathcal{N}^{\otimes}_{I\infty} \times \mathcal{N}^{\otimes}_{J\infty} &\simeq \mathcal{N}^{\otimes}_{(I\cap J)\infty} \\ \mathrm{Res}^G_H \mathcal{N}^{\otimes}_{I\infty} &\simeq \mathcal{N}^{\otimes}_{\mathrm{Res}^G_H I\infty} \\ \mathrm{Ind}^G_H \mathcal{N}^{\otimes}_{I\infty} &\simeq \mathcal{N}^{\otimes}_{\mathrm{Ind}^G_H I\infty} \\ \mathrm{CoInd}^G_H \mathcal{N}^{\otimes}_{I\infty} &\simeq \mathcal{N}^{\otimes}_{\mathrm{CoInd}^G_H I\infty}. \end{aligned}$$

In particular, norms of I-commutative algebras are $\mathrm{CoInd}^G_H I$-commutative algebras, and when I, J are unital, we have

$$\underline{\mathrm{CAlg}}^{\otimes}_I \underline{\mathrm{CAlg}}^{\otimes}_J(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_{I\vee J}(\mathcal{C}). \tag{7}$$

Remark 1.5. By Theorem B and Corollary F, every *unital* weak $\mathcal{N}_\infty$-operad I can canonically be expressed as
$$\mathcal{N}^{\otimes}_{I\infty} \simeq \mathrm{Norm}^{\otimes}_{R(I)} \otimes \mathbb{E}^{\otimes}_{\infty,\mathcal{F}_I}$$
for some canonically determined $R(I) \in \mathrm{Transf}_G$ and $\mathcal{F}_I \in \mathrm{Fam}_G$, where $\mathbb{E}^{\otimes}_{\infty,\mathcal{F}}$ parameterizes unital G objects whose H-values have compatible multiplications whenever $H \in \mathcal{F}$. The interpretation of Eq. (7) applied to this decomposition is that I-commutative algebras may be constructed as interchanging pairs of commutative algebra structures on fibers and unital norms, each indexed by I. ◁

In view of Theorem E, a simple combinatorial argument (carried out as Corollary 4.58) gives the following inductive strategy for constructing C_{p^n}-commutative algebras *one norm at a time*.

Corollary G. *Fix p prime and $n \geq 1$. For all $k < n$, write*
$$I^k_{k-1} := \mathbb{F}^{\simeq}_G \cup \mathrm{Hom}(C_{p^n}/C_{p^k}, C_{p^n}/C_{p^{k+1}}) \subset \mathbb{F}_{C_p}$$
Then, I^k_{k-1} is a weak indexing system, and
$$\underline{\mathrm{CAlg}}^{\otimes}_{I^n_{n-1}} \cdots \underline{\mathrm{CAlg}}^{\otimes}_{I^1_0} \underline{\mathrm{CAlg}}^{\otimes} \mathcal{C} \simeq \underline{\mathrm{CAlg}}^{\otimes}_{C_{p^n}} \mathcal{C}.$$

We offer various additional corollaries in Sections 4.7 and 5.5 concerning lifts of various functors in equivariant homotopy theory to functors between categories of I-commutative algebras; included among these are equivariant factorization homology and equivariant algebraic K-theory. We go on to state a family of conjectures concerning further properties of equivariant higher algebra in Section 6.3.

1.2. Notation and conventions.

1.3. Acknowledgements.