THE BALMER SPECTRA OF SPECTRAL STACKS

1. Introduction

The theory of connective spectral algebraic geometry (developed in [Lur17]) is, unlike the theory of nonconnective spectral algebraic geometry, very similar to the classical theory of algebraic geometry. A connective spectral scheme $\mathcal{X}$ behaves much like the underlying classical scheme X, with the higher homotopy sheaves $\pi_k \mathcal{O}_{\mathcal{X}}$ functioning as added nilpotent homotopy-theoretic directions. For instance, $\tau_{\leqslant k}\mathcal{O}_{\mathcal{X}}$ is a square-zero extension of $\tau_{\leqslant k-1}\mathcal{O}_{\mathcal{X}}$. In light of this, one might expect the *homotopy-theoretic* data of $\mathcal{X}$ to, in some sense, be orthogonal to the *algebro-geometric* data of its underlying classical scheme X. Our goal in this document is to prove that a version of this philosophy can be made precise.

If $\mathcal{C}$ is a symmetric monoidal presentable stable ∞-category, we will write $\mathrm{Spec}(\mathcal{C})$ to denote the Balmer spectrum ([Bal05]) of $\mathcal{C}$. The topological space underlying $\mathrm{Spec}(\mathcal{C})$ classifies the tensor ideals of $\mathcal{C}$, which in turn describes when an object of $\mathcal{C}$ can be constructed from another object by retracts, cofiber sequences, and tensor products. The main result of this document roughly shows that if $f : \mathcal{X} \to \mathcal{Y}$ is a proper flat morphism of sufficiently nice spectral algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ which satisfy a regularity condition, and $|X|$ and $|Y|$ denote the topological space associated to the underlying stacks of $\mathcal{X}$ and $\mathcal{Y}$, then $\mathrm{Spec}(\mathrm{QCoh}(\mathcal{X})^\omega)$ is homeomorphic to the fiber product $|X| \times_{|Y|} \mathrm{Spec}(\mathrm{QCoh}(\mathcal{Y})^\omega)$ (in topological spaces). A more precise statement is as follows (see Theorem 2.7 for a statement in slightly more generality).

Theorem 1.1 (Theorem 2.7)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be Noetherian connective quasi-geometric spectral algebraic stacks, such that $i^*i_*\mathcal{O}_X$ (resp. $i^*i_*\mathcal{O}_Y$) is a perfect $\mathcal{O}_X$-module (resp. perfect $\mathcal{O}_Y$-module), where $i : X \to \mathcal{X}$ and $i : Y \to \mathcal{Y}$ are the canonical morphisms. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism such that:*

- *X and Y are of finite cohomological dimension;*
- *f is flat, and the pushforward functor f_* preserves perfect objects;*
- *$\mathcal{O}_{\mathcal{Y}}$ is in the thick subcategory of $\mathrm{QCoh}(\mathcal{Y})^\omega$ generated by $f_*(\mathcal{O}_{\mathcal{X}})$.*

Then there is a homeomorphism

$$\mathrm{Spec}(\mathrm{QCoh}(\mathcal{X})^\omega) \cong \mathrm{Spec}(\mathcal{D}(X)^\omega) \times_{\mathrm{Spec}(\mathcal{D}(Y)^\omega)} \mathrm{Spec}(\mathrm{QCoh}(\mathcal{Y})^\omega).$$

Theorem 1.1 does indeed espouse the philosophy discussed in the beginning of this section. For the sake of philosophical discussion, suppose that the regularity condition on $\mathcal{Y}$ could be removed. Setting $\mathcal{Y} = \mathrm{Spec}(\mathbb{S})$ in Theorem 1.1 then shows (using the main result of [Hal16]) that if $\mathcal{X}$ is a (nice enough) proper flat spectral algebraic space, then $\mathrm{Spec}(\mathrm{QCoh}(\mathcal{X})^\omega)$ is homeomorphic to $|X| \times_{|\mathrm{Spec}(\mathbf{Z})|} \mathrm{Spec}(\mathrm{Sp}^\omega)$. The map $\mathrm{Spec}(\mathrm{Sp}^\omega) \to |\mathrm{Spec}(\mathbf{Z})|$ is surjective, and the fiber over each prime $(p) \in |\mathrm{Spec}(\mathbf{Z})|$ is the $\mathbf{Z}$-indexed collection of the thick subcategories of p-local spectra of type n. None of this information is seen by $|X|$, and it is in this sense $\mathrm{Spec}(\mathrm{QCoh}(\mathcal{X})^\omega)$ is composed of the two "orthogonal" factors of the product $|X| \times_{|\mathrm{Spec}(\mathbf{Z})|} \mathrm{Spec}(\mathrm{Sp}^\omega)$.

As a consequence of Theorem 2.7, we recover a restricted version of [Aok20, Theorem I] as Corollary 2.10.

2. Background

We introduce the following definition, motivated by [HP14].

Definition 2.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of quasi-geometric spectral algebraic stacks. Say that f is cohomologically proper if the pushforward f_* takes perfect objects of $\mathrm{QCoh}(\mathcal{X})$ to perfect objects of $\mathrm{QCoh}(\mathcal{Y})$.

Remark 2.2. By [Lur17, Proposition 9.1.5.2], every compact object of $\mathrm{QCoh}(\mathcal{X})$ is perfect if $\mathcal{X}$ is a quasi-geometric spectral algebraic stack, but the converse is true if and only if $\mathcal{O}_{\mathcal{X}} \in \mathrm{QCoh}(\mathcal{X})$ is compact.

Example 2.3. In light of [Lur17, Theorem 6.1.3.2], any morphism of spectral Deligne-Mumford stacks which is proper, locally of finite presentation, and finite Tor-amplitude is cohomologically proper.

Lemma 2.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of quasi-geometric spectral algebraic stacks. Suppose further that $\mathcal{O}_{\mathcal{X}}$ is a compact object of $\mathrm{QCoh}(\mathcal{X})$. Then f_* is cohomologically proper if and only if $f_*\mathcal{O}_{\mathcal{X}}$ is a perfect object of $\mathrm{QCoh}(\mathcal{Y})$.*

Proof. Clearly, if f_* is cohomologically proper, then $f_*\mathcal{O}_{\mathcal{X}}$ is a perfect object of $\mathrm{QCoh}(\mathcal{Y})$. To show the converse, observe that these assumptions on f allow us to use [Lur17, Proposition 9.1.5.8] in order to apply [Lur17, Proposition 9.1.5.7], yielding that f_* preserves small colimits. In particular, since $f_*\mathcal{O}_{\mathcal{X}}$ is assumed to be perfect, the functor f_* preserves perfect objects. □

Before we state the main result, we need another definition, motivated by [BL14].

Definition 2.5. A quasi-geometric spectral algebraic stack $\mathcal{X}$ with underlying stack X is said to be weakly regular if the morphism $X \to \mathcal{X}$ is cohomologically proper.

Remark 2.6. It follows from [BL14, Proposition 1.5] that if $\mathcal{X}$ is Noetherian, then $\mathcal{X}$ is weakly regular if $i^*i_*\mathcal{O}_X$ is a perfect $\mathcal{O}_X$-module, where $i : X \to \mathcal{X}$ is the canonical morphism. If $\mathcal{X} = \mathrm{Spec}(R)$ is affine and $\pi_0(R)$ is regular, then this amounts to saying that $\pi_0(R) \otimes_R \pi_0(R)$ is a perfect $\pi_0(R)$-module. This is *not* true for $R = \mathbb{S}$, but is true for $R = \mathrm{bo}, \mathrm{tmf}, \mathrm{BP}\langle n\rangle$.

We can finally state the main theorem of this document.

Theorem 2.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of weakly regular quasi-geometric spectral algebraic stacks. Suppose that:*

- *X and Y are of finite cohomological dimension;*
- *f is flat and cohomologically proper;*
- *$\mathcal{O}_{\mathcal{Y}}$ is in the thick subcategory of $\mathrm{QCoh}(\mathcal{Y})^\omega$ generated by $f_*(\mathcal{O}_{\mathcal{X}})$.*

Then there is a homeomorphism

$$\mathrm{Spec}(\mathrm{QCoh}(\mathcal{X})^\omega) \cong \mathrm{Spec}(\mathcal{D}(X)^\omega) \times_{\mathrm{Spec}(\mathcal{D}(Y)^\omega)} \mathrm{Spec}(\mathrm{QCoh}(\mathcal{Y})^\omega).$$

Remark 2.8. The assumptions in Theorem 2.7 are satisfied in many cases. By definition, every weakly perfect stack (in the sense of [Lur17, Section 9.4.3]) satisfies the property that $\mathcal{O}_{\mathcal{X}}$ is a compact object of $\mathrm{QCoh}(\mathcal{X})$; the class of weakly perfect stacks contains, for instance, all quasicompact quasiseparated spectral algebraic spaces (see [Lur17, Proposition 9.6.1.1]). The condition being flat is relatively mild, and Remark 2.3 provides many examples of cohomologically proper morphisms. For instance, every morphism of weakly regular quasicompact quasiseparated spectral algebraic spaces, whose underlying algebraic spaces are of finite cohomological dimension, which is proper, flat, and locally of finite presentation satisfies the hypotheses of Theorem 2.7.

Corollary 2.9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of weakly regular quasi-geometric spectral algebraic stacks. Suppose that:*

- *X and Y have quasi-finite and separated diagonals;*

- *X and Y of finite cohomological dimension;*
- *f is flat and cohomologically proper;*
- *$\mathcal{O}_{\mathcal{Y}}$ is in the thick subcategory of $\mathrm{QCoh}(\mathcal{Y})^\omega$ generated by $f_*(\mathcal{O}_{\mathcal{X}})$.*

Then there is a homeomorphism between the Balmer spectrum $\mathrm{Spec}(\mathrm{QCoh}(\mathcal{X})^\omega)$ and $|X| \times_{|Y|} \mathrm{Spec}(\mathrm{QCoh}(\mathcal{Y})^\omega)$.

Proof. Since X is a quasicompact algebraic stack with quasi-finite and separated diagonal, we can run the proof of [Hal16, Theorem 1.2] using [Hal16, Theorem 1.1] to conclude that there is a homeomorphism between the Balmer spectrum of $\mathrm{Spec}(\mathcal{D}(X)^\omega)$ and $|X|$ (the part of the proof dealing with the *homeomorphism* does not utilize tameness). We conclude by Theorem 2.7. $\square$

Recall that if $\mathcal{C}$ is an ∞-category, the ∞-category $\mathcal{C}_{\mathrm{fil}}$ of filtered objects in $\mathcal{C}$ is the functor category $\mathrm{Fun}(\mathbf{Z}, \mathcal{C})$, where $\mathbf{Z}$ is viewed as a partially ordered abelian group with the standard ordering, and the monoidal structure is via Day convolution. The ∞-category $\mathcal{C}_{\mathrm{gr}}$ of graded objects in $\mathcal{C}$ is the functor category $\mathrm{Fun}(\mathbf{Z}^{\mathrm{ds}}, \mathcal{C})$, where $\mathbf{Z}^{\mathrm{ds}}$ is the integers viewed as a discrete abelian group, and the monoidal structure is again via Day convolution.

Corollary 2.10. *Let $\mathcal{Y}$ be a Noetherian connective quasi-geometric spectral algebraic stack such that $i^*i_*\mathcal{O}_Y$ is a perfect $\mathcal{O}_Y$-module, and such that Y is of finite cohomological dimension. Let $\mathfrak{S}$ denote the Sierpiński space; then there are homeomorphisms*

$$\mathrm{Spec}(\mathrm{QCoh}(\mathcal{Y})^\omega_{\mathrm{fil}}) \cong \mathfrak{S} \times \mathrm{Spec}(\mathrm{QCoh}(\mathcal{Y})^\omega),\ \mathrm{Spec}(\mathrm{QCoh}(\mathcal{Y})^\omega_{\mathrm{gr}}) \cong \mathrm{Spec}(\mathrm{QCoh}(\mathcal{Y})^\omega).$$

Proof. Let $\mathcal{X} = \mathbf{A}^1/\mathbb{G}_m \times \mathcal{Y}$, where $\mathbf{A}^1$ is the flat affine line and $\mathbb{G}_m$ is the flat multiplicative group. By [Mou19], there is an equivalence $\mathrm{QCoh}(\mathcal{X})^\omega \simeq \mathrm{QCoh}(\mathcal{Y})^\omega_{\mathrm{fil}}$. (In the graded case, one replaces $\mathcal{X}$ by $\mathrm{B}\mathbb{G}_m \times \mathcal{Y}$.) The structure morphism $\mathcal{X} \to \mathcal{Y}$ is flat and preserves perfect objects. Moreover, the underlying stack X is isomorphic to $\mathbf{A}^1/\mathbb{G}_m \times Y$, and therefore X is of finite cohomological dimension (since the same is true of Y). Finally, since $f_*(\mathcal{O}_{\mathcal{X}}) \cong \mathcal{O}_{\mathcal{Y}}$, it is certainly true that $\mathcal{O}_{\mathcal{Y}}$ is in the thick subcategory of $\mathrm{QCoh}(\mathcal{Y})^\omega$ generated by $f_*(\mathcal{O}_{\mathcal{X}})$. By Theorem 2.7 (or, rather, by Corollary 2.9, proved below), we conclude that there are homeomorphisms

$$\mathrm{Spec}(\mathrm{QCoh}(\mathcal{Y})^\omega_{\mathrm{fil}}) \cong |\mathbf{A}^1/\mathbb{G}_m \times Y| \times_{|Y|} \mathrm{Spec}(\mathrm{QCoh}(\mathcal{Y})^\omega),$$
$$\mathrm{Spec}(\mathrm{QCoh}(\mathcal{Y})^\omega_{\mathrm{gr}}) \cong |\mathrm{B}\mathbb{G}_m \times Y| \times_{|Y|} \mathrm{Spec}(\mathrm{QCoh}(\mathcal{Y})^\omega).$$

To conclude, it suffices to observe that there are homeomorphisms $|\mathbf{A}^1/\mathbb{G}_m \times Y| \cong |\mathbf{A}^1/\mathbb{G}_m| \times |Y|$, $|\mathrm{B}\mathbb{G}_m \times Y| \cong |Y|$, and $|\mathbf{A}^1/\mathbb{G}_m| \cong \mathfrak{S}$. $\square$

3. Proof of Theorem 2.7

We begin by recalling some background from [Aok20, Section 3.3].

Definition 3.1. Say that an ∞-category $\mathcal{C}$ is a tt-∞-category if it is an idempotent complete stable ∞-category with an $\mathbf{E}_2$-monoidal structure whose associated tensor product is exact in each variable. A radical ideal of $\mathcal{C}$ is a full replete stable ∞-category I of $\mathcal{C}$ such that:

- If $X \oplus Y \in I$ for some $X, Y \in \mathcal{C}$, then $X, Y \in I$;
- If $X \in \mathcal{C}$ and $Y \in I$, then $X \otimes Y \in I$;
- If $X \in \mathcal{C}$ is such that $X^{\otimes n} \in I$ for some $n \geqslant 0$, then $X \in I$.

Let $\sqrt{X}$ denote the smallest radical ideal containing an object $X \in \mathcal{C}$. Let $\mathrm{Zar}(\mathcal{C})$ denote the partially ordered set of radical ideals of the form $\sqrt{X}$ where $X \in \mathcal{C}$. It is a distributive lattice, known as the Zariski lattice of $\mathcal{C}$. The Balmer spectrum of $\mathcal{C}$ is the spectrum (in the order-theoretic sense) of $\mathrm{Zar}(\mathcal{C})$. The functor sending a tt-∞-category to its Zariski lattice is contravariant, as is the functor sending a distributive lattice to its spectrum.

Definition 3.2. A support of $\mathcal{C}$ is a pair (L, s) consisting of a distributive lattice L and a function $s : \mathrm{h}\mathcal{C} \to L$ such that:

(a) s sends finite direct sums to joins;
(b) s sends finite tensor products to meets;
(c) if $X \to Y \to Z$ is a cofiber sequence, then $s(X) \vee s(Y) = s(X) \vee s(Z) = s(Y) \vee s(Z)$.

Proposition 3.3 ([Aok20, Proposition 3.9])**.** *Let $\mathcal{C}$ be a tt-∞-category. Then $\mathrm{Zar}(\mathcal{C})$ is a distributive lattice, and the pair $(\mathrm{Zar}(\mathcal{C}), X \mapsto \sqrt{X})$ is an initial object in the category of supports of $\mathcal{C}$.*

We now turn to the proof of Theorem 2.7.

Lemma 3.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of spectral stacks, and let $\mathcal{G} \in \mathrm{QCoh}(X)^\omega$. Let $\mathcal{C}$ be the subcategory of $\mathrm{QCoh}(\mathcal{X})^\omega$ spanned by those $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})^\omega$ such that $\sqrt{f_*(\mathcal{F} \otimes \mathcal{G})} = \sqrt{f_*(\mathcal{F}) \otimes f_*(\mathcal{G})}$. Let $\mathcal{C}'$ denote the subcategory of $\mathrm{QCoh}(\mathcal{X})^\omega$ spanned by those $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})^\omega$ such that $\sqrt{f_*(\mathcal{F})} = \sqrt{f_* f^* f_*(\mathcal{F})}$. Then $\mathcal{C}$ and $\mathcal{C}'$ are thick subcategories of $\mathrm{QCoh}(\mathcal{X})^\omega$.*

Proof. Suppose $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is a cofiber sequence in $\mathrm{QCoh}(\mathcal{X})^\omega$ such that $\mathcal{F}', \mathcal{F}'' \in \mathcal{C}$ (resp. $\mathcal{F}, \mathcal{F}' \in \mathcal{C}'$); we must show that $\mathcal{F} \in \mathcal{C}$ (resp. $\mathcal{F} \in \mathcal{C}$). Since f_* preserves cofiber sequences, we obtain cofiber sequences

$$f_*(\mathcal{F}' \otimes \mathcal{G}) \to f_*(\mathcal{F} \otimes \mathcal{G}) \to f_*(\mathcal{F}'' \otimes \mathcal{G}),\ f_*(\mathcal{F}') \otimes f_*(\mathcal{G}) \to f_*(\mathcal{F}) \otimes f_*(\mathcal{G}) \to f_*(\mathcal{F}'') \otimes f_*(\mathcal{G}).$$

It follows that

$$\sqrt{f_*(\mathcal{F}) \otimes f_*(\mathcal{G})} = \sqrt{f_*(\mathcal{F}') \otimes f_*(\mathcal{G})} \vee \sqrt{f_*(\mathcal{F}'') \otimes f_*(\mathcal{G})} = \sqrt{f_*(\mathcal{F}' \otimes \mathcal{G})} \vee \sqrt{f_*(\mathcal{F}'' \otimes \mathcal{G})} = \sqrt{f_*(\mathcal{F} \otimes \mathcal{G})};$$

therefore, $\mathcal{F} \in \mathcal{C}$, as desired. (Similarly, there are cofiber sequences

$$f_*(\mathcal{F}') \to f_*(\mathcal{F}) \to f_*(\mathcal{F}''),\ f_* f^* f_*(\mathcal{F}') \to f_* f^* f_*(\mathcal{F}) \to f_* f^* f_*(\mathcal{F}''),$$

and therefore

$$\sqrt{f_* f^* f_*(\mathcal{F})} = \sqrt{f_* f^* f_*(\mathcal{F}')} \vee \sqrt{f_* f^* f_*(\mathcal{F}'')} = \sqrt{f_*(\mathcal{F}')} \vee \sqrt{f_*(\mathcal{F}'')} = \sqrt{f_*(\mathcal{F})};$$

this implies that $\mathcal{F} \in \mathcal{C}'$.) $\square$

Lemma 3.5. *Suppose $\mathcal{X}$ is a weakly regular quasi-geometric spectral algebraic stack such that the underlying stack X has finite cohomological dimension. Then there is a morphism $\mathrm{Spec}(\mathrm{QCoh}(\mathcal{X})^\omega) \to \mathrm{Spec}(\mathcal{D}(X)^\omega)$ of topological spaces.*

Proof. It suffices to define a morphism $\mathrm{Zar}(\mathcal{D}(X)^\omega) \to \mathrm{Zar}(\mathrm{QCoh}(\mathcal{X})^\omega)$. By Proposition 3.3, it suffices to define a support function $s : \mathrm{h}\mathcal{D}(X)^\omega \to \mathrm{Zar}(\mathrm{QCoh}(\mathcal{X})^\omega)$; we claim that $s(\mathcal{F}_0) = \sqrt{i_*(\mathcal{F}_0)}$ does the trick. Since i_* preserves direct sums and cofiber sequences, the function s satisfies conditions (a) and (c) of Definition 3.2; it remains to show that $\sqrt{i_*(\mathcal{F}_0 \otimes \mathcal{G}_0)} = \sqrt{i_*(\mathcal{F}_0)} \wedge \sqrt{i_*(\mathcal{G}_0)}$, i.e., $\sqrt{i_*(\mathcal{F}_0 \otimes \mathcal{G}_0)} = \sqrt{i_*(\mathcal{F}_0) \otimes i_*(\mathcal{G}_0)}$. Fix $\mathcal{G}_0 \in \mathcal{D}(X)^\omega$; by Lemma 3.4, the subcategory $\mathcal{C}$ of $\mathcal{D}(X)^\omega$ spanned by those $\mathcal{F}_0$ such that $\sqrt{i_*(\mathcal{F}_0 \otimes \mathcal{G}_0)} = \sqrt{i_*(\mathcal{F}_0) \otimes i_*(\mathcal{G}_0)}$ is thick. Since X has finite cohomological dimension, [Lur17, Proposition 9.1.5.3] implies that it suffices to show that $\mathcal{C}$ contains $\mathcal{O}_X$, i.e., that $\sqrt{i_*(\mathcal{O}_X) \otimes i_*(\mathcal{G}_0)} = \sqrt{i_*(\mathcal{G}_0)}$.

The projection formula (applicable in this generality by [Lur17, Proposition 9.1.5.7]) implies that $i_*(\mathcal{O}_X) \otimes i_*(\mathcal{G}_0) \simeq i_* i^* i_*(\mathcal{G}_0)$. By Lemma 3.4, the subcategory $\mathcal{C}'$ of $\mathcal{D}(X)^\omega$ spanned by those $\mathcal{G}_0$ such that $\sqrt{i_* i^* i_*(\mathcal{G}_0)} = \sqrt{i_*(\mathcal{G}_0)}$ is a thick subcategory of $\mathcal{D}(X)^\omega$, so it suffices to show that $\mathcal{C}'$ contains $\mathcal{O}_X$. In other words, it suffices to show that $\sqrt{i_* i^* i_*(\mathcal{O}_X)} = \sqrt{i_*(\mathcal{O}_X)}$.

We first claim that $\mathcal{O}_X$ is in the thick subcategory of $\mathcal{D}(X)^\omega$ generated by $i^* i_* \mathcal{O}_X$. Indeed, it suffices to observe that $\mathcal{O}_X$ is a retract of $i^* i_* \mathcal{O}_X$. Pulling back the morphism $\mathcal{O}_{\mathcal{X}} \to i_* \mathcal{O}_X$ along i^* produces the morphism $\mathcal{O}_X \to i^* i_* \mathcal{O}_X$, and the morphism $i^* i_* \mathcal{O}_X \to \mathcal{O}_X$ is adjoint to the identity on $i_*(\mathcal{O}_X)$. (The key point is that $i^* i_* \mathcal{O}_X$ is, indeed, in $\mathcal{D}(X)^\omega$; this is not true without the hypothesis of weak regularity.)

Since $\mathcal{O}_X$ is in the thick subcategory of $\mathcal{D}(X)^\omega$ generated by $i^* i_* \mathcal{O}_X$, the thick subcategory of $\mathrm{QCoh}(\mathcal{X})^\omega$ generated by $i_* i^* i_* \mathcal{O}_X$ contains $i_*(\mathcal{O}_X)$ (because i preserves compact objects, by virtue of weak regularity), so $i_*(\mathcal{O}_X) \in \sqrt{i_* i^* i_* \mathcal{O}_X}$. Moreover, because the thick subcategory of $\mathcal{D}(X)^\omega$ generated by $\mathcal{O}_X$ is the entirety of $\mathcal{D}(X)^\omega$, it contains $i^* i_* \mathcal{O}_X$; therefore, $i_* i^* i_* \mathcal{O}_X$ is in the thick subcategory of $\mathrm{QCoh}(\mathcal{X})^\omega$ generated by $i_* \mathcal{O}_X$ (again because i preserves compact objects), so $i_* i^* i_* \mathcal{O}_X \in \sqrt{i_* \mathcal{O}_X}$ (and hence $\sqrt{i_*(\mathcal{O}_X)} = \sqrt{i_* i^* i_* \mathcal{O}_X}$), as desired. □

Proof of Theorem 2.7. Let $\mathcal{X}$ be a quasi-geometric stack satisfying the hypotheses of the theorem. Define a map $p : \mathrm{Zar}(\mathrm{QCoh}(\mathcal{Y})^\omega) \otimes_{\mathrm{Zar}(\mathcal{D}(Y)^\omega)} \mathrm{Zar}(\mathcal{D}(X)^\omega) \to \mathrm{Zar}(\mathrm{QCoh}(\mathcal{X})^\omega)$ as follows. The symmetric monoidal functor $f^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ defines a morphism $\mathrm{Zar}(\mathrm{QCoh}(\mathcal{Y})^\omega) \to \mathrm{Zar}(\mathrm{QCoh}(\mathcal{X})^\omega)$ of distributive lattices. The proof of Lemma 3.5 yields morphisms $i_* : \mathrm{Zar}(\mathcal{D}(X)^\omega) \to \mathrm{Zar}(\mathrm{QCoh}(\mathcal{X})^\omega)$ and $i_* : \mathrm{Zar}(\mathcal{D}(Y)^\omega) \to \mathrm{Zar}(\mathrm{QCoh}(\mathcal{Y})^\omega)$. We claim that the resulting diagram

$$\begin{tikzcd} \mathrm{Zar}(\mathcal{D}(Y)^\omega) \arrow[r] \arrow[d] & \mathrm{Zar}(\mathcal{D}(X)^\omega) \arrow[d] \\ \mathrm{Zar}(\mathrm{QCoh}(\mathcal{Y})^\omega) \arrow[r] & \mathrm{Zar}(\mathrm{QCoh}(\mathcal{X})^\omega) \end{tikzcd} \tag{3.1}$$

commutes; this yields the desired morphism p. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f_0"'] & \mathcal{X} \arrow[d, "f"] \\ Y \arrow[r, "i"] & \mathcal{Y}. \end{tikzcd} \tag{3.2}$$

The two ways of going around the diagram (3.1) are given on a generator for the radical ideal by $f^* i_*, i_* f_0^* : \mathcal{D}(Y) \to \mathrm{QCoh}(\mathcal{X})$. There is a natural transformation $f^* i_* \to i_* f_0^*$ which is an equivalence if the diagram (3.2) is homotopy Cartesian. This, in turn, occurs when f is flat.

We shall now use the universal property of $\mathrm{Zar}(\mathcal{C})$ in Proposition 3.3 to show that there is a map $q : \mathrm{Zar}(\mathrm{QCoh}(\mathcal{X})^\omega) \to \mathrm{Zar}(\mathrm{QCoh}(\mathcal{Y})^\omega) \otimes_{\mathrm{Zar}(\mathcal{D}(Y)^\omega)} \mathrm{Zar}(\mathcal{D}(X)^\omega)$. It will be clear from the construction of q that it is inverse to p.

We need to define a function $s : \mathrm{h}\,\mathrm{QCoh}(\mathcal{X})^\omega \to \mathrm{Zar}(\mathrm{QCoh}(\mathcal{Y})^\omega) \otimes_{\mathrm{Zar}(\mathcal{D}(Y)^\omega)} \mathrm{Zar}(\mathcal{D}(X)^\omega)$ satisfying the conditions of Definition 3.2. Suppose $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})^\omega$. The pushforward $f_* \mathcal{F}$ is *a priori* only a quasicoherent sheaf on $\mathcal{Y}$, but cohomological properness of $\mathcal{X}$ implies that $f_* \mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^\omega$. This yields a radical ideal $\sqrt{f_* \mathcal{F}} \in \mathrm{Zar}(\mathrm{QCoh}(\mathcal{Y})^\omega)$. Pullback along the morphism $i : X \to \mathcal{X}$ produces a map $\mathrm{Zar}(\mathrm{QCoh}(\mathcal{X})^\omega) \to \mathrm{Zar}(\mathcal{D}(X)^\omega)$. We therefore obtain a function $s : \mathrm{h}\,\mathrm{QCoh}(\mathcal{X})^\omega \to \mathrm{Zar}(\mathrm{QCoh}(\mathcal{Y})^\omega) \otimes_{\mathrm{Zar}(\mathcal{D}(Y)^\omega)} \mathrm{Zar}(\mathcal{D}(X)^\omega)$ given by

$$s(\mathcal{F}) = \sqrt{f_* \mathcal{F}} \otimes \sqrt{i^* \mathcal{F}}$$

for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})^\omega$. We claim that s is a support function. Proposition 3.3 then yields the desired morphism q.

Since f_* and i^* both preserve finite direct sums and cofiber sequences, the function s satisfies conditions (a) and (c) of Definition 3.2. It remains to show that s sends finite tensor products to meets. Since i^* is symmetric monoidal, the map $\mathrm{h}\,\mathrm{QCoh}(\mathcal{X})^\omega \to \mathrm{Zar}(\mathcal{D}(X)^\omega)$ sending $\mathcal{F}$ to $\sqrt{i^* \mathcal{F}}$ sends finite tensor products to meets. It remains to show that the map $\mathrm{h}\,\mathrm{QCoh}(\mathcal{X})^\omega \to \mathrm{Zar}(\mathrm{QCoh}(\mathcal{Y})^\omega)$ sending $\mathcal{F}$ to $\sqrt{f_* \mathcal{F}}$ sends finite tensor products to meets. It suffices to show that if $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}(\mathcal{X})^\omega$, then $\sqrt{f_*(\mathcal{F} \otimes \mathcal{G})} = \sqrt{f_*(\mathcal{F}) \otimes f_*(\mathcal{G})}$. By Lemma 3.4, the subcategory $\mathcal{C}$ of $\mathrm{QCoh}(\mathcal{X})^\omega$ for which $\sqrt{f_*(\mathcal{F} \otimes \mathcal{G})} = \sqrt{f_*(\mathcal{F}) \otimes f_*(\mathcal{G})}$ is closed under finite colimits and retracts.

It therefore suffices to show that $\mathcal{O}_\mathcal{X} \in \mathcal{C}$. (Indeed, it then follows that $\mathcal{C}$ contains every perfect object of $\mathrm{QCoh}(\mathcal{X})^\omega$. But this is the entirety of $\mathrm{QCoh}(\mathcal{X})^\omega$, since perfect and compact objects coincide by virtue of [Lur17, Proposition 9.1.5.3] and the assumption that $\mathcal{O}_{\mathcal{X}}$ is compact.) To show that $\mathcal{O}_{\mathcal{X}} \in \mathcal{C}$, it suffices to argue that $\sqrt{f_*(\mathcal{O}_{\mathcal{X}}) \otimes \mathcal{F}} = \sqrt{\mathcal{F}}$ for any $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^\omega$. This, in turn, follows from the assumption that $\mathcal{O}_{\mathcal{Y}}$ is in the thick subcategory of $\mathrm{QCoh}(\mathcal{Y})^\omega$ generated by $f_*(\mathcal{O}_{\mathcal{X}})$. □

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