ON CONNECTIVITY OF SPACES OF EQUIVARIANT CONFIGURATIONS

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Abstract. We provide conditions on a locally smooth G-manifold under which its nonempty spaces of equivariant configurations $\mathrm{Conf}_S^G(X)$ are d-connected for all finite G-sets S. We use this to show that $\mathbb{E}_{dV}$-algebras in a G-symmetric monoidal $(d-1)$-category canonically lift to $\mathbb{E}_{\infty V}$-algebras.

Throughout this paper, we fix G a Lie group.

Definition 1. If $H \subset G$ is a closed subgroup and $S \in \mathbb{F}_H$ a finite H-set, we let

$$\mathrm{Conf}_S^H(X) \subset \mathrm{Map}^H(S, X)$$

be the (topological) subspace of H-equivariant embeddings $S \hookrightarrow M$. $\triangleleft$

Nonequivariantly, the homotopy type of configurations spaces in X is a rich source of homeomorphism-invariants of X. In this paper, we study some rudiments of an equivariant lift of this in the smooth setting. Namely, in Section 1, we supply sufficient conditions for a smooth G-manifold M such that its nonempty configurations spaces $\mathrm{Conf}_S^G(M)$ are all d-connected.

We have a particular application in mind; the structure spaces of the *little V-disks operad* are smooth G-manifolds, and connectivity statements of G-operads translate to structural statements about their algebras (see [Ste24a]). For instance, in Section 2, we prove a strengthening of the following theorem.

Theorem 2. *Suppose G is finite. If $\mathcal{C}$ is a G-symmetric monoidal $(d-1)$-category and V a real orthogonal G-representation, then the forgetful functor*

$$\mathbf{Alg}_{\mathbb{E}_{\infty V}}(\mathcal{C}) \to \mathbf{Alg}_{\mathbb{E}_{dV}}(\mathcal{C})$$

is an equivalence of $(d-1)$-categories.

In particular, $\mathbb{E}_{\infty V}$ is a weak $\mathcal{N}_\infty$-operad, so [Ste24a] and Theorem 2 provide a *homotopical incomplete Mackey functor* model for $\mathbb{E}_{dV}$-algebras in Cartesian G-symmetric monoidal $(d-1)$-categories and CHLL provides a *bi-incomplete Tambara functor* model for $\mathbb{E}_{dV}$-rings in the setting of homotopical incomplete Mackey functors valued in a $(d-1)$-category.

1. Equivariant configuration spaces in locally smooth manifolds

Definition 3 ([Bre72, § IV])**.** If M is a smooth manifold with a continuous G-action, we say that the action is *locally smooth* if, for each point $x \in M$, there exists a real orthogonal $\mathrm{stab}_G(x)$-representation V_x and a trivializing open neighborhood

$$x \in \coprod_{G/\mathrm{stab}_G(x)} V_x \overset{\circ}{\hookrightarrow} M,$$

where for a topological H-space X, we write $\coprod_{G/\mathrm{stab}_G(x)} X := G \times_H X$ as a topological G-space. In this case, we say that M with its action is a *locally smooth G-manifold.* $\triangleleft$

Smooth actions on manifolds admit well-behaved tubular neighborhoods; for example, [Bre72, Cor V.2.4] proves that smooth actions are locally smooth. On the other hand, if M is a locally smooth G-manifold, then the inclusion $M_{(H)} \hookrightarrow M$ of points with orbit isomorphic to G/H is a locally closed topological submanifold [Bre72, Thm IV.3.3], which is smooth if M is smooth [Bre72, Cor VI.2.5].

We begin this section in Section 1.1 by proving the following.

Theorem 4 (equivariant Fadell-Neuwirth fibration)**.** *Fix M a locally smooth G-manifold, $S, T \in \mathbb{F}_G$ a pair of finite G-sets, and $\iota : S \hookrightarrow M$ a G-equivariant configuration. The following is a homotopy-Cartesian square:*

$$\begin{tikzcd} \mathrm{Conf}^G_T(M - \iota(S)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Conf}^G_{S \sqcup T}(M) \arrow[d, "U"] \\ \{\iota\} \arrow[r, hook] & \mathrm{Conf}^G_S(M) \end{tikzcd}$$

Thus the long exact sequence in homotopy for $T = G/H$ yields means for computing homotopy groups of $\mathrm{Conf}^G_S(M)$ inductively on the cardinality of the orbit set $|S_G|$, with inductive step hinging on homotopy of

$$\mathrm{Conf}^G_{G/H}(M - \iota(S)) \simeq (M - \iota(S))_{(H)}.$$

We denote by $[\mathcal{O}_G]$ the subconjugacy lattice of closed subgroups of G, and we let

$$\mathrm{Istrp}(M) = \{\mathrm{stab}_x(G) \mid x \in M\} \subset [\mathcal{O}_G]$$

be the full subposet spanned by conjugacy classes (H) for which $M_{(H)}$ is nonempty. We are inspired to make the following definition.

Definition 5. A locally smooth G-manifold M is

- $\geq d$*-dimensional at each orbit type* if $M_{(H)}$ is $\geq d$-dimensional for each $(H) \in \mathrm{Istrp}(M)$;
- $(d-2)$*-connected at each orbit type* if $M_{(H)}$ is $(d-2)$-connected for each $(H) \in \mathrm{Istrp}(M)$. $\triangleleft$

In Section 1.2, we use Theorem 4 to prove the following.

Theorem 6. *If a locally smooth G-manifold M is $\geq d$-dimensional and $(d-2)$-connected at each orbit type, then for all finite G-sets $S \in \mathbb{F}_G$, the configuration space $\mathrm{Conf}^G_S(M)$ is either empty or $(d-2)$-connected.*

In order to identify applications of this theorem, we give sufficient conditions for M to be $(d-2)$-connected at each orbit type. Note by repeatedly applying [Bre72, Thm IV.3.1] that the subspace $M_{\leq (H)} \subset M$ of orbits mapping to G/H is a closed submanifold. In Section 1.3, we use this to prove the following.

Proposition 7. *Suppose that M is a smooth G-manifold satisfying the following conditions:*

(a) *M is $\geq d$-dimensional at each orbit type.*
(b) *$M_{\leq (H)}$ is $(d-2)$-connected for each H.*
(c) $\mathrm{codim}(M_{\leq (K)} \hookrightarrow M_{\leq (H)}) \geq d$ *for each $(K) \leq (H)$.*
(d) $\mathrm{Istrp}(M)$ *is finite (e.g. G compact and M finite type, c.f. [Bre72, Thm IV.10.5]).*

Then M is $(d-2)$-connected at each orbit type.

1.1. **A Fadell-Neuwirth fibration for equivariant configurations.** Our strategy for Theorem 4 mirrors that of Knudsen in the notes [Knu18]. In particular, we would like to use Quillen's theorem B [Qui73], which requires us to construct $\mathrm{Conf}^H_S(M)$ as a classifying space. In fact, there is a general scheme to do this:

Lemma 8 ([DI04, Thm 2.1], via [Knu18, Thm 4.0.2])**.** *If $\mathcal{B}$ is a topological basis for X such that all elements of $\mathcal{B}$ are weakly contractible, then the canonical map*

$$|\mathcal{B}| = \mathrm{hocolim}_{\mathcal{B}^*} \to X$$

is a weak equivalence, where on the left $\mathcal{B}$ is considered as a poset under inclusion.

To use this, define an elementwise-contractible basis for $\mathrm{Conf}^G_S(M)$ by

$$\widetilde{\mathcal{B}}^G_S(M) := \left\{ (X, \sigma) \;\middle|\; \exists (V_x) \in \prod_{[x] \in \mathrm{Orb}_S} \mathbf{Rep}^{\mathrm{orth}}_{\mathbb{R}}(\mathrm{stab}_G([x])), \ \text{ s.t. } \coprod_{[x]}^{S} V_x \simeq X \subset M, \ \ \sigma : S \xrightarrow{\sim} \pi_0(U) \right\},$$

where for all tuples $(Y_x) \in \prod_{[x] \in \mathrm{Orb}_S} \mathbf{Top}_{\mathrm{stab}_G([x])}$, we write

$$\coprod_{[x]}^{S} Y_x := \coprod_{[x] \in \mathrm{Orb}(S)} \left(G \times_{\mathrm{stab}_G([x])} Y_x \right) \in \mathbf{Top}_G$$

for the *indexed disjoint union of* Y_x. We fix $\mathcal{B}^G_S(M) \subset \widetilde{\mathcal{B}}^G_S(M)$ the smaller basis consisting of open sets (X,σ) possessing neighborhoods $(X,\sigma) \subset (X',\sigma)$ such that the associated embeddings factor as

$$\begin{tikzcd} \coprod_U^S D(V_U)^\circ \arrow[r, phantom, "\simeq"] \arrow[d, hook, "\exists"'] & \coprod_U^S V_U \arrow[d, hook, "X"] \\ V'_U \arrow[r, hook, "X'"] & M \end{tikzcd} \tag{1}$$

where $D(V_U)^\circ$ denotes the open V_U-disk; that is, open sets in $\mathcal{B}^G_S(M)$ consist of collections of configurations possessing a fixed common neighborhood resembling disjoint unions of real orthogonal representations, subject to the condition that there is "space on all sides" of the neighborhood. This is functorial in two ways:

- given a summand inclusion $S \hookrightarrow T \sqcup S$, the forgetful map $\mathrm{Conf}^G_{T\sqcup S}(M) \to \mathrm{Conf}^G_S(M)$ preserves basis elements, inducing a map $\mathcal{B}^G_{T\sqcup S}(M) \to \mathcal{B}^G_S(M)$.
- any open embedding $\iota : M \hookrightarrow N$ induces a map $\mathrm{Conf}^G_T(M) \hookrightarrow \mathrm{Conf}^G_T(N)$ preserving basis elements, inducing a map $\mathcal{B}^H_S(M) \to \mathcal{B}^H_S(N)$.

To summarize, we've observed the proof of following lemma.

Lemma 9. *Given $H \subset G$ and $S,T \in \mathbb{F}_H$, there is an equivalence of arrows*

$$\begin{tikzcd} \left|\mathcal{B}^G_{T\sqcup S}(M)\right| \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Conf}^G_{T\sqcup S}(M) \arrow[d] \\ \left|\mathcal{B}^G_S(M)\right| \arrow[r, phantom, "\simeq"] & \mathrm{Conf}^G_S(M) \end{tikzcd}$$

Thus we can characterize the homotopy fiber of U using Quillen's theorem B and the following.

Proposition 10. *For $(X_S,\sigma_S) \le (X'_S,\sigma'_S) \in \mathcal{B}^G_S(M)$, and an S-configuration $\mathbf{x} \in X_S$, we have a diagram*

$$\begin{tikzcd} \mathcal{B}^G_T(M-\mathbf{x}) & \mathcal{B}^G_T(M-\overline{X}_S) \arrow[l, "\varphi"] \arrow[d, phantom, "\wr"] & \mathcal{B}^G_T(M-\overline{X'}_S) \arrow[l] \arrow[ll, bend right, "\varphi"'] \arrow[d, phantom, "\wr"] \\ & ((X_S,\sigma_S)\downarrow U) & \left((X'_S,\sigma'_S)\downarrow U\right) \arrow[l] \end{tikzcd}$$

such that the maps φ induce weak equivalences on classifying spaces.

Proof. The maps φ are each induced by the open inclusions $M - \overline{X}_S \hookrightarrow M - \mathbf{x}$, so the top horizontal arrows commute. The equivalences $\mathcal{B}^G_T(M-X_S) \simeq ((X_S,\sigma_S)\downarrow U)$ simply follow by unwinding definitions. Thus we're left with proving that φ induces an equivalence on classifying spaces

$$\begin{tikzcd} \mathrm{Conf}^G_T(M-\mathbf{x}) \arrow[d, phantom, "\wr"] & \mathrm{Conf}^G_T(M-X_s) \arrow[l] \arrow[d, phantom, "\wr"] \\ \left|\mathcal{B}^G_T(M-\mathbf{x})\right| & \left|\mathcal{B}^G_T(M-X_s)\right| \arrow[l] \end{tikzcd}$$

By Eq. (1), it suffices to prove that the map $\mathrm{Conf}^G_T(V - D(V)) \to \mathrm{Conf}^G_T(V - \{0\})$ is a weak equivalence, which follows by the standard linearl (hence equivariant) deformation retract of each onto a thickening of the sphere $S(V) \subset V$ of points of norm 2. □

We are ready to conclude our equivariant homotopical lift of [FN62, Thm 1].

Proof of Theorem 4. By the above analysis, we may replace our diagram with a homotopy equivalent diagram given by the geometric realiztion of the following diagram of posets, and prove that it is homotopy Cartesian

$$\begin{tikzcd} \mathcal{B}^G_T(M-\iota(S)) \arrow[r] \arrow[d] & \mathcal{B}^G_{T\sqcup S}(M) \arrow[d] \\ \{\iota\} \arrow[r, hook] & \mathcal{B}^G_S(M) \end{tikzcd}$$

By Quillen's theorem B [Qui73, Thm B], it suffices to prove two statements:

- for all basis elements (X_S, σ_S), The canonical map $((X_s, \sigma_s) \downarrow U) \to \mathcal{B}_T^G(M - \iota(S))$ induces a weak equivalence on classifying spaces, and
- for all inclusions of basis elements $(X_S, \sigma_S) \subset (X'_S, \sigma'_S)$, the canonical map $\left((X'_S, \sigma'_S) \downarrow U\right) \to ((X_S, \sigma_S) \downarrow U)$ induces a weak equivalence on classifying spaces.

In fact, both statements follow immediately from Proposition 10, with the second using two-out-of-three. □

1.2. **Proof of the main theorem in topology.** To prove Theorem 6, we begin with a lemmas.

Lemma 11. *If M is a locally smooth G-manifold which is at least d-dimensional and $(d-2)$-connected at each orbit type and $\iota : G/H \hookrightarrow M$ an embedded orbit, then $M - \iota(G/H)$ is at least d-dimensional and $(d-2)$-connected at each orbit type.*

Proof. We have

$$(M - \iota(G/H))_{(K)} = \begin{cases} M_{(K)} & G/K \neq G/H \\ M_{(H)} - \iota(G/H) & G/K = G/h, \end{cases}$$

so the only nontrivial case is $H = K$, in which case we're tasked with verifying that the complement of a discrete set of points in a d-dimensional $(d-2)$-connected manifold is $(d-2)$-connected. This is a well known classical fact in algebraic topology which follows quickly from the Blakers-Massey theorem. □

Proof of Theorem 6. If $d-2 < 0$, there is nothing to prove, so assume that $d - 2 \geq 0$. We induct on $|S_G|$ with base case 1, i.e. with $S = G/H$. In this case, $\mathrm{Conf}^G_{G/H}(M) = M_{(H)}$ is $(d-2)$-connected by assumption.

For induction, fix some $S \sqcup G/H \in \mathbb{F}_G$ and inductively assume the theorem when $|T_G| \leq |S_G|$. Then, note that $\mathrm{Conf}^G_S(M)$ is $(d-2)$-connected by assumption and $M - \iota(S)$ is $\geq d$-dimensional and $(d+2)$-connected at each orbit by Lemma 11, so $\mathrm{Conf}^G_{G/H}(M - \iota(S))$ $(d-2)$-connected by the inductive hypothesis. Thus Theorem 4 expresses $\mathrm{Conf}^G_{S \sqcup G/H}(M)$ as the total space of a homotopy fiber sequence with connected base and fiber, so it is connected. Furthermore, examining the long exact sequence associated with Theorem 4, we find that

$$\begin{tikzcd} 0 \arrow[rr] \arrow[d, phantom, "\wr"] & & \pi_k \mathrm{Conf}^G_{S \sqcup G/H}(M) \arrow[rr] \arrow[dr] & & 0 \arrow[d, phantom, "\wr"] \\ \pi_k \mathrm{Conf}^G_{G/H}(M - \iota(S)) \arrow[urr] & & & & \pi_k \mathrm{Conf}^G_S(M) \end{tikzcd}$$

is exact for $0 < k \leq d - 2$; hence $\mathrm{Conf}^G_{S \sqcup G/H}(M)$ is $(d-2)$-connected, completing the induction. □

1.3. **Some sufficient conditions for connectivity at each orbit.** We begin with the following observation:

Observation 12. If M satisfies the conditions of Proposition 7, then $M_{\leq (H)}$ does as well. ◁

We will strengthen Proposition 7. Pick an order on $\mathrm{Istrp}(M) = (G/H_1, \ldots, G/H_n, G/G)$, and write

$$\begin{aligned} M_k &= M - \bigcup_{i<k} M_{\leq (H_i)} \\ \widetilde{M}_k &= M_{\leq (H_k)} - \bigcup_{i<k} M_{\leq (H_k)} \cap M_{\leq (H_i)} \\ &= M_{\leq (H_k)} - \bigcup_{\substack{(K) \leq (H_k) \cap (H_i) \\ i<k}} M_{\leq (K)} \end{aligned}$$

Lemma 13. *For all k, the space M_k is $(d-2)$-connected.*

Proof. We induct in two ways:

- First, we inductively assume we have proved the lemma at full strength when G is replaced with any proper subgroup $H \subsetneq G$ such that $G/H \in \mathrm{Istrp}(M)$; since $\mathrm{Istrp}(M)$ is finite, this begins with the base case in which case there are no such proper subgroups.
- Second, we inductively assume that we have proved the lemma for all $k' < k$; this begins with the base case that $k = 1$, in which case we have $M_1 = M = M_{\leq (G)}$, which is $(d-2)$-connected by assumption.

Under these assumptions, note that $\widetilde{M}_{k-1} \subset M_{k-1}$ is a $(d-2)$-connected closed submanifold of codimension $\geq d$ in a $(d-2)$-connected smooth manifold with complement is M_k. Thus it possesses a tubular neighborhood $\widetilde{M}_{k-1} \subset \tau(\widetilde{M}_{k-1}) \subset M_{k-1}$, and "hemmed gluing" presents a homotopy pushout square

$$\begin{tikzcd}
\widetilde{M}_{k-1} \arrow[r] \arrow[d, "\tilde{\iota}"] & M_k \arrow[d, "\iota"] \arrow[dl, phantom, "\ulcorner"] \\
\partial\tau(\widetilde{M}_{k-1}) \arrow[r] & M_{k-1}
\end{tikzcd}$$

The boundary $\partial\tau\left(\tilde{M}_{k-1}\right)$ is the total space of a c-sphere bundle over a $(d-2)$-connected space, where

$$c = \operatorname{codim}(M_{\leq(H_k)} \hookrightarrow M) - 1 > d - 2.$$

The long exact sequence in homotopy reads

$$\begin{tikzcd}
\pi_1(S^c) \arrow[r] & \pi_1\left(\partial\tau\left(\widetilde{M}_{k-1}\right)\right) \arrow[r] & \pi_1\left(\widetilde{M}_{k-1}\right) \arrow[r] \arrow[dr] & 0 \arrow[r] & \pi_0\left(\partial\tau\left(\widetilde{M}_{k-1}\right)\right) \arrow[r] & \pi_0\left(\widetilde{M}_{k-1}\right) \arrow[r] & 0 \\
& & & \pi_0(S^c) \arrow[u, phantom, "\wr"] \arrow[ur] & & &
\end{tikzcd}$$

so $\partial\tau\widetilde{M}_{k-1}$ is connected, and when $d-2 \geq 1$, $\partial\tau\widetilde{M}_{k-1}$ is simply connected. Furthermore, at degree $0 < \ell \leq (d-2)$ the Gysin sequence reads

$$\begin{tikzcd}
0 \arrow[r] & H^\ell(\partial\tau\widetilde{M}_{k-1}) \arrow[r] \arrow[dr] & 0 \\
H^\ell(\widetilde{M}_{k-1}) \arrow[u, phantom, "\wr"] \arrow[ur] & & H^{\ell-c}(\widetilde{M}_{k-1}) \arrow[u, phantom, "\wr"]
\end{tikzcd}$$

so $\partial\tau\widetilde{M}_{k-1}$ has vanishing cohomology in degrees $0 < \ell \leq d-2$. Hurewicz' theorem then implies that $\partial\widetilde{M}_{k-1}$ is $(d-2)$-connected when $d - 2 \neq 1$.

In particular, this together with $(d-3)$-connectivity of the homotopy fiber S^c implies that $\tilde{\iota}$ is a $(d-2)$-connected map, so its homotopy pushout ι is $(d-2)$-connected. Since M_{k-1} is a $(d-2)$-connected space by assumption, this implies that M_k is $(d-2)$-connected, completing the induction. □

Proof of Proposition 7. By Observation 12 it suffices to prove that $M_{(G)}$ is $(d-2)$-connected. This is precisely Lemma 13 when $k = n+1$. □

Warning 14. Neither the conditions of Proposition 7 or of Theorem 6 are stable under restrictions; indeed, for $G = C^2$ and $[C_2]$ a C_2-torsor, the example $[C_2] \cdot D^n$ satisfies the conditions of Proposition 7 for $d = n$, but its underlying manifold does not satisfy the conditions of Theorem 6 for any d, as it is not connected. We will rectify this in the setting of real orthogonal G-representation by introducing stronger sufficient conditions which themselves are stable under restriction. ◁

2. Representations, homotopy-coherent algebra, and configuration spaces

In homotopy-coherent algebra, a prominent role is played by the operads $\mathbb{E}_1 = \mathcal{A}_\infty$ and $\mathbb{E}_\infty$, whose algebras are *homotopy-coherently associative algebras* and *homotopy-coherently commutative algebras*, respectively. Dunn's celebrated "additivity theorem" proved non-homotopically [Dun88] (later made homotopical by Lurie [HA, Thm 5.1.2.2]) that an object possessing n-interchanging $\mathbb{E}_1$-structures may equivalently be presented as an algebra over the $\mathbb{E}_n$*-operad*, whose space of k-ary operations is weakly equivalent to the ordered configuration space $\mathrm{Conf}_k(\mathbb{R}^n)$. Thus, after Dunn and Lurie, a higher-categorical version of the Eckmann-Hilton argument may be phrased as stating that $\mathbb{E}_n$*-algebras in* $(n-1)$*-categories canonically lift to* $\mathbb{E}_\infty$*-algebras*; Lurie showed that this is equivalent to the statement that $\mathrm{Conf}_k(\mathbb{R}^n)$ is $(n-2)$-connected for all n,k [HA, Cor 5.1.1.7], which was a half-century old fact of manifold topology due to [FN62].

We would like to lift this to equivariant higher algebra using the equivariant little disks G-operads $\mathbb{E}_V$; these appear in [Hor19], where they are shown to have S-ary operation space

$$\mathbb{E}_V(S) \simeq \mathrm{Conf}_S^H(V)$$

for all $S \in \mathbb{F}_H$. Thus we are compelled to seek a representation theoretic context lifting the assumptions of Proposition 7. We propose the following.

Definition 15. We say V *has d-codimensional fixed points* if $\left|V^H\right|, \left|V^K/V^H\right| \in \{0\} \cup [d,\infty]$ for all $K \subset H \subset G$. ◁

When $G = e$, this is equivalent to simply being d-dimensional.

Proposition 16. *If a real orthogonal G-representation V has d-codimensional fixed points, then the smooth G-manifold $V - \{0\}$ is at least d-dimensional and $(d-2)$-connected at each orbit type.*

Proof. We may write V as a filtered (homotopy) colimit $V = \bigcup_i V_i$ with V_i a finite dimensional real orthogonal G-representation with $\min(i,d)$-codimensional fixed points; then, if V_i is $(i-2)$-connected for each i, taking a colimit, this implies that V is d-connected. Hence it suffices to prove this in the case we that V is finite dimensional.

In this case, G acts smoothly on V, and we make the following observations:

(a) $V_{(H)} = V^H - \bigcup_{K \leq (H)} V^K$ is either empty or $\left|V^H\right| \geq d$-dimensional.
(b) $V_{\leq (H)} = V^H_G$ is contractible, hence it is $(d-2)$-connected.
(c) $\operatorname{codim}(V_{\leq (K)} \hookrightarrow V^*_{\leq (H)}) = \left|V^H\right| - \left|V^K\right| = \left|V^H/V^K\right| \geq d$ by assumption.
(d) $\operatorname{Istrp}(V)$ is finite since V is finite dimensional.

Thus Proposition 7 applies, proving the proposition. □

Corollary 17. *If V has d-codimensional fixed points, then for all closed subgroups $H \subset G$ and finite H-sets $S \in \mathbb{F}_H$, $\operatorname{Conf}^H_S(V)$ is $(d-2)$-connected or empty.*

Proof. We begin by noting

$$\operatorname{Conf}^H_S(V) = \begin{cases} \operatorname{Conf}^H_{S - *_H}(\operatorname{Res}^G_H(V - \{0\})) & S^H \neq \varnothing, \\ \operatorname{Conf}^H_S(\operatorname{Res}^G_H(V - \{0\})) & \text{otherwise.} \end{cases}$$

so it suffices to show $\operatorname{Conf}^H_S(\operatorname{Res}^G_H(V - \{0\}))$ to be $(d-2)$-connected or empty. Noting that the condition of having d-codmimensional fixed points is restriction-stable, this follows by Theorem 6 and Proposition 16. □

Remark 18. Let $G = C_{p^N}$ be the p^Nth cyclic group for some $N \in \mathbb{N} \cup \{\infty\}$. Then, we have

$$\operatorname{Conf}^{C_{p^N}}_{C_{p^N}/C_{p^M}}(V) = V^{C_{p^M}} - V^{C_{p^{M-1}}} \simeq S(V^{C_{p^{M-1}}}) \times S(V^{C_{p^M}}/V^{C_{p^{M-1}}});$$

when V embeds C_{p^N}/C_{p^M}. In particular, this has non-vanishing homotopy group in degrees $\left|V^{C_{p^{M-1}}}\right| - 1$ and $\left|V^{C_{p^M}/C_{p^{M-1}}}\right| - 1$. Thus when $G = C_{p^N}$, if V does not have d-codimensional fixed points, then there exists some $S \in \mathbb{F}_H$ such that $\operatorname{Conf}^H_S(V)$ is neither $(d-2)$-connected nor empty. In particular Corollary 17 is sharp among connectivity bounds using fixed point codimension. ◁

To state a corollary, we define the weak indexing system

$$\mathbb{F}_{AV} = \left\{ S \in \mathbb{F}_H \mid \operatorname{Conf}^H_S(V) \neq \varnothing \right\}.$$

as in [Ste24a; Ste24b]. Our main algebraic corollary is the following.

Theorem 2'. *If V has d-codimensional fixed points and $\mathcal{C}$ is a G-symmetric monoidal $(d-1)$-category, then*

$$U : \operatorname{CAlg}_{AV}(\mathcal{C}) \to \mathbf{Alg}_{\mathbb{E}_V}(\mathcal{C})$$

is an equivalence of $(d-1)$-categories.

Proof. By [Ste24a], this is equivalent to the property that $\mathbb{E}_V$ is a $(d-2)$-connected G-operad, i.e. its nonempty structure spaces are $(d-2)$-connected. By [Hor19], these structure spaces are $\operatorname{Conf}^H_S(V)$, so the result follows from Corollary 17. □

In particular, note that $\left|k \cdot V^H\right| = k\left|V^H\right|$ and $\left|k \cdot V^K / k \cdot V^H\right| = k \cdot \left|V^K/V^H\right|$; hence if V has d-codmensional fixed points, kV has kd-codimensional fixed points. All representations have 1-codimensional fixed points, so dV has d-codimensional fixed points; hence Theorem 2' specializes to Theorem 2.

Remark 19. Theorem 2' is significantly stronger than Theorem 2; indeed, we may choose $G = C_p$, fix a generator $x \in C_p$, and let λ_i denote the irreducible 2-dimensional real orthogonal C_p-representation on whom x acts by rotation at an angle of $\frac{2\pi i}{p}$. Then, when $d < p/2$, the (nontrivial) representation $V = d \oplus \bigoplus_{1=i}^d \lambda_i$ has d-codimensional fixed points, but it contains only one copy of each of its nontrivial summands, so it can't be expressed as a direct sum of two nontrivial representations. ◁

Nevertheless, we specialize the following corollaries to dV for readability. The first yields a simple canonical $\mathrm{RO}(G)$-graded incomplete Tambara structures on $\mathbb{E}_{2V}$-ring spectra, and it follows from Theorem 2 in combination with CHLL.

Corollary 20. *If V is a real orthogonal G-representation with positive-dimensional fixed points and (I, AV) a compatible pair of indexing systems (e.g. I complete), then there are factorizations*

$$\begin{tikzcd}
 & \mathrm{Tamb}_{I,AV}\left(\mathbf{Ab}^{\mathrm{RO}(G)}\right) \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{Tamb}_{I,AV}\left(\mathbf{Ab}^{\mathbb{Z}}\right) \arrow[d, phantom, "\wr"] \\
 & \mathrm{CAlg}_{AV}\left(\mathbf{Ab}^{\mathrm{RO}(G)}\right) \arrow[r] \arrow[d, "U"] & \mathrm{CAlg}_{AV}\left(\mathbf{Ab}^{\mathbb{Z}}\right) \arrow[d, "U"'] \\
\mathbf{Alg}_{\mathbb{E}_{2V}}\left(\underline{\mathrm{Sp}}_G\right) \arrow[uur, bend left] \arrow[ur, bend left] \arrow[r, "\underline{\pi}_\star"] \arrow[rr, bend right, "\underline{\pi}_*"] & \mathrm{Mack}_I(\mathbf{Ab})^{\mathrm{RO}(G)} \arrow[r] & \mathrm{Mack}_I(\mathbf{Ab})^{\mathbb{Z}}
\end{tikzcd}$$

i.e. the stable homotopy groups of an $\mathbb{E}_{2V}$-ring spectrum have natural AV-Tambara structures.

Finally, we acquire incomplete Mackey structures on $\mathbb{E}_{(n+2)V}$-monoidal n-categories.

Corollary 21. *$\mathbb{E}_{(n+2)V}$-monoidal n-categories canonically lift to AV-symmetric monoidal n categories, i.e.*

$$U : \mathbf{Cat}^{\otimes}_{AV,n} \to \mathbf{Cat}^{\otimes}_{\mathbb{E}_{(n+2)V},n}$$

is an equivalence of $(n+1)$-categories. In particular, when $V = \rho$, the forgetful functor

$$U : \mathbf{Cat}^{\otimes}_{G,n} \to \mathbf{Cat}^{\otimes}_{\mathbb{E}_{(n+2)\rho},n}$$

is an equivalence of 2-categories.

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