

YOU CAN CONSTRUCT G -COMMUTATIVE ALGEBRAS ONE NORM AT A TIME

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ABSTRACT. We define the category of G -operads and the hierarchy of *generalized N_∞ -operads*, which are G -suboperads of Comm_G^\otimes . We exhibit an isomorphism between the category of generalized N_∞ -operads and the self-join poset

$$\text{Op}_G^{GN_\infty} \simeq \text{Ind} - \text{Sys}_G \star \text{Ind} - \text{Sys}_G,$$

where $\text{Ind} - \text{Sys}_G$ is the poset of *indexing systems* in G . This recognizes generalized N_∞ -operads as parameterizing *some commutative multiplicative transfers and possibly a commutative multiplication*. Indeed, their algebras in semiadditive Cartesian categories are incomplete Mackey functors and their algebras in Mackey functors recover incomplete Tambara functors when they are N_∞ operads, i.e. when they contain \mathbb{E}_∞ .

After this, we discuss some in-progress research. Namely, we construct a *Boardman-Vogt tensor product* of G -operads and demonstrate that tensor products of generalized N_∞ operads correspond with joins in $\text{Ind} - \text{Sys}_G \star \text{Ind} - \text{Sys}_G$ i.e. there is an $N_{(IV)_\infty}$ -monoidal equivalence

$$\text{Alg}_{N_{I_\infty}} \text{Alg}_{N_{J_\infty}} C \simeq \text{Alg}_{N_{(IV)_\infty}} C$$

for all $N_{(IV)_\infty}$ -monoidal categories C , allowing G -commutative structures to be constructed “one norm at a time.”

Foreword. The following are notes prepared for a casual talk in the [zygotop](#) seminar concerning research which is currently in-progress. The reader should read with the understanding that they are particularly error-prone, as the non-cited results herein amount to the communication of a pre-draft of a paper in a casual setting. The reader should also henceforth implicitly insert the text ∞ - before the words operad and category.

1. INTRODUCTION

In [Dre71], the concept of a *Mackey functor* was introduced; this structure was described as consisting of functors $M_I : \mathcal{O}_G \rightarrow \mathbf{Mod}_R$ and $M_R : \mathcal{O}_G^{\text{op}} \rightarrow \mathbf{Mod}_R$ which agree on \mathcal{O}_G^\approx and satisfying the *double coset formula*

$$R_J^H I_K^H = \prod_{x \in [J \backslash H / K]} I_{J \cap x K x^{-1}}^I \cdot \text{conj}_X R_{x^{-1} J x \cap K}$$

for all $J, K \subset H$, where $R_J^K := M_R(G/J \rightarrow G/K)$ and similar for I . The ur-example of this is the assignment $H \mapsto \mathbf{Rep}_H$ with covariant functoriality Ind and contravariant functoriality Res . This was repackaged and generalized into the modern definition of the *category of C -valued G -Mackey functors*

$$\mathcal{M}_G(C) := \text{Fun}^\times(\text{Span}(\mathbb{F}_G), C),$$

where \mathbb{F}_G denotes the category of finite G -sets.

In parallel, the concept of *transfer maps in group cohomology* was being developed in [Evens], later lifted to genuine equivariant cohomology in [Greenlees], and finally developed as a functor

$$N_H^G : \text{Sp}_H \rightarrow \text{Sp}_G$$

in [HHR16], which played a crucial role in the solution to the Kervaire invariant one problem. These were noted in [HH16] to satisfy the conditions of a *Symmetric monoidal Mackey functor*, a notion they distinguished from their notion of *G -symmetric monoidal categories* due to coherence issues.

In the broad program announced in [Bar+16], the correct notion of *G -symmetric monoidal G - ∞ -categories* (henceforth *G -symmetric monoidal categories*) was introduced:

Definition 1.1. Let C have finite products. Then, the category of G -commutative monoids in C is

$$\text{CMon}_G(C) := \mathcal{M}_G(C).$$

The category of G -symmetric monoidal categories is $\text{CMon}_G(\mathbf{Cat})$.

We similarly define the *category of small G -categories* as

$$\mathbf{Cat}_G := \text{Fun}(O_G^{\text{op}}, \mathbf{Cat}) \simeq \mathbf{Cat}_{/O_G^{\text{op}}}^{\text{cocart}},$$

where the equivalence is the *straightening-unstraightening construction* of [HTT], and $O_G^{\text{op}} \subset \mathbb{F}_G$ denotes the full subcategory of transitive G -sets, henceforth referred to as the *orbit category*. We may informally summarize the structure of a G -symmetric monoidal category $C^\otimes \in \mathbf{CMon}_G(\mathbf{Cat})$ as consisting of, for every conjugacy class (H) of G , a category with Weyl group action $C_H \in \mathbf{Cat}^{BW_G H}$, as well as functors

$$\begin{aligned} \otimes_H^2 : C_H^2 &\rightarrow C_H, \\ N_K^H : C_K &\rightarrow C_H, \\ \text{Res}_K^H : C_H &\rightarrow C_K \end{aligned}$$

for all subconjugacy classes (K) of (H) . These are supplied with coherent data recognizing them as associative, commutative, unital, and compatible with each other and the Weyl group action. The maps Res encode an underlying G -category C of C^\otimes , and N_K^H is pronounced “the norm from K to H .”

Given C^\otimes a G -symmetric monoidal category, we may informally define a *G -commutative monoid* to be a tuple of objects $(X_H) \in \prod_{G/H \in O_G} C_H$ satisfying

$$X_H \simeq \text{Res}_H^G X_G$$

together with structure maps

$$\begin{aligned} \mu_H^2 : X_H^{\otimes 2} &\rightarrow X_H \\ \text{tr}_K^H : N_K^H X_K &\rightarrow X_H, \end{aligned}$$

for all $H \subset K$, together with coherent associativity, commutativity, and unitality data. We may intuitively view these data as altogether specifying that these structure maps jointly construct a contractible space of maps

$$X^{\otimes S} \rightarrow X_H$$

for all finite H -sets $S \in \mathbb{F}_H$, where

$$X^{\otimes S} := \bigotimes_{H/K \in \text{Orb}(S)} N_K^H X_K.$$

The map tr_K^H is pronounced “the transfer from K to H .” When $C^\otimes = M_G(C)^\otimes$ with the *HHR norm* G -symmetric monoidal structure of [HH16], these are called *G -Tambara functors valued in C* .

This talk concerns various relaxations of the notion of G -commutative algebras. Namely, we will define a symmetric monoidal closed category Op_G of (colored) G -operads, whose internal hom $\underline{\mathbf{Alg}}_O(C)^\otimes$ is called the *operad of algebras under pointwise tensors*, and whose tensor product is called the *Boardman-Vogt tensor product*.

We are particularly interested in \mathcal{N}_∞ operads, which interpolate between \mathbb{E}_∞ and the G -operad Comm_G which encodes G -commutative algebras by adding a subset of the transfers parameterized by Comm_G . These transfers are structured:

Definition 1.2. A *G -transfer system* is a core-preserving wide subcategory $O_G^\approx \subset T \subset O_G$ which is closed under base change, i.e. for any diagram in O_G

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow \alpha' & & \downarrow \alpha \\ U' & \longrightarrow & V' \end{array}$$

with $U \hookrightarrow V' \times_{U'} V$ a summand inclusion (pullback taken in \mathbb{F}_G) and $\alpha \in T$, we have $\alpha' \in T$.

An *indexing system* is a subcategory $I \subset \mathbb{F}_G$ induced by a transfer system under taking coproducts. A *generalized indexing system* is a core-preserving subcategory $I \subset \mathbb{F}_G$ which is either an indexing system or is attained by one by removing all non-isomorphisms of trivial H -sets. The poset of indexing systems under inclusion is denoted $\text{Ind} - \text{Sys}_G$, and the poset of generalized indexing systems is denoted $\text{Ind} - \text{Sys}_G$.

It is not hard to see that there is an equivalence of posets

$$\widehat{\text{Ind} - \text{Sys}_G} \simeq \text{Ind} - \text{Sys}_G \star \text{Ind} - \text{Sys}_G,$$

and in particular, generalized indexing systems decompose into two different join-stable copies of indexing systems, depending on whether maps of trivial G -sets are included.

Given a generalized indexing system I , we will construct an operad called $N_{I\infty}^\otimes$ encoding precisely the maps tr_K^H such that $K \hookrightarrow H$ is in I , as well as encoding the map μ_H if and only if I is an indexing system. The main theorem of this talk characterizes the tensor products of generalized N_∞ operads.

Theorem A. *There is a fully faithful and symmetric monoidal inclusion*

$$N_{(-)\infty}^\otimes : \widehat{\text{Ind} - \text{Sys}_G} \xhookrightarrow{\Pi} \text{Op}_G^\otimes$$

whose image consists of the G -suboperads of Comm_G^\otimes , and when restricted to the indexing systems has image consisting of G -operads \mathcal{O}^\otimes possessing diagrams $\mathbb{E}_\infty^\otimes \subset \mathcal{O}^\otimes \subset \text{Comm}_G^\otimes$. In particular, for \mathcal{C}^\otimes an $N_{(I\vee J)\infty}$ -monoidal category, there is a canonical $N_{(I\vee J)\infty}$ -monoidal equivalence

$$\underline{\text{Alg}}_{N_{I\infty}}^\otimes \underline{\text{Alg}}_{N_{J\infty}}^\otimes \mathcal{C} \simeq \underline{\text{Alg}}_{N_{(I\vee J)\infty}}^\otimes \mathcal{C}.$$

We say an inclusion of subgroup $H \subset K$ is *atomic* if it is proper and there exist no chains of proper subgroup inclusions $H \subset J \subset K$. More generally, we say that a conjugacy class $(H) \in \text{Conj}(G)$ is an *atomic subclass* of (K) if there exists an atomic inclusion $\tilde{H} \subset \tilde{K}$ with $\tilde{H} \in (H)$ and $\tilde{K} \in (K)$, and we say that (K) is atomic if the canonical inclusion $1 \hookrightarrow K$ is atomic.

Given $(H) \subset (K)$ an atomic subclass, we refer to the N^∞ -operad corresponding to the minimal index system containing the inclusion $H \hookrightarrow K$ as $N^\infty(H, K)$. When $(H) = (1)$, we instead simply write $N^\infty(K)$.

Corollary B. *Let $1 = G_n \subset G_{n-1} \subset \dots \subset G_0 = G$ be a maximal subgroup series of a finite group, and let \mathcal{C} be a G -symmetric monoidal category. Then, there exists a canonical G -symmetric monoidal equivalence*

$$\underline{\text{Alg}}_{N^\infty(G_1, G_0)}^\otimes \cdots \underline{\text{Alg}}_{N^\infty(G_n, G_{n-1})}^\otimes \mathcal{C} \simeq \underline{\text{CAlg}}_G^\otimes \mathcal{C}$$

Furthermore, if $G \simeq H \times J$, then

$$\underline{\text{CAlg}}_H^\otimes \underline{\text{CAlg}}_J^\otimes \mathcal{C} \simeq \underline{\text{CAlg}}_G^\otimes \mathcal{C}.$$

Remark. One may worry about the comparison between models for G -operads, as our notion of N_∞ -operads is ostensibly embedded deep within the world of G - ∞ -operads, which are not known to be equivalent to the ∞ -category presented by the graph model structure or by genuine G operads.

However, some work has been done to simplify the story of N_∞ operads in the model story; in particular, by [Rub21, Thm 2.16, 3.6], the full ∞ -category of the ∞ -category of *genuine* G -operads is equivalent to $\text{Ind} - \text{Sys}_G$ via a functor A which sits in a commutative diagram

$$\begin{array}{ccc} \text{Op}_G^{\text{gen}, N_\infty} & \xrightarrow{N|_{N_\infty}} & \text{Op}_G^{N_\infty} \\ & \searrow A & \downarrow A \\ & & \text{Ind} - \text{Sys}_G \end{array}$$

where we use that the functor N of [BP21] is canonically ∞ -categorical when restricted to full subcategories of Op_G^{gen} which happen to be 1-categories and map to a 1-subcategory of Op_G . Both functors named A are equivalences (c.f. [Ex 2.4.7]Nardin), and hence $N|_{N_\infty}$ is an equivalence.

2. THE IDEAS

2.1. Fibrous patterns. In order to precisely define G -operads, the most efficient way will be to go through the technology of *algebraic patterns*, a concept first defined by German mathematician Honyi Chu and the Norwegian mathematician Rune Haugseng, who generally referred to them using the letter \mathcal{O} .

Definition 2.1. An *algebraic pattern* is an ∞ -category \mathcal{J} , together with a factorization system $(\mathcal{J}^{\text{int}}, \mathcal{J}^{\text{act}})$ of \mathcal{J} and a full subcategory $\mathcal{J}^{\text{el}} \subset \mathcal{J}^{\text{int}}$. The *category of algebraic patterns* is the full subcategory

$$\text{AlgPatt} \subset \text{Fun}(D, \text{Cat})$$

spanned by algebraic patterns, where $D := \bullet \rightarrow \bullet \rightarrow \bullet \leftarrow \bullet$.

Maps in \mathcal{J}^{int} and \mathcal{J}^{act} are pronounced *inert* and *active maps*, and objects of \mathcal{J}^{el} are pronounced *elementary objects*. For instance, \mathbb{F}_* , together with its inert and active maps as defined in [HA, § 2] and elementary objects $\{\langle 1 \rangle\}$ determines an algebraic pattern. In analogy with [HA, § 2], we will use these to develop a notion of operads, called *fibrous patterns*.

Definition 2.2. Let \mathcal{J} be an algebraic pattern. A *fibrous \mathcal{J} -pattern* is a map of algebraic patterns $\pi : \mathcal{O} \rightarrow \mathcal{J}$ such that

- (1) \mathcal{O} has π -cocartesian lifts for inert morphisms of \mathcal{J} ,
- (2) (Segal condition for colors) For every active morphism $\omega : V_0 \rightarrow V_1$ in \mathcal{J} , the functor

$$O_{V_0} \rightarrow \lim_{\alpha \in \mathcal{J}_{V_1/}^{\text{el}}} O_{\omega_{\alpha,!} V_1}^{\simeq}$$

induced by cocartesian transport along ω_{α} is an equivalence, where $\omega_{(-)} : \mathcal{J}_{Y/}^{\text{el}} \rightarrow \mathcal{J}_{X/}^{\text{int}}$ is the inert morphism appearing in the inert-active factorization of $\alpha \circ \omega$, and

- (3) (Segal condition for multimorphism) for every active morphism $\omega : V_1 \rightarrow V_2$ in \mathcal{J} and all objects $X_i \in \mathcal{O}_{\mathcal{J}_{V_i/}}$, the commutative square

$$\begin{array}{ccc} \text{Map}_{\mathcal{O}}(X_0, X_1) & \longrightarrow & \lim_{\alpha \in \mathcal{J}_{V_1/}^{\text{el}}} \text{Map}_{\mathcal{O}}(X_0, \omega_{\alpha,!} X_1) \\ \downarrow & & \downarrow \\ \text{Map}_{\mathcal{J}}(V_0, V_1) & \longrightarrow & \lim_{\alpha \in \mathcal{J}_{O_1/}^{\text{el}}} \text{Map}_{\mathcal{J}}(O_0, \omega_{\alpha,!} O_1) \end{array}$$

is cartesian.

A fibrous \mathcal{J} -pattern $\pi : \mathcal{C} \rightarrow \mathcal{J}$ is a *Segal \mathcal{J} -category* if π is a cocartesian fibration. The category of fibrous \mathcal{J} -patterns is the full subcategory

$$\text{Fbrs}(\mathcal{J}) \subset \text{AlgPatt}_{/\mathcal{J}}$$

spanned by fibrous patterns, and the category of Segal \mathcal{J} -categories is the full subcategory of

$$\text{Seg}_{\mathcal{J}}(\text{Cat}) \subset \text{Cat}_{/\mathcal{J}}^{\text{cocart}}$$

spanned by Segal \mathcal{J} -categories.

We state one technical lemma:

Lemma 2.3. *All of the inclusions*

$$\text{Seg}(\mathcal{J}) \rightarrow \text{Fbrs}(\mathcal{J}) \hookrightarrow \text{AlgPatt}_{/\mathcal{J}} \rightarrow \text{Cat}_{/\mathcal{J}} \rightarrow \text{Cat}$$

have left adjoints; in particular, the full subcategory $\text{Fbrs}(\mathcal{J}) \subset \text{AlgPatt}_{/\mathcal{J}}$ is localizing.

We refer to the left adjoint $\text{Env} : \text{Fbrs}(\mathcal{J}) \rightarrow \text{Seg}(\mathcal{J})$ as the *Segal envelope*, and we use it analogously to the *symmetric monoidal envelope*, reducing the question of characterizing maps of fibrous patterns into Segal \mathcal{J} -categories into simply a question of characterizing maps of Segal \mathcal{J} -categories, which is much simpler.

Example 2.4:

Definition 2.5. Given the data of \mathcal{X} a category, $\mathcal{X}_b, \mathcal{X}_f$ wide subcategories, and $\mathcal{X}_0 \subset \mathcal{X}_b$ a full subcategory, we define the *span pattern* $\text{Span}_{b,f}(\mathcal{X}; \mathcal{X}_0)$ to have:

- underlying category $\text{Span}_{b,f}(\mathcal{X})$ whose objects are objects in \mathcal{X} and whose morphisms $X \rightarrow Z$ are spans

$$X \xleftarrow{B} Y \xrightarrow{F} Z$$

with $B \in \mathcal{X}_b$ and $F \in \mathcal{X}_f$.

- inert morphisms $\mathcal{X}_b^{\text{op}} \subset \text{Span}(\mathcal{X})$.
- active morphisms $\mathcal{X}_f \subset \text{Span}(\mathcal{X})$.
- Elementary objects $\mathcal{X}_0^{\text{el}} \subset \mathcal{X}_b^{\text{op}}$.

Then, for instance we have the following:

Theorem 2.6 ([BHS22]). *Pullback along the inclusion $\mathbb{F}_* \hookrightarrow \text{Span}(\mathbb{F})$ induces an equivalence on the categories of fibrous patterns and Segal categories.*

2.2. G-operads and I-operads. There is an adjunction

$$\text{Tot} : \mathbf{Cat}_G \rightleftarrows \mathbf{Cat} : \text{CoFr}^G$$

where Tot takes the total category of a cocartesian fibration and $\text{CoFr}^G(C)$ is classified by functor categories

$$\text{CoFr}^G(C)_H := \text{Fun}(O_H^{\text{op}}, C)$$

with functoriality dictated by pullback. In particular, the *G-category of small G-categories* $\underline{\mathbf{Cat}}_G := \text{CoFr}^G(C)$ has G-fixed points given by \mathbf{Cat} .

Remark. Elmendorf's theorem may be reinterpreted in this language as the statement that the *G-category of G-spaces* $\underline{\mathcal{S}}_G$ is G-cofreely generated by \mathcal{S} .

Let $\mathbb{F}_G := \text{CoFr}^G(\mathbb{F})$ and let $\mathbb{F}_{G,*} := \text{CoFr}^G(\mathbb{F}_*)$. Then, there is an equivariant lift of [theorem 2.6](#).

Theorem 2.7 ([BHS22]). *Pullback along the composition $\mathbb{F}_{G,*} \hookrightarrow \text{Span}(\text{Tot}\mathbb{F}_G) \xrightarrow{U} \text{Span}(\mathbb{F}_G)$ induces an equivalence on the categories of fibrous patterns and Segal categories, where \mathbb{F}_G is the category of G-sets.*

Definition 2.8. The category of G-operads is the category of fibrous patterns

$$\text{Op}_G := \text{Fbrs}(\text{Span}(\mathbb{F}_G)).$$

If O, \mathcal{P} are G-operads, the category of *O-algebras in \mathcal{P}* is the functor category of algebraic patterns

$$\mathbf{Alg}_O(\mathcal{P}) := \text{Fun}_{\text{AlgPatt}}(O, \mathcal{P}).$$

We may equivalently characterize *O-algebras in \mathcal{P}* as functors which preserve cocartesian lifts of inert morphisms. In order to identify G-operads, we use the following exercise in category theory which was carried out in [BHS22, § 5.2].

Proposition 2.9. *An identity-on-objects functor $\pi : O \rightarrow \text{Span}(\mathbb{F}_G)$ is a G-operad if and only if it satisfies the following conditions:*

- (1) *O has π -cocartesian lifts for inert morphisms of $\text{Span}(\mathbb{F}_G)$.*
- (2) *For every map of G-sets $S \rightarrow T$, the inert morphisms $\{U \leftarrow T \mid U \in \text{Orb}(T)\}$ induce equivalences*

$$\text{Map}_O(S, T) \simeq \prod_{U \in \text{Orb}(T)} \text{Map}_O(S, U).$$

Furthermore, a cocartesian fibration $\pi : O \rightarrow \text{Span}(\mathbb{F}_G)$ is a Segal $\text{Span}(\mathbb{F}_G)$ -category if and only if it unstraightens to a G-symmetric monoidal category.

We may further reorganize this through the following elementary lemma about G-sets.

Lemma 2.10. *The assignment $\varphi : T \mapsto \text{Ind}_H^G T \rightarrow G/H$ underlies an equivalence of categories*

$$\mathbb{F}_H \simeq (\mathbb{F}_G)_{/G/H}.$$

Write $\Sigma_G \simeq \text{CoFr}^G(\mathbb{F}^\infty)$. By applying [lemma 2.10](#) and taking cores of slice categories, we construct a forgetful functor

$$O_{\text{sseq}} : \text{Op}_G^{\text{one-object}} \rightarrow \text{Fun}(\text{Tot}\Sigma_G, \mathcal{S})$$

with value on $S \in \mathbb{F}_H$ given by $\pi_O^{-1}(\text{Ind}_H^G S \rightarrow G/H)$. We refer to $O(S) := O_{\text{sseq}}(S)$ as the *space of S-ary operations*. This functor is further analyzed in [section 3.1](#), where e.g. it is shown to be conservative.

An early observation about genuine equivariant homotopy coherent algebraic structures is that the structure of transfers *does not come canonically* from an \mathbb{E}_∞ -structure; that is, $\mathbb{E}_\infty \in \text{Op}_G$ is not terminal. The failure of \mathbb{E}_∞ to be terminal is parameterized by the category of *generalized N^∞ -operads*:

Definition 2.11. Write $\text{Comm}_G^\otimes := (\text{Span}(\mathbb{F}_G) = \text{Span}(\mathbb{F}_G))$ for the terminal G -operad. A G -operad \mathcal{O}^\otimes is a *generalized N^∞ -operad* if the unique morphism $\mathcal{O}^\otimes \rightarrow \text{Comm}_G^\otimes$ is a monomorphism, i.e. it has one object and

$$\mathcal{O}(S) \in \{*, \emptyset\}$$

for all $S \in \mathbb{F}_H$.

A generalized N^∞ operad $\mathcal{N}_{\infty I}$ is an N^∞ operad if it admits a map

$$\mathbb{E}_\infty \rightarrow \mathcal{O}^\otimes,$$

i.e. $\mathcal{O}(S) \simeq *$ whenever $S \in \mathbb{F}_H$ has trivial H -action.

Write $\text{Op}_G^{GN^\infty}$ for the full subcategory consisting of generalized N_∞ -operads. The following proposition is an exercise in category theory, and establishes that a map to an N_∞ operad is a *property*, not a structure.

Proposition 2.12. *Given $\mathcal{N}_{I_\infty} \in \text{Op}_G^{GN^\infty}$ a generalized N_∞ operad, the forgetful functor*

$$\text{Op}_{G,/\mathcal{N}_{I_\infty}} \rightarrow \text{Op}_G$$

is fully faithful.

Proof idea. It is equivalent to prove that $\text{Map}(\mathcal{O}, \mathcal{N}_{I_\infty}) \in \{*, \emptyset\}$ for all $\mathcal{O} \in \text{Op}_G$. In fact, there is a localizing (1-) subcategory $N : \text{Op}_{1,G} \hookrightarrow \text{Op}_G$ consisting of operads whose structure spaces are discrete, and whose localization functor $h : \text{Op}_G \rightarrow \text{Op}_{1,G}$ takes π_0 of the structure spaces. \mathcal{N}_{I_∞} evidently lies in $\text{Op}_{1,G}$, so we have

$$\text{Map}_{\text{Op}_G}(\mathcal{O}, \mathcal{N}_{I_\infty}) \simeq \text{Hom}_{\text{Op}_{1,G}}(h\mathcal{O}, \mathcal{N}_{I_\infty}).$$

Hence it suffices to check that the latter set is empty or contractible. This is easy to see in $\text{Op}_{1,G}$, since $\text{Hom}(-, *)$ and $\text{Hom}(-, \emptyset)$ are always either empty or contractible. \square

In particular, this implies that $\text{Op}_G^{GN^\infty}$ is a poset, so we'd like to identify this poset. There is a functor

$$A : \text{Op}_G \rightarrow \widehat{\text{Ind} - \text{Sys}_G}$$

called the *admissible sets* with value over G/H given by

$$A(\mathcal{O})_{/G/H} := \{S \rightarrow G/H \mid \pi_{\mathcal{O}}^{-1}(S \rightarrow G/H) \neq \emptyset\}$$

and extended to general G -sets by coproducts. The following proposition is an exercise in category theory (see [NS22, Ex 2.4.7], or the original references [BH15; GW18; Rub21]):

Proposition 2.13. *The restricted functor*

$$A : \text{Op}_G^{GN^\infty} \rightarrow \widehat{\text{Ind} - \text{Sys}_G}$$

is an equivalence of categories.

We denote by $\mathcal{N}_{(-)\infty}$ the composite functor

$$\mathcal{N}_{(-)\infty} : \widehat{\text{Ind} - \text{Sys}_G} \xrightarrow{A^{-1}} \text{Op}_G^{GN^\infty} \hookrightarrow \text{Op}_G$$

Using this, we finally define *I-operads*.

Definition 2.14. Let I be a generalized indexing system. Then, the *category of I-operads* is the slice category

$$\text{Op}_I := \text{Op}_{G,/\mathcal{N}_{\infty I}^\otimes}.$$

Given $\mathcal{O}^\otimes, \mathcal{P}^\otimes \in \text{Op}_I$, the *category of \mathcal{O} -algebras in \mathcal{P}* is the full subcategory

$$\text{Alg}_{\mathcal{O}}(C) \subset \text{Fun}_{/\mathcal{N}_{\infty I}^\otimes}(\mathcal{O}^\otimes, C^\otimes)$$

spanned by maps of I -operads.

Remark. The notation $\text{Alg}_{\mathcal{O}}(C)$ does not include I . This presents no problem; indeed, by [proposition 2.12](#), the categories of \mathcal{O} -algebras in \mathcal{P} considered over various indexing systems (including the terminal one, i.e. in G -operads) are canonically equivalent to one another.

A useful property of these are that G operads *fibred* over \mathcal{O}^\otimes have an intrinsic description in terms of \mathcal{O} . We may state these in the language of fibrous patterns.

Proposition 2.15 ([BHS22, Cor 4.1.17]). *Let O be a fibrous \mathcal{J} -pattern. Then, the pushforward functor $\pi_! : \text{AlgPatt}_{/O} \rightarrow \text{AlgPatt}_{/\mathcal{J}}$ preserves fibrous patterns, and the associated functor*

$$\pi_! : \text{Fbrs}(O) \rightarrow \text{Fbrs}(\mathcal{J})_{/O}$$

is an equivalence of categories.

In particular, the category of I -operads is covariantly functorial in I , and it possesses an intrinsic expression along the lines of ??.

Example 2.16:

Let $\mathcal{F} \subset O_G$ be a *family*, i.e. a collection of subgroups of G closed under sub-conjugation. Then, $\mathcal{F} \cup O_G^\sim$ is a transfer system, and we denote by $\mathcal{I}_{\mathcal{F}}$ the corresponding indexing system.

Let V be a real orthogonal G -representation, let \mathcal{F}_V is the family consisting of subgroups H such that $V^H \neq *$, and let $\mathcal{I}_V := \mathcal{I}_{\mathcal{F}_V}$. Then, there is an \mathcal{I}_V -operad \mathbb{E}_V of *little V -disks*, which may be informally understood to have S -ary operations the H -equivariant embeddings $S \hookrightarrow V$:

$$\mathbb{E}_V(S) \simeq \text{Conf}_H(S, V).$$

This along with a computation of the G -symmetric monoidal envelope was carried out in ??. These participate in *equivariant infinite loop space theory*, in the sense that there is a fully faithful embedding

$$\{V - \text{loop spaces}\} \hookrightarrow \mathbf{Alg}_{\mathbb{E}_V}(S_G)$$

with image given by the \mathbb{E}_V spaces satisfying a grouplike condition, up to model categorical weirdness. See [GM11] for details.

2.3. The BV tensor product. By lemma 2.3, the category of algebraic patterns has a cartesian monoidal structure such that the *underlying category* functor $U : \text{AlgPatt}^\times \rightarrow \mathbf{Cat}^\times$ is symmetric monoidal.

Definition 2.17. The category of *symmetric monoidal algebraic patterns* is $\mathbf{CMon}(\text{AlgPatt})$.

By [HA, § 2.2], a symmetric monoidal structure on \mathcal{J} endows on the slice category $\text{AlgPatt}_{/\mathcal{J}}^\otimes$ a symmetric monoidal structure, which we may view as taking O, \mathcal{P} to the tensor product

$$O \times \mathcal{P} \rightarrow \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}.$$

Definition 2.18. The *Boardman-Vogt symmetric monoidal category of fibrous \mathcal{J} -patterns* is the localized symmetric monoidal structure

$$\text{Fbrs}(\mathcal{J})^\otimes \hookrightarrow \text{AlgPatt}_{/\mathcal{J}}^\times.$$

We may view the tensor product of fibrous \mathcal{J} -patterns as yielding the localized composite

$$O \otimes \mathcal{P} := L_{\text{Fbrs}}(O \times \mathcal{P} \rightarrow \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}).$$

Note that the category \mathbb{F}_G has finite products, and any indexing system \mathcal{I} is closed under products. In particular, this endows $i : \mathcal{N}_{\mathcal{I}^\infty}^\otimes \rightarrow \text{Span}(\mathbb{F}_G)$ with the structure of a map of symmetric monoidal algebraic patterns under $\text{Span}(\times)$.

Definition 2.19. The *Boardman-Vogt symmetric monoidal category of I -operads* is

$$\text{Op}_I^\otimes := \text{Fbrs}(\mathcal{N}_{\mathcal{I}^\infty})$$

Proposition 2.20. *Given an inclusion $i : \mathcal{N}_{\mathcal{I}^\infty} \hookrightarrow \mathcal{N}_{\mathcal{J}^\infty}$, pushforward along i yields a functor*

$$i_! : \text{Op}_I^\otimes \rightarrow \text{Op}_{\mathcal{J}}^\otimes$$

realizing Op_I as a symmetric monoidal colocalizing subcategory of $\text{Op}_{\mathcal{J}}$.

The verification of this comes down to the following fact, which follows from the results of [HA, § 2.2.2], and is almost generalized by [Bar23, p. 2.37].

Lemma 2.21. *Given $f : X \rightarrow Y$ a map of commutative algebra objects in \mathcal{C} a symmetric monoidal category, the associated functor $f_! : \mathcal{C}_{/X} \rightarrow \mathcal{C}_{/Y}$ lifts to a canonical symmetric monoidal functor between the slice symmetric monoidal categories.*

We may “see” this fact by staring at the following commutative diagram:

$$\begin{array}{ccccc}
 & & X \otimes X & \longrightarrow & X \\
 & \nearrow & \downarrow & & \downarrow \\
 A \otimes B & & & & \\
 & \searrow & Y \otimes Y & \longrightarrow & Y
 \end{array}$$

The BV tensor product satisfies a mapping-out property; namely, we review in [section 3.3](#) the construction due to [\[NS22, § 5.3\]](#) of the operad $\underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes}(Q)$, and we prove the following theorem.

Theorem 2.22. *There is a natural equivalence of operads*

$$\underline{\mathbf{Alg}}_{\mathcal{O} \otimes \mathcal{P}}^{\otimes} Q \simeq \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes} \underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes} Q$$

realizing $\mathbf{Alg}_{\mathcal{P}}^{\otimes}(-)$ as an internal hom for the BV tensor product.

2.4. Summary of the argument. We would like to construct an equivalence $\mathcal{N}_{I\infty} \otimes \mathcal{N}_{J\infty} \simeq \mathcal{N}_{(I \vee J)\infty}$. Let’s begin with the special case $I \subset J$; in this case, we can say something stronger.

Proposition 2.23. *If \mathcal{O} is a one-object G -operad, then the map $\mathcal{N}^{\infty}(I) \rightarrow \mathcal{N}^{\infty}(I) \otimes \mathcal{O}$ is an I -equivalence; in particular, $\mathcal{N}^{\infty}(I)$ is \otimes -idempotent.*

To prove this, we use [\[NS22, Cor 5.3.9\]](#); in particular, they generalize [\[HA\]](#) to verify that any of the following conditions are true of $\mathbf{Alg}_{\mathcal{N}^{\infty}(I)}^{\otimes}(C)$, which recognize it as I -cocartesian:

Lemma 2.24. *The following are equivalent:*

- (1) *For all one-object I -operads \mathcal{O} , the forgetful functor $\mathbf{Alg}^{\mathcal{O}}(C) \rightarrow C$ is an equivalence.*
- (2) *For all maps $f : S \rightarrow T$ in I , the action map $f_{\otimes} : C_S \rightarrow C_T$ is left adjoint to the pullback map $f^* : C_S \rightarrow C_T$.*

We prove this in ???. Having proved this, we acquire a (unique) diagram

$$\begin{array}{ccc}
 \mathcal{N}_{I\infty} & & \\
 \searrow & \nearrow & \\
 & \mathcal{N}_{I\infty} \otimes \mathcal{N}_{J\infty} & \xrightarrow{\varphi} \mathcal{N}_{(I \vee J)\infty} \otimes \mathcal{N}_{(I \vee J)\infty} = \mathcal{N}_{(I \vee J)\infty} \\
 \nearrow & \nwarrow & \\
 \mathcal{N}_{J\infty} & &
 \end{array}$$

and we are tasked with proving that φ is an equivalence. An unfortunate fact is that the functor $U : \mathbf{Op}_{I \vee J} \rightarrow \mathbf{Op}_I \times \mathbf{Op}_J$ doesn’t appear to be conservative in general. Our strategy will come down to trying *really hard* to make it conservative. We do so via the following two lemmas, proved as [lemmas 3.5](#) and [3.6](#).

Lemma 2.25. *Denote by $i : I \cup J \subset I \vee J$ the (non-indexing system) union of subcategories. Then, the pullback*

$$\mathbf{Fbrs}(\mathbf{Span}(I \cup J)) \rightarrow \mathbf{Op}_I \times \mathbf{Op}_J$$

is conservative and symmetric monoidal. In particular, U reflects equivalences between $I \vee J$ -operads in the image of $L_{\mathbf{Fbrs}} i_!$.

Lemma 2.26. *There is an equivalence $\mathcal{N}_{(I \vee J)\infty} \simeq L_{\mathbf{Fbrs}} i_! \mathbf{Span}(I \cup J)$.*

Proof of theorem A. By the above argument, it suffices to prove that φ is an equivalence; in fact, by [lemmas 2.25](#) and [2.26](#) and symmetry it suffices to prove that the localized functor

$$\iota_J^* \mathcal{N}_{I \cap J \infty} \otimes \mathcal{N}_{J\infty} \rightarrow \iota_J^* \mathcal{N}_{I \vee J}$$

is an equivalence. But $\iota_J^* \mathcal{N}_{I\infty} \simeq \mathcal{N}_{I \cap J \infty}$, so the above is the inclusion $\mathcal{N}_{I \cap J \infty} \otimes \mathcal{N}_{J\infty} \rightarrow \mathcal{N}_{J\infty}$, which is an equivalence by [proposition 2.23](#). \square

3. TECHNICAL NONSENSE

3.1. Passing to monads is conservative. Our arguments will be reminiscent of [SY19, § 2.3-2.4]. Let $\mathbf{Fbrs}_\bullet(\mathcal{I})$ denote the full subcategory of fibrous patterns whose associated maps $\mathcal{O}^{\text{el}} \rightarrow \mathcal{I}^{\text{el}}$ are equivalences. Define the functor $(-)_{\text{sseq}}$ to be the composite

$$\mathbf{Fbrs}_\bullet(\mathcal{I}) \xrightarrow{\varphi} \mathbf{Fun}(\mathbf{Ar}^{\text{act}}(\mathcal{I}), \mathcal{S}) \rightarrow \mathbf{Fun}(\underline{\Sigma}_{\mathcal{I}}, \mathcal{S})$$

where $\underline{\Sigma}_{\mathcal{I}} \subset \mathbf{Ar}^{\text{act}}(\mathcal{I})$ is the full subcategory of active arrows whose targets are elementary objects.

Lemma 3.1 (C.f. [SY19, Prop 2.3.6]). *The functor $(-)_{\text{sseq}}$ is conservative.*

Proof. Suppose $f : \mathcal{O} \rightarrow \mathcal{P}$ induces an equivalence $f_{\text{sseq}} : \mathcal{O}_{\text{sseq}} \simeq \mathcal{P}_{\text{sseq}}$. By the definition of fibrous patterns, this implies that $\varphi(f)$ is an equivalence.

Note that $\text{Env}_{\mathcal{I}}^{\mathcal{A}} f = (-) \times_{\mathcal{O}} \mathbf{Ar}^{\text{act}}(\mathcal{O})$ is identity on objects, so it is essentially surjective; the natural transformation $\varphi(f)$ precisely specifies the action of $\text{Env}_{\mathcal{I}}^{\mathcal{A}} f$ on morphisms, so $\text{Env}_{\mathcal{I}}^{\mathcal{A}} f$ is an equivalence. Since $\text{Env}_{\mathcal{I}}^{\mathcal{A}}$ is fully faithful, this implies that f is an equivalence. \square

We now specialize to the case $\mathcal{I} = \text{Span}(I)$. Note that $\underline{\Sigma}_{\text{Span}(\mathbb{F}_G)} \simeq \underline{\Sigma}_G$, where $\underline{\Sigma}_G \simeq \text{CoFr}^G \Sigma$. Furthermore, $\underline{\Sigma}_{\text{Span}(I)} \rightarrow \underline{\Sigma}_G$ is fully faithful with image spanned by I -admissible H -sets; we refer to this as $\underline{\Sigma}_I$. Hence we may translate lemma 3.1 to the following:

Proposition 3.2. *The forgetful functor*

$$(-)_{\text{sseq}} : \mathbf{Op}_I \rightarrow \mathbf{Fun}(\underline{\Sigma}_I, \mathcal{S})$$

sending $\mathcal{O}(S) := \pi_O^{-1}(\text{Ind}_H^G S \rightarrow G/H)$ for all $S \in \mathbb{F}_H \cap I$ is conservative.

Remark. The genuine model structure $\text{Sym}_\bullet^G(\mathbf{sSet})$ of [BP22] exists and presents $\mathbf{Fun}(\text{Tot} \underline{\Sigma}_G, \mathcal{S})$; the ∞ -category of Genuine G -operads are then algebras over a monad on $\mathbf{Fun}(\text{Tot} \underline{\Sigma}_G, \mathcal{S})$ which are explicitly defined in [BP21]. In this setting, lemma 3.1 amounts to a verification of one of the two Barr-Beck conditions expressing U as monadic (cf [HA, Thm 4.7.3.5]), and hence we view it as a step in the direction of proving that these two models are equivalent.

We say that a G -operad \mathcal{O}^\otimes is *reduced* if $\mathcal{O}(T) = *$ whenever T is empty or a transitive H set. Let \mathcal{O}^\otimes be a reduced G -operad, \mathcal{C} a G -symmetric monoidal category, and $X : \text{triv}^\otimes \rightarrow \mathcal{C}^\otimes$ a G -object. Denote by $X_{\text{sseq}} \in \mathbf{Fun}_G(\underline{\Sigma}_G, \mathcal{C})$ the functor of G -categories underlying the adjunct map of G -symmetric monoidal categories to X . We can use this to characterize the monad associated with an operad.

We say that a symmetric monoidal category is *distributive* if the action maps $f_\otimes : \mathcal{C}_S \rightarrow \mathcal{C}_T$ preserve coproducts separately in each variable (see [NS22]).

Proposition 3.3. *Let \mathcal{O} be a reduced G -operad and let \mathcal{C}^\otimes be a distributive G -symmetric monoidal category. Then, the forgetful map $\underline{\text{Alg}}_{\mathcal{O}}(\mathcal{C}) \rightarrow \mathcal{C}$ is monadic, and the associated monad $T_{\mathcal{O}}$ acts on $X \in \mathcal{C}$ as*

$$T_{\mathcal{O}}X := \text{colim } X_{\text{sseq}}.$$

In particular, we have

$$(T_{\mathcal{O}}X)^H \simeq \coprod_{\substack{K \subset H \\ S \in \mathbb{F}_K}} \left(\mathcal{O}(S) \otimes X^{\otimes \text{Ind}_K^H S} \right)_{h \in \text{Aut}_K S},$$

where for all $S' \in \mathbb{F}_H$, we write

$$X^{\otimes S'} := \bigotimes_{U \in \text{Orb}(S')} N_U^H X_U.$$

Proof. Monadicity is precisely [NS22, Cor 5.1.5] when $\mathcal{T} = \mathcal{O}_G$, so it suffices to compute the associated monad in this case. Note that $X_{\text{sseq}}(S) \simeq \mathcal{O}(S) \otimes X^{\otimes S}$, so the computation of $(T_{\mathcal{O}}X)^H$ follows immediately from the statement $T_{\mathcal{O}}X \simeq \text{colim } X_{\text{sseq}}$, so it suffices to prove this statement.

By [NS22, Rem 4.3.6], the left adjoint $\text{Fr} : \mathcal{C} \rightarrow \mathbf{Alg}_O(\mathcal{C})$ is computed on X by G -operadic left Kan extension of the corresponding map $\text{triv}^\otimes \xrightarrow{X} \mathcal{C}^\otimes$ along the canonical inclusion $\text{triv}^\otimes \rightarrow \mathcal{O}^\otimes$; the underlying G -functor of this is computed by the G -left Kan extension

$$\begin{array}{ccc} \Sigma_G & \xlongequal{\quad} & \text{Env}_O \text{triv} \xrightarrow{X} \mathcal{C} \\ \downarrow & & \downarrow \quad \searrow \text{Fr } X \\ *_G & \xlongequal{\quad} & \mathcal{O} \end{array}$$

I.e. by the indexed colimit

$$T_O X \simeq \text{colim } X_{\text{sseq}}.$$

□

Suppose \mathcal{C} is a finitely cocomplete Cartesian closed category, and let $\text{CoFr}^G(\mathcal{C})$ be the G -category of G -coefficient systems valued in \mathcal{C} , and write $\mathcal{C}_G := \text{CoFr}^G(\mathcal{C})^G \simeq \text{Fun}(\mathcal{O}_G^{\text{op}}, \mathcal{C})$. By [NS22, Prop 3.2.5] (noting that all colimits involved are finite), the G -Cartesian G -symmetric monoidal structure on $\text{CoFr}^G(\mathcal{C})$ is distributive. Using Elmendorf's theorem, we apply this to \mathcal{S}_G :

Corollary 3.4. *Let \mathcal{O} be a reduced G -operad. Then, the functor $\mathbf{Alg}_{(-)}(\underline{\mathcal{S}}_G) : \text{Op}_G^{\text{Red}} \rightarrow \mathbf{Cat}$ is conservative.*

Proof. All but the final statement follow by the above analysis. Suppose $\varphi : \mathcal{O} \rightarrow \mathcal{P}$ induces an equivalence on $\mathbf{Alg}_{\mathcal{O}}(\mathcal{S}_G) \rightarrow \mathbf{Alg}_{\mathcal{P}}(\mathcal{S})$.

Then φ induces a natural equivalence $T_{\mathcal{O}, \mathcal{S}_G} \Rightarrow T_{\mathcal{P}, \mathcal{S}_G}$ respecting the summand decomposition in [proposition 3.3](#). Choosing X a set with at least 2 points, we find that $n_S \cdot \mathcal{O}(S) \rightarrow n_S \cdot \mathcal{P}(S)$ is an equivalence for some $n_S > 0$ and all S ; this implies that $\mathcal{O}(S) \rightarrow \mathcal{P}(S)$ is an equivalence for all S , i.e. φ_Σ is an equivalence. By [lemma 3.1](#), this implies φ is an equivalence. □

3.2. The conservativity lemmas. We have two conservativity lemmas to prove.

Lemma 3.5. *Denote by $i : I \cup J \subset I \vee J$ the (non-indexing system) union of subcategories. Then, the pullback*

$$\text{Fbrs}(\text{Span}(I \cup J)) \rightarrow \text{Op}_I \times \text{Op}_J$$

is conservative. In particular, \mathcal{U} reflects equivalences between $I \vee J$ -operads in the image of $L_{\text{Fbrs}} i$.

Proof. Passing to the underlying symmetric sequences yields a diagram

$$\begin{array}{ccc} \text{Fbrs}(\text{Span}(I \cup J)) & \xrightarrow{i^*} & \text{Op}_I \times \text{Op}_J \\ \downarrow & & \downarrow \\ \text{Fun}(\Sigma_I \cup \Sigma_J, \mathcal{S}) & \xrightarrow{\quad} & \text{Fun}(\Sigma_I, \mathcal{S}) \times \text{Fun}(\Sigma_J, \mathcal{S}) \end{array}$$

The left vertical arrow is conservative by [proposition 3.2](#). Note that $\Sigma_I \cup \Sigma_J \simeq \Sigma_I \amalg_{\Sigma_{I \cap J}} \Sigma_J$, so the bottom vertical arrow is simply the inclusion

$$\text{Fun}(\Sigma_I, \mathcal{S}) \times_{\text{Fun}(\Sigma_{I \cap J}, \mathcal{S})} \text{Fun}(\Sigma_J, \mathcal{S}) \hookrightarrow \text{Fun}(\Sigma_I, \mathcal{S}) \times \text{Fun}(\Sigma_J, \mathcal{S}),$$

which is conservative. Hence the diagonal composite is conservative, implying that i^* is conservative as well. □

The second is essentially similar. Note that $\text{Env}_I \text{Span}(J) \simeq \mathbb{F}_J^{\amalg}$ for all $J \subset I$, and that

$$(1) \quad \mathbb{F}_J^{\amalg} \coprod_{\mathbb{F}_{I \cap J}^{\amalg}} \mathbb{F}_I^{\amalg} \simeq \mathbb{F}_{I \vee J}^{\amalg},$$

where the coproduct is taken in the category of G -symmetric monoidal categories. We use this:

Lemma 3.6. *The canonical map $L_{\text{Fbrs}} i! \text{Span}(I \cup J) \rightarrow \mathcal{N}_{(I \vee J)^\infty}$ is an equivalence.*

Proof. By [corollary 3.4](#), it suffices to prove that the induced map

$$\mathbf{Alg}_{\mathcal{N}_{(I \vee J)}}(\mathcal{S}_G) \rightarrow \mathbf{Alg}_{L_{\mathbf{Fbrs}!} \text{Span}(I \cup J)}(\mathcal{S}_G) \simeq \mathbf{Alg}_{\text{Span}(I \cup J)}(i^* \mathcal{S}_G)$$

is an equivalence. Unwinding definitions, this is equivalent to proving that the following diagram is cartesian:

$$\begin{array}{ccc} \text{Fun}_G^{\otimes}(\mathbb{F}_{I \vee J}, \underline{\mathcal{S}}_G) & \longrightarrow & \text{Fun}_G^{\otimes}(\mathbb{F}_I, \underline{\mathcal{S}}_G) \\ \downarrow & & \downarrow \\ \text{Fun}_G^{\otimes}(\mathbb{F}_J, \underline{\mathcal{S}}_G) & \longrightarrow & \text{Fun}^{\times}(\mathbb{F}_{I \cap J}, \underline{\mathcal{S}}_G) \end{array}$$

In fact, this is precisely (1). □

3.3. The BV tensor product on fibrous patterns is closed. Need to remove all of the relative products

Definition 3.7. Let \mathcal{J} be a symmetric monoidal algebraic pattern. Then, a *bifunctor of fibrous \mathcal{J} -patterns* is a diagram in $\mathbf{Fbrs}(\mathcal{J})$

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{P} & \longrightarrow & \mathcal{Q} \\ \downarrow & & \downarrow \\ \mathcal{J} \times \mathcal{J} & \longrightarrow & \mathcal{J} \end{array}$$

Let $F : \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \rightarrow \mathcal{J}^{\otimes}$ be a bifunctor of fibrous \mathcal{J} -patterns and let $C^{\otimes} \in \mathbf{Fbrs}(\mathcal{J})$ be a fibrous \mathcal{J} -pattern. The following construction generalizes [\[NS22, § 5.3\]](#).

Construction 3.8. Define $P : \mathcal{O}^{\otimes} \times_{\mathcal{J}^{\text{el}}} \text{Ar}(\mathcal{J}^{\text{el}}) \rightarrow \mathcal{O}^{\otimes}$ by cocartesian pushforward. We have a diagram

$$\mathcal{O}^{\otimes} \xleftarrow{\pi} \mathcal{O}^{\otimes} \times \text{Ar}(\mathcal{J}^{\text{el}}) \times \mathcal{P}^{\otimes} \xrightarrow{P \times \text{id}} \mathcal{O}^{\otimes} \mathcal{P}^{\otimes} \xrightarrow{F} \mathcal{J}^{\otimes}.$$

and an associated push-pull adjunction

$$L_{\mathbf{Fbrs}} F_!(P \times \text{id})_! \pi^* : \mathbf{Fbrs}(\mathcal{O}) \rightleftarrows \mathbf{Fbrs}(\mathcal{J}) : \pi_*(P \times \text{id})^* F^*.$$

We verify that this adjunction exists in [lemma 3.11](#), and we define $\underline{\mathbf{Alg}}_{\mathcal{J}}^{\otimes}(\mathcal{P}; C) \rightarrow \mathcal{O}^{\otimes}$ to be $\pi_*(P \times \text{id})^* F^*(C^{\otimes})$.

Products of equivalences are equivalences; this proves the following lemma.

Lemma 3.9. *External products of strong Segal morphisms are strong Segal morphisms.*

The proof of the following lemma is precisely that of [\[CH21, Lem 9.4\]](#).

Lemma 3.10. *Fibrous patterns are strong Segal morphisms.*

The following is an exercise in category theory:

Lemma 3.11. *Fix $(\mathcal{O}, \mathcal{O}') \in \mathbf{Cat}_{\mathcal{J}} \times \mathbf{Cat}_{\mathcal{J}}$. Then,*

- (1) $f \times f' : \mathcal{O} \times \mathcal{O}' \rightarrow \mathcal{J} \times \mathcal{J}$ is a (strong-, iso-) Segal morphism if and only if f and f' are (strong-, iso-) Segal morphisms.
- (2) $\pi_{\mathcal{O} \times \mathcal{O},*}$ preserves fibrous patterns (resp. Segal categories) if and only if $\pi_{\mathcal{O},*}$ and $\pi_{\mathcal{O}',*}$ is preserves fibrous patterns (Segal categories).
- (3) $\mathcal{O} \times \mathcal{O}'$ is a fibrous $\mathcal{J} \times \mathcal{J}$ -pattern (resp. Segal $\mathcal{J} \times \mathcal{J}$ -category) if and only if \mathcal{O} and \mathcal{O}' are fibrous \mathcal{J}, \mathcal{J} patterns (Segal $\mathcal{J} \times \mathcal{J}$ -categories).

In particular, the morphisms $F, P \times \text{id}, \pi$ above are strong Segal morphisms and π_* preserves fibrous patterns and Segal categories.

Proof. For (1), note that the associated functor

$$\mathcal{O}_{X/}^{\text{el}} \times \mathcal{O}'_{X'}^{\text{el}} \rightarrow \mathcal{J}_{fX/}^{\text{el}} \times \mathcal{J}'_{f'X'}^{\text{el}}$$

is the product $f_{X/}^{\text{el}} \times f'_{X'}^{\text{el}}$, so this follows by noting that products commute with limits and that a product of functors is an equivalence if and only if the factors are equivalences.

Is (2) true???

For (3), this amounts to checking that a morphism is $\pi \times \pi'$ -cocartesian if and only if it's a product of π and π' -cocartesian arrows and commuting limits past products in [\[BHS22, Def 4.1.2\]](#). □

The following lemma follows immediately from [lemma 3.11](#).

Lemma 3.12. *Suppose C is a Segal \mathcal{F} -category. Then, $\underline{\mathbf{Alg}}_{\mathcal{F}}^{\otimes}(\mathcal{P}; C)$ is a Segal O -category.*

The resulting fibrous is pronounced “the fibrous \mathcal{F} -pattern of G -equivariant O -algebras in C .”
We specialize to the case that $\mathcal{F}^{\otimes} = O^{\otimes}$, in which case we write

$$\underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes}(C) := \underline{\mathbf{Alg}}_{\mathcal{F}}^{\otimes}(\mathcal{P}; C).$$

Then, the above diagram instead reads as

$$\mathcal{F} \xleftarrow{\pi} \mathcal{F} \times_G \mathbf{Ar}(\mathcal{F}^{\text{el}}) \times_G \mathcal{P}^{\otimes} \xrightarrow{P \times \text{id}} \mathcal{F} \times_{\mathcal{F}^{\text{el}}} \mathcal{P}^{\otimes} \xrightarrow{F} \mathcal{F}.$$

So that the left adjoint is computed by the fibrous localization of the map $Q \times_G \mathcal{P} \rightarrow \text{Comm}_G^{\otimes}$ in the following:

$$\begin{array}{ccc} \pi^*(P \times \text{id})_! Q & \simeq & Q \times_{\mathcal{F}^{\text{el}}} \mathcal{P} \\ \downarrow & \swarrow \pi_Q \times \text{id} & \\ \mathcal{F} \times_{\mathcal{F}^{\text{el}}} \mathcal{P} & & \\ \downarrow \text{id} \times \pi_Q & \searrow F & \\ \mathcal{F}^{\text{el}} \times_{\mathcal{F}^{\text{el}}} \mathcal{F}^{\text{el}} & \xrightarrow{\otimes} & \mathcal{F}^{\text{el}} \end{array}$$

in fact, by definition, this is precisely $Q \otimes_{\mathcal{F}} \mathcal{P}$. This concludes the proof of [theorem 2.22](#).

As a sanity check, we verify that our construction matches that of [\[NS22, § 5.3\]](#). In fact, pullback along the map $\varphi : \mathbb{F}_{G,*} \rightarrow \text{Span}(\mathbb{F}_G)$ carries the category of diagrams of fibrous patterns

$$\begin{array}{ccc} O^{\otimes} \times_G \mathcal{P}^{\otimes} & \longrightarrow & Q^{\otimes} \\ \downarrow & & \downarrow \\ \mathbb{F}_{G,*} \times_G \mathbb{F}_{G,*} & \longrightarrow & \mathbb{F}_{G,*} \end{array}$$

equivalently the category of bifunctors.

In [\[NS22, § 5.3\]](#), the following properties were verified for a

Proposition 3.13. *Let $F : O^{\otimes} \times_G \mathcal{P}^{\otimes} \rightarrow \mathcal{F}^{\otimes}$ be a bifunctor of G -operads and let $C^{\otimes} \rightarrow \mathcal{F}^{\otimes}$ be a functor of G -operads.*

- (1) *If O has one object, then the underlying G -category of $\underline{\mathbf{Alg}}_{\mathcal{F}}^{\otimes}(\mathcal{P}; C)$ is the usual G -category $\underline{\mathbf{Alg}}_{\mathcal{F}}(\mathcal{P}; C)$.*
- (2) *If C^{\otimes} is \mathcal{F} -monoidal, then $\underline{\mathbf{Alg}}_{\mathcal{F}}^{\otimes}(\mathcal{P}; C) \underline{\mathbf{Alg}}_O^{\otimes}(C)$ is O -monoidal, and there is a O -monoidal lift $\underline{\mathbf{Alg}}_{\mathcal{F}}^{\otimes}(\mathcal{P}; C) \rightarrow C^{\otimes}$ to the forgetful functor.*

3.4. An I -symmetric monoidal category is cocartesian if and only if unital algebra structures are canonical.

Define the category $\Gamma_G^* := \text{CoFr}^G(\Gamma^*)$. Given C an I -coproduct complete G -category, define the functor $C^{\text{II}} \rightarrow \Gamma_G^*$ to satisfy the following equivalence:

$$\text{Map}_{\text{Span}(\mathbb{F}_G)}(K, C^{\text{II}}) \simeq \text{Map}(K \times_{\mathbb{F}_{G,*}} \Gamma_G^*, C).$$

An object of C^{II} may be viewed as $S_+ \rightarrow G/H_+$ a pointed H -set and $(C_s)_{U \in \text{Orb}(S)}$ an S -tuple of elements of C ; a morphism in C^{II} $(C_s) \rightarrow (D_t)$ may be viewed as a map $(S_+ \rightarrow G/H_+) \xrightarrow{f} (T_+ \rightarrow G/J_+)$ in \mathbb{F}_G together with a map

$$f_U : \coprod_{V \in f^{-1}(U)} N_V^U C_V \rightarrow D_U$$

for all $U \in \text{Orb}(T)$. Unwinding definitions, we find the following lemma.

Lemma 3.14. *A morphism $f : (C_s)_{s \in S} \rightarrow (D_t)_{t \in T}$ is π -cocartesian if and only if f_U is an equivalence for all $U \in \text{Orb}(T)$. In particular, f is inert if and only if the following conditions are satisfied:*

- (1) *The projected morphism $\pi(f) : S \rightarrow T$ is inert.*
- (2) *The associated map $C_{f^{-1}(U)} \rightarrow D_U$ is an equivalence for all $U \in \text{Orb}(T)$.*

Having characterized this, we may draw a diagram of Cartesian squares

$$\begin{array}{ccccc} \mathcal{O} & \xhookrightarrow{\iota} & \mathcal{O}_{\Gamma}^{\otimes} & \longrightarrow & \mathcal{O}^{\otimes} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \mathcal{O}_G^{\text{op}} & \hookrightarrow & \Gamma_G^* & \longrightarrow & \mathbb{F}_{G,*} \end{array}$$

Note that the objects of $\mathcal{O}_{\Gamma}^{\otimes}$ consist of triples $(S_+ \rightarrow G/H, U, X)$ where $U \in \text{Orb}(S)$ and $X \in \mathcal{O}_S$, and the image of ι is equivalent to the triples where $S_+ \simeq G/K$ for some $K \subset H$ (hence $U = S$).

Note that cocartesian transport along inert morphism $U_+ \hookrightarrow S_+$ induces an equivalence

$$\text{Map}_{\mathcal{O}_{\Gamma}^{\otimes}}(Y, (S_+ \rightarrow G/H, U, X)) \simeq \text{Map}_{\mathcal{O}_{\Gamma}^{\otimes}}(Y, (U_+ \rightarrow G/H, U, X_U)).$$

In particular, ι witnesses \mathcal{O} as a *colocalizing subcategory*, with localization functor

$$R(S_+ \rightarrow G/H, U, X) \simeq (U_+ \rightarrow G/H, U, X).$$

We use this in the following lemma characterizing \mathcal{O} -algebras in \mathcal{C}^{II} .

Lemma 3.15. *TFAE for a functor $A : \mathcal{O}_{\Gamma}^{\otimes} \rightarrow \mathcal{C}$.*

- (1) *The corresponding map $\mathcal{O}^{\otimes} \rightarrow \mathcal{C}^{\text{II}}$ is a map of I-operads.*
- (2) *For all morphisms α in $\mathcal{O}_{\Gamma}^{\otimes}$ whose image in \mathcal{O}^{\otimes} is inert, $A(\alpha)$ is an equivalence in \mathcal{C} .*
- (3) *If $f : (S_+ \rightarrow G/H_+, U, X) \rightarrow (U_+ \rightarrow G/H_+, U, X_U)$ is a cocartesian lift of the inert morphism, then $A(f)$ is an equivalence.*
- (4) *A is left Kan extended from \mathcal{O} .*

Furthermore, every functor $F : \mathcal{O} \rightarrow \mathcal{C}$ admits a left Kan extension along $\mathcal{O} \hookrightarrow \mathcal{O}_{\Gamma}^{\otimes}$; in particular, the forgetful functor $\underline{\text{Alg}}_{\mathcal{O}}(\mathcal{C}) \rightarrow \underline{\text{Fun}}_{\mathcal{C}}(\mathcal{O}, \mathcal{C})$ is an equivalence.

Proof. (1) \iff (2) follows immediately from ?? . (2) \iff (3) is immediate by definition. (3) \iff (4) is the computation of left Kan extension along the inclusion of a colocalizing subcategory. The pointwise formula for left Kan extension is precisely the composition $RF : \mathcal{O}_{\Gamma}^{\otimes} \rightarrow \mathcal{C}$. \square

We would additionally like to characterize I -symmetric monoidal functors into \mathcal{C}^{II} . This follows quickly from [lemma 3.15](#).

Lemma 3.16. *TFAE for a map of I-operads $\varphi : \mathcal{O}^{\otimes} \rightarrow \mathcal{C}^{\text{II}}$:*

- (1) *φ is a map of I-symmetric monoidal categories.*
- (2) *The underlying G -functor $F : \mathcal{O} \rightarrow \mathcal{C}$ preserves I-indexed coproducts.*

In particular, restriction yields an equivalence

$$\text{Fun}_I^{\otimes}(\mathcal{O}^{\otimes}, \mathcal{C}^{\text{II}}) \xrightarrow{\sim} \text{Fun}_I^{\text{II}}(\mathcal{O}, \mathcal{C}).$$

Theorem 3.17 (C.f. [\[HA, Prop 2.4.3.9\]](#)). *The following are equivalent for $\mathcal{C}^{\otimes} \in \text{CMon}_I(\mathbf{Cat})$.*

- (1) *For all unital I-operads \mathcal{O}^{\otimes} , the forgetful functor $\underline{\text{Alg}}_{\mathcal{O}}(\mathcal{C}) \rightarrow \underline{\text{Fun}}_{\mathcal{C}}(\mathcal{O}, \mathcal{C})$ is an equivalence.*
- (2) *The forgetful functor $\text{CAlg}_I(\mathcal{C}) \rightarrow \mathcal{C}$ is an equivalence.*
- (3) *For all morphisms $f : S \rightarrow T$ in \mathcal{I} , the action map $f_{\otimes} : \mathcal{C}_S \rightarrow \mathcal{C}_T$ is left adjoint to the pullback $f^* : \mathcal{C}_T \rightarrow \mathcal{C}_S$.*
- (4) *There is an I-symmetric monoidal equivalence $\mathcal{C}^{\otimes} \simeq \mathcal{C}^{\text{II}}$ extending the identity on \mathcal{C} .*

Proof of theorem 3.17. (1) \implies (2) by choosing $\mathcal{O} = \mathcal{N}_{I\infty}$. (2) \implies (3) is precisely [\[NS22, Thm 5.3.9\]](#), noting that The forgetful functor $\text{CAlg}_I(\mathcal{C}) \rightarrow \mathcal{C}$ is I -symmetric monoidal by construction. (3) \implies (4) follows by applying [lemma 3.16](#) to the identity functor in the case $\mathcal{O} = \mathcal{C}$. (4) \implies (1) is precisely [lemma 3.15](#). \square

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