ON THE JAMES AND HILTON–MILNOR SPLITTINGS, & THE METASTABLE EHP SEQUENCE

SANATH DEVALAPURKAR AND PETER HAINE

Abstract. This note provides modern proofs of some classical results in algebraic topology, such as the James Splitting, the Hilton–Milnor Splitting, and the metastable EHP sequence. We prove the James and Hilton–Milnor Splittings in the maximal generality of any ∞-category with finite limits and pushouts in which pushouts squares remain pushouts after basechange along an arbitrary morphism (i.e., Mather's Second Cube Lemma holds). Working in this generality shows that the James and Hilton–Milnor splittings hold in profinite spaces and motivic spaces over arbitrary base schemes. The latter extends results of Wickelgren and Williams, who prove the James Splitting over a perfect field. We also give two proofs of the metastable EHP sequence in the setting of ∞-topoi: the first is a new, non-computational proof that only utilizes basic connectedness estimates involving the James filtration and the Blakers–Massey Theorem, while the second reduces to the classical computational proof.

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1. Introduction

A classical result of James shows that given a pointed space X, the homotopy type $\Sigma\Omega\Sigma X$ given by suspending the loopspace on the suspension of X splits as a wedge sum

$$\Sigma\Omega\Sigma X \simeq \bigvee_{i\geq 1} \Sigma X^{\wedge i}$$

Date: April 20, 2020.

of suspensions of smash powers of X [6; 18]. Hilton and Milnor proved a related splitting result [14; 15; 23, Theorem 3]: given pointed spaces X and Y, they showed that there is a homotopy equivalence

$$\Omega\Sigma(X \vee Y) \simeq \Omega\Sigma X \times \Omega\Sigma Y \times \Omega\left(\bigvee_{i,j\geq 1} \Sigma X^{\wedge i} \wedge Y^{\wedge j}\right) .$$

The first objective of this note is to provide clear, modern, and non-computational proofs of the James and Hilton–Milnor Splittings. The only property particular to the ∞-category of spaces that our proofs utilize is *Mather's Second Cube Lemma* [21], which asserts that pushout squares remain pushouts after basechange along an arbitrary morphism (see § 2.1). Hence the James and Hilton–Milnor Splittings hold in any ∞-category where we can make sense of suspensions, loops, wedge sums, and smash products, and have access to Mather's Second Cube Lemma:

Theorem 1.1 (James Splitting; Theorem 2.9 and Corollary 2.10)**.** *Let $\mathcal{X}$ be an ∞-category with finite limits and pushouts, and assume that Mather's Second Cube Lemma holds in $\mathcal{X}$. Then for every pointed object $X \in \mathcal{X}_*$, there is a natural equivalence*

$$\Sigma\Omega\Sigma X \simeq \Sigma X \vee \Sigma(X \wedge \Omega\Sigma X) .$$

If $\mathcal{X}_*$ has countable coproducts, then there is a natural equivalence

$$\Sigma\Omega\Sigma X \simeq \bigvee_{i\geq 1} \Sigma X^{\wedge i} .$$

Theorem 1.2 (Hilton–Milnor Splitting; Theorem 3.1 and Corollary 3.2)**.** *Let $\mathcal{X}$ be an ∞-category with finite limits and pushouts, and assume that Mather's Second Cube Lemma holds in $\mathcal{X}$. Then for every pair of pointed objects $X, Y \in \mathcal{X}_*$ there is an equivalence*

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \wedge \Omega Y) .$$

If $\mathcal{X}_*$ has countable coproducts, then there is an equivalence

$$\Omega\Sigma(X \vee Y) \simeq \Omega\Sigma X \times \Omega\Sigma Y \times \Omega\left(\bigvee_{i,j\geq 1} \Sigma X^{\wedge i} \wedge Y^{\wedge j}\right) .$$

It might seem that knowing that the James and Hilton–Milnor Splittings hold in this level of generality is of dubious advantage; the settings in which one is most likely to want to apply these splittings are the ∞-category **Spc** of spaces (where the results are already known), or an ∞-topos (where the results follows immediately from the results for **Spc**). However, algebraic geometry provides an example that does not immediately follow from the result for spaces: motivic spaces. The obstruction is that the ∞-category of motivic spaces over a scheme is not an ∞-topos; since motivic localization almost never commutes with taking loops, knowing the James and Hilton–Milnor Splittings in the ∞-topos of Nisnevich sheaves does not allow one to deduce that they hold in motivic spaces.

Wickelgren and Williams prove [34, Theorem 1.5] that the James Splitting holds for motivic spaces over a perfect field. The reason for the restriction on the base is because their proof relies on Morel's unstable $\mathbf{A}^1$-connectivity Theorem [24, Theorems 5.46 and 6.1], which implies that motivic localization commutes with loops [2, Theorem 2.4.1; 24, Theorem 6.46] over perfect fields. However, the unstable $\mathbf{A}^1$-connectivity property does not hold for higher-dimensional bases [2, Remark 3.3.5; 3], so a different method is needed if one wants to prove James and Hilton–Milnor Splittings for motivic spaces over more general bases. This is where our generalization pays off: work of Hoyois [17, Proposition 3.15] shows that, in particular, Mather's Second Cube Lemma holds in motivic spaces over an *arbitrary* base scheme. Therefore, Theorems 1.1 and 1.2 apply in this setting (see Examples 2.11 and 3.3).

The second goal of this note is to give a modern construction of the metastable EHP sequence in an ∞-topos $\mathcal{X}$. For every pointed object $X \in \mathcal{X}_*$, the James Splitting provides *Hopf maps*

$$h_n \colon \Omega\Sigma X \to \Omega\Sigma(X^{\wedge n}) \ .$$

Provided that X is connected, there is also a *James filtration* $\{\mathrm{J}_m(X)\}_{m \geq 0}$ on $\Omega\Sigma X$, and, moreover, the composite

$$\mathrm{J}_{n-1}(X) \longrightarrow \Omega\Sigma X \xrightarrow{h_n} \Omega\Sigma X^{\wedge n} \tag{1.3}$$

is trivial. The sequence (1.3) is not a fiber sequence in general[1], but is in the *metastable range*:

Theorem 1.4 (metastable EHP sequence; Theorem 4.32)**.** *Let $\mathcal{X}$ be an ∞-topos, $k \geq 0$ an integer, and $X \in \mathcal{X}_*$ a pointed k-connected object. Then for every integer $n \geq 1$, the morphism $\mathrm{J}_{n-1}(X) \to \mathrm{fib}(h_n)$ is $((n+1)(k+1)-3)$-connected.*

We note here that a morphism is m-connected in our terminology if and only if it is $(m+1)$-connected in the classical terminology (see Warning 4.7).

We provide two proofs of Theorem 1.4. The first proof is new and non-computational; it only makes use of some basic connectedness estimates involving the James filtration and the Blakers–Massey Theorem. In the second proof we simply note that Theorem 1.4 for a general ∞-topos follows immediately from the claim for the ∞-topos of spaces. In the case of spaces, we provide a computational proof; we include this second proof because we found it surprisingly difficult to find the computational proof we were familiar with in the literature.

1.1. **Linear overview.** We have written this note with two audiences in mind: the student interested in seeing proofs of Theorems 1.1, 1.2 and 1.4 in the classical setting of spaces, and the expert homotopy theorist interested in applying these results to more general contexts such as motivic spaces or profinite spaces. The student can always take $\mathcal{X}$ to be the ∞-category of spaces, and the expert can safely skip the background sections provided for the student. We also note that this text should still be accessible to the reader familiar with homotopy (co)limits but unfamiliar with higher categories, since all we use in our proofs are basic manipulations of homotopy (co)limits.

Section 2 is dedicated to proving Theorem 1.1. In § 2.1, we provide background on Mather's Second Cube Lemma and the universality of pushouts. In § 2.2, we provide a proof of the James Splitting. Our proof is roughly the same as proofs presented elsewhere [16; 30, §17.2; 35], but it seems that the generality of the argument we present here is not very well-known.

Section 3 provides a quick proof of Theorem 1.2. Again, shadows of the proof we provide appear in the literature [10; 11, §2 & 3; 30, §17.8], but it seems that the generality of the proof has not been completely internalized by the community. Using work of Wickelgren [33, Corollary 3.2], we also give an application to describe the motivic space $\Omega\Sigma(\mathbf{P}^1 \smallsetminus \{0,1,\infty\})$ in terms of smash powers of $\mathbf{G}_m$ with $\mathbf{P}^1$ (Example 3.3).

Section 4 is dedicated to proving Theorem 1.4. In § 4.1, we begin by recalling the basics of connectedness and the Blakers–Massey Theorem in an ∞-topos. In § 4.2, we provide the background on the James construction needed to understand the statement of Theorem 1.4, as well as some connectedness estimates we need to prove Theorem 1.4. In § 4.3, we give a refinement of the James Splitting in terms of the James filtration. In § 4.4, we first provide a proof of Theorem 1.4 using the Blakers–Massey Theorem (which we have not seen elsewhere), and then record for posterity what we imagine is the standard computational proof of Theorem 1.4.

[1] When $\mathcal{X}$ is the ∞-category of spaces and X is a sphere, James and Toda proved that, roughly, the sequence (1.3) becomes a fiber sequence after *p-localization*. See [19; 20; 31] for a precise statement.

Acknowledgements. We are grateful to Tom Bachmann for pointing out that colimits are universal in motivic spaces. We are indebted to André Joyal for pointing out some errors in an earlier version as well as alerting us to our misuse of terminology regarding connectivity. The second-named author gratefully acknowledges support from both the MIT Dean of Science Fellowship and the National Science Foundation Graduate Research Fellowship under Grant #112237.

1.2. **Notation & background.** In this subsection we set the basic notational conventions that we use throughout this note as well as recall a bit of relevant background.

Notation 1.5. Let $\mathcal{X}$ be an ∞-category. If $\mathcal{X}$ has a terminal object, we write $* \in \mathcal{X}$ for the terminal object and $\mathcal{X}_*$ for the ∞-category of pointed objects in $\mathcal{X}$. If $\mathcal{X}_*$ has coproducts and $X, Y \in \mathcal{X}_*$, we write $X \vee Y$ for the coproduct of X and Y in $\mathcal{X}_*$. If $\mathcal{X}_*$ has coproducts and products, note that there is a natural comparison morphism $X \vee Y \to X \times Y$ induced by the morphisms

$$(\mathrm{id}_X, *) \colon X \to X \times Y \qquad \text{and} \qquad (*, \mathrm{id}_Y) \colon Y \to X \times Y \ .$$

We say that a morphism $f \colon X \to Y$ in $\mathcal{X}_*$ is *null* if f factors through the zero object $*$.

Recollection 1.6. Let $\mathcal{X}$ be an ∞-category with finite products and pushouts, and $X, Y \in \mathcal{X}_*$ pointed objects of $\mathcal{X}$. The *smash product* $X \wedge Y$ of X and Y is the cofiber

$$\begin{tikzcd} X \vee Y \arrow[r] \arrow[d] & X \times Y \arrow[d] \\ * \arrow[r] & X \wedge Y \end{tikzcd}$$

of the comparison morphism $X \vee Y \to X \times Y$.

Recollection 1.7. Let $\mathcal{X}$ be an ∞-category with pushouts and a terminal object. The *suspension* of an object $X \in \mathcal{X}$ is the pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd} .$$

Recollection 1.8. Let $\mathcal{X}$ be an ∞-category with finite limits. The *loop object* of a pointed object $X \in \mathcal{X}_*$ is the pullback

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

in $\mathcal{X}_*$.

We also make repeated use of the following easy fact. The unfamiliar reader should consult [4, §2; 26].

Lemma 1.9. *Let $\mathcal{X}$ be an ∞-category with pushouts and*

$$\begin{tikzcd} X_1 & X_0 \arrow[l] \arrow[r] & X_2 \\ W_1 \arrow[u] \arrow[d] & W_0 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & W_2 \arrow[u] \arrow[d] \\ Y_1 & Y_0 \arrow[l] \arrow[r] & Y_2 \end{tikzcd} \tag{1.10}$$

a commutative diagram in $\mathcal{X}$. Then the colimit of the diagram (1.10) *exists and is equivalent to both of the following two iterated pushouts:*

(1.9.1) *Form the pushout of the rows of* (1.10)*, then take the pushout of the resulting span*

$$X_1 \overset{X_0}{\sqcup} X_2 \longleftarrow W_1 \overset{W_0}{\sqcup} W_2 \longrightarrow Y_1 \overset{Y_0}{\sqcup} Y_2 \ .$$

(1.9.2) *Form the pushout of the columns of* (1.10)*, then take the pushout of the resulting span*

$$X_1 \overset{W_1}{\sqcup} Y_1 \longleftarrow X_0 \overset{W_0}{\sqcup} Y_0 \longrightarrow X_2 \overset{W_2}{\sqcup} Y_2 \ .$$

2. The James Splitting

In this section, we present a proof of the James Splitting which holds in any ∞-category with finite limits and pushouts, where pushout squares remain pushouts after basechange along an arbitrary morphism. The argument we give roughly follows the argument Hopkins' gave in his course on stable homotopy theory in the setting of spaces [16, Lecture 4, §3]; Hopkins attributes this proof to James [18; 19; 20] and Ganea [9].

2.1. **Universal pushouts and Mather's Second Cube Lemma.** The key property utilized in the proofs we present of the James and Hilton–Milnor Splittings is that pushout squares are preserved by arbitrary basechange. This implies that, in particular, the James and Hilton–Milnor Splittings hold in any ∞-topos, but also in other situations (such as motivic spaces). In this subsection, we provide the categorical context that we work in for the rest of the paper and give a convenient reformulation of the stability of pullbacks under basechange in terms of Mather's Second Cube Lemma (Lemma 2.5).

Recollection 2.1. Let $\mathcal{I}$ be an ∞-category and let $\mathcal{X}$ be an ∞-category with pullbacks and all $\mathcal{I}$-shaped colimits. We say that *$\mathcal{I}$-shaped colimits in $\mathcal{X}$ are universal* if $\mathcal{I}$-shaped colimits in $\mathcal{X}$ are stable under pullback along any morphism. That is, for every diagram $F\colon \mathcal{I} \to \mathcal{X}$ and pair of morphisms $\operatorname{colim}_{i\in\mathcal{I}} F(i) \to Z$ and $Y \to Z$ in $\mathcal{X}$, the natural morphism

$$\operatorname*{colim}_{i\in\mathcal{I}}(F(i) \times_Z Y) \to \Big(\operatorname*{colim}_{i\in\mathcal{I}} F(i)\Big) \times_Z Y$$

is an equivalence.

Example 2.2. Let $0 \leq n \leq \infty$, and let $\mathcal{X}$ be an n-topos. One of the Giraud–Lurie axioms for n-topoi guarentees that all small colimits in $\mathcal{X}$ are universal [HTT, Theorem 6.1.0.6 & Proposition 6.4.1.5]. In particular, small colimits in the category **Set** of sets and the ∞-category **Spc** of spaces are universal.

Example 2.3 (motivic spaces)**.** Let S be a scheme. The ∞-category $\mathbf{H}(S)$ of *motivic spaces over S* is defined as the $\mathbf{A}^1$-localization of the ∞-topos $\mathrm{Sh}_{\mathrm{nis}}(\mathbf{Sm}_S)$ of sheaves of spaces on the category $\mathbf{Sm}_S$ of smooth schemes of finite type over S equipped with the Nisnevich topology. Concretely, $\mathbf{H}(S)$ is the full subcategory of $\mathrm{Sh}_{\mathrm{nis}}(\mathbf{Sm}_S)$ spanned by those Nisnevich sheaves $\mathcal{F}$ on $\mathbf{Sm}_S$ with the property that for every smooth S-scheme X, the projection $\mathrm{pr}_1\colon X \times_S \mathbf{A}^1_S \to X$ induces an equivalence

$$\mathrm{pr}_1^*\colon \mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}(X \times_S \mathbf{A}^1_S) \ .$$

The inclusion $\mathbf{H}(S) \subset \mathrm{Sh}_{\mathrm{nis}}(\mathbf{Sm}_S)$ admits a left adjoint $\mathrm{L}_{\mathrm{mot}}\colon \mathrm{Sh}_{\mathrm{nis}}(\mathbf{Sm}_S) \to \mathbf{H}(S)$ called *motivic localization*. Motivic localization preserves finite products, but not all finite limits. Moreover, the ∞-category $\mathbf{H}(S)$ is not an ∞-topos (see [29, Remark 3.5; 27, §4.3]), and it is not immediately clear from the construction if any colimits are universal in $\mathbf{H}(S)$. Nonetheless, Hoyois has shown that all small colimits are universal in $\mathbf{H}(S)$ [17, Proposition 3.15].

Example 2.4 (profinite spaces)**.** We say that a space X is *π-finite* if X is truncated, has finitely many connected components, and $\pi_i(X,x)$ is finite for every integer $i \geq 1$ and point $x \in X$. Write $\mathbf{Spc}_\pi \subset \mathbf{Spc}$ for the full subcategory spanned by the π-finite spaces and $\mathrm{Pro}(\mathbf{Spc}_\pi)$ for the ∞-category of *profinite spaces*. Infinite coproducts in $\mathrm{Pro}(\mathbf{Spc}_\pi)$ are not universal (see [SAG, Warning E.6.0.9]), however, finite colimits and geometric realizations of simplicial objects are universal in $\mathrm{Pro}(\mathbf{Spc}_\pi)$ [SAG, Theorem E.6.3.1 & Corollary E.6.3.2].

The following result gives a reformulation of what it means for pushouts to be universal in terms of Mather's Second Cube Lemma, which Mather originally proved in the ∞-category of spaces [21, Theorem 25].

Lemma 2.5. *Let $\mathcal{X}$ be an ∞-category with pullbacks and pushouts. The following conditions are equivalent:*

(2.5.1) *Pushouts in $\mathcal{X}$ are universal.*

(2.5.2) Mather's Second Cube Lemma *holds in $\mathcal{X}$: Given a commutative cube*

$$\begin{tikzcd}
A_0 \arrow[rr] \arrow[dd] \arrow[dr] & & A_2 \arrow[dd] \arrow[dr] & \\
& A_1 \arrow[rr] \arrow[dd] & & A_3 \arrow[dd] \\
B_0 \arrow[rr] \arrow[dr] & & B_2 \arrow[dr] & \\
& B_1 \arrow[rr] & & B_3
\end{tikzcd}$$

in $\mathcal{X}$ where the bottom horizontal face is a pushout square and all vertical faces are pullback squares, the top horizontal square is a pushout square.

Proof. The implication (2.5.1) $\Rightarrow$ (2.5.2) is immediate. To see that (2.5.2) $\Rightarrow$ (2.5.1), suppose that we are given a pushout square

$$\begin{tikzcd}
B_0 \arrow[r] \arrow[d] & B_2 \arrow[d] \\
B_1 \arrow[r] & B_3 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{2.6}$$

in $\mathcal{X}$ and morphisms $f\colon B_3 \to Z$ and $g\colon Y \to Z$ in $\mathcal{X}$. For each $i \in \{0,1,2,3\}$, define $A_i \coloneq B_i \times_Z Y$, so that all the vertical squares in the diagram

$$\begin{tikzcd}
A_0 \arrow[rr] \arrow[dd] \arrow[dr] & & A_2 \arrow[dd] \arrow[dr] & & \\
& A_1 \arrow[rr] \arrow[dd] & & A_3 \arrow[dd] \arrow[r] & Y \arrow[dd] \\
B_0 \arrow[rr] \arrow[dr] & & B_2 \arrow[dr] & & \\
& B_1 \arrow[rr] & & B_3 \arrow[r] & Z
\end{tikzcd} \tag{2.7}$$

are pullbacks. Since the bottom horizontal square of the cube in (2.7) is a pushout, (2.5.2) implies that the top horizontal square is also a pushout. Thus the pushout square (2.6) remains a pushout after base change along an arbitrary morphism, as desired. □

Since the main results of this note are about pointed objects, we make the following mildly abusive convention:

Convention 2.8. We say that an ∞-category $\mathcal{X}$ *has universal pushouts* if $\mathcal{X}$ has finite limits and pushouts, and pushouts in $\mathcal{X}$ are universal.

2.2. **The James Splitting.** The James Splitting, originally proven in [18], provides a splitting of the space $\Omega\Sigma X$ after a single suspension. The goal of this subsection is to provide a proof of the James Splitting that only relies on the universality of pushouts and a few elementary computations involving the interaction between forming suspensions, loop objects, and smash products. The James Splitting gives us access to generalized Hopf invariants in this very general setting, and implies the stable Snaith splitting for $\Omega\Sigma X$ [28].

Theorem 2.9 (James Splitting)**.** *Let $\mathcal{X}$ be an ∞-category with universal pushouts. For every pointed object $X \in \mathcal{X}_*$, there is a natural equivalence*

$$\Sigma\Omega\Sigma X \simeq \Sigma X \vee \Sigma(X \wedge \Omega\Sigma X) .$$

Using the fact that $\Sigma(X \wedge \Omega\Sigma X) \simeq X \wedge \Sigma\Omega\Sigma X$ (Lemma 2.23) and iterating the equivalence of Theorem 2.9 yields the following:

Corollary 2.10 (James Splitting, redux)**.** *Let $\mathcal{X}$ be an ∞-category with universal pushouts. If $\mathcal{X}_*$ has countable coproducts, then for any pointed object $X \in \mathcal{X}_*$ there is a natural equivalence*

$$\Sigma\Omega\Sigma X \simeq \bigvee_{i \geq 1} \Sigma X^{\wedge i} .$$

Example 2.11. Let S be a scheme. Since colimits are universal in the ∞-category $\mathbf{H}(S)$ of motivic spaces over S (Example 2.3), Theorem 2.9 and Corollary 2.10 imply that for any pointed motivic space $X \in \mathbf{H}(S)_*$ we have S^1-James Splittings

$$\Sigma\Omega\Sigma X \simeq \Sigma X \vee \Sigma(X \wedge \Omega\Sigma X) \qquad \text{and} \qquad \Sigma\Omega\Sigma X \simeq \bigvee_{i \geq 1} \Sigma X^{\wedge i} .$$

The right-hand splitting $\Sigma\Omega\Sigma X \simeq \bigvee_{i\geq 1} \Sigma X^{\wedge i}$ generalizes the base of the motivic James Splitting of Wickelgren and Williams [34, Theorem 1.5] from a perfect field to an arbitrary scheme.

The James Splitting gives rise to the *Hopf maps* that appear in the metastable EHP sequence (see § 4).

Construction 2.12. Let $\mathcal{X}$ be an ∞-category with pushouts and X a pointed object of $\mathcal{X}$. For each integer $n \geq 1$, we define the *Hopf map* $h_n \colon \Omega\Sigma X \to \Omega\Sigma X^{\wedge n}$ as the adjoint to the collapse map

$$\Sigma\Omega\Sigma X \simeq \bigvee_{i \geq 1} \Sigma X^{\wedge i} \to \Sigma X^{\wedge n}$$

induced by the James Splitting of Corollary 2.10.

Remark 2.13. There is another suspension in motivic homotopy theory, given by smashing with the multiplicative group scheme $\mathbf{G}_m$. One would like an analogue of Corollary 2.10 in $\mathbf{H}(S)$ for $\mathbf{G}_m$-suspensions. For $S = \mathrm{Spec}(\mathbf{R})$, Betti realization defines a functor

$$\mathbf{H}(\mathrm{Spec}(\mathbf{R})) \to \mathbf{Spc}_{\mathrm{C}_2}$$

to C_2-spaces which sends S^1 to the circle with trivial C_2-action, and $\mathbf{G}_m$ to the sign representation circle S^σ. Even though Betti realization is not an equivalence, it closely ties $\mathbf{R}$-motivic homotopy theory with C_2-equivariant homotopy theory. In [13], Hill studies the *signed James construction* in C_2-equivariant unstable homotopy theory, and shows that an analogue of Corollary 2.10 holds for $\Omega^\sigma\Sigma^\sigma X$ after suspending by the regular representation sphere $\mathrm{S}^\rho = \mathrm{S}^1 \wedge \mathrm{S}^\sigma$. This might lead one to hope that there is an analogue of Hill's result in motivic homotopy theory which proves the James Splitting for $\Omega_{\mathbf{G}_m}\Sigma_{\mathbf{G}_m} X$ after $\mathbf{P}^1$-suspension; at the moment, we are not aware of such a result.

Before we prove Theorem 2.9, we need a few preliminary results. First, we give a more convenient expression for $\Sigma\Omega\Sigma X$ as the cofiber of the projection $\mathrm{pr}_2\colon X \times \Omega\Sigma X \to X$. This expression for $\Sigma\Omega\Sigma X$ is an immediate consequence of the following:

Lemma 2.14. *Let $\mathcal{X}$ be an ∞-category with universal pushouts. For every pointed object $X \in \mathcal{X}_*$, the square*

$$\begin{tikzcd} X \times \Omega\Sigma X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_2"'] & \Omega\Sigma X \arrow[d] \\ \Omega\Sigma X \arrow[r] & * \end{tikzcd}$$

is a pushout square.

Proof. Apply Mather's Second Cube Lemma to the cube

$$\begin{tikzcd} X \times \Omega\Sigma X \arrow[rr, "\mathrm{pr}_2"] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd] & & \Omega\Sigma X \arrow[dr] \arrow[dd] & \\ & \Omega\Sigma X \arrow[rr] \arrow[dd] & & * \arrow[dd] \\ X \arrow[rr] \arrow[dr] & & * \arrow[dr] & \\ & * \arrow[rr] & & \Sigma X . \end{tikzcd} \tag{2.15}$$

Note that the bottom face of (2.15) is a pushout by the definition of the suspension ΣX, and the vertical faces are pullbacks by the definition of the object ΩX and the fact that $*$ is the terminal object of $\mathcal{X}$. $\square$

Corollary 2.16. *Let $\mathcal{X}$ be an ∞-category with universal pushouts. For every pointed object $X \in \mathcal{X}_*$, there is a natural equivalence*

$$\mathrm{cofib}(\mathrm{pr}_2\colon X \times \Omega\Sigma X \to \Omega\Sigma X) \simeq \Sigma\Omega\Sigma X \ .$$

Next, we give a convenient expression for the term $\Sigma(X \wedge \Omega\Sigma X)$ in the James Splitting as the pushout of the span

$$X \longleftarrow X \times \Omega\Sigma X \longrightarrow \Omega\Sigma X \ .$$

Our proof of this appeals to the following fact, which follows immediately from the definitions.

Lemma 2.17. *Let $\mathcal{X}$ be an ∞-category with pushouts and a terminal object, and let $X, Y \in \mathcal{X}_*$ be pointed objects of $\mathcal{X}$. Then the square*

$$\begin{tikzcd} X \vee Y \arrow[r, "(*, \mathrm{id}_Y)"] \arrow[d, "(\mathrm{id}_X, *)"'] & Y \arrow[d] \\ X \arrow[r] & * \end{tikzcd}$$

is a pushout square.

Proposition 2.18. *Let $\mathcal{X}$ be an ∞-category with finite limits and pushouts. Then for every pair of pointed objects $X, Y \in \mathcal{X}$, there is a pushout square*

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & Y \arrow[d] \\ X \arrow[r] & \Sigma(X \wedge Y) \ , \end{tikzcd}$$

where the morphisms $X \to \Sigma(X \wedge Y)$ and $Y \to \Sigma(X \wedge Y)$ are null.

Proof. Let C denote the pushout $X \sqcup^{X \times Y} Y$; we desire to show that $C \simeq \Sigma(X \wedge Y)$. We apply Lemma 1.9 to the commutative diagram

$$\begin{tikzcd} * & \arrow[l] * \arrow[r] & * \\ X \arrow[u] \arrow[d, equal] & \arrow[l] X \vee Y \arrow[r] \arrow[u] \arrow[d] & Y \arrow[u] \arrow[d, equal] \\ X & \arrow[l] X \times Y \arrow[r] & Y\ . \end{tikzcd} \tag{2.19}$$

Appealing to Lemma 2.17, taking pushouts of the rows of (2.19) results in the span

$$C \longleftarrow * \longrightarrow * \ ,$$

which has pushout C. Alternatively, since the smash product $X \wedge Y$ is the cofiber of the comparison morphism $X \vee Y \to X \times Y$, taking pushouts of the columns of (2.19) results in the span

$$* \longleftarrow X \wedge Y \longrightarrow * \ . \tag{2.20}$$

By definition, the pushout of the span (2.20) is the suspension $\Sigma(X \wedge Y)$, so Lemma 1.9 shows that

$$C \simeq \Sigma(X \wedge Y)\ .$$

To conclude the proof, note that it follows from the definitions that the induced morphisms

$$X \to \Sigma(X \wedge Y) \qquad \text{and} \qquad Y \to \Sigma(X \wedge Y)$$

factor through the zero object $* \in \mathcal{X}_*$. □

Proposition 2.18 also provides a general formula for the cofiber $\operatorname{cofib}(\mathrm{pr}_2 \colon X \times Y \to Y)$ that allows us to relate the expressions for $\Sigma\Omega\Sigma X$ and $\Sigma(X \wedge \Omega\Sigma X)$ from Corollary 2.16 and Proposition 2.18, respectively.

Corollary 2.21. *Let $\mathcal{X}$ be an ∞-category with finite limits and pushouts. Then, for every pair of pointed objects $X, Y \in \mathcal{X}_*$:*

(2.21.1) *There is a natural equivalence* $\operatorname{cofib}(\mathrm{pr}_2 \colon X \times Y \to Y) \simeq \Sigma X \vee \Sigma(X \wedge Y)$.
(2.21.2) *There a natural equivalence* $\Sigma(X \times Y) \simeq \Sigma(X \wedge Y) \vee \Sigma X \vee \Sigma Y$.

Proof. Consider the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & Y \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] \arrow[d] & \Sigma(X \wedge Y) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r] \arrow[d] & \Sigma Y \vee \Sigma(X \wedge Y) \arrow[d] \\ * \arrow[r] & \Sigma X \vee \Sigma(X \wedge Y) \arrow[r, tail] & \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)\ , \end{tikzcd} \tag{2.22}$$

where the top-left square is the pushout square of Proposition 2.18. Since the maps $X \to \Sigma(X \wedge Y)$ and $Y \to \Sigma(X \wedge Y)$ are null, the diagram (2.22) commutes and the bottom-left and top-right squares of (2.22) are pushout squares. This proves (2.21.1). To prove (2.21.2), note that the bottom-right square in the diagram (2.22) is a pushout. □

Corollaries 2.16 and 2.21 now combine to give the James Splitting.

Proof of Theorem 2.9. Combining Corollary 2.16 with Corollary 2.21 in the case that $Y = \Omega\Sigma X$ we see that there are natural equivalences

$$\begin{aligned} \Sigma\Omega\Sigma X &\simeq \mathrm{cofib}(\mathrm{pr}_2 \colon X \times \Omega\Sigma X \to \Omega\Sigma X) \\ &\simeq \Sigma X \vee \Sigma(X \wedge \Omega\Sigma X) \ . \end{aligned}$$

□

The splitting $\Sigma\Omega\Sigma X \simeq \bigvee_{i\geq 1} \Sigma X^{\wedge i}$ of Corollary 2.10 is immediate from Theorem 2.9 combined with the following elementary fact:

Lemma 2.23. *Let $\mathcal{X}$ be an ∞-category with universal pushouts. For every pair of pointed objects $X, Y \in \mathcal{X}_*$, there is a natural equivalence*

$$\Sigma(X \wedge Y) \simeq X \wedge \Sigma Y \ .$$

Proof. Since pushouts in $\mathcal{X}$ are universal and colimits commute, the squares

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{id}_X \times *"] \arrow[d, "\mathrm{id}_X \times *"'] & X \times * \arrow[d, "\mathrm{id}_X \times *"] \\ X \times * \arrow[r, "\mathrm{id}_X \times *"'] & X \times \Sigma Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \vee Y \arrow[r, "\mathrm{id}_X \vee *"] \arrow[d, "\mathrm{id}_X \vee *"'] & X \vee * \arrow[d, "\mathrm{id}_X \vee *"] \\ X \vee * \arrow[r, "\mathrm{id}_X \vee *"'] & X \vee \Sigma Y \end{tikzcd}$$

are both pushouts in $\mathcal{X}_*$. By the definition of the smash product and the facts that colimits commute and $X \wedge * \simeq *$, we see that

$$\begin{aligned} X \wedge \Sigma Y &= \mathrm{cofib}(X \vee \Sigma Y \to X \times \Sigma Y) \\ &\simeq \mathrm{cofib}\left((X \vee *) \overset{X \vee Y}{\sqcup} (X \vee *) \to (X \times *) \overset{X \times Y}{\sqcup} (X \times *) \right) \\ &\simeq (X \wedge *) \overset{X \wedge Y}{\sqcup} (X \wedge *) \\ &\simeq \Sigma(X \wedge Y) \ . \end{aligned}$$

□

2.3. **Ganea's Lemma.** Since the method of proof is similar to the arguments in this section, we close with the following lemma of Ganea [8, Theorem 1.1]. This will not be used in the sequel.

Lemma 2.24. *Let $\mathcal{X}$ be an ∞-category with universal pushouts. Let $f \colon X \to Y$ be a morphism in $\mathcal{X}_*$, and write $i \colon \mathrm{fib}(f) \to X$ for the induced morphism from the fiber of f. Then there is a natural equivalence*

$$\mathrm{fib}(\mathrm{cofib}(i) \to Y) \simeq \Sigma(\Omega Y \wedge \mathrm{fib}(f)) \ .$$

Proof. By Proposition 2.18, it suffices to show that the square

$$\begin{tikzcd} \Omega Y \times \mathrm{fib}(f) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \mathrm{fib}(f) \arrow[d] \\ \Omega Y \arrow[r] & \mathrm{fib}(\mathrm{cofib}(g) \to Y) \end{tikzcd}$$

is a pushout. Consider the diagram

$$
\begin{tikzcd}
\Omega Y \times \mathrm{fib}(f) \arrow[rr, "\mathrm{pr}_2"] \arrow[dr, "\mathrm{pr}_1"] \arrow[dd, "\mathrm{pr}_2"'] & & \mathrm{fib}(f) \arrow[dr] \arrow[dd] & & \\
& \Omega Y \arrow[rr] \arrow[dd] & & \mathrm{fib}(\mathrm{cofib}(g) \to Y) \arrow[r] \arrow[dd] & * \arrow[dd] \\
\mathrm{fib}(f) \arrow[rr, "i"] \arrow[dr] & & X \arrow[dr] & & \\
& * \arrow[rr] & & \mathrm{cofib}(g) \arrow[r] & Y
\end{tikzcd}
\tag{2.25}
$$

and note that each vertical square is a pullback square. The bottom horizontal square in (2.25) is a pushout square by definition, so the assumption that pushouts in $\mathcal{X}$ are universal implies that the top horizontal square is a pushout as well. $\square$

3. The Hilton–Milnor Splitting

The main result of this section is the following:

Theorem 3.1 (Hilton–Milnor Splitting)**.** *Let $\mathcal{X}$ be an ∞-category with universal pushouts and $X, Y \in \mathcal{X}_*$. Then there is an equivalence*

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \wedge \Omega Y) \ .$$

Before giving the proof of Theorem 3.1, we discuss some applications. Theorem 3.1 and Corollary 2.10 imply:

Corollary 3.2. *Let $\mathcal{X}$ be an ∞-category with universal pushouts. If $\mathcal{X}_*$ has countable coproducts, then for every pair of pointed objects $X, Y \in \mathcal{X}_*$ there is an equivalence*

$$\Omega\Sigma(X \vee Y) \simeq \Omega\Sigma X \times \Omega\Sigma Y \times \Omega\left(\bigvee_{i,j \geq 1} \Sigma X^{\wedge i} \wedge Y^{\wedge j}\right) \ .$$

Example 3.3. Let S be a scheme. Since colimits are universal in the ∞-category $\mathbf{H}(S)$ of motivic spaces over S (Example 2.3), Theorem 3.1 implies that for any pointed motivic spaces $X, Y \in \mathbf{H}(S)_*$ we have equivalences

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \wedge \Omega Y)$$

and

$$\Omega\Sigma(X \vee Y) \simeq \Omega\Sigma X \times \Omega\Sigma Y \times \Omega\left(\bigvee_{i,j \geq 1} \Sigma X^{\wedge i} \wedge Y^{\wedge j}\right) \ .$$

The Hilton–Milnor Splitting allows us to express the motivic space $\mathbf{P}^1 \smallsetminus \{0, 1, \infty\}$ in terms of smash powers of $\mathbf{G}_m$ with $\mathbf{P}^1$:

Example 3.4. Let K be a field of characteristic 0. Using the Morel–Voevodsky motivic purity Theorem [25, §3, Theorem 2.23], Wickelgren [33, Corollary 3.2] showed that there is an equivalence

$$\Sigma(\mathbf{P}^1_K \smallsetminus \{0, 1, \infty\}) \simeq \Sigma(\mathbf{G}_m \vee \mathbf{G}_m)$$

in the ∞-category $\mathbf{H}(K)$ of motivic spaces over $\mathrm{Spec}(K)$. Since $\Sigma \mathbf{G}_m \simeq \mathbf{P}^1_K$, it follows from Example 3.3 that we have equivalences

$$\begin{aligned}
\Omega\Sigma(\mathbf{P}^1_K \smallsetminus \{0,1,\infty\}) &\simeq \Omega\Sigma(\mathbf{G}_m \vee \mathbf{G}_m) \\
&\simeq \Omega\Sigma\mathbf{G}_m \times \Omega\Sigma\mathbf{G}_m \times \Omega\Sigma\left(\bigvee_{i,j\geq 1} \mathbf{G}_m^{\wedge(i+j)}\right) \\
&\simeq \Omega\mathbf{P}^1_K \times \Omega\mathbf{P}^1_K \times \Omega\Sigma\left(\bigvee_{i,j\geq 1} \mathbf{G}_m^{\wedge(i+j)}\right)
\end{aligned}$$

in $\mathbf{H}(K)$. This may be further expanded by iterated applications of Theorem 3.1, and, as in the classical setting, gives an expression for $\Omega\Sigma(\mathbf{P}^1_K \smallsetminus \{0,1,\infty\})$ as a direct product of copies of $\Omega(\mathbf{G}_m^{\wedge n} \wedge \mathbf{P}^1_K)$ indexed over a basis for the free Lie algebra on two generators [32, Section XI.6].

We now turn to the proof of the Hilton–Milnor Splitting. We first show that there is a fiber sequence

$$\Sigma(\Omega Y \wedge \Omega X) \longrightarrow X \vee Y \longrightarrow X \times Y \ . \tag{3.5}$$

We then show that the sequence (3.5) splits after taking loops. To do this, we construct a section

$$\Omega(X \times Y) \to \Omega(X \vee Y) \ ,$$

and use the fact that a fiber sequence of group objects with a section splits on the level of underlying objects. After proving that (3.5) is a fiber sequence we give a quick review of group objects and deduce Theorem 3.1 from the Splitting Lemma (Lemma 3.12).

We start with the following observation:

Lemma 3.6. *Let $\mathcal{X}$ be an ∞-category with finite limits and $X, Y \in \mathcal{X}_*$. Then there is a natural equivalence*

$$\mathrm{fib}((\mathrm{id}_X, *)\colon X \to X \times Y) \simeq \Omega Y \ .$$

Next, we prove the existence of the fiber sequence (3.5).

Lemma 3.7. *Let $\mathcal{X}$ be an ∞-category with universal pushouts and $X, Y \in \mathcal{X}_*$. Then there is a natural equivalence*

$$\mathrm{fib}(X \vee Y \to X \times Y) \simeq \Sigma(\Omega Y \wedge \Omega X) \ .$$

Proof. Write $F := \mathrm{fib}(X \vee Y \to X \times Y)$. By Proposition 2.18, it suffices to show that there is a pushout square

$$\begin{tikzcd}
\Omega Y \times \Omega X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \Omega X \arrow[d] \\
\Omega Y \arrow[r] & F \ .
\end{tikzcd}$$

Consider the diagram

$$
\begin{tikzcd}
\Omega Y \times \Omega X \arrow[rr, "\mathrm{pr}_2"] \arrow[dr, "\mathrm{pr}_1"] \arrow[dd] & & \Omega X \arrow[dr] \arrow[dd] & & & \\
& \Omega Y \arrow[rr] \arrow[dd] & & F \arrow[rr] \arrow[dd] & & * \arrow[dd] \\
* \arrow[rr] \arrow[dr] & & Y \arrow[dr] & & & \\
& X \arrow[rr] & & X \vee Y \arrow[rr] & & X \times Y .
\end{tikzcd}
\tag{3.8}
$$

The right-most vertical square in (3.8) is a pullback by definition, and the front and right vertical squares in the cube appearing in (3.8) are pullback squares by Lemma 3.6. The back and left vertical squares in the cube appearing in (3.8) are pullback squares by the Gluing Lemma for pullback squares. The bottom horizontal square in (3.8) is a pushout square by definition, so the assumption that pushouts in $\mathcal{X}$ are universal implies that the top horizontal square is a pushout as well. $\square$

3.1. **Reminder on group objects & the Splitting Lemma.** In order to split the fiber sequence (3.5) after taking loops, we need a few basic facts about group objects which we review now.

Definition 3.9. Let $\mathcal{X}$ be an ∞-category. A *group object* in $\mathcal{X}$ is a simplicial object $G_\bullet \colon \boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{X}$ such that

(3.9.1) For each integer $n \geq 0$ and partition $[n] = S \cup S'$ such that $S \cap S' = \{s\}$ consists of a single element, the induced square

$$
\begin{tikzcd}
G_n \arrow[r] \arrow[d] & G_\bullet(S') \arrow[d] \\
G_\bullet(S) \arrow[r] & G_\bullet(\{s\})
\end{tikzcd}
$$

is a pullback square in $\mathcal{X}$.

(3.9.2) The object G_0 is a terminal object of $\mathcal{X}$.

In this case, we call $G_1 \in \mathcal{X}$ the *underlying object* of $G_\bullet$. The face map $d_1 \colon G_1 \times G_1 \simeq G_2 \to G_1$ provides a multiplication on G_1 with unit given by the degeneracy map $s_0 \colon * \simeq G_0 \to G_1$.

We write $\mathrm{Grp}(\mathcal{X}) \subset \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{X})$ for the full subcategory spanned by the group objects.

The key example of a group object is loops on a pointed object. As a simplicial object, ΩX can be written as the *Čech nerve* of the basepoint $* \to X$; since we need to discuss Čech nerves in § 4.1, we recall the definition here.

Recollection 3.10. Let $\mathcal{X}$ be an ∞-category with pullbacks, and let $e \colon W \to X$ be a morphism in $\mathcal{X}$. The *Čech nerve* $\check{\mathrm{C}}_\bullet(e)$ of e is the simplicial object

$$
\cdots \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; W \underset{X}{\times} W \underset{X}{\times} W \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; W \underset{X}{\times} W \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; W
$$

in $\mathcal{X}$, where $\check{\mathrm{C}}_n(e)$ is the $(n+1)$-fold fiber product of W over X, each degeneracy map is a diagonal morphism, and each face map is a projection. Note that the morphism $e \colon W \to X$ defines a natural augmentation $\check{\mathrm{C}}_\bullet(e) \to X$.

Lemma 3.11. *Let $\mathcal{X}$ be an ∞-category with finite limits and $X \in \mathcal{X}_*$. Then ΩX naturally admits the structure of a group object of $\mathcal{X}$.*

Proof. Let $\mathrm{U}_\bullet(X)$ denote the Čech nerve of the basepoint $* \to X$. Since $\mathrm{U}_0(X) \simeq *$, [HTT, Proposition 6.1.2.11] shows that the Čech nerve $\mathrm{U}_\bullet(X)$ is a group object of $\mathcal{X}$. Since $\mathrm{U}_1(X) \simeq \Omega X$, it follows that the loop functor $\Omega\colon \mathcal{X}_* \to \mathcal{X}_*$ factors as the composite

$$\mathcal{X}_* \xrightarrow{\mathrm{U}_\bullet} \mathrm{Grp}(\mathcal{X}) \longrightarrow \mathcal{X}_*$$

of the functor given by the assignment $X \mapsto \mathrm{U}_\bullet(X)$ followed by the forgetful functor $\mathrm{Grp}(\mathcal{X}) \to \mathcal{X}_*$. $\square$

We leave the following Splitting Lemma as an amusing exercise for the reader.

Lemma 3.12 (Splitting Lemma)**.** *Let $\mathcal{X}$ be an ∞-category with finite limits, let*

$$A \longrightarrow B \xrightarrow{p} C$$

be a fiber sequence of group objects in $\mathcal{X}$, and write $m\colon B \times B \to B$ for the multiplication on B. For any section $s\colon C \to B$ of p on the level of underlying pointed objects of $\mathcal{X}$, the composite

$$A \times C \xrightarrow{i \times s} B \times B \xrightarrow{m} B$$

is an equivalence in $\mathcal{X}_*$.

We can now prove Theorem 3.1.

Proof of Theorem 3.1. By Lemmas 3.7 and 3.11, there is a fiber sequence

$$\Omega\,\mathrm{fib}(X \vee Y \to X \times Y) \longrightarrow \Omega(X \vee Y) \longrightarrow \Omega X \times \Omega Y \tag{3.13}$$

of group objects of $\mathcal{X}$. Note that the map $\Omega(X \vee Y) \to \Omega X \times \Omega Y$ has a section defined by the composite

$$\Omega X \times \Omega Y \xrightarrow{\Omega i_1 \times \Omega i_2} \Omega(X \vee Y) \times \Omega(X \vee Y) \xrightarrow{\quad m \quad} \Omega(X \vee Y)\ ,$$

where $i_1\colon X \to X \vee Y$ and $i_2\colon Y \to X \vee Y$ are the coproduct insertions, and m is the multiplication coming from the group structure on $\Omega(X \vee Y)$. By Lemma 3.12 the fiber sequence (3.13) splits, so applying Lemma 3.7 we see that there are equivalences

$$\begin{aligned}\Omega(X \vee Y) &\simeq \Omega X \times \Omega Y \times \Omega\,\mathrm{fib}(X \vee Y \to X \times Y) \\ &\simeq \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \wedge \Omega Y)\ .\end{aligned}$$

$\square$

4. The metastable EHP sequence

In classical algebraic topology, the *metastable EHP sequence* is the statement that the composite

$$X \longrightarrow \Omega\Sigma X \xrightarrow{h_2} \Omega\Sigma X^{\wedge 2}$$

is a fiber sequence in a range depending on the connectedness of X, known as the *metastable range*. Here the first map $X \to \Omega\Sigma X$ is the unit and h_2 is the Hopf map (Construction 2.12). For the higher Hopf maps $h_n\colon \Omega\Sigma X \to \Omega\Sigma X^{\wedge n}$, there is an analogous fiber sequence in a range

$$\mathrm{J}_{n-1}(X) \longrightarrow \Omega\Sigma X \xrightarrow{h_n} \Omega\Sigma X^{\wedge n}\ ,$$

where $\mathrm{J}_{n-1}(X)$ is the $(n-1)^{\mathrm{st}}$ piece of the *James filtration* on $\Omega\Sigma X$.

This section is dedicated to a non-computational proof of the metastable EHP sequence in an ∞-topos that only makes use of the Blakers–Massey Theorem and some basic connectedness results (Theorem 4.32). To explain what we mean by 'a fiber sequence in a range' and the connectedness estimates we need, in § 4.1 we review the basics of connectedness in an ∞-topos. In § 4.2 we review the James filtration. In § 4.3 we refine the James Splitting to a splitting $\Sigma\,\mathrm{J}_n(X) \simeq \bigvee_{i=1}^{n} \Sigma X^{\wedge i}$. In § 4.4, we give our non-computational proof of the metastable EHP sequence via the Blakers–Massey Theorem, and also record a computational proof for posterity.

4.1. **Connectedness and the Blakers–Massey Theorem.** In this subsection, we review the basic properties of k-truncated and k-connected morphisms in an ∞-topos that we need in order to make sense of the metastable EHP sequence in this setting. We also recall the Blakers–Massey Theorem (Theorem 4.12) and Freudenthal Suspension Theorem (Corollary 4.13) in an ∞-topos, since our proof of the metastable EHP sequence relies on these results.

The reader interested in the details of results discussed here should consult [HTT, §6.5.1; 1, §3.3] for connectedness results, and [1] for the Blakers–Massey Theorem.

Definition 4.1. Let $\mathcal{X}$ be an ∞-topos. For each integer $k \geq -2$, define *k-truncatedness* for morphisms in $\mathcal{X}$ recursively as follows.

(4.1.1) A morphism f is (-2)-*truncated* if f is an equivalence.
(4.1.2) For $k \geq -1$, a morphism $f\colon X \to Y$ is *k-truncated* if the diagonal $\Delta_f\colon X \to X \times_Y X$ is $(k-1)$-truncated.

An object $X \in \mathcal{X}$ is *k-truncated* if the unique morphism $X \to *$ is k-truncated.

Write $\mathcal{X}_{\leq k} \subset \mathcal{X}$ for the full subcategory spanned by the k-truncated objects. The inclusion $\mathcal{X}_{\leq k} \subset \mathcal{X}$ admits a left adjoint which we denote by $\tau_{\leq k}\colon \mathcal{X} \to \mathcal{X}_{\leq k}$.

Example 4.2. Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology τ, and let $k \geq -2$ be an integer. Then a sheaf $\mathcal{F} \in \mathrm{Sh}_\tau(\mathcal{C})$ of spaces on $\mathcal{C}$ with respect to τ is k-truncated if and only if $\mathcal{F}(c)$ is a k-truncated space for every $c \in \mathcal{C}$. That is, $\mathcal{F}$ is k-truncated if and only if $\mathcal{F}$ is a sheaf of k-truncated spaces.

Remark 4.3. If $\mathcal{X}$ is an ∞-topos, then the full subcategory $\mathcal{X}_{\leq 0}$ spanned by the 0-truncated objects is an ordinary topos, i.e., a category of sheaves of sets on a Grothendieck site.

Recollection 4.4. Let $\mathcal{X}$ be an ∞-topos. A morphism $f\colon X \to Y$ in $\mathcal{X}$ is an *effective epimorphism* if the augmentation $\check{\mathrm{C}}_\bullet(f) \to Y$ exhibits Y as the colimit of the Čech nerve of f (see Recollection 3.10). Equivalently, f is an effective epimorphism if and only if $\tau_{\leq 0}(f)\colon\ \tau_{\leq 0}(X) \to \tau_{\leq 0}(Y)$ is an effective epimorphism in the ordinary topos $\mathcal{X}_{\leq 0}$ of 0-truncated objects of $\mathcal{X}$ [HTT, Proposition 7.2.1.14].

Example 4.5. A morphism $f\colon X \to Y$ in the ∞-topos $\mathbf{Spc}$ of spaces is an effective epimorphism if and only if $\pi_0(f)\colon\ \pi_0(X) \to \pi_0(Y)$ is a surjection of sets.

Definition 4.6. Let $\mathcal{X}$ be an ∞-topos. For each integer $k \geq -2$, define *k-connectedness* for morphisms in $\mathcal{X}$ recursively as follows.

(4.6.1) Every morphism is (-2)-*connected*.
(4.6.2) For $k \geq -1$, a morphism $f\colon X \to Y$ is *k-connected* if f is an effective epimorphism and the diagonal $\Delta_f\colon X \to X \times_Y X$ is $(k-1)$-connected.

An object $X \in \mathcal{X}$ is *k-connected* if the unique morphism $X \to *$ is k-connected.

In particular, a morphism $f\colon X \to Y$ is (-1)-connected if and only if f is an effective epimorphism.

Warning 4.7. Our conventions for connectedness follow those of Anel, Biedermann, Finster, and Joyal [1, §3.3]. For $k \geq 0$, a homotopy type X is k-connected in our sense if and only if X is k-connected in the classical terminology [7, §6.7; 12, p. 346; 22, Chapter 10, §4]. In particular, X is 0-connected if and only if X is path-connected. This convention *differs* from the classical one for maps: an n-connected map in our sense is an $(n+1)$-connected map in the classical sense.

Comparing to Lurie's terminology [HTT, §6.5.1], an object or morphism is n-connect*ed* in our sense if and only if it is $(n+1)$-connect*ive* in Lurie's sense. One of the benefits of our terminological choice is that the constant factors in many connectedness estimates are eliminated (see Theorem 4.12 and Corollary 4.13).

The following basic properties of k-connected morphisms are proven in [HTT, §6.5.1].

Proposition 4.8. *Let $\mathcal{X}$ be an ∞-topos and $k \geq -2$ be an integer.*

(4.8.1) *The class of k-connected morphisms in $\mathcal{X}$ is stable under composition.*
(4.8.2) *The class of k-connected morphisms in $\mathcal{X}$ is stable under pushout along any morphism.*
(4.8.3) *The class of k-connected morphisms in $\mathcal{X}$ is stable under pullback along any morphism.*
(4.8.4) *The class of k-connected objects in $\mathcal{X}$ is stable under finite products.*
(4.8.5) *Given a morphism $f\colon X \to Y$ in $\mathcal{X}$ with a section $s\colon Y \to X$, the morphism f is $(k+1)$-connected if and only if the section s is k-connected.*
(4.8.6) *An object $X \in \mathcal{X}$ is k-connected if and only if the k-truncation $\tau_{\leq k}(X)$ of X is terminal in $\mathcal{X}$.*

In the ∞-topos of spaces, the following connectedness estimates are usually done by appealing to cell structures. Such arguments are unavailable in an arbitrary ∞-topos, so we deduce these connectedness estimates from Proposition 4.8.

Proposition 4.9. *Let $\mathcal{X}$ be an ∞-topos, $X, Y \in \mathcal{X}_*$ pointed objects, and $k, \ell \geq 0$ integers. If X is k-connected and Y is ℓ-connected, then:*

(4.9.1) *The induced morphism $X \vee Y \to X \times Y$ is $(k+\ell)$-connected.*
(4.9.2) *The smash product $X \wedge Y$ is $(k+\ell+1)$-connected.*
(4.9.3) *For each postive integer n, the n-fold smash product $X^{\wedge n}$ is $(n(k+1)-1)$-connected.*

Proof. First, (4.9.1) follows from the fact that the basepoints $* \to X$ and $* \to Y$ are $(k-1)$-connected and $(\ell-1)$-connected (4.8.5), respectively, and a general fact about pushout-products of connected morphisms [1, Corollary 3.3.7(4)].

Now we prove (4.9.2). Since $(k+\ell)$-connected morphisms are stable under pushout (4.8.2), by (4.9.1) and the pushout square

$$\begin{tikzcd} X \vee Y \arrow[r] \arrow[d] & X \times Y \arrow[d] \\ * \arrow[r] & X \wedge Y \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

defining the smash product $X \wedge Y$, we see that the basepoint $* \to X \wedge Y$ is $(k+\ell)$-connected. Hence $X \wedge Y$ is $(k+\ell+1)$-connected (4.8.5).

Finally, (4.9.3) follows from (4.9.2) by induction. $\square$

Now we record a convenient fact about the interaction between connectedness and pullbacks that we need in out proof of the metastable EHP sequence.

Proposition 4.10. *Let $\mathcal{X}$ be an ∞-topos, $\ell \geq -2$ be an integer, and*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & C \arrow[d, dashed, "c"] & B \arrow[l, "g"'] \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & C' & B' \arrow[l, "g'"] \end{tikzcd}$$

be a commutative diagram in $\mathcal{X}$. If a and b are ℓ-connected and c is $(\ell+1)$-connected, then the induced morphism on pullbacks $A \times_C B \to A' \times_{C'} B'$ is ℓ-connected.

Proof. Since ℓ-connected morphisms are stable under composition, by factoring the induced morphism $A \times_C B \to A' \times_{C'} B'$ as a composite of induced morphisms

$$A \times_C B \to A' \times_{C'} B \to A' \times_{C'} B' \ ,$$

it suffices to prove the claim in the special case $B = B'$ and the morphism $b\colon B \to B'$ is the identity. To prove the claim when b is the identity, first write $(\mathrm{id}_A, f)\colon A \to A \times_{C'} C$ for the section of the projection $A \times_{C'} C \to A$ induced by the commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & C \arrow[d, "c"] \\ A \arrow[r, "f'a"'] & C' \end{tikzcd} .$$

Consider the following commutative diagram of pullback squares

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \times_{C'} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A' \times_{C'} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "(\mathrm{id}_A, f)"'] & A \times_{C'} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A' \times_{C'} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "c"] \\ & A \arrow[r, "a"'] & A' \arrow[r, "f'"'] & C' \end{tikzcd} ,$$

and notice that the composite middle horizontal map $A \to C$ is f. Since $c\colon C \to C'$ is $(\ell+1)$-connected, the projection $A \times_{C'} C \to A$ is $(\ell+1)$-connected (4.8.3). Hence the section $(\mathrm{id}_A, f)\colon A \to A \times_{C'} C$ is ℓ-connected (4.8.5). Consequently the induced morphism $A \times_C B \to A \times_{C'} B$ is ℓ-connected. Now note that since $a\colon A \to A'$ is ℓ-connected, the induced morphism $A \times_{C'} B \to A' \times_{C'} B$ is ℓ-connected. Hence the composite morphism $A \times_C B \to A' \times_{C'} B$ is ℓ-connected, as desired. $\square$

In particular, Proposition 4.10 shows that the class of ℓ-connected morphisms in an ∞-topos is closed under finite products. Setting $B = B' = *$ in Proposition 4.10 we deduce:

Corollary 4.11. *Let $\mathcal{X}$ be an ∞-topos, $\ell \geq -2$ be an integer, and*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & C \arrow[d, "c"] \\ A' \arrow[r, "f'"'] & C' \end{tikzcd}$$

be a commutative square in $\mathcal{X}$. If a is ℓ-connected and c is $(\ell+1)$-connected, then for every point $x\colon * \to C$, the induced morphism $\mathrm{fib}_x(f) \to \mathrm{fib}_{cx}(f')$ on fibers is ℓ-connected.

We conclude this subsection by recalling the Blakers–Massey and Freudenthal Suspension Theorems in the setting of ∞-topoi.

Theorem 4.12 (Blakers–Massey [1, Corollary 4.3.1])**.** *Let $\mathcal{X}$ be an ∞-topos and let*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d] \\ B \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & D \end{tikzcd}$$

be a pushout square in $\mathcal{X}$. If f is k-connected and g is ℓ-connected, then the induced morphism $A \to B \times_D C$ is $(k+\ell)$-connected.

As in the classical setting, applying the Blakers–Massey Theorem to the pushout defining the suspension immediately implies the Freudenthal Suspension Theorem.

Corollary 4.13 (Freudenthal Suspension Theorem)**.** *Let $\mathcal{X}$ be an ∞-topos, and $X \in \mathcal{X}_*$ a pointed k-connected object. Then the unit morphism $X \to \Omega\Sigma X$ is $2k$-connected.*

4.2. **The James construction.** In this section, we recall some facts about the James filtration. Classically, the James filtration $\{\mathrm{J}_n(X)\}_{n\geq 0}$ provides a multiplicative filtration on the free monoid $\mathrm{J}(X)$ on a pointed space X, in the homotopical sense. At the point-set level, $\mathrm{J}(X)$ can be presented as the free topological monoid on X, and $\mathrm{J}_n(X)$ can be identified the subspace of words of length at most n in $\mathrm{J}(X)$. Concatenation of words then supplies $\{\mathrm{J}_n(X)\}_{n\geq 0}$ with the structure of a filtered monoid. Since the trivial monoid and trivial group coincide, if X is connected, then the free monoid $\mathrm{J}(X)$ on X coincides with the free group $\Omega\Sigma X$ on X.

In a general ∞-category, we can define the James construction as follows. This definition is provided in [5, Section 3] in the context of homotopy type theory; the arguments made in [5, Section 3] are formal and valid in any ∞-topos.

Construction 4.14 (James construction)**.** Let $\mathcal{X}$ be an ∞-category with finite products and pushouts, and let $X \in \mathcal{X}_*$ be a pointed object of $\mathcal{X}$. For each integer $n \geq 0$ we define a pointed object $\mathrm{J}_n(X) \in \mathcal{X}_*$ as well as morphisms

$$i_n \colon \mathrm{J}_n(X) \to \mathrm{J}_{n+1}(X) \qquad \text{and} \qquad \alpha_n \colon X \times \mathrm{J}_n(X) \to \mathrm{J}_{n+1}(X)$$

in $\mathcal{X}_*$ recursively as follows.

(4.14.1) We define $\mathrm{J}_0(X) :- *$, $\mathrm{J}_1(X) :- X$, the morphism $i_0 \colon * \to X$ is the basepoint, and the morphism $\alpha_0 \colon X \times * \to X$ is the projection $\mathrm{pr}_1 \colon X \times * \xrightarrow{\sim} X$.

(4.14.2) For $n \geq 2$, we define $\mathrm{J}_n(X)$, i_{n-1}, and α_{n-1} by the pushout square

$$\begin{tikzcd} X \times \mathrm{J}_{n-2}(X) \overset{\mathrm{J}_{n-2}(X)}{\sqcup} \mathrm{J}_{n-1}(X) \arrow[r] \arrow[d] & \mathrm{J}_{n-1}(X) \arrow[d, "i_{n-1}"] \\ X \times \mathrm{J}_{n-1}(X) \arrow[r, "\alpha_{n-1}"'] & \mathrm{J}_n(X) \end{tikzcd} \tag{4.15}$$

where: the top horizontal morphism is induced by the universal property of the pushout by the commutative square

$$\begin{tikzcd} \mathrm{J}_{n-2}(X) \arrow[r, "i_{n-2}"] \arrow[d, "(*,\mathrm{id})"'] & \mathrm{J}_{n-1}(X) \arrow[d, equal] \\ X \times \mathrm{J}_{n-2}(X) \arrow[r, "\alpha_{n-2}"'] & \mathrm{J}_{n-1}(X) \end{tikzcd},$$

and the left vertical morphism is induced by the universal property of the pushout by the commutative square

$$\begin{tikzcd} \mathrm{J}_{n-2}(X) \arrow[r, "i_{n-2}"] \arrow[d, "(*,\mathrm{id})"'] & \mathrm{J}_{n-1}(X) \arrow[d, "(*,\mathrm{id})"] \\ X \times \mathrm{J}_{n-2}(X) \arrow[r, "\mathrm{id} \times i_{n-2}"'] & X \times \mathrm{J}_{n-1}(X) \end{tikzcd}.$$

For each positive integer n, define a morphism $a_n\colon X^{\times n} \to \mathrm{J}_n(X)$ as the composite

$$X^{\times n} \simeq X^{\times n-1} \times \mathrm{J}_1(X) \xrightarrow{\mathrm{id} \times \alpha_1} X^{\times n-2} \times \mathrm{J}_2(X) \xrightarrow{\mathrm{id} \times \alpha_2} \cdots \longrightarrow X \times \mathrm{J}_{n-1}(X) \xrightarrow{\alpha_{n-1}} \mathrm{J}_n(X) \ .$$

Finally, define $\mathrm{J}(X) \coloneq \operatorname{colim}_{n \geq 0} \mathrm{J}_n(X)$.

Definition 4.16. Let $\mathcal{X}$ be an ∞-category with finite products and pushouts, and let $X \in \mathcal{X}_*$ be a pointed object of $\mathcal{X}$. For each integer $n \geq 0$ define a morphism $u_n\colon \mathrm{J}_n(X) \to \Omega\Sigma X$ recursively as follows. The morphism u_0 is the basepoint, and the morphism $u_1\colon X \to \Omega\Sigma X$ is the unit. For $n \geq 2$, the morphism u_n is induced by the commutative square

$$\begin{tikzcd} X \times \mathrm{J}_{n-2}(X) \overset{\mathrm{J}_{n-2}(X)}{\sqcup} \mathrm{J}_{n-1}(X) \arrow[r] \arrow[d] & \mathrm{J}_{n-1}(X) \arrow[d, "u_{n-1}"] \\ X \times \mathrm{J}_{n-1}(X) \arrow[r, "m \circ (u_1 \times u_{n-1})"'] & \Omega\Sigma X \end{tikzcd}$$

where $m\colon \Omega\Sigma X \times \Omega\Sigma X \to \Omega\Sigma X$ is the group multiplication.

The morphisms u_n induce a morphism $u\colon \mathrm{J}(X) \to \Omega\Sigma X$.

Theorem 4.17 ([5, Section 6])**.** *Let $\mathcal{X}$ be an ∞-topos and $X \in \mathcal{X}_*$ a pointed object. If X is 0-connected, then the induced map $u\colon \mathrm{J}(X) \to \Omega\Sigma X$ is an equivalence.*

Brunerie [5] gives an elementary proof of the following connectedness estimate.

Lemma 4.18 ([5, Proposition 4])**.** *Let $\mathcal{X}$ be an ∞-topos, $k \geq 0$ an integer, and $X \in \mathcal{X}_*$ a pointed k-connected object. Then the morphism $i_{n-1}\colon \mathrm{J}_{n-1}(X) \to \mathrm{J}_n(X)$ is $(n(k+1)-2)$-connected.*

Corollary 4.19. *Let $\mathcal{X}$ be an ∞-topos, $k \geq 0$ an integer, and $X \in \mathcal{X}_*$ a pointed k-connected object. Then for all integers $n \geq 0$, the object $\mathrm{J}_n(X)$ is k-connected.*

Proof. If $n = 0$, then the claim is clear since $\mathrm{J}_0(X) = *$. If $n > 0$, then by Lemma 4.18 the morphisms $i_0, \ldots, i_{n-1}$ are all $(k-1)$-connected. Hence the basepoint

$$i_{n-1} \cdots i_0\colon * \to \mathrm{J}_n(X)$$

is $(k-1)$-connected; equivalently, $\mathrm{J}_n(X)$ is k-connected (4.8.5). □

Lemma 4.20 ([5, Proposition 6])**.** *Let $\mathcal{X}$ be an ∞-topos, $k \geq 0$ an integer, and $X \in \mathcal{X}_*$ a pointed k-connected object. Then the morphism $u_n\colon \mathrm{J}_n(X) \to \Omega\Sigma X$ is $((n+1)(k+1)-2)$-connected.*

4.3. **Splitting the James construction.** The purpose of this subsection is to prove the following refinement of Corollary 2.10, which we use in our proof of the metastable EHP sequence (Theorem 4.32).

Proposition 4.21. *Let $\mathcal{X}$ be an ∞-category with universal pushouts, and let $X \in \mathcal{X}_*$. Then there is a splitting*

$$\Sigma\, \mathrm{J}_n(X) \simeq \bigvee_{1 \leq i \leq n} \Sigma X^{\wedge i} \ . \tag{4.22}$$

If $\mathcal{X}$ is an ∞-topos and X is 0-connected, then under the map $\Sigma u_n\colon \Sigma\, \mathrm{J}_n(X) \to \Sigma\Omega\Sigma X$, the splitting (4.22) is an equivalence onto the first n factors of the splitting $\Sigma\Omega\Sigma X \simeq \bigvee_{i \geq 1} \Sigma X^{\wedge i}$ of Corollary 2.10.

The proof of Proposition 4.21 requires some preliminaries. We need to relate the cofiber of i_n to smash powers of X; before doing so we need some preparatory lemmas.

Lemma 4.23. *Let $\mathcal{X}$ be an ∞-category with finite products and pushouts, and let $X, Y \in \mathcal{X}_*$. Then there is a cofiber sequence*

$$Y \longrightarrow \mathrm{cofib}((\mathrm{id}_X, *) \colon X \to X \times Y) \longrightarrow X \wedge Y \ .$$

Proof. There is a map of cofiber sequences

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d] & X \arrow[r] \arrow[d, "(\mathrm{id}_X, *)"] & * \arrow[d] \\ X \vee Y \arrow[r] & X \times Y \arrow[r] & X \wedge Y \end{tikzcd} \qquad (4.24)$$

where the leftmost vertical map is the coproduct insertion. The cofiber of the coproduct insertion $X \to X \vee Y$ is Y, and the cofiber of the basepoint $* \to X \wedge Y$ is $X \wedge Y$. To conclude, note that taking vertical cofibers in (4.24) results in a cofiber sequence. ☐

The following is a straightforward application of Lemma 1.9.

Lemma 4.25. *Let $\mathcal{X}$ be an ∞-category with pushouts and a terminal object and let*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

be a commutative square in $\mathcal{X}_*$. Then there is a natural equivalence

$$\mathrm{cofib}\left(Y \sqcup^X Z \to W\right) \simeq \mathrm{cofib}\left(\mathrm{cofib}(X \to Z) \to \mathrm{cofib}(Y \to W)\right) \ .$$

We are now ready to show that $\mathrm{cofib}(i_n) \simeq X^{\wedge n+1}$.

Proposition 4.26. *Let $\mathcal{X}$ be an ∞-category with universal pushouts, and let $X \in \mathcal{X}_*$. Then for each integer $n \geq 0$, there is a natural equivalence*

$$\mathrm{cofib}(i_n \colon \mathrm{J}_n(X) \to \mathrm{J}_{n+1}(X)) \simeq X^{\wedge n+1} \ .$$

Moreover, the composite

$$X^{\times n+1} \xrightarrow{a_n} \mathrm{J}_{n+1}(X) \longrightarrow X^{\wedge n+1}$$

is equivalent to the canonical map $X^{\times n+1} \to X^{\wedge n+1}$.

Proof. We prove the claim by induction on n. For the base case, note that since the morphism i_0 is the basepoint $* \to X$, the cofiber of i_0 is X. For the inductive step we assume that $\mathrm{cofib}(i_n) \simeq X^{\wedge n+1}$ and show that $\mathrm{cofib}(i_{n+1}) \simeq X^{\wedge n+2}$. From the defining pushout square (4.15), we see that

$$\mathrm{cofib}(i_{n+1}) \simeq \mathrm{cofib}\left(X \times \mathrm{J}_n(X) \sqcup^{\mathrm{J}_n(X)} \mathrm{J}_{n+1}(X) \to X \times \mathrm{J}_{n+1}(X)\right) \ .$$

Applying Lemma 4.25 shows that

$$\mathrm{cofib}(i_{n+1}) \simeq \mathrm{cofib}\left(\mathrm{cofib}(\mathrm{J}_n(X) \to X \times \mathrm{J}_n(X)) \to \mathrm{cofib}(\mathrm{J}_{n+1}(X) \to X \times \mathrm{J}_{n+1}(X))\right) \ ,$$

where the map of cofibers is induced by the map $i_n \colon \mathrm{J}_n(X) \to \mathrm{J}_{n+1}(X)$. By Lemma 4.23, there is a cofiber sequence

$$X \to \mathrm{cofib}(\mathrm{J}_n(X) \to X \times \mathrm{J}_n(X)) \to X \wedge \mathrm{J}_n(X) \ ;$$

moreover, the map $i_n\colon \mathrm{J}_n(X) \to \mathrm{J}_{n+1}(X)$ induces a map of cofiber sequences

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & \mathrm{cofib}(\mathrm{J}_n(X) \to X \times \mathrm{J}_n(X)) \arrow[r] \arrow[d] & X \wedge \mathrm{J}_n(X) \arrow[d, "\mathrm{id}_X \wedge i_n"] \\
X \arrow[r] & \mathrm{cofib}(\mathrm{J}_{n+1}(X) \to X \times \mathrm{J}_{n+1}(X)) \arrow[r] & X \wedge \mathrm{J}_{n+1}(X)\ .
\end{tikzcd}
\tag{4.27}
$$

Since the leftmost vertical map is the identity, taking vertical cofibers in the map of cofiber sequences (4.27) produces an equivalence between the vertical cofibers of the middle and right vertical maps. Since the cofiber of the middle vertical map is $\mathrm{cofib}(i_{n+1})$, we find that

$$\mathrm{cofib}(i_{n+1}) \simeq \mathrm{cofib}\left(\mathrm{id}_X \wedge i_n\colon X \wedge \mathrm{J}_n(X) \to X \wedge \mathrm{J}_{n+1}(X)\right)\ .$$

Since pushouts in $\mathcal{X}$ are universal we have a natural equivalence

$$\mathrm{cofib}(\mathrm{id}_X \wedge i_n\colon X \wedge \mathrm{J}_n(X) \to X \wedge \mathrm{J}_{n+1}(X)) \simeq X \wedge \mathrm{cofib}(i_n\colon \mathrm{J}_n(X) \to \mathrm{J}_{n+1}(X))$$

By the inductive hypothesis, $\mathrm{cofib}(i_n) \simeq X^{\wedge n+1}$, so $\mathrm{cofib}(i_{n+1}) \simeq X^{\wedge n+2}$, as desired. $\square$

Next we split the term

$$\Sigma\Big(X \times \mathrm{J}_{n-1}(X) \overset{\mathrm{J}_{n-1}(X)}{\sqcup} \mathrm{J}_n(X)\Big)$$

in the pushout square (4.15) defining $\Sigma\,\mathrm{J}_{n+1}(X)$ and prove Proposition 4.21.

Lemma 4.28. *Let $\mathcal{X}$ be an ∞-category with universal pushouts, $X \in \mathcal{X}_*$, and $n \geq 1$ an integer. Then there is a natural equivalence.*

$$\Sigma\Big(X \times \mathrm{J}_{n-1}(X) \overset{\mathrm{J}_{n-1}(X)}{\sqcup} \mathrm{J}_n(X)\Big) \simeq \Sigma(X \wedge \mathrm{J}_{n-1}(X)) \vee \Sigma X \vee \Sigma\,\mathrm{J}_n(X)\ .$$

Proof. Since suspension preserves pushouts, we have a pushout square

$$
\begin{tikzcd}
\Sigma\,\mathrm{J}_{n-1}(X) \arrow[r] \arrow[d] & \Sigma\,\mathrm{J}_n(X) \arrow[d] \\
\Sigma(X \times \mathrm{J}_{n-1}(X)) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma\Big(X \times \mathrm{J}_{n-1}(X) \sqcup^{\mathrm{J}_{n-1}(X)} \mathrm{J}_n(X)\Big)\ .
\end{tikzcd}
\tag{4.29}
$$

Under the equivalence

$$\Sigma(X \times \mathrm{J}_{n-1}(X)) \simeq \Sigma(X \wedge \mathrm{J}_{n-1}(X)) \vee \Sigma X \vee \Sigma\,\mathrm{J}_{n-1}(X)$$

of Corollary 2.21, the left vertical map in (4.29) is the coproduct insertion. Hence on pushouts we see that

$$\Sigma\Big(X \times \mathrm{J}_{n-1}(X) \overset{\mathrm{J}_{n-1}(X)}{\sqcup} \mathrm{J}_n(X)\Big) \simeq \Sigma(X \wedge \mathrm{J}_{n-1}(X)) \vee \Sigma X \vee \Sigma\,\mathrm{J}_n(X)\ . \qquad \square$$

Proof of Proposition 4.21. We prove the claim by induction on n. The base case where $n = 1$ is obvious. For the inductive step, assume that $n \geq 1$ and $\Sigma\,\mathrm{J}_n(X) \simeq \bigvee_{i=1}^{n} \Sigma X^{\wedge n}$. By Proposition 4.26 we have a cofiber sequence

$$\mathrm{J}_n(X) \xrightarrow{i_n} \mathrm{J}_{n+1}(X) \longrightarrow X^{\wedge n+1}\ ,$$

so the inductive hypothesis and the duals of Lemmas 3.11 and 3.12, it suffices to define a retraction

$$r_n\colon \Sigma\,\mathrm{J}_{n+1}(X) \to \Sigma\,\mathrm{J}_n(X)$$

of the map Σi_n.

We construct the retractions $r_n\colon \Sigma\,\mathrm{J}_{n+1}(X) \to \Sigma\,\mathrm{J}_n(X)$ inductively. For the base case, the retraction $r_0\colon \Sigma X \to *$ of Σi_0 is the unique morphism. For the inductive step, assume that

$n \geq 1$ and we have constructed a retraction $r_{n-1} \colon \Sigma \mathrm{J}_n(X) \to \Sigma \mathrm{J}_{n-1}(X)$ of Σi_{n-1}; we use this to construct a retraction r_n of Σi_n. Since suspension preserves pushouts, suspending the defining pushout square (4.15) yields a pushout square

$$
\begin{tikzcd}
\Sigma\Big(X \times \mathrm{J}_{n-1}(X) \overset{\mathrm{J}_{n-1}(X)}{\sqcup} \mathrm{J}_n(X)\Big) \arrow[r] \arrow[d] & \Sigma \mathrm{J}_n(X) \arrow[d, "\Sigma i_n"] \\
\Sigma(X \times \mathrm{J}_n(X)) \arrow[r, "\Sigma\alpha_n"'] & \Sigma \mathrm{J}_{n+1}(X) \ .
\end{tikzcd}
\tag{4.30}
$$

In order to define a retraction of Σi_n, it suffices to define a retraction of the left vertical map in (4.30), i.e., it suffices to define a retraction

$$\Sigma(X \times \mathrm{J}_n(X)) \to \Sigma\Big(X \times \mathrm{J}_{n-1}(X) \overset{\mathrm{J}_{n-1}(X)}{\sqcup} \mathrm{J}_n(X)\Big) \ .$$

By Corollary 2.21 and Lemma 4.28, we have equivalences

$$\Sigma(X \times \mathrm{J}_n(X)) \simeq \Sigma(X \wedge \mathrm{J}_n(X)) \vee \Sigma X \vee \Sigma \mathrm{J}_n(X)$$

and

$$\Sigma\Big(X \times \mathrm{J}_{n-1}(X) \overset{\mathrm{J}_{n-1}(X)}{\sqcup} \mathrm{J}_n(X)\Big) \simeq \Sigma(X \wedge \mathrm{J}_{n-1}(X)) \vee \Sigma X \vee \Sigma \mathrm{J}_n(X) \ .$$

Moreover, the left vertical map in (4.30) is induced by the suspensions of the identity on X, identity on $\mathrm{J}_n(X)$, and the map $i_n \colon \mathrm{J}_{n-1}(X) \to \mathrm{J}_n(X)$. Under the identifications

$$\Sigma(X \wedge \mathrm{J}_{n-1}(X)) \simeq X \wedge \Sigma \mathrm{J}_{n-1}(X) \qquad \text{and} \qquad \Sigma(X \wedge \mathrm{J}_n(X)) \simeq X \wedge \Sigma \mathrm{J}_n(X)$$

of Lemma 2.23, we see that the map

$$\mathrm{id}_X \wedge r_{n-1} \colon \Sigma(X \wedge \mathrm{J}_{n-1}(X)) \simeq X \wedge \Sigma \mathrm{J}_{n-1}(X) \longrightarrow X \wedge \Sigma \mathrm{J}_n(X) \simeq \Sigma(X \wedge \mathrm{J}_n(X))$$

is a retraction of $\Sigma(\mathrm{id}_X \wedge i_{n-1})$. Hence the map

$$(\mathrm{id}_X \wedge r_{n-1}) \vee \mathrm{id} \vee \mathrm{id} \colon \Sigma(X \wedge \mathrm{J}_n(X)) \vee \Sigma X \vee \Sigma \mathrm{J}_n(X) \longrightarrow \Sigma(X \wedge \mathrm{J}_{n-1}(X)) \vee \Sigma X \vee \Sigma \mathrm{J}_n(X)$$

supplies the desired retraction of the left vertical map in (4.30). ☐

4.4. **Proofs of the metastable EHP sequence.** In this subsection, we present two proofs of the metastable EHP sequence in the setting of ∞-topoi. Before making a precise statement of the main result, we need the following easy lemma.

Lemma 4.31. *Let $\mathcal{X}$ be an ∞-topos, $X \in \mathcal{X}_*$ be a pointed 0-connected object, and $n \geq 1$ an integer. Then the composite*

$$\mathrm{J}_{n-1}(X) \xrightarrow{u_n} \Omega\Sigma X \xrightarrow{h_n} \Omega\Sigma X^{\wedge n}$$

is null, where h_n is the Hopf map of Construction 2.12.

Proof. It suffices to prove the corresponding statement on adjoints: in other words, we need to show that the composite

$$\Sigma \mathrm{J}_{n-1}(X) \xrightarrow{\Sigma u_n} \Sigma\Omega\Sigma X \longrightarrow \Sigma X^{\wedge n}$$

is null. This is an immediate consequence of Proposition 4.21. ☐

We can now state the metastable EHP sequence.

Theorem 4.32 (metastable EHP sequence)**.** *Let $\mathcal{X}$ be an ∞-topos, $k \geq 0$ an integer, and $X \in \mathcal{X}_*$ a pointed k-connected object. Then for every integer $n \geq 1$, the morphism $\mathrm{J}_{n-1}(X) \to \mathrm{fib}(h_n)$ induced by Lemma 4.31 is $((n+1)(k+1)-3)$-connected.*

Remark 4.33. Theorem 4.32 implies the metastable EHP sequence of Asok–Wickelgren–Williams for ∞-topoi of hypersheaves on an ordinary site with enough points [2, Proposition 3.1.4].

The first proof of Theorem 4.32 we present is *internal* to ∞-topoi, and only uses basic facts about connectedness and the James construction, as well as the Blakers–Massey Theorem. The second reduces to the ∞-topos **Spc** of spaces, then uses the homology Whitehead Theorem and Serre spectral sequence to give a calculational proof of the metastable EHP sequence in the classical setting. Both perspectives are valuable, and we present the second here in part because the calculational proof of the metastable EHP sequence does not seem to be easy to locate in the literature.

Internal proof of Theorem 4.32. First we show that it suffices to prove the claim where we replace $\mathrm{fib}(h_n)$ by the fiber of the morphism $\mathrm{J}_n(X) \to X^{\wedge n}$. Observe that we have a commutative square

$$\begin{tikzcd} \mathrm{J}_n(X) \arrow[r] \arrow[d, "u_n"'] & X^{\wedge n} \arrow[d] \\ \Omega\Sigma X \arrow[r, "h_n"'] & \Omega\Sigma X^{\wedge n} \end{tikzcd} , \tag{4.34}$$

where the right vertical morphism is the unit. Since X is k-connected, the morphism

$$u_n \colon \mathrm{J}_n(X) \to \Omega\Sigma X$$

is $((n+1)(k+1)-2)$-connected (Lemma 4.20) and $X^{\wedge n}$ is $(n(k+1)-1)$-connected (4.9.3). By the Freudenthal Suspension Theorem (Corollary 4.13) the unit morphism $X^{\wedge n} \to \Omega\Sigma X^{\wedge n}$ is $2(n(k+1)-1)$-connected. Since $n \geq 1$, we have that

$$2(n(k+1)-1) \geq (n+1)(k+1)-2 ,$$

so that both of the vertical morphisms in (4.34) are $((n+1)(k+1)-2)$-connected. Applying Corollary 4.11 to the square (4.34), we see that the induced morphism on horizontal fibers

$$\mathrm{fib}(\mathrm{J}_n(X) \to X^{\wedge n}) \to \mathrm{fib}(h_n)$$

is $((n+1)(k+1)-3)$-connected. Therefore, to prove that the morphism $\mathrm{J}_{n-1}(X) \to \mathrm{fib}(h_n)$ is $((n+1)(k+1)-3)$-connected, it suffices to show that the induced morphism

$$\mathrm{J}_{n-1}(X) \to \mathrm{fib}(\mathrm{J}_n(X) \to X^{\wedge n}) \tag{4.35}$$

is $((n+1)(k+1)-3)$-connected.

Since X is k-connected, $\mathrm{J}_{n-1}(X)$ is k-connected (Corollary 4.19) and the morphism

$$i_{n-1} \colon \mathrm{J}_{n-1}(X) \to \mathrm{J}_n(X)$$

is $(n(k+1)-2)$-connected (Lemma 4.18). Applying the Blakers–Massey Theorem (Theorem 4.12) to the cofiber sequence

$$\begin{tikzcd} \mathrm{J}_{n-1}(X) \arrow[r, "i_{n-1}"] \arrow[d] & \mathrm{J}_n(X) \arrow[d] \\ * \arrow[r] & X^{\wedge n} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

provided by Proposition 4.26, we see that morphism (4.35) is $((n+1)(k+1)-3)$-connected. $\square$

Computational proof of Theorem 4.32. Let $m \coloneq (n+1)(k+1)-3$; we need to show that the map

$$\mathrm{J}_{n-1}(X) \to \mathrm{fib}(h_n)$$

is m-connected. The following two facts allow us to reduce to proving the claim in the case that $\mathcal{X} = \mathbf{Spc}$.

(1) If the conclusion of Theorem 4.32 holds for the ∞-topos $\mathbf{Spc}$, then it holds for any presheaf ∞-topos.
(2) If $L\colon \mathcal{Y} \to \mathcal{X}$ is a left exact left adjoint between ∞-topoi, then L preserves: suspensions, loops, smash products, fibers, the James construction, and m-connectedness (this last fact is [HTT, Proposition 6.5.1.16]).

Now we prove the claim for $\mathcal{X} = \mathbf{Spc}$. The claim is trivial if $n = 1$, so assume that $n \geq 2$. Since X is k-connected by assumption, the smash power $X^{\wedge n}$ is $(nk+n-1)$-connected (4.9.3). Since $\Omega\Sigma X^{\wedge n}$ is simply-connected, the Serre spectral sequence for (integral) homology has E^2-page

$$E^2_{p,q} = \mathrm{H}_p(\Omega\Sigma X^{\wedge n}; \mathrm{H}_q(\mathrm{fib}(h_n))) \cong \mathrm{H}_p(\Omega\Sigma X^{\wedge n}) \otimes \mathrm{H}_q(\mathrm{fib}(h_n)) \ .$$

Since

$$\mathrm{H}_p(\Omega\Sigma X^{\wedge n}) \cong \bigoplus_{i \geq 0} \widetilde{\mathrm{H}}_p(X^{\wedge n})^{\otimes i} \ ,$$

and $\widetilde{\mathrm{H}}_p(X^{\wedge n})^{\otimes i}$ becomes nontrivial in degree $i(nk+n+2)$, we find that

$$\mathrm{H}_p(\Omega\Sigma X^{\wedge n}) \cong \mathrm{H}_p(X^{\wedge n}) \qquad \text{for} \qquad p < 2(nk+n+2) \ .$$

In particular, $E^2_{p,0} = \mathrm{H}_p(X^{\wedge n})$ for $p < 2(nk+n+2)$. Consequently, the Serre spectral sequence has no nontrivial differentials off bidegrees $(p,0)$ with $p < 2(nk+n+2)$.

The E^2-page of this spectral sequence is very simple if $p+q < (n+1)(k+1)$: in this range, $E^2_{p,q}$ vanishes unless one of p or q is zero, in which case

$$E^2_{p,0} = \mathrm{H}_p(X^{\wedge n}) \qquad \text{and} \qquad E^2_{0,q} = \mathrm{H}_q(\mathrm{fib}(h_n))$$

(note that $(n+1)(k+1) \leq 2(nk+n+2)$). Recall that for $p < 2(nk+n+2)$, the Serre spectral sequence has no nontrivial differentials off bidegrees $(p,0)$. Moreover, for $q < (n+1)(k+1)-1$, are also no nontrivial differentials with target in bidegree $(0,q)$. Consequently, for

$$p+q < (n+1)(k+1)-2 \ ,$$

we find that the Serre spectral sequence collapses at the E^2-page, and therefore that

$$\widetilde{\mathrm{H}}_*(\Omega\Sigma X) \cong \widetilde{\mathrm{H}}_*(\mathrm{fib}(h_n)) \oplus \widetilde{\mathrm{H}}_*(\Omega\Sigma X^{\wedge n}) \qquad \text{for} \qquad * < (n+1)(k+1)-2 \ .$$

The map $\mathrm{J}_{n-1}(X) \to \mathrm{fib}(h_n)$ then induces a homology equivalence in degrees $< (n+1)(k+1)-2$. We conclude by the homology Whitehead Theorem. $\square$

Remark 4.36. In the case that $n = 2$ and $\mathcal{X} = \mathbf{Spc}$, the computational proof of the metastable EHP sequence given here reduces to the proof presented in [2, Proposition 3.1.2].

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