

Homotopy-coherent interchange and equivariant little disk operads

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A question you might happen upon

Interchange and
little V-disks

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In trace methods for Real algebraic K-theory, THH has a Real analogue:¹

$$\begin{array}{ccc} \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) & & \mathrm{Alg}_{\mathbb{E}_\sigma}(\mathrm{Sp}_{C_2}) \\ \mathrm{THH} \downarrow & \xrightarrow{\text{“Reality”}} & \downarrow \mathrm{THR} \\ \mathrm{Sp} & & \mathrm{Sp}_{C_2} \end{array}$$

In this, σ is the sign representation and \mathbb{E}_σ -algebras are a genuine-equivariant version of rings with anti-involution.

Question (c.f. Dotto-Moi-Patchkoria-Reeh² '17)

What algebraic structure does THR of highly structured C_2 -ring spectra have?

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¹ Emanuele Dotto. *Stable real K-theory and real topological Hochschild homology*. 2012. arXiv: 1212.4310 [math.AT].

² Dotto_Real.

How to construct structure on THH

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Observation

THH can be given a symmetric monoidal structure, so we may lift

$$\begin{array}{ccccc} \mathrm{Alg}_{\mathcal{O} \otimes \mathbb{E}_1}(\mathrm{Sp}) & \simeq & \mathrm{Alg}_{\mathcal{O}} \mathrm{Alg}_{\mathbb{E}_1}^{\otimes}(\mathrm{Sp}) & \dashrightarrow & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) \\ & \searrow & \downarrow U & & \downarrow U \\ & & \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) & \xrightarrow{\mathrm{THH}} & \mathrm{Sp} \end{array}$$

Theorem (Dunn³ '88, Lurie⁴ '09)

$\mathbb{E}_n \simeq \mathbb{E}_{n-1} \otimes \mathbb{E}_1$; hence THH takes \mathbb{E}_n -rings to \mathbb{E}_{n-1} -rings.

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³Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258.

⁴DAG-VI.

How to construct structure on THR

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Observation (S.⁵)

THR can be given a C_2 -symmetric monoidal structure, so we may lift

$$\begin{array}{ccccc} \mathrm{Alg}_{\mathcal{O} \otimes \mathbb{E}_\sigma}(\mathrm{Sp}_{C_2}) & \simeq & \mathrm{Alg}_{\mathcal{O}} \mathrm{Alg}_{\mathbb{E}_\sigma}^{\otimes}(\mathrm{Sp}_{C_2}) & \dashrightarrow & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}_{C_2}) \\ & & \downarrow U & & \downarrow U \\ & & \mathrm{Alg}_{\mathbb{E}_\sigma}(\mathrm{Sp}_{C_2}) & \xrightarrow{\mathrm{THR}} & \mathrm{Sp}_{C_2} \end{array}$$

Conjecture

$$\mathbb{E}_V \otimes \mathbb{E}_W \simeq \mathbb{E}_{V \oplus W}$$

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⁵Natalie Stewart. *On tensor products with equivariant commutative operads*. 2025. arXiv: 2504.02143 [math.AT].

Statement of the additivity theorem

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For this talk, all terms are defined ∞ -categorically.

Theorem (S.⁵)

Given V, W orthogonal G -representations, we have

$$\mathbb{E}_V \otimes \mathbb{E}_W \simeq \mathbb{E}_{V \oplus W};$$

hence there is an equivalence of ∞ -categories

$$\mathrm{Alg}_{\mathbb{E}_V} \mathrm{Alg}_{\mathbb{E}_W}^{\otimes} (\mathrm{Sp}_G) \simeq \mathrm{Alg}_{\mathbb{E}_{V \oplus W}} (\mathrm{Sp}_G).$$

Corollary

THR of $\mathbb{E}_{V \oplus \sigma}$ -rings has a natural \mathbb{E}_V -ring structure.

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⁵Natalie Stewart. *On homotopical additivity of equivariant little disks operads (forthcoming)*. 2025.

Myriad versions of this theorem

- 0 May '72:⁶ $C_n \otimes C_m$ and C_{n+m} agree on **connected spaces**.
- 1 Dunn '88:⁷ $C_1^{\otimes n} \simeq C_n$ w.r.t. a **point-set** tensor product.
- 2 Brinkmeier '00:⁸ $C_n \otimes C_m \simeq C_{n+m}$ w.r.t. a **point-set** tensor product.
- 3 Rourke-Sanderson '00:⁹ $D_V \otimes D_W$ and $D_{V \oplus W}$ agree on **G-connected G-spaces**.
- 4 Lurie '09:¹⁰ $\mathbb{E}_n^{\otimes} \overset{\text{BV}}{\otimes} \mathbb{E}_m^{\otimes} \simeq \mathbb{E}_{n+m}^{\otimes}$ with respect to a **homotopical** tensor product.
- 5 Fiedorowicz-Vogt '15:¹¹ Dunn & Brinkmeier's result extends to **cofibrant \mathbb{E}_n -operads**.
- 6 Szczesny '24:¹² $D_V \otimes D_W \simeq D_{V \oplus W}$ w.r.t. a **point-set** tensor product.
- 7 S. '25: $\mathbb{E}_V^{\otimes} \overset{\text{BV}}{\otimes} \mathbb{E}_W^{\otimes} \simeq \mathbb{E}_{V \oplus W}^{\otimes}$ with respect to a **homotopical** tensor product.

⁶J. P. May. *The geometry of iterated loop spaces*. Vol. Vol. 271. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1972, pp. viii+175.

⁷Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258.

⁸Brinkmeier.

⁹Colin Rourke and Brian Sanderson. "Equivariant configuration spaces". In: *J. London Math. Soc. (2)* 62.2 (2000), pp. 544–552.

¹⁰DAG-VI.

¹¹Zbigniew Fiedorowicz and Rainer M. Vogt. *An Additivity Theorem for the Interchange of E_n Structures*. 2013. arXiv: 1102.1311 [math.AT].

¹²Ben Szczesny. *Equivariant Framed Little Disk Operads are Additive*. 2024. arXiv: 2410.20235 [math.AT].

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A heavily abridged history of G - ∞ -categorical operads

- 0 Hill-Hopkins '16:¹³ G -commutative monoids are **semi-Mackey functors**.
- 1 Nardin-Shah '22:¹⁴ G -operads are a type of fibration over a G -categorical version $\underline{\mathbb{F}}_{G,*}$ of Segal's category Γ^{op} .
- 2 Barkan-Haugsgeng-Steinebrunner '22:¹⁵ G -operads are also **fibrations over the effective Burnside 2-category $\text{Span}(\mathbb{F}_G)$** .
- 3 S. '25:¹⁶ homotopy-coherent interchange is corepresented by **BV tensor products** and G -operads are monadic over G -symmetric sequences.
- 4 S. '25:¹⁷ Algebras in **(co)cartesian G -symmetric monoidal structures** have concrete descriptions and $\mathcal{N}_{I\infty} \otimes \mathcal{N}_{J\infty} \simeq \mathcal{N}_{IV\infty}$

¹³Michael A. Hill and Michael J. Hopkins. *Equivariant symmetric monoidal structures*. 2016. arXiv: 1610.03114 [math.AT].

¹⁴Denis Nardin and Jay Shah. *Parametrized and equivariant higher algebra*. 2022. arXiv: 2203.00072 [math.AT].

¹⁵Shaul Barkan, Rune Haugseng, and Jan Steinebrunner. *Envelopes for Algebraic Patterns*. 2022. arXiv: 2208.07183 [math.CT].

¹⁶Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT].

¹⁷Natalie Stewart. *On tensor products with equivariant commutative operads*. 2025. arXiv: 2504.02143 [math.AT].

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How to prove equivariant Dunn additivity

$$\begin{array}{ccccccc}
 \mathbb{P}_V^{\otimes} \overset{BV}{\otimes} \mathbb{P}_W^{\otimes} & \xleftarrow{L_{\mathbb{P}}} & \mathbb{P}_V^{\otimes} \times \mathbb{P}_W^{\otimes} & \xrightarrow{M_{\mathbb{P}}} & \mathbb{P}_V^{\otimes} \wr \mathbb{P}_W^{\otimes} & \xrightarrow{\tilde{\varphi}_{\mathbb{P}}} & \mathbb{P}_{V|W}^{\otimes} \hookrightarrow \mathbb{P}_{V \oplus W}^{\otimes} \\
 \alpha_V \otimes \alpha_W \downarrow & & \alpha_V \times \alpha_W \downarrow & & \alpha_V \wr \alpha_W \downarrow & & \alpha_{V|W} \downarrow & \downarrow \alpha_{V \oplus W} \\
 \mathbb{E}_V^{\otimes} \overset{BV}{\otimes} \mathbb{E}_W^{\otimes} & \xleftarrow{L_{\mathbb{E}}} & \mathbb{E}_V^{\otimes} \times \mathbb{E}_W^{\otimes} & \xrightarrow{M_{\mathbb{E}}} & \mathbb{E}_V^{\otimes} \wr \mathbb{E}_W^{\otimes} & \xrightarrow{\varphi_{\mathbb{E}}} & \mathbb{E}_{V \oplus W}^{\otimes} = \mathbb{E}_{V \oplus W}^{\otimes}
 \end{array}$$

- 1 Define “G-operads” as a localizing subcategory of “G-preoperads.”
- 2 Define G-preoperad models \times and \wr for $\overset{BV}{\otimes}$, reduce to showing that $\varphi_{\mathbb{E}}$ is an L_{Op_G} -equivalence.
- 3 Define “approximations” α , an approximated “surjection to image” $\tilde{\varphi}_{\mathbb{P}}$ onto “decomposable little disks,” verify that $\alpha_{V|W}$ is an approximation by lifting Dunn’s argument about decomposability of little disks.
- 4 Use ∞ -category theory to reduce to showing $\tilde{\varphi}_{\mathbb{P}}$ is an L_{Op_G} -equivalence.
- 5 Verify that $\tilde{\varphi}_{\mathbb{P}}$ by lifting Dunn’s argument about uniqueness of decompositions.

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A quasi-definition of G-operads

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Definition

- A **G-preoperad** is a functor $\mathcal{O}^\otimes \rightarrow \text{Span}(\mathbb{F}_G)$ with cocartesian lifts over backwards maps.
- A **G-operad** is required to satisfy “Segal conditions.”
- An **\mathcal{O} -algebra in \mathcal{C}^\otimes** is a functor preserving cocartesian arrows :

$$\begin{array}{ccc} \mathcal{O}^\otimes & \xrightarrow{\quad \varphi \quad} & \mathcal{C}^\otimes \\ & \searrow \quad \swarrow & \\ & \text{Span}(\mathbb{F}_G) & \end{array}$$

$$\begin{array}{c} \text{Op}_G \quad \xleftarrow{L_{\text{Op}_G}} \quad \text{POp}_G \\ \quad \quad \quad \perp \quad \quad \quad \nearrow \\ \text{Cat}_{\infty, / \text{Span}(\mathbb{F}_G)}^{\text{Backwards-cocart}} \end{array}$$

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Sidebar: S -ary spaces in Mike's G -operads

When considering the *homotopy theory* of Mike's G -operads, we usually define *graph subgroups*: given $H \subset G$ and $S \in \mathbb{F}_H$ define

$$\Gamma_S = \{ (h, \rho_S(h)) \mid h \in H, \rho_S: H \rightarrow \Sigma_{|S|} \} \subset G \times \Sigma_S.$$

Definition

The S -ary structure space of \mathcal{O} is the fixed points

$$\mathcal{O}(S) = \mathcal{O}(|S|)^{\Gamma_S}.$$

A map of Mike's G -operads $\varphi: \mathcal{O} \rightarrow \mathcal{P}$ is a *weak equivalence* if $\mathcal{O}(S) \rightarrow \mathcal{P}(S)$ is an equivalence for all $S \in \mathbb{F}_H$.

Our theory of G -operads should factor through this...

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The underlying G -symmetric sequence

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Construction

Given $\mathcal{O}^\otimes \in \mathbf{Op}_G$, $H \subset G$, and $S \in \mathbb{F}_G$, define the **structure space**

$$\begin{array}{ccc} \mathcal{O}(S) & \xrightarrow{\quad} & \mathbf{Mor}(\mathcal{O}^\otimes) \\ \downarrow & \lrcorner & \downarrow \\ \{\mathrm{Ind}_H^G S = \mathrm{Ind}_H^G S \rightarrow [G/H]\} & \longrightarrow & \mathbf{Mor}(\mathrm{Span}(\mathbb{F}_G)) \end{array}$$

Proposition (S.¹⁸)

If \mathcal{O}^\otimes has “one color” then is it conservatively identified by $(\mathcal{O}(S))_{\substack{H \subset G \\ S \in \mathbb{F}_H}}.$

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¹⁸Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT].

Example

The **little V-disks G-operad** has S-ary structure space given by **H-equivariant configurations of S in V**:

$$\mathbb{E}_V(S) := \text{Emb}^{H, \text{Affine}}(S \cdot D(V), D(V)) \simeq \text{Conf}_S^H(V).$$

Example

Given a G-symmetric monoidal category $\mathcal{C}^\otimes : \text{Span}(\mathbb{F}_G) \rightarrow \text{Cat}_\infty$, its unstraightening is a G-operad. Given $X \in \mathcal{C}(G/G)$, there is an **endomorphism G-operad** with

$$\text{End}_X(S) \simeq \text{Map}_{\mathcal{C}(G/H)} \left((\text{Res}_H^G X)^{\otimes S}, \text{Res}_H^G X \right)$$

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In particular, an \mathbb{E}_V -algebra in \mathcal{C} consists of an object $X \in \mathcal{C}_G$ and homotopy-coherently compatible maps

$$\mathrm{Conf}_S^H(V) \rightarrow \mathrm{Map}_{\mathcal{C}(G/H)} \left((\mathrm{Res}_H^G X)^{\otimes S}, \mathrm{Res}_H^G X \right).$$

Example (Horev-Klang-Zou¹⁹ '20)

Let $\underline{\mathcal{S}}_G^{G-\times}$ be the *cartesian structure* on G -spaces. Then, for all $X \in \mathcal{S}_G$, we have $\Omega^V X \in \mathrm{Alg}_{\mathbb{E}_V}(\mathcal{S}_G)$.

Example (loc. cit.)

Let $\underline{\mathrm{Sp}}_G^{\otimes}$ be the HHR G -symmetric monoidal structure. If $f: \Omega^V X \rightarrow \underline{\mathrm{Pic}}(\underline{\mathrm{Sp}}_G)$ is a **V-loop map**, then $\mathrm{Th}(f) \in \mathrm{Alg}_{\mathbb{E}_V}(\mathrm{Sp}_G)$.

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¹⁹Jeremy Hahn, Asaf Horev, Inbar Klang, Dylan Wilson, and Fofing Zou. *Equivariant nonabelian Poincaré duality and equivariant factorization homology of Thom spectra*. 2024. arXiv: 2006.13348 [math.AT].

Modelling G-operadic constructions, two ways

Model categorist: **(co)fibrantly replace**, then apply a construction to monoids in G-symmetric sequences.

∞ -categorist: apply a G-preoperadic construction, then **L_{Op_G} -localize**.

Shared goal: model corepresenting object for **pairings** (aka interchanging algebras, bifunctors, etc.) akin to May.²⁰

$$\begin{array}{ccc} \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} & \xrightarrow{\text{"pairing"}} & \mathcal{Q}^{\otimes} \\ \downarrow \pi & & \downarrow \pi \\ \text{Span}(\mathbb{F}_G) \times \text{Span}(\mathbb{F}_G) & \xrightarrow{\wedge} & \text{Span}(\mathbb{F}_G) \end{array}$$

Today, we are ∞ -categorists.

²⁰May_pairings.

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The Boardman-Vogt tensor product

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Definition

$$\mathcal{O}^{\otimes} \overset{\text{BV}}{\otimes} \mathcal{P}^{\otimes} := L_{\text{Op}_G} \left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \longrightarrow \text{Span}(\mathbb{F}_G)^2 \xrightarrow{\wedge} \text{Span}(\mathbb{F}_G) \right)$$

$\text{Alg}_{\mathcal{O}}(\mathcal{C})$ lifts to a “pointwise” G -symmetric monoidal category $\underline{\text{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$.

Proposition (S.²¹)

$(-) \overset{\text{BV}}{\otimes} \mathcal{O}^{\otimes}$ is left adjoint to $\underline{\text{Alg}}_{\mathcal{O}}^{\otimes}(-)$, so

$$\text{Alg}_{\mathcal{O}} \underline{\text{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \simeq \text{Alg}_{\mathcal{O} \otimes \mathcal{P}}(\mathcal{C}).$$

Also, have “wreath” operator \wr and natural L_{Op_G} -equivalences

$$\mathcal{O}^{\otimes} \overset{\text{BV}}{\otimes} \mathcal{P}^{\otimes} \leftarrow \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \rightarrow \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}.$$

²¹Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT].

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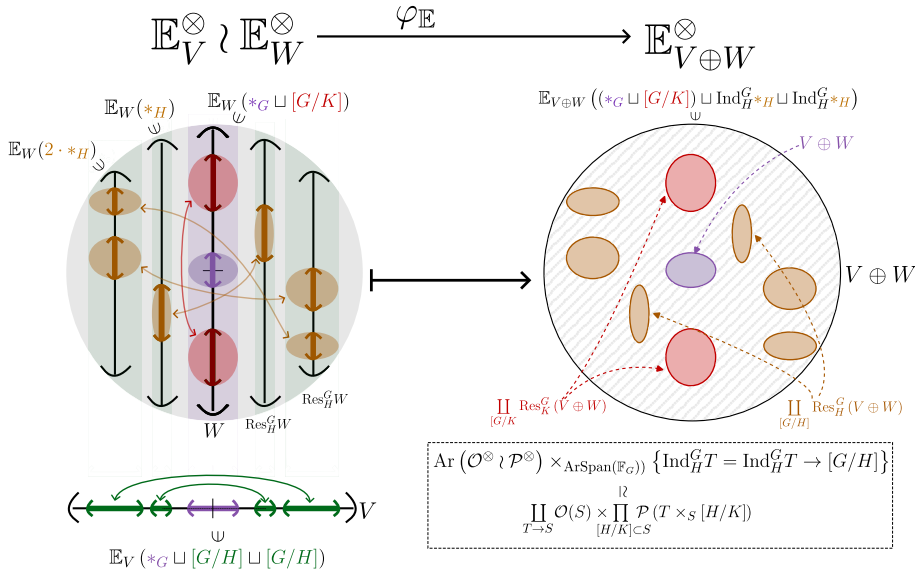
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Sidebar on G - ∞ -categories

Need theory of “approximations.” We use *equivariant higher category theory*.

Definition

A G -category is a coefficient system of ∞ -categories $\mathcal{C}: \mathcal{O}_G^{\text{op}} \rightarrow \text{Cat}$.

Example

The terminal G -category has $*_G(G/H) = *$.

Definition

A G -functor $\mathcal{C} \rightarrow \mathcal{D}$ is a natural transformation $F: \mathcal{C} \Rightarrow \mathcal{D}$. A G -object is a G -functor $X: *_G \rightarrow \mathcal{C}$.

The corresponding theory of G -(co)limits, G -adjunctions formalizes “ H -set indexed (co)products.” Moral for approximations: **model (reflective) localizations of the underlying G -category**.

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Preoperad models for 1-colored G -operads

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Given $\mathcal{P}^\otimes \in \mathbf{POp}_G$, we let $\mathcal{P}_{/P}^{\text{act}}$ be the ∞ -category of arrows $P' \rightarrow P$ projecting to a forward map $T = T \rightarrow S$.

Definition

A map of G -preoperads $\alpha: \mathcal{P}^\otimes \rightarrow \mathcal{O}^\otimes$ with \mathcal{O}^\otimes a “one color” G -operad is a **weak approximation** if

- 1 The G -category of colors $U\mathcal{P}$ **has a terminal G -object**, and
- 2 For all $P \in \mathcal{P}^\otimes$ and $T \rightarrow \pi(P)$, **the map of spaces**

$$B\left(\mathcal{P}_{/P}^{\text{act}} \times_{\mathbb{F}_{G,/\pi(P)}} \{T \rightarrow \pi(P)\}\right) \rightarrow \prod_{[H/K] \subset \pi(P)} \mathcal{O}(T \times_{\pi(P)} [H/K])$$

is a weak equivalence.

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Proposition (Harpaz²² '19 + reinterpretation)

If $\alpha: \mathcal{P}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ is a weak approximation, pullback is fully faithful

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{S}_G) \subset \mathrm{Alg}_{\mathcal{P}}(\mathcal{S}_G)$$

with image the \mathcal{P} -monoids whose “color” G -functors $U\mathcal{P} \rightarrow \underline{\mathcal{S}}_G$ are constant.

Weak approximations can be made to have many colors; a weak approximation α is a **strong approximation** if $U\mathcal{P} \rightarrow U\mathcal{O}$ is an equivalence.

Proposition (S.²³)

$\mathrm{Alg}_{(-)}(\mathcal{S}_G)$ detects L_{Op_G} -equivalences when $U\mathcal{P} \rightarrow U\mathcal{O}$ is an equivalence; in particular, **strong approximations are L_{Op_G} -equivalences**.

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²²Yonatan Harpaz. *Little cubes algebras and factorization homology (course notes)*.

²³Natalie Stewart. *On tensor products with equivariant commutative operads*. 2025. arXiv: 2504.02143 [math.AT].

The L_{Op_G} -conservativity argument, in short

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Proof sketch.

Examine the free \mathcal{O} - G -space monad:²⁴

$$(T_{\mathcal{O}}X)^H \simeq \coprod_{S \in \mathbb{F}_H} \left(\mathcal{O}(S) \times (X^S)^H \right)_{h \text{Aut}_H S}.$$

The inclusion $\text{Aut}_H(S) \subset \text{End}_H(S) = (S^S)^H$ yields natural splitting

$$(T_{\mathcal{O}}S)^H \simeq \mathcal{O}(S) \sqcup \text{Junk}.$$

Use monadicity of $\text{Alg}_{\mathcal{O}}(\mathcal{S}_G) \rightarrow \mathcal{S}_G$ and conservativity of $(\mathcal{O}(S))_{S \in \mathbb{F}_H}^{HCG}$. □

²⁴Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT].

Proposition (Dugger-Isaksen²⁵ '01)

If X is a topological space and $\mathfrak{D} \subset \mathcal{P}(X)$ a basis of contractible open subsets, then we get a weak equivalence

$$B\mathfrak{D} \xrightarrow{\sim} X$$

Corollary

Let $\mathfrak{D}_S^H(V) \subset \text{Conf}_S^H(V)$ be the basis of configurations in affinely $\coprod_S D(V)$ -shaped invariant subspaces of $D(V)$. We get a weak equivalence

$$B\mathfrak{D}_S^H(V) \xrightarrow{\sim} \text{Conf}_S^H(V) \simeq \mathbb{E}_V(S).$$

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²⁵ Daniel Dugger and Daniel C. Isaksen. "Topological hypercovers and A^1 -realizations". In: *Math. Z.* 246.4 (2004), pp. 667–689.

$$\begin{array}{ccccccc}
 \mathbb{P}_V^\otimes \overset{\text{BV}}{\otimes} \mathbb{P}_W^\otimes & \xleftarrow{L_{\mathbb{P}}} & \mathbb{P}_V^\otimes \times \mathbb{P}_W^\otimes & \xrightarrow{M_{\mathbb{P}}} & \mathbb{P}_V^\otimes \wr \mathbb{P}_W^\otimes & \xrightarrow{\tilde{\varphi}_{\mathbb{P}}} & \mathbb{P}_{V|W}^\otimes \hookrightarrow \mathbb{P}_{V \oplus W}^\otimes \\
 \alpha_V \otimes \alpha_W \downarrow & & \alpha_V \times \alpha_W \downarrow & & \alpha_V \wr \alpha_W \downarrow & & \alpha_{V|W} \downarrow \quad \quad \quad \downarrow \alpha_{V \oplus W} \\
 \mathbb{E}_V^\otimes \overset{\text{BV}}{\otimes} \mathbb{E}_W^\otimes & \xleftarrow{L_{\mathbb{E}}} & \mathbb{E}_V^\otimes \times \mathbb{E}_W^\otimes & \xrightarrow{M_{\mathbb{E}}} & \mathbb{E}_V^\otimes \wr \mathbb{E}_W^\otimes & \xrightarrow{\varphi_{\mathbb{E}}} & \mathbb{E}_{V \oplus W}^\otimes = \mathbb{E}_{V \oplus W}^\otimes
 \end{array}$$

We define a G -preoperad \mathbb{P}_V^\otimes such that $\mathbb{P}_{V/P}^{\text{act}} \simeq \mathfrak{D}_S^H(V)$, yielding a weak approximation $\alpha_V: \mathbb{P}_V^\otimes \rightarrow \mathbb{E}_V^\otimes$. Then, we define a \mathbb{P} -Dunn map fitting into the above diagram.

Here, $\mathbb{P}_{V|W}^\otimes$ is the “ G -preoperadic image, i.e. “decomposed little disks.”

We’re tasked with verifying that $\varphi_{\mathbb{E}} \circ M_{\mathbb{E}}$ induces an equivalence

$$\text{Alg}_{\mathbb{E}_{V \oplus W}}(\mathcal{S}_G) \xrightarrow{\sim} \text{Alg}_{\mathbb{E}_V \times \text{Alg}_{\mathbb{E}_W}}(\mathcal{S}_G) \xleftarrow{\sim} \text{Alg}_{\mathbb{E}_V \otimes \mathbb{E}_W}(\mathcal{S}_G)$$

$$\begin{array}{ccccccc}
 \mathbb{P}_V^{\otimes} \otimes^{\text{BV}} \mathbb{P}_W^{\otimes} & \xleftarrow{L_{\mathbb{P}}} & \mathbb{P}_V^{\otimes} \times \mathbb{P}_W^{\otimes} & \xrightarrow{M_{\mathbb{P}}} & \mathbb{P}_V^{\otimes} \wr \mathbb{P}_W^{\otimes} & \xrightarrow{\tilde{\varphi}_{\mathbb{P}}} & \mathbb{P}_{V|W}^{\otimes} \xrightarrow{\sim} \mathbb{P}_{V \oplus W}^{\otimes} \\
 \alpha_V \otimes \alpha_W \downarrow & & \alpha_V \times \alpha_W \downarrow & & \alpha_V \wr \alpha_W \downarrow & & \alpha_{V|W} \downarrow & \alpha_{V \oplus W} \downarrow \\
 \mathbb{E}_V^{\otimes} \otimes^{\text{BV}} \mathbb{E}_W^{\otimes} & \xleftarrow{L_{\mathbb{E}}} & \mathbb{E}_V^{\otimes} \times \mathbb{E}_W^{\otimes} & \xrightarrow{M_{\mathbb{E}}} & \mathbb{E}_V^{\otimes} \wr \mathbb{E}_W^{\otimes} & \xrightarrow{\varphi_{\mathbb{E}}} & \mathbb{E}_{V \oplus W}^{\otimes} = \mathbb{E}_{V \oplus W}^{\otimes}
 \end{array}$$

- 1 $L_{\mathbb{P}}$ and $L_{\mathbb{E}}$ are L_{Op_G} -equivalences by fiat.
- 2 A variant of Harpaz's strategy²⁶ shows that $M_{\mathbb{P}}$ and $M_{\mathbb{E}}$ are L_{Op_G} -equivalences.
- 3 Simple ∞ -category theory shows that $\alpha_V \times \alpha_W$ is a weak approximation.
- 4 A variant of Dunn's strategy²⁷ shows that $\alpha_{V|W}$ is a weak approximation.
- 5 Explicit 1-category theory shows that $\tilde{\varphi}_{\mathbb{P}}$ is a strong approximation.
- 6 Routine bookkeeping then shows that $\text{Alg}_{\mathbb{E}_{V \oplus W}}(\mathcal{S}_G) \xrightarrow{\sim} \text{Alg}_{\mathbb{E}_V \times \mathbb{E}_W}(\mathcal{S}_G)$.

²⁶Yonatan Harpaz. *Little cubes algebras and factorization homology* (course notes).

²⁷Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258.

Stalkwise-linearizable tangential structures

Define $\mathbb{E}_{B_G^{\text{linTop}(n)}}^{\otimes}$ to have colors the linearizable G -actions on \mathbb{R}^n and operations the *topological* embeddings.

Given a G -space X with stalkwise-linearizable equivariant \mathbb{R}^n -bundle $T_{\bullet}: X \rightarrow B_G^{\text{linTop}(n)}$, we define the assembly

$$\mathbb{E}_X^{\otimes} := \text{Lop}_G \left(\mathbb{E}_{B_G^{\text{linTop}(n)}}^{\otimes} \times_{B_G^{\text{linTop}(n)}^{G-\sqcup}} X^{G-\sqcup} \right).$$

Theorem (S.²⁸)

There is a natural equivariant colimit expression of operads

$$\underline{\text{colim}}_{x \in X} \mathbb{E}_{T_x}^{\otimes} \xrightarrow{\sim} \mathbb{E}_X^{\otimes}$$

²⁸ Natalie Stewart. *On homotopical additivity of equivariant little disks operads (forthcoming)*. 2025.

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Corollary (S.)

Given stalkwise-linearizable equivariant \mathbb{R}^k -bundles $X \rightarrow B_G^{\text{lin}}\text{Top}(n)$ and $Y \rightarrow B_G^{\text{lin}}\text{Top}(m)$, fixing the direct sum structure

$$X \times Y \longrightarrow B_G^{\text{lin}}\text{Top}(n) \times B_G^{\text{lin}}\text{Top}(m) \xrightarrow{\oplus} B_G^{\text{lin}}\text{Top}(n+m),$$

there is an equivalence.

$$\mathbb{E}_X^{\otimes} \overset{\text{BV}}{\otimes} \mathbb{E}_Y^{\otimes} \xrightarrow{\sim} \mathbb{E}_{X \times Y}^{\otimes}.$$

Other cases that are covered by the original argument:

- (Equivariant) swiss cheese
- Stratified little disks

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Let $\underline{\mathrm{Disk}}_G^{X\text{-fr}, \sqcup} \subset \underline{\mathrm{Mfld}}_G^{X\text{-fr}, \sqcup}$ be the G -symmetric monoidal full category whose objects are disjoint unions of X -framed G -disks and whose mapping spaces are X -framed equivariant disk embeddings.

Corollary (S., c.f. Dwyer-Hess-Knudsen²⁹ '19)

There is a G -symmetric monoidal equivalence

$$\underline{\mathrm{Disk}}_G^{X\text{-fr}, \sqcup} \square \underline{\mathrm{Disk}}_G^{Y\text{-fr}, \sqcup} \simeq \underline{\mathrm{Disk}}_G^{X \times Y\text{-fr}, \sqcup}.$$

Here, \square is the box product of semi-Mackey functors valued in Cat_∞ .

²⁹William Dwyer, Kathryn Hess, and Ben Knudsen. *Configuration spaces of products*. 2018. arXiv: 1710.05093 [math.AT].

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This presentation was made in Beamer, with figures via tikz-cd and Inkscape, presented via Impressive. The tex is on my website. The title slide is $H_{C_2}^{\star}(*_{C_2}; \mathbb{Z})$ [3].

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