YOU CAN CONSTRUCT G-COMMUTATIVE ALGEBRAS ONE NORM AT A TIME

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ABSTRACT. We define the category of G-operads and the hierarchy of $\mathcal{N}-\infty$*-operads*, which are suboperads of the terminal G-operad Comm_G containing $\mathbb{E}_\infty$. We exhibit an isomorphism between the category of $\mathcal{N}^\infty$ operads and the poset of *indexing systems*, which are nice subcategories of $\underline{\mathbb{F}}_G$; this witnesses $\mathcal{N}^\infty$ operads as describing *commutative multiplication with restriction and some multiplicative transfers*. Indeed, their algebras in Cartesian categories are incomplete Mackey functors and their algebras in Mackey functors are incomplete Tambara functors.

After this, we discuss some in-progress research. Namely, we construct a *Boardman-Vogt tensor product* of G-operads, and prove that (nonempty) tensor products of $\mathcal{N}^\infty$ operads correspond with joins of indexing systems, i.e. there is an $\mathcal{N}^\infty(I \cup J)$-monoidal equivalence

$$\mathbf{Alg}^{\mathcal{N}^\infty(I)}\mathbf{Alg}^{\mathcal{N}^\infty(J)}C \simeq \mathbf{Alg}^{\mathcal{N}^\infty(I\cup J)}C$$

for all $\mathcal{N}^\infty(I \cup J)$-monoidal categories C, allowing G-commutative structures to be constructed "one transfer at a time."

Foreword. The following are notes prepared for a casual talk in the zygotop seminar concerning research which is currently in-progress cite . Though I will attempt to confine these notes to their own proofs, citations to the literature, and well-marked conjecture, the reader should read with the understanding that they are particularly error-prone.

1. INTRODUCTION

In [Dre71], the concept of a *Mackey functor* was introduced; this structure was described as consisting of functors $M_I : \mathcal{O}_G \to \mathbf{Mod}_R$ and $M_R : \mathcal{O}_G^{\mathrm{op}} \to \mathbf{Mod}_R$ which agree on $\mathcal{O}_G^{\simeq}$ and satisfying the *double coset formula*

$$R_J^H I_K^H = \prod_{x\in[J\backslash H/K]} I^J_{J\cap xKx^{-1}} \cdot \mathrm{conj}_X R_{x^{-1}Jx\cap K}$$

for all $J, K \subset H$, where $R_J^K := M_R(G/J \to G/K)$ and similar for I. The ur-example of this is the assignment $H \mapsto \mathbf{Rep}_H(R)$ with covariant functoriality Ind and contravariant functoriality Res. This was repackaged and generalized into the modern definition of the *category of C-valued G-Mackey functors*

$$\mathcal{M}_G(C) := \mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_G), C),$$

where $\mathbb{F}_G$ denotes the category of finite G-sets.

In parallel, the concept of *transfer maps in group cohomology* was being developed in **[Evens]**, later lifted to genuine equivariant cohomology in **[Greenlees]**, and finally developed as a functor

$$N_H^G : \mathrm{Sp}_H \to \mathrm{Sp}_G$$

in [HHR16], which played a crucial role in the solution to the Kervaire invariant one problem. These were noted in [HH16] to satisfy the conditions of a *Symmetric monoidal Mackey functor*, a notion they distinguished from their notion of *G-symmetric monoidal categories* due to coherence issues.

In the broad program announced in [Bar+16], the correct notion of *G-symmetric monoidal G-∞-categories* (henceforth G-symmetric monoidal categories) was introduced:

Definition 1.1. Let C have finite products. Then, the category of G-commutative monoids in C is

$$\mathrm{CMon}_G(C) := \mathcal{M}_G(C).$$

The category of G-symmetric monoidal categories is $\mathrm{CMon}_G(\mathbf{Cat})$.

We similarly define the *category of small G-categories* as

$$\mathbf{Cat}_G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Cat}) \simeq \mathbf{Cat}^{\mathrm{cocart}}_{/\mathcal{O}_G^{\mathrm{op}}},$$

where the equivalence is the *straightening-unstraightening construction* of [HTT]. This has an adjunction

$$\mathrm{Tot} : \mathbf{Cat}_G \rightleftarrows \mathbf{Cat} : \mathrm{CoFr}^G$$

where Tot takes the total category of a cocartesian fibration and $\mathrm{CoFr}^G(\mathcal{C})$ is classified by functor categories

$$\mathrm{CoFr}^G(\mathcal{C})_H := \mathrm{Fun}(\mathcal{O}_H^{\mathrm{op}}, \mathcal{C})$$

with functoriality dictated by pullback. In particular, the *G-category of small G-categories* $\underline{\mathbf{Cat}}_G := \mathrm{CoFr}^G(\mathcal{C})$ has G-fixed points given by **Cat**.

Let $\underline{\mathbb{F}}_{G,*} := \mathrm{CoFr}^G(\mathbb{F}_*)$. We may understand an object in $\underline{\mathbb{F}}_{G,*}$ as a map of finite G-sets $S \to U$ where U is an orbit (recognizing S as induced from H if $U \simeq G/H$), and a map $\begin{pmatrix} S \\ \downarrow \\ U \end{pmatrix} \to \begin{pmatrix} S' \\ \downarrow \\ U' \end{pmatrix}$ as a span

$$\begin{tikzcd} S \arrow[d] & T \arrow[l] \arrow[r] \arrow[d] & S' \arrow[d] \\ U & V \arrow[l] \arrow[r, equal] & U' \end{tikzcd}$$

whose associated map $T \to S \times_U U'$ is a summand inclusion. We say that such maps are *active* if they are forwards, and *inert* if they are backwards.

Definition 1.2. A *G-operad* is a functor $\pi : \mathcal{O}^\otimes \to \mathrm{Tot}\underline{\mathbb{F}}_{G,*}$ such that

(1) $\mathcal{O}^\otimes$ has π-cocartesian lifts for inert morphisms with specified domains,
(2) the π-cocartesian lifts induce equivalences on the categories of colors

$$\mathcal{O}^\otimes_{(S \to U)} \simeq \prod_{W \in \mathrm{Orb}(S)} \mathcal{O}^\otimes_{(W = W)}.$$

(3) For any morphism $\psi : \begin{pmatrix} S \\ \downarrow \\ U \end{pmatrix} \to \begin{pmatrix} S' \\ \downarrow \\ U' \end{pmatrix}$ in $\mathrm{Tot}\underline{\mathbb{F}}_{G,*}$, pair $(x,y) \in \mathcal{O}_{(S \to U)} \times \mathcal{O}_{(S' \to U')}$ and collection of cocartesian edges $\{y \to y_W \mid W \in \mathrm{Orb}(S)\}$ lying over the inert morphisms $S \hookleftarrow W = W$, the induced map

$$\mathrm{Map}^{\psi}_{\mathcal{O}^\otimes}(x,y) \to \prod_{W \in \mathrm{Orb}(S)} \mathrm{Map}^{\psi|_W}_{\mathcal{O}^\otimes}(x, y_W)$$

is an equivalence.

Morphisms of G-operads are morphisms over $\mathrm{Tot}\underline{\mathbb{F}}_{G,*}$ preserving cocartesian lifts for inert morphisms.

This is a straightforward, but heavy, generalization of the ∞-operads of [HA] to the equivariant world, and we suggest the interested reader consult [BHS22] for a less heavy variant or [NS22] for the original. In particular, postcomposition along the inclusion functor $\mathbb{F} \hookrightarrow \mathrm{Tot}\underline{\mathbb{F}}_{G,*}$ induces a fully faithful functor $\mathrm{Op} \hookrightarrow \mathrm{Op}_G$.

An early observation about genuine equivariant homotopy coherent algebraic structures is that the structure of transfers *does not come canonically* from an $\mathbb{E}_\infty$-structure; that is, $\mathbb{E}_\infty \in \mathrm{Op}_G$ is not terminal. The failure of $\mathbb{E}_\infty$ to be terminal is parameterized by the category of *N^∞-operads*:

Definition 1.3. Write $\mathrm{Comm}_G^\otimes := \left(\mathrm{Tot}\underline{\mathbb{F}}_{G,*} = \mathrm{Tot}\underline{\mathbb{F}}_{G,*}\right)$ for the terminal G-operad. A G-operad $\mathcal{O}^\otimes$ is *subterminal* if the unique morphism $\mathcal{O}^\otimes \to \mathrm{Comm}_G^\otimes$ is a monomorphism, i.e. $\mathcal{O}_U^\otimes \simeq *$ for all U and $\mathrm{Map}^{\psi}_{\mathcal{O}}(x,y) \in \{*, \varnothing\}$ for all $\psi : \pi(x) \to \pi(y)$.

An N^∞ *operad* is a subterminal G-operad $\mathcal{O}^\otimes$ admitting a map $\mathbb{E}_\infty \to \mathcal{O}^\otimes$.

Write $\mathcal{N}_G^\infty \subset \mathrm{Op}_G$ for the full subcategory consisting of N^∞-operads, and write $\widehat{\mathcal{N}}_G^\infty := \mathcal{N}_G^\infty \cup \left\{\mathcal{O}_{\mathrm{triv}}^{otimes}\right\}$ The following proposition is an easy exercise in category theory:

Proposition 1.4. *The category $\widehat{\mathcal{N}}_G^\infty$ is a poset, i.e. all of its mapping spaces are contractible or empty.*

In ref , we endow Op_G with a *Boardman-Vogt symmetric monoidal structure*, satisfying the universal property that

$$\mathbf{Alg}^{\mathcal{O}\otimes\mathcal{P}}(\mathcal{C}) \simeq \mathbf{Alg}^{\mathcal{O}}\mathbf{Alg}^{\mathcal{P}}(\mathcal{C}).$$

We would like to characterize the tensor products of these, but to do so, we need a candidate, which are called *indexing systems*.

Definition 1.5. An *indexing system* is a core-containing subcategory $\mathcal{O}_G^{\simeq} \hookrightarrow I \hookrightarrow \mathcal{O}_G$ which is closed under base change, i.e. for any

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\alpha'"] & V \arrow[d, "\alpha"] \\ U' \arrow[r] & V' \end{tikzcd}$$

with $U \hookrightarrow V' \times_{U'} V$ a summand inclusion and $\alpha \in I$, we have $\alpha' \in I$. The poset of indexing systems under inclusion is denoted $\mathrm{Ind}-\mathrm{Sys}_G$, and the poset of indexing systems with an added initial object is denoted $\widehat{\mathrm{Ind}-\mathrm{Sys}}_G$.

Given an indexing system, there is a corresponding full subcategory of $\underline{\mathbb{F}}_{G,*}$ which happens to have the structure of a G-operad. We call this functor $\mathcal{N}^\infty(-) : \widehat{\mathrm{Ind}-\mathrm{Sys}}_G \to \mathrm{Op}_G$, with value on $\varnothing$ given by $\mathcal{O}^{\otimes}_{\mathrm{triv}}$.

Theorem A. *The functor $\mathcal{N}^\infty(-) : \widehat{\mathrm{Ind}-\mathrm{Sys}}_G \to \mathrm{Op}_G$ is fully faithful with image $\widehat{\mathcal{N}}^\infty_G$. Furthermore, this functor is symmetric monoidal for the cocartesian structure on $\widehat{\mathrm{Ind}-\mathrm{Sys}}_G$ and the BV tensor product on Op_G; this supplies a canonical equivalence*

$$\mathbf{Alg}^{\mathcal{N}^\infty(I)}\mathbf{Alg}^{\mathcal{N}^\infty(J)}\mathcal{C} \simeq \mathbf{Alg}^{\mathcal{N}^\infty(I\cup J)}\mathcal{C}$$

for all indexing systems I, J.

We say an inclusion of subgroup $H \subset K$ is *atomic* if it is proper and there exist no chains of proper subgroup inclusions $H \subset J \subset K$. More generally, we say that a conjugacy class $(H) \in \mathrm{Conj}(G)$ is an *atomic subclass* of (K) if there exists an atomic inclusion $\tilde{H} \subset \tilde{K}$ with $\tilde{H} \in (H)$ and $\tilde{K} \in (K)$, and we say that (K) is atomic if the canonical inclusion $1 \hookrightarrow K$ is atomic.

Given $(H) \subset (K)$ an atomic subclass, we refer to the $\mathcal{N}^\infty$-operad corresponding to the minimal index system containing the inclusion $H \hookrightarrow K$ as $\mathcal{N}^\infty(H, K)$. When $(H) = (1)$, we instead simply write $\mathcal{N}^\infty(K)$.

Corollary B. *Let $1 = G_n \subset G_{n-1} \subset \cdots \subset G_0 = G$ be a maximal subgroup series of a finite group, and let $\mathcal{C}$ be a G-symmetric monoidal category. Then, there exists a canonical G-symmetric monoidal equivalence*

$$\mathbf{Alg}^{\mathcal{N}^\infty(G_1,G_0)} \cdots \mathbf{Alg}^{\mathcal{N}^\infty(G_n,G_{n-1})\mathcal{C}\simeq\mathrm{CAlg}_G\mathcal{C}}.$$

Furthermore, if $G \simeq H \times J$, then

$$\mathrm{CAlg}_H\mathrm{CAlg}_J\mathcal{C} \simeq \mathrm{CAlg}_G\mathcal{C}.$$

Remark. One may worry about the comparison between models for G-operads, as our notion of N^∞-operads is ostensibly embedded deep within the world of G-∞-operads, which are not known to be equivalent to the ∞-category presented by the graph model structure or by genuine G operads. However, by ref, all notions of N^∞ operads coincide .

2. The Boardman-Vogt tensor product

2.1. The G-symmetric monoidal envelope.

2.2. The internal hom on G-operads.

2.3. Day convolution and the categorical Fourier transform.

3. Commutative operads

3.1. Tensor products of subterminal operads.

Proposition 3.1. *Let $\mathcal{O}^\otimes$ be a G-operad. The following are equivalent:*

(1) $\mathcal{O}^\otimes$ is $\mathcal{N}^\infty$.
(2) the functor $\mathrm{Env}\mathcal{O} \to \underline{\mathbb{F}}_G^{\amalg}$ is an equifibrous core-preserving G-symmetric monoidal subcategory.
(3) the functor $\mathrm{Env}\mathcal{O} \to \underline{\mathbb{F}}_G^{\amalg}$ is a wide subcategory.

Proof. Obvious. □

Endow $\mathrm{CMon}_G(\mathbf{Cat})_{/\underline{\mathbb{F}}_G^{\amalg}}$ with the sliced Day convolution symmetric monoidal structure. Let $\mathrm{CMon}_G(\mathbf{Cat})^{\mathrm{ecs}}_{\underline{\mathbb{F}}_G^{\amalg}} \subset \mathrm{CMon}_G(\mathbf{Cat})_{/\underline{\mathbb{F}}_G^{\amalg}}$ be the full subcategory spanned by equifibrous core-preserving G-symmetric monoidal subcategories of $\underline{\mathbb{F}}_G^{\amalg}$

Proposition 3.2. $\mathrm{CMon}_G(\mathbf{Cat})^{\mathrm{eqw}}_{\underline{\mathbb{F}}_G^{\amalg}}$ *is closed under binary tensor products.*

Proof. Note that $\underline{\mathbb{F}}_G^{\amalg}$ is idempotent under the Day convolution tensor product, so it suffices to prove that the *external* tensor product $\mathcal{C} \times \mathcal{D} \to \underline{\mathbb{F}}_G^{\amalg} \times \underline{\mathbb{F}}_G^{\amalg}$ is an equifibrous core-preserving subcategory.[1] In fact, by the pointwise formula for left Kan extensions, it suffices to prove that products and colimits in $\mathrm{Ar}(\mathrm{CMon}_G\mathbf{Cat})$ which are pointwise equifibrous core-preserving subcategories of $\underline{\mathbb{F}}_G^{\amalg}$ are taken to inclusions of equifibrous core-preserving subcategories.

It is easy to verify this for products and coproducts, so we must verify this for geometric realizations . By the argument due to Thomas Blom in the AT discord , the core-preserving property is preserved under sifted colimits, and the hom-spaces are sifted colimits of the hom spaces. Hence it suffices to prove that the geometric realization of a summand inclusion in sSet is a summand inclusion in $\mathcal{S}$. But this is just simple homotopy theory. Geometric realizations should just always preserve equifibrous summand inclusions. □

Remark. The equifibrous wide condition is necessary, but i doubt the 1-category condition is necessary.

3.2. Synthesis.

Proof of theorem A. By groth , image of N infty , and closure of N infty , the sub-poset $\widehat{\mathcal{N}}_G^\infty \subset \mathrm{Op}_G$ is closed under tensor products, so by HA citation , it is endowed with a canonical symmetric monoidal structure. It remains to prove that this structure is cocartesian. Clearly $\mathcal{O}_{\mathrm{triv}}$ is both the initial object and the unit, so it suffices to prove that binary tensor products are computed by the join of indexing systems.

In fact, the universal property for the BV tensor product constructs a diagram in $\mathcal{N}_G^\infty$

$$\begin{tikzcd} \mathcal{N}^\infty(I) \arrow[dr] \arrow[drr] & & \\ & \mathcal{N}^\infty(I) \otimes \mathcal{N}^\infty(J) \arrow[r] & \mathcal{N}^\infty(I \cup J) = \mathcal{N}^\infty(I) \cup \mathcal{N}^\infty(J) \\ \mathcal{N}^\infty(J) \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

so that $\mathcal{N}^\infty(I), \mathcal{N}^\infty(J) \subset \mathcal{N}^\infty(I) \otimes \mathcal{N}^\infty(J) \subset \mathcal{N}^\infty(I) \cup \mathcal{N}^\infty(J)$. By the universal property for the join, this guarantees that the map $\mathcal{N}^\infty(I) \otimes \mathcal{N}^\infty(J) \to \mathcal{N}^\infty(I \cup J)$ is an equivalence. □

Proof of corollary B. The first statement is immediate. For the second statement, by the structure theorem for finitely generated Abelian groups, it suffices to prove this for C_{p^n}, i.e. to prove that there are exactly two indexing systems for C_{p^n}. □

[1] Argue this using G-symmetric monoidality of the grothendieck construction. Alternatively, are cocartesian symmetric monoidal categories idempotent?

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