YOU CAN CONSTRUCT G-COMMUTATIVE ALGEBRAS ONE NORM AT A TIME

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Abstract. We define the category of G-operads and the hierarchy of *generalized $\mathcal{N}_\infty$-operads*, which are G-suboperads of $\mathrm{Comm}_G^{\otimes}$. We exhibit an isomorphism between the category of subterminal operads and the self-join poset

$$\mathrm{Op}_G^{\mathcal{N}_\infty} \simeq \mathrm{Ind}-\mathrm{Sys}_G \star \mathrm{Ind}-\mathrm{Sys}_G,$$

where $\mathrm{Ind}-\mathrm{Sys}_G$ is the poset of *indexing systems* in G. This generalized $\mathcal{N}_\infty$-operads as parameterizing *some commutative multiplicative transfers and possibly a commutative multiplication*. Indeed, their algerbas in semiadditive Cartesian categories are incomplete Mackey functors and their algebras in Mackey functors recover incomplete Tambara functors when they are N_∞ *operads*, i.e. when they contain $\mathbb{E}_\infty$.

After this, we discuss some in-progress research. Namely, we construct a *Boardman-Vogt tensor product* of G-operads, and tensor products of genereralized $\mathcal{N}_\infty$ operads correspond with joins in $\mathrm{Ind}-\mathrm{Sys}_G \star \mathrm{Ind}-\mathrm{Sys}_G$ i.e. there is an $\mathcal{N}_{(I\vee J)\infty}$-monoidal equivalence

$$\mathbf{Alg}^{\mathcal{N}_{I\infty}} \mathbf{Alg}^{\mathcal{N}_{J\infty}} \mathcal{C} \simeq \mathbf{Alg}^{\mathcal{N}_{(I\vee J)\infty}} \mathcal{C}$$

for all $\mathcal{N}_{(I\vee J)\infty}$-monoidal categories $\mathcal{C}$, allowing G-commutative structures to be constructed "one norm at a time."

Foreword. The following are notes prepared for a casual talk in the zygotop seminar concerning research which is currently in-progress cite . Though I will attempt to confine these notes to their own proofs, citations to the literature, and well-marked conjecture, the reader should read with the understanding that they are particularly error-prone.

1. Introduction

In [Dre71], the concept of a *Mackey functor* was introduced; this structure was described as consisting of functors $M_I : \mathcal{O}_G \to \mathbf{Mod}_R$ and $M_R : \mathcal{O}_G^{\mathrm{op}} \to \mathbf{Mod}_R$ which agree on $\mathcal{O}_G^{\simeq}$ and satisfying the *double coset formula*

$$R_J^H I_K^H = \prod_{x \in [J\backslash H/K]} I^J_{J\cap xKx^{-1}} \cdot \mathrm{conj}_X R_{x^{-1}Jx\cap K}$$

for all $J, K \subset H$, where $R_J^K := M_R(G/J \to G/K)$ and similar for I. The ur-example of this is the assignment $H \mapsto \mathbf{Rep}_H(R)$ with covariant functoriality Ind and contravariant functoriality Res. This was repackaged and generalized into the modern definition of the *category of $\mathcal{C}$-valued G-Mackey functors*

$$\mathcal{M}_G(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_G), \mathcal{C}),$$

where $\mathbb{F}_G$ denotes the category of finite G-sets.

In parallel, the concept of *transfer maps in group cohomology* was being developed in [**Evens**], later lifted to genuine equivariant cohomology in [**Greenlees**], and finally developed as a functor

$$N_H^G : \mathrm{Sp}_H \to \mathrm{Sp}_G$$

in [HHR16], which played a crucial role in the solution to the Kervaire invariant one problem. These were noted in [HH16] to satisfy the conditions of a *Symmetric monoidal Mackey functor*, a notion they distinguished from their notion of *G-symmetric monoidal categories* due to coherence issues.

In the broad program announced in [Bar+16], the correct notion of *G-symmetric monoidal G-∞-categories* (henceforth G-symmetric monoidal categories) was introduced:

Definition 1.1. Let $\mathcal{C}$ have finite products. Then, the category of G-commutative monoids in $\mathcal{C}$ is

$$\mathrm{CMon}_G(\mathcal{C}) := \mathcal{M}_G(\mathcal{C}).$$

The category of G-symmetric monoidal categories is $\mathrm{CMon}_G(\mathbf{Cat})$.

We similarly define the *category of small G-categories* as

$$\mathbf{Cat}_G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Cat}) \simeq \mathbf{Cat}^{\mathrm{cocart}}_{/\mathcal{O}_G^{\mathrm{op}}},$$

where the equivalence is the *straightening-unstraightening construction* of [HTT]. We may summarize the structure $\mathcal{C}^{\otimes} \in \mathrm{CMon}_G(\mathbf{Cat})$ a G-symmetric monoidal category, as consisting of, for every conjugacy class (H) of G, a category with Weyl group action $\mathcal{C}_H \in \mathbf{Cat}^{BW_G H}$, as well as functors

$$\begin{aligned} \otimes_H^2 &: \mathcal{C}_H^2 \rightarrow \mathcal{C}_H, \\ N_H^K &: \mathcal{C}_H \rightarrow \mathcal{C}_K, \\ \mathrm{Res}_H^K &: \mathcal{C}_K \rightarrow \mathcal{C}_H \end{aligned}$$

which are associative, commutative, unital, and compatible with each other and the Weyl group action, together with coherence. The maps Res encode an underlying G-category $\mathcal{C}$ of $\mathcal{C}^{\otimes}$, and N_H^K is pronounced "the norm from H to K."

Given $\mathcal{C}$ a G-symmetric monoidal category, we may informally define a *G-commutative monoid* to be a tuple of objects $(X_H)_{H \in \mathcal{O}_G}$ satisfying

$$X_H \simeq \mathrm{Res}_H^G X_G$$

together with structure maps

$$\begin{aligned} \otimes_H^2 &: X_H^{\otimes 2} \rightarrow X_H \\ \mathrm{tr}_H^K &: N_H^K X_H \rightarrow X_K, \end{aligned}$$

for all $H \subset K$. The map tr_H^K is pronounced "the transfer from H to K."

This talk concerns various relaxations of the notion of G-commutative algebras. Namely, we will define a symmetric monoidal closed category Op_G of *(colored) G-operads*, whose internal hom $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ is called the *operad of algebras under pointwise tensors*, and whose tensor product is called the *Boardman-Vogt tensor product*.

We will define $\mathcal{N}_\infty$ *operads*, which interpolate between $\mathbb{E}_\infty$ and the G-operad Comm_G which encodes G-commutative algebras by adding a subset of the transfers parameterized by Comm_G:

Definition 1.2. A *G-transfer system* is a core-preserving wide subcategory $\mathcal{O}_G^{\simeq} \subset T \subset \mathcal{O}_G$ which is closed under base change, i.e. for any diagram in $\mathcal{O}_G$

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\alpha'"] & V \arrow[d, "\alpha"] \\ U' \arrow[r] & V' \end{tikzcd}$$

with $U \hookrightarrow V' \times_{U'} V$ a summand inclusion (pullback taken in $\mathbb{F}_G$) and $\alpha \in T$, we have $\alpha' \in T$.

An *indexing system* is a subcategory $\mathcal{I} \subset \underline{\mathbb{F}}_G$ induced by a transfer system under taking coproducts. A *generalized indexing system* is a core-preserving subcategory $\mathcal{I} \subset \underline{\mathbb{F}}_G$ which is either an indexing system or is attained by one by removing all non-isomorphisms of trivial G-sets. The poset of indexing systems under inclusion is denoted $\mathrm{Ind}-\mathrm{Sys}_G$, and the poset of generalized indexing systems is denoted .

It is not hard to see that there is an equivalence of posets

$$\widehat{\mathrm{Ind}-\mathrm{Sys}}_G \simeq \mathrm{Ind}-\mathrm{Sys}_G \star \mathrm{Ind}-\mathrm{Sys}_G,$$

and in particular, generalized indexing systems decompose into two different join-stable copies of indexing systems, depending on whether maps of trivial G-sets are included.

The main theorem of this talk follows:

Theorem A. *There is a fully faithful and symmetric monoidal inclusion*

$$\mathcal{N}_{(-)\infty} : \widehat{\mathrm{Ind}-\mathrm{Sys}}_G^{\amalg} \hookrightarrow \mathrm{Op}_G^{\otimes}$$

whose image consists of the suboperads of Comm_G, *and when restricted to the indexing systems has image consisting of operads* $\mathcal{O}$ *possessing diagrams* $\mathbb{E}_\infty \subset \mathcal{O} \subset \mathrm{Comm}_G$. *In particular, for* $\mathcal{C}$ *an* $\mathcal{N}_{(I\vee J)\infty}$*-monoidal category, there is a canonical* $\mathcal{N}_{(I\vee J)\infty}$*-monoidal equivalence*

$$\mathbf{Alg}^{\mathcal{N}_{I\infty}}\mathbf{Alg}^{\mathcal{N}_{J\infty}}\mathcal{C} \simeq \mathbf{Alg}^{\mathcal{N}_{(I\vee J)\infty}}\mathcal{C}.$$

We say an inclusion of subgroup $H \subset K$ is *atomic* if it is proper and there exist no chains of proper subgroup inclusions $H \subset J \subset K$. More generally, we say that a conjugacy class $(H) \in \mathrm{Conj}(G)$ is an *atomic subclass* of (K) if there exists an atomic inclusion $\tilde{H} \subset \tilde{K}$ with $\tilde{H} \in (H)$ and $\tilde{K} \in (K)$, and we say that (K) is atomic if the canonical inclusion $1 \hookrightarrow K$ is atomic.

Given $(H) \subset (K)$ an atomic subclass, we refer to the $\mathcal{N}^\infty$-operad corresponding to the minimal index system containing the inclusion $H \hookrightarrow K$ as $\mathcal{N}^\infty(H,K)$. When $(H) = (1)$, we instead simply write $\mathcal{N}^\infty(K)$.

Corollary B. *Let* $1 = G_n \subset G_{n-1} \subset \cdots \subset G_0 = G$ *be a maximal subgroup series of a finite group, and let* $\mathcal{C}$ *be a* G*-symmetric monoidal category. Then, there exists a canonical* G*-symmetric monoidal equivalence*

$$\mathbf{Alg}^{\mathcal{N}^\infty(G_1,G_0)}\cdots\mathbf{Alg}^{\mathcal{N}^\infty(G_n,G_{n-1})}\mathcal{C} \simeq \mathrm{CAlg}_G\mathcal{C}.$$

Furthermore, if $G \simeq H \times J$, *then*

$$\mathrm{CAlg}_H\mathrm{CAlg}_J\mathcal{C} \simeq \mathrm{CAlg}_G\mathcal{C}.$$

Remark. One may worry about the comparison between models for G-operads, as our notion of N^∞-operads is ostensibly embedded deep within the world of G-∞-operads, which are not known to be equivalent to the ∞-category presented by the graph model structure or by genuine G operads. However, in addition to being too complicated to work with, the model of ref, all notions of N^∞ operads coincide .

2. The ideas

2.1. Fibrous patterns.

Definition 2.1. An *algebraic pattern* is an ∞-category $\mathcal{O}$, together with a factorization system $(\mathcal{O}^{\mathrm{int}}, \mathcal{O}^{\mathrm{act}})$ of $\mathcal{O}$ and a full subcategory $\mathcal{O}^{\mathrm{el}} \subset \mathcal{O}^{\mathrm{int}}$. The *category of algebraic patterns* is the full subcategory

$$\mathrm{AlgPatt} \subset \mathrm{Fun}(D, \mathbf{Cat})$$

spanned by algebraic patterns, where $D := \bullet \to \bullet \to \bullet \leftarrow \bullet$.

Maps in $\mathcal{O}^{\mathrm{int}}$ and $\mathcal{O}^{\mathrm{act}}$ are pronounced *inert and active maps*, and objects of $\mathcal{O}^{\mathrm{el}}$ are pronounced *elementary objects*. For instance, $\mathbb{F}_*$, together with its inert and active maps as defined in [HA, § 2] and elementary objects $\{\langle 1\rangle\}$ determines an algebraic pattern. In analogy with [HA, § 2], we will use these to develop a notion of operads, called *fibrous patterns*.

Definition 2.2. Let $\mathcal{O}$ be an algebraic pattern. A *fibrous* $\mathcal{O}$*-pattern* is a map of algebraic patterns $\pi : \mathcal{P} \to \mathcal{O}$ such that

(1) $\mathcal{P}$ has π-cocartesian lifts for inert morphisms of $\mathcal{O}$,
(2) (Segal condition for colors) For every active morphism $\omega : O_0 \to O_1$ in $\mathcal{O}$, the functor

$$\mathcal{P}^{\simeq}_{O_0} \to \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O_1/}} \mathcal{P}^{\simeq}_{\omega_{\alpha,!}O_1}$$

induced by cocartesian transport along ω_α is an equivalence, where $\omega_{(-)} : \mathcal{O}^{\mathrm{el}}_{Y/} \to \mathcal{O}^{\int}_{X/}$ is the inert morphism appearing in the inert-active factorization of $\alpha \circ \omega$, and
(3) (Segal condition for multimorphism) for every active morphism $\omega : O_1 \to O_2$ in $\mathcal{O}$ and all objects $X_i \in \mathcal{P}_{O_i}$, the commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{P}}(X_0, X_1) \arrow[r] \arrow[d] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O_1/}} \mathrm{Map}_{\mathcal{P}}(X_0, \omega_{\alpha,!}X_1) \arrow[d] \\
\mathrm{Map}_{\mathcal{O}}(O_0, O_1) \arrow[r] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O_1/}} \mathrm{Map}_{\mathcal{O}}(O_0, \omega_{\alpha,!}O_1)
\end{tikzcd}$$

is cartesian.

A fibrous O-pattern $\pi : \mathcal{P} \to O$ is a *Segal O-category* if π is a cocartesian fibration. The category of fibrous O-patterns is the full subcategory

$$\mathrm{Fbrs}(O) \subset \mathrm{AlgPatt}_{/O}$$

spanned by fibrous patterns, and the category of Segal O-∞-category is the full subcategory of

$$\mathrm{Seg}_O(\mathbf{Cat}) \subset \mathrm{Fbrs}(O) \times_{\mathbf{Cat}_{/O}} \mathbf{Cat}_{/O}^{\mathrm{cocart}}$$

spanned by Segal O-categories.

We state one technical lemma:

Lemma 2.3. *All of the inclusions*

$$\mathrm{Seg}(O) \to \mathrm{Fbrs}(O) \hookrightarrow \mathrm{AlgPatt}_{/O} \to \mathbf{Cat}_{/O} \to \mathbf{Cat}$$

have left adjoints; in particular, the full subcategory $\mathrm{Fbrs}(O) \subset \mathrm{AlgPatt}_{/O}$ *is localizing.*

We refer to the left adjoint $\mathrm{Env} : \mathrm{Fbrs}(O) \to \mathrm{Seg}(O)$ as the *Segal envelope*, and we use it analogously to the *symmetric monoidal envelope*, reducing the question of characterizing maps of fibrous patterns into Segal O-categories into simply a question of characterizing maps of Segal O-categories, which is much simpler.

Example 2.4:

Definition 2.5. Given the data of $\mathcal{X}$ a category, $\mathcal{X}_b, \mathcal{X}_f$ wide subcategories, and $\mathcal{X}_0 \subset \mathcal{X}_b$ a full subcategory, we define the *span pattern* $\mathrm{Span}_{b,f}(\mathcal{X}; \mathcal{X}_0)$ to have:

- underlying category $\mathrm{Span}_{b,f}(\mathcal{X})$ whose objects are objects in $\mathcal{X}$ and whose morphisms $X \to Z$ are spans
$$X \xleftarrow{B} Y \xrightarrow{F} Z$$
with $B \in \mathcal{X}_b$ and $F \in \mathcal{X}_f$.
- inert morphisms $\mathcal{X}_b^{\mathrm{op}} \subset \mathrm{Span}(\mathcal{X})$.
- active morphisms $\mathcal{X}_f \subset \mathrm{Span}(\mathcal{X})$.
- Elementary objects $\mathcal{X}_0^{\mathrm{el}} \subset \mathcal{X}_b^{\mathrm{op}}$.

Then, for instance we have the following:

Theorem 2.6 ([BHS22]). *Pullback along the inclusion* $\mathbb{F}_* \hookrightarrow \mathrm{Span}(\mathbb{F})$ *induces an equivalence on the categories of fibrous patterns and Segal categories.*

2.2. **G-operads and $\mathcal{I}$-operads.** There is an adjunction

$$\mathrm{Tot} : \mathbf{Cat}_G \rightleftarrows \mathbf{Cat} : \mathrm{CoFr}^G$$

where Tot takes the total category of a cocartesian fibration and $\mathrm{CoFr}^G(C)$ is classified by functor categories

$$\mathrm{CoFr}^G(C)_H := \mathrm{Fun}(O_H^{\mathrm{op}}, C)$$

with functoriality dictated by pullback. In particular, the *G-category of small G-categories* $\underline{\mathbf{Cat}}_G := \mathrm{CoFr}^G(C)$ has G-fixed points given by $\mathbf{Cat}$.

Remark. Elmendorf's theorem may be reinterpreted in this language as the statement that the *G-category of G-spaces* $\mathcal{S}_G$ is cofreely generated by $\mathcal{S}$.

Let $\underline{\mathbb{F}}_G := \mathrm{CoFr}^G(\mathbb{F})$ and let $\underline{\mathbb{F}}_{G,*} := \mathrm{CoFr}^G(\mathbb{F}_*)$. Then, there is an equivariant lift of ref :

Theorem 2.7 ([BHS22]). *Pullback along the composition* $\underline{\mathbb{F}}_{G,*} \hookrightarrow \mathrm{Span}(\mathrm{Tot}\underline{\mathbb{F}}_G) \xrightarrow{U} \mathrm{Span}(\mathbb{F}_G)$ *induces an equivalence on the categories of fibrous patterns and Segal categories, where* $\mathbb{F}_G$ *is the category of G-sets.*

Definition 2.8. The *category of G-operads* is the category of fibrous patterns

$$\mathrm{Op}_G := \mathrm{Fbrs}(\mathrm{Span}(\mathbb{F}_G)).$$

A good sanity check is to verify that the category of G-symmetric monoidal categories agrees with the category of Segal $\mathrm{Span}(\mathbb{F}_G)$-categories; after some argumentation, one finds that the Segal conditions associated with the unstraightening of a cocartesian fibration over $\mathrm{Span}(\mathbb{F}_G)$ are precisely the condition that the unstraightened functor preserves products in $\mathrm{Span}(\mathbb{F}_G)$.

This is a straightforward, but heavy, generalization of the ∞-operads of [HA] to the equivariant world, and we suggest the interested reader consult [BHS22] for a less heavy variant or [NS22] for the original. In particular, postcomposition along the inclusion functor $\mathbb{F} \hookrightarrow \mathrm{Tot}\underline{\mathbb{F}}_{G,*}$ induces a fully faithful functor $\mathrm{Op} \hookrightarrow \mathrm{Op}_G$.

An early observation about genuine equivariant homotopy coherent algebraic structures is that the structure of transfers *does not come canonically* from an $\mathbb{E}_\infty$-structure; that is, $\mathbb{E}_\infty \in \mathrm{Op}_G$ is not terminal. The failure of $\mathbb{E}_\infty$ to be terminal is parameterized by the category of *generalized N^∞-operads*:

Definition 2.9. Write $\mathrm{Comm}_G^\otimes := (\mathrm{Span}(\mathbb{F}_G) = \mathrm{Span}(\mathbb{F}_G))$ for the terminal G-operad. A G-operad $\mathcal{O}^\otimes$ is a *generalized $\mathcal{N}^\infty$-operad* if the unique morphism $\mathcal{O}^\otimes \to \mathrm{Comm}_G^\otimes$ is a monomorphism, i.e. $\mathcal{O}_U^\otimes \simeq *$ for all U and $\mathrm{Map}_{\mathcal{O}}^{\psi}(x, y) \in \{*, \varnothing\}$ for all $\psi : \pi(x) \to \pi(y)$.

A generalized $\mathcal{N}^\infty$ operad $\mathcal{N}_{\infty I}$ is an N^∞ *operad* if it admits a map

$$\mathbb{E}_\infty \to \mathcal{O}^\otimes.$$

Write $\mathrm{Op}_G^{GN\infty}$ for the full subcategory consisting of generalized $\mathcal{N}_\infty$-operads. The following proposition is an exercise in category theory, and establishes that a map to an $\mathcal{N}_\infty$ operad is a *property*, not a structure.

Proposition 2.10. *Given $\mathcal{N}_{I\infty} \in \mathrm{Op}_G^{GN\infty}$ a generalized $\mathcal{N}_\infty$ operad, the forgetful functor*

$$\mathrm{Op}_{G,/\mathcal{N}_{I\infty}} \to \mathrm{Op}_G$$

is fully faithful.

Proof idea. It is equivalent to prove that $\mathrm{Map}(\mathcal{O}, \mathcal{N}_{I\infty}) \in \{*, \varnothing\}$ for all $\mathcal{O} \in \mathrm{Op}_G$ In fact, there is a localizing (1-) subcategory $N : \mathrm{Op}_{1,G} \hookrightarrow \mathrm{Op}_G$ consisting of operads whose structure spaces are discrete, and whose localization functor $h : \mathrm{Op}_G \to \mathrm{Op}_{1,G}$ takes π_0 of the structure spaces. $\mathcal{N}_{I\infty}$ evidently lies in $\mathrm{Op}_{1,G}$, so we have

$$\mathrm{Map}_{\mathrm{Op}_G}(\mathcal{O}, \mathcal{N}_{I\infty}) \simeq \mathrm{Hom}_{\mathrm{Op}_{1,G}}(h\mathcal{O}, \mathcal{N}_{I\infty}).$$

Hence it suffices to check that the latter set is empty or contractible. This is easy to see in $\mathrm{Op}_{1,G}$, since $\mathrm{Hom}(-, *)$ and $\mathrm{Hom}(-, \varnothing)$ are always either empty or contractible. $\square$

In particular, this implies that $\mathrm{Op}_G^{GN\infty}$ is a poset, so we'd like to identify this poset. There is a functor

$$A : \mathrm{Op}_G \to \widehat{\mathrm{Ind} - \mathrm{Sys}}_G$$

called the *admissible sets* with value over G/H given by

$$A(\mathcal{O})_{/(G/H)} := \left\{S \to G/H \mid \pi_{\mathcal{O}}^{-1}(S \to G/H) \neq \varnothing\right\}$$

and extended to general G-sets by coproducts. The following proposition is an exercise in category theory (see [NS22, Ex 2.4.7], or the original references [BH15; GW18; Rub21]):

Proposition 2.11. *The restricted functor*

$$A : \mathrm{Op}_G^{GN\infty} \to \widehat{\mathrm{Ind} - \mathrm{Sys}}_G$$

is an equivalence of categories.

We denote by $\mathcal{N}_{(-)\infty}$ the composite functor

$$\mathcal{N}_{(-)\infty} : \widehat{\mathrm{Ind} - \mathrm{Sys}}_G \xrightarrow{A^{-1}} \mathrm{Op}_G^{GN\infty} \hookrightarrow \mathrm{Op}_G$$

Using this, we finally define *I-operads*.

Definition 2.12. Let I be a generalized indexing system. Then, the *category of I-operads* is the slice category

$$\mathrm{Op}_I := \mathrm{Op}_{G,/\mathcal{N}^{\otimes}_{\infty I}}.$$

Given $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes} \in \mathrm{Op}_I$, the *category of $\mathcal{O}$-algebras in $\mathcal{P}$* is the full subcategory

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}) \subset \mathrm{Fun}_{/\mathcal{N}^{\otimes}_{\infty I}}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$$

spanned by maps of $\mathcal{I}$-operads.

Remark. The notation $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$ does not include $\mathcal{I}$. This presents no problem; indeed, by proposition 2.10, the categories of $\mathcal{O}$-algebras in $\mathcal{P}$ considered over various indexing systems (including the terminal one, i.e. in G-operads) are canonically equivalent to one another.

Example 2.13:
Let $\mathcal{F} \subset \mathcal{O}_G$ be a *family*, i.e. a collection of subgroups of G closed under sub-conjugation. Then, $\mathcal{F} \cup \mathcal{O}_G^{\simeq}$ is a transfer system, and we denote by $\mathcal{I}_{\mathcal{F}}$ the corresponding indexing system.

Let V be a real orthogonal G-representation, let $\mathcal{F}_V$ is the family consisting of subgroups H such that $V^H \neq *$, and let $\mathcal{I}_V := \mathcal{I}_{\mathcal{F}_V}$. Then, there is an I_V-operad $\mathbb{E}_V$ of *little V-disks*, which may be informally understood to have

$$\pi^{-1}_{\mathbb{E}_V}(\mathrm{Ind}_H^G T \to G/H) := \mathrm{Conf}_H(T, V)$$

the space of H-equivariant embeddings of $T \hookrightarrow V$ (c.f. [Hor19]). These participate in *equivariant infinite loop space theory*, in the sense that there is an equivalence

$$\mathbf{Alg}_{\mathbb{E}_V}(\mathcal{S}_G) \simeq \{V - \textit{loop spaces}\}\,;$$

see Guillou-May for details.

2.3. **The BV tensor product.** By ref , the category of algebraic patterns has a cartesian monoidal structure.

Definition 2.14. The category of *symmetric monoidal algebraic patterns* is $\mathrm{CMon}(\mathrm{AlgPatt})$.

A symmetric monoidal structure on $\mathcal{O}$ endows on the slice category $\mathrm{AlgPatt}^{\otimes}_{/\mathcal{O}}$ a symmetric monoidal structure, which we may view as taking $\mathcal{P}, \mathcal{P}'$ to the tensor product

$$\mathcal{P} \times \mathcal{P}' \to \mathcal{O} \times \mathcal{O} \to \mathcal{O}.$$

Definition 2.15. The *Boardman-Vogt symmetric monoidal category of fibrous $\mathcal{O}$-patterns* is the localized symmetric monoidal structure

$$\mathrm{Fbrs}(\mathcal{O})^{\otimes} \leftrightarrows \mathrm{AlgPatt}^{\otimes}_{/\mathcal{O}}.$$

We may view the tensor product of fibrous $\mathcal{O}$-patterns as yielding the localized composite

$$\mathcal{O} \otimes \mathcal{P}' := L_{\mathrm{Fbrs}}(\mathcal{P} \times \mathcal{P}' \to \mathcal{O} \times \mathcal{O} \to \mathcal{O}).$$

Note that the category $\mathbb{F}_G$ has finite products, and any indexing system $\mathcal{I}$ is closed under products. In particular, this endows $i : \mathcal{N}^{\otimes}_{\mathcal{I}\infty} \to \mathrm{Span}(\mathbb{F}_G)$ with the structure of a map of symmetric monoidal algebraic patterns under the so it has a cartesian monoidal structure. By cite , the forgetful functor $\mathrm{Fbrs}(\mathcal{P}) \to \mathrm{Fbrs}(\mathcal{O})_{/\mathcal{P}}$ is an equivalence, so we may use this to define the BV tensor product of I-operads.

Definition 2.16. The *Boardman-Vogt symmetric monoidal category of I-operads* is

$$\mathrm{Op}^{\otimes}_{\mathcal{I}} := \mathrm{Fbrs}(\mathcal{N}_{\mathcal{I}\infty})$$

The following proposition is easy:

Proposition 2.17. *Given an inclusion $i : \mathcal{N}_{\mathcal{I}\infty} \hookrightarrow \mathcal{N}_{\mathcal{J}\infty}$, pushforward along i yields a functor*

$$i_! : \mathrm{Op}^{\otimes}_{\mathcal{I}} \to \mathrm{Op}^{\otimes}_{\mathcal{J}}$$

realizing $\mathrm{Op}_{\mathcal{I}}$ as a symmetric monoidal colocalizing subcategory of $\mathrm{Op}_{\mathcal{J}}$.

The verification of this comes down to the following fact:

Lemma 2.18. *Given $f : X \to Y$ a map of commutative algebra objects in $\mathcal{C}$ a symmetric monoidal, the associated functor $f_! : \mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ lifts to a canonical symmetric monoidal functor between the slice symmetric monoidal categories.*

Given $\mathcal{O}, \mathcal{P} \in \mathrm{Op}_{\mathcal{I}}$, their BV tensor product has a mapping out property:

Proposition 2.19. *The category $\mathbf{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\mathcal{Q})$ is equivalent to the category of commutative diagrams of algebraic patterns*

$$\begin{tikzcd} \mathcal{O} \times \mathcal{P} \arrow[r] \arrow[d, "\pi_{\mathcal{O}} \times \pi_{\mathcal{P}}"] & \mathcal{Q} \arrow[d, "\pi_{\mathcal{Q}}"] \\ \mathcal{N}^{\otimes}_{\mathcal{I}\infty} \times \mathcal{N}^{\otimes}_{\mathcal{I}\infty} \arrow[r, "\otimes"] & \mathcal{N}_{\mathcal{I}\infty} \end{tikzcd}$$

An $\mathcal{I}$-operad called the *pointwise tensor product* on $\mathbf{Alg}_{\mathcal{P}}(\mathcal{Q})$ was constructed in [NS22]. By argument......... , this implies the following proposition:

Proposition 2.20. *There is a natural equivalence*

$$\mathbf{Alg}_{\mathcal{O}\otimes\mathcal{P}}\mathcal{Q} \simeq \mathbf{Alg}_{\mathcal{O}}\mathbf{Alg}^{\otimes}_{\mathcal{P}}\mathcal{Q}$$

realizing $-\otimes\mathcal{P}$ as left adjoint to $\mathbf{Alg}^{\otimes}_{\mathcal{P}}(-)$.

2.4. **Summary of the argument.** We would like to construct an equivalence $\mathcal{N}_{I\infty} \otimes \mathcal{N}_{J\infty} \simeq \mathcal{N}_{(I\vee J)\infty}$. Let's begin with the special case $I \subset J$; in this case, we can say something stronger.

Proposition 2.21. *If $\mathcal{O}$ is a one-object G-operad, then the map $\mathcal{N}^{\infty}(I) \to \mathcal{N}^{\infty}(I) \otimes \mathcal{O}$ is an I-equivalence; in particular, $\mathcal{N}^{\infty}(I)$ is $\otimes$-idempotent.*

To prove this, we use [NS22, Cor 5.3.9]; in particular, they generalize [HA] to verify that any of the following conditions are true of $\mathbf{Alg}^{\otimes}_{\mathcal{N}^{\infty}(I)}(\mathcal{C})$, and we verify that the conditions are equivalent in ref .

Lemma 2.22. *The following are equivalent:*

(1) *The forgetful functor $\mathrm{CAlg}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*
(2) *For all one-object I-operads $\mathcal{O}$, the forgetful functor $\mathbf{Alg}^{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*
(3) *The I-restricted operad is cocartesian*

Having proved this, we acquire a (unique) diagram

$$\begin{tikzcd} \mathcal{N}_{I\infty} \arrow[dr] \arrow[drr] & & \\ & \mathcal{N}_{I\infty} \otimes \mathcal{N}_{J\infty} \arrow[r, "\varphi"] & \mathcal{N}_{(I\vee J)\infty} \otimes \mathcal{N}_{(I\vee J)\infty} = \mathcal{N}_{(I\vee J)\infty} \\ \mathcal{N}_{J\infty} \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

and we are tasked with proving that φ is an equivalence. An unfortunate fact is that the functor $U : \mathrm{Op}_{I\vee J} \to \mathrm{Op}_I \times \mathrm{Op}_J$ doesn't appear to be conservative in general. Our strategy will come down to trying *really hard* to make it conservative. We do so via the following two lemmas, proved as ref .

Lemma 2.23. *Denote by $i : I \cup J \subset I \vee J$ the (non-indexing system) union of subcategories. Then, the pullback*

$$\mathrm{Fbrs}(\mathrm{Span}(\mathcal{I} \cup \mathcal{J})) \to \mathrm{Op}_I \times \mathrm{Op}_J$$

is conservative. In particular, U reflects equivalences between $\mathcal{I} \vee \mathcal{J}$-operads in the image of $L_{\mathrm{Fbrs}} i_!$.

Lemma 2.24. *There is an equivalence $\mathcal{N}_{(I\vee J)\infty} \simeq L_{\mathrm{Fbrs}} i_! \,\mathrm{Span}(I \cup J)$.*

Proof of theorem A. By the above argument, it suffices to prove that φ is an equivalence; in fact, by lemmas 2.23 and 3.6 and symmetry it suffices to prove that the localized functor

$$\iota_J^* \mathcal{N}_{I\cap J\infty} \otimes \mathcal{N}_{J\infty} \to \iota_J^* \mathcal{N}_{I\vee J}$$

is an equivalence. But $\iota_J^* \mathcal{N}_{I\infty} \simeq \mathcal{N}_{I\cap J\infty}$, so the above is the inclusion $\mathcal{N}_{I\cap J\infty} \otimes \mathcal{N}_{J\infty} \to \mathcal{N}_{J\infty}$, which is an equivalence by proposition 2.21. □

3. Technical nonsense

3.1. **Passing to monads is conservative.** Our arguments will be reminiscent of [SY19, § 2.3-2.4]

Given $\mathcal{P} \to \mathcal{O}$ a fibrous pattern, we define

$$\mathrm{Ar}^{\simeq}_{\mathrm{act/el}}(\mathcal{O}) \subset \mathrm{Ar}(\mathcal{O})$$

to be the core of the full subcategory of the arrow category consisting of active maps with elementary codomain, and we define

$$\mathcal{P}_{\Sigma} := \mathrm{Ar}(\mathcal{P}) \times_{\mathrm{Ar}(\mathcal{O})} \mathrm{Ar}^{\simeq}_{\mathrm{act/el}}(\mathcal{O}),$$

which we view as the *associated symmetric sequence.*

Lemma 3.1 (C.f. [SY19, Prop 2.3.6]). *Let* $\mathrm{Fbrs}_{\bullet}(\mathcal{O})$ *denote the full subcategory of fibrous patterns whose associated maps* $\mathcal{P}^{\mathrm{el}} \to \mathcal{O}^{\mathrm{el}}$ *are equivalences. Then, the functor*

$$(-)_{\Sigma} : \mathrm{Fbrs}_{\bullet}(\mathcal{O}) \to \mathrm{Fun}\left(\mathrm{Ar}^{\simeq}_{\mathrm{act/el}}(\mathcal{O}), \mathcal{S}\right)$$

is conservative.

Proof. Just look at the Segal condition for fibrous patterns □

In the case $\mathcal{O} = \mathrm{Span}(\mathbb{F}_G)$, note that an element of $\mathrm{Ar}_{\mathrm{act/el}}(\mathrm{Span}(\mathbb{F}_G))$ is precisely a map of G-sets $S \to G/H$; but in fact, there is a unique H-set T and equivalence $\mathrm{Ind}_H^G T \simeq S$ over G/H, highlighting an equivalence $\mathbb{F}_{G,/G/H} \simeq \mathbb{F}_H$. Hence we have

$$\mathrm{Ar}_{\mathrm{act/el}}(\mathrm{Span}(\mathbb{F}_G)) \simeq \mathrm{Tot}\underline{\mathbb{F}}_G,$$

and $\mathrm{Ar}^{\simeq}_{\mathrm{act/el}}(\mathrm{Span}(\mathbb{F}_G)) \simeq (\mathrm{Tot}\underline{\mathbb{F}}_G)^{\simeq}$ Setting $\overline{\underline{\Sigma}}_G := (\mathrm{Tot}\underline{\mathbb{F}}_G)^{\simeq}$, the above lemma asserts that

$$(-)_{\Sigma} : \mathrm{Op}_G \to \mathrm{Fun}(\overline{\underline{\Sigma}}_G, \mathcal{S})$$

is conservative.

Remark. Let $\underline{\Sigma}_G := \mathrm{CoFr}^G(\mathbb{F}^{\simeq})$, so that $\overline{\underline{\Sigma}}_G \simeq \left(\mathrm{Tot}\underline{\Sigma}_G\right)^{\simeq}$. Then, the above lemma implies that the evident forgetful functor $U : \mathrm{Op}_G \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_G, \mathcal{S})$ is conservative. The *genuine model structure* $\mathrm{Sym}^G_{\bullet}(\mathrm{sSet})$ of [BP22] exists and presents $\mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_G, \mathcal{S})$; this model category has a *composition product* for which monoids are a model for *genuine G-operads*, which are not known to be equivalent to G-operads.

In this setting, lemma 3.1 amounts to a verification of one of the two Barr-Beck conditions expressing U as *monadic* (cf [HA, Thm 4.7.3.5]); if one can verify that U creates spit geometric realizations and characterize the associated monad along the lines of [BP22, § A], then they may prove that one-object genuine G-operads are equivalent to one-object G-operads.

We say that a G-operad $\mathcal{O}$ is *reduced* if $\mathcal{O}_{\Sigma}(\mathrm{Ind}_H^G T \to G/H) = *$ whenever T is empty or an orbit. In this setting, we can characterize the *monad* associated with an operad:

Proposition 3.2. *Let* $\mathcal{O}$ *be a reduced G-operad and let* $\mathcal{C} \in \mathrm{CAlg}_G(\mathrm{Pr}^L_G)$ *be a presentably G-symmetric monoidal category. Then, the forgetful map* $\underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ *is monadic, and the associated monad* $T_{\mathcal{O}}$ *acts on* $X \in \mathcal{C}$ *as*

$$(T_{\mathcal{O}}X)_H \simeq \coprod_{\substack{J \supset K \subset H \\ S \in \mathbb{F}_J}} \left(\mathcal{O}(S) \otimes X^{\otimes\left(\mathrm{Ind}_K^H \mathrm{Res}_K^J S\right)}\right)_{h\,\mathrm{Aut}_J S},$$

where for all $S' \in \mathbb{F}_H$*, we write*

$$X^{\otimes S'} := \bigotimes_{U \in \mathrm{Orb}(S')} N_U^H X_U.$$

Reference what presentability is and prove this

Corollary 3.3. *The functor* $\mathbf{Alg}_{(-)}(\mathcal{S}_G) : \mathrm{Op}_G^{\mathrm{Red}} \to \mathbf{Cat}_G$ *is conservative.*

3.2. **The conservativity lemmas.** We have two conservativity lemmas to prove. The first is easier:

Lemma 3.4. *Denote by $i : I \cup J \subset I \vee J$ the (non-indexing system) union of subcategories. Then, the pullback*

$$\mathrm{Fbrs}(\mathrm{Span}(\mathcal{I} \cup \mathcal{J})) \to \mathrm{Op}_I \times \mathrm{Op}_J$$

is conservative. In particular, U reflects equivalences between $\mathcal{I} \vee \mathcal{J}$-operads in the image of $L_{\mathrm{Fbrs}} i_!$.

Proof. Passing to the underlying symmetric sequences yields a diagram

$$\begin{tikzcd} \mathrm{Fbrs}(\mathrm{Span}(I \cup J)) \arrow[r, "i^*"] \arrow[d, tail] & \mathrm{Op}_I \times \mathrm{Op}_J \arrow[d] \\ \mathrm{Fun}(I \cup J, \mathcal{S}) \arrow[r, tail] & \mathrm{Fun}(I, \mathcal{S}) \times \mathrm{Fun}(J, \mathcal{S}) \end{tikzcd}$$

The diagonal functor is a composite of two conservative arrows by **??**, so it is conservative, and hence i^* is conservative. □

The second will take a bit more work. Note that the Segal conditions for Segal $\mathrm{Span}(I \cup J)$-categories are a *Union* of those of Segal $\mathrm{Span}(I)$-categories and Segal $\mathrm{Span}(J)$-categories. That is,

Lemma 3.5. *The following diagram of categories is cartesian:*

$$\begin{tikzcd} \mathrm{Seg}_{\mathrm{Span}(I \cup J)}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Seg}_{\mathrm{Span}(I)}(\mathcal{C}) \arrow[d] \\ \mathrm{Seg}_{\mathrm{Span}(J)}(\mathcal{C}) \arrow[r] & \mathrm{Seg}_{\mathrm{Span}(I \cap J)}(\mathcal{C}) \end{tikzcd}$$

In particular, all but the top left are simply categories of product preserving functors. We use this:

Lemma 3.6. *There is an equivalence $\mathcal{N}_{(I \vee J)\infty} \simeq L_{\mathrm{Fbrs}} i_! \,\mathrm{Span}(I \cup J)$.*

Proof. The functor $L_{\mathrm{Fbrs}} i_! \,\mathrm{Span}(I \cup J)$ is left adjoint to i^*, so it suffices by lemma to verify that the following square is cartesian:

$$\begin{tikzcd} \mathrm{Fun}^{\times}(\mathrm{Span}(I \vee J), \mathcal{S}) \arrow[r] \arrow[d] & \mathrm{Fun}^{\times}(\mathrm{Span}(I), \mathcal{S}) \arrow[d] \\ \mathrm{Fun}^{times}(\mathrm{Span}(J), \mathcal{S}) \arrow[r] & \mathrm{Fun}^{\times}(\mathrm{Span}(I \cap J, \mathcal{S}) \end{tikzcd}$$

The property that this square is cartesian is witnessed by the equivalence

$$\mathrm{Span}(I \vee J) \simeq \mathrm{Span}(I) \coprod_{\mathrm{Span}(I \cap J)} \mathrm{Span}(J),$$

with pushout taken in the category of Cartesian categories and product preserving functors. □

3.3. **Identifying cocartesian symmetric monoidal structures.**

3.4. **The pointwise tensor product is an internal hom.**

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