LOOP GROUPS AND THEIR REPRESENTATIONS

AN EXERCISE IN CONFUSION

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Abstract. We will give an introduction to the representation theory of loop groups of compact Lie groups: we will discuss what positive energy representations are, why they exist, how to construct them (via a Schur-Weyl style construction and a Borel-Weil style construction), and how to show that they don't depend on choices. Motivation will come from both mathematics and quantum mechanics.

These are preliminary notes for my talk at Juvitop in Fall 2019 about positive energy representations of loop groups. The reader should be careful with the veracity of some of the statements in these notes; I've tried to emphasize the "big picture", and focus less on the details (in part because of time, but mostly because I do not understand them yet).

1. Introduction

The title of my talk is supposed to be "Intertwining of Positive Energy Representations after Pressley-Segal", and the sole purpose is to be to explain Theorem 13.4.3 of [PS86]. To make sure I don't fail in this task, let me state this result.

Theorem 1.1. *Let G be a simple*[1] *simply-connected compact Lie group. Then any positive energy representation E of the loop group LG admits a projective intertwining action of* $\mathrm{Diff}^+(S^1)$.

Hopefully this theorem means nothing to you at the moment — if it does, you presumably don't need to sit through the rest of this talk, because what I'd like to do in the remainder of my time is to explain all the components of this theorem, and then sketch a proof (time permitting/if I've timed stuff correctly).

Here's a rough sketch of what Theorem 1.1 is about. The representation theory of a semisimple compact Lie group G is very well-behaved: one has the Peter-Weyl theorem, which allows one to provide any finite-dimensional G-representation with a G-invariant Hermitian inner product, and this inner product decomposes the representation into a direct sum of irreducibles. Moreover, the irreducibles are in bijection with dominant weights, where the representation associated to a dominant weight is given (by Borel-Weil) as the global sections of a line bundle associated to a homogeneous space of G (a particular flag variety).

Most representations of loop groups will not satisfy analogues of this property, so we'd like to hone down on the ones which do. These are the "positive energy representations"; these essentially satisfy properties necessary to be able to write down highest/lowest weight vectors. Theorem 1.1 then states that positive energy representations are preserved under reparametrizations of the circle (which give automorphisms of the loop group LG). One can therefore think of Theorem 1.1 as a consistency result.

Before proceeding, I'd like to give some motivation for caring about the representation theory of loop groups. One motivation comes from the connection between representation theory and homotopy theory. The Atiyah-Segal completion theorem relates representations of a compact Lie group G to G-equivariant K-theory, and one hopes that representations of the loop group LG is related to G-equivariant elliptic cohomology. Another motivation comes from the hope that geometry on the free loop space LM of a manifold M is supposed to correspond to correspond to "higher-dimensional geometry" over M. (For instance, if M has a Riemannian metric, then one is supposed to think of the scalar curvature of LM at a loop as the integral of the Ricci curvature of g over the loop.) In light

[1] I think you can remove this assumption, but I'm including it just to be safe.

of this hope, it is rather pacifying to have a strong analogy between representation theory of compact Lie groups and of loop groups. In fact, all of these motivations are related by a story that still seems to be mysterious at the moment.

There's also motivation from physics for studying the representation theory of loop groups. The wavefunction of a free particle on the circle S^1 must be an L^2-function on S^1 (because the probability of finding the particle somewhere on the circle is 1). There is an action of the loop group $L\mathrm{U}(1)$ on $L^2(S^1; \mathbf{C})$ given by pointwise multiplication (a pair $\gamma : S^1 \to \mathrm{U}(1)$ and $f \in L^2(S^1; \mathbf{C})$ is sent to the L^2-function $f_\gamma(z) = \gamma(z)f(z)$). In particular, $L\mathrm{U}(1)$ gives a lot of automorphisms of the Hilbert space $L^2(S^1; \mathbf{C})$; this is relevant to quantum mechanics, where observables are (Hermitian) operators on the Hilbert space of states. Having a particularly (mathematically) natural source of symmetries is useful. In [Seg85], Segal in fact says: "In fact it is not much of an exaggeration to say that the mathematics of two-dimensional quantum field theory is almost the same thing as the representation theory of loop groups".

2. Representations of loop groups

Definition 2.1. Let G be a compact connected Lie group. The loop group $LG = \mathrm{Map}_{C^\infty}(S^1, G)$ is the group of smooth unbased loops in G.

There will be a lot of circles floating around, and so we will distinguish these by subscripts. Some of these will be denoted by $\mathbf{T}$, for "torus".

Remark 2.2. The loop group LG is an infinite-dimensional Lie group, and it has an action of S^1 by rotation. We will denote this "rotation" circle by $\mathbf{T}_{\mathrm{rot}}$. This action will turn out to be very useful shortly.

The action of $\mathbf{T}_{\mathrm{rot}}$ allows one to consider the semidirect product $LG \rtimes \mathbf{T}_{\mathrm{rot}}$. The following proposition is then an exercise in manipulating symbols:

Proposition 2.3. *An action of $LG \rtimes \mathbf{T}_{\mathrm{rot}}$ on a vector space V is the same data as an action R of $\mathbf{T}_{\mathrm{rot}}$ on V and an action U of LG on V satisfying*

$$R_\theta U_\gamma R_\theta^{-1} = U_{R_\theta \gamma}.$$

Most interesting representations U of LG on a vector space V satisfy the property that $U_\gamma U_{\gamma'} = c(\gamma, \gamma')U_{\gamma\gamma'}$, where $c(\gamma, \gamma') \in \mathbf{C}^\times$. This is precisely:

Definition 2.4. A *projective representation* of LG on a Hilbert space V is a continuous homomorphism $LG \to P\mathrm{U}(V)$.

Remark 2.5. Why Hilbert spaces? From a mathematical perspective, this is because Hilbert spaces are well-behaved infinite-dimensional vector spaces. From a physical perspective, this is because Hilbert spaces are spaces of states. In fact, this also explains why most interesting representations are projective: the state of a quantum system is not a vector in the Hilbert space, but rather a vector in the projectivization of the Hilbert space. This is just the statement that shifting the wavefunction by a phase does not affect physical observations.

Since $P\mathrm{U}(V) \simeq K(\mathbf{Z}, 2)$, we see that equivalence classes of projective representations of LG are in bijection with elements of $\mathrm{H}^2(LG; \mathbf{Z})$. There is a central extension

$$1 \to \mathbf{T}_{\mathrm{central}} \to \mathrm{U}(V) \to P\mathrm{U}(V) \to 1,$$

and so any projective representation ρ of LG determines a central extension

$$1 \to \mathbf{T}_{\mathrm{central}} \to \widetilde{LG}_\rho \to LG \to 1.$$

Conversely, any central extension of LG gives rise to a projective representation of LG. In particular:

Remark 2.6. Isomorphism classes of central extensions of LG are in bijection with elements of $\mathrm{H}^2(LG; \mathbf{Z})$. If G is simple and simply-connected, then $\mathrm{H}^2(LG; \mathbf{Z}) \cong \mathrm{H}^3(G; \mathbf{Z}) \cong \mathbf{Z}$.

Definition 2.7. Let G be a simple and simply-connected compact Lie group. The *universal central extension* $\widetilde{LG}$ of LG is the central extension corresponding to the generator of $\mathrm{H}^2(LG; \mathbf{Z}) \cong \mathbf{Z}$.

The following result is key.

Theorem 2.8 ([PS86, Theorem 4.4.1])**.** *There is a unique action of* $\mathrm{Diff}^+(\mathbf{T}_{\mathrm{rot}})$ *on* $\widetilde{LG}$ *which covers the action on* LG*. Moreover,* $\widetilde{LG}$ *deserves to be called "universal", because there is a unique map of extensions from* $\widetilde{LG}$ *to any other central extension of* LG*.*

Remark 2.9. As a consequence, the action of $\mathbf{T}_{\mathrm{rot}}$ on LG lifts canonically to $\widetilde{LG}$. Every projective unitary representation of LG with an intertwining action of $\mathbf{T}_{\mathrm{rot}}$ is equivalently a unitary representation of $\widetilde{LG} \rtimes \mathbf{T}_{\mathrm{rot}}$. For the remainder of this talk, we will abusively say write "representation of LG" to mean a representation of $\widetilde{LG} \rtimes \mathbf{T}_{\mathrm{rot}}$.

Notation. It is a little inconvenient to constantly keep writing $\widetilde{LG} \rtimes \mathbf{T}_{\mathrm{rot}}$, so we will henceforth denote it by $\widetilde{LG}^+$. The subgroup $\mathbf{T}_{\mathrm{rot}}$ of $\widetilde{LG}^+$ is also known as the "energy circle" (for reasons to be explained below).

One of the nice properties of tori is that their representations take on a particularly simple form, thanks to the magic of Fourier series. The action of S^1 on a finite-dimensional vector space is the same data as a $\mathbf{Z}$-grading. The case of topological vector spaces is slightly more subtle: if S^1 acts on a topological vector space V, then one can consider the closed "weight" subspace V_n of V where the action of S^1 is by the character[2] $z \mapsto z^{-n}$. Then the direct sum $\bigoplus_{n \in \mathbf{Z}} V_n$ is a dense subspace of V; it is known as the subspace of *finite energy* vectors in V. This is simply the usual weight decomposition adapted to the topological setting.

Definition 2.10. The action of S^1 on a topological vector space V is said to satisfy the *positive energy condition* if the weight subspace $V_n = 0$ for $n < 0$. Equivalently, the action of S^1 is represented by $e^{-iA\theta}$, where A is an operator with positive spectrum.

Remark 2.11. The motivation for this definition comes from quantum mechanics: the wavefunction of a free particle on a circle is e^{inx} (up to normalization), and requiring that the energy (which is essentially the weight n) to be positive is mandated by physics.

Definition 2.12. A representation of LG (which, recall, means a representation of $\widetilde{LG}^+$) is said to satisfy the *positive energy condition* if it satisfies the positive energy condition when viewed as a representation of the energy/central circle $\mathbf{T}_{\mathrm{rot}}$.

Remark 2.13. It doesn't make sense for a representation of LG to be positive energy if you take "representation of LG" to mean a literal representation of LG; one needs to interpret that phrase as meaning a representation of $\widetilde{LG}^+$.

We can now see the utility of Theorem 1.1: the positive energy condition involves the canonical parametrization of the circle, and to ensure that our definition would agree with that of an alien civilization's, we should ensure that the pullback f^*V of any positive energy representation V of LG along an orientation-preserving diffeomorphism $f \in \mathrm{Diff}^+(S^1)$ (or, rather, $\mathrm{Diff}^+(\mathbf{T}_{\mathrm{rot}})$) is another positive energy representation. That is precisely the content of Theorem 1.1.

In the introduction, we said that positive energy representations of loop groups are supposed to satisfy analogues of many properties of representations of compact Lie group. To make that statement precise, we need to introduce some definitions that impose sanity conditions on the representations we'd want to study.

Definition 2.14. Let V be a representation of a topological group G (possibly infinite-dimensional). Then V is said to be:

- *irreducible* if it has no closed G-invariant subspace;

[2] Some conventions are different: the action might be by $z \mapsto z^n$. We're following [PS86].

- *smooth* if the following condition is satisfied: let V_{sm} denote the subspace of vectors $v \in V$ such that the orbit map $G \to V$ sending g to gv is continuous; then V_{sm} is dense in V.

Two G-representations V and W are *essentially equivalent* if there is a continuous G-equivariant map $V \to W$ which is injective and has dense image.

Warning 2.15. Essential equivalence is *not* an equivalence relation!

Then:

Theorem 2.16 ([PS86, Theorem 9.3.1])**.** *Let V be a smooth positive energy representation of LG. Then up to essential equivalence:*

- *V is completely reducible into a discrete direct sum of irreducible representations;*
- *V is unitary;*
- *V extends to a holomorphic projective representation of $LG_{\mathbf{C}}$;*
- *V admits a projective intertwining action of $\mathrm{Diff}^+(S^1)$, where this S^1 is the energy/rotation circle. (This is Theorem 1.1.)*

The proof of this result takes up the bulk of the second part of Pressley-Segal.

Remark 2.17. The group G includes into LG as the subgroup of constant loops. Let G be simple and simply-connected. If T is a maximal torus of G, then one has $\mathbf{T}_{\mathrm{rot}} \times T \times \mathbf{T}_{\mathrm{central}} \subseteq \widetilde{LG}^+$. Consequently, if V is a representation of $\widetilde{LG}^+$, then V can be decomposed (up to essential equivalence) as a $\mathbf{T}_{\mathrm{rot}} \times T \times \mathbf{T}_{\mathrm{central}}$-representation:

$$V = \bigoplus_{(n,\lambda,h) \in \mathbf{T}_{\mathrm{rot}}^\vee \times T^\vee \times \mathbf{T}_{\mathrm{central}}^\vee} V_{(n,\lambda,h)}.$$

Here, n is the energy of V; λ is a weight of V (regarded as a representation of T); and h is a character of $\mathbf{T}_{\mathrm{central}}$. If V is irreducible, then $\mathbf{T}_{\mathrm{central}}$ must act by scalars by Schur's lemma, and so only one value of h can occur; this is called the *level* of V. It turns out that if V is a smooth positive energy representation, then each weight space $V_{n,\lambda,h}$ is finite-dimensional. In fact, a representation of LG of level h is the same as a representation of $\widetilde{LG}_h \rtimes \mathbf{T}_{\mathrm{rot}}$, where $\widetilde{LG}_h$ is the central extension of LG corresponding by Remark 2.6 to $h \in \mathbf{Z} \cong \mathrm{H}^2(LG; \mathbf{Z})$.

Remark 2.18. By Remark 2.17, an irreducible positive energy representation V of LG is uniquely determined by the level h and its lowest energy subspace V_0: the representation V is generated as a $\widetilde{LG}^+$-representation by V_0.

Remark 2.19. The level h is an element of $\mathrm{H}^2(LG; \mathbf{Z}) \cong \mathrm{H}^3(G; \mathbf{Z}) \cong \mathrm{H}^4(BG; \mathbf{Z})$, i.e., is a map $BG \to K(\mathbf{Z}, 4)$. This $K(\mathbf{Z}, 4)$ is closely tied to the twisting $K(\mathbf{Z}, 4) \to B\mathrm{GL}_1(\mathrm{tmf})$ of tmf.

As a side note, we observe the following:

Proposition 2.20. *Let V be a smooth positive energy representation of LG. Then V is irreducible as a representation of $\widetilde{LG}$.*

Proof. Assume V is not irreducible as a $\widetilde{LG}$-representation. Projection onto a proper $\widetilde{LG}$-invariant summand defines a bounded self-adjoint operator $T : V \to V$ which commutes with $\widetilde{LG}$, but not with the action (say R) of $\mathbf{T}_{\mathrm{rot}}$. Then define for each $n \in \mathbf{Z}$ the bounded operator

$$T_n = \int_{\mathbf{T}_{\mathrm{rot}}} z^n R_z T R_z^{-1} dz.$$

Clearly T_n commutes with $\widetilde{LG}$, and T_n sends the weight space V_m to V_{m+n}. Because T does not commute with $\mathbf{T}_{\mathrm{rot}}$, the operator T_n must be nontrivial for at least one $n < 0$. Suppose that m is the lowest energy of V (i.e., the smallest m such that the weight space $V_m \neq 0$; because V is of positive energy, $m \geq 0$ — but that doesn't matter for now). Then $T_n(V_m) = 0$ if $n < 0$. Since V is irreducible as a representation of $\widetilde{LG}^+$, it is generated as a representation by V_m. But then $T_n(V) = 0$ for all $n < 0$. The adjoint to T_n is T_{-n}, and so $T_n(V) = 0$ for all $n \neq 0$.

This implies that T commutes with the action of $\mathbf{T}_{\mathrm{rot}}$ (which is a contradiction). Indeed, the T_n are the Fourier coefficients of the loop $S^1 \to \mathrm{End}(V)$ sending z to $R_z T R_z^{-1}$, so we find that this loop must be constant. Consequently, T must commute with the action of $\mathbf{T}_{\mathrm{rot}}$, as desired. □

3. A proof sketch of Theorem 1.1

The goal of this section is to go through the proof of Theorem 1.1. As with all proofs in representation theory, we may first reduce to the irreducible case. Before arguing the general case, I want to make an observation.

Observation 3.1. Recall that Schur-Weyl duality sets up a one-to-one correspondence between representations of $\mathrm{SU}(n)$ and representations of the symmetric groups, by studying the decomposition of the tensor power $V^{\otimes d}$ of the standard representation V under the action of Σ_d.

One may hope that some analogue of Observation 3.1 is true for representations of loop groups. Suppose we could construct a giant representation of $L\mathrm{SU}(n)$ whose h-fold tensor product contains all the irreducible positive energy representations of level h, such that this big representation admits an intertwining action of $\mathrm{Diff}^+(S^1)$. Then (with a little bit of work), we would obtain an intertwining action of $\mathrm{Diff}^+(S^1)$ on all irreducible positive representations of $L\mathrm{SU}(n)$, which would prove Theorem 1.1 in this particular case. We would like to then reduce from the case of a general G to the case of $\mathrm{SU}(n)$. The Peter-Weyl theorem says that a simply-connected G is a closed subgroup of $\mathrm{SU}(n)$ for some n, so that gives us some hope that a technique like this might work. This isn't exactly the approach that's taken in [PS86], but something very close to it is in fact what is done. I haven't had time to flesh out the details of the approach outlined above (nor have I found a reference which takes this approach[3] — except for maybe [Was98], but I've gotten very confused over whether certain results in that paper are stated for a general compact Lie group or only for $\mathrm{SU}(n)$), but I think it might be interesting.

Here is how the story is supposed to go for $L\mathrm{SU}(n)$.

Construction 3.2. Let $G = \mathrm{SU}(n)$. Define $H = L^2(S^1, V)$, where V is the standard representation. Let $\mathrm{H}^2(S^1, V) \subseteq H$ denote the Hardy space of L^2-functions on S^1 with only nonnegative Fourier coefficients, and let P denote the projection of H onto $\mathrm{H}^2(S^1, V)$. Then $H = PH \oplus P^\perp H$. The *Fock space* $\mathcal{F}_P$ is then defined as the (Hilbert space completion of the) tensor product

$$\mathcal{F}_P = \bigwedge(PH \oplus \overline{P^\perp H}) \cong \bigoplus_{i,j} \Lambda^i(PH) \oplus \Lambda^j(\overline{P^\perp H}),$$

where $\overline{V}$ denotes the complex conjugate space to V. The Fock space turns out to be the "giant representation" we were after: it's the fundamental representation of $L\mathrm{SU}(n)$.

Remark 3.3. The Fock space is familiar from quantum mechanics: it is the state space of free fermionic particles and antiparticles in the Hilbert space H. More precisely, the subspace $\Lambda^i(PH) \oplus \Lambda^j(\overline{P^\perp H})$ consists of i fermionic particles and j fermionic antiparticles. This explains why we take the conjugate space to $P^\perp H$: it is so that the antiparticles have positive energy.

A loop on G acts on H by pointwise multiplication, and $f \in \mathrm{Diff}^+(S^1)$ acts on H by sending $\xi : S^1 \to V$ to $\xi(f^{-1}(z)) \cdot |(f^{-1})'(z)|^{1/2}$. (The square root factor is a normalization factor to ensure unitarity of the action.) In fact, this gives an action of $LG \rtimes \mathrm{Diff}^+(S^1)$ on H, and one can ask when this descends to a projective representation of $LG \rtimes \mathrm{Diff}^+(S^1)$ on the Fock space $\mathcal{F}_P$. Segal wrote down a *quantization condition* for when a unitary operator on H descends to a projective transformation of $\mathcal{F}_P$: namely, u descends to $\mathcal{F}_P$ if and only if the commutator $[u, P]$ is Hilbert-Schmidt[4]. One checks

[3]Probably because it doesn't work for some obvious reason.

[4]Recall that a bounded operator A on a Hilbert space is Hilbert-Schmidt if $\mathrm{Tr}(A^*A)$ is finite.

that the action of $LG \rtimes \mathrm{Diff}^+(S^1)$ on H satisfies Segal's quantization criterion, and so descends to a projective representation of $LG \rtimes \mathrm{Diff}^+(S^1)$ on the Fock space $\mathcal{F}_P$.

Almost by definition, the action of $S^1 = \mathbf{T}_{\mathrm{rot}}$ on $\mathcal{F}_P$ is of positive energy, and so $\mathcal{F}_P$ is a representation of positive energy. It turns out that:

Theorem 3.4 ([PS86, Section 10.6] and [Was98, Chapter I.5])**.** *The irreducible summands of $\mathcal{F}_P^{\otimes h}$ give all the irreducible positive energy representations of $L\mathrm{SU}(n)$ of level h.*

Remark 3.5. The space denoted $\mathcal{H}$ in that section of [PS86] is the Hilbert space completion of the Fock space $\mathcal{F}_P$.

Peter will expand on this construction of the irreducible level h representations of $L\mathrm{SU}(n)$ next week in his talk about the Segal-Sugawara construction.

This approach isn't exactly the one taken in [PS86], but something essentially like it is. The first reduction comes from:

Lemma 3.6. *Let V and W be positive energy representations of $\widetilde{LG}$. Suppose that V is irreducible, and that $V \oplus W$ admits an intertwining action of $\mathrm{Diff}^+(S^1)$. Then V admits an intertwining action of $\mathrm{Diff}^+(S^1)$.*

We will prove this shortly; first, we will indicate how to use this to prove the general case.

Remark 3.7. It suffices to prove by Lemma 3.6 that for every irreducible positive energy representation V of LG, there is some G' and an embedding $i : LG \to LG'$ where Theorem 1.1 is true for G', and an irreducible representation V' of LG' such that V is a summand of i^*V'.

To use this reduction, we first prove that Theorem 1.1 is true for a class of Lie groups G. In fact:

Theorem 3.8. *Theorem 1.1 is true if G is simple, simply-connected, and simply-laced*[5]*.*

The proof of this result is quite similar to that of Theorem 3.4: one constructs the analogue of the Fock space for LG (which, like in the $\mathrm{SU}(n)$ case, has an intertwining action of $\mathrm{Diff}^+(S^1)$), and then shows that every irreducible positive energy representation is a summand of some twist of this representation of LG.

Construction 3.9. Let ΩG denote the *based* loop space of G, regarded as the homogeneous quotient $LG/G \simeq LG_{\mathbf{C}}/L^+G_{\mathbf{C}}$. Then $\mathrm{H}^2(\Omega G; \mathbf{Z}) \cong \mathrm{H}^3(G; \mathbf{Z}) \cong \mathbf{Z}$ (because G is simple and simply-connected), so every integer gives rise to a complex line bundle on ΩG. The holomorphic sections Γ of the line bundle corresponding to the generator is called the *basic representation* of LG.

Example 3.10. If $G = \mathrm{SU}(n)$, then[6] Γ is the Fock space described above.

Then:

Proposition 3.11 ([PS86, Proposition 9.3.9])**.** *Let G be a simple, simply-connected, and simply-laced Lie group. Then any irreducible positive energy representation of level h of LG is a summand in $i_h^*\Gamma$, where $i_h : LG \to LG$ is the map induced by the degree h map $S^1 \to S^1$.*

I don't yet understand how, but Γ is supposed to admit an intertwining action of $\mathrm{Diff}^+(S^1)$ via the "blip construction". If we assume this, then combining Proposition 3.11 with Lemma 3.6 shows that Theorem 1.1 is true for LG when G is simply-laced (and simple and simply-connected).

According to Remark 3.7, it now suffices to show:

Proposition 3.12. *For every irreducible positive energy representation V of LG, there is a simply-laced G' and an embedding $i : LG \to LG'$, as well as an irreducible representation V' of LG' such that V is a summand of i^*V'.*

[5]Recall that G is simply-laced if all its nonzero roots have the same length; in other words, if the Dynkin diagram of G does not have multiple edges (so the Dynkin diagram is of ADE type).

[6]I think.

This is proved in [PS86, Lemma 13.4.4] in the following manner.

One first classifies all the irreducible representations of LG. Using the loop group analogue of Schur-Weyl duality worked well when $G = \mathrm{SU}(n)$, but that won't do in the general case. Instead, one utilizes a loop group analogue of Borel-Weil (see [Seg85, Section 4.2]). Recall how this works for Lie groups: fix a maximal torus T of G; then, for every antidominant weight λ of T (i.e., $\langle h_\alpha, \lambda\rangle \leq 0$ for every positive root α), there is an associated line bundle $\mathcal{L}_\lambda$ on $G/T \cong G_{\mathbf{C}}/B^+$. The holomorphic sections of $\mathcal{L}_\lambda$ is an irreducible representation of G of lowest weight λ, and all irreducible representations of G arise this way.

In the loop group case, one again begins by fixing a maximal torus T of G (one should think of $\mathbf{T}_{\mathrm{rot}} \times T \times \mathbf{T}_{\mathrm{central}}$ as a maximal torus of LG). Consider the homogeneous space LG/T. There is a fiber sequence

$$G/T \to LG/T \to \Omega G,$$

and the set of isomorphism classes of complex line bundles on LG/T is $\mathrm{H}^2(LG/T;\mathbf{Z}) \cong \mathrm{H}^2(\Omega G;\mathbf{Z}) \oplus \mathrm{H}^2(G/T;\mathbf{Z}) = \mathbf{Z} \oplus \widehat{T}$, where $\widehat{T}$ is the character group of T. In particular, line bundles on LG/T are indexed by $(h, \lambda) \in \mathbf{Z} \oplus \widehat{T}$.

Theorem 3.13 (Borel-Weil for loop groups; [PS86, Theorem 9.3.5])**.** *One has:*

- *The space $\Gamma(\mathcal{L}_{h,\lambda})$ of holomorphic sections is zero or irreducible of positive energy of level h; moreover, every projective irreducible representation of LG arises this way.*
- *The space $\Gamma(\mathcal{L}_{h,\lambda})$ is nonzero if and only if (h, λ) is antidominant*[7]*, i.e.,*

$$0 \geq \lambda(h_\alpha) \geq -\frac{h}{2}\langle h_\alpha, h_\alpha\rangle$$

for each positive coroot h_α of G. (In particular, λ is antidominant as a weight of $T \subseteq G$.)

The upshot is that irreducible representations correspond to antidominant weights. To prove Proposition 3.12, it suffices to show that all antidominant weights of LG are restrictions of antidominant weights of LG' for some simply-laced G'. The argument now proceeds case-by-case, as G ranges over all simple simply-connected simply-laced compact Lie groups. The proof is not very enlightening, so I will not talk about it.

Appendix A. Random thoughts

I'm just spitballing here; I don't think I'll say any of this in the talk.

Minta told me that Segal gave another proof of Theorem 1.1 via conformal field theories (which Arun discussed last time) in [Seg04], but I don't understand it. I'd like to believe that there is another proof (maybe closely related?) coming from studying TQFTs. Namely, one important example of a TQFT is that of Chern-Simons theory. Recall that the Chern-Simons 3-form associated to a connection A on a 3-manifold M is the 3-form $\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$, whose exterior derivative is $\mathrm{Tr}(F \wedge F)$, where F is the curvature of A, and Tr is induced from the (positive-definite) Killing form on $\mathfrak{g}$ associated via the Chern-Weil homomorphism $\mathbf{R} \cong \mathrm{H}^4(BG;\mathbf{R}) \to \mathrm{Sym}^2(g^*)^G$ to the generator. The Chern-Simons action

$$S = \frac{k}{4\pi}\int_{M^3} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

defines for every integer $k \in \mathbf{Z}$ a functional on the moduli of principal G-bundles with connection on M^3. The integer k is the called the *level*.

This action defines a *topological* quantum field theory. It is quite well-behaved, and one expects it to be an extended topological quantum field theory. In particular, the cobordism hypothesis says that this should be determined by a dualizable object in some symmetric monoidal $(\infty, 3)$-category. I don't think that the value of Chern-Simons theory on a point is known (but there are multiple candidates). However, a general rule of extended TQFTs is that the value of an extended TQFT on the circle is the

[7] Recall that if G is the simply-laced group $\mathrm{SU}(n)$, then the weight lattice is $\bigoplus_{1\leq i\leq n+1} \mathbf{Z}\chi_i/\mathbf{Z}\sum_i \chi_i$, and the roots are $\chi_i - \chi_j$ with $i \neq j$. The positive roots corresponding to the usual Borel of upper-triangular matrices are $\chi_i - \chi_j$ for $i < j$. Therefore, $(h, \lambda = \lambda_1, \cdots, \lambda_n)$ is antidominant if λ is antidominant, i.e., $\lambda_1 \leq \cdots \leq \lambda_n$, and if $\lambda_n - \lambda_1 \leq h$.

center (i.e., Hochschild (co)homology) of the value on a point. In particular, the value of Chern-Simons theory on the circle is the center of its value on the point, and it seems to be the case that its value on the circle (if G is simply-connected, at least) is the category $\mathrm{Rep}^k(LG)$ of positive energy representations of LG of level k. In other words, the reduction of Chern-Simons theory to a 2-dimensional TQFT (i.e., the TQFT $Z_{\mathrm{CS}}(- \times S^1)$) is supposed to be $\mathrm{Rep}^k(LG)$. This is called the *Verlinde TQFT*. I would guess that the flow of information goes the other way, but perhaps this perspective can be used to prove some of the statements above about the positive energy representations of loop groups.

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