CONNECTIVITY OF SPACES OF EQUIVARIANT CONFIGURATIONS AND $\mathbb{E}_V$-ALGEBRAIC WIRTHMÜLLER MAPS

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Abstract. We provide conditions on a locally smooth G-manifold under which its spaces of equivariant configurations $\mathrm{Conf}_S^G(X)$ are n-connected. We use this to give simple sharp conditions under which the fibers of the (finite-index) $\mathbb{E}_V$-algebra Wirthmüller maps are n-connected; as an easy application, we use this to construct incomplete Tambara structures on the homotopy groups of $\mathbb{E}_{2V}$-ring spectra.

Throughout this paper, we fix G a Lie group.

Definition 1. If $H \subset G$ is a closed subgroup, $S \in \mathbb{F}_H$ an H-set with finitely many orbits, and X a topological space with G-action, we denote the subspace of H-equivariant embeddings $S \hookrightarrow M$ by

$$\mathrm{Conf}_S^H(X) \subset \mathrm{Map}^H(S, X). \qquad \triangleleft$$

Nonequivariantly, the homotopy type of configurations spaces in X is a rich source of homeomorphism-invariants of X [LS04]. In this paper, we study some rudiments of an equivariant lift of this in the smooth setting. Namely, in Section 1, we supply sufficient conditions for a smooth G-manifold M such that its nonempty configurations spaces $\mathrm{Conf}_S^G(M)$ are all d-connected.

We have a particular application in mind; the structure spaces of the *little V-disks operad* are configuration spaces in smooth G-manifolds, and connectivity statements of G-operads translate to structural statements about their algebras (see [Ste24]). For instance, in Section 2, we prove a sharp strengthening of the following theorem, where $\mathbb{E}_V^{\otimes}$ is the proper equivariant little V-disks operad (with finite-index transfers and restrictions) of [Ste25c].

Theorem A. *If $\mathcal{C}$ is a G-symmetric monoidal $(d-1)$-category and V a real orthogonal G-representation, then the forgetful functor*

$$\mathrm{Alg}_{\mathbb{E}_{\infty V}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_{dV}}(\mathcal{C})$$

is an equivalence of $(d-1)$-categories.

In particular, $\mathbb{E}_{\infty V}$ is a weak $\mathcal{N}_\infty$-operad, so [Ste25d] and Theorem A provide a *homotopical incomplete Mackey functor* model for $(d-2)$-truncated $\mathbb{E}_{dV}$-algebras in Cartesian G-symmetric monoidal ∞-categories and [CHLL24, Thm B] provides a *bi-incomplete Tambara functor* model for $\mathbb{E}_{dV}$-rings in the setting of $(d-2)$-truncated homotopical incomplete Mackey functors.

In Section 3 we study individual arities; in view of the semiadditive closure theorem of [CLL24], the following measures the defect preventing Theorem A from holding for arbitrary G-spaces.

Theorem B. *Suppose $H \subset G$ is a compact closed subgroup, V an orthogonal G-representation, and $S \in \mathbb{F}_H$ a finite H-set admitting an embedding into $\mathrm{Res}_H^G V$. The following conditions of (H, S, V, n) are equivalent.*

(i) The S-indexed $\mathbb{E}_V$-H-space Wirthmüller map

$$W_{X,S} \colon X^{\sqcup S} \to X^{\times S}$$

is n-connected for all $X \in \mathrm{Alg}_{\mathbb{E}_{\mathrm{Res}_H^G V}}(\underline{\mathcal{S}}_H)$;

(ii) The space $\mathrm{Conf}_S^H(V)$ is n-connected; and

(iii) The following conditions hold:

(a) If $\left|S^H\right| \geq 2$, then $\dim V^H \geq n+2$, and

(b) For all orbits $[H/K] \subset S$ and intermediate subgroups $K \subset J \subset H$, we have

$$\dim V^K \geq \dim V^J + n + 2.$$

Date: January 21, 2025.

Here, the $[H/K]$-*indexed Wirthmüller map* in a G-category is the comparison map $\mathrm{Ind}_K^H X \to \mathrm{CoInd}_K^H X$, which is adjunct to the map $X \to \mathrm{Res}_K^H \mathrm{CoInd}_K^H X$ picked out under Mackey's double coset formula by the identity double coset; for instance, the *Wirthmüller isomorphism theorem* (for finite-index inclusions) states that this is an equivalence in the Spanier-Whitehead G-category.

The S-indexed Wirthmüller map is a combination of this with the ordinary semiadditive norm map, and is centered in [CLL24; Nar16]; in a pointed G-category, writing $X^{\sqcup S} := \coprod_{S=\coprod_i [H/K_i]} \mathrm{Ind}_{K_i}^H \mathrm{Res}_{K_i}^H X$ and similar for $X^{\times S}$, the S-Wirthmüller map is classified by the diagonal matrix

$$W_{X,S} = \begin{bmatrix} W_{\mathrm{Res}_{K_1}^H X, [H/K_i]} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & W_{\mathrm{Res}_{K_\ell^H} X, [H/K_\ell]} \end{bmatrix}$$

1. Configuration spaces in smooth G-manifolds

Definition 2 ([Bre72, § IV])**.** If M is a smooth manifold with a continuous G-action, we say that the action is *locally smooth* if, for each point $x \in M$, there exists a real orthogonal $\mathrm{stab}_G(x)$-representation V_x and a trivializing open neighborhood

$$x \in \coprod_{G/\mathrm{stab}_G(x)} V_x \hookrightarrow M,$$

where for a topological H-space X, we write $\coprod_{G/\mathrm{stab}_G(x)} X := G \times_H X$ as a topological G-space. In this case, we say that M with its action is a *locally smooth G-manifold.* ◁

Smooth actions on manifolds admit well-behaved tubular neighborhoods; for example, [Bre72, Cor VI.2.4] proves that smooth actions are locally smooth. On the other hand, if M is a locally smooth G-manifold, then the inclusion $M_{(H)} \hookrightarrow M$ of points with orbit isomorphic to G/H is a locally closed topological submanifold [Bre72, Thm IV.3.3], which is smooth if M is smooth [Bre72, Cor VI.2.5].

We begin this section in Section 1.1 by proving the following.

Theorem 3 (Equivariant Fadell-Neuwirth fibration)**.** *Fix M a locally smooth G-manifold, $S, T \in \mathbb{F}_G$ a pair of orbit-finite G-sets, and $\iota\colon S \hookrightarrow M$ a G-equivariant configuration. The following is a homotopy-Cartesian square:*

$$\begin{tikzcd} \mathrm{Conf}_T^G(M - \iota(S)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Conf}_{S \sqcup T}^G(M) \arrow[d, "U"] \\ \{\iota\} \arrow[r, hook] & \mathrm{Conf}_S^G(M) \end{tikzcd}$$

Remark 4. In the case that T and S satisfy the condition that, for all $x \in T$ and $y \in S$, $\mathrm{stab}_G(x)$ and $\mathrm{stab}_G(y)$ are non-conjugate, then the argument [Hil12, Thm 2.9] shows that U is (up to $\mathrm{Aut}(S)$-equivariant homeomorphism) just given by the projection $\mathrm{Conf}_S^G(X) \times \mathrm{Conf}_T^G(X) \to \mathrm{Conf}_S^G(X)$ for any *topological* G-space X; however, this does not reduce our analysis to the non-equivariant case, as $\mathrm{Conf}^G_{n \cdot [G/H]}(X)$ is not a non-equivariant configuration space. ◁

Thus the long exact sequence in homotopy for $T = G/H$ yields means for computing homotopy groups of $\mathrm{Conf}_S^G(M)$ inductively on the cardinality of the orbit set $|S_G|$, with inductive step hinging on homotopy of

$$\mathrm{Conf}_{G/H}^G(M - \iota(S)) \simeq (M - \iota(S))_{(H)}.$$

We denote by $[\mathcal{O}_G]^{\mathrm{op}}$ the subconjugacy lattice of closed subgroups of G, and we let

$$\mathrm{Istrp}(M) = \{\mathrm{stab}_x(G) \mid x \in M\} \subset [\mathcal{O}_G]^{\mathrm{op}}$$

be the full subposet spanned by conjugacy classes (H) for which $M_{(H)}$ is nonempty. We are inspired to make the following definition.

Definition 5. A locally smooth G-manifold M is

- $\geq d$*-dimensional at each orbit type* if $M_{(H)}$ is $\geq d$-dimensional for each $(H) \in \mathrm{Istrp}(M)$;

- $(d-2)$-*connected at each orbit type* if $M_{(H)}$ is $(d-2)$-connected for each $(H) \in \mathrm{Istrp}(M)$. $\triangleleft$

In Section 1.2, we use Theorem 3 to prove the following.

Theorem 6. *If G is compact, and a locally smooth G-manifold M is $\geq d$-dimensional and $(d-2)$-connected at each orbit type, then for all finite G-sets $S \in \mathbb{F}_G$, the configuration space $\mathrm{Conf}^G_S(M)$ is either empty or $(d-2)$-connected.*

In order to identify applications of this theorem, we give sufficient conditions for M to be $(d-2)$-connected at each orbit type. Note by repeatedly applying [Bre72, Thm IV.3.1] that the subspace $M_{\supseteq(H)} \subset M$ consisting of points of orbit type $[G/K]$ a quotient of $[G/H]$ is a disjoint union of closed submanifolds. In Section 1.3, we use this to prove the following.

Proposition 7. *Suppose that M is a smooth G-manifold satisfying the following conditions:*
 (a) M is $\geq d$-dimensional at each orbit type.
 (b) $M_{\supseteq(H)}$ is $(d-2)$-connected for each conjugacy class $(H) \in \mathrm{Istrp}(M)$.
 (c) $\mathrm{codim}(M_{\supseteq(K)} \hookrightarrow M_{\supseteq(H)}) \geq d$ for each $(H) \subseteq (K)$ in $\mathrm{Istrp}(M)$.
 (d) $\mathrm{Istrp}(M)$ is finite (e.g. G compact and M finite type, c.f. [Bre72, Thm IV.10.5]).
Then M is $(d-2)$-connected at each orbit type.

1.1. **A Fadell-Neuwirth fibration for equivariant configurations.** Our strategy for Theorem 3 mirrors that of Knudsen in the notes [Knu18]. In particular, we would like to use Quillen's theorem B [Qui73], which requires us to construct $\mathrm{Conf}^H_S(M)$ as a classifying space. In fact, there is a general scheme to do this:

Lemma 8 ([DI04, Thm 2.1], via [Knu18, Thm 4.0.2])**.** *If $\mathcal{B}$ is a topological basis for X such that all elements of $\mathcal{B}$ are weakly contractible, then the canonical map*
$$|\mathcal{B}| = \mathrm{hocolim}_{\mathcal{B}} * \to X$$
is a weak equivalence, where on the left $\mathcal{B}$ is considered as a poset under inclusion.

To use this, define an elementwise-contractible basis for $\mathrm{Conf}^G_S(M)$ by
$$\widetilde{\mathcal{B}}^G_S(M) := \left\{ (X,\sigma) \;\middle|\; \exists (V_x) \in \prod_{[x]\in \mathrm{Orb}_S} \mathbf{Rep}^{\mathrm{orth}}_{\mathbb{R}}(\mathrm{stab}_G([x])), \text{ s.t. } \coprod_{[x]}^{S} V_x \simeq X \subset M,\ \sigma : S \xrightarrow{\sim} \pi_0(U) \right\},$$
where for all tuples $(Y_x) \in \prod_{[x]\in \mathrm{Orb}_S} \mathrm{Top}_{\mathrm{stab}_G([x])}$, we write
$$\coprod_{[x]}^{S} Y_x := \coprod_{[x]\in \mathrm{Orb}(S)} \left(G \times_{\mathrm{stab}_G([x])} Y_x \right) \in \mathrm{Top}_G$$
for the *indexed disjoint union of Y_x*. We fix $\mathcal{B}^G_S(M) \subset \widetilde{\mathcal{B}}^G_S(M)$ the smaller basis consisting of open sets (X,σ) possessing neighborhoods $(X,\sigma) \subset (X',\sigma)$ such that the associated embeddings factor as

(1)
$$\begin{tikzcd} \coprod_U^S D(V_U)^\circ \arrow[r, phantom, "\simeq"] \arrow[d, hook, "\exists"'] & \coprod_U^S V_U \arrow[d, hook, "X"] \\ V'_U \arrow[r, hook, "X'"] & M \end{tikzcd}$$

where $D(V_U)^\circ$ denotes the open unit V_U-disk; that is, open sets in $\mathcal{B}^G_S(M)$ consist of collections of configurations possessing a fixed common neighborhood resembling disjoint unions of real orthogonal representations, subject to the condition that there is "space on all sides" of the neighborhood. This is functorial in two ways:

- given a summand inclusion $S \hookrightarrow T \sqcup S$, the forgetful map $\mathrm{Conf}^G_{T\sqcup S}(M) \to \mathrm{Conf}^G_S(M)$ preserves basis elements, inducing a map $\mathcal{B}^G_{T\sqcup S}(M) \to \mathcal{B}^G_S(M)$.
- any open embedding $\iota : M \hookrightarrow N$ induces a map $\mathrm{Conf}^G_T(M) \hookrightarrow \mathrm{Conf}^G_T(N)$ preserving basis elements, inducing a map $\mathcal{B}^H_S(M) \to \mathcal{B}^H_S(N)$.

To summarize, we've observed the proof of following lemma.

Lemma 9. *Given $H \subset G$ and $S, T \in \mathbb{F}_H$, there is an equivalence of arrows*

$$\begin{tikzcd} \left|\mathcal{B}^G_{T \sqcup S}(M)\right| \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Conf}^G_{T \sqcup S}(M) \arrow[d] \\ \left|\mathcal{B}^G_S(M)\right| \arrow[r, phantom, "\simeq"] & \mathrm{Conf}^G_S(M) \end{tikzcd}$$

Thus we can characterize the homotopy fiber of U using Quillen's theorem B and the following.

Proposition 10. *For $(X_S, \sigma_S) \le (X'_S, \sigma'_S) \in \mathcal{B}^G_S(M)$, and an S-configuration $\mathbf{x} \in X_S$, we have a diagram*

$$\begin{tikzcd} \mathcal{B}^G_T(M - \mathbf{x}) & \mathcal{B}^G_T(M - \overline{X}_S) \arrow[l, "\varphi"'] \arrow[d, phantom, "\wr"] & \mathcal{B}^G_T(M - \overline{X'}_S) \arrow[l] \arrow[ll, bend right, "\varphi"'] \arrow[d, phantom, "\wr"] \\ & ((X_S, \sigma_S) \downarrow U) & \left((X'_S, \sigma'_S) \downarrow U\right) \arrow[l] \end{tikzcd}$$

such that the maps φ induce weak equivalences on classifying spaces.

We will power this with the following observation:

Observation 11. Recall that an embedding of topological G-spaces $f : Y \hookrightarrow Z$ is a G-*isotopy equivalence* if there exists another G-equivariant embedding $g : Z \hookrightarrow Y$ and a pair of G-equivariant isotopies $gf \sim \mathrm{id}_Z$, $fg \sim \mathrm{id}_Y$. If $f : Y \to Z$ is a G-isotopy equivalence, then postcomposition with f induces a $G \times \Sigma_n$-isotopy equivalence $\mathrm{Conf}_n(Y) \hookrightarrow \mathrm{Conf}_n(Z)$; indeed, postcomposition with f and g induce G-equivariant embeddings, and postcomposition with the isotopies $gf \sim \mathrm{id}_Z$, $fg \sim \mathrm{id}_Y$ yields equivariant isotopies $\mathrm{Conf}_n(g) \circ \mathrm{Conf}_n(f) \sim \mathrm{Conf}_n(gf) \sim \mathrm{Conf}_n(\mathrm{id}_Z) \sim \mathrm{id}_{\mathrm{Conf}_n(Z)}$ and similar for fg.

In particular, the vertical arrows in the following diagram are isotopy equivalences

$$\begin{tikzcd} \mathrm{Conf}^H_S(X) \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Conf}_{|S|}(X)^{\Gamma_S} \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Map}^G\left(G \times \Sigma_{|S|}/\Gamma_S, \mathrm{Conf}_{|S|}(X)\right) \arrow[d] \\ \mathrm{Conf}^H_S(Y) \arrow[r, phantom, "\simeq"] & \mathrm{Conf}_{|S|}(Y)^{\Gamma_S} \arrow[r, phantom, "\simeq"] & \mathrm{Map}^G\left(G \times \Sigma_{|S|}/\Gamma_S, \mathrm{Conf}_{|S|}(Y)\right) \end{tikzcd}$$

where $\Gamma_S = \{(h, \rho_S(h)) \mid h \in H\} \subset G \times \Sigma_{|S|}$ is the graph subgroup corresponding with an H-set S with action map $\rho_S : H \to \Sigma_{|S|}$. Hence f induces a homotopy equivalence $\mathrm{Conf}^H_S(X) \xrightarrow{\sim} \mathrm{Conf}^H_S(Y)$. ◁

Proof of Proposition 10. The maps φ are each induced by the open inclusions $M - \overline{X}_S \hookrightarrow M - \mathbf{x}$, so the top horizontal arrows commute. The equivalences $\mathcal{B}^G_T(M - \overline{X_S}) \simeq ((X_S, \sigma_S) \downarrow U)$ simply follow by unwinding definitions. Thus we're left with proving that φ induces an equivalence on classifying spaces

$$\begin{tikzcd} \mathrm{Conf}^G_T(M - \mathbf{x}) \arrow[d, phantom, "\wr"] & \mathrm{Conf}^G_T(M - \overline{X_s}) \arrow[l] \arrow[d, phantom, "\wr"] \\ \left|\mathcal{B}^G_T(M - \mathbf{x})\right| & \left|\mathcal{B}^G_T(M - \overline{X_s})\right| \arrow[l] \end{tikzcd}$$

By Observation 11, it suffices to show that $M - \overline{X_S} \hookrightarrow M - \mathbf{x}$ is a G-isotopy equivalence. In fact, by Eq. (1), it suffices to prove that the inclusion $f : V - D(V) \hookrightarrow V - \{0\}$ is a G-isotopy equivalence. But this is easy; scaling is equivariant, so we may define the G-equivariant embedding $g : V - \{0\} \to V - D(V)$ by $g(x) = \frac{1+|x|}{|x|} \cdot x$. Then, each of the equivariant isotopies $gf \sim \mathrm{id}, fg \sim \mathrm{id}$ can be taken as restrictions of $h(t, x) = \frac{1-t+|x|}{|x|} \cdot x$. □

We are ready to conclude our equivariant homotopical lift of [FN62, Thm 1].

Proof of Theorem 3. By the above analysis, we may replace our diagram with a homotopy equivalent diagram given by the geometric realiztion of the following diagram of posets, and prove that it is homotopy Cartesian

$$\begin{tikzcd} \mathcal{B}^G_T(M-\iota(S)) \arrow[r] \arrow[d] & \mathcal{B}^G_{T\sqcup S}(M) \arrow[d] \\ \{\iota\} \arrow[r, hook] & \mathcal{B}^G_S(M) \end{tikzcd}$$

By Quillen's theorem B [Qui73, Thm B], it suffices to prove two statements:

- for all basis elements (X_S,σ_S), The canonical map $((X_s,\sigma_s)\downarrow U)\to \mathcal{B}^G_T(M-\iota(S))$ induces a weak equivalence on classifying spaces, and
- for all inclusions of basis elements $(X_S,\sigma_S)\subset(X'_S,\sigma'_S)$, the canonical map $\left((X'_S,\sigma'_S)\downarrow U\right)\to((X_S,\sigma_S)\downarrow U)$ induces a weak equivalence on classifying spaces.

In fact, both statements follow immediately from Proposition 10, with the second using two-out-of-three. ☐

1.2. **Proof of the main theorem in topology.** To prove Theorem 6, we begin with a lemma.

Lemma 12. *If M is a locally smooth G-manifold which is at least d-dimensional and $(d-2)$-connected at each orbit type and $\iota: G/H\hookrightarrow M$ an embedded orbit, then $M-\iota(G/H)$ is at least d-dimensional and $(d-2)$-connected at each orbit type.*

Proof. We have

$$(M-\iota(G/H))_{(K)} = \begin{cases} M_{(K)} & G/K\neq G/H \\ M_{(H)}-\iota(G/H) & G/K = G/h, \end{cases}$$

so the only nontrivial case is $H=K$, in which case we're tasked with verifying that the complement of a discrete set of points in a d-dimensional $(d-2)$-connected manifold is $(d-2)$-connected. This is a well known classical fact in algebraic topology which follows quickly from the Blakers-Massey theorem. ☐

Proof of Theorem 6. If $d-2<0$, there is nothing to prove, so assume that $d-2\geq 0$. We induct on $|S_G|$ with base case 1, i.e. with $S=G/H$. In this case, $\mathrm{Conf}^G_{G/H}(M)=M_{(H)}$ is $(d-2)$-connected by assumption.

For induction, fix some $S\sqcup G/H\in\mathbb{F}_G$ and inductively assume the theorem when $|T_G|\leq|S_G|$. Then, note that $\mathrm{Conf}^G_S(M)$ is $(d-2)$-connected by assumption and $M-\iota(S)$ is $\geq d$-dimensional and $(d+2)$-connected at each orbit by Lemma 12, so $\mathrm{Conf}^G_{G/H}(M-\iota(S))$ $(d-2)$-connected by the inductive hypothesis. Thus Theorem 3 expresses $\mathrm{Conf}^G_{S\sqcup G/H}(M)$ as the total space of a homotopy fiber sequence with connected base and fiber, so it is connected. Furthermore, examining the long exact sequence associated with Theorem 3, we find that

$$\begin{tikzcd} 0 \arrow[rr] \arrow[d, phantom, "\wr"] & & \pi_k\mathrm{Conf}^G_{S\sqcup G/H}(M) \arrow[rr] \arrow[dr] & & 0 \arrow[d, phantom, "\wr"] \\ \pi_k\mathrm{Conf}^G_{G/H}(M-\iota(S)) \arrow[urr] & & & & \pi_k\mathrm{Conf}^G_S(M) \end{tikzcd}$$

is exact for $0<k\leq d-2$; hence $\mathrm{Conf}^G_{S\sqcup G/H}(M)$ is $(d-2)$-connected, completing the induction. ☐

Remark 13. The above proof argues a sharper statement if we fix a particular G-set S admitting an embedding into M, so that $\mathrm{Istrp}(S)\subset\mathrm{Istrp}(M)$; we have shown that whenever M is a smooth G-manifold with $M_{(H)}$ being $\geq d$-dimensional and $(d-2)$-connected for each $(H)\in\mathrm{Istrp}(S)$, the space $\mathrm{Conf}^G_S(M)$ is $(d-2)$-connected. ◁

1.3. **Some sufficient conditions for connectivity at each orbit.** We begin with the following observations:

Observation 14. If M satisfies the conditions of Proposition 7, then $M_{\supseteq(H)}$ does as well. ◁

Observation 15. If M satisfies the conditions of Proposition 7 for $d-2\geq 0$, then in particular, the orbit space M_G are a union of path-connected topological spaces $\left(M_{\supseteq(H)}\right)_G$ along path-connected intersections, so M_G is path-connected. In particular, by [Bre67, Thm IV.3.1], M has a *principle orbit type*, i.e. a minimal element of $(H_{\min})\in\mathrm{Istrp}(M)$. Hence we have underlying space

$$M^e = M_{(H_{\min})},$$

so the underlying space M^e is $\geq d$-dimensional and $(d-2)$-connected. ◁

We will strengthen Proposition 7. Pick an order on $\mathrm{Istrp}(M) = (G/H_1, \dots, G/H_n, G/G)$, and write

$$\begin{aligned} M_k &= M - \bigcup_{i<k} M_{\supseteq (H_i)} \\ \widetilde{M}_k &= M_{\supseteq (H_k)} - \bigcup_{i<k} M_{\supseteq (H_k)} \cap M_{\supseteq (H_i)} \\ &= M_{\supseteq (H_k)} - \bigcup_{\substack{(K) \supseteq (H_k H_i) \\ i<k}} M_{\supseteq (K)} \end{aligned}$$

Lemma 16. *For all k, the space M_k is $(d-2)$-connected.*

Proof. We induct in two ways:

- First, we inductively assume we have proved the lemma at full strength when G is replaced with any proper subgroup $H \subsetneq G$ such that $[G/H] \in \mathrm{Istrp}(M)$; since $\mathrm{Istrp}(M)$ is finite, this begins with the base case in which case there are no such proper subgroups.
- Second, we inductively assume that we have proved the lemma for all $k' < k$; this begins with the base case that $k = 1$, in which case we have $M_1 = M$, which is $(d-2)$-connected by Observation 15.

Under these assumptions, note that $\widetilde{M}_{k-1} \subset M_{k-1}$ is a $(d-2)$-connected closed submanifold of codimension $\geq d$ in a $(d-2)$-connected smooth manifold with complement is M_k. Thus it possesses a tubular neighborhood $\widetilde{M}_{k-1} \subset \tau(\widetilde{M}_{k-1}) \subset M_{k-1}$, and "hemmed gluing" presents a homotopy pushout square

$$\begin{tikzcd} \partial\tau\widetilde{M}_{k-1} \arrow[r] \arrow[d, "\tilde{\iota}"] & M_k \arrow[d, "\iota"] \\ \widetilde{M}_{k-1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & M_{k-1} \end{tikzcd}$$

The boundary $\partial\tau\left(\tilde{M}_{k-1}\right)$ is the total space of a c-sphere bundle over a $(d-2)$-connected space, where

$$c = \mathrm{codim}(M_{\leq (H_k)} \hookrightarrow M) - 1 > d - 2.$$

The long exact sequence in homotopy reads

$$\begin{tikzcd} \pi_1(S^c) \arrow[r] & \pi_1\left(\partial\tau\left(\widetilde{M}_{k-1}\right)\right) \arrow[r] & \pi_1\left(\widetilde{M}_{k-1}\right) \arrow[r] \arrow[dr] & 0 \arrow[r] \arrow[d, phantom, "\wr"] & \pi_0\left(\partial\tau\left(\widetilde{M}_{k-1}\right)\right) \arrow[r] & \pi_0\left(\widetilde{M}_{k-1}\right) \arrow[r] & 0 \\ & & & \pi_0(S^c) \arrow[ur] & & & \end{tikzcd}$$

so $\partial\tau\widetilde{M}_{k-1}$ is connected, and when $d-2 \geq 1$, $\partial\tau\widetilde{M}_{k-1}$ is simply connected. Furthermore, at degree $0 < \ell \leq (d-2)$ the Gysin sequence in integral homology reads

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, phantom, "\wr"] & H^\ell(\partial\tau\widetilde{M}_{k-1}) \arrow[r] \arrow[dr] & 0 \arrow[d, phantom, "\wr"] \\ H^\ell(\widetilde{M}_{k-1}) \arrow[ur] & & H^{\ell-c}(\widetilde{M}_{k-1}) \end{tikzcd}$$

so $\partial\tau\tilde{M}_{k-1}$ has vanishing cohomology in degrees $0 < \ell \leq d-2$. Hurewicz' theorem then implies that $\partial\widetilde{M}_{k-1}$ is $(d-2)$-connected.

In particular, this together with $(d-3)$-connectivity of the homotopy fiber S^c implies that $\tilde{\iota}$ is a $(d-2)$-connected map, so its homotopy pushout ι is $(d-2)$-connected. Since M_{k-1} is a $(d-2)$-connected space by assumption, this implies that M_k is $(d-2)$-connected, completing the induction. □

We conclude a strengthening of Proposition 7.

Proposition 17. *Suppose that $M_{(\supseteq H)}$ is $\geq d$-dimensional, $\mathrm{Istrp}(M)$ is finite, and for all inclusions $(H) \subset (K)$ in $\mathrm{Istrp}(M)$, the following conditions are satisfied:*

(b) $M_{\supseteq (K)}$ *is $(d-2)$-connected, and*

(c) $\mathrm{codim}\left(M_{\supseteq (K)} \hookrightarrow M_{\supseteq (H)}\right) \geq d$.

Then, $M_{(H)}$ is $(d-2)$-connected.

Proof. By Observation 14 we may assume $H = G$. This is precisely what is shown in Lemma 16 when $k = n+1$. □

Warning 18. Neither the conditions of Proposition 7 or of Theorem 6 are stable under restrictions for general G-manifolds; for instance, let $G = C_2 \times C_2$ be the Klein 4 group, and $H, H' \subset G$ a pair of distinct order-2 subgroups. Write σ, σ' for the inflated orthogonal G-representations from the sign representation of H and H'. Then for any $n \in \mathbb{N}$, define the $(2n-2)$-dimensional smooth G-manifold

$$M := (n-2)\cdot\sigma \oplus n \sqcup S\left((2n-1)\cdot\sigma'\right),$$

Note that M satisfies Proposition 7 for n, but its restriction to H is not $(n-2)$-connected at each orbit type when $n-2 \geq 0$, as $\left(\mathrm{Res}_H^G M\right)_{(e)} \simeq S^{2n-1} \sqcup D^n$ is not connected. ◁

2. Representations, homotopy-coherent algebra, and configuration spaces

In homotopy-coherent algebra, a prominent role is played by the operads $\mathbb{E}_1 = \mathcal{A}_\infty$ and $\mathbb{E}_\infty$, whose algebras are *homotopy-coherently associative algebras* and *homotopy-coherently commutative algebras*, respectively. Dunn's celebrated "additivity theorem" proved non-homotopically [Dun88] (later made homotopical by Lurie [HA, Thm 5.1.2.2]) that an object possessing n-interchanging $\mathbb{E}_1$-structures may equivalently be presented as an algebra over the $\mathbb{E}_n$*-operad*, whose space of k-ary operations is weakly equivalent to the ordered configuration space $\mathrm{Conf}_k(\mathbb{R}^n)$. Thus, after Dunn and Lurie, a higher-categorical version of the Eckmann-Hilton argument may be phrased as stating that $\mathbb{E}_n$*-algebras in* $(n-1)$*-categories canonically lift to* $\mathbb{E}_\infty$*-algebras*; Lurie showed that this is equivalent to the statement that $\mathrm{Conf}_k(\mathbb{R}^n)$ is $(n-2)$-connected for all n, k [HA, Cor 5.1.1.7], which was a half-century old fact of manifold topology due to [FN62].

We would like to lift this to equivariant higher algebra using the equivariant little disks G-operads $\mathbb{E}_V$; these appear in [Hor19], where they are shown to have S-ary operation space

$$\mathbb{E}_V(S) \simeq \mathrm{Conf}_S^H(V)$$

for all $S \in \mathbb{F}_H$. Thus we are compelled to seek a representation theoretic context lifting the assumptions of Proposition 7. We propose the following.

Definition 19. We say V *has* d*-codimensional fixed points* if $\left|V^H\right|, \left|V^K/V^H\right| \in \{0\} \cup [d, \infty]$ for all $K \subset H \subset G$. ◁

When $G = e$, this is equivalent to simply being d-dimensional.

Proposition 20. *If a real orthogonal G-representation V has d-codimensional fixed points, then the smooth G-manifold $V - \{0\}$ is at least d-dimensional and $(d-2)$-connected at each orbit type.*

Proof. We may write V as a filtered (homotopy) colimit $V = \bigcup_i V_i$ with V_i a finite dimensional real orthogonal G-representation with $\min(i,d)$-codimensional fixed points; then, if V_i is $(i-2)$-connected for each i, taking a colimit, this implies that V is d-connected. Hence it suffices to prove this in the case we that V is finite dimensional.

In this case, G acts smoothly on V, and we make the following observations:

(a) $(V - \{0\})_{(H)} = \bigcup_{H' \in (H)} \left(V^{H'} - \bigcup_{H' \subset K} V^K\right) - \{0\}$ is either empty or $\max_{H' \in (H)} \left|V^{H'}\right| \geq d$-dimensional.

(b) $V_{\leq (H)} = \bigcup_{H' \in (H)} V^H$ is a union of contractible spaces along contractible intersections, so it is contractible and $\geq d$-dimensional; by the same argument as Lemma 12, $(V - \{0\})_{(H)} = V_{(H)} - \{0\}$ is $(d-2)$-connected.

(c) when $(K), (H) \in \mathrm{Istrp}(M)$,

$$\begin{aligned}
\mathrm{codim}((V-\{0\})_{\leq(K)} \hookrightarrow (V-\{0\})_{\leq(H)}) &= \mathrm{codim}(V_{\leq(K)} \hookrightarrow V_{\leq(H)}) \\
&= \min_{\substack{H' \subset K \\ V^{H'} \neq V^K}} \left(\left|V^{H'}\right| - \left|V^K\right|\right) \\
&= \min_{\substack{H' \subset K \\ V^{H'} \neq V^K}} \left|V^H/V^K\right| \\
&\geq d
\end{aligned}$$

by assumption, and it is nonzero since $V_{(H)}$ is nonempty.

(d) $\mathrm{Istrp}(V)$ is finite since V is finite dimensional.

Thus Proposition 7 applies, proving the proposition. □

Remark 21. Replacing Proposition 7 with Proposition 17, we find that $V_{(H)}$ is $(d-2)$-connected if $\left|V^K/V^H\right| \in \{0\} \cup [d,\infty]$ for all $K \subset G$. ◁

Corollary 22. *If V has d-codimensional fixed points, then for all closed subgroups $H \subset G$ and finite H-sets $S \in \mathbb{F}_H$, $\mathrm{Conf}_S^H(V)$ is $(d-2)$-connected or empty.*

Proof. We begin by noting

$$\mathrm{Conf}_S^H(V) = \begin{cases} \mathrm{Conf}^H_{S-*_H}(\mathrm{Res}_H^G(V-\{0\})) & S^H \neq \varnothing, \\ \mathrm{Conf}_S^H(\mathrm{Res}_H^G(V-\{0\})) & \text{otherwise.} \end{cases}$$

so it suffices to show $\mathrm{Conf}_S^H(\mathrm{Res}_H^G(V-\{0\}))$ to be $(d-2)$-connected or empty. Noting that the condition of having d-codmimensional fixed points is restriction-stable, this follows by Theorem 6 and Proposition 20. □

In fact, we have a converse to this.

Proposition 23. *If a finite-index inclusion of subgroups $K \subset H$ has $V^H \hookrightarrow V^K$ a proper inclusion of codimension $< d$, then $\mathrm{Conf}^G_{[G/K]}(V) \simeq V_{(K)}$ is not $(d-2)$-connected.*

Proof. This never occurs when V is 0-dimensional. If V^G is $0 < c < d$-dimensional, then we may directly see $\mathrm{Conf}^G_{2\cdot *_G}(V) = \mathrm{Conf}_2(V^G) = S^{c-1}$ is not $(d-2)$-connected, as it has nontrivial π_{c-1}. Thus we assume that V^G is $\geq d$-dimensional, so that V^H is $\geq d$-dim for all H.

Fix $c := \min_{K \subsetneq H' \in \mathrm{Istrp}(V)} \mathrm{codim}\left(V^{H'} \hookrightarrow V^K\right)$. We may replace V with the real orthogonal G-representation $V^K = V_{(\geq K)}$. We're left with proving that $V_{(K)} = V - \bigcup_{K \subsetneq H' \in \mathrm{Istrp}(V)} V^{H'}$ is not $(d-2)$-connected. Pick an order $(H_i)_{1 \leq i \leq n}$ on $\mathrm{Istrp}(V) - \{(K)\}$ so that $c = \mathrm{codim}\left(V^{H'} \hookrightarrow V^K\right)$, and set the notation

$$V_\ell := V - \bigcup_{i=1}^{\ell-1} V^{H_i}$$
$$\widetilde{V}_\ell := V^{H_\ell} - \bigcup_{i=1}^{\ell-1} V^{H_i} \cap V^{H_\ell}$$

so that $V_1 = V \simeq *$ and $V_{n+1} = V_{(K)}$. Furthermore, note that $V_2 = V - V^{H_1} \simeq V^{H_1} \times S\left(V/V^{H_1}\right) \simeq S^{c-1}$; in particular, its reduced homology is

$$\widetilde{H}_m(V_2) = \begin{cases} \mathbb{Z} & n = c-1; \\ 0 & \text{otherwise.} \end{cases}$$

We argue via induction on ℓ that V_ℓ $\widetilde{H}_m(V_{\ell-1}) = 0$ when $m < c-1$ and that $\widetilde{H}_{c-1}(V_\ell)$ is nontrivial. The end of this induction implies the proposition; indeed, if $c-1 = 0$ then this directly implies that $V_{n+1} = V_{(H)}$ has at least two path components, and if $c-1 > 0$, then Hurewicz' theorem will imply that

$$\pi_{c-1}(V_{(K)})_{\mathbf{Ab}} = \pi_{c-1}(V_{n+1})_{\mathbf{Ab}} \simeq \widetilde{H}_{c-1}(V_{n+1}) \neq 0.$$

The base case $\ell = 2$ is satisfied by the above computation of $\widetilde{H}_m(V_2)$, so we inductively assume the statement is true for $\ell - 1$. Write $c_\ell := \mathrm{codim}\left(V^{H_\ell} \hookrightarrow V\right)$. Note that the normal bundle of $V^{H_\ell} \subset V$ is a trivial D^{c_ℓ}-bundle; this restricts to the (trivial) normal bundle of $\widetilde{V}_{\ell-1} \subset V_{\ell-1}$, so the bounding $S^{c_\ell - 1}$ sphere bundle $\partial\tau\widetilde{V}_{\ell-1} \to V_\ell$ is trivial. Thus "hemmed gluing" presents a homotopy pushout square

$$\begin{tikzcd} S^{c_{\ell-1}} \times \widetilde{V}_{\ell-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V_\ell \arrow[d] \\ \widetilde{V}_{\ell-1} \arrow[r] & V_{\ell-1} \end{tikzcd}$$

If $c_\ell > c$, the left vertical arrow (hence the right vertical arrow) is a homology isomorphism in degrees $\leq c-1$, proving the inductive step. Furthermore, if $c_\ell = c$, then the vertical arrows are homology isomorphisms in degrees $\leq c-2$ and the associated map $\widetilde{H}_c(S^{c-1} \times \widetilde{V}_{\ell-1}) \to \widetilde{H}_c(\widetilde{V}_{\ell-1})$ is an isomorphism. This implies that $H_m(\widetilde{V}_\ell) = 0$ when $m < c-1$ and the Mayer-Vietoris sequence restricts to a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] \arrow[dr] & \mathbb{Z} \oplus \widetilde{H}_{c-1}\left(\widetilde{V}_{\ell-1}\right) \arrow[r] \arrow[d, phantom, "\wr"] & \widetilde{H}_{c-1}\left(\widetilde{V}_{\ell-1}\right) \oplus \widetilde{H}_{c-1}\left(V_\ell\right) \arrow[r] & \widetilde{H}_{c-1}\left(V_{\ell-1}\right) \arrow[r] & 0, \\
& \widetilde{H}_{c-1}\left(S^{c-1} \times \widetilde{V}_{\ell-1}\right) \arrow[ur] & & &
\end{tikzcd}$$

so that $\widetilde{H}_{c-1}(V_\ell) \neq 0$, as desired. □

To state a corollary, we define the weak indexing system

$$\mathbb{F}_{AV} = \left\{ S \in \mathbb{F}_H \mid \mathrm{Conf}_S^H(V) \neq \varnothing \right\}.$$

as in [Ste24; Ste25d]. Our main algebraic corollary uses this to marry real representation theory, algebraic topology, homotopy theory, equivariant higher category theory, and equivariant higher algebra.

Theorem A'. *Let G be a finite group and V a real orthogonal G-representation. Then, the following conditions are equivalent:*

(a) V has d-codimensional fixed points.
(b) For all subgroups $H \subset G$ and finite H-sets S, the space $\mathrm{Conf}_S^H(V)$ is empty or $(d-2)$-connected.
(c) The G-operad $\mathbb{E}_V^\otimes$ is $(d-2)$-connected.[1]
(d) The forgetful functor

$$U : \mathrm{CMon}_{AV}(\mathcal{S}) \to \mathrm{Mon}_{\mathbb{E}_V}(\mathcal{S})$$

is an equivalence of $(d-1)$-categories.
(e) For all G-symmetric monoidal $(d-1)$-categories, the forgetful functor

$$U : \mathrm{CAlg}_{AV}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_V}(\mathcal{C})$$

is an equivalence of $(d-1)$-categories.

Proof. The equivalence (a) $\iff$ (b) is Corollary 22 and Proposition 23. By [Hor19], the structure spaces $\mathbb{E}_V(S)$ is $\mathrm{Conf}_S^H(V)$, so (b) $\iff$ (c) by definition. The equivalences (c) $\iff$ (d) $\iff$ (e) are recorded in [Ste25d]. □

In particular, note that $\left|k \cdot V^H\right| = k\left|V^H\right|$ and $\left|k \cdot V^K / k \cdot V^H\right| = k \cdot \left|V^K / V^H\right|$; hence if V has d-codmensional fixed points, kV has kd-codimensional fixed points. All representations have 1-codimensional fixed points, so dV has d-codimensional fixed points; hence Theorem A' specializes to Theorem A.

Remark 24. Theorem A' is significantly stronger than Theorem A; indeed, we may choose $G = C_p$, fix a generator $x \in C_p$, and let λ_i denote the irreducible 2-dimensional real orthogonal C_p-representation on whom x acts by rotation at an angle of $\frac{2\pi i}{p}$. Then, when $d \leq p/2$, the (nontrivial) representation $V = d \oplus \bigoplus_{1=i}^d \lambda_i$ has d-codimensional fixed points, but it contains only one copy of each of its nontrivial summands, so it can't be expressed as a direct sum of two copies of a nontrivial representation. ◁

Nevertheless, we specialize the following corollaries to dV for readability. The first yields a natural AV-Tambara structure on the 0th homotopy groups of an $\mathbb{E}_{2V}$-ring spectrum, and it follows from Theorem A in combination with [CHLL24, Thm 4.3.6].[2]

Corollary 25. *If V is an orthogonal G-representation, then there is a factorization*

$$\begin{tikzcd}
\mathrm{Alg}_{\mathbb{E}_{2V}}(\underline{\mathrm{Sp}}_G) \arrow[r, dashed] \arrow[d, "U"'] & \mathrm{CAlg}_{AV}\underline{\mathrm{Mack}}_G(\mathbf{Ab}) \arrow[r, phantom, "\simeq"] & \mathrm{Tamb}_{AV}(\mathrm{Set}) \arrow[d, "U"] \\
\mathrm{Sp}_G \arrow[rr, "\underline{\pi}_0"'] & & \mathrm{Mack}_G(\mathbf{Ab})
\end{tikzcd}$$

Finally, we acquire incomplete Mackey structures on $\mathbb{E}_{(n+2)V}$-monoidal n-categories.

[1] Recall from [Ste25d] that a G-operad $\mathcal{O}^\otimes$ is $(d-2)$-*connected* if its nonempty structure spaces $\mathcal{O}(S)$ are $(d-2)$-connected.
[2] In the case $V^G \neq 0$, AV is an indexing category, so we could simply reference the earlier work of [Cha24; San23].

Corollary 26. *$\mathbb{E}_{(n+2)V}$-monoidal n-categories canonically lift to AV-symmetric monoidal n categories, i.e.*

$$U : \mathrm{Cat}^{\otimes}_{AV,n} \to \mathrm{Cat}^{\otimes}_{\mathbb{E}_{(n+2)V},n}$$

is an equivalence of $(n+1)$-categories. In particular, when $V = \rho$, the forgetful functor

$$U : \mathrm{Cat}^{\otimes}_{G,n} \to \mathrm{Cat}^{\otimes}_{\mathbb{E}_{(n+2)\rho},n}$$

is an equivalence of 2-categories.

3. Index-local criteria for connectivity

In [Ste25b] we characterized not just connectivity of G-operads, but connectivity *at a weak indexing system*. We will completely characterize this for $\mathbb{E}_V^{\otimes}$; first, an index-by-index criterion.

Proposition 27. *Conditions (ii) and (iii) of Theorem B are equivalent.*

To begin, we may replace V with $\mathrm{Res}^G_H V$ in order to assume $H = G$. We introduce a middle step.

Lemma 28. $\mathrm{Conf}^G_S(V)$ *is $(d-2)$-connected if and only if:*

(a) If $[G/H] \in \mathrm{Orb}(S)$ then $V_{(H)}$ is $(d-2)$-connected, and

(b) if $2 \cdot *_G \subset S$, then $\dim V^G \geq d$.

Proof. The backwards direction is Theorem 6. If (2) is not satisfied, then the splitting

$$\mathrm{Conf}^G_S(V) \simeq \mathrm{Conf}^G_{S^G}(V) \times \mathrm{Conf}^G_{S-S^G}(V)$$

reduces this to verifying that $\mathrm{Conf}^G_{S^G}(V) \simeq \mathrm{Conf}^G_{|S^G|}\left(\mathbb{R}^{\dim V^G}\right)$ is not $(d-2)$-connected when $\dim V^G < d$ and $\left|S^G\right| > 1$, which is well-known. For the remaining case, assume (1) is not satisfied. Given X a space, let

$$\mathrm{Conn}(X) := \min\{n \in \mathbb{Z}_{\geq -1} \mid X \text{ is } n\text{-connected}\}$$

Let $c := \min_{[G/H]\in\mathrm{Orb}(S)} \mathrm{Conn}(V_{(H)})$ and chose an order $([G/H_1], \dots, [G/H_n]) = \mathrm{Orb}(S)$ so that $\mathrm{Conn}(V_{(H_1)}) = c$. We will verify inductively that c is the minimal index m at which $\pi_m \mathrm{Conf}^G_{\coprod_{i\leq \ell}[G/H_i]}(V) \neq 0$, with the case $\ell = 1$ satisfied by assumption.

The fact that $\pi_m \mathrm{Conf}^G_{\coprod_{i\leq\ell}[G/H_i]}(V) \simeq 0$ when $m < c$ follows from an identical argument to Theorem 6. Furthermore, the sequence of Theorem 3 ends as

$$\cdots \longrightarrow \pi_c \mathrm{Conf}^G_{[G/H_\ell]}(V) \longrightarrow \pi_c \mathrm{Conf}^G_{\sqcup_{i\leq \ell}[G/H_i]}(V) \longrightarrow \mathrm{Conf}^G_{\sqcup_{i\leq\ell-1}[G/H_i]}(V) \to 0;$$

thus $\mathrm{Conf}^G_{\sqcup_{i\leq \ell-1}[G/H_i]}(V)$ implies that $\mathrm{Conf}^G_{\sqcup_{i\leq\ell}[G/H_i]}(V)$ is nonzero. ∎

Proof of Proposition 27. We are left with verifying that condition (a) of Proposition 27 and Lemma 28 are equivalent in the case $G = H$; the forward direction is Remark 21 and the backwards direction is Proposition 23. ∎

Remark 29. Since codimension is additive under composition, it suffices to verify (a) in the case that there exist no intermediate subgroup inclusions $K \subset J' \subset J$; equivalently, we may pick elements $x \in H - K$ and verify the codimension condition for $K \subset \langle K, x\rangle$. ◁

We may use this to *spread out* connectivity across weak indexing systems. First, the Wirthmüller connectivity side.

Lemma 30. *Let $\mathcal{O}^{\otimes}$ be a unital G-operad. Then, the collection of arities*

$$\underline{\mathbb{F}}^{\mathrm{n-conn}}_{\mathcal{O}} := \left\{ S \mid \forall X \in \mathrm{Alg}_{\mathbb{E}_{\mathrm{Res}^G_H V}}(\underline{\mathcal{S}}_H, W_{X,S} \colon X^{\sqcup S} \to X^{\times S} \text{ is } n\text{-connected} \right\}$$

is a unital weak indexing system.

Proof. By [Ste25a], this is equivalent to the semiadditive locus of $\underline{\mathrm{Alg}}_{\mathbb{E}_V}(\underline{\mathcal{S}}_G)$, which is a unital weak indexing system by [Ste25b]. ∎

Now, we will show that the configuration connectivity side spreads out as well.

Proposition 31. *Let $\underline{\mathbb{F}}^{\mathrm{n-conn}(V)} \subset \underline{\mathbb{F}}_G$ be the collection containing each S such that $\mathrm{Conn}_S^H(V)$ is n-connected. Then, $\underline{\mathbb{F}}^{\mathrm{n-conn}(V)}$ is a unital weak indexing system.*

Before showing this, we will see how it proves Theorem B.

Proof of Theorem B. By Proposition 27, it suffices to show that $\underline{\mathbb{F}}^{\mathrm{n-conn}(V)} = \underline{\mathbb{F}}^{\mathrm{n-conn}}_{\mathbb{E}_V}$; in fact, Proposition 31 verifies that

$$\underline{\mathbb{F}}^{\mathrm{n-conn}(V)} = \left\{ S \mid \mathrm{Bor}_{I_S}^G \mathbb{E}_V^{\otimes} \text{ is } n\text{-connected} \right\},$$

and the identification of the right hand side with $\underline{\mathbb{F}}^{\mathrm{n-conn}}_{\mathbb{E}_V}$ is a straightforward application a specialization of a purely operad-theoretic result [Ste25a]. ◻

For the rest of this article, we verify Proposition 31 via a series of small claims.

Lemma 32. *Let V be an orthogonal G-representation.*

1. *For all H, $\varnothing_H \in \mathbb{F}_H^{\mathrm{n-conn}(V)}$.*
2. *The collection of elements of $\underline{\mathbb{F}}^{\mathrm{n-conn}(V)}$ with $n \in \{0\} \cup [2, \infty)$ fixed points is closed under coproducts.*
3. *If $S \in \mathbb{F}_K^{\mathrm{n-conn}(V)}$ and $[H/K] \in \mathbb{F}_H^{\mathrm{n-conn}(V)}$, then $\mathrm{Ind}_K^H S \in \mathbb{F}_K^{\mathrm{n-conn}(V)}$.*
4. *If $V_{(H)}$ and $V_{(K)}$ are $(n-2)$-connected, then $V_{(H\cap K)}$ is $(n-2)$-connected.*
5. *$\underline{\mathbb{F}}^{\mathrm{n-conn}(V)}$ is closed under restriction.*

Proof. (1) follows because $\mathrm{Conf}^H_{\varnothing_H}(V)$ is contractible; (2) follows by unwinding Proposition 27.

For (3), by (2), it suffices to prove the statement in the case that $S = [K/J]$. In this case, fix some $x \in H - J$, and note that it suffices to prove that $\mathrm{codim}\left(V^{\langle J,x\rangle} \hookrightarrow V^J\right) \geq n$. In fact, if $x \in K$, then this follows by Corollary 22 for $[K/J]$, so assume that $x \notin K$. Then, we get an intersection diagram

$$\begin{tikzcd} V^{\langle K,x\rangle} \arrow[r, hook, "b"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & V^K \arrow[d, hook] \\ V^{\langle J,x\rangle} \arrow[r, hook, "a"] & V^J \end{tikzcd} \tag{2}$$

where arrows are decorated with their codimension. By Corollary 22, it suffices to prove that $c \geq n$, and by linear algebra, $c \geq b$, so it suffices to prove that $b \geq n$. By assumption, V embeds $[H/K]$, so $b \geq 0$; since $[H/K] \in \mathbb{F}_H^{\mathrm{n-conn}(V)}$, we have $b \geq n$.

The proof of (4) is similar. Let $x \in G$ be an element not contained in $H \cap K$ satisfying the property that x does not stabilize $V^{H\cap K}$. Writing $J := H \cap K$, we're tasked with proving that

$$\mathrm{codim}(V^{\langle J,x\rangle} \hookrightarrow V^J) \geq n.$$

We may assume without loss of generality that $x \notin K$, so we once again form Eq. (2) and find that $a \geq n$. To prove (5), we use Lemma 28 and note that the orbit types $[H/J]$ appearing in $\mathrm{Res}_H^G S$ are precisely those isomorphic to $[H/H \cap gKg^{-1}]$ for $[G/K] \subset S$. ◻

Proof of Proposition 31. First note that $*_H, \varnothing_H \in \mathbb{F}_H^{\mathrm{n-conn}(V)}$ by Lemma 32, and $\mathbb{F}_H^{\mathrm{n-conn}(V)}$ is closed under restriction by Lemma 32, so it suffices to prove that $\underline{\mathbb{F}}^{\mathrm{n-conn}(V)}$ is closed under self-indexed coproducts. Fix some $S \in \mathbb{F}_H^{\mathrm{n-conn}(V)}$ and $(T_K) \in \mathbb{F}_S^{\mathrm{n-conn}(V)}$. We check the conditions of Proposition 27, starting with (b).

If $2 \cdot *_H \subset \coprod_K^S T_K$, then either $2 \cdot *_H \subset S$ or $S^H = *$ and $2 \cdot *_H \subset T_H$. In either case, Proposition 27 applied to either S or T_H confirms that $\dim V^H \geq d$, so (b) holds.

Second, fix some $[H/K] \in \mathrm{Orb}(\coprod_K^S T_K)$. Then, there is a factorization $K \subset J \subset H$ with $[H/J] \in \mathrm{Orb}(S)$ and $[J/K] \in \mathrm{Orb}(S)$, which follows from Lemma 28. ◻

Acknowledgements

The author would like to thank Ben Knudsen and Mike Hopkins for helpful conversations on this topic. In particular, Ben Knudsen made the author aware of the Theorem B approach to the Fadell-Neuwirth fibration, which greatly contributed to the simplicity of this article.

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