CONNECTIVITY OF EQUIVARIANT CONFIGURATION SPACES AND $\mathbb{E}_V$-ALGEBRAIC WIRTHMÜLLER MAPS

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ABSTRACT. We provide conditions on a locally smooth G-manifold under which its spaces of equivariant configurations $\mathrm{Conf}_S^G(X)$ are n-connected. This specializes to give simple sharp conditions under which the fibers of the (finite-index) $\mathbb{E}_V$-algebra Wirthmüller maps are n-connected.

Throughout this paper, we fix G a Lie group.

Definition 1. If $H \subset G$ is a closed subgroup, $S \in \mathbb{F}_H$ an H-set with finitely many orbits, and X a topological space with G-action, we denote the subspace of H-equivariant embeddings $S \hookrightarrow M$ by

$$\mathrm{Conf}_S^H(X) \subset \mathrm{Map}^H(S, X).$$

◀

Nonequivariantly, the homotopy type of configurations spaces in X is a rich source of homeomorphism-invariants of X [LS04]. In this paper, we study some rudiments of an equivariant lift of this in the smooth setting. Namely, in Section 1, we supply sufficient conditions for a smooth G-manifold M such that its nonempty configurations spaces $\mathrm{Conf}_S^G(M)$ are all d-connected.

We have a particular application in mind: the structure spaces of the *little V-disks operad* are configuration spaces in smooth G-manifolds, and connectivity statements of G-operads translate to structural statements about their algebras. In particular, we prove the following, where $\mathbb{E}_V$ is the *finite-index proper equivariant little V-disks G-operad* of [Ste25c].[1]

Theorem A. *Suppose $H \subset G$ is a compact closed subgroup, V an orthogonal G-representation, and $S \in \mathbb{F}_H$ a finite H-set admitting an embedding into $\mathrm{Res}_H^G V$. The following conditions of (H, S, V, n) are equivalent.*

  (i) Both of the following are satisfied:
    (a) If $\left|S^H\right| \geq 2$, then $\dim V^H \geq n + 2$, and
    (b) For all orbits $[H/K] \subset S$ and intermediate subgroups $K \subset J \subset H$, we have

$$\dim V^K \geq \dim V^J + n + 2;$$

  (ii) the space $\mathrm{Conf}_S^H(V)$ is n-connected; and
  (iii) the S-indexed $\mathbb{E}_V$-H-space Wirthmüller map

$$W_{S,(X_K)} : \coprod_K^S X_K \to \prod_K^S X_K$$

  is n-connected for all $(X_K) \in \prod_{[H/K] \subset S} \mathrm{Alg}_{\mathbb{E}_{\mathrm{Res}_K^G V}}(\underline{\mathcal{S}}_K)$.

Here, the $[H/K]$-*indexed Wirthmüller map* in a G-category is the comparison map $\mathrm{Ind}_K^H X \to \mathrm{CoInd}_K^H X$, which is adjunct to the map $X \to \mathrm{Res}_K^H \mathrm{CoInd}_K^H X \simeq \prod_{g \in [K \backslash H/K]} \mathrm{CoInd}_{gKg^{-1}}^H \mathrm{Res}_{gKg^{-1}}^H X$ which projects to the identity when $g = e$ and 0 otherwise. For instance, the *Wirthmüller isomorphism theorem* (for finite-index inclusions) states that this is an equivalence in the equivariant Spanier-Whitehead category [Wir75].

The S-indexed Wirthmüller map is a combination of this with the ordinary semiadditive norm map, and is centered in [CLL24; Nar16]; in a pointed G-category, writing $\coprod_{K_i}^H X_{K_i} := \coprod_{S = \coprod_i [H/K_i]} \mathrm{Ind}_{K_i}^H X_{K_i}$ and similar for

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[1] In the case that G is finite, this recovers all other notions of $\mathbb{E}_V$.

$\prod_{K_i}^H X_{KM_i}$, the S-Wirthmüller map is classified by the diagonal matrix

$$W_{X,S} = 1 \begin{bmatrix} W_{X_{K_1},[H/K_1]} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & W_{X_{K_\ell},[H/K_\ell]} \end{bmatrix}$$

We briefly mention an easy corollary to this in equivariant stable homotopy theory; define

$$\mathrm{Sp}_{G,[d,d+n]} := \left\{ X \in \mathrm{Sp}_G \ \middle| \ \forall \ell \notin [a,b], H \subset G, \ \ \pi_\ell^H(X) \simeq * \right\} \subset \mathrm{Sp}_G.$$

Corollary B. *Fix $0 \le n, d \le \infty$ and an orthogonal G-representation V such that, for all $K \subset H \subset G$, we have*

$$\dim V^G, \ \dim\left(V^K/V^H\right) \in \{0\} \cup [n,\infty].$$

Then, the forgetful functor

$$U \colon \mathrm{Alg}_{\mathbb{E}_{\infty V}}\left(\underline{\mathrm{Sp}}_{G,[d,d+n-2]}\right) \longrightarrow \mathrm{Alg}_{\mathbb{E}_V}\left(\underline{\mathrm{Sp}}_{G,[d,d+n-2]}\right)$$

is an equivalence.

1. Configuration spaces in smooth G-manifolds

Definition 2 ([Bre72, § IV]). If M is a smooth manifold with a continuous G-action, we say that the action is *locally smooth* if, for each point $x \in M$, there exists a real orthogonal $\mathrm{stab}_G(x)$-representation V_x and a trivializing open neighborhood

$$x \in \coprod_{G/\mathrm{stab}_G(x)} V_x \lhook\joinrel\longrightarrow M,$$

where for a topological H-space X, we write $\coprod_{G/\mathrm{stab}_G(x)} X := G \times_H X$ as a topological G-space. In this case, we say that M with its action is a *locally smooth G-manifold*. ◁

Smooth actions on manifolds admit well-behaved tubular neighborhoods; for example, [Bre72, Cor VI.2.4] proves that smooth actions are locally smooth. On the other hand, if M is a locally smooth G-manifold, then the inclusion $M_{(H)} \hookrightarrow M$ of points with orbit isomorphic to G/H is a locally closed topological submanifold [Bre72, Thm IV.3.3], which is smooth if M is smooth [Bre72, Cor VI.2.5].

We begin this section in Section 1.1 by proving the following.

Theorem 3 (Equivariant Fadell-Neuwirth fibration). *Fix M a locally smooth G-manifold, $S, T \in \mathbb{F}_G$ a pair of orbit-finite G-sets, and $\iota \colon S \hookrightarrow M$ a G-equivariant configuration. The following is a homotopy-Cartesian square:*

$$\begin{array}{ccc} \mathrm{Conf}_T^G(M - \iota(S)) & \longrightarrow & \mathrm{Conf}_{S \sqcup T}^G(M) \\ \downarrow & \lrcorner & \downarrow U \\ \{\iota\} & \longrightarrow & \mathrm{Conf}_S^G(M) \end{array}$$

Thus the long exact sequence in homotopy for $T = G/H$ yields means for computing homotopy groups of $\mathrm{Conf}_S^G(M)$ inductively on the cardinality of the orbit set $|S_G|$, with inductive step hinging on homotopy of

$$\mathrm{Conf}_{G/H}^G(M - \iota(S)) \simeq (M - \iota(S))_{(H)}.$$

Remark 4. In the case that T and S satisfy the condition that, for all $x \in T$ and $y \in S$, $\mathrm{stab}_G(x)$ and $\mathrm{stab}_G(y)$ are non-conjugate, U is actually a trivial fiber bundle [BQV23]. Furthermore, defining the *unordered*

configuration space $C_S^G(M) := \mathrm{Conf}_S^G(M)/\mathrm{Aut}_G(S)$, we acquire a splitting of principal bundles

$$
\begin{array}{ccccc}
\mathrm{Conf}_S^G(M) & \simeq & \prod_{(H)\subset G} \mathrm{Conf}_{S_{(H)}}^G(M) & \simeq & \prod_{(H)\subset G} \mathrm{Conf}_{S_{(H)}}^G\left(M_{(H)}\right) \\
\downarrow & & \downarrow & & \downarrow \\
C_S^G(M) & \simeq & \prod_{(H)\subset G} C_{S_{(H)}}^G(M) & \simeq & \prod_{(H)\subset G} C_{S_{(H)}/G}\left(M_{(H)}/G\right)
\end{array}
$$

Unfortunately, this does not obviate Theorem 3, as it appears to often be more convenient to determine invariants of $\mathrm{Conf}_{n\cdot[G/H]}^G(M)$ inductively using Theorem 3 than to determine those invariants on $C_{S_{(H)}/G}\left(M_{(H)}/G\right)$ and pull back along the defining principal $\mathrm{Aut}_G(n\cdot[G/H]) \simeq W_G H \wr \Sigma_n$-bundle. ◄

We denote by $[\mathcal{O}_G]^{\mathrm{op}}$ the subconjugacy lattice of closed subgroups of G, and we let

$$\mathrm{Istrp}(M) = \{\mathrm{stab}_x(G) \mid x \in M\} \subset [\mathcal{O}_G]^{\mathrm{op}}$$

be the full subposet spanned by conjugacy classes (H) for which $M_{(H)}$ is nonempty. We are inspired to make the following definition.

Definition 5. Fix a G-set S with finitely many orbits. A locally smooth G-manifold M is

- $\geq d$-*dimensional at the isotropy of* S if $M_{(H)}$ is $\geq d$-dimensional for each $[G/H] \subset S$;
- $(d-2)$-*connected at the isotropy of* S if $M_{(H)}$ is $(d-2)$-connected for each $[G/H] \subset S$. ◄

In Section 1.2, we observe that Theorem 3 implies the following.

Corollary 6. *Fix G a compact Lie group, M a locally smooth G-manifold, and S a G-set with finitely many orbits. If M is $\geq d$-dimensional and $(d-2)$-connected at the isotropy of S, then $\mathrm{Conf}_S^G(M)$ is $(d-2)$-connected.*

In order to identify applications of this theorem, we give sufficient conditions for M to be $(d-2)$-connected at each orbit type. Note by repeatedly applying [Bre72, Thm IV.3.1] that the subspace $M_{\supseteq(H)} \subset M$ consisting of points of orbit type $[G/K]$ a quotient of $[G/H]$ is a disjoint union of closed submanifolds. In Section 1.3, we use this to prove the following.

Proposition 7. *Suppose that M is a smooth G-manifold satisfying the following conditions:*

- *(a) M is $\geq d$-dimensional at the isotropy of S.*
- *(b) $M_{\supseteq(H)}$ is $(d-2)$-connected whenever $[G/H] \subset S$.*
- *(c) $\mathrm{codim}(M_{\supseteq(K)} \hookrightarrow M_{\supseteq(H)}) \geq d$ for each $[G/H] \subset S$ and $H \subset K$.*
- *(d) $\mathrm{Istrp}(M)$ is finite (e.g. G compact and M finite type, c.f. [Bre72, Thm IV.10.5]).*

Then M is $(d-2)$-connected at the isotropy of S.

1.1. A Fadell-Neuwirth fibration for equivariant configurations.

Our strategy for Theorem 3 mirrors that of Knudsen in the notes [Knu18]. In particular, we would like to use Quillen's theorem B [Qui73], which requires us to construct $\mathrm{Conf}_S^H(M)$ as a classifying space. In fact, there is a general scheme to do this:

Lemma 8 ([DI04, Thm 2.1], via [Knu18, Thm 4.0.2]). *If $\mathcal{B}$ is a topological basis for X such that all elements of $\mathcal{B}$ are weakly contractible, then the canonical map*

$$|\mathcal{B}| = \mathrm{hocolim}_{\mathcal{B}} * \to X$$

is a weak equivalence, where on the left $\mathcal{B}$ is considered as a poset under inclusion.

To use this, define an elementwise-contractible basis for $\mathrm{Conf}_S^G(M)$ by

$$\widetilde{\mathcal{B}}_S^G(M) := \left\{ (X,\sigma) \,\middle|\, \exists (V_x) \in \prod_{[x]\in\mathrm{Orb}_S} \mathbf{Rep}_{\mathbb{R}}^{\mathrm{orth}}(\mathrm{stab}_G([x])), \text{ s.t. } \coprod_{[x]}^S V_x \simeq X \subset M, \; \sigma : S \xrightarrow{\sim} \pi_0(U) \right\},$$

where for all tuples $(Y_x) \in \prod_{[x]\in\mathrm{Orb}_S} \mathrm{Top}_{\mathrm{stab}_G([x])}$, we write

$$\coprod_{[x]}^S Y_x := \coprod_{[x]\in\mathrm{Orb}(S)} \left(G \times_{\mathrm{stab}_G([x])} Y_x \right) \in \mathrm{Top}_G$$

for the *indexed disjoint union of Y_x*. We fix $\mathcal{B}_S^G(M) \subset \widetilde{\mathcal{B}}_S^G(M)$ the smaller basis consisting of open sets (X, σ) possessing neighborhoods $(X, \sigma) \subset (X', \sigma)$ such that the associated embeddings factor as

(1)

$$
\begin{array}{ccc}
\coprod\limits_U^S D(V_U)^\circ & \simeq & \coprod\limits_U^S V_U \\
{\scriptstyle \exists} \downarrow & & \downarrow {\scriptstyle X} \\
V_U' & \hookleftarrow X' \longrightarrow & M
\end{array}
$$

where $D(V_U)^\circ$ denotes the open unit V_U-disk; that is, open sets in $\mathcal{B}_S^G(M)$ consist of collections of configurations possessing a fixed common neighborhood resembling disjoint unions of real orthogonal representations, subject to the condition that there is "space on all sides" of the neighborhood. This is functorial in two ways:

- given a summand inclusion $S \hookrightarrow T \sqcup S$, the forgetful map $\mathrm{Conf}_{T \sqcup S}^G(M) \to \mathrm{Conf}_S^G(M)$ preserves basis elements, inducing a map $\mathcal{B}_{T \sqcup S}^G(M) \to \mathcal{B}_S^G(M)$.
- any open embedding $\iota : M \hookrightarrow N$ induces a map $\mathrm{Conf}_T^G(M) \hookrightarrow \mathrm{Conf}_T^G(N)$ preserving basis elements, inducing a map $\mathcal{B}_S^H(M) \to \mathcal{B}_S^H(N)$.

To summarize, we've observed the proof of following lemma.

Lemma 9. *Given $H \subset G$ and $S, T \in \mathbb{F}_H$, there is an equivalence of arrows*

$$
\begin{array}{ccc}
\left| \mathcal{B}_{T \sqcup S}^G(M) \right| & \simeq & \mathrm{Conf}_{T \sqcup S}^G(M) \\
\downarrow & & \downarrow \\
\left| \mathcal{B}_S^G(M) \right| & \simeq & \mathrm{Conf}_S^G(M)
\end{array}
$$

Thus we can characterize the homotopy fiber of U using Quillen's theorem B and the following.

Proposition 10. *For $(X_S, \sigma_S) \leq (X_S', \sigma_S') \in \mathcal{B}_S^G(M)$, and an S-configuration $\mathbf{x} \in X_S$, we have a diagram*

$$
\begin{array}{ccc}
\mathcal{B}_T^G(M - \mathbf{x}) & \xleftarrow{\quad \varphi \quad} \mathcal{B}_T^G(M - \overline{X}_S) & \longleftarrow \mathcal{B}_T^G(M - \overline{X}'_S) \\
& \wr \downarrow & \wr \downarrow \\
& ((X_S, \sigma_S) \downarrow U) & \longleftarrow \left((X_S', \sigma_S') \downarrow U \right)
\end{array}
$$

such that the maps φ induce weak equivalences on classifying spaces.

We will power this with the following observation:

Observation 11. Recall that an embedding of topological G-spaces $f : Y \hookrightarrow Z$ is a *G-isotopy equivalence* if there exists another G-equivariant embedding $g : Z \hookrightarrow Y$ and a pair of G-equivariant isotopies $gf \sim \mathrm{id}_Z$, $fg \sim \mathrm{id}_Y$. If $f : Y \to Z$ is a G-isotopy equivalence, then postcomposition with f induces a $G \times \Sigma_n$-isotopy equivalence $\mathrm{Conf}_n(Y) \hookrightarrow \mathrm{Conf}_n(Z)$; indeed, postcomposition with f and g induce G-equivariant embeddings, and postcomposition with the isotopies $gf \sim \mathrm{id}_Z$, $fg \sim \mathrm{id}_Y$ yields equivariant isotopies $\mathrm{Conf}_n(g) \circ \mathrm{Conf}_n(f) \sim \mathrm{Conf}_n(gf) \sim \mathrm{Conf}_n(\mathrm{id}_Z) \sim \mathrm{id}_{\mathrm{Conf}_n(Z)}$ and similar for fg.

In particular, the vertical arrows in the following diagram are isotopy equivalences

$$
\begin{array}{ccccc}
\mathrm{Conf}_S^H(X) & \simeq & \mathrm{Conf}_{|S|}(X)^{\Gamma_S} & \simeq & \mathrm{Map}^G\left(G \times \Sigma_{|S|}/\Gamma_S, \mathrm{Conf}_{|S|}(X) \right) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Conf}_S^H(Y) & \simeq & \mathrm{Conf}_{|S|}(Y)^{\Gamma_S} & \simeq & \mathrm{Map}^G\left(G \times \Sigma_{|S|}/\Gamma_S, \mathrm{Conf}_{|S|}(Y) \right)
\end{array}
$$

where $\Gamma_S = \{(h, \rho_S(h)) \mid h \in H\} \subset G \times \Sigma_{|S|}$ is the graph subgroup corresponding with an H-set S with action map $\rho_S : H \to \Sigma_{|S|}$. Hence f induces a homotopy equivalence $\mathrm{Conf}_S^H(X) \xrightarrow{\sim} \mathrm{Conf}_S^H(Y)$.   ◂

Proof of Proposition 10. The maps φ are each induced by the open inclusions $M - \overline{X}_S \hookrightarrow M - \mathbf{x}$, so the top horizontal arrows commute. The equivalences $\mathcal{B}_T^G(M - \overline{X_S}) \simeq ((X_S, \sigma_S) \downarrow U)$ simply follow by unwinding definitions. Thus we're left with proving that φ induces an equivalence on classifying spaces

$$\begin{array}{ccc} \mathrm{Conf}_T^G(M - \mathbf{x}) & \longleftarrow & \mathrm{Conf}_T^G(M - \overline{X}_s) \\ \wr\wr & & \wr\wr \\ \left|\mathcal{B}_T^G(M - \mathbf{x})\right| & \longleftarrow & \left|\mathcal{B}_T^G(M - \overline{X_s})\right| \end{array}$$

By Observation 11, it suffices to show that $M - \overline{X_S} \hookrightarrow M - \mathbf{x}$ is a G-isotopy equivalence. In fact, by Eq. (1), it suffices to prove that the inclusion $f : V - D(V) \hookrightarrow V - \{0\}$ is a G-isotopy equivalence. But this is easy; scaling is equivariant, so we may define the G-equivariant embedding $g : V - \{0\} \to V - D(V)$ by $g(x) = \frac{1+|x|}{|x|} \cdot x$. Then, each of the equivariant isotopies $gf \sim \mathrm{id}, fg \sim \mathrm{id}$ can be taken as restrictions of $h(t, x) = \frac{1-t+|x|}{|x|} \cdot x$. $\qquad\square$

We are ready to conclude our equivariant homotopical lift of [FN62, Thm 1].

Proof of Theorem 3. By the above analysis, we may replace our diagram with a homotopy equivalent diagram given by the geometric realiztion of the following diagram of posets, and prove that it is homotopy Cartesian

$$\begin{array}{ccc} \mathcal{B}_T^G(M - \iota(S)) & \longrightarrow & \mathcal{B}_{T \sqcup S}^G(M) \\ \downarrow & & \downarrow \\ \{\iota\} & \longhookrightarrow & \mathcal{B}_S^G(M) \end{array}$$

By Quillen's theorem B [Qui73, Thm B], it suffices to prove two statements:

- for all basis elements (X_S, σ_S), The canonical map $((X_s, \sigma_s) \downarrow U) \to \mathcal{B}_T^G(M - \iota(S))$ induces a weak equivalence on classifying spaces, and
- for all inclusions of basis elements $(X_S, \sigma_S) \subset (X_S', \sigma_S')$, the canonical map $\left((X_S', \sigma_S') \downarrow U\right) \to ((X_S, \sigma_S) \downarrow U)$ induces a weak equivalence on classifying spaces.

In fact, both statements follow immediately from Proposition 10, with the second using two-out-of-three. $\quad\square$

1.2. **Connectivity at various orbits.** To prove Corollary 6, we begin with a lemma.

Lemma 12. *If M is a locally smooth G-manifold which is at least d-dimensional and $(d-2)$-connected at the isotropy of S and $\iota : [G/H] \hookrightarrow M$ a configuration, then $M - \iota(G/H)$ is at least d-dimensional and $(d-2)$-connected at the isotropy of S.*

Proof. We have

$$(M - \iota(G/H))_{(K)} = \begin{cases} M_{(K)} & G/K \neq G/H \\ M_{(H)} - \iota(G/H) & G/K = G/h, \end{cases}$$

so the only nontrivial case is $H = K$, in which case we're tasked with verifying that the complement of a discrete set of points in a d-dimensional $(d-2)$-connected manifold is $(d-2)$-connected. This is a well known fact in algebraic topology which follows quickly from the Blakers-Massey theorem. $\qquad\square$

Proof of Corollary 6. If $d - 2 < 0$, there is nothing to prove, so assume that $d - 2 \geq 0$. We induct on $|S_G|$ with base case 1, i.e. with $S = G/H$. In this case, $\mathrm{Conf}_{G/H}^G(M) = M_{(H)}$ is $(d-2)$-connected by assumption.

For induction, fix some $S \sqcup G/H \in \mathbb{F}_G$ and inductively assume the theorem when $|T_G| \leq |S_G|$. Then, note that $\mathrm{Conf}_S^G(M)$ is $(d-2)$-connected by assumption and $\mathrm{Conf}_{G/H}^G(M - \iota(S))$ $(d-2)$-connected by the inductive hypothesis combined with Lemma 12. Thus Theorem 3 expresses $\mathrm{Conf}_{S \sqcup G/H}^G(M)$ as the total space of a homotopy fiber sequence with connected base and fiber, so it is connected. Furthermore, examining the long exact sequence associated with Theorem 3, we find that

$$\begin{array}{ccccc} 0 & \longrightarrow & \pi_k \mathrm{Conf}_{S \sqcup G/H}^G(M) & \longrightarrow & 0 \\ \wr\wr & \nearrow & & \searrow & \wr\wr \\ \pi_k \mathrm{Conf}_{G/H}^G(M - \iota(S)) & & & & \pi_k \mathrm{Conf}_S^G(M) \end{array}$$

is exact for $0 < k \leq d - 2$; hence $\mathrm{Conf}_{S \sqcup G/H}^G(M)$ is $(d-2)$-connected, completing the induction. $\qquad\square$

1.3. **Some sufficient conditions for connectivity at various orbits.** We begin with the following observations:

Observation 13. If M satisfies the conditions of Proposition 7, then $M_{\supseteq(H)}$ does as well.   ◄

Observation 14. If M satisfies the conditions of Proposition 7 for $d - 2 \geq 0$, then in particular, the orbit space M_G are a union of path-connected topological spaces $\left(M_{\supseteq(H)}\right)_G$ along path-connected intersections, so M_G is path-connected. In particular, by [Bre67, Thm IV.3.1], M has a *principle orbit type*, i.e. a minimal element of $(H_{\min}) \in \mathrm{Istrp}(M)$. Hence we have underlying space

$$M^e = M_{(H_{\min})},$$

so the underlying space M^e is $\geq d$-dimensional and $(d-2)$-connected.   ◄

We will strengthen Proposition 7. Pick an order on $\mathrm{Istrp}(M) = (G/H_1, \ldots, G/H_n, G/G)$, and write

$$M_k = M - \bigcup_{i<k} M_{\supseteq(H_i)}$$

$$\widetilde{M}_k = M_{\supseteq(H_k)} - \bigcup_{i<k} M_{\supseteq(H_k)} \cap M_{\supseteq(H_i)}$$

$$= M_{\supseteq(H_k)} - \bigcup_{\substack{(K)\supseteq(H_k H_i) \\ i<k}} M_{\supseteq(K)}$$

Lemma 15. *For all k, the space M_k is $(d-2)$-connected.*

Proof. We induct in two ways:

- First, we inductively assume we have proved the lemma at full strength when G is replaced with any proper subgroup $H \subsetneq G$ such that $[G/H] \in \mathrm{Istrp}(M)$; since $\mathrm{Istrp}(M)$ is finite, this begins with the base case in which case there are no such proper subgroups.
- Second, we inductively assume that we have proved the lemma for all $k' < k$; this begins with the base case that $k = 1$, in which case we have $M_1 = M$, which is $(d-2)$-connected by Observation 14.

Under these assumptions, note that $\widetilde{M}_{k-1} \subset M_{k-1}$ is a $(d-2)$-connected closed submanifold of codimension $\geq d$ in a $(d-2)$-connected smooth manifold with complement is M_k. Thus it possesses a tubular neighborhood $\widetilde{M}_{k-1} \subset \tau(\widetilde{M}_{k-1}) \subset M_{k-1}$, and "hemmed gluing" presents a homotopy pushout square

$$\begin{array}{ccc} \partial\tau\widetilde{M}_{k-1} & \longrightarrow & M_k \\ \downarrow{\scriptstyle \tilde\iota} & \ulcorner & \downarrow{\scriptstyle \iota} \\ \widetilde{M}_{k-1} & \longrightarrow & M_{k-1} \end{array}$$

The boundary $\partial\tau\left(\tilde{M}_{k-1}\right)$ is the total space of a c-sphere bundle over a $(d-2)$-connected space, where

$$c = \mathrm{codim}(M_{\leq(H_k)} \hookrightarrow M) - 1 > d - 2.$$

The long exact sequence in homotopy reads

$$\pi_1(S^c) \longrightarrow \pi_1\left(\partial\tau\left(\widetilde{M}_{k-1}\right)\right) \longrightarrow \pi_1\left(\widetilde{M}_{k-1}\right) \longrightarrow 0 \longrightarrow \pi_0\left(\partial\tau\left(\widetilde{M}_{k-1}\right)\right) \longrightarrow \pi_0\left(\widetilde{M}_{k-1}\right) \longrightarrow 0$$
$$\searrow \quad \underset{\pi_0(S^c)}{\wr} \quad \nearrow$$

so $\partial\tau\widetilde{M}_{k-1}$ is connected, and when $d-2 \geq 1$, $\partial\tau\widetilde{M}_{k-1}$ is simply connected. Furthermore, at degree $0 < \ell \leq (d-2)$ the Gysin sequence in integral homology reads

$$\begin{array}{ccccc} 0 & \longrightarrow & H^\ell(\partial\tau\widetilde{M}_{k-1}) & \longrightarrow & 0 \\ {\scriptstyle \wr} & \nearrow & & \searrow & {\scriptstyle \wr} \\ H^\ell(\widetilde{M}_{k-1}) & & & & H^{\ell-c}(\widetilde{M}_{k-1}) \end{array}$$

so $\partial\tau\tilde{M}_{k-1}$ has vanishing cohomology in degrees $0 < \ell \leq d - 2$. Hurewicz' theorem then implies that $\partial\widetilde{M}_{k-1}$ is $(d-2)$-connected.

In particular, this together with $(d-3)$-connectivity of the homotopy fiber S^c implies that $\widetilde{\iota}$ is a $(d-2)$-connected map, so its homotopy pushout ι is $(d-2)$-connected. Since M_{k-1} is a $(d-2)$-connected space by assumption, this implies that M_k is $(d-2)$-connected, completing the induction. $\qquad\square$

We conclude the following equivalent statement of Proposition 7.

Proposition 16. *Suppose that $M_{(\supseteq H)}$ is $\geq d$-dimensional, $\mathrm{Istrp}(M)$ is finite, and for all inclusions $(H) \subset (K)$ in $\mathrm{Istrp}(M)$, the following conditions are satisfied:*

(b) $M_{\supseteq(K)}$ is $(d-2)$-connected, and

(c) $\mathrm{codim}\left(M_{\supseteq(K)} \hookrightarrow M_{\supseteq(H)}\right) \geq d$.

Then, $M_{(H)}$ is $(d-2)$-connected.

Proof. By Observation 13 we may assume $H = G$. This is precisely what is shown in Lemma 15 when $k = n+1$. $\qquad\square$

Warning 17. Neither the conditions of Proposition 7 or of Corollary 6 are stable under restrictions for general G-manifolds; for instance, let $G = C_2 \times C_2$ be the Klein 4 group, and $H, H' \subset G$ a pair of distinct order-2 subgroups. Write σ, σ' for the inflated orthogonal G-representations from the sign representation of H and H'. Then for any $n \in \mathbb{N}$, define the $(2n-2)$-dimensional smooth G-manifold

$$M := (n-2) \cdot \sigma \oplus n \sqcup S\left((2n-1) \cdot \sigma'\right),$$

Note that M satisfies Proposition 7 for n, but its restriction to H is not $(n-2)$-connected at each orbit type when $n - 2 \geq 0$, as $\left(\mathrm{Res}_H^G M\right)_{(e)} \simeq S^{2n-1} \sqcup D^n$ is not connected. $\qquad\triangleleft$

2. REPRESENTATIONS, HOMOTOPY-COHERENT ALGEBRA, AND CONFIGURATION SPACES

2.1. Connectivity of $\mathrm{Conf}_S^H(V)$.

We begin by verifying that Conditions (i) and (ii) are equivalent.

Proposition 18. *If a real orthogonal G-representation V satisfies the conditions of Theorem A, then the smooth G-manifold $V - \{0\}$ is at least d-dimensional and $(d-2)$-connected at the isotropy of S.*

Proof. We may write V as a filtered (homotopy) colimit $V = \bigcup_i V_i$ with V_i a finite dimensional real orthogonal G-representation with $\min(i, d)$-codimensional fixed points; then, if V_i is $(i-2)$-connected for each i, taking a colimit, this implies that V is d-connected. Hence it suffices to prove this in the case we that V is finite dimensional.

In this case, G acts smoothly on V, and we make the following observations:

(a) $(V - \{0\})_{(H)} = \bigcup_{H' \in (H)} \left(V^{H'} - \bigcup_{H' \subset K} V^K\right) - \{0\}$ is either empty or $\max_{H' \in (H)} \left|V^{H'}\right| \geq d$-dimensional.

(b) $V_{\leq(H)} = \bigcup_{H' \in (H)} V^H$ is a union of contractible spaces along contractible intersections, so it is contractible and $\geq d$-dimensional; by the same argument as Lemma 12, $(V - \{0\})_{(H)} = V_{(H)} - \{0\}$ is $(d-2)$-connected.

(c) when $(K), (H) \in \mathrm{Istrp}(M)$,

$$\begin{aligned}
\mathrm{codim}((V - \{0\})_{\leq(K)} \hookrightarrow (V - \{0\})_{\leq(H)}) &= \mathrm{codim}(V_{\leq(K)} \hookrightarrow V_{\leq(H)}) \\
&= \min_{\substack{H' \subset K \\ V^{H'} \neq V^K}} \left(\left|V^{H'}\right| - \left|V^K\right|\right) \\
&= \min_{\substack{H' \subset K \\ V^{H'} \neq V^K}} \left|V^H / V^K\right| \\
&\geq d
\end{aligned}$$

by assumption, and it is nonzero since $V_{(H)}$ is nonempty.

(d) $\mathrm{Istrp}(V)$ is finite since V is finite dimensional.

Thus Proposition 7 applies, proving the proposition. $\qquad\square$

Proposition 19. *If a finite-index inclusion of subgroups $K \subset H$ has $V^H \hookrightarrow V^K$ a proper inclusion of codimension $< d$, then $\mathrm{Conf}_{[G/K]}^G(V) \simeq V_{(K)}$ is not $(d-2)$-connected.*

Proof. This never occurs when V is 0-dimensional. If V^G is $0 < c < d$-dimensional, then we may directly see $\mathrm{Conf}^G_{2 \cdot \ast_G}(V) = \mathrm{Conf}_2(V^G) = S^{c-1}$ is not $(d-2)$-connected, as it has nontrivial π_{c-1}. Thus we assume that V^G is $\geq d$-dimensional, so that V^H is $\geq d$-dim for all H.

Fix $c := \min_{K \subsetneq H' \in \mathrm{Istrp}(V)} \mathrm{codim}\left(V^{H'} \hookrightarrow V^K\right)$. We may replace V with the real orthogonal G-representation $V^K = V_{(\geq K)}$. We're left with proving that $V_{(K)} = V - \bigcup_{K \subsetneq H' \in \mathrm{Istrp}(V)} V^{H'}$ is not $(d-2)$-connected. Pick an order $(H_i)_{1 \leq i \leq n}$ on $\mathrm{Istrp}(V) - \{(K)\}$ so that $c = \mathrm{codim}\left(V^{H'} \hookrightarrow V^K\right)$, and set the notation

$$V_\ell := V - \bigcup_{i=1}^{\ell-1} V^{H_i}$$

$$\widetilde{V}_\ell := V^{H_\ell} - \bigcup_{i=1}^{\ell-1} V^{H_i} \cap V^{H_\ell}$$

so that $V_1 = V \simeq \ast$ and $V_{n+1} = V_{(K)}$. Furthermore, note that $V_2 = V - V^{H_1} \simeq V^{H_1} \times S\left(V/V^{H_1}\right) \simeq S^{c-1}$; in particular, its reduced homology is

$$\widetilde{H}_m(V_2) = \begin{cases} \mathbb{Z} & n = c-1; \\ 0 & \text{otherwise.} \end{cases}$$

We argue via induction on ℓ that V_ℓ $\widetilde{H}_m(V_{\ell-1}) = 0$ when $m < c-1$ and that $\widetilde{H}_{c-1}(V_\ell)$ is nontrivial. The end of this induction implies the proposition; indeed, if $c - 1 = 0$ then this directly implies that $V_{n+1} = V_{(H)}$ has at least two path components, and if $c - 1 > 0$, then Hurewicz' theorem will imply that

$$\pi_{c-1}(V_{(K)})_{\mathbf{Ab}} = \pi_{c-1}(V_{n+1})_{\mathbf{Ab}} \simeq \widetilde{H}_{c-1}(V_{n+1}) \neq 0.$$

The base case $\ell = 2$ is satisfied by the above computation of $\widetilde{H}_m(V_2)$, so we inductively assume the statement is true for $\ell - 1$. Write $c_\ell := \mathrm{codim}\left(V^{H_\ell} \hookrightarrow V\right)$. Note that the normal bundle of $V^{H_\ell} \subset V$ is a trivial D^{c_ℓ}-bundle; this restricts to the (trivial) normal bundle of $\widetilde{V}_{\ell-1} \subset V_{\ell-1}$, so the bounding $S^{c_\ell-1}$ sphere bundle $\partial \tau \widetilde{V}_{\ell-1} \to V_\ell$ is trivial. Thus "hemmed gluing" presents a homotopy pushout square

$$\begin{array}{ccc} S^{c_\ell-1} \times \widetilde{V}_{\ell-1} & \longrightarrow & V_\ell \\ \downarrow & \lrcorner & \downarrow \\ \widetilde{V}_{\ell-1} & \longrightarrow & V_{\ell-1} \end{array}$$

If $c_\ell > c$, the left vertical arrow (hence the right vertical arrow) is a homology isomorphism in degrees $\leq c-1$, proving the inductive step. Furthermore, if $c_\ell = c$, then the vertical arrows are homology isomorphisms in degrees $\leq c-2$ and the associated map $\widetilde{H}_c(S^{c-1} \times \widetilde{V}_{\ell-1}) \to \widetilde{H}_c(\widetilde{V}_{\ell-1})$ is an isomorphism. This implies that $H_m(\widetilde{V}_\ell) = 0$ when $m < c-1$ and the Mayer-Vietoris sequence restricts to a short exact sequence

$$0 \longrightarrow \mathbb{Z} \oplus \widetilde{H}_{c-1}\left(\widetilde{V}_{\ell-1}\right) \longrightarrow \widetilde{H}_{c-1}\left(\widetilde{V}_{\ell-1}\right) \oplus \widetilde{H}_{c-1}(V_\ell) \longrightarrow \widetilde{H}_{c-1}(V_{\ell-1}) \longrightarrow 0,$$
$$\searrow \quad \underset{\widetilde{H}_{c-1}\left(S^{c-1} \times \widetilde{V}_{\ell-1}\right)}{\overset{\shortparallel}{}} \quad \nearrow$$

so that $\widetilde{H}_{c-1}(V_\ell) \neq 0$, as desired. $\qquad \square$

Corollary 20. *Conditions (i) and (ii) are equivalent.*

Proof. We begin by noting

$$\mathrm{Conf}^H_S(V) = \begin{cases} \mathrm{Conf}^H_{S-\ast_H}(\mathrm{Res}^G_H(V - \{0\})) & S^H \neq \varnothing, \\ \mathrm{Conf}^H_S(\mathrm{Res}^G_H(V - \{0\})) & \text{otherwise.} \end{cases}$$

so it suffices to show $\mathrm{Conf}^H_S(\mathrm{Res}^G_H(V - \{0\}))$ to be $(d-2)$-connected or empty. Noting that the condition of having d-codmimensional fixed points is restriction-stable, this follows by Corollary 6 and Proposition 18. $\quad \square$

2.2. **Connectivity of Wirthmüller maps.** The following result of [Ste25a] is central.

Proposition 21 ([Ste25a])**.** *Let I be a unital weak indexing system. Then, the following conditions are equivalent:*

(a) *for all $S \in \underline{\mathbb{F}}_I$, the space $\mathcal{O}(S)$ is n-connected.*
(b) *for all $S \in \underline{\mathbb{F}}_I$ and S-tuples $(X_K) \in \underline{\mathrm{Alg}}_{\mathbb{E}_V}(\underline{\mathcal{S}}_G)_S$, the Wirthmüller map*

$$W_{S,(X_K)} \colon \coprod_K^S X_K \to \prod_K^S X_K$$

is n-connected.

(c) *for all $S \in \underline{\mathbb{F}}_I$ and S-tuples $(X_K) \in \underline{\mathrm{Alg}}_{\mathbb{E}_V}(\underline{\mathcal{S}}_{G,\leq n})_S$, the Wirthmüller map*

$$W_{S,(X_K)} \colon \coprod_K^S X_K \to \prod_K^S X_K$$

is an equivalence.

Corollary 22. *Let $\mathcal{O}^\otimes$ be a unital G-operad. Then, the collection of arities*

$$\underline{\mathbb{F}}_{\mathcal{O}}^{n-\mathrm{conn}} := \left\{ S \ \Big| \ \forall (X_K) \in \underline{\mathrm{Alg}}_{\mathbb{E}_V}(\underline{\mathcal{S}}_G)_S, \ W_{S,(X_K)} \text{ is } n\text{-connected} \right\}$$

is a unital weak indexing system.

Proof. This follows by combining Proposition 21 with [Ste25b]. $\qquad\square$

The condition that $\mathbb{E}_V(S)$ is n-connected cuts out a unital weak indexing system as well.

Proposition 23. *Let $\underline{\mathbb{F}}^{n-\mathrm{conn}(V)} \subset \underline{\mathbb{F}}_G$ be the collection containing each S such that $\mathrm{Conn}_S^H(V)$ is n-connected. Then, $\underline{\mathbb{F}}^{n-\mathrm{conn}(V)}$ is a unital weak indexing system.*

Before showing this, we will see how it proves Theorem A.

Proof of Theorem A. After Corollary 20, it suffices to show that Condition (i) and Proposition 16 are equivalent, i.e. $\underline{\mathbb{F}}^{n-\mathrm{conn}(V)} = \underline{\mathbb{F}}_{\mathbb{E}_V}^{n-\mathrm{conn}}$; indeed, Proposition 23 verifies that $S \in \underline{\mathbb{F}}^{n-\mathrm{conn}(V)}$ if and only if $\underline{\mathbb{F}}_{I_S} \subset \underline{\mathbb{F}}^{n-\mathrm{conn}(V)}$, which together with Proposition 21 implies that

$$\underline{\mathbb{F}}^{n-\mathrm{conn}(V)} = \left\{ S \ \Big| \ \forall \ T \in \underline{\mathbb{F}}_{I_S}, \ \mathbb{E}_V(T) \text{ is } n\text{-connected} \right\} = \underline{\mathbb{F}}_{\mathbb{E}_V}^{n-\mathrm{conn}}. \qquad\square$$

For the rest of this section, we verify Proposition 23 via a series of small claims.

Lemma 24. *Let V be an orthogonal G-representation.*

(1) *For all H, $\varnothing_H \in \mathbb{F}_H^{n-\mathrm{conn}(V)}$.*
(2) *$S \in \mathbb{F}_H^{n-\mathrm{conn}(V)}$ if and only if $2 \cdot *_H \in \mathbb{F}_H^{n-\mathrm{conn}(V)}$ whenever $\left|S^H\right| \geq 2$ $[H/K] \in \mathbb{F}_H^{n-\mathrm{conn}(V)}$ whenever $[H/K] \subset S$.*
(3) *If $S \in \mathbb{F}_K^{n-\mathrm{conn}(V)}$ and $[H/K] \in \mathbb{F}_H^{n-\mathrm{conn}(V)}$, then $\mathrm{Ind}_K^H S \in \mathbb{F}_K^{n-\mathrm{conn}(V)}$.*
(4) *If $V_{(H)}$ and $V_{(K)}$ are $(n-2)$-connected, then $V_{(H \cap K)}$ is $(n-2)$-connected.*
(5) *$\underline{\mathbb{F}}^{n-\mathrm{conn}(V)}$ is closed under restriction.*

Proof. (1) follows by contractibility of $\mathrm{Conf}_{\varnothing_H}^H(V)$; (2) follows by unwinding Corollary 20.

For (3), by (2), it suffices to prove the statement in the case that $S = [K/J]$. In this case, fix some $x \in H - J$, and note that it suffices to prove that $\mathrm{codim}\left(V^{\langle J,x \rangle} \hookrightarrow V^J\right) \geq n$. In fact, if $x \in K$, then this follows by Corollary 20 for $[K/J]$, so assume that $x \notin K$. Then, we get an intersection diagram

(2)
$$\begin{array}{ccc} V^H \longrightarrow V^{\langle K,x \rangle} & \xrightarrow{\ b\ } & V^K \\ \downarrow \quad\quad\ \lrcorner & & \downarrow \\ V^{\langle J,x \rangle} & \xrightarrow{\ a\ } & V^J \end{array}$$

where arrows are decorated with their codimension. We're tasked with proving that $a \geq n$, and by linear algebra, $a \geq b$, so it suffices to prove that $b \geq n$. By assumption, V embeds $[H/K]$, so $b > 0$; since $[H/K] \in \mathbb{F}_H^{\mathrm{n-conn}(V)}$, we have $b \geq n$ by Corollary 20.

The proof of (4) is similar. Let $x \in G$ be an element not contained in $H \cap K$ satisfying the property that x does not stabilize $V^{H \cap K}$. Writing $J := H \cap K$, we're tasked with proving that

$$\mathrm{codim}(V^{\langle J, x \rangle} \hookrightarrow V^J) \geq n.$$

We may assume without loss of generality that $x \notin K$, so we once again form Eq. (2) and find that $a \geq n$. To prove (5), note that the orbit types $[H/J]$ appearing in $\mathrm{Res}_H^G S$ are precisely those isomorphic with $J = H \cap gKg^{-1}$ with $[G/K] \subset S$, so this follows from (4) and Corollary 20. $\qquad\square$

Proof of Proposition 23. First note that $*_H, \varnothing_H \in \mathbb{F}_H^{\mathrm{n-conn}(V)}$ by Lemma 24, and $\mathbb{F}_H^{\mathrm{n-conn}(V)}$ is closed under restriction by Lemma 24, so it suffices to prove that $\underline{\mathbb{F}}^{\mathrm{n-conn}(V)}$ is closed under self-indexed coproducts. Fix some $S \in \mathbb{F}_H^{\mathrm{n-conn}(V)}$ and $(T_K) \in \mathbb{F}_S^{\mathrm{n-conn}(V)}$, and write $T := \coprod_K^S T_K$.

By Lemma 24, we're faced with the case $T = 2 \cdot *_H$ or $T = [H/J]$ for $K \neq H$. In the former case, it follows that either $S = 2 \cdot *_H = T$ or $S = *_H$ and $T_H = 2 \cdot *_H = T$, so $T \in \underline{\mathbb{F}}^{\mathrm{n-conn}(V)}$. In the latter case, $S = [H/K]$ and $T = [K/J]$, and the result follows from Lemma 24. $\qquad\square$

2.3. A corollary about $\mathbb{E}_V$-ring spectra.

Definition 25. Let $\mathrm{Cat}^{\mathrm{st},trex}$ denote the ∞-category whose objects are stable ∞-categories with t-structure and whose morphisms are right t-exact functors. The *∞-category of fiberwise-stable I-symmetric monoidal ∞-categories with t-structure* is the full subcategory

$$\mathrm{Cat}_I^{\mathrm{st},trex,\otimes} \subset \mathrm{CMon}_I\big(\mathrm{Cat}^{\mathrm{st},trex}\big).$$

whose restriction functors are additionally left t-exact. ◄

In this case, $\mathcal{C}_{\geq a} \hookrightarrow \mathcal{C}$ is a well-defined G-symmetric monoidal subcategory and $\mathcal{C}_{\geq a} \to \mathcal{C}_{[a,b]}$ a well-defined G-symmetric monoidal localization.

Proposition 26. *Suppose $\mathcal{C}$ is a fiberwise-stable I-symmetric monoidal ∞-category with t-structure and $\mathcal{O}^{\otimes}$ a k-connected G-operad. Then, the forgetful functor*

$$\mathrm{CAlg}_{A\mathcal{O}}\big(\mathcal{C}_{[d,d+k]}\big) \to \mathrm{Alg}_{\mathcal{O}}\big(\mathcal{C}_{[d,d+k]}\big)$$

is an equivalence.

Proof. It suffices to note that $\mathcal{C}_{H,[d,d+k]}$ is a k-category and apply [Ste25b]. $\qquad\square$

Example 27. For all $H \subset G$, we give $\mathrm{Sp}_H = \big(\underline{\mathrm{Sp}}_G\big)_H$ the homotopy t-structure, whose 0-connectives agree with slice 0-connectives, (see [Wil17] in the case $n = 0$). Its restriction functors are evidently t-exact and its binary tensors and norms are right t-exact [HHR16, Prop 4.26, Prop 4.33]. Corollary B follows immediately by combining Theorem A and Proposition 26. ◄

3. Low-dimensional computations

Fix an orthogonal G-representation V and a chain of subgroups $K \subset H \subset G$. Define

$$X := \big(\mathrm{Res}_H^G V\big)_{(K)}/H.$$

Identifying $\mathrm{Aut}_H(n \cdot [H/K]) \simeq N_H K \wr \Sigma_n$, the long exact sequence in homotopy associated with the quotient map $\mathrm{Conf}_{n \cdot [H/K]}^H(V) \to C_n(X)$ reads

(3)

$$0 \longrightarrow \pi_\ell \mathrm{Conf}_{n \cdot [H/K]}^H(V) \longrightarrow \pi_\ell C_n(X) \longrightarrow 0$$

$$0 \longrightarrow \pi_1 \mathrm{Conf}_{n \cdot [H/K]}^H(V) \longrightarrow \pi_1 C_n(X)$$

$$N_H K \wr \Sigma_n \longleftarrow \pi_0 \mathrm{Conf}_{n \cdot [H/K]}^H(V) \longrightarrow \pi_0 C_n(X) \longrightarrow 0$$

In particular low-dimensional examples, we can compute $\pi_\ell C_n(X)$ explicitly.

Example 28. Suppose V is trivial. Then,

$$\mathrm{Conf}_S^G(V) \simeq \begin{cases} \mathrm{Conf}_{S^G}(V^G) & S = n \cdot *_G; \\ \varnothing & \text{otherwise.} \end{cases}$$

More generally, if $N \subset G$ is a normal subgroup, then

$$\mathrm{Conf}_S^G\left(\mathrm{Infl}_{G/N}^G V\right) \simeq \begin{cases} \mathrm{Conf}_{S^N}(V) & S = S^N; \\ \varnothing & \text{otherwise.} \end{cases}$$

◄

Example 29. Suppose V is 1-dimensional and nontrivial, so that $V^G = 0$. Given $x \in V - V^G$, the orbit-stabilizer theorem implies that $\mathrm{stab}_G x \subset G$ is an index-2 subgroup, it's not too hard to see that $V \simeq \mathrm{Infl}_{G/\mathrm{stab}_G x}^G \sigma$.

Moreover, note that $C_n(\sigma - \{0\}/C_2) \simeq C_n(\mathbb{R}^1)$ is contractible. In particular, applying Eq. (3) yields that $\mathrm{Conf}_S^G(V)$ is nonempty if and only if $\left|S^G\right| \leq 1$ and $S - S^G = \mathrm{Infl}_{G/\mathrm{stab}_G x}^G n \cdot *_{G/\mathrm{stab}_G x}$, in which case $\mathrm{Conf}_S^G(V)$ is an $\mathrm{Aut}_G(S)$-torsor.

◄

Example 30. Suppose $G = C_n$ for $n \neq 0$, ω is a complex nth root of unity, and λ_ω is the 2-dimensional orthogonal G-representation on which a distinguished generator $x \in C_n$ acts by ω. Then, $\lambda_k \simeq \mathrm{Infl}_{C_n/C_n^{\mathrm{order}(\omega)}}^{C_n} \lambda_\omega$, so by Example 28 we may assume without loss of generality that ω is primitive.

Moreover, given a divisor $m|n$, we have $\mathrm{Res}_{C_m}^{C_n} \lambda_\omega \simeq \lambda_{\omega^{n/m}}$, so it suffices to consider C_n-equivariant configurations. In this case, we have $\lambda_{\omega,(C_n)} = \lambda - \{0\}$, on which C_n acts freely with orbit space homeomorphic to the annulus $\mathbb{R}^2 - \{0\}$. It is known that the fundamental group of $C_k(\mathbb{R}^2 - \{0\})$ is the *cylindrical braid group* on k letters [DH98], i.e. the braid group associated with the Coxeter graph B_k, as in

$$t \xrightarrow{\ \ 4\ \ } g_1 \xrightarrow{\quad} \cdots \xrightarrow{\quad} g_{k-1};$$

that is,

$$\pi_0 C_k(\lambda_{\omega,(C_n)}/C_n) \simeq \mathrm{CBr}_k := \langle t, g_1, \ldots, g_{n-1} \mid t g_1 t g_1 = g_1 t g_1 t, \ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \ g_i g_j = g_j g_i \text{ if } |i - j| \geq 2 \rangle.$$

Writing elements of $C_n \wr \Sigma_k$ as $(c_1, \ldots, c_k; \sigma)$, the Deck map $\pi_1 C_k(\lambda_{\omega,(C_n)}) = \mathrm{CBr}_k \longrightarrow C_n \wr \Sigma_k$ then takes

$$g_i \mapsto (0, \ldots, 0, 0; \tau_i),$$
$$t \mapsto (0, \ldots, 0, 1; \tau_1 \tau_2 \cdots \tau_{n-1}),$$

where τ_i is the ith simple transposition. Combining everything we've found, we have

$$\mathrm{Conf}_S^G(V) \simeq \begin{cases} B \ker\left(\mathrm{CBr}_{|S - S^{C_n}|/n} \to C_n \wr \Sigma_{|S - S^{C_n}|/n}\right) & \left|V^{C_n}\right| \leq 1 \\ \varnothing & \text{otherwise.} \end{cases}$$

◄

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