

Kan Seminar Notes

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This will be a rough collection of live-L^AT_EXed notes covering the Kan seminar talks given in Fall 2021. I'll make no promises that the contents of this are readable, or without significant clerical error. Exercise skepticism, and don't use these as a replacement for the papers. Last update: October 25, 2021.

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1 Gabrielle Li: Serre, Cohomologie modulo 2 des complexes d'Eilenberg Mac Lane (i)

This talk was delivered September 15, 2021 by Gabrielle Li. Throughout, $H^*(-) := H^*(-; \mathbb{F}_2)$.

1.1 Steenrod operations

The *Steenrod operations* are a family of cohomology operations $Sq^n : H^*(X) \rightarrow H^{*+n}(X)$ such that:

- (1) Each Sq^n is natural in X .
- (2) Each Sq^n is stable: $Sq^n(\Sigma X) = \Sigma Sq^n(X)$.
- (3) When $|x| = n$, $Sq^n(x) = x \cup x$.
- (4) $Sq^0 = \text{id}$.

We give a basis for these:

Definition 1.1. A sequence $I = (i_1) \subset \mathbb{Z}_{>0}$ is *admissible* if $i_k \geq 2i_{k+1}$ for each k . We define the *degree* $n(I) := \sum i_k$ and the *excess* $e(I) = \sum (i_k - i_{k+1}) = 2i_1 - n(I)$ (padding with zeros).

1.2 Borel's theorem

Let $F \hookrightarrow E \rightarrow B$ be a Serre fibration. Recall that, in the cohomological Serre spectral sequence, we have transgression morphisms $\tau : E_r^{0,r-1} \rightarrow E_r^{r,0}$, whose domain is a subset of $H^{r-1}(F)$ and whose codomain is a quotient of $H^r(B)$. This is an additive relation between $H^{r-1}(F)$ and $H^r(B)$. We say that $x \in H^{r-1}(F)$ is *transgressive* if it survives to the r page.

We hold off on proving the following proposition until the next talk:

Proposition 1.2. τ commutes with Steenrod operations.

We need a bit more language to use this:

Definition 1.3. For a space X , an ordered family of elements $(x_i) \subset H^*(X)$ is a *simple system of generators* if:

- (1) Each x_i is homogeneous.
- (2) The increasing products $x_{i_1} \cdots x_{i_j}$ (for $i_k < i_{k+1}$) form a \mathbb{F}_2 -basis of $H^*(X)$.

The following examples are important:

Example 1.4:

$\mathbb{F}_2[x_1, x_2, \dots]$ has simple system of generators $(x_j^{2^i})$. Similar systems apply to the exterior algebra $E[x]$ and the truncated polynomial algebra $\mathbb{F}_2[x]/(x^{2^i})$.

We're finally ready to state our theorem:

Theorem 1.5 (Borel). *Given a fibration $F \hookrightarrow E \rightarrow B$ satisfying the following properties:*

- (1) $E_2^{s,t} = H^s(B) \otimes H^t(F)$ (for instance, when B is 1-connected and $H^*(B), H^*(F)$ are f.g.).
- (2) $H^i(E) = 0$ for $i > 0$.
- (3) $H^*(F)$ have a simple system of transgressive generators (x_i) .

Then, $H^(B)$ is a polynomial algebra generated (independently) by the any choice of representatives $y_i \in H^*(B)$ which map to $\tau(x_i)$ in $E_*^{*,0}$.*

Note that, whenever $H^*(F)$ is a polynomial algebra generated by z_i , we know that $H^*(F)$ has a simple system of generators $z_i^{2^r}$. In order to use this, we introduce a bit of notation:

Notation. $L(a, r) := \{2^{r-1}a, 2^{r-2}a, \dots, 2a, a\}$.

Note that $z_i^{2^r} = \text{Sq}^{L(n_i, r)}(z_i)$. Hence

$$\tau\left(z_i^{2^r}\right) = \text{Sq}^{L(n_i, r)} t_i$$

where $t_i := \tau(z_i)$. Hence $H^*(B)$ is a polynomial algebra generated by $\text{Sq}^{L(n_i, r)}(z_i)$.

1.3 Performing the calculation

We will use Borel's theorem soon, but first, a lemma:

Lemma 1.6. *An admissible sequence $J = \{j_1, \dots, j_k\}$ with $e(J) < q - 1$. Then, we may define a sequence*

$$J' := \{2^{r-1}s_J, 2^{r-2}s_J, \dots, s_J, j_1, j_2, \dots, j_k\},$$

where $s_J = q - 1 + n(J)$. Then, J' is admissible, with $e(J') < q$; furthermore, all admissible sequences of excess $< q$ arise this way.

The reversal is surprisingly easy; simply take the longest prefix satisfying $j_1 = 2j_2 = \dots = 2^i j_i$.

We will need a few more constructions to prepare for the calculation:

- (1) There is a fibration $K(\mathbb{F}_2, q - 1) \hookrightarrow E \rightarrow K(\mathbb{F}_2, q)$ where E is contractible.
- (2) By Hurewicz, $H^q(K(\mathbb{F}_2, q)) = \mathbb{F}_2$, with a generator that we call u_q .

Theorem 1.7. *$H^*(K(\mathbb{Z}/2, q), \mathbb{Z}/2)$ is a polynomial algebra (independently) generated by $\text{Sq}^I(u_q)$ where I runs over the admissible sequences of excess $e(I) < q$.*

Proof. We prove this via induction. The $q = 1$ case is easy, as we have $K(\mathbb{F}_2, 1) = \mathbb{RP}^\infty$, and $H^*(\mathbb{RP}^\infty) = \mathbb{F}_2[u_q]$ via the usual computation.

For the inductive step, assume we've proven the theorem for $q - 1$. We use the fibration from (1). For an admissible sequence J , let

$$S_J := |\text{Sq}^J(u_{q-1})| = q - 1 + n(J).$$

We have transgression additive relation $H^{q-1}(K(\mathbb{F}_2, q - 1)) \rightsquigarrow H^q(K(\mathbb{F}_2, q))$. Note that the transgression sends $\tau(u_{q-1}) = u_q$ (this will be justified later). Using our trick,

$$\tau(\text{Sq}^J(u_{q-1})) = \text{Sq}^J u_q.$$

By Borel, the $H^*(K(\mathbb{F}_2, q))$ is generated by $\text{Sq}^{L(s_J, r)} \text{Sq}^J u_q = \text{Sq}^{L(s_J, r)J} u_q = \text{Sq}^I u_q$, where I is an admissible sequence with $e(I) < q$, and all such I are generated this way. \square

The other computations are routine and similar.

2 Weixiao Lu: Serre, Cohomologie modulo 2 des complexes d'Eilenberg Mac Lane (ii)

This talk was delivered September 17, 2021 by Weixiao Lu. We'll first cover some preliminaries.

2.1 Preliminaries

Theorem 2.1 (Serre spectral sequence). *Let $F \hookrightarrow E \xrightarrow{p} B$ be a Serre fibration. Then, there is a spectral sequence*

$$E_2^{s,t} = H^s(B; H^t(p^{-1}(-); G)) \implies H^{s+t}(E; G).$$

If $\pi_1(B)$ acts trivially on $H^n(p^{-1}(-))$, then

$$E_2^{2,t} = H^s(B; H^t(F; G)).$$

Proof sketch. If F^*C^* is a filtered cochain complex, we have an SS,

$$E_0^{s,t} = \text{gr}^s(C^{s+t}) \implies H^{s+t}(C^*).$$

Assume B is a CW complex with n -skeleton B^n . Then, $E_n := p^{-1}(B^n)$. We have $F^*S^*(E) = S^*(E, E_{s-1}) = \ker(S^*(E) \rightarrow S^*(E_{s-1}))$, which gives the right E_0 page. \square

In any upper-right quadrant SS, we have a transgression morphism $d^n : E_n^{0,n-1} \rightarrow E_n^{n,0}$. Note that $E_n^{0,n-1} \subset E_{n-1}^{0,n-1} \subset \dots \subset H^{n-1}(F)$. The transgressive elements of $H^{n-1}(F)$ map to some quotient of $H^n(B)$.

We can create a diagram

$$\begin{array}{ccc} H^n(B, b) & \xrightarrow{p^*} & H^n(E, F) \\ \downarrow \sim & \nearrow & \nwarrow \partial \\ H^n(B) & & H^{n-1}(F) \end{array}$$

Theorem 2.2 (Transgression theorem). *The transgression relation coincides with this diagram.*

This comes down to how the Serre SS was constructed.

Proposition 2.3. *The Steenrod square Sq_i “commutes” with transgression in the sense that any $x \in H^{n-1}(F; \mathbb{Z}/2)$ transgressive has $\text{Sq}^i x$ transgressive, and $\tau(\text{Sq}^i x) = \text{Sq}^i(\tau x)$.*

Proof. Recall that a functor is stable iff it commutes with coboundary operators, so Sq_1 commutes with coboundary operators. Further, recall that it's natural. Hence the following diagram commutes, so Sq^i “commutes with the transgression relation” (is a morphism of cospans):

$$\begin{array}{ccccc} & & H^{n+i}(E, F) & & \\ & \nearrow p^* & \uparrow \text{Sq}^i & \nwarrow \partial & \\ H^{n+i}(B) & & & & H^{n+i-1}(F) \\ \uparrow \text{Sq}^i & & & & \uparrow \text{Sq}^i \\ H^n(B) & \nearrow p^* & H^n(E, F) & \nwarrow \partial & H^{n-1}(F) \end{array}$$

\square

Recall that for G a f.g. Abelian group,

1. $H^*(K(G \times H; q)) = H^*(K(G; q)) \otimes H^*(K(H; q))$.
2. $H^*(K(\mathbb{F}_2; q)) = \mathbb{F}_2[\text{Sq}^I u_q \mid I \text{ admissible, s.t. } e(I) < q]$.
3. $H^*(K(\mathbb{F}_2; q)) = \mathbb{F}_2[\text{Sq}^I u_q \mid I \text{ admissible, s.t. } e(I) < q, 1 \text{ does not appear in } i]$.
4. $H^*(K(\mathbb{F}_2^h; q)) = \mathbb{F}_2[\text{Sq}^I u_q, \text{Sq}^J k_{q+1}]$ where $k_{q+1} \in H^{q+1}(K(\mathbb{F}_2^h, q))$ for admissibles $e(I) < q, e(J) \leq q$ where no Sq^1 term appears in both Sq^I and Sq^J . This comes from a fibration [fill in from notes later](#).
5. $H^*(K(\mathbb{F}_{p^h}; q)) = \mathbb{Z}/2$ for p odd with $q > 0$.

Remark. We have a different choice of generators related to universal classes, but as graded \mathbb{F}_2 -algebras,

$$H^*(K(\mathbb{F}_{2^h}; q)) \simeq H^*(K(\mathbb{F}_2; q)).$$

We will aim towards the following theorem:

Theorem 2.4. *For all $n > 1$, there are infinitely many indices i at which $\pi_i(S^n)$ has nonzero 2-torsion.*

Our tool will be Poincaré series. The accents in Poincaré's name are to be understood from here on out.

2.2 Poincaré series

For L_* a finite type graded k -vector space, define the series

$$L(t) = \sum_{n \in \mathbb{N}} \dim L^n t^n \in \mathbb{Z}[[t]].$$

This is called the *Poincaré series*, called $\theta(G; q; t)$ in the case of $H^*(K(G; q))$.

Example 2.5:

For $L^* = \mathbb{Z}/2[u]$, we have

$$L(t) = \frac{1}{1 - t^m}.$$

Note that $(N^* \otimes M^*)(t) = L(t)M(t)$. Hence $L'^* = k[u_1, \dots]$ with finite type has

$$L(t) = \prod_{n \geq 1} \frac{1}{1 - t^{\deg u_i}}$$

which converges t -adically.

Hence

$$\theta(\mathbb{F}_2, q, t) = \prod_{e(I) < q} \frac{1}{1 - t^{\deg(\text{Sq}^I u_q)}} = \prod_{e(I) < q} \frac{1}{1 + tq + n(I)}.$$

We can give this another combinatorial description:

Proposition 2.6.

$$\theta(\mathbb{F}_2, q, t) = \prod_{n_1 \geq n_2 \geq \dots \geq n_{q-1} \geq 0} \frac{1}{1 - t^{2^{n_1} + \dots + 2^{n_{q-1}} + 1}}.$$

The radius of convergence of this is 1 considered as a complex power series. We can continue to analyze this series along these lines:

Theorem 2.7.

$$\lim_{x \rightarrow \infty} \frac{\log_2 \theta(\mathbb{F}_2, q, 1 - 2^{-x})}{x^q / q!} = 1.$$

In general there is an essential singularity at 1. Serre used this replacement to reign it in, but we won't work with it very explicitly.

2.3 Applications

Theorem 2.8. *Suppose X is a 1-connected space satisfying the following conditions:*

1. $H_*(X; \mathbb{Z})$ is of finite type.
2. $H_i(X; \mathbb{F}_2) = 0$ for $i \gg 0$.

Then, for infinitely many indices i , $\pi_i(X)$ has a subspace isomorphic to \mathbb{Z} or $\mathbb{Z}/2$.

This directly implies Theorem 2.4 once you know that only finitely many homotopy groups of spheres are infinite.

We see this using a whitehead tower

$$\begin{array}{ccc}
 & \cdots & \\
 & \downarrow & \\
 & X_{n+1} & \\
 & \downarrow & \searrow \\
 & X_n & \longrightarrow X \\
 & \downarrow & \nearrow \\
 & X_{n-1} & \\
 & \downarrow & \\
 & \cdots &
 \end{array}$$

where X_n is n -connected, and a π_i iso to X and X_{n-1} for $i > n$. We'll use another piece of machinery, seen by the Serre SS directly.

Lemma 2.9. *For $F \hookrightarrow E \rightarrow B$ a Serre fibration with B simply connected, $B(t)F(t) \geq E(t)$.*

Proof of Theorem 2.8. Otherwise, there is some largest q with $\pi_q(X) \otimes \mathbb{Z}/2 \neq 0$. Then, there is some j smallest such that $H_j(X; \mathbb{Z}/2) \neq 0$. Then, $\pi_j(X) \otimes \mathbb{Z}/2 \neq 0$.

In the whitehead tower, $X_q \rightarrow X_{q-1}$ is trivial on $\pi_*(-) \otimes \mathbb{Z}/2$, so $H^*(X_q, \mathbb{Z}/2)$ is trivial. Using the fibration $X_q \hookrightarrow X_{q-1} \rightarrow K(\pi_q(X), q)$ from the whitehead tower, we must have $H^*(X_{q-1}) = H^*(K(\pi_q(X), q))$. Then,

$$X_{q-1}(t) = \theta(\pi_q(x), q, t).$$

Further, the fibrations in the whitehead series imply that

$$X_{i+1}(t) \leq X_i(t)\theta(\pi_{i+1}(X), i, t)$$

for each i , Chaining these together forever, what we get is

$$\theta(\pi_q(X), q, t) \leq X_1(t)\theta(\pi_2(X), 1, t) \cdots \theta(\pi_{q-1}(x), q-2, t).$$

Note that $X_1(t)$ is a polynomial, so bounded on $[0, 1]$. Applying our asymptotic bound on θ yields a contradiction. \square

3 Zihong Chen: Moore, Semi-simplicial complexes and Postnikov systems

This talk was delivered September 20, 2021 by Zihong (Peter) Chen.

3.1 Review of simplicial sets

The talk began with a very brief review of simplicial sets: let Δ be the category of finite ordered sets and order preserving maps. Recall that such maps are generated by distinguished maps $\delta_i : [n] \rightarrow [n+1]$ and $s_i : [n+1] \rightarrow [n]$, called the *face and degeneracy maps*.

Definition 3.1. A *simplicial set* is a functor $X : \Delta^n \rightarrow \mathbf{Set}$.

The morphism set is completely characterised by their images on face and degeneracy maps, which must satisfy a collection of combinatorial relations, which I won't write down here.

Example 3.2:

The *standard n -simplex* is given by the representable functor $\Delta[n] := \text{Hom}(-, [n])$.

By Yoneda's lemma, $X_n = \text{Hom}(\Delta[n], X)$, where $X_n = X([n])$.

Example 3.3:

If $X \in \mathbf{Top}$, the singular simplicial set $\text{Sing}(X)$ is familiar. It participates in an adjunction, with left adjoint $|\cdot|$ the *Geometric realization*.

Example 3.4:

Define the *i th face* $\delta_i : \Delta[n-1] \rightarrow \Delta[n]$. The *i th horn* is $\bigvee_i^n := \bigcup_{k \neq i} \delta_i$. The *boundary* is $\partial\Delta[n] = \bigcup_i \delta_i$.

This allows us to define the combinatorial equivalent of a topological space:

Definition 3.5. A simplicial set X is a *Kan complex* if every morphism $\bigvee_k^n \rightarrow X$ factors through $\Delta[n] \rightarrow X$; you can *fill any horn* (not necessarily uniquely).

A morphism $p : E \rightarrow B$ is a *Kan fibration* if it has the right lifting property against horn inclusions:

$$\begin{array}{ccc} \bigvee_k^n & \longrightarrow & E \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \Delta[n] & \longrightarrow & B \end{array}$$

Examples of this include $\text{Sing}(X)$, and any simplicial group (which we won't prove).

Definition 3.6. For X a Kan complex, define the *path components* $\pi_0(X) = X_0 / \sim$ where $x \sim y$ if there exists some p with $d_1 p = x$ and $d_0 p = y$.

This is in fact an equivalence relation: you can do this via horn filling, which was drawn on the board, but which I will not spell out. We can define higher homotopy groups after defining the internal hom:

Definition 3.7. For $A \subset X$ and $B \subset Y$, define the *mapping object*

$$\text{Map}((X, A), (Y, B)) = \text{Hom}(\Delta[n] \times (X, A), (Y, B))$$

i.e. the maps $\Delta[n] \times X \rightarrow Y$ restricting to a map $\Delta[n] \times A \rightarrow B$. The maps $\Delta[i] \rightarrow \mathbf{Set}$ form a covariant functor, so this is a contravariant functor, i.e. a simplicial set.

We use the following Theorem of Kan:

Theorem 3.8 (Kan). *If Y, B are Kan complexes, then so is $\text{Map}((X, A), (Y, B))$.*

We finally define homotopy groups.

Definition 3.9. If X is a Kan complex, define $\pi_n(X, x) := \pi_0(\text{Map}((\Delta[n], \partial\Delta[n]), (X, x)))$. A Kan complex is $K(\Pi, n)$ if $\pi_q(X, x) = \Pi$ when $q = n$ and 0 otherwise.¹

We will use these to decompose Kan complexes.

3.2 Postnikov systems

Let $\Delta[q]_n$ be the n -skeleton of $\Delta[q]$. For X a Kan complex, define the complex $X^{(n)}$ via

$$X_q^{(n)} = X_q / \sim \quad x \sim y \iff x|_{\Delta[q]_n} = y|_{\Delta[q]_n}.$$

The maps are induced by X . We have the following properties:

1. $X^{(n)}$ is a Kan complex.
2. There is a quotient Kan fibration $X^{(n)} \xrightarrow{p} X^{(k)}$ if $n > k$.
3. $\pi_q(X^{(n)}, x) = 0$ if $q > n$.
4. $p_* : \pi_q(X^{(n)}, x) \xrightarrow{\sim} \pi_q(X^{(k)}, x)$ is an iso if $n \geq k \geq q$.

As in topology, Kan fibrations induce LES of homotopy groups; hence the fiber $F^{(n+1)} \hookrightarrow X^{(n+1)} \xrightarrow{p} X^{(n)}$ is a $K(\pi_{n+1}(X), x+1)$. We finally give this a name:

Definition 3.10. $(X^0, X^{(1)}, \dots)$ is called the *natural Postnikov system* of X .

This motivates a question: How far is X from $\prod_n K(\pi_n, n)$? It's always a colimit, but we'll measure how complex it is in the following section.

The idea is that $K(\pi_{n+1}, n+1) \hookrightarrow X^{(n+1)} \rightarrow X^{(n+1)}$ will be seen as something like a “principal $K(\pi_{n+1}, n+1)$ -bundle.” We will construct something like a “classifying space” $\overline{W}(K(\pi_{n+1}, n+1)) = K(\pi_{n+1}, n+2)$, and derive algebraic invariants from this. Let's actually do this now:

3.3 Principally twisted cartesian products

Definition 3.11. A *principally twisted Cartesian product* (PTCP) with simplicial group G and base G is written

$$E(T) = G \times_T B$$

where $E(T)_n = G_n \times B_n$ with degeneracy maps all the same, except that

$$\partial_0(g, b) = (T(b) \cdot d_0 g, d_0 b)$$

and T is a *twisting function* $B_q \rightarrow G_{q-1}$ for $q \geq 1$.

This is a combinatorial version of *holonomy*, as per a comment from Prof. Miller.

Definition 3.12. A PTCP is of *type* (W) if $B_0 = \{b_0\}$ and

$$\partial_0|_{\{e_q\} \times B_q} : [e_q] \times B_q \xrightarrow{\sim} E(T)_{q-1}$$

is an iso. Let \int be its inverse.

Theorem 3.13. If $G \times_T B$, $G' \times_{T'} B'$, and $\gamma : G \rightarrow G'$ is a morphism of simplicial group, then there exists a unique γ -equivariant map $\theta : G \times_T B \rightarrow G' \times_{T'} B'$ and *Some condition holds of θ -fill in later*.

I couldn't follow this part; use \int to construct this “upwards” from b_0 , or something like that.

Corollary 3.14. A PTCP of type (W) with group G is unique, if it exists.

¹This *actually* has a requirement of minimality, but we handwave this away.

Theorem 3.15. *If $E(T)$ is PTCP of type (W) , it is contractible.*

They do exist! We can construct them by $B := \overline{W}(G)$, $W(G) = G \times_{T(G)} \overline{W}(G)$, where $\overline{W}_n(G) = G_{n-1} \times \cdots \times G_0$ for $n \geq 1$, and terminal for $n = 0$. [put face and degen maps here](#). It has twisting function

$$T(G)[g_n, \dots, g_0] = g_n.$$

It can be checked explicitly that this is type (W).²

Corollary 3.16. *Every PTCP with group G is by*

$$B \xrightarrow{\pi} \overline{W}(G)$$

with $\pi(b) = [T(b), T(\partial_0 b), \dots, T(\partial_0^{n-1} b)]$.

[This is a simplicial version of the bar construction??](#)

This allows us to explicitly construct $K(\pi, n)!$ Define $K(\pi, 0)$ to be π in each degree and $\partial_i s_i$ all identity. Define $K(\pi, n) = \overline{W}(K(\pi, n-1))$ inductively. We can see this is in fact a $K(\pi_1)$ via a fibration

$$K(\pi, n) \rightarrow W(K(\pi, n)) \rightarrow \overline{W}(K(\pi, n)),$$

where we know $W(*)$ to be contractible.

The main technical result follows:

Lemma 3.17. *Suppose there is no nontrivial morphism $\pi_1 \rightarrow \text{Aut}(\pi_n)$. Then, $X^{(n)}$ is a PTCP with group $K(\pi_{n+1}, n+1)$.³*

To handwave, the idea for this is that minimal Kan fibrations are fiber bundles. Given the π_1 assumption, the structure group is $K(\pi_{n+1}, n+1)$. Then, a “principal G -bundle” is the same thing as a PTCP, in some intuitive way.

We can define the k -invariants via the fibrations $K(\pi_{n+1}, n+1) \hookrightarrow X^{(n+1)} \rightarrow X^{(n)}$: there is a universal class

$$u \in H^{n+2}(K(\pi_{n+1}, n+2))$$

and via the map $X^{(n+1)} \xrightarrow{f^{n+2}} \overline{W}(K(\pi_{n+1}, n+1)) = K(\pi_{n+1}, n+2)$, we can define k -invariants as $(f^{n+2})^* u = k^{n+2}$.

²This was written down in class.

³Per a comment of Prof. Miller, we only need simplicity, not total nontriviality of morphisms $\pi_1 \rightarrow \text{Aut}(\pi_n)$.

4 Dylan Pentland: Borel, La cohomologie modulo 2 de certains espaces homogenes

This talk was delivered September 22, 2021 by Dylan Pentland.

4.1 Motivation and prerequisites

Characteristic classes We have a functor

$$\mathrm{Bun}_{O(n)} : \mathbf{Top}^{\mathrm{op}} \rightarrow \mathbf{Set}$$

sending X to the isomorphism classes of principal $O(n)$ bundles mod isomorphisms. We know that this is representable, i.e. expressible as $\mathrm{Bun}_{O(n)}(-) = \mathrm{Hom}(-, \mathrm{BO}(n))$ (in the homotopy category).

Definition 4.1. A *characteristic class* is a natural transformation $\mathrm{Bun}_{O(n)} \Rightarrow H^i(-)$, where coefficients are understood mod 2. By the Yoneda lemma, this is the same thing as an element of $H^i(\mathrm{BO}(n))$.

We're going to characterize these via a cohomology computation. The main theorem is as follows: let $Q(n) \subset O(n)$ be the diagonal matrices. From this inclusion, we get a projection $\mathrm{BQ}(n) \xrightarrow{p} \mathrm{BO}(n)$, which yields an induced map

$$\rho^* : H^*(\mathrm{BO}(n)) \rightarrow H^*(\mathrm{BQ}(n)) \simeq \mathbb{F}_2[x_1, \dots, x_n].$$

Theorem 4.2. *The map ρ^* satisfies the following properties:*

- ρ^* is injective.
- the image of ρ^* is $\mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_{in}}$.
- $p^*(w_i) = e_i$.

The splitting principle We give a modern POV on this:

Theorem 4.3. *Let X be paracompact and $E \rightarrow X$ a bundle. There is an undiced bundle $f : \mathrm{Fl}(E) \rightarrow X$ so that*

$$f^* : H^*(X) \rightarrow H^*(\mathrm{Fl}(E))$$

*is injective, and f^*E splits into a direct sum of line bundles.*

This winds up telling you the injectivity of Theorem 4.2, but not the image statement (only a containment). Either way, the proof is not much easier.

Spectral sequences breaking down We'll keep some assumptions about finite type cohomology. Dylan stated the requirement of simply connected spaces or principal G -bundle.⁴

Theorem 4.4. *Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration. The associated Serre spectral sequence (SSS) is trivial if and only if $H^*(E) \rightarrow H^*(F)$ is surjective. In this case, we say that F is totally non-homologous to zero, and we have the following properties:*

- p^* is injective.
- $P(E) = P(B) \cdot P(F)$.

The condition is called totally non-homologous to zero because the dual condition $H_*(F) \hookrightarrow H_*(E)$ makes sense for this name.

⁴Haynes had some comments about this; principality is not enough in general. There's secretly some connectedness condition.

4.2 Cohomology of $\mathrm{BO}(n)$

Outline of the proof of Theorem 4.2 Let $F_n = O(N)/Q(N)$. We will use the fibration

$$F_n \hookrightarrow \mathrm{BQ}(n) \xrightarrow{p} \mathrm{BO}(n).$$

We call this fibration (\star) . We follow the following steps:

- (1) $H^*(F_n) = \langle H^1(F_n) \rangle$ so that $P(F_n) = (1-t) \cdots (1-t)^n \cdot (1-t)^{-n}$.
- (2) The SSS for (\star) is trivial, so $P(\mathrm{BQ}(n)) = P(\mathrm{BO}(n))P(F_n)$, giving injectivity.
- (3) $\mathrm{im} \rho^* \subset \mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_{in}}$, and dimensions yield that this is an equality.

General rule of theorem: every spectral sequence written today will be trivial.

Step 1: cohomology of F_n . We use induction via the fibration

$$F_{n-1} \hookrightarrow F_n \rightarrow \mathbb{P}^{n-1}$$

Lemma 4.5. $\dim H^1(F_n) \geq n-1$.

$$\text{Write } {}^n E_r = \bigoplus_{s+t=n} E_r^{s,t}.$$

Proof. Recall the fibration $F_n \hookrightarrow \mathrm{BSQ}(n) \rightarrow \mathrm{BSO}(n)$. The base space is simply connected, so

$$E_2^{1,0} = H^1(\mathrm{BSO}(n), H^0(F_n)) = 0.$$

Hence

$$\dim^1 E_2 \geq \dim^1 E_\infty = \dim H^1(\mathrm{BSQ}(n)) = n-1.$$

This implies that $E_2^{0,1} = H^1(F_n)$. □

Proposition 4.6. $P(F_n) = (1-t) \cdots (1-t^n)(1-t)^{-n}$ and $H^*(F_n) = \langle H^1(F_n) \rangle$.

Proof. Return to the fibration from the beginning of the fibration. We know the Poincaré polynomial for projective space, and we just have to prove that the SSS is trivial.⁵ Write $H^*(F_n) \xrightarrow{i^*} H^*(F_{n-1})$. Assume both claims for $n-1$, so $\dim H^1(F_{n-1}) = n-2$. Note the following:

- $\dim E_2^{1,0} = \dim H^1(\mathbb{P}^{n-1}) = 1$.
- $\dim E_2^{0,1} = \dim H^0(\mathbb{P}^{n-1}, H^1(F_n)) \leq n-2$.

Look at $\dim H^1(F_n) = {}^1 E_\infty \leq n-1$; combined with our previous bound, we have $\dim H^1(F_n) = n-1$. This implies that ${}^1 E_n = {}^1 E_\infty$ since they have equal dimensions. This implies that $E_2^{0,1}$ are cocycles for differentials.

Further, note that $\mathrm{im} i^*|_{\deg 1} = E_\infty^{0,1} = H^1(F_{n-1})$. Since cohomology of the codomain is generated in degree 1, this implies that i^* is surjective, so the SSS is trivial. This implies the Poincaré polynomial is as we said it is, by a familiar technique. □

Step 2: triviality of the SSS of (\star) .

Proposition 4.7. *The SSS for (\star) is trivial.*

Proof. Note that $\dim^1 E_2 = \dim H^1(\mathrm{BO}(n), H^0(F_n)) + \dim H^0(\mathrm{BO}(n), H^1(F_n))$. The first is equal to 1, and the second is $\leq n+1$, and the second is $\leq n-1$, so the total is $\leq n$.

Now look at $\dim {}^1 E_\infty \leq \dim^1 E_2$, which is an equality for dimension reasons. We have $\dim^1 E_2 \geq \dim^1 E_\infty$, and hence $H^0(\mathrm{BO}(n), H^1(F_n)) = H^1(F_n)$. For reasons relating to generation at degree 1, we also have $H^0(\mathrm{BO}(n), H^k(F_n)) = H^k(F_n)$. Hence $H^*(\mathrm{BQ}(n)) \twoheadrightarrow H^*(F_n)$. Hence the SSS is trivial. □

This allows us to immediately compute the Poincaré polynomial

$$P(\mathrm{BO}(n)) = \frac{1}{(1-t)(1-t^2) \cdots (1-t^n)}.$$

⁵In particular, $P(\mathbb{P}^{n-1}) = \frac{1-t^n}{1-t}$, so since passing to the associated graded preserves graded dimension, triviality implies that the Poincaré series are multiplicative, and we can prove the Poincaré series computation inductively. I'll skip this.

Step 3: containment of the image of ρ^* in the symmetric polynomials. Combinatorics exists:

Lemma 4.8. $P(\mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_{in}}) = P(\mathrm{BO}(n))$.

Use Schur polynomials. Back to the topology.

Proposition 4.9. $\mathrm{in} \rho^* \subset \mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_{in}}$.

Proof. $\Sigma_{in} = N_n/Q(n)$. Write down the classifying space fibration:

$$Q(n) \hookrightarrow EQ(n) \rightarrow BQ(n)$$

N_n acts on this, and acts on polynomials by permuting the generators in $\mathbb{F}_2[x_1, \dots, x_n]$. The normalizer N_n also acts on

$$O(n) \rightarrow EO(n) \rightarrow \mathrm{BO}(n).$$

the action on $\mathrm{BO}(n)$ is homotopically trivial⁶, which we could use... Instead, we know the groups, so we can check concretely that this acts trivially on the cohomology, which gives the image containment. \square

Hidden step 4: talking about $p^*(w_i) = e_i$. Once we know the $p^*(w_i) = e_i$ statement, universal relations on Steifel-Whitney classes come down to relations on $H^*(\mathrm{BO}(n))$.

⁶This is a general fact stated by Haynes, which comes down to some categories thing I didn't catch...

5 Mikayel Mkrtchyan: Milnor, The Steenrod algebra and its dual

This talk was delivered September 24, 2021 by Mikayel Mkrtchyan.

5.1 Refresher on the Steenrod algebra and Hopf algebras

For the next 48 minutes or so, we set $p = 2$ and work with the mod-2 Steenrod algebra. Recall that the *Steenrod algebra* \mathcal{A}^* is a graded commutative algebra of mod-2 cohomology operations generated as an algebra by $\text{Sq}^n \in \mathcal{A}^n$ for $n \geq 1$. For a finite sequence $I = (i_1, \dots, i_r)$, define

$$\text{Sq}^I = \text{Sq}^{i_1} \dots \text{Sq}^{i_r}.$$

Recall that a sequence I is admissible if $a_i \geq 2a_{i-1}$. We have a basis made of these:

Theorem 5.1 (Serre-Cartan). *The set $\{\text{Sq}^I \mid I \text{ admissible}\}$ is an \mathbb{F}_2 -basis for \mathcal{A}^* .*

This is proved via the following relation:

Theorem 5.2 (Adem relation). *For all $0 < n < 2m$,*

$$\text{Sq}^n \text{Sq}^m = \sum_{k=1}^{n/2} \binom{m-k-1}{n-2k} \text{Sq}^{n+m-k} \text{Sq}^k,$$

and these generate all relations in a presentation of \mathcal{A}^ .*⁷

As a preview of what's to come, define \mathcal{A}_* to be the dual coalgebra.

Theorem 5.3. *\mathcal{A}^* is a graded connected Hopf algebra, and its dual satisfies $\mathcal{A}_* \simeq \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$.*

As an application, we'll show that all elements in $\mathcal{A}^{>0}$ are nilpotent.

The base field of \mathbb{F}_2 is to be understood.

Definition 5.4. A connected graded Hopf algebra is a graded associative algebra B^* s.t. $B_0 = \mathbb{F}_2$, endowed with a coassociative *comultiplication* map

$$B^* \xrightarrow{\psi} B^* \otimes B^*$$

s.t. $\psi(b) = b \otimes 1 + 1 \otimes b + \sum b'_i \otimes b''_i$ for all $b \in B^{>0}$.

Projecting to the 0th graded part is the “augmentation” (counit), and you can define the antipode uniquely given this data.

5.2 Coalgebra structure on the Steenrod algebra

We want to define a map $\psi : \mathcal{A}^* \rightarrow \mathcal{A}^* \otimes \mathcal{A}^*$ extending the map

$$\psi(\text{Sq}^n) = \sum_{i+j=n} \text{Sq}^i \otimes \text{Sq}^j.$$

We will prove that this is well defined using the Cartan formula

$$\text{Sq}^n(a \times b) = \sum_{i+j=n} \text{Sq}^i(a) \times \text{Sq}^j(b).$$

We'll also use the following

⁷The presentation statement was said verbally but not written.

Lemma 5.5. *Fix some n . There exists a space U with finite-type cohomology and a class $u \in H^*(U)$ such that*

$$\sigma : \mathcal{A}^* \rightarrow H^*(U)$$

given by $a \mapsto a \cdot u$ is injective on $\mathcal{A}^{\leq n}$.

Proof sketch. This is given by $U = K(\mathbb{Z}/2, n+1)$ with $u \in H^{n+1}(K(\mathbb{Z}/2, n+1))$. Use the result of Gabi's talk. \square

Lemma 5.6. *There exists a lift*

$$\begin{array}{ccc} T(\{\text{Sq}^n\}) & \longrightarrow & \mathcal{A}^* \\ \downarrow \psi & \nearrow & \\ \mathcal{A}^* \otimes \mathcal{A}^* & & \end{array}$$

Proof. Choose (U, u, σ) as in the lemma. Form the diagram

$$\begin{array}{ccc} T(\{\text{Sq}^n\}) & \longrightarrow & \mathcal{A}^* \\ \downarrow \psi & \nearrow & \downarrow \text{act on } u \times u \\ \mathcal{A}^* \otimes \mathcal{A}^* & \xrightarrow{\sigma \otimes \sigma} & H^*(U) \otimes H^*(U) \simeq H^*(U \times U). \end{array}$$

Then, note that $\sigma \otimes \sigma$ is injective at degrees $\leq n$, and this commutes, hence we can choose a lift in lower degrees; this allows you to define it degree-wise by picking high enough n . \square

Corollary 5.7. \mathcal{A}^* is a cocommutative Hopf algebra.⁸

5.3 Housekeeping

Suppose X is a finite CW complex. This is routine, formal, and not talked about explicitly.

1. We have an action

$$\mathcal{A}^* \otimes H^* \rightarrow H^*.$$

2. This yields a dual operation

$$H_* \otimes \mathcal{A}_* \rightarrow H_*.$$

3. We can dualize this:

$$\lambda : H^* \rightarrow H^* \otimes \mathcal{A}_*,$$

since the homology and \mathcal{A} are both finite type.

4. Note that λ makes H^* into a \mathcal{A}_* -comodule.

5. The following proof was omitted:

Lemma 5.8. λ is an \mathbb{F}_2 -algebra homomorphism.

Let's work an example.

Example 5.9:

Let $X := \mathbb{RP}^\infty = K(\mathbb{Z}/2, 1)$, with $u \in H^1(X)$.^a

Lemma 5.10.

$$\text{Sq}^n(x^{2^m}) = \begin{cases} x^{2^{m+1}} & n = 2^m \\ 0 & \text{otherwise} \end{cases}$$

Proof sketch. Define $\text{Sq} := \sum_i \text{Sq}_i$. Note that $\text{Sq}(u) = u + u^2$, so $\text{Sq}(u^{2^m}) = u^{2^m} + u^{2^{m+1}}$. \square

⁸He called it coassociative, but I omit this as this is the convention for Hopf algebras in general.

Corollary 5.11. $\lambda : H^*(X) \rightarrow H^*(X) \otimes \mathcal{A}_*$ is given by

$$\lambda(u) = \sum_k u^{2^j} \otimes \zeta_k$$

where $\langle \zeta_i, \text{Sq}^I \rangle = 0$ unless $I = I_k := (2^{i-1}, 2^{k-2}, \dots, 1, 0)$.

I'm lagging a bit behind, so expect this next bit to be choppy.

^aWe'll see why we don't have to care that X is finite.

5.4 Algebra structure on the dual Steenrod algebra

Let I be an admissible sequence, and define

$$\gamma(I) = (i_1 - 2i_2, i_2 - 2i_3, \dots, i_r, 0).$$

Let R be a sequence.

Proposition 5.12. Let I, J be admissible sequences of the same degree. Then,

$$\langle \zeta^{\gamma(J)}, \text{Sq}^I \rangle = \begin{cases} 1 & I = J \\ 0 & I < J \end{cases}$$

where $<$ denotes the lexicographic order.⁹

Proof. We prove this by induction. Let $J = (a_j, \dots, a_k, 0)$ and similar for I and b . define

$$J' = (a_1 - 2^{k-1}, a_2 - 2^{k-2}, \dots, 0).$$

Then,

$$\gamma(J) = \gamma(J') + (\text{a } 1 \text{ in the } k\text{th spot}).$$

Hence

$$\zeta^{\gamma(J)} = \zeta^{\gamma(J')} \cdot \zeta_k,$$

so that

$$\langle \zeta^{\gamma(J)}, \text{Sq}^I \rangle = \langle \zeta^{\gamma(J)} \otimes \zeta_k, \psi(\text{Sq}^I) \rangle = \langle \zeta^{\gamma(J')} \otimes \zeta_k, \sum \text{Sq}^{I_1} \otimes \text{Sq}^{I_2} \rangle.$$

If you work out the nitty gritty, this concludes the proof by induction. □

Corollary 5.13. $\mathcal{A}_* \simeq \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$.

Proof. The last proposition proved that $\zeta^{\gamma(J)}$ form an \mathbb{F}_2 -basis, which is exactly equivalent to \mathcal{A}_* being a polynomial algebra in ζ_i . □

We now characterize the comultiplication.

Theorem 5.14. The comultiplication map $\varphi_* : \mathcal{A}_* \rightarrow \mathcal{A}_* \otimes \mathcal{A}_*$ is given by

$$\zeta_n \mapsto \sum_{k \geq 0} \zeta_{n-k}^{2^k} \otimes \zeta_k.$$

This is some measure of the basis we gave being nice.

Proof. We have coassociativity:

$$\begin{array}{ccc} H^* & \longrightarrow & H^* \otimes \mathcal{A}_* \\ \downarrow & & \downarrow \lambda \otimes \text{id} \\ H^* \otimes \mathcal{A}_* & \xrightarrow{\text{id} \otimes \varphi_*} & H^* \otimes \mathcal{A}_* \otimes \mathcal{A}_* \end{array}$$

We perform a diagram chase for $X := \mathbb{RP}^\infty$. □

⁹It hasn't been mentioned what happens when $I > J$.

5.5 Positive-degree homogeneous elements of the Steenrod algebra are nilpotent

Define $J_n \subset \mathcal{A}_*$ by $(\zeta_1^{2k}, \zeta_2^{k-1}, \dots, \zeta_{k-1}^2, \zeta_{k+1}, \dots)$. Observe that $\varphi_*(J_n) \subset J_n \otimes \mathcal{A}_*$ by our characterization of the Milnor diagonal, and hence \mathcal{A}_*/J_n is a Hopf algebra quotient of \mathcal{A}_* of finite dimension. By duality, this corresponds with a f.d. Hopf subalgebra, and expanding n threatens to swallow \mathcal{A}_* :

Corollary 5.15. *\mathcal{A}^* is the union of its finite dimensional Hopf subalgebras.*

By degree arguments, a positive dimension homogeneous element either is nilpotent or spans an infinite dimensional Hopf subalgebra, so this gives the nilpotency statement.

5.6 A sketch of the $p > 2$ case

In the odd p case, we have Lens spaces instead of \mathbb{RP}^∞ , and there are more cohomology elements:

Theorem 5.16.

$$\mathcal{A}_*^p = \mathbb{F}_p[\zeta_1, \dots, \zeta_i] \otimes \bigwedge^* [\tau_0, \tau_1, \dots].$$

6 Preston Cranford: Thom, Quelques proprietes globales des varietes differentiables (i)

This talk was delivered September 29, 2021 by Preston Cranford.¹⁰

6.1 Motivation and overview

We're motivated by the following question

Problem 6.1 (Steenrod's problem). *Let K be a finite polyhedron. Given $z \in H_r(K)$ (over \mathbb{F}_2 or \mathbb{Z}), does there exist compact M and map $f : M \rightarrow K$ s.t. $f_*[M] = z$?*

If so, say that z is *realized* by M . We will use the notation V^n for the n -manifold, and W^p the manifold realizing a class. We henceforth restrict to \mathbb{F}_2 coefficients. Our result will be that whenever $2p \leq n$, all z are realizable.

We'll follow the following outline:

- Thom spaces, classes,...
- We show z is realizable iff another class of a Thom space is realizable.
- We will study $MO(k)$ and show this has the homotopy type of a product of \mathbb{F}_2 -Eilenberg Mac Lane spaces.

6.2 Thom spaces and realizability

We'll use the following:

Definition 6.2. Let $G \subset O(n)$ be a distinguished subgroup with a faithful representation. A G -structure on M is a principal G -subbundle of the frame bundle on M . The $O(k)$ structure is associated with a metric on M .

Recall that we have a universal bundle $EO(k) \xrightarrow{p} BO(k)$ where $EO(k)$ is weakly contractible. Let $AO(k)$ be the mapping cylinder, and let $MO(k)$ be $AO(k)/(EO(k) \sim *)$.¹¹ See footnote.

Recall the following: construct the map

$$\varphi^* : H^{r-k}(BO(k)) \rightarrow H^r(AO(k), EO(k)) \simeq H^r(MO(k)).$$

Theorem 6.3 (Thom isomorphism theorem). *The map φ^* is an isomorphism.*

For unit class $w \in H^0(BO(k))$, call $\varphi^*(w) = u$ the *fundamental class* of $MO(k)$.

We'll prove the following:

Theorem 6.4. *Let V^n be a closed n -manifold. Then, $z \in H_{n-k}(V^n)$ is realizable by W^{n-k} if and only if its Poincaré dual $u \in H^k(V^n)$ is induced by some $f : U^n \rightarrow MO(k)$.*

Proof. (\implies) Suppose we have $z \in H_{n-k}(V^n)$ realized by $i : W^{n-k} \rightarrow V^n$. Let $N \rightarrow W^{n-k}$ be a normal tubular neighborhood of W^{n-k} , which has an $O(k)$ -structure.¹² Define a filler via pullback¹³

$$\begin{array}{ccc} N & \dashrightarrow & VO(k) \\ \downarrow & \lrcorner & \downarrow \\ W^{n-k} & \longrightarrow & BO(k) \end{array}$$

¹⁰I came in a bit late, so expect the beginning of these to be choppy and potentially subtly wrong.

¹¹This was corrected by Haynes to more modern notation, noting that this is wrong. $EO(k)$ here was supposed to be the universal sphere bundle. Haynes suggested the Thom space as the quotient of the universal disk bundle by the universal sphere bundle, and that the homotopical definition here was more general than necessary.

¹²He went over how to construct this locally, but I won't repeat it, since it's routine and I'm catching up a bit.

¹³There's notational confusion; $AO(k)$ henceforth means the mapping cone of the universal sphere bundle, and $VO(k)$ is the total space of the universal vector bundle.

Explicitly, this takes geodesic ball fibers to ball fibers. Taking cohomology, we have

$$\begin{array}{ccc} H^k(N, \partial N) & \longleftarrow & H^n(\mathrm{VO}(n), \mathrm{EO}(n)) \\ \uparrow & & \uparrow \\ H^0(W^{n-k}) & \longleftarrow & H^0(\mathrm{BO}(n)) \end{array}$$

We have a collapse map, where a is the point at infinity:

$$\begin{array}{ccccccc} & & N & \longrightarrow & W^{n-k} & \longrightarrow & \mathrm{BO}(n) \longrightarrow \mathrm{MO}(k) \\ & \nearrow & \uparrow & & & & \uparrow \\ & & \partial n & \longrightarrow & & & \{a\} \\ & \nwarrow & & & & & \nwarrow \\ V^n/N & \longrightarrow & & & & & \{a\} \end{array}$$

I am very confused. The following diagram was on the board, but I can't decipher what it means:

$$\begin{array}{ccccc} \text{p.d. to } z & \longleftarrow & \varphi^*(w_0) & \longleftarrow & U \\ & & \uparrow & & \uparrow \\ & & w_0 & \longleftarrow & w_{O(k)} \end{array}$$

He commented on the other direction after this. □

We can identify the cohomology of Thom space as an ideal: $H^*(\mathrm{MO}(k)) = w_k H^*(\mathrm{BO}(k))$, recalling that $H^*(\mathrm{BO}(k)) = \mathbb{F}_2[w_1, \dots, w_k]$.

Recall that we have the bundle $\mathrm{VO}(n) \rightarrow \mathrm{BO}(n)$, and this expresses the fundamental group

$$\pi_1(\mathrm{MO}(k)) = \pi_1(\mathrm{BO}(k)) / \mathrm{im}(\pi_1(\mathrm{VO}(k)) \rightarrow \pi_1(\mathrm{BO}(k))) = \mathbb{F}_2 / \mathbb{F}_2 = 0.$$

Hence $\mathrm{MO}(k)$ is simply connected. Recall the following:

Theorem 6.5. *If X, Y are simply connected and $f : X \rightarrow Y$ is a map s.t. f^* is an iso for $r < k$ and monic at $r = k$, then there is a map $g : X_k \rightarrow Y_k$ that is a homotopy equivalence.*

This allows you to prove the following theorme:

Theorem 6.6. *There is a $2k$ -equivalence $\mathrm{MO}(k) \rightarrow K(\mathbb{F}_2, k)^{e_1} \times \dots \times K(\mathbb{F}_2, 2k)^{e_{2R}}$.*

7 Swapnil Garg: Thom, Quelques proprietes globales des varieties differentiables (ii)

This talk was delivered October 1, 2021 by Swapnil Garg.

7.1 Transversality and tubular neighborhoods

For the duration of this talk, V^n is a smooth manifold, and N^{p-q} is a smooth, compact manifold.

Definition 7.1. A map $f : V^n \rightarrow M^p$ is transversal to $N^{p-q} \subset M^p$ at a point $y \in N^{p-q}$ if for all $x \in f^{-1}(y)$, the map $DF : T_x V^n \rightarrow T_y M^p \rightarrow T_y M^p / T_y N^{p-q}$.

If d is tranverse at all Y , then we say f is *transversal to N^{p-q}* .

In this case, $f^{-1}(p - q)$ is a smooth manifold with normal bundle $f^{-1}(\text{normal bundle of } N^{p-q})$.

Let $p : T \rightarrow N^{p-q}$ be a tubular neighborhood.

Definition 7.2. Let H be the group of diffeomorphisms $A \in \text{Aut } T$ s.t.

- (a) A is the identity on ∂T .
- (b) A preserves $p^{-1}(y)$ for all $y \in N^{p-q}$.

Defining the distance between diffeomorphisms by taking some supremum over the maps, derivatives, inverse, etc., H is a metrizable space.

Let $y \in N^{p-q}$ be a point, and $y \in X, X'$ balls of radius r, r' around y in N^{p-q} . These can be chosen small enough to trivialize the normal bundle. Let $D = p^{-1}(X) = X \times B^q$ and similarly for D' .

There is a diagram

$$\begin{array}{ccccc} V^n & \xrightarrow{f} & M & & \\ & & \uparrow & & \\ & & N & & \\ & & \uparrow & & \\ V|_{f^{-1}(D)} & \xrightarrow{f} & D' & \xrightarrow{k} & B^q \end{array}$$

If $A \circ f$ has a critical value at y , i.e. the derivative is non-surjective, then $k \circ A \circ f$ has a critical value at $k(y) = 0$ (the center of B^q).

Let $\sigma_i \subset H$ be the set of automorphisms A s.t. $k \circ A \circ f|_{f^{-1}(D) \cap K_i}$ has a critical value at 0. We'll characterize this:

Lemma 7.3. *The set σ_i is closed and has no interior points in H .*

Hence the generic diffeomorphism is regular on K_i .

Proof. Closedness. For $A \notin \sigma_i$, we want to construct a ball around A . If the absolute value of the determinant of the $q \times q$ Jacobian of A is bounded below by 2ε , then under perturbing A s.t. the Jacobian changes by at most ε , the map remains regular. Hence σ_i is closed.

Empty interior. We invoke Sard's theorem. Take $A \in \sigma_i$. Assuming f is C^n , there exists a regular value c of $K \circ A \circ f$ which is arbitrarily close to 0. Take G_1 to be the identity on B^q , G_0 a diffeomorphism of B^q with $G_0(c) = 0$, and G_t to be a homotopy between them with varying t . Let $E(y, z) = (y, G_{d(y)}(z))$ with $d(x) = 0$, $d(\partial x') = 1$ and $d \in C^\infty$. Take $k \circ E \circ A \circ f$; this has a regular value at 0, which allows you to conclude.¹⁴ \square

Varying K_i , the countable intersection of $(\sigma_i)^c$ is dense. We get a meaningful result out of this:

Theorem 7.4. *A C^n map $f : V^n \rightarrow M^p$ can be perturbed to be transversal to $N^{p-q} \subset M^p$.*

We'll use this to talk about cobordism.

¹⁴Not sure how this finishes.

7.2 Cobordism and L -equivalence

Definition 7.5. Let V, V' be oriented compact manifolds of dimension k . The manifolds V and V' are *cobordant* if there exists a $k+1$ -manifold X with $\partial(X) = V' \amalg -V$.¹⁵¹⁶

Suppose $W^{n-k} \subset V^n \subset \mathbb{R}^{n+m}$. For $x \in W^{n-k}$, let $H(kx)$ be the k -dimensional vector space tangent to V^n and tangent to W^{n-k} inside \mathbb{R}^{n+m} . This yields a map $W^{n-k} \rightarrow \text{Gr}_k(\mathbb{R}^{n+m})$. Taking $m \rightarrow \infty$, we have a map $\text{Gr}_k(\mathbb{R}^{n+m}) \rightarrow \text{BO}(k)$.

For N a tubular neighborhood of W^{n-k} in V^n , we get a map

$$\begin{array}{ccc} N & \longrightarrow & D(\text{EO}(k)) \\ \downarrow & & \downarrow \\ W^{n-k} & \xrightarrow{g} & \text{BO}(k) \end{array}$$

Collapsing $V - N$ to a point gives a map

$$V \xrightarrow{f} D(\text{EO}(k))/S(\text{EO}(k)) = \text{MO}(k).$$

whose homotopy class turns out to be independent of the Riemannian metric chosen to define it.

We define a stronger equivalence:

Definition 7.6. Let $W_i^{n-k} \subset V^n$ be submanifolds. We say that they are *L -equivalent* with respect to V^n if there exists a submanifold $X^{n-k+1} \subset V^n \times [0, 1]$ such that $\partial X^{n-k+1} = W_0^{n-k} \sqcup W_1^{n-k}$.¹⁷

If W_i^{n-k} are L -equivalent, they generate homotopic maps $V^n \rightarrow \text{MO}(k)$. The main theorem of Thom's in this area shows that this is a bijection:

Theorem 7.7. *The above map $L_{n-k}(V^n) \rightarrow [V^n, \text{MO}(k)]$ is a bijection.*¹⁸

There is some confusion about how this depends on V^n . Taking V^n the sphere spectrum, taking a limit yields that the Cobordism ring corresponds with the stable homotopy groups of the Thom spectrum:

Lemma 7.8. $L_k(S^n) \simeq \mathfrak{N}^k$ is a bijection if $n \geq 2k + 2$. A similar statement holds in the oriented case.

Theorem 7.9. $\mathfrak{N}^k \simeq L_k(S^{n+k}) \simeq \pi_{n+k}(\text{MO}(n))$ for $n \geq k + 2$. Similarly, $\Omega^k \simeq \pi_{n+k}(\text{MSO}(n))$.

7.3 Other remarks

Note that we know the stable homotopy groups of $\text{MO}(n)$; they are $\mathbb{F}_2^{d(k)}$, as per Preston's talk.

We can further argue that cobordant manifolds have the same Steifel-Whitney numbers. We argue via a big diagram:

$$\begin{array}{ccc} S^{n+k} & \xrightarrow{f_{W'}} & \text{MO}(n) \xrightarrow{F} K(n+k, \mathbb{F}_2) \\ & & \downarrow \text{proj} \\ V_\omega & \longrightarrow & \text{Gr}_k \end{array}$$

we can trace elements of the cohomology and make an argument there. **I couldn't quite follow.**

¹⁵The sign indicates reversal of orientation.

¹⁶He drew the pair of pants cobordism as an example here.

¹⁷He drew a helpful picture; $V^n \times [0, 1]$ is pictured as a solid cylinder, whose ends are disk copies of V^n , in which W^i are embedded circles. An L -equivalence is pictured as a cylinder bounded by each W_i , on opposite disk faces.

¹⁸I use my own notation here' Swapnil followed Thom's notation.

8 Haoshuo Fu: Hirzebruch, Topological methods in algebraic geometry

This talk was delivered on October 4, 2021 by Haoshuo Fu. The topic of this talk will be the index theorem.

8.1 The signature of a manifold

Definition 8.1. Let M be an oriented closed manifold of dimension $4k$. There is a symmetric bilinear form

$$\begin{aligned}\phi_M : H^{2k}(M, \mathbb{R}) \otimes H^{2k}(M, \mathbb{R}) &\rightarrow \mathbb{R} \\ x \otimes y &\mapsto \langle x \cup y, [M] \rangle.\end{aligned}$$

The *signature* of M is the signature

$$\tau(M) = (\# \text{ pos. eigenvals. of } \phi_M) - (\# \text{ neg. eigenvals. of } \phi_M).$$

Example 8.2:

In the case $M = \mathbb{CP}^{2k}$, then $H^{2k}(\mathbb{CP}^{2k}) = \mathbb{R}$, we have $\varphi = I$, so $\tau(\mathbb{CP}^{2k}) = 1$.

We can characterize these:

Proposition 8.3. *The following hold:*

- $\tau(-M) = -\tau(M)$.
- $\tau(M \sqcup N) = \tau(M) + \tau(N)$.
- $\tau(M^{4k} \times N^{4\ell}) = \tau(M^{4k}) \cdot \tau(N^{4\ell})$.
- If $M^{4k} = \partial W^{4k+1}$, then $\tau(M^{4k}) = 0$.

Proof sketch. For the third bullet, note that¹⁹

$$h^{2k+2\ell}(M^{4k} \times N^{4\ell}) = \bigoplus_{i \in \mathbb{Z}} H^{2k+i}(M^{4k}) \oplus H^{2k-i}(N^{4\ell}) =: \bigoplus_{i \in \mathbb{Z}} V(i).$$

When $i = 0$, we have $\phi_{M \times N}|_{V(0)} = \phi_M \otimes \phi_N$. When $i > 0$, letting A be the pairing of $V(i)$ and $V(-i)$, we have

$$\phi_{M \times N}|_{V(i) \oplus V(-i)} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}.$$

Noting that $\det(\lambda I - \phi_{M \times N}|_{V(i) \oplus V(-i)}) = \det(\lambda^2 I - A^2)$ allows one to conclude.

The last bullet it proved via Poincaré duality; given the morphism of exact sequences

$$\begin{array}{ccccc} H^{2k}(w) & \xrightarrow{i^*} & H^{2k}(M) & \xrightarrow{\delta} & H^{2k+1}(W, M) \\ \downarrow & & \downarrow & & \downarrow \\ H_{2k+1}(W, M) & \longrightarrow & H_{2k}(M) & \longrightarrow & H_{2k}(W) \end{array}$$

the snake lemma yields a short exact sequence

$$0 \rightarrow \text{im } i_* \rightarrow H^{2k}(M) \rightarrow \text{im } \delta \rightarrow 0.$$

Note that $\dim \text{im } \delta = \dim \text{im } i^*$, so $\text{im } i^*$ is a subspace of $H^{2k}(M)$ of half dimension. We have $\phi_M|_{\text{im } i^* \otimes \text{im } i^*} = 0$, and subsequently, $\phi_M = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, giving the fourth bullet point. \square

Corollary 8.4. τ yields a homomorphism $\tau : \Omega_{4*} \rightarrow \mathbb{Z}$.

¹⁹I omit the proof of the first two, as they're obvious.

8.2 Review of the structure of the cobordism ring

Take notation from the previous two talks, on Thom. Note that there is a (pulled back) map

$$\begin{array}{ccc} \xi_n \oplus e & \longrightarrow & \xi_{n+1} \\ \downarrow & & \downarrow \\ \mathrm{BSO}(n) & \longrightarrow & \mathrm{BSO}(n+1). \end{array}$$

Taking Thom spaces yields a morphism $\Sigma \mathrm{MSO}(n) = \mathrm{Th}(\xi_n \oplus e) \rightarrow \mathrm{MSO}(n+1)$. This yields a prespectrum MSO .²⁰

Previous results yield an iso of graded Abelian groups

$$\pi_*(\mathrm{MSO}) = \mathrm{colim} \pi_{*+n}(\mathrm{MSO}(n)) = \Omega_*.$$

We can give MSO a ring spectrum structure via maps

$$\begin{array}{ccc} \xi_m \oplus \xi_n & \longrightarrow & \xi_{m+n} \\ \downarrow & & \downarrow \\ \mathrm{BSO}(m) \times \mathrm{BSO}(n) & \longrightarrow & \mathrm{BSO}(m+n) \end{array}$$

and a similar trick to before.

Recall the rational Hurewicz theorem:

Theorem 8.5 (Rational Hurewicz theorem). *Let X be a simply connected space with $\pi_i(X) \otimes \mathbb{Q} = 0$ for all $0 \leq i \leq n$. Then, the Hurewicz map*

$$h : \pi_i(X) \otimes \mathbb{Q} \rightarrow \tilde{H}_i(X; \mathbb{Q})$$

is an isomorphism for $0 \leq i \leq 2n$.

Applying this to the Thom spectrum, noting that we have we have

$$\tilde{H}^i(\mathrm{MSO}(n); \mathbb{Q}) = 0 \quad i \leq n$$

we can compute rational stable homotopy groups of MSO by computing homology in degree $\leq 2n$.²¹

$$\pi_*(\mathrm{MSO}) \otimes \mathbb{Q} \xrightarrow{\sim} H_*(\mathrm{MSO}; \mathbb{Q}).$$

We may combine this with the Thom isomorphism

$$H^i(\mathrm{BSO}(n); \mathbb{Q}) \xrightarrow{\sim} H^{i+n}(\mathrm{MSO}(n); \mathbb{Q})$$

by taking a limit in order to yield an iso

$$H^*(\mathrm{BSO}(n); \mathbb{Q}) \xrightarrow{\sim} H^*(\mathrm{MSO}; \mathbb{Q}).$$

Taking udals yields an iso

$$H_*(\mathrm{MSO}; \mathbb{Q}) \xrightarrow{\sim} H_*(\mathrm{BSO}(n); \mathbb{Q}).$$

Hence the signature actually yields a homomorphism

$$\tau : H_*(\mathrm{BSO}; \mathbb{Q}) \rightarrow \mathbb{Q},$$

i.e. an element $L \in H^*(\mathrm{BSO}; \mathbb{Q})$. We can write this in homogeneous parts as $L_n \in H^{4n}(\mathrm{BSO}; \mathbb{Q})$.

A diagram chase yields that

$$\langle L_N(\tau_M), [M] \rangle = \tau(M).$$

²⁰I would just call this a sequential spectrum.

²¹This is true of any ring spectrum, according to Haynes.

This is powerful, if we can determine the structure of L_N . We do so in a little while, but first, a bit more structure.

There is a commutative diagram

$$\begin{array}{ccc} H_*(\mathrm{BSO}; \mathbb{Q}) \otimes H(\mathrm{BSO}; \mathbb{Q}) & \longrightarrow & H_*(\mathrm{BSO}; \mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathbb{Q} \otimes \mathbb{Q} & \xlongequal{\quad} & \mathbb{Q} \end{array}$$

yielding a comultiplication

$$H^*(\mathrm{BSO}; \mathbb{Q}) \rightarrow H^*(\mathrm{BSO}; \mathbb{Q}) \otimes H^*(\mathrm{BSO}; \mathbb{Q})$$

sending $L \mapsto L \otimes L$. The Whitney sum formula yields that $p \mapsto p \otimes p$ as well.

8.3 Multiplicative sequences

We henceforth fix B a unital ring, P_i indeterminants with $P_0 = 1$ and $\deg P_i = i$.

Definition 8.6. A sequence $(K_n(P_1, \dots, P_n) \in B[P_1, \dots]_n)$ of polynomials of degree n is called a *multiplicative sequence*, or an *m-sequence*, if in the notation

$$\sum_{i=0}^{\infty} P_i \zeta^i = \left(\sum_j P'_j \zeta^j \right) \left(\sum_k P''_k \zeta^k \right)$$

we have

$$\sum_{i=0}^{\infty} K_i(P_1, \dots, P_n) \zeta^i = \left(\sum_j K_j(P'_1, \dots, P'_j) \zeta^j \right) \left(\sum_k K_k(P''_1, \dots, P''_k) \zeta^k \right).$$

Note that (L_n) is an m -sequence.²² We will use computational tools for multiplicative sequences to work with this:

Theorem 8.7. *The sequence (K_n) is uniquely determined by*

$$Q(\xi) = K(1 + \xi) = \sum_{i=0}^{\infty} b_i \zeta^i$$

where $b_i = K_i(1, 0, \dots, 0)$.

Proof. Note that

$$\sum_{i=0}^{\infty} P_i \zeta^i = \prod_{i=1}^m (1 + \beta_i \zeta)$$

for some β_i via the splitting principle. Hence

$$\sum_{i=0}^m K_i(P_1, \dots, P_i) \zeta^i \prod_{i=1}^m Q(\beta_i \zeta).$$

This determines each K_i . □

Let's work an example.

Example 8.8:

Recall that $H^*(\mathbb{CP}^{2k}; \mathbb{Q}) = \mathbb{Q}[h]/h^{2k+1}$. We can use some algebraic geometry^a to show that

$$\sum L_i \zeta^i = Q(h^i \zeta)^{2k+1}.$$

²²He justified this, but I couldn't follow it.

This shows that the coefficient of ζ^k in $Q(\zeta^{2k+1})$ is 1. We will conclude with this:

Claim. $Q(\zeta)$ is unique and $Q(\zeta) = \frac{\sqrt{\zeta}}{\sinh \sqrt{\zeta}}$.

Proof. Note that, in the notation $Q(\zeta) = \sum_{i=0}^{\infty} b_i \zeta^i$, we have $(2k+1)b_k + \text{lower terms} = 1$. By induction, b_k is unique, and

$$\int \frac{1}{\zeta^{k+1}} \left(\frac{\sqrt{\zeta}}{\sinh \sqrt{\zeta}} \right)^{2k+1} d\zeta = 1.$$

□

^aI'm not going to copy the alg geo out of laziness.

This allows us to conclude the index theorem:

Theorem 8.9 (Hirzbruch's signature theorem). *If (L_n) is the m -sequence related to $\frac{\sqrt{\zeta}}{\sinh \sqrt{\zeta}}$, then*

$$\langle L_n(\tau_M), \sqrt{M} \rangle = \tau(M) \in \mathbb{Z}.$$

In particular, we have

$$L_1 = \frac{1}{3}p_i \quad L_2 = \frac{1}{45}(7p_2 - p_1^2).$$

9 Natalie Stewart: Milnor, On manifolds homeomorphic to the 7-sphere

This talk was delivered on October 6, 2021 by Natalie Stewart. These notes were prepared in advance.

9.1 Motivation: the generalized Poincaré conjecture

The following conjecture was proposed by Poincaré in 1904:

Conjecture 9.1 (Poincaré conjecture). *Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.*

This problem alluded solution for nearly a century. Before then, analogous conjectures were proved; that closed 2-manifolds homotopy equivalent to S^2 (henceforth *topological homotopy 2-spheres*) must be homeomorphic to the 2-sphere followed from classification of surfaces. That homotopy n -spheres are homeomorphic to S^n was proved for $n \geq 5$ in 1966 by M. H. A. Newman using a technique pioneered by Stephen Smale called *PL engulfing* [6].

This motivates a class of conjectures;

Conjecture 9.2 (Generalized Poincaré conjecture). *Given a category of manifolds \mathcal{C} , all homotopy n -spheres in \mathcal{C} are isomorphic.*

In the case where $\mathcal{C} = \mathbf{Diff}$ is the category of smooth manifolds, the first known counterexample was due to Milnor, who constructed a family of at least 7 pairwise nondiffeomorphic smooth structures on the topological 7-sphere [4]. Those which are not diffeomorphic to the usual smooth structure on S^7 are called *exotic 7-spheres*. We will focus on his original construction in this note and the corresponding lecture, culminating in a proof of the following theorem:

Theorem 9.3. *There are at least 3 diffeomorphism classes of exotic spheres.*

Later work of Milnor and Kervaire computed that there are exactly 27 diffeomorphism classes of exotic 7-spheres, and that exotic spheres exist in many dimensions ≥ 7 , by using h -cobordism [3]. This breaks down in dimension ≤ 4 , as the h -cobordism theorem fails, so this fails to classify smooth 4 or 3 spheres. The following chart summarizes the status of the generalized Poincaré conjecture:

category	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n \geq 7$
Top	True	True	True	True	True	True
PL	True	Open	True	True	True	True
Diff	True	Open ($\mathbf{PL}_{\leq 4} \simeq \mathbf{Diff}_{\leq 4}$)	True	True	False	Often false

With that context in mind, we now move on to define the invariants Milnor used in his construction.

9.2 Milnor's λ invariant

Throughout this note, set

$$H^*(X) := H^*(X; \mathbb{Z}).$$

For the duration of this section, fix a closed compact orientable 7-manifold M satisfying

$$H^3(M) = H^4(M) = 0. \tag{1}$$

By Thom [7], any closed compact oriented 7-manifold bounds a compact oriented 8-manifold (with boundary);²³ suppose that M is bounded by B , with orientation class $[B] \in H_8(B)$. Condition (1) implies that the inclusion

$$\iota : H^4(B, M) \rightarrow H^4(B)$$

²³This comes down to the identification $\Omega^* = \pi_*(\mathbf{MSO})$ and explicit computations of the low-degree homotopy groups of this spectrum via the $2k$ -equivalence of its k -th space with products of Eilenberg Mac Lane spaces.

is an isomorphism, so we may define the first Pontryagin number of B in this setting to be

$$q(B) := \langle [B], (\iota^{-1}p_1)^2 \rangle$$

Recall that the *index* $\tau(B)$ is defined to be the index of the quadratic form $\alpha \mapsto \langle [B], \alpha^2 \rangle$ on $H^4(B, M)/(\text{torsion})$, i.e. the number of positive terms minus the number of negative terms in a real diagonalization of the form.

We will build an invariant on M out of $q(B)$ and $\tau(B)$, then prove that it's independent of choice of B . First, we take a quick digression into properties of closed 8-manifolds, which may fail in the non-closed case.

Recall the following corollary of the Hirzebruch signature theorem [2]:

Theorem 9.4 (Hirzebruch signature theorem, 8-dimensional case). *For C a closed oriented 8-manifold and $[C] \in H_8(C)$ a fundamental class, we have*

$$\tau(C) = \langle [C], \frac{1}{45} (7p_2(C) - p_1^2(C)) \rangle.$$

In particular, the signature theorem implies that

$$2q(C) - \tau(C) = 2(q(C) + 45\tau(C)) = 2 \cdot 7p_2(C) \equiv 0 \pmod{7}.$$

Since B bounds M , a nontrivial space, there is no guarantee that the analogue to this equality holds in our setting. In fact, failure of this equality is our main tool:

Definition 9.5. For M a closed 7-manifold bounded by an 8-manifold B , define the invariant

$$\lambda_B(M) := 2q(B) - \tau(B) \in \mathbb{F}_7.$$

This should worry the reader; it appears to depend on B . We assuage this fear via the following theorem:

Theorem 9.6. *The residue class $\lambda_B(M) \in \mathbb{F}_7$ doesn't depend on the choice of the manifold B .*

We henceforth simplify notation to

$$\lambda(M) := \lambda_B(M).$$

We will this to prove Theorem 9.3; for each $z \in \mathbb{F}_7$, we will construct a homeomorphism 7-sphere M with $\lambda(M) = z$.

Proof of Theorem 9.6. Suppose B, B' both bound M . Let B'' be the oriented manifold given by B' with opposite orientation. Defined the closed 3-manifold

$$C := B \cup_M B''$$

It is enough to prove the following equations:

$$\tau(C) = \tau(B) - \tau(B') \tag{2}$$

$$q(C) = q(B) - q(B'), \tag{3}$$

since then the different choices of $\lambda(M)$ differ by $q(C) - \tau(C) \equiv 0 \pmod{7}$.

We first prove (2). The co-inclusion morphisms yield a commutative square

$$\begin{array}{ccc} H^n(C, M) & \xrightarrow{h} & H^n(B, M) \oplus H^n(B'', M) \\ \downarrow \iota_C & & \downarrow \iota_B \oplus \iota_{B''} \\ H^n(C) & \xrightarrow{k} & H^n(B) \oplus H^n(B'') \end{array}$$

compatible with the Kronecker pairing. The top horizontal arrow is always an isomorphism. The vertical arrows are clearly isos when $n = 4$ by (1), so k is an isomorphism in that case as well. Let $\alpha := \iota_C h^{-1}(\alpha^2, \alpha''^2) \in H^4(C)$ be an element. Then, we may diagram chase:

$$\begin{aligned} \langle [C], \alpha^2 \rangle_{H^8(C)} &= \langle [C], \iota_C h^{-1}(\alpha^2, \alpha''^2) \rangle_{H^8(C)} \\ &= \langle h \iota_C^{-1} [C], (\alpha^2, \alpha''^2) \rangle_{H^8(B, M) \oplus H^8(B'', M)} \\ &= \langle ([B], [B'']), (\alpha^2, \alpha''^2) \rangle_{H^8(B, M) \oplus H^8(B'', M)} \\ &= \langle [B], \alpha^2 \rangle_{H^8(B, M)} + \langle [B''], \alpha''^2 \rangle_{H^8(B'', M)}. \end{aligned}$$

Hence the quadratic form of C is the direct sum of that of B and B'' , and hence it's the direct sum of that of B and the negative of that of B' . This yields equation (2).

For equation (3), note²⁴ that

$$kp_1(C) = p_1(B) \oplus p_1(B'')$$

Hence $q(C) = q(B) - q(B'')$ by an analogous argument to (2). \square

Now that we know that λ is well defined, note that Pontryagin classes and indices both switch sign when reversing orientation; this yields the following technical lemma:

Lemma 9.7. *Reversing the orientation of M multiplies $\lambda(M)$ by -1 . Hence any M possessing $\lambda(M) \neq 0$ has no orientation reversing diffeomorphism onto itself.*

We want to use λ ; our strategy will begin with the construction of a convenient family of spaces with easily computable λ invariants.

9.3 The construction of Milnor's exotic spheres

One candidate for exotic spheres is the restriction of 4-plane bundles over S^4 to their associated 3-sphere bundles; these are always 7-dimensional manifolds, and we can classify them explicitly via the *clutching construction*:

Construction 9.8 (the clutching construction). Consider S^4 as the union of the upper and lower hemispheres D_+ and D_- along the equator $S^3 \subset S^4$. For a map $f : S^3 \rightarrow \mathrm{SO}(4)$, construct the 4-plane bundle B_f by gluing $D_+ \times \mathbb{R}^4$ to $D_- \times \mathbb{R}^4$ along $(x, v)_+ \sim (x, f(x)(v))_-$ for $x \in S^3$. The following theorem is well known [1]:

Theorem 9.9. *This identification descends to an isomorphism $\pi_3(\mathrm{SO}(4)) \xrightarrow{\sim} \mathrm{Vec}_{\mathbb{R},4}(S^3)$.*

The group $\mathrm{SO}(4)$ has universal cover $\pi : S^3 \rightarrow \mathrm{SO}(4)$ given by $\pi(u, w)(v) = uvw$, written using quaternionic multiplication.²⁵ Hence there is an isomorphism $\mathrm{Vec}_{\mathbb{R},4}(S^3) \simeq \pi_3(\mathrm{SO}(4)) \simeq \mathbb{Z}^2$.

Remark. This construction is easy to picture topologically, but not very good for determining the differentiable structure as written. Taking a hemmed gluing is more suitable; we can instead replace D_+, D_- with \mathbb{R}^4 , glued along $\mathbb{R}^4 - \{0\}$ via a modified stereographic projection

$$(u, v) \mapsto (u', v') = \left(\frac{u}{|u|^2}, \frac{u^h v u^j}{|u|^{i+j}} \right)$$

The associated 4-plane bundle of this is isomorphic to the previously described bundle. Further, this describes the differentiable structure explicitly, and restricting to the associated sphere bundle, the transition function of the differentiable structure has the same formula.

Let $f_{h,j} : S^3 \rightarrow \mathrm{SO}(4)$ correspond with the pair $(h, -j)$. Let

$$\begin{array}{ccc} S^3 & \hookrightarrow & E_{h,j} \\ & & \downarrow \xi_{h,j} \\ & & S^4 \end{array}$$

be the corresponding 3-sphere bundle. For each odd integer k , let M_k be the total space of $E_{h,j}$ where h and j are determined by the equations $h + j = 1$ and $h - j = k$. These will be our candidates; we will show that they are homeomorphic to S^7 , usually with nontrivial λ invariant. First we tackle the homeomorphism, via techniques from Morse theory.

²⁴One can see this via representability; pushing forward the iso

$$[C, \mathrm{BU}(n)] \simeq [B \cup_M B'', \mathrm{BU}(n)] \simeq [B, \mathrm{BU}(n)] \coprod_{[M, \mathrm{BU}(n)]} [B'', \mathrm{BU}(n)]$$

along the second Chern class morphism $\mathrm{BU}(n) \rightarrow K(\mathbb{Z}, n)$ and applying this to the complexification of tangent bundles yields the desired statement after noting that the second Chern class of a bundle on M is trivial by (1).

²⁵This convention is nonstandard, but agrees with Milnor.

9.4 M_k is homeomorphic to S^7 : a Morse theoretic sketch

Consider the following hypothesis on a closed manifold M :

- (H) *There exists a differentiable function $f : M \rightarrow \mathbb{R}$ having only two critical points, where each are nondegenerate.*

We prove the following:

Theorem 9.10. *A manifold M satisfying hypothesis (H) is homeomorphic to S^7 .*

Proposition 9.11. *M_k satisfies hypothesis (H), and hence it is homeomorphic to S^7 .*

The proofs of these are largely irrelevant to each other, so I'll sketch the concrete statement first.

Proof sketch for Proposition 9.11. We can define the function f in local coordinates, compatibly with transition functions:

$$f(u, v) = \frac{\Re(v)}{\sqrt{1 + |u|^2}}$$

$$f(u', v') = \frac{\Re(u'/v')}{\sqrt{1 + |u'/v'|^2}}$$

where $\Re(\cdot)$ is the real part of a quaternion. The reader can verify that this has exactly two critical points, each nondegenerate, at $(u, v) = (0, \pm 1)$. \square

Proof sketch for Theorem 9.10. Suppose $f : M \rightarrow \mathbb{R}$ is a function witnessing hypothesis (H). Normalize f so that $f(x_0) = 0$ and $f(x_1) = 1$. According to Morse [5], one can take local coordinates v_1, \dots, v_n in a neighborhood V of x_0 so that

$$f(x) = v_1^2 + \dots + v_n^2 \quad \text{on } V.$$

One may define a Riemannian metric on V via $ds^2 = dv_1^2 + \dots + dv_n^2$, and extend this to one on all of M .

The gradient of f defines a vector field on M ; consider the differential equation

$$\frac{dx}{dt} = \frac{\nabla f}{|\nabla f|^2}.$$

This equation has solutions. Suppose $x_a(t)$ is a solution equation; note that

$$f(x_a(t)).$$

so that $x_a(t) = (a_1^{1/2}(t), \dots, a_n^{1/2}(t))$ on V . Map the unit sphere of \mathbb{R}^n into M via the map

$$a \mapsto x_a(t).$$

This defines a diffeomorphism of the open n -cell onto $M - \{x_1\}$. Adding a single point yields the theorem. \square

9.5 Completing the proof: calculating λ invariants

We refer to elements of $H^4(S^4)$ as elements of \mathbb{Z} , but we will find that signs don't matter, as we always square these elements when using them.. We first show that these generate all of the possible λ invariants:

Lemma 9.12. *The λ invariant of M_k is as follows:*

- (i) $p_1(\xi_{h,j}) = \pm 2(h-j)\iota.$
- (ii) $\lambda(M_k) = k^2 - 1.$

Proof. Part (i). First, note that $\xi_{h,j} \oplus \xi_{h',j'}$ is stably isomorphic to $\xi_{h+h',j+j'}$: writing local coordinates²⁶ of $\xi_{h+h',j+j'} \oplus e$ as (u, v, w) on the preimage of D^+ with transition function

$$(u, v, w) \mapsto (u', v', w') = (u, u^{h'} u^h v u^h u^{j'}, w)$$

and the coordinates on $\xi_{h,j} \oplus \xi_{h',j'}$ similarly as (x, y, z) with transition function

$$(x, y, z) \mapsto (x', y', z') = (x, x^h y x^j, x^{h'} y x^{j'}),$$

there is an iso $\xi_{h+h',j+j'} \oplus e \rightarrow \xi_{h,j} \oplus \xi_{h',j'}$ given by the identity above D^+ and the map

$$(u', v', w') \mapsto (x', x'^{-h'} v' x'^{-j'}, x'^{h'} w' x'^{j'})$$

over D_- . We can check that this is compatible with transition functions:

$$\begin{array}{ccc} (\xi_{h+j} \oplus e)|_{D_+ \cap D_-} & \xrightarrow{\quad\quad\quad} & (\xi_{h+j} \oplus e)|_{D_+ \cap D_-} \\ \downarrow & & \downarrow \\ (u, v, w) \mapsto (u, u^{h+h'} v u^{j+j'}, w) & & \\ \downarrow & \downarrow & \\ (x, y, z) \mapsto (x, x^h y x^j, x^{h'} y x^{j'}) & & \\ \downarrow & & \downarrow \\ \xi_{h+h',j+j'}|_{D_+ \cap D_-} & \xrightarrow{\quad\quad\quad} & \xi_{h+h',j+j'}|_{D_+ \cap D_-} \end{array}$$

Hence the Whitney sum formula yields $p_1(\xi_{h+h',j+j'}) = p_1(\xi_{h,j} + \xi_{h',j'})$, i.e. $p_1(\xi_{i,j})$ is linear in h and j . Further, there is an isomorphism $\xi_{h,j} \rightarrow \xi_{-j,-h}$; hence $p_1(\xi_{h,j}) = ah + bj = -bh - aj$ for some constants a, b , i.e. $p_1(\xi_{h,j}) = c(h - j)$ for some constant c .

To compute the constant c , it suffices to compute an example. The case $(h, j) = (1, 0)$ corresponds with the ordinary quaternionic Hopf fibration, i.e. the sphere bundle associated with the tautological quaternionic line bundle

$$\begin{array}{ccc} S^3 & \hookrightarrow & S^7 \\ \downarrow & & \downarrow \\ \mathbb{H} & \hookrightarrow & \mathbb{H}^2 \\ & & \downarrow T_{\mathbb{H}} \\ & & \mathbb{H}\mathbb{P}^1 \\ & & \parallel \\ & & S^4 \end{array}$$

For any complex vector bundle ξ , one has $\xi \otimes \mathbb{C} = \xi \oplus \bar{\xi}$, and $c_i(\bar{\xi}) = (-1)^i c_i(\xi)$. In the case that ξ is 2-dimensional with vanishing first Chern class (e.g. if $H^2(B(\xi)) = 0$), the Whitney sum formula then yields

$$1 + p_1(\xi) = (1 + c_2(\xi))^2 = 1 + 2c_2(\xi)$$

and hence $p_1(\xi) = 2c_2(\xi)^2$. Hence it is enough to compute $c_2(T_{\mathbb{H}})$. We compute this via the comparison map

$$g : \mathbb{CP}^2 \rightarrow \mathbb{HP}^1$$

sending $[a + bi : c + di] \mapsto [a + bi + cj + dk]$. This yields a pullback square

$$\begin{array}{ccc} \mathbb{H}^2 \times_{\mathbb{HP}^1} \mathbb{CP}^2 & \longrightarrow & \mathbb{H}^2 \\ \downarrow g^* T_{\mathbb{H}} & & \downarrow T_{\mathbb{H}} \\ \mathbb{CP}^2 & \xrightarrow{g} & \mathbb{HP}^1 \end{array}$$

²⁶Here and elsewhere, e refers to a trivial 1-dimensional vector bundle.

In fact, there is a diagram of real vector bundles

$$\begin{array}{ccccc}
\mathbb{C}^4 & & \xrightarrow{\phi} & & \mathbb{H}^2 \\
& \searrow \text{!} & & & \downarrow T_{\mathbb{H}} \\
& T_{\mathbb{C}}^2 & g^*(\mathbb{H}^2) & \longrightarrow & \mathbb{H}^2 \\
& & \downarrow g^* T_{\mathbb{H}} & & \downarrow T_{\mathbb{H}} \\
& & \mathbb{CP}^2 & \xrightarrow{g} & \mathbb{HP}^1
\end{array}$$

where

$$\phi([a + bi : c + di], w + xi, y + zi) = ([a + bi + cj + dk], w + xi + yj + zk)$$

Note that ϕ is fiberwise-injective, and hence the canonical map $\mathbb{C}^4 \rightarrow g^*(\mathbb{H}^2)$ is fiberwise-injective. Both have fiber of real dimension 4, so this must be an isomorphism of real vector bundles, i.e. $g^*T_{\mathbb{H}} = T_{\mathbb{C}}^2$. Then, naturality of Chern classes implies that

$$g^*c_2(T_{\mathbb{H}}) = c_2(T_{\mathbb{C}}^2) = c_1(T_{\mathbb{C}})^2$$

is a generator of $H^4(\mathbb{CP}^2)$. In turn, since $g : \mathbb{CP}^2 \rightarrow \mathbb{HP}^1$ is a $\mathbb{CP}^1 = S^2$ bundle, the Gysin sequence yields

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H^1(\mathbb{HP}^1) & \longrightarrow & H^4(\mathbb{HP}^1) & \xrightarrow{g^*} & H^4(\mathbb{CP}^2) \longrightarrow H^2(\mathbb{HP}^1) \longrightarrow \cdots \\
& & \parallel & & & & \parallel \\
& & 0 & & & & 0
\end{array}$$

and g^* is an isomorphism. Hence $c_2(T_{\mathbb{H}})$ is a generator of $H^4(S^4)$, i.e. $p_1(\xi_{1,0}) = p_1(T_{\mathbb{H}}) = \pm 2$, determining the constant c .

Part (ii). Note that, for any smooth fiber bundle $F \hookrightarrow E \rightarrow B$, one has $TE = (TB)^* \oplus (TF)_*$, by decomposition of the tangent space at a point into vectors tangent and normal to the copy of F containing it.

In particular, applying this to $TE_{h,j}$ and along with the Whitney sum formula yields

$$p_1(E_{h,j}) = \xi_{h,j}^* p_1(\xi_{h,j}) + \iota_* p_1(S^3) = \xi_{h,j}^* p_1(\xi_{h,j}).$$

Again applying the Gysin sequence yields an isomorphism

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H^7(S^4) & \longrightarrow & H^7(E_{h,j}) & \xrightarrow{\xi_{h,j}^*} & H^4(S^4) \longrightarrow H^8(E_{h,j}) \longrightarrow \cdots \\
& & \parallel & & & & \parallel \\
& & 0 & & & & 0
\end{array}$$

i.e. $p_1(E_{h,j}) = \pm 2(h-j)$. Pick an orientation of $E_{h,j}$ (and hence B_k) so that this is positive. Then, $\tau(B_k) = 1$, and

$$q(B_k) = \langle [B_k], 4k^2 \rangle = 4k^2.$$

Hence we have

$$\lambda(M_k) = 8k^2 - 1 = k^2 - 1 \pmod{7},$$

as desired. \square

This allows us to finally conclude Theorem 9.3:

Proof of Theorem 9.3. Whenever $k \neq k' \pmod{7}$, we have $\lambda(M_k) \neq \lambda(M_{k'})$, and hence M_k is not diffeomorphic to $M_{k'}$. Hence $\{M_k\}$ yields at least 7 diffeomorphism classes of spaces; by Proposition 9.11, these are all homeomorphism 7-spheres, so there are at least 7 diffeomorphism classes of homeomorphism 7-spheres. \square

10 Serina Hu: Brown, Cohomology theories

This talk was delivered on October 9, 2021 by Serina Hu.

10.1 Statement of Brown representability, and easy corollaries

Let \mathcal{C} be a homotopy category of pointed connected CW complexes.²⁷ A contravariant functor $H : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}_*$ is *representable* if there is some $Y \in \mathcal{C}$ and a natural isomorphism

$$[-, Y] \simeq H(-).$$

Representing objects are unique up to iso, in this case, pointed homotopy.

Theorem 10.1. *A functor $H : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}_*$ is representable if and only if the following holds:*

- (e) (Mayer-Vietoris axiom). The following hold:²⁸
 - $H(*) = \{*\}$.
 - If $X = X_1 \cup X_2$, $X_1, X_2, X_1 \cap X_2 \in \mathcal{C}$ and $(X_1, X_1 \cap X_2)$ and $(X_2, X_1 \cap X_2)$ have the HEP, and there exists $u_i \in H(X_i)$ s.t. $H(j_1)u_1 = H(j_2)u_2$, then there exists some $v \in H(X)$ s.t. $H(i_1)v = u_1$ and $H(i_2)v = u_2$, in the following diagram:

$$\begin{array}{ccccc}
 & & H(X) & & \\
 & \swarrow^{H(i_1)} & & \searrow_{H(i_2)} & \\
 H(X_1) & & & & H(X_2) \\
 & \searrow_{H(j_1)} & & \swarrow_{H(j_2)} & \\
 & & H(X_1 \cap X_2) & &
 \end{array}$$

- (w) (wedge) For spheres S_α^n with $\iota_\beta : S_\beta^n \hookrightarrow \bigvee S_\alpha^n$, the induced map

$$H\left(\bigvee S_\alpha^n\right) \rightarrow \prod H(S_\alpha^n)$$

is an iso.

- (l) (limit) For $X = \bigcup X_n$ a filtration with $X_j \in \mathcal{C}$ and $X_n^n = X^n$, we have

$$\text{colim } H(\iota_n) : H(X) \rightarrow \text{colim } H(X_n)$$

is a surjection.

We get some free results:

Corollary 10.2. *Singular cohomology in degree n is representable, i.e. there exists some $K(A, n)$ s.t.*

$$[-, K(A, n)] \simeq H^n(-, A).$$

We can chain these together into a spectral statement:

Corollary 10.3. *Let $\{H^q, \omega^q\}$ be a cohomology theory with coefficients in A .²⁹ Then, there exists an Ω -spectrum $\{Y_q, h_q\}$ representing H^q .³⁰*

Let's move on to actually talking about and motivating the axioms.

²⁷I will add an asterisk to \mathbf{Set}_* , against Serina's notation.

²⁸I think this can be summarized by saying H takes weak pushouts to weak pullbacks (which is often the axiom used), but I'm not 100% sure of that.

²⁹There was confusion here, but I think this is ok notation for a cohomology theory whose codomain category is $(A - \text{mod})^{\mathbb{Z}}$.

³⁰This means that the structure maps have adjoint maps given by equivalences $Y_q \simeq \omega Y_{q+1}$ expressing this as an infinite loop space, and these structure maps intertwine the cohomology theory:

$$\omega^q T_q = T_q \omega^{q+1}.$$

10.2 Motivation for the axioms of a Brown functor: homotopy colimits

Recall the (e) axiom; this is satisfied by representable functors, since given a diagram

$$\begin{array}{ccc}
 & [X, Y] & \\
 H(i_1) \swarrow & & \searrow H(i_2) \\
 [X_1, Y] & & [X_2, Y] \\
 H(j_1) \searrow & & \swarrow H(j_2) \\
 & [X_1 \cap X_2, Y] &
 \end{array}$$

we can form the homotopy pushout $X = X_1 \sqcup_{X_1 \cap X_2} X_2$ ³¹. The point of forcing the HEP is that, in this case, homotopy pushouts are simply traditional pushouts.³², and use this universal property to construct the desired element.

Similarly, the (l) axiom can be phrased via “mapping telescopes,” i.e. sequential homotopy colimits; one can verify this in the representable case by simply noting that the mapping telescope of such an inclusion is X , and hence the universal property of homotopy colimits allows one to construct the desired element.³³

Let’s move on to a sketch of the proof.

10.3 Sketch of the proof of Brown representability

We’ll develop this in steps:

Step (1). Fix $Y \in \mathcal{C}$, $u \in H(Y)$, and define $T(u) : [-, Y] \rightarrow H(-)$ mapping f to the pushforward $H(f)u$. Our goal will be to pick appropriate Y, u making this an iso. We start with a lemma, reducing the work drastically:

Lemma 10.4. *If $T(u) : [S^n, Y] \simeq H(S^n)$, then this is actually a natural equivalence on all of $X \in \mathcal{C}$.*

Step (2). We will inductively construct a sequence $Y_i, u_i \in H(Y_i)$ such that:

- (i) $Y_{n-1} \hookrightarrow Y_n$.
- (ii) $H(i_n)u_n = u_{n-1}$.
- (iii) $T(u_n) : [S^m, Y_n] \rightarrow H(S^m)$ is surjective for all m , and bijective for $m \leq n$.

For the base case $n = 0$, we just need surjectivity. Let $g_\alpha^m \in H(S^m)$ be elements, and let $Y = \bigvee S_\alpha^m$. Then, fix $H(Y_0) = \prod H(S_\alpha^m)$. If $h_\alpha^m : S^m \xrightarrow{\sim} S_\alpha^m \hookrightarrow Y_0$. Pick u_0 s.t. $H(h_\alpha^m)(u_0) = g_\alpha^m = T(u_0)(h_\alpha^m)$.³⁴

For the inductive step, suppose we’ve gotten such a pair $u_{n-1} \in H(Y_{n-1})$. We know that $T(u_{n-1}) : [S^m, Y_{n-1}] \rightarrow H(S^m)$ is bijective for $m \leq n-1$. We want to modify this only in degrees $\geq n$ s.t. the map at degree n is also a bijection.

Consider the kernel of $[S^n, Y_{n-1}] \rightarrow H(S^n)$.³⁵ Let $A = \bigvee S_\beta^n$, $f = \bigvee f_\beta : A \rightarrow Y_{n-1}$, and let Y_n be the mapping cone of this.³⁶ We have the following:

Claim. *The sequence of pointed sets*

$$H(Y_n) \xrightarrow{H(i)} H(Y_{n-1}) \xrightarrow{H(f)} H(A)$$

is exact.

³¹I don’t know notation to use the homotopy pushout.

³²This is true based on model category nonsense; homotopy colimits along cofibrations are just traditional colimits, assuming the index category is simple and/or reedy (really, that it admits a model structure whose homotopy equivalences are simply pointwise homotopy equivalences).

³³I think one can spend months on homotopy colimits and not fully understand them. Lord knows I have. Sorry if this section is confusing.

³⁴Haynes commented that this is a huge use of the axiom of choice. Just another reason to be pro-choice I suppose...

³⁵By “kernel,” we mean preimage of the basepoint.

³⁶Serina drew a picture of a wizard hat, aka a mapping cone, here.

Proof. Mapping cones are homotopy pushouts:

$$\begin{array}{ccccc}
 & & H(Y_n) & & \\
 & \swarrow & & \searrow & \\
 & & H(i_1) & & H(i_2) \\
 H(\overline{X}) & & & & H(\overline{Y}) = H(Y_{n-1}) \\
 & \searrow & & \swarrow & \\
 & & H(j_1) & & H(j_2) \\
 & & H(\overline{A}) = H(A) & &
 \end{array}$$

Use axiom (e).³⁷

□

If $u_{n-1} \in \ker(H(f))$, let $\iota_B : S_\beta^n \hookrightarrow A$ represent it. There must be some $u_n \in H(Y_n)$ s.t. $H(\iota_n)u_n = u_{n-1}$, by exactness.

For surjectivity of $T(u_n)$, just push along ι_n :

$$\begin{array}{ccc}
 [S^m, Y_{n-1}] & & \\
 \downarrow \iota_n & \searrow T(u_{n-1}) & \\
 [S^m, Y_n] & \xrightarrow{T(u_n)} & H(S^m)
 \end{array}$$

since the upper right arrow is surjective, the lower right arrow is as well.

For injectivity of $m < n$, a similar argument applies; for $m = n$, an element in the kernel yields a nullhomotopic map

$$S^n \xrightarrow{f_\beta} Y_{n-1} \hookrightarrow Y_n$$

for each β ; hence $\ker(T_{u_{p-1}}) \subset \ker \iota_{p*}$, implying injectivity.

Step (3). Take $Y = \operatorname{colim} Y_n$, and use axiom (l) to finish.

11 Tristan Yang: Milnor, On the cobordism ring and a complex analogue

This talk was delivered on October 13, 2021 by Tristan Yang.

The talk is a trojan horse. We're talking about the stable homotopy category and the Adams spectral sequence, and we may get to the complex cobordism stuff if we have time.

11.1 Stable homotopy theory

To pass to the stable context, we will define spectra. The goal will be “inverting” the suspension functor. We'll construct a functor

$$\Sigma^\infty : \operatorname{ho} \mathbf{Top}_* \rightarrow \operatorname{ho} \mathbf{Sp}$$

into the stable homotopy category, along with a suspension functor

$$\Sigma : \operatorname{ho} \mathbf{Sp} \rightarrow \operatorname{ho} \mathbf{Sp}$$

extending the suspension on $\operatorname{ho} \mathbf{Top}_*$, in the sense that $\Sigma \Sigma^\infty = \Sigma^\infty \Sigma$.

Let's actually define our category.

³⁷I missed this argument

Definition 11.1. The category \mathbf{hoSp} has as objects the sequences of homotopy types $\{Y_k \in \mathbf{hoTop}_*\}$ together with homotopy classes of maps $\Sigma Y_k \rightarrow Y_{k+1}$. Define the suspension spectrum $\Sigma^\infty Y = \{\Sigma^k Y\}$ with the obvious identities. For Y a finite CW complex, define the class of maps

$$[\Sigma^\infty Y, X]_n := \operatorname{colim}_k [\Sigma^k Y, X_k].$$

Example 11.2:

An example includes the sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$, and note that

$$\pi_n(X) := \operatorname{colim}_k \pi_{n+k}(X_k)$$

is naturally identified with the maps $[\mathbb{S}, X]$.

Example 11.3:

Brown representability yields one direction of a bijection between Ω -spectra (spectra with adjoint structure maps that are equivalences).^a This for instance yields the *Eilenberg-Mac Lane* Ω -spectra, which we denote $H\mathbb{F}_p$, and represents cohomology with \mathbb{F}_p coefficients.

^aTristan mentioned that every spectra is equivalent to an Ω -spectra. I assume that this is the normal fibrant replacement in the stable model structure. I'm a bit confused as we've only defined homotopy types of spectra, so this would mean every such homotopy type *is* an Ω -spectra (the distinction is point-set, and we haven't defined spectra in a point-set way).

The stable homotopy category satisfies the following axioms:

- (i) there is a functor $\Sigma^\infty : \mathbf{hoTop}_* \rightarrow \mathbf{hoSp}$.
- (ii) there is a suspension auto-equivalence $\Sigma : \mathbf{hoSp} \rightarrow \mathbf{hoSp}$ s.t. $\Sigma\Sigma^\infty = \Sigma^\infty\Sigma$.
- (iii) The category \mathbf{hoSp} is additive (it is enriched over \mathbf{Ab} and has all finite products and coproducts, which agree). To give the enrichment, apply suspensions twice and use the normal (homotopy) Abelian group structure.
- (iv) The category \mathbf{hoSp} is triangulated.³⁸
- (v) There is a symmetric monoidal structure \otimes on \mathbf{hoSp} extending \wedge , having.³⁹

With the last axiom in mind, we can define generalized homology corresponding with a spectrum to be

$$\tilde{E}_n(X) := \pi_n(E \otimes X).$$

With this in mind, we can define the Adams spectral sequence:

11.2 The Adams spectral sequence

Our goal is to compute $[Y, X]_*$. The idea is that we might be able to compute this via the homology. We can't do this super naively, but maybe we can get a bit of a foothold this way.

Let's take a special case. Suppose $X = H\mathbb{F}_p$. Then,

$$\begin{aligned} [Y, H\mathbb{F}_p] &\simeq H^*Y \\ &= \operatorname{Hom}_{\mathbb{F}_p}(H_*Y, \mathbb{F}_p) \\ &= \operatorname{Hom}_{A_*}(H_*Y, \mathbb{F}_p \otimes A_*) \\ &= \operatorname{Hom}_*(H_*Y, H_*H\mathbb{F}_p). \end{aligned}$$

³⁸There are distinguished triangles, generalizing LESs, which are nice. Details not given, and I don't have time to write out the axioms, which are not easy to get meaning out of at first anyways.

³⁹This is actually monoidal closed.

The general case is harder. We can get a bit more foothold by just noting that

$$\left[Y, \bigoplus \Sigma^{|b_i|} H\mathbb{F}_p \right] \simeq \text{Hom}(H_* Y, A_* \otimes H_* X) \quad \text{whenever } H_* X \text{ has basis } \{b_i\}$$

noting that we can equivalently write

$$\bigoplus \Sigma^{|b_i|} H\mathbb{F}_p \simeq H\mathbb{F}_p \otimes X.$$

Warning. This doesn't generalize to arbitrary homology theories: the iso $E_*(E \otimes X) \simeq E_* E \otimes_{E_*} E_* X$, which requires a flatness condition. We only really care about \mathbb{F}_p -(co)homology in this talk, so we won't worry about this.

We can form a tower

$$\begin{array}{ccccc} X & \longleftarrow & \overline{H\mathbb{F}_p} \otimes X & \longleftarrow & \cdots \\ \downarrow & \nearrow & \downarrow & \nearrow & \\ H\mathbb{F}_p \otimes X & & H\mathbb{F}_p \otimes \overline{H\mathbb{F}_p} \otimes X & & \end{array}$$

where the dashed lines have degree +1. Taking $[Y, -]$ of this tower yields a diagram

$$\begin{array}{ccccc} [Y, X] & \longleftarrow & [Y, \overline{H\mathbb{F}_p} \otimes X] & \longleftarrow & \cdots \\ \downarrow & \nearrow & \downarrow & \nearrow & \\ [Y, H\mathbb{F}_p \otimes X] & & [Y, H\mathbb{F}_p \otimes \overline{H\mathbb{F}_p} \otimes X] & & \end{array}$$

which rolls (via direct sums) into an exact couple. The associated spectral sequence is called the *Adams spectral sequence*, or the *ASS*.⁴⁰

Proposition 11.4. *The ASS has E_2 page given by*⁴¹

$$E_2^{s,t} = \text{Ext}_{A_*}^{s,t}(H_* Y, H_* X).$$

We have a convergence theorem:

Theorem 11.5. *The Adams Spectral sequence converges to the p -completion of homotopy:*

$$E_r^{s,t} \implies ([Y, X]_{t-s})_p^\wedge$$

Intuition about the completion. The given filtration $X \leftarrow X_1 \leftarrow \cdots$ is not actually exhaustive: $X_\infty := \lim X_n$ does not in general vanish.

We can take cofibers to get a new filtration:

$$\begin{array}{ccccccc} & & & & & & X \\ & & & & & \swarrow & \downarrow \\ X & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \longrightarrow X_\infty \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & Z_1 & \longrightarrow & Z_2 & \longrightarrow & \cdots \longrightarrow Z_\infty = 0 \end{array}$$

and note that the two yield the same SS. □

⁴⁰This is not my choice of abbreviation.

⁴¹Tristan used the notation $\text{Ext}^s(-)_t$ here.

11.3 Applications to complex cobordism

Define the spectrum MU to be $(*, \text{MU}(1), \Sigma \text{MU}(1), \text{MU}(2), \dots)$. We have the following, which comes from the Thom iso:

Proposition 11.6. $H_* \text{MU} = \mathbb{F}_p[b_1, \dots]$ as an \mathbb{F}_2 -module, where $|b_i| = 2i$.

Recall that $A_* \simeq \mathbb{F}_p[\xi_i] \otimes \bigwedge(\tau_i)$, where $|\xi_i| = 2p^i - 2$ and $|\tau_i| = 2p^i - 1$. Let the first summand be P .

Proposition 11.7. $H_* \text{MU}$ is a P -comodule.

Proof. This follows by noting that all elements of $H_* \text{MU}$ have even degree, so odd-degree elements of A_* coact trivially. \square

We can do better:

Proposition 11.8. As an A_* -comodule,

$$H_* \text{MU} \simeq P \otimes \mathbb{F}_p[b_i \mid i \neq p^h - 1].$$

Proof. $H_* \text{MU} \rightarrow H_* \text{MU} \otimes P \rightarrow P$ surjects onto P (check this). The Thom iso map $\tilde{H}_{*+2} \mathbb{CP}^\infty \rightarrow H_* \text{MU}$ has codomain $\mathbb{F}_p[\beta_0, \beta_1, \dots]$. \square

We ran out of time here. The upshot is that we can compute that the E_2 page has only even degrees, which is enough to show that there's no odd torsion.

12 Carina Hong: Dyer and Lashof, A topological proof of the Bott periodicity theorem

This talk was delivered on October 15, 2021 by Carina Hong.

We will prove Bott periodicity, which we now state. Define $O := \text{colim } O(n)$ and similar for the other classic groups.

Theorem 12.1 (Bott periodicity). $\pi_n(O) \simeq \pi_{n+4}(\text{Sp}) \simeq \pi_{n+8}(O)$; further, the groups on low degree are given by

π_n	$n = 0$	1	2	3	4	5	6	7
	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}	0	0	0	\mathbb{Z}

Similarly, $\pi_n(U) = \pi_{n+2}(U)$.

Our strategy will be a collection of homotopy equivalences exhibiting various classical groups as loop spaces of each other.⁴²

Our goal will be proving the following theorem, which is the hard step in complex Bott periodicity:

Theorem 12.2. *There exists a homology-isomorphism $\text{BU} \rightarrow \Omega \text{SU}$ of H -spaces.*

We need to give some definitions to work this out fully.

12.1 Background on our classical groups and on H -spaces

Definition 12.3. An H -space is a space X together with a multiplication map $\mu : X \times X \rightarrow X$ and a unit $e : * \rightarrow X$ exhibiting the usual unit laws.⁴³

Now that we have that out of the way, let's define some groups. We define the space⁴⁴

$$\text{BU} := U / U \times U = \text{colim } U(2n) / (U(n) \times U(n)).$$

We want to give an H -space structure on these. Define the H -space structure on ΩSU via pointwise multiplication.⁴⁵ Define the H -space structure on BU by an action of the *linear isometries operad*:

$$\mu(T_1, T_2) : \mathbb{C}^{\infty \oplus 2} \xrightarrow{\xi^{-1} \oplus \xi^{-1}} \mathbb{C}^{\infty \oplus 4} \xrightarrow{T_1 \oplus T_2} \mathbb{C}^{\infty \oplus 4} \xrightarrow{\text{id} \oplus \gamma \oplus \text{id}} \mathbb{C}^{\infty \oplus 4} \xrightarrow{\xi \oplus \xi} \mathbb{C}^{\infty \oplus 2}$$

The board work of this talk is too fast to be reasonably live-texed. Proceed with caution, as there will be many holes.

12.2 Characterizing the map $\text{BU} \rightarrow \Omega \text{SU}$

We're going to prove that our map is a homology iso. We need the following to do:

Claim. $H_*(\text{BU}) = \mathbb{Z}[z_2, z_4, \dots]$, where z_{2i} are the images of an additive basis of $H_*(\mathbb{CP}^\infty)$.

In addition to this, we need to compute $H_*(\Omega \text{SU})$, which reduces to computing $H_*(\text{SU})$ via the following dual to a special case of Borel's theorem.

Theorem 12.4 (Dual to Borel's theorem). *If X is an H -space s.t. $H_*(X)$ is a transgressively generated exterior algebra generated in odd degrees by (γ_{2n+i}) , then $H_*(\Omega X)$ is a polynomial algebra generated by transgressions of γ_{2n+1} .*

⁴²I am behind in copying this, so I missed the constructions of the maps here.

⁴³I think this only needs to hold up to homotopy sometimes, and for the way we'll use these.

⁴⁴There was contention on how to define the maps of this colimit.

⁴⁵According to Haynes, this is equivalent to doing loop composition.

We can compute this via the following diagram:

$$\begin{array}{ccc}
\mathbb{CP}^{n-1} & \longrightarrow & \Omega \mathrm{SU}(n) \\
\downarrow i & & \downarrow \Omega i \\
\mathbb{CP}^n & \longrightarrow & \Omega \mathrm{SU}(n+1) \\
\downarrow \rho & & \downarrow \Omega \pi \\
S^{2n} & \xrightarrow{h} & \Omega S^{2n+1}
\end{array}$$

You can show that $\Omega \pi \circ \alpha = i$ is trivial, so that $\Omega \pi \circ \alpha$ factors as $h \circ \rho$. The colimit \hat{h} of h is a homeomorphism.^{46,47}

The upshot of this is the following computation:

Claim. $H_*(\mathrm{SU}(n+1)) \simeq \bigwedge [\gamma_3, \gamma_5, \dots, \gamma_{2n+1}]$.

This is true for $\mathrm{SU}(2) = S^3$, so you can prove it inductively from there. The key is to use the cohomological SSS, which collapses at the second page, for degree reasons, and use some Hopf algebra technology to turn this into a homology statement.

Taking a colimit yields that $H_*(\mathrm{SU})$ is an exterior algebra which, via Theorem 12.4, yields $H_*(\Omega \mathrm{SU})$ is abstractly isomorphic to $H_*(\mathrm{BU})$.⁴⁸

⁴⁶I think this is what \hat{h} means, but not sure.

⁴⁷Haynes suggests that you can use this to inductively compute $H_*(\omega \mathrm{SU}(n))$ and take a colimit, noting that the cofiber sequence on the left is meaningful for the interpretation of the transgressions that you use in such a computation.

⁴⁸I'm confused about how to pull that our map is a H_* -isomorphism; once you do this, use simplicity coming from the H -space structure to conclude that it's a weak equivalence.

13 Eunice Sukarto: Atiyah, K-theory (no notes today)

This talk was delivered on October 20, 2021 by Eunice Sukarto.

I woke up far too tired to take good notes on this today. See Atiyah's book, or any standard notes on K theory.

14 Gabrielle Li: Adams and Atiyah, K-theory and the Hopf invariant

This talk was delivered on October 22, 2021 by Gabrielle Li.

14.1 Motivation and review from K-theory

Our motivation is the usual one: a structure of a real division algebra on \mathbb{R}^n yields a structure of S^{n-1} as an H -space, which yields a Hopf invariant 1 element in $\pi_{2n-1}(S^n)$. It looked approximately like the [nlab reference](#).

We review some facts from Eunice's talk:

Proposition 14.1. $\tilde{K}^0(S^n)$ is \mathbb{Z} for even n and 0 for odd n , and it's generated by the bott element $\wedge^{n/2}b$ where $b = H^* - 1$, with $b^2 = 0$. In either case, $\tilde{K}^0(S^n)$ has trivial cup product.

We also have the splitting principle.

14.2 The statement of the Hopf invariant 1 problem

Let $f : S^{2n-1} \rightarrow S^n$ be a map (really, an element of $\pi_{2n-1}(S^n)$). Form the homotopy cofiber

$$S^{2n-1} \rightarrow S^n \rightarrow Cf.$$

The LES in reduced K theory yields

$$\tilde{K}^0(S^{n+1}) \rightarrow \tilde{K}^0(S^{2n}) \rightarrow \tilde{K}^0(Cf) \rightarrow \tilde{K}^0(S^n) \rightarrow 0 = K^0(s^{2n-1})$$

Let v be a generator of $\tilde{K}^0(S^n)$ and β a preimage of this in the cofiber. Let α be the image of a generator of $\tilde{K}^0(S^n)$ in the cofiber.

Note that $i^*(\beta^2) = i^*(\beta)^2 = 0$. Hence β^2 is in the image of the boundary map, i.e. there is some h s.t. $\beta^2 = h\alpha$.

Definition 14.2. The *Hopf invariant* of an element $f \in \pi_{2n-1}(S^n)$ is $h \in \tilde{K}^0(Cf)$, as defined above.

Note that whenever n is odd, $\beta^2 = 0$ by graded commutativity.⁴⁹

If n is even and $\beta' = \beta + m\alpha$, then $\beta'^2 = \beta^2 + 2m\alpha\beta$. Note that

$$i^*(\alpha\beta) = i^*(\alpha)i^*(\beta) = 0$$

and hence $\alpha\beta = i\alpha$ for some i , so that $i\alpha\beta = i^2\alpha = \alpha\beta^2 = h\alpha^2$, which is then 0, by graded commutativity and torsion-freeness. Hence this invariant is well-defined.

We can state the theorem:

Theorem 14.3 (Hopf invariant one problem). *If $n \neq 1, 2, 4$, then there is no Hopf invariant 1 element in $\pi_{2n-1}(S^n)$.*

We'll now develop some techniques to prove this.

14.3 Adams operations

We'll define some cohomology operations, which extend exterior powers: recall that there is an exterior power of a vector bundle $\wedge^k V$ which is fiberwise given by the exterior power of vector spaces. This descends to a natural endomorphism of K -theory λ^k .⁵⁰

Recall from the theory of symmetric polynomials that there is an expression of $x_1^k + \dots + x_n^k = s_k(\sigma_1, \dots, \sigma_k)$ where σ_i are elementary symmetric polynomials. We can define the Adams operations:

⁴⁹This is true mod 2 by graded-commutativity. To get it to a total statement needs some torsion free statement, which is not immediately clear to me.

⁵⁰This is not multiplicative, right?

Definition 14.4. The ring endomorphism $\psi^k : \tilde{K}^0(K) \rightarrow \tilde{K}^0(X)$ by $E \mapsto s_k(\lambda^1(E), \dots, \lambda^k(E))$.

These have nice properties:

Proposition 14.5 (Properties of the Adams operations). *The following hold for ψ^k the k th Adams operations:*

- (1) ψ^k is natural.
- (2) For a line bundle L , $\psi^k(L) = L^{\otimes k}$.
- (3) Each ψ^k is a ring homomorphism.
- (4) $\psi^k \psi^l = \psi^{kl}$, so in particular, Adams operations commute.
- (5) For prime p , $\psi^p(E) = E^p \text{ mod } p$.

Proof. (1) is clear.

For (2), we prove the slightly stronger statement that $\psi^k(L_1 \oplus L_2) = L_1^{\otimes k} \oplus L_2^{\otimes k}$. We claim that $\lambda^k(L_1 \oplus L_2) = \sigma_k(L_1, L_2)$ since $\lambda^k(L_i) = 0$ whenever $k \geq 2$.⁵¹ Slotting this in yields $\psi^k = s_k(\sigma_1(L_1, L_2), \dots, \sigma_k(L_1, L_2)) = L_1^k \oplus L_2^k$.

For (3)'s additivity, simply apply the splitting principle and naturality; pulling back the Adams operations along the splitting of the thing you're applying to yields the right splitting of the result of the Adams operations.

We skipped (4). For (5), apply the splitting principle again. \square

These are particularly nice on the K -theory of the sphere:

Lemma 14.6. *On $\tilde{K}^0(S^{2m})$, we have*

$$\psi^k(a) = k^m a.$$

Proof. Fix $m = 1$ and $b = H^* - 1$. Then,

$$\psi^k(b) = (H^*)^k - 1 = (b + 1)^k - 1 = kb.$$

Now, work inductively, and assume it's true for $m - 1$. Write the product map

$$\tilde{K}^0(S^2) \otimes \tilde{K}^0(S^{2m-2}) \rightarrow \tilde{K}^0(S^{2m}).$$

Note that this maps b to kb and $\lambda^{n-1}b \mapsto k^{m-1}b$; then, the ring homomorphism structure yields that $\lambda^m b \mapsto k^m b$. \square

We've assembled enough tools to prove our main theorem.

14.4 Proof of the hopf invariant 1 theorem

Proof of Theorem 14.3. As mentioned earlier, we can assume n is even, i.e. $n = 2m$. Recall that $\psi^k(\alpha) = k^{2m}\alpha$ and $\psi^k(\beta) = k^m\beta = \mu_k\alpha$.

Use commutativity of ψ^3 and ψ^2 ; this expresses

$$6^m\beta + 3^m\mu_2\alpha + 2^m\mu_3\alpha = \psi^2\psi^3(\beta) = \psi^3\psi^2(\beta) = 5^m\beta + 2^m\mu_3\alpha + 3^{2m}\mu_2\alpha.$$

We can cancel additively, then recall that torsion-freeness allows us to cancel α ; this implies that

$$3^m(1 - 3^m)\mu_2 = 2^m(1 - 2^m)\mu_3,$$

and in particular, 2^m divides $3^m(1 - 3^m)\mu_2$; however, noting that $\mu^2(\beta) = 2^m\beta + \mu_2 \equiv \beta^2 \pmod{2}$, so $\mu_2 \equiv h \pmod{2}$. Hence a Hopf invariant one implies that 2^m divides $3^m(1 - 3^m)$, which is only true when $m = 1, 2, 4$, i.e. when $n = 2, 4, 8$. \square

⁵¹This is trivial when $k \geq 2$.

15 Weixiao Lu: Quillen, The spectrum of an equivariant cohomology ring

This talk was delivered on October 27, 2021 by Weixiao Lu.

16 Dylan Pentland: Quillen, The cohomology and K-theory of the general linear groups over a finite field

This talk was delivered on October 29, 2021 by Dylan Pentland.

17 Zihong Chen: Quillen, Higher Algebraic K-theory I

This talk was delivered on Nov 3, 2021 by Zihong (Peter) Chen.

18 Mikayel Mkrtchyan: Griffiths and Morgan, Rational homotopy theory and differential forms

This talk was delivered on Nov 5, 2021 by Mikayel Mkrtchyan.

19 Preston Cranford: Quillen, Homotopical Algebra

This talk was delivered on November 10, 2021 by Preston Cranford.

20 Swapnil Garg: Hill, Hopkins, Ravenel, On the nonexistence of elements of Kervaire invariant one

This talk was delivered on November 12, 2021 by Swapnil Garg.

21 Haoshuo Fu: Bousfield, Localization of spectra with respect to homology

This talk was delivered on November 17, 2021 by Haoshuo Fu.

22 Natalie Stewart: May, The Geometry of Iterated Loop Spaces

22.1 Operads and their algebras

Motivation: little cubes One of the most important structures in algebraic topology is the *n*th homotopy group, i.e. a group structure on the pointed set

$$[S^n, X] = [*, \Omega^n X] = \pi_0 \Omega^n X.$$

When we construct such a group structure, we tend to make a non-canonical choice concerning how to compose *n*-spheres, constructing a magma structure on $\Omega^n X$, which fails to be unital, associative, or commutative on the nose, but which descends to a group structure on homotopy classes of maps, i.e. $\pi_0 \Omega^n X$. The construction usually goes as follows:

Construction 22.1 (Homotopy groups). We construct a group structure on $\Omega^n X = \text{Maps}((I^n, \partial I^n), (X, *))$. Such a map α is drawn in such a diagram: [little cubes diagram](#). The composition is $\alpha \circ \beta$ by “shrinking” each map by half rectilinearly in one direction, then sticking them together, as drawn in the following diagram: [little cubes diagram](#). Associativity is witness by a homotopy between diagrams: [little cubes diagram](#). Unitality is witnessed by a similar homotopy, and in the case $n \geq 2$, we have the *Eckmann-Hilton argument*, which is another such homotopy of diagrams: [little cubes diagram](#).

A good mantra to take is to *take homotopy as late as possible*. Even this construction was defined point-set, then the group we reference is gotten via a formal process (taking homotopy) in a category which is not very concrete (ho \mathbf{Top}_*).

We can obey this mantra even better: we could have defined the composition of two maps in any way “homotopic” to the way we did, so there is actually a coherent family of ways of compose two maps, as well as ways to compose *m* many maps: if $C_n(m)$ is the space of rectilinear embeddings of *m* disjoint *n*-cubes into an *n*-cube, then all of our choices are jointly summarized by an \mathbb{N} -indexed collection of maps

$$C_n(m) \times (\Omega^n X)^m \rightarrow \Omega^n X$$

compatibly with the “slotting in” composition maps

$$C_n(m) \times \prod_{i=1}^m C_n(k_i) \rightarrow C_n(k_1 + \cdots + k_m).$$

We will formalize this, along with equivariance, through the notion of an *operad*.

Operads and algebras For the duration of this section, fix $(\mathcal{V}, I, \otimes)$ a cocomplete symmetric monoidal closed category. We define the notion formalizing a *compatible collection of symmetric *n*-ary operations*.

Definition 22.2. An *operad* (sometimes called a *symmetric operad*) O in \mathcal{V} is the data:

- for each $n \in \mathbb{N}$, an object $O(n)$,
- for each index n , a right-action of the symmetric group $S_n \rightarrow \mathcal{V}(O(n), O(n))$, and
- for each tuple (n_1, \dots, n_k) with $n = \sum_i n_i$, a *composition* morphism

$$\gamma : O(k) \otimes O(n_1) \otimes \cdots \otimes O(n_k) \rightarrow O(n),$$

- an distinguished *identity* element $I \rightarrow O(1)$,

subject to the following conditions:

- (i) (Composition of operations is S_* -equivariant) the composition γ coequalizes the following pair:

$$O(k) \otimes \bigotimes O(n_i) \rightrightarrows O(k) \otimes \bigotimes O(n_i) \xrightarrow{\gamma} O(n)$$

where one of the arrows acts on $O(k)$ via the prescribed action, and the other acts by permuting tensor powers.⁵²

⁵²In this setting, “coequalizes” means that the precomposition of either action before γ yields the same result, i.e. you can either permute tensor powers or act on the space $O(k)$.

- (ii) (Composition of operations is associative) for tuples (d_1, \dots, d_k) and (e_1, \dots, e_j) , defining $f_i := \gamma(d_i, e_{j_1+\dots+j_{i-1}+1}, \dots, e_{j_1+\dots+j_s})$ we have

$$\gamma(\gamma(c; d_1, \dots, d_k); e_1, \dots, e_j) = \gamma(c; f_1, \dots, f_k).$$

- (iii) (Composition of operations is unital)

$$\gamma(1; d) = d$$

for all $d \in O(j)$ and

$$\gamma(c; 1^k)$$

for $c \in O(k)$ and $1^k = (1, \dots, 1) \in O(1)^{\otimes k}$.

A *morphism of operads* is an S_* -equivariant morphism of graded objects $f_* : O(*) \rightarrow O'(*)$ which is unital and compatible with composition:

$$f(1) = 1$$

$$f(\gamma(\theta; \theta_1, \dots, \theta_n)) = \gamma(f(\theta); f(\theta_1), \dots, f(\theta_n)).$$

This has a graphical calculus. **Draw some trees sometime.**

Operads are defined as a blueprint for a type of algebraic object. To understand the significance of these, we need to understand how to make objects that comply with this blueprint. To do so, we need an example and a definition.

Example 22.3:

Let $X \in \mathcal{V}$ be an object. Then, the *endomorphism operad on X* , denoted \mathcal{E}_X , has:

- Objects $\mathcal{E}_X(n) := \mathcal{V}(X^{\otimes n}, X)$,
- symmetric action induced by the symmetric action on $X^{\otimes n}$,
- composition given by composition of endomorphism

$$\gamma(\theta; \theta_1, \dots, \theta_n) : X^{\otimes n} = X^{\otimes n_1} \otimes \dots \otimes X^{\otimes n_k} \xrightarrow{\otimes \theta_i} X^{\otimes k} \xrightarrow{\theta} X.$$

It's easy to verify that this is an operad.

Definition 22.4. Let O be an operad. An *algebra over O* is an object X and a morphism of operads

$$O \rightarrow \mathcal{E}_X.$$

That is, the set $O(n)$ represents the *possible n -ary operations of an algebraic theory*, and an algebra assigns these to particular n -ary operations on an object, with behavior dictated by the symmetry, composition, and unitality. We can flesh this out via an example:

Example 22.5:

The *commutative monoid operad* Comm has objects $\text{Comm}(n) = I$, composition morphisms induced by the identity $I = I$, and trivial symmetric action. It's algebras are canonically identified with *commutative monoid objects in \mathcal{V}* , with the unique map $M^{\otimes n} \rightarrow M$ given by the unique n -ary operation given by repeated application of the monoid law.

Hence we've subsumed the construction of commutative rings/algebras, commutative monoid, topological commutative monoid, etc. We can get around the commutativity condition by adding extra elements:

Example 22.6:

Suppose \mathcal{V} has finite coproducts. The *monoid operad* Assoc has objects $\text{Assoc}(n) = \coprod_{S_n} I$, with shuffling comultiplicands, with composition chosen uniquely by equivariance, and with identity given by the identity map $I = I$.

Hence we've subsumed monoids. One such example of this will be a monoid in a category of endofunctors, which in turn will describe all operad algebras:

22.2 Monad algebras are operad algebras

Bad Definition 22.7. Fix \mathcal{C} a category. A *monad in \mathcal{C}* is a monoid in the monoidal category of endofunctors $(\mathcal{C}^{\mathcal{C}}, \circ, \text{id})$.

This is a bad definition for two reasons: it takes nontrivial time to unroll, and it doesn't tell us how to define monad algebras, which are what we really care about. We could talk about these as *modules over an algebra over an operad* where the operad is Assoc in an endofunctor category, but this is inefficient. We might as well unroll our definition into something readable:

Good Definition 22.8. Fix \mathcal{C} a category. A *monad in \mathcal{C}* is the data:

1. An endofunctor $O : \mathcal{C} \Rightarrow \mathcal{C}$,
2. A *multiplication* natural transformation $\mu : O \circ O \Rightarrow O$, and
3. A *unit* natural transformation $\eta : \text{id}_{\mathcal{C}} \Rightarrow O$.

subject to associativity and unitality relations:

$$\begin{array}{ccc} O^3 X & \xrightarrow{\mu \otimes \text{id}} & O^2 X \\ \downarrow \text{id} \otimes \mu & & \downarrow \mu \\ O^2 X & \xrightarrow{\mu} & OX \end{array} \quad \begin{array}{ccc} OX & \xrightarrow{\text{id} \otimes \eta} & O^2 X \\ \eta \otimes \text{id} \downarrow & \searrow & \downarrow \mu \\ O^2 X & \xrightarrow{\mu} & OX \end{array}$$

A *algebra over a monad O* is an object $X \in \mathcal{C}$ together with an action $OX \rightarrow X$ which is associative and unital:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & OX \\ & \searrow & \downarrow a \\ & & X \end{array} \quad \begin{array}{ccc} O^2 X & \xrightarrow{\text{id} \otimes a} & OX \\ \downarrow \mu & & \downarrow a \\ OX & \xrightarrow{a} & X \end{array}$$

The following establishes that monads are strongly related to adjoint functors:

Example 22.9:

Suppose that $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ is a pair of adjoint functors. Then, there is a monad structure on the endofunctor RL on \mathcal{D} , whose multiplication is induced by the counit $\varepsilon : LR \Rightarrow \text{id}$:

$$RLRL \xrightarrow{R \circ \varepsilon \circ L} RL$$

with unit given by the adjunction unit.

However, we can go the other direction: for O a monad and X an object, there is an O -algebra structure on OX given by exactly the structure of O , and this is functorial; one can verify that this yields a pair of adjoint functors:

$$\begin{array}{ccc} & \text{Free}_O & \\ \mathcal{C} & \xrightarrow{\quad} & O(\mathcal{C}) \\ & \text{U} & \\ & \perp & \end{array}$$

where U is the forgetful functor taking an operad algebra and $O(\mathcal{C})$ is the category of O -algebras.⁵³

Definition of the monad associated with an operad.

22.3 Little cubes and approximation, and a handwave of the recognition principle

22.4 The two-sided bar construction

22.5 The recognition principle

⁵³For $O = RL$ induced by a pair of adjoint functors, it is not always the case that the free-forgetful adjunction is identified with (L, R) . There is a recognition theorem of when this is true, called *Beck's monadicity theorem*; the condition is that the right adjoint is conservative and preserves certain coequalizers.

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