ON TENSOR PRODUCTS OF EQUIVARIANT COMMUTATIVE OPERADS

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Abstract. We lift the Boardman-Vogt tensor product to a symmetric monoidal closed G-∞-category $\underline{\mathrm{Op}}_G^{\otimes}$ of G-operads, whose underlying ∞-category consists of Nardin-Shah's *$\mathcal{O}_G$-∞-categories*. This possesses a fully faithful G-right adjoint

$$\mathcal{N}_{(-)\infty} : \underline{\mathrm{wIndex}}_G \hookrightarrow \underline{\mathrm{Op}}_G$$

with image the *G-poset of weak $\mathcal{N}_\infty$-G-operads* whose left adjoint constructs the *arity support* weak indexing system.

We precisely characterize the weak $\mathcal{N}_\infty$-G-operads whose tensor powers are weak $\mathcal{N}_\infty$, called the *aE-unital weak $\mathcal{N}_\infty$-G-operads*. We show that the G-subcategory

$$\underline{\mathrm{wIndex}}_G^{aE-\mathrm{uni},\otimes} \hookrightarrow \underline{\mathrm{Op}}_G^{\otimes}$$

is symmetric monoidal and combinatorially characterize its tensor products; in particular, the full G-subcategory of *unital weak $\mathcal{N}_\infty$-G-operads* is cocartesian symmetric monoidal, i.e. its tensor products are joins of (unital) weak indexing systems.

As a special case, we recognize Blumberg-Hill's $\mathcal{N}_\infty$-operads as a symmetric monoidal sub-poset $\mathrm{Index}_G^{\vee} \subset \mathrm{wIndex}_G^{\mathrm{uni},\vee}$ confirming a conjecture of Blumberg-Hill. In particular, for I, J unital weak indexing systems and $\mathcal{C}$ an $I \vee J$-symmetric monoidal ∞-category, we construct a canonical $I \vee J$-symmetric monoidal equivalence

$$\underline{\mathrm{CAlg}}_I^{\otimes} \underline{\mathrm{CAlg}}_J^{\otimes} \mathcal{C} \simeq \underline{\mathrm{CAlg}}_{I\vee J}^{\otimes} \mathcal{C}.$$

Contents

Introduction

Summary of main results. Fix G a finite group. G-equivariantly homotopy-coherent algebraic structures are naturally founded in the notion of homotopical *G-commutative monoids* (i.e. *G (semi-)Mackey functors*). In the context of this paper, the *∞-category*[1] *of G-commutative monoids* in an ∞-category $\mathcal{D}$ will refer to the ∞-category of product-preserving functors

$$\mathrm{CMon}_G(\mathcal{D}) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_G), \mathcal{D}),$$

where $\mathrm{Span}(\mathbb{F}_G)$ is the effective Burnside 2-category of G and $\mathbb{F}_G$ is the 1-category of finite G-sets. The ∞-category of *small G-symmetric monoidal ∞-categories* is $\mathbf{Cat}_G^{\otimes} := \mathrm{CMon}_G(\mathbf{Cat})$, where $\mathbf{Cat}$ denotes the ∞-category of small ∞-categories.[2]

Given $\mathcal{C}^{\otimes} \in \mathbf{Cat}_G^{\otimes}$ a G-symmetric monoidal ∞-category, the product-preserving functor

$$\iota_H : \mathrm{Span}(\mathbb{F}) \xrightarrow{* \mapsto G/H} \mathrm{Span}(\mathbb{F}_G)$$

constructs a symmetric monoidal ∞-category $\mathcal{C}_H^{\otimes} := \iota_H^* \mathcal{C}^{\otimes}$ whose underlying ∞-category $\mathcal{C}_H$ is the *value* of $\mathcal{C}^{\otimes}$ on the orbit G/H.[3] For all subgroups $K \subset H \subset G$, the covariant and contravariant functoriality of $\mathcal{C}^{\otimes}$ then yield symmetric monoidal *restriction* and *norm* functors

$$\mathrm{Res}_K^H : \mathcal{C}_H^{\otimes} \to \mathcal{C}_K^{\otimes},$$
$$N_K^H : \mathcal{C}_K^{\otimes} \to \mathcal{C}_H^{\otimes}.$$

We use this structure to encode *algebras* in $\mathcal{C}^{\otimes}$, for which we need a notion of *G-operads*.

Various notions of G-operad have been introduced for this. In Section 2.3 we introduce an ∞-category Op_G of *$\mathcal{O}_G$-∞-operads* (henceforth G-operads) equivalent to that of [NS22]. Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_G$ a G-operad and $S \in \mathbb{F}_H$ an H-set for some $H \subset G$, we construct a *space of S-ary operations* $\mathcal{O}(S)$, together with *operadic composition maps*

$$\mathcal{O}(S) \otimes \bigotimes_{H/K_i \in \mathrm{Orb}(S)} \mathcal{O}(T_i) \to \mathcal{O}\left(\coprod_{H/K_i \in \mathrm{Orb}(S)} \mathrm{Ind}_{K_i}^H T_i \right), \tag{1}$$

operadic restriction maps

$$\mathcal{O}(S) \to \mathcal{O}(\mathrm{Res}_K^H S), \tag{2}$$

[1] In this paper we will call ∞-categories *∞-categories* and 0-truncated ∞-categories *1-categories*. We hope this prevents avoidable confusion in older readers.

[2] When the underlying ∞-category is a 1-category, these differ from Hill-Hopkins' *symmetric monoidal Mackey functors* only by asserting some reasonable coherence diagrams for the double coset formula isomorphisms; see Section 5.2 for details.

[3] In this paper, "orbits" refer to transitive G-sets, i.e. objects of the orbit category $\mathcal{O}_G \subset \mathbf{Set}_G$ spanned by transitive G-sets.

and *equivariant symmetric group action*

$$\mathrm{Aut}_H(S) \times \mathcal{O}(S) \to \mathcal{O}(S). \tag{3}$$

Eqs. (2) and (3) together ascend to a structure of a *G-symmetric sequence*; we go on to show in Corollary 2.78 that this structure is *monadic* over G-symmetric sequences under a reducedness assumption.

Definition. We say that $\mathcal{O}^\otimes$ *has at least one color* if $\mathcal{O}(*_H)$ is nonempty for all subgroups $H \subset G$, and we say $\mathcal{O}^\otimes$ *has at most one color* if $\mathcal{O}(*_H) \in \{*, \varnothing\}$ for all $H \subset G$. We say that $\mathcal{O}^\otimes$ *has one color* if it has at least one color and at most one color. ◁

When $\mathcal{O}^\otimes$ has one color, an $\mathcal{O}$-algebra in the G-symmetric monoidal ∞-category $\mathcal{C}^\otimes$ can intuitively be viewed as a tuple $(X_H \in \mathcal{C}_H^{BW_G(H)})_{G/H \in \mathcal{O}_G}$ satisfying $X_K \simeq \mathrm{Res}_K^H X_H$, together with $\mathcal{O}(S)$-actions

$$\mu_S : \mathcal{O}(S) \otimes X_H^{\otimes S} \to X_H \tag{4}$$

for all $S \in \mathbb{F}_H$ and $H \subset G$, homotopy-coherently compatible with the maps Eqs. (1) to (3), where we write

$$X_H^{\otimes S} := \bigotimes_{H/K \in \mathrm{Orb}(S)} N_K^H \mathrm{Res}_K^H X_H.$$

for the *indexed tensor powers* in $\mathcal{C}^\otimes$.

Example. There exists a terminal G-operad $\mathrm{Comm}_G^\otimes$, which is characterized up to (unique) equivalence by the property that $\mathrm{Comm}_G(S)$ is contractible for all $S \in \mathbb{F}_H$; its algebras are endowed with contractible spaces of maps $X_H^{\otimes S} \to X_H$ for all $S \in \mathbb{F}_H$, as well as coherent homotopies witnessing their compatibility. We see in Section 5.2 that Comm_G-algebras present a homotopical lift of Hill-Hopkins' *G-commutative monoids*, though we prefer to reserve this name for the Cartesian case, following the convention of [HA]. ◁

In this paper, we are concerned with indexed tensor products of $\mathcal{O}$-algebras as well as $\mathcal{P}$-algebras in the resulting G-symmetric monoidal ∞-category. Mirroring the nonequivariant case, we will accomplish this by realizing an operad of $\mathcal{O}$-algbras in $\mathcal{P}$ as the *internal hom* with respect to a symmetric monoidal structure on the ∞-category of G-operads.

In order to characterize this tensor product, we will relate it to a tensor product on the category of G-symmetric monoidal ∞-categories. In Section 1.1 we define the ∞-category of *G-∞-categories* to be

$$\mathbf{Cat}_G \coloneqq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Cat}).$$

As a weakening of the notion of a G-symmetric monoidal ∞-category, we define a *symmetric monoidal G-∞-category* to be a commutative monoid object in $\mathbf{Cat}_G$. The restriction structure between the ∞-categories $\mathrm{CMon}_G(\mathcal{C})$ is summarized defining a G-∞-category $\underline{\mathrm{CMon}}_G(\mathcal{C})$ with the values

$$(\underline{\mathrm{CMon}}_G(\mathcal{C}))_H \coloneqq \mathrm{CMon}_H(\mathcal{C}).$$

Our first theorem establishes a symmetric monoidal structure on $\underline{\mathrm{CMon}}_G(\mathbf{Cat}) \coloneqq \underline{\mathbf{Cat}}_G^\otimes$ satisfying an analogous universal property to [GGN15, Thm 5.1], this time based on G-∞-category of coefficient systems

$$(\underline{\mathrm{Coeff}}_G\mathcal{C})_H \coloneqq \mathrm{Fun}(\mathcal{O}_H^{\mathrm{op}}, \mathcal{C}).$$

Theorem A. *If $\mathcal{C}$ is a presentably symmetric monoidal ∞-category, then there exists a unique presentably symmetric monoidal structure $\underline{\mathrm{CMon}}_G^{\otimes-\mathrm{mode}}(\mathcal{C})$ on $\underline{\mathrm{CMon}}_G(\mathcal{C})$ such that the free G-commutative monoid G-functor*

$$\underline{\mathrm{Coeff}}_G\mathcal{C} \to \underline{\mathrm{CMon}}_G(\mathcal{C})$$

possesses a (necessarily unique) symmetric monoidal structure.

In Section 1.3, we generalize Theorem A to *G-presentable ∞-categories*, e.g. as developed in [Hil24]. We use this to define the coherences on a *Boardman-Vogt symmetric monoidal structure on G-operads.*

Theorem B. *There exists a unique symmetric monoidal structure $\underline{\mathrm{Op}}_G^\otimes$ on $\underline{\mathrm{Op}}_G$ attaining a (necessarily unique) symmetric monoidal structure on the fully faithful G-functor*

$$\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\amalg}} : \underline{\mathrm{Op}}_G^\otimes \to \underline{\mathbf{Cat}}_{G,/\underline{\mathbb{F}}_G^{\sqcup}}^{\otimes-\mathrm{mode}},$$

Furthermore, $\underline{\mathrm{Op}}_G^\otimes$ satisfies the following properties.

(1) *In the case $G = e$ is the trivial group, there is a canonical symmetric monoidal equivalence $\mathrm{Op}_e^{\otimes} \simeq \mathrm{Op}_{\infty}^{\otimes}$, under the symmetric monoidal structure of [BS24a]; in particular, the underlying tensor product is equivalent to that of [BV73; HM23; HA].*

(2) *The underlying tensor functor $- \overset{BV}{\otimes} \mathcal{O} : \mathrm{Op}_G \to \mathrm{Op}_G$ possesses a right adjoint $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(-)$, whose underlying G-∞-category is the G-∞-category of algebras $\underline{\mathbf{Alg}}_{\mathcal{O}}(-)$; the associated ∞-category is the ∞-category of algebras $\mathbf{Alg}_{\mathcal{O}}(-)$.*

(3) *The $\overset{BV}{\otimes}$-unit of $\mathrm{Op}_G^{\otimes}$ is the G-operad $\mathrm{triv}_G^{\otimes}$ defined in [NS22]; hence $\underline{\mathbf{Alg}}_{\mathrm{triv}_G}^{\otimes}(\mathcal{O}) \simeq \mathcal{O}^{\otimes}$.*

(4) *When $\mathcal{C}^{\otimes}$ is a G-symmetric monoidal ∞-category, $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is a G-symmteric monoidal ∞-category; furthermore, when $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a map of G-operads, the pullback G-functor*

$$\underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \to \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$$

is G-symmetric monoidal; in particular, if $\mathcal{O}^{\otimes}$ has one object, then pullback along the canonical map $\mathrm{triv}_G^{\otimes} \to \mathcal{P}^{\otimes}$ presents the unique natural transformation of operads

$$\underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \to \mathcal{C}^{\otimes},$$

and this is G-symmetric monoidal when $\mathcal{C}$ is G-symmetric monoidal.

(5) *When $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a G-symmetric monoidal functor, the induced lax G-symmetric monoidal functor*

$$\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{D})$$

is G-symmetric monoidal.

Remark. In analogy to [BV73], in Observation 2.32 we interpret algebras over the BV-tensor product $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ in a G-symmetric monoidal category $\mathcal{C}^{\otimes}$ as *bifunctors of G-operads $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathcal{C}^{\otimes}$*; unwinding definitions in the case $\mathcal{C}^{\otimes}$ is G-symmetric monoidal, we interpret these as *interchanging pairs of $\mathcal{O}$- and $\mathcal{P}$-algebras structures on an object of $\mathcal{C}$* in Observation 5.19; we show that this fully determines $\overset{\mathrm{BV}}{\otimes}$ in Corollary 4.5.

Furthermore, by Yoneda's lemma, the G-operad $\underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C})$ itself is determined by the property that its $\mathcal{O}$-algebras are interchanging pairs of $\mathcal{O}$- and $\mathcal{P}$-algebra structures on an object in $\mathcal{C}$; we show in Philosophical remark 4.1 that G-symmetric monoidal ∞-categories are determined by their underlying G-operads, so this fully determines $\underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C})$ as a G-symmetric monoidal ∞-category.

Lastly, in Proposition 4.19 we show that, under the *G-symmetric monoidal envelope* equivalence $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Fun}_G^{\otimes}(\mathrm{Env}\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$, the G-symmetric monoidal structure on algebras corresponds with the *pointwise G-symmetric monoidal structure* of [NS22, § 3.3]; intuitively indexed tensor products of $\mathcal{O}$-algebras are simply indexed tensor products of their underlying H-objects with the "diagonal" $\mathcal{O}$-algebra structure. ◁

Remark. After this introduction, we replace $\mathcal{O}_G$ with an atomic orbital ∞-category $\mathcal{T}$ for the remainder of the paper; we prove Theorem B as well as other theorems in this introduction in this setting, greatly generalizing the stated results, solely at the cost of ease of exposition. ◁

Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_G^{\mathrm{oc}}$ a G-operad with one color and $\psi : T \to S$ a map of finite H-sets, we also define the *space of multimorphisms*[4]

$$\mathrm{Mul}_{\mathcal{O}}^{\psi}(T; S) := \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T \times_S U).$$

We then define the subcategory[5] $A\mathcal{O} \subset \mathbb{F}_G$ of *$\mathcal{O}$-admissible maps* by

$$A\mathcal{O} := \left\{ \psi : T \to S \;\middle|\; \mathrm{Mul}_{\mathcal{O}}^{\psi}(T; S) \neq \varnothing \right\} \subset \mathbb{F}_G.$$

[4] We only make the assumption that $\mathcal{O}^{\otimes}$ has one color for ease of exposition; throughout the remainder of text following the introduction, we will not make this assumption.

[5] Throughout this paper, we say *subobject* to mean monomorphism in the sense of [HTT, § 5.5.6]; in the case the ambient ∞-category is a 1-category, this agrees with the traditional notion.

In the case our objects are in the ∞-category **Cat** of small ∞-categories, we call this a *subcategory*; in the case that the containing ∞-category is a 1-category, this is canonically expressed as a *core-preserving wide subcategory of a full subcategory*, i.e. it is a *replete subcategory*. Hence it is uniquely determined by its morphisms, so we will implicitly identify subcategories of $\mathcal{C}$ a 1-category with their corresponding subsets of $\mathrm{Mor}(\mathcal{C})$.

In essence, taking tensor products of Eq. (4) yields an action

$$\mathrm{Mul}_{\mathcal{O}}^{\psi}(T;S) \otimes X_H^{\otimes T} \to X_H^{\otimes S},$$

and $A\mathcal{O}$ consists of the *pairs of equivariant arities* over which this produces structure on X.

The fact that $\varnothing$ accepts no maps from nonempty sets potentially obstructs construction of maps as in Eqs. (1) and (2), so $A\mathcal{O}$ can't be an *arbitrary* subcategory. In order to state restrictions on $A\mathcal{O}$, we introduce some terminology; we say that a G-operad $\mathcal{O}^{\otimes}$ is *E-unital* if

$$\mathcal{O}(\varnothing_V) = \begin{cases} * & \mathcal{O}(*_V) \neq \varnothing; \\ \varnothing & \mathcal{O}(*_V) = \varnothing. \end{cases}$$

We say that $\mathcal{O}^{\otimes}$ is *unital* if it is E-unital and has at least one color, and we say that $\mathcal{O}^{\otimes}$ is *reduced* if it is E-unital and has one color. More generally, we say that $\mathcal{O}^{\otimes}$ is *almost E-unital* (henceforth aE-unital) if whenever $S \in \mathbb{F}_V$ is noncontractible and $\mathcal{O}(S) \neq \varnothing$, we have $\mathcal{O}(\varnothing_V) = *$, and we say that $\mathcal{O}^{\otimes}$ is *almost-unital* if it is almost E-unital and has at least one color. We denote the full subcategory spanned by unital G-operads by $\mathrm{Op}_G^{\mathrm{uni}} \subset \mathrm{Op}_G$.

Theorem C. *The following posets are each equivalent:*

1. *The poset* $\mathrm{Sub}_{\mathrm{Op}_G}(\mathrm{Comm}_G)$ *of* sub-commutative G-operads.
2. *The poset* $\mathrm{Op}_{G,0}$ *of* G-0-operads.
3. *The essential image* $A(\mathrm{Op}_G) \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_G)$
4. *The sub-poset* $\mathrm{wIndexCat}_G \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_G)$ *spanned by subcategories $I \subset \mathbb{F}_G$ which are closed under base change and automorphisms and satisfy the Segal condition that*
$$T \to S \in I \qquad \iff \qquad \forall U \in \mathrm{Orb}(S), \quad T \times_S U \to U \in I$$
5. *The sub-poset* $\mathrm{wIndex}_G \subset \mathrm{FullSub}_G(\underline{\mathbb{F}}_G)$ *spanned by full G-subcategories $\mathcal{C} \subset \underline{\mathbb{F}}_G$ which are closed under self-indexed coproducts and have $*_H \in \mathcal{C}_H$ whenever $\mathcal{C}_H \neq \varnothing$.*

Furthermore, there are a equalities of sub-posets

$$\begin{aligned} \mathrm{Index}_G \simeq \mathrm{IndCat} = \mathrm{wIndexCat}_{G,\geq A\mathbb{E}_\infty} = \mathrm{wIndexCat}^{\mathrm{uni}}_{G,\geq A\mathbb{E}_\infty} = A\mathrm{Op}^{\mathrm{uni}}_{G,\geq \mathbb{E}_\infty} &\subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_G), \\ \mathrm{wIndexCat}_G^{\mathrm{uni}} = A\mathrm{Op}_G^{\mathrm{uni}} &\subset \mathrm{wIndex}_G \\ \mathrm{wIndexCat}_G^{a\mathrm{uni}} = A\mathrm{Op}_G^{a\mathrm{uni}} &\subset \mathrm{wIndex}_G \\ \mathrm{wIndexCat}_G^{E\mathrm{uni}} = A\mathrm{Op}_G^{E\mathrm{uni}} &\subset \mathrm{wIndex}_G \\ \mathrm{wIndexCat}_G^{aE\mathrm{uni}} = A\mathrm{Op}_G^{aE\mathrm{uni}} &\subset \mathrm{wIndex}_G. \end{aligned}$$

where Index_G *denotes the* indexing systems *of [BH15; BP21; GW18; Rub21a] and the remaining notation is that of [St24a].*

References. The equivalence between Poset (4) and Poset (5) is handled in [St24a]; nevertheless, the composite map from Poset (1) to Poset (5) is shown to be furnished by the *self-indexed symmetric monoidal envelope* in Example 2.72. We then characterize the image of A, constructing an equivalence between Poset (3) and Poset (4) in Proposition 2.41 and Corollary 3.10.

Poset (2) and Poset (3) are shown to be equivalent in Corollary 3.10 by constructing a fully faithful right adjoint to

$$A : \mathrm{Op}_G \rightleftarrows \mathrm{wIndex}_G : \mathcal{N}_{(-)\infty}. \tag{5}$$

whose image is the G-0-operads. Along the way, in Remark 3.9 we show that Poset (1) and Poset (2) are equivalent as subcategories. Finally, the remaining identites follow by Observation 3.11 □

Under the assumption that $\mathcal{O}^{\otimes}$ is reduced, by [St24a], the information of $A\mathcal{O}$ may be understood as specifying for which subgroup inclusions $K \hookrightarrow H$ over which $\mathcal{O}^{\otimes}$ prescribes a multiplication

$$N_K^H X_K \to X_H,$$

and for which subgroup inclusions $K_1 \neq \cdots \neq K_n \hookrightarrow H$, $\mathcal{O}^{\otimes}$ additionally prescribes a twisted multiplication

$$X_H \otimes \bigotimes_i N_{K_i}^H X_{K_i} \to X_H$$

We call the operads $\mathcal{N}^{\otimes}_{I\infty}$ constructed by Eq. (5) *weak* $\mathcal{N}_\infty$*-operads*. In general, by Theorem C, we find that a slice category $\mathrm{Op}_{G,/\mathcal{O}^\otimes} \to \mathrm{Op}_G$ is a full subcategory if and only if $\mathcal{O}^\otimes$ is a weak $\mathcal{N}_\infty$-operad, in which case we write

$$\mathrm{Op}_I := \mathrm{Op}_{G,/\mathcal{N}^{\otimes}_{I\infty}} \simeq A^{-1}(\mathrm{wIndex}_{G,\leq I});$$

explicitly, a map $\mathcal{P}^\otimes \to \mathcal{N}^{\otimes}_{I\infty}$ is a *property* of $\mathcal{P}^\otimes$, and this property is the arity support condition $A\mathcal{P} \leq I$.

We may understand $\mathcal{N}^{\otimes}_{I\infty}$ in a hands-on manner in a number of ways; for instance, it is constructed explicitly in Proposition 2.41. On the other hand, the equivalence between conditions Poset (2) and Poset (4) of Theorem C shows that $\mathcal{N}^{\otimes}_{I\infty}$ is uniquely identified by the property

$$\mathcal{N}_{I\infty}(S) = \begin{cases} * & \mathrm{Ind}_H^G S \to G/H \quad \text{is in } I; \\ \varnothing & \text{otherwise.} \end{cases} \tag{6}$$

Alternatively, we may see this indirectly using the existence of free G-operads on symmetric sequences (see Corollary 2.78).

There are many weak $\mathcal{N}_\infty$-G-operads of interest outside of the world of $\mathcal{N}_\infty$-G-operads/

Example. Given $\mathcal{F} \subset \mathcal{O}_G^{\mathrm{op}}$ a G-family, the G-operad $\mathrm{triv}_{\mathcal{F}}^\otimes := \mathcal{N}^{\otimes}_{I^{\mathrm{triv}}_{cF}\infty}$ is characterized by a natural equivalence

$$\underline{\mathbf{Alg}}^{\otimes}_{\mathrm{triv}_{\mathcal{F}}}(\mathcal{C}) = \mathrm{Bor}_{\mathcal{F}}^G(\mathcal{C}^\otimes),$$

in Proposition 2.62, where $\mathrm{Bor}_{\mathcal{F}}^G$ is the $\mathcal{F}$*-Borelification* discussed in Section 3.3. ◁

Example. Given $\mathcal{F} \subset \mathcal{O}_G^{\mathrm{op}}$ a G-family, in Section 3.4 we characterize the unital $\mathcal{N}_\infty$-G-operad $\mathbb{E}^{\otimes}_{\mathcal{F}\infty} := \mathcal{N}^{\otimes}_{I^\infty_{\mathcal{F}}\infty}$ by a natural equivalence

$$\mathbf{Alg}_{\mathbb{E}_{\mathcal{F}\infty}}(\mathcal{C}) \simeq \mathrm{CAlg}(\Gamma^{\mathcal{F}}\mathcal{C}) \times_{(\Gamma^{\mathcal{F}}\mathcal{C})^{1/}} \mathcal{C}_G^{1/}$$

where $\Gamma^{\mathcal{F}}\mathcal{C}^\otimes$ is the symmetric monoidal ∞-category of $\mathcal{F}$*-objects*

$$\Gamma^{\mathcal{F}}\mathcal{C}^\otimes \simeq \lim_{V \in \mathcal{F}^{\mathrm{op}}} \mathcal{C}_V^\otimes. \qquad ◁$$

We say a real orthogonal G-representation V is a *weak universe* if it admits an equivalence $V \simeq V \oplus V$.

Example. Given V a weak G-universe, we verify in Section 3.4 that the homotopy type $\mathbb{E}_V^\otimes$ of the little V-disks G-operad is a weak $\mathcal{N}_\infty$-operad whose arity support is computed by

$$S \in (A\mathbb{E}_V)_{[G/H]} \qquad \iff \qquad \exists\ H-\text{equivariant embedding } S \not\hookrightarrow V.$$

In particular, if λ is a nontrivial irreducible C_p-representation, we use this to compute $A\mathbb{E}^{\otimes}_{\infty\lambda}$ in Section 3.4, verifying that $\mathbb{E}^{\otimes}_{\infty\lambda}$ is *not* an $\mathcal{N}_\infty$-operad in the sense of [BH15]. In particular, since $\mathbb{E}_V$ is also modelled by the V-Steiner operad, there are point-set models for $\mathbb{E}_V$-actions on V-fold loop spaces [GM17, p 9]). ◁

Given I an aE-unital weak indexing system, in Theorem 1.43 and Corollary 1.72, we characterize the ∞*-category of* I*-commutative monoids* in $\mathcal{C}$ a complete ∞-category as

$$\mathrm{CMon}_I(\mathcal{C}) := \mathbf{Alg}_{\mathcal{N}_{I\infty}}(\mathcal{C}^\times) \simeq \mathrm{Fun}^\times(\mathrm{Span}_I(\mathbb{F}_G), \mathcal{C}),$$

where $\mathrm{Span}_I(\mathbb{F}_G) \subset \mathrm{Span}(\mathbb{F}_G)$ is the subcategory whose forward maps are in I; we define the ∞*-category of* I*-symmetric monoidal* ∞*-categories* as

$$\mathbf{Cat}_I^\otimes := \mathrm{CMon}_I(\mathbf{Cat}).$$

We also show in Proposition 2.47 that I-symmetric monoidal ∞-categories have underlying I-operads; for $\mathcal{C} \in \mathbf{Cat}_I^\otimes$, we define the ∞-category of I*-commutative algebras* in $\mathcal{C}$ as

$$\mathrm{CAlg}_I(\mathcal{C}) := \mathbf{Alg}_{\mathcal{N}_{I\infty}}(\mathcal{C}).$$

We show in Corollary 3.19, that analogs of Theorem B hold for I-commutative algebra objects in I-symmetric monoidal categories.

We would like to use this to characterize indexed tensor product of I-commutative algebras. To that end, for any I-operad $\mathcal{O}^\otimes$, note that the forgetful functor $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}^{I-\times}$ preserves indexed tensor products, preserves indexed cartesian products, and reflects isomorphisms; these together imply that whenever I-indexed tensor products in $\mathcal{C}^\otimes$ are I-indexed products, the same is true for $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$. This completely characterises the I-symmetric monoidal structure by the following result.

Theorem D. *When I is aE-unital, there are fully faithful embeddings $(-)^{I-\sqcup}, (-)^{I-\times}$ making the following commute:*

$$\begin{tikzcd} \mathbf{Cat}_I^{\sqcup} \arrow[r, hook, "(-)^{I-\sqcup}"] \arrow[dr, "U"'] & \mathbf{Cat}_I^{\otimes} \arrow[d, dashed, "U"] & \mathbf{Cat}_I^{\times} \arrow[l, hook', "(-)^{I-\times}"'] \arrow[dl, "U"] \\ & \mathbf{Cat}_I & \end{tikzcd}$$

The image of $(-)^{I-\sqcup}$ is spanned by the I-symmetric monoidal ∞-categories whose indexed tensor products are indexed coproducts and the image of $(-)^{I-\times}$ is spanned by those whose indexed tensor products are indexed products.

We will show that I-indexed tensor products in $\underline{\mathrm{CAlg}}_I^{\otimes}\mathcal{C}$ are indexed coproducts, and that this completely characterizes $\mathcal{N}_{I\infty}^{\otimes}$. The crucial step uses Theorem D to reduce this to a category theoretic criterion, namely *I-semiadditivity.*

Theorem E. *Let $\mathcal{O}^{\otimes}$ be a G-operad. Then, the following properties are equivalenent.*

(a) The G-∞-category $\underline{\mathbf{Alg}}_{\mathcal{O}}\mathcal{S}_G$ is $A\mathcal{O}$-semiadditive.
(b) The unique map $\mathcal{O}^{\otimes} \to \mathcal{N}_{A\mathcal{O}\infty}^{\otimes}$ is an equivalence and $A\mathcal{O}$ is aE-unital.

Furthermore, $\underline{\mathrm{CAlg}}_I^{\otimes}\mathcal{C}$ is I-cocartesian for any I-symmetric monoidal ∞-category $\mathcal{C}$ and aE-unital weak indexing system I.

We say that an I-operad $\mathcal{O}^{\otimes}$ is *reduced* if the (unique) map $\mathcal{O}^{\otimes} \to \mathcal{N}_{I\infty}$ induces equivalences

$$\mathcal{O}(S) \simeq \mathcal{N}_{I\infty}(S) \qquad \forall \quad S \in \mathbb{F}_H \quad \text{ empty or contractible}$$

(c.f. Eq. (6)). We completely characterize algebras in cocartesian I-symmetric monoidal categories in Theorem 4.10, and from this Theorem E entirely characterizes the tensor products of reduced I-operads with $\mathcal{N}_{I\infty}^{\otimes}$ in the almost-E-unital setting.

Corollary F. *$\mathcal{N}_{I\infty}^{\otimes} \otimes \mathcal{N}_{I\infty}^{\otimes}$ is a weak $\mathcal{N}_\infty$-operad if and only if I is aE-unital. In this case, if $\mathcal{O}^{\otimes}$ is a reduced I-operad, then the unique map*

$$\mathcal{O}^{\otimes} \otimes \mathcal{N}_{I\infty}^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$$

is an equivalence.

One additional corollary to this is the infinitary homotopical version of Dunn's additivity result [Dun88] in the genuine equivariant setting.

Corollary G (Equivariant infinitary Dunn additivity)**.** *Let G be a finite group and V, W real orthogonal G-representations satisfying at least one of the following conditions:*

(a) V, W are weak G-universes, or
(b) the canonical map $\mathbb{E}_V^{\otimes} \simeq \mathbb{E}_{V\oplus W}^{\otimes}$ is an equivalence.

Then the canonical map

$$\mathbb{E}_V^{\otimes} \overset{BV}{\otimes} \mathbb{E}_W^{\otimes} \to \mathbb{E}_{V\oplus W}^{\otimes}$$

is an equivalence; equivalently, for any G-symmetric monoidal category $\mathcal{C}$, the pullback functors

$$\mathbf{Alg}_{\mathbb{E}_V}\underline{\mathbf{Alg}}_{\mathbb{E}_W}^{\otimes}(\mathcal{C}) \leftarrow \mathbf{Alg}_{\mathbb{E}_{V\oplus W}}(\mathcal{C}) \rightarrow \mathbf{Alg}_{\mathbb{E}_W}\underline{\mathbf{Alg}}_{\mathbb{E}_V}^{\otimes}(\mathcal{C})$$

are equivalences.

Corollary F immediately characterizes many tensor products of weak $\mathcal{N}_\infty$-operads, since $\mathcal{N}_{I\infty}$ is a J-operad whenever $I \leq J$. We go on to completely characterize indexed tensor products of almost-E-unital weak $\mathcal{N}_\infty$-operads, affirming Conjecture 6.27 of [BH15].

Theorem H. *The functor $\mathcal{N}_{(-)\infty}^{\otimes} : \mathrm{wIndex}_G \to \mathrm{Op}_G$ lifts to a fully faithful G-right adjoint*

$$\begin{tikzcd} \underline{\mathrm{wIndex}}_G \arrow[r, hook, bend right, "\mathcal{N}_{(-)\infty}^{\otimes}"'] \arrow[r, phantom, "\top"] & \underline{\mathrm{Op}}_G \arrow[l, bend right, "A"'] \end{tikzcd}$$

whose restriction $\underline{\mathrm{wIndex}}_G^{aE\mathrm{uni}} \subset \underline{\mathrm{Op}}_G$ *is symmetric monoidal. Furthermore, the resulting tensor product on* $\mathrm{wIndex}_G^{aE\mathrm{uni},\otimes}$ *is computed by the Borelified join*

$$I \otimes J = \mathrm{Bor}^G_{\mathrm{cSupp}(I\cap J)}(I \vee J);$$

in particular, when I *and* J *are almost-*E*-unital weak indexing systems, we have*

$$\begin{aligned}
\mathcal{N}^{\otimes}_{I\infty} \otimes \mathcal{N}^{\otimes}_{J\infty} &\simeq \mathcal{N}^{\otimes}_{(I\vee J)\infty} \otimes \mathrm{triv}^{\otimes}_{\mathrm{cSupp}(I\cap J)} \\
\mathcal{N}^{\otimes}_{I\infty} \times \mathcal{N}^{\otimes}_{J\infty} &\simeq \mathcal{N}^{\otimes}_{(I\cap J)\infty} \\
\mathrm{Res}^G_H \mathcal{N}^{\otimes}_{I\infty} &\simeq \mathcal{N}^{\otimes}_{\mathrm{Res}^G_H I\infty} \\
\mathrm{CoInd}^G_H \mathcal{N}^{\otimes}_{I\infty} &\simeq \mathcal{N}^{\otimes}_{\mathrm{CoInd}^G_H I\infty}.
\end{aligned}$$

Hence norms of I*-commutative algebras are* $\mathrm{CoInd}^G_H I$*-commutative algebras, and when* I, J *are almost-unital, we have*

$$\underline{\mathrm{CAlg}}^{\otimes}_I \underline{\mathrm{CAlg}}^{\otimes}_J(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_{I\vee J}(\mathcal{C}). \tag{7}$$

Remark. The reader interesting in computing tensor products of aE-unital weak $\mathcal{N}_\infty$-operads may benefit from reading the combinatorial characterization of joins of weak indexing systems in terms of *closures* in [St24a]. $\triangleleft$

Along the way, we quickly acquire various corollaries in equivariant higher algebra. FOr instance, in **??** we use Corollary G to define iterated Real topological Hochschild homology for $\mathbb{E}_V$-algebras whenever V has infinitely many σ-summands, and compute it as a colimit when $V = \infty\rho$. We go on in Corollary 5.3 to lift Bonventre's genuine operadic nerve to a conservative functor of ∞-categories, and verify in Proposition 5.5 that it restricts to an equvivalence between the two categories of discrete G-operads, giving traditional presentations for all of our objects in the G-1-categorical setting.

Notation and conventions. We assume that the reader is familiar with the technology of higher category theory and higher algebra as developed in [HTT] and [HA, § 2-3], though we encourage the reader to engage with such technologies via a "big picture" perspective akin to that of [Gep19, § 1-2] and [Hau23, § 1-3].

Throughout this paper, we frequently describe conditions which may be satisfied by objects parameterized over some ∞-category $\mathcal{T}$. If P is a property, in the instance where there exists a Borelification adjunction

$$\mathrm{Bor}^{\mathcal{T}}_{\mathcal{F}} : \mathcal{C}_{\mathcal{T}} \rightleftarrows \mathcal{C}_{\mathcal{F}} : E^{\mathcal{T}}_{\mathcal{F}}$$

along family inclusions $\mathcal{F} \subset \mathcal{T}$, we say that $X \in \mathcal{C}_{\mathcal{T}}$ is E-P when there exists some $\overline{X} \in \mathcal{F}_{\mathcal{F}}$ which is P such that $X \simeq E^{\mathcal{T}}_{\mathcal{F}}\overline{X}$. We say that X is *almost* E-P (or aE-P) if $\mathcal{C}_{\mathcal{F}}$ has a terminal object $*_{\mathcal{F}}$ for all $\mathcal{F}$, and there is a pushout expression

$$X \simeq *_{\mathcal{F}'} \sqcup_{*_{\mathcal{F}}} *_{\mathcal{F}'}$$

for some $\mathcal{F}' \subset \mathcal{F}$; we say that X is *almost* P (or a-P) if it's almost E-P and $\mathcal{F}' = \mathcal{T}$ in the above.

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1. Equivariant symmetric monoidal categories

1.1. **Recollections on $\mathcal{T}$-∞-categories.** In this section, we briefly summarize some relevant elements of parameterized and equivariant higher category theory. Of course, this theory has advanced far past that which is summarized here; for instance, further details can be found in the work of Barwick-Dotto-Glasman-Nardin-Shah [BDGNS16a; BDGNS16b; Nar16; Sha22; Sha23], Cnossen-Lenz-Linskens [CLL23a; CLL23b; CLL24; Lin24; LNP22], and Hilman [Hil24].

We view this setting of *atmoic orbital ∞-categories* as a natural axiomatic home for higher algebra centered around the Burnside category (see [Nar16, § 4]), generalizing the orbit categories of families of subgroups of finite groups. The reader who is exclusively interested in equivariant homotopy theory is encouraged to assume every atomic orbital ∞-category is the orbit category of a family of subgroups of a finite group.