18.715 — REPRESENTATION THEORY

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ABSTRACT. 18.715 is being taught by Professor Bjorn Poonen in Fall 2016. I am planning on taking notes throughout the semester. I'll add drawings (if any) by putting them on a notepad, scanning them as images, and finally inserting them in the appropriate spots. This means that they might be messy. Any mistakes in these notes are only mine, and not Professor Poonen's.

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1. Introduction

Office hours are Monday 4:30 - 5:30 pm, Thursday 4:30 - 5:30 pm, and Friday from 3:30-4:30 pm.

Fix a field k. A representation of a group G is a k-vector space V with an action of G i.e. with a group homomorphism $G \to \mathrm{Aut}(V)$. For example, if $V = k^n$, then $\mathrm{Aut}(V) = \mathrm{GL}_n(k)$. You care about representations because you can prove stuff, such as:

Theorem 1.1 (Burnside's $p^a q^b$ theorem)**.** *If G is a group with $|G| = p^a q^b$ where p, q are prime, then G is solvable.*

We're also going to talk about representations of quivers Q. A quiver's a bunch of arrows. A representation of Q is a vector space for each vertex of Q, and for each edge, give a linear map between the vector spaces. The third kind of representations we're going to talk about are representations of Lie algebras $\mathfrak{g}$, to be defined later.

The goal of the whole subject is to classify the representations in each case. There are some things that you can do for all three things at once. We convert everything into questions about modules and algebras. To each object, we can associate an associative k-algebra A such that representations of the object is the same as a vector space V equipped with a k-algebra homomorphism $A \to \mathrm{End}(V)$. This will be called a representation of A, or a left A-module. For example, if we want to study representations of a group G, this associated A is the group algebra $k[G]$. For a quiver, this A is called a path algebra. For a lie algebra, this A is the universal enveloping algebra. Let's now get started. This'll be a "boring" lecture because it's mainly definitions today.

Definition 1.2. Let k be a field (not necessarily algebraically closed). A k-algebra is a k-vector space A with a k-bilinear map $A \times A \to A$. (Here k-bilinear means that $(x_1 + x_2)y = x_1 y + x_2 y$ and $x(y_1 + y_2) = xy_1 + xy_2$ and $(cx)y = x(cy) = x(cy)$ for $x, ... \in A$ and $c \in k$.)

Notice that this isn't necessarily an associate or commutative multiplication (or even with 1). If $B \subseteq A$ is a k-basis, then $A \times A \to A$ is uniquely determined by its values on $B \times B$.

Definition 1.3. A is associative if the multiplication is associate and there is a multiplicative identity. Again, you only have to check on the basis of A.

Example 1.4. Trivial example: $A = \{0\}$. Also, $A = k$. The polynomial algebra $A = k[x, y]$ whose basis is $\{x^i, y^j | (i, j) \in \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}\}$. Here, x and y commute.

Another example is $A = \operatorname{End} V$ for a vector space V, where multiplication is composition. For example, $V = k^n$, then $\operatorname{End} V = \mathrm{M}_n(k)$.

There's also the free associative algebra $k\langle x, y\rangle$. The basis consists of words in x and y, where you stick a bunch of letters together. The scalars commute with everything, but x and y do not commute. What's the multiplication? Multiplication of basis elements is given by concatenating the words.

An extremely important example for a group G is $A = k[G]$, the group algebra, whose basis is G and where the multiplication of basis elements is the multiplication in G. A general element of this looks like $\sum_{g\in G} a_g g$ and the a_gs are elements of k that are zero for all but finitely many g. You multiply with the distributive law.

The Hamilton quaternion algebra $\mathbf{H}$ is the associative algebra over $\mathbf{R}$ with basis $1, i, j, k$ and multiplication $i^2 = j^2 = k^2 = -1$, $ij = k = -jk$, $jk = i = -kj$ and $ki = j = -ik$.

For a non-associative example, we have Lie algebra. Consider $\mathfrak{sl}_2 = \{X \in M_2(K) | \operatorname{Tr}(X) = 0\}$ where $[X, Y] = XY - YX$.

Definition 1.5. A homomorphism of associative k-algebras $f : A \to B$ is a linear map on the underlying vector spaces that respects the multiplication, i.e. $f(1) = 1$ and $f(xy) = f(x)f(y)$. If there is a k-algebra homomorphism $g : B \to A$ such that $fg = \mathrm{id}_B$ and $gf = \mathrm{id}_A$, then f is called an isomorphism.

We'll now define what a representation of an associative k-algebra is.

Definition 1.6. Modules, also called representations in this context, are the following. Let A be an associative k-algebra. A representation of A is a k-vector space V with a k-algebra homomorphism $A \to \operatorname{End}(V)$. This is also called an A-module, always assumed to be a *left* A-module.

Also, recall that a representation of G is a k-vector space V with a group homomorphism $A \to \operatorname{Aut}(V)$.

Example 1.7. Again, $V = 0$, called the zero representation. There's the regular representation $V = A$ with $A \to \operatorname{End}(A)$ given by $a \mapsto (x \mapsto ax)$. If $A = k$, then an A-module is just a k-vector space.

The most important example is when $A = k[G]$.

Proposition 1.8 (Emmy Noether's interpretation of group representations)**.** *A $k[G]$-module is a representation of G over k.*

Proof. If A is a $k[G]$-module, we have k-algebra homomorphism $k[G] \to \operatorname{End}(V)$. Take the unit groups of both sides, giving $k[G]^\times \to \operatorname{End}(V)^\times = \operatorname{Aut}(V)$. Now, the elements of G are units of $k[G]$, so we get a composition $G \to k[G]^\times \to \mathrm{Aut}(V)$. Going the other way, if you have $\rho : G \to \mathrm{Aut}(V)$, define the scalar multiplication $k[G] \to \mathrm{End}(V)$ by extending what the basis elements are. So:

$$\left(\sum a_g g\right) v := \sum a_g \rho(g)(v).$$

$\square$

Definition 1.9. Let V and W be A-algebras where A is an associative k-algebra. A homomorphism $f : V \to W$ is a k-linear map such that $f(av) = af(v)$. These are also called "intertwining operator", especially when you're talking about group representations. The set of all possible homomorphisms $V \to W$ is denoted $\mathrm{Hom}_A(V, W)$, which is a k-vector space. (It's also clearly a representation of A.) You can also consider $\mathrm{End}_A(V)$, and this is an associative k-algebra, and isomorphisms etc.

Definition 1.10. Anything you do with abelian groups, you can also do with modules. For example:

- If V and W are A-representations, then $V \oplus W$ is the direct sum of k-vector spaces and define $a(v \oplus w) = av \oplus aw$.
- You can also talk about subrepresentations of a representation V of A. This is a subspace W of V such that $aW \subseteq W$ for all $a \in A$. It's enough to check this on the bases of A and W.
- A representation V of A is irreducible (or simple) if it has *exactly* two sub representations 0 and V. In particular, 0 is *not* irreducible.
- A representation $V \neq 0$ is indecomposable if it is not isomorphic to a direct sum of nonzero representations.
- V is faithful if $\rho : A \to \mathrm{End}\, V$ is injective.

Let's do some examples of these.

Example 1.11. Let $A = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\} \subseteq M_2(k)$, which is a 3-dimensional associative algebra. Let $V = \{\begin{pmatrix} * \\ * \end{pmatrix}\}$ with ordinary vector multiplication. The subrepresentations of V are of the form 0, V, and $\{\begin{pmatrix} * \\ 0 \end{pmatrix}\}$. How do you know there are no others? If W contains $\{\begin{pmatrix} a \\ b \end{pmatrix}\}$, then $b \neq 0$. Then W contains anything obtained from $\begin{pmatrix} a \\ b \end{pmatrix}$, in particular $\begin{pmatrix} b \\ 0 \end{pmatrix}$. These two matrices span the whole of V, so $W = V$. Also, V is indecomposable because it cannot be written as the direct sum of subrepresentations. However, V is not irreducible. Is V faithful? Well, $\mathrm{End}(V)$ is $M_2(k)$, and the map $A \to \mathrm{End}(V)$ is the inclusion, so V is faithful.

Theorem 1.12 (Schur's lemma)**.** *Suppose you have a nonzero homomorphism $\phi : V \to W$ of representations.*

(1) *If V is irreducible, then ϕ is injective.*
(2) *If W is irreducible, then ϕ is surjective.*
(3) *If both are irreducible, then ϕ is an isomorphism.*

Proof. Look at the kernel of ϕ, which is a subrepresentation of V, and the image of ϕ, which is a subrepresentation of W. □

Corollary 1.13 (Another version of Schur's lemma)**.** *Over an algebraically closed field $k = \overline{k}$, let A be an associative k-algebra, where V is a finite-dimensional irreducible representation of A. Every homomorphism $V \to V$ of representations of A is just scalar multiplication by some element of k i.e. $v \mapsto \lambda v$. In other words,* $\mathrm{End}_A(V) = k$.

Proof. Let λ be an eigenvalue of ϕ. This exists because λ is a root of the characteristic polynomial, which always exists since k is algebraically closed. Consider the endomorphism $\phi - \lambda \mathrm{id}_V$ of V. This kills an eigenvector (which again always exists), so it's not an isomorphism. By Schur's lemma, $\phi - \lambda \mathrm{id}_V$ is the zero homomorphism. Therefore, $\phi = \lambda \mathrm{id}_V$. □

2. More about Schur's lemma, ideals, quivers, and path algebras

Let's talk about Schur's lemma. We need finite-dimensionality, for the following reason. Suppose V has infinite dimension. Let A be a nontrivial field extension of k (for example, $A = k(t)$, the rational function field). Let $V = A$. Then $\mathrm{End}_A(V) = M_1(A) := A$, which is not k (it's a lot bigger than A). The same argument shows that we need k to be algebraically closed, because otherwise A can be any finite field extension.

Corollary 2.1 (of Schur's lemma)**.** *Let $k = \overline{k}$, A be a commutative associative k-algebra, and V an irreducible finite dimensional A-representation. Then* $\dim_k(V) = 1$.

Proof. If $b \in A$, then $\phi_b : V \to V$ that sends $v \mapsto bv$ is an A-module endomorphism, because $\phi_b(av) = a\phi_b(v)$, and this is true because A is commutative. Schur's lemma says that ϕ_b is multiplication by some scalar in k. Thus every subspace of V is a subrepresentation, because it's going to be respected by all the scalar multiplications of elements by A. Since V is irreducible, so the only subrepresentations are 0 or V. This means that V is one-dimensional. □

Example 2.2. Suppose $k = \overline{k}$. Then $A = k[x] = k\langle x\rangle$. All representations of A are finite-dimensional, but this is just a k-vector space V with an endomorphism $\phi_x : V \to V$, given by $v \mapsto xv$. Each irreducible finite-dimensional representation is isomorphic to k with an endomorphism that is multiplication by some $\lambda \in k$, which is determined by the representation. There are more indecomposable representations, however. Each indecomposable representation is isomorphic to k^n with one endomorphism, i.e., multiplication by a $n \times n$-matrix in Jordan canonical form (because changing basis corresponds to changing conjugacy classes) with λs on the diagonal, with $\lambda \in k$ uniquely determined.

Definition 2.3. Let A be an associative k-algebra. A left ideal is a subspace $I \subseteq A$ such that $aI \subseteq I$ for all $a \in A$, with the same thing for right ideals. There's also two-sided ideals, which are ideals that are both left and right ideals. If $\phi : A \to B$ is a k-algebra homomorphism, the kernel is a two-sided ideal.

Definition 2.4. A is said to be simple if $A \neq 0$ and its only two-sided ideals are 0 and A.

Example 2.5. For example, $A = \mathrm{M}_n(k)$ is simple. There are left ideals, though.

Definition 2.6. If S is a subset of A, then write $\langle S\rangle$ be the two-sided ideal it generates. This thing is the smallest two-sided ideal containing S, or as the k-span of $\{asb | a, b \in k \text{ and } s \in S\}$. This comes up especially when constructing quotient algebras.

Example 2.7. If $I \subseteq A$ is a two-sided ideal, then A/I is another associative k-algebra. As a special case of this, if $f_1, \cdots, f_r \in k\langle x_1, \cdots, x_n\rangle$, then we can form $k\langle x_1, \cdots, x_n\rangle/\langle f_1, \cdots, f_r\rangle$. For such $A := k\langle x_1, \cdots, x_n\rangle/\langle f_1, \cdots, f_r\rangle$, what is an A-module? Without quotienting by the ideal, it's just a k-vector space V with $T_1, \cdots, T_n : V \to V$. But when we quotient by $\langle f_1, \cdots, f_r$, there's an extra condition that $f_i(T_1, \cdots, T_n) = 0$ for all $1 \leq i \leq r$ because of the way the endomorphisms are defined. Let's look at some examples of this.

Consider $k\langle x, y\rangle/\langle xy - yx\rangle \simeq k[x, y]$. We forced x and y to commute. Another example is the *Weyl algebra* $W := k\langle x, y\rangle/\langle yx - xy - 1\rangle$, where we suppose the characteristic of k is 0. We'll look at a particular module for W.

Let $V = k[t] \ni g$ with $xg = tg$ and $yg = \frac{d}{dt}g$. The relation holds, because $(yx - xy - 1)g = \frac{d}{dt}(tg) - t\frac{d}{dt}g - g = 0$. In fact, it turns out that (left as an exercise) V is a faithful W-module, so $W \simeq \{\sum_{i=0}^{n} h_i(t)\frac{d^i}{dt^i} | n \in \mathbf{Z}_{\geq 0}, h_0, \cdots, h_n \in k[t]\}$ by $x \mapsto t$ and $y \mapsto d/dt$. This is called the algebra of polynomial differential operators.

2.1. **Quivers.**

Definition 2.8. A quiver Q consists of sets I of vertices and E of edges equipped with a map $E \mapsto V \times V$ that takes h to h' and h'', thought of as a map $h : h' \to h''$.

Definition 2.9. A representation of Q consists of a vector space V_i over some fixed field k for every $i \in I$ and a linear map $x_h : V_{h'} \to V_{h''}$ for each edge $h \in E$. (There's absolutely no constraints.) Given two representations of the same quiver Q, specified by $(V_i), (x_h)$ and $(W_i), (y_h)$, a homomorphism is a collection of linear maps $f_i : V_i \to W_i$ such that

$$\begin{tikzcd} V_{h'} \arrow[r, "x_h"] \arrow[d, "f_{h'}"] & V_{h''} \arrow[d, "f_{h''}"] \\ W_{h'} \arrow[r, "y_h"] & W_{h''} \end{tikzcd}$$

commutes for each $h \in E$ and $h', h'' \in I$. You can now get isomorphisms, etc.

Example 2.10. What is a finite-dimensional representation of the trivial quiver with one point and one edge from the point to itself? This is just a vector space with an endomorphism, i.e., k^n with a single $n \times n$-matrix. If you have (k^n, X) and (k^n, Y), they are isomorphic if and only if there is an isomorphism $f : k^n \to k^n$ such that $Y \circ f = f \circ X$, i.e., X and Y are conjugate to one another.

Example 2.11. Consider the quiver with two points and two edges in the same direction. Are the representations $(k, \mathrm{id}, \times 2)$ and $(k, \mathrm{id}, \times 3)$ isomorphic? They aren't because you have to give two isomorphisms $f, g : k \to k$ such that $f \circ 1 = 1 \circ g$ and $f \circ 3 = 2 \circ g$, which cannot happen.

Our goal is to start with a quiver Q and a ground field k, and construct an associative k-algebra P_Q such that a representation of Q is the same thing as a representation of P_Q.

Definition 2.12. A path in Q is a sequence of composable edges. More precisely, it's a (finite) sequence of vertices $(i_0, \cdots, i_n)$ and a sequence of edges $e_1, \cdots, e_n$ for some $n \in \mathbf{Z}_{\geq 0}$ such that each e_r goes from i_{r-1} to i_r.

Example 2.13. If you fix one vertex $i \in I$, let p_i be the path starting and ending at i with zero edges. It doesn't go anywhere.

Definition 2.14. The path algebra P_Q has basis {paths in Q}, and multiplication of basis elements is concatenation of paths that are composable, and zero if they aren't composable. Then P_Q is an associative k-algebra except that maybe it has no 1.

Proposition 2.15. *If Q has only finitely many vertices, then $\sum_{i\in I} p_i$ is a* 1 *in the algebra.*

Proof. Let q be any path, i.e., a basis element of P_Q, say ending at j. Then:

$$p_i q = \begin{cases} q & \text{if } p_i \text{ is the endpoint of } q, \text{ i.e., } i = j \\ 0 & \text{else} \end{cases}$$

So $(\sum_{i\in I} p_i)q = q$ because of the above. Similarly, if you multiply q on the right, then $q \sum p_i = q$. □

Example 2.16. Let's look at the trivial quiver with one point and one edge from the point to itself. Each in path in this is specified by some $n \in \mathbf{Z}_{\geq 0}$. Therefore, $P_Q \simeq k[x]$ because the basis consists of powers of x, which in turn are specified by some $n \in \mathbf{Z}_{\geq 0}$. Notice that a representation of Q is a representation of $k[x]$, as we saw above.

Proposition 2.17. *A representation of Q is the same thing as a representation of P_Q.*

Proof. If you have $(V_i), (x_h)$ a representation of Q, then take this to the P_Q-module $\bigoplus_{i\in I} V_i$ where each path acts as a composition of x_h. If you start with a P_Q-module V, take this to $(p_i V)$, (length 1 path) because $p_i V$ picks out a vector space. □

2.2. **Lie algebras.**

Example 2.18. Let A be an assocative k-algebra. Define $[a, b] = ab - ba$ for $a, b \in A$. It measures the noncommutativity of A. What properties does this pairing have?

(1) $[,]$ is k-bilinear.
(2) $[a, a] = 0$ for all $a \in A$.
(3) The Jacobi identity: $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$.

It turns out that all other identities that this pairing satisfies are consequences of these. One such identity is $[a, b] = -[b, a]$, also known as skew-symmetric. In the presence of bilinearity, $[a, a] = 0$ implies $[a, b] = -[b, a]$. The converse is almost true, and it is true if the characteristic of the field you're working in is not 2. To see this, if $[a, a] = 0$, then $0 = [a + b, a + b] = [a, a] + [a, b] + [b, a] + [b, b]$. Let's try to go the other way. If $[a, b] = -[b, a]$, then $[a, a] = -[a, a]$ by setting $b = a$, so $2[a, a] = 0$, but you can't divide by 2 if you're in a field whose characteristic is zero.

Definition 2.19. A Lie algebra is a vector space $\mathfrak{g}$ equipped with a k-bilinear pairing $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that:

(1) $[a, a] = 0$ for all $a \in A$.
(2) The Jacobi identity: $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$.

Note that $[,]$ is usually not associative, nor does it even have a 1.

Example 2.20. Any vector space V with $[,] = 0$. This is called an abelian Lie algebra. If $\mathfrak{g}$ is a Lie algebra, and $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, then $\mathfrak{h}$ is another Lie algebra. It's called a Lie subalgebra.

Example 2.21. If you have a k-algebra A, a derivation of A is a k-linear map (not an algebra homomorphism, nor even an A-module homomorphism!) $D : A \to A$ such that $D(ab) = D(a)b + aD(b)$. If D and E are derivations of A, then so is $[D, E] := D \circ E - E \circ D$. Then $\mathrm{Der}(A) = \{\text{all derivations of } A\}$ with this bracket is a Lie algebra.

3. Lie algebras

We'll begin with an optional topic (so we're going to use things you're not supposed to know about yet). Let $\mathcal{M}$ be a C^∞-manifold over $\mathbf{R}$. Define $C^\infty(\mathcal{M})$ be the collection of C^∞-functions $\mathcal{M} \to \mathbf{R}$. Let $t \in T_p\mathcal{M}$ be a tangent vector. We can get the directional derivative $D_t f \in \mathbf{R}$. Suppose you had a C^∞-vector field X by associating to any point in $\mathcal{M}$ a vector. Then we can take the directional derivative at every point, so we get a whole new function out. This defines a derivation $D_X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ given by taking $f \mapsto (p \mapsto D_{X(p)} f)$. This shouldn't be too surprising. It turns out that this map from $\{C^\infty\text{-vector fields on } \mathcal{M}\} \to \mathrm{Der}(C^\infty(\mathcal{M}))$ is an isomorphism. You can transport the Lie bracket on $\mathrm{Der}(C^\infty(\mathcal{M}))$ to get the notion of a bracket of two vector fields.

What does this have to do with Lie groups? Here's another optional topic. Let G be a Lie group (a manifold + a group such that the structures are compatible, i.e., such that the group operations are C^∞ or continuous or etc.). This group has an identity called $e \in G$. The Lie algebra of G is defined to be the vector space $\mathfrak{g} = \mathrm{Lie}(G) := T_e G$ with the following bracket. We can transport any tangent vector in $T_e G$ to any other point by multiplying by an element of G, so we can get a whole vector field that is (clearly) translation invariant. This means that there's an isomorphism $T_e G \simeq \{\text{left-invariant vector fields on } G\}$. This turns out to be a Lie subalgebra of $\{C^\infty\text{-vector fields on } G\}$, so it's closed under the Lie bracket as defined above. This defines the bracket on $\mathfrak{g}$.

Now let's move on to topics that don't require knowledge outside the prereqs. Consider $A = \mathrm{End}(V)$, with $[a, b] = ab - ba$. This is a Lie algebra, denoted $\mathfrak{gl}(V)$. If V is k^n, this is also written $\mathfrak{gl}_n$ or $\mathfrak{gl}(n)$. In the case when V is finite-dimensional, say over $\mathbf{R}$, then $\mathfrak{gl}(V)$ is the Lie algebra of the Lie group $\mathrm{GL}(V)$.

Theorem 3.1 (Ado's theorem)**.** *Every finite-dimensional Lie algebra is a Lie subalgebra of $\mathfrak{gl}(V)$ for some finite-dimensional V. (What conditions on the field? Poonen isn't sure, but he thinks that it's true over all fields. Someone suggests that the field should be of characteristic 0, so Poonen writes that we can assume that it's of characteristic 0. LATER CLARIFIED: it's true over any field.)*

Proof. Skipped, doesn't belong in this course. □

Definition 3.2. A homomorphism of Lie algebras is a k-linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $\phi([a, b]) = [\phi(a), \phi(b)]$.

You can interpret Lie algebras as "groups of infinitesimal transformations". We're going to be using $\mathbf{R}[\varepsilon]/(\varepsilon^2)$, that is sometimes called the *algebra of dual numbers*. The elements of this algebra are simply of the form $a + b\varepsilon$ where $a, b \in \mathbf{R}$. It's kinda like complex numbers. You should think of ε as being infinitesimal; something so small that $\varepsilon^2 = 0$. If $f(x) \in \mathbf{R}[x]$, then for any $a \in R$, $f(a + \varepsilon) = f(a) + f'(a)\varepsilon$.

Let's consider $\mathfrak{gl}_2(\mathbf{R})$; note that this is just all 2×2-matrices. We want to view this as $\{1 + \varepsilon A | A \in \mathrm{M}_2(\mathbf{R})\} \subseteq \mathrm{GL}_2(\mathbf{R}[\varepsilon]/(\varepsilon^2))$. This is invertible because of the usual way we invert $1/(1 + x)$. The inverse of $1 + \varepsilon A$ is just $1 - \varepsilon A$. What is this map $\mathfrak{gl}_2(\mathbf{R}) \to \{1 + \varepsilon A | A \in \mathrm{M}_2(\mathbf{R})\}$? This just takes $A \mapsto 1 + \varepsilon A$. This is a homomorphism from $\mathfrak{gl}_2(\mathbf{R})$ under addition to $\{1 + \varepsilon A | A \in \mathrm{M}_2(\mathbf{R})\}$ under multiplication. Why is this a homomorphism? Consider $(1 + \varepsilon A)(1 + \varepsilon B) = 1 + \varepsilon(A + B) + O(\varepsilon^2) = 1 + \varepsilon(A + B)$. It's therefore a homomorphism, and obviously an isomorphism of vector spaces (not of Lie algebras, because the bracket on $\{1 + \varepsilon A | A \in \mathrm{M}_2(\mathbf{R})\}$ wasn't really specified). These should be considered as matrices that are very close to the identity.

There's a group homomorphism $\mathrm{GL}_2(\mathbf{R}) \xrightarrow{\det} \mathbf{R}^\times = \mathrm{GL}_1(\mathbf{R})$, which induces $\{1 + \varepsilon A | A \in \mathrm{M}_2(\mathbf{R})\} \mapsto \{1 + \varepsilon x | x \in \mathbf{R}\}$. This is simply a map $\mathfrak{gl}_2(\mathbf{R}) \to \mathfrak{gl}_1(\mathbf{R}) \simeq \mathbf{R}$ where the latter $\mathbf{R}$ is the vector space of real numbers under addition. This should be thought of as the derivative of the determinant. What is this map? An element of $\{1 + \varepsilon A | A \in \mathrm{M}_2(\mathbf{R})\}$ is of the form $\begin{pmatrix} 1 + \varepsilon a & \varepsilon b \\ \varepsilon c & 1 + \varepsilon d \end{pmatrix}$, and this is mapped to (check this) $1 + \varepsilon(a + d)$. But this just induces the trace $\mathfrak{gl}_2(\mathbf{R}) \to \mathfrak{gl}_1(\mathbf{R}) \simeq \mathbf{R}$! This is actually true even for $\mathfrak{gl}_n(\mathbf{R})$.

Recall that $\mathrm{SL}_2(\mathbf{R}) := \ker(\mathrm{GL}_2(\mathbf{R}) \xrightarrow{\det} \mathbf{R}^\times)$, so $\mathfrak{sl}_2(\mathbf{R}) := \ker(\mathfrak{gl}_2(\mathbf{R}) \xrightarrow{\mathrm{Tr}} \mathbf{R})$, so it consists of 2×2-matrices of trace 0. The dimension of this is 3, because $\mathfrak{gl}_2(\mathbf{R})$ is 4-dimensional, and the trace map is surjective. The basis of $\mathfrak{sl}_2(\mathbf{R})$ is $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. These can have zero determinant because, as said above, $\mathfrak{gl}_2(\mathbf{R})$ consists of all matrices, so we don't need $\mathfrak{sl}_2(\mathbf{R})$ to consist of matrices with determinant 1.

Definition 3.3. A representation of a Lie algebra $\mathfrak{g}$ is a vector space V with a homomorphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(V) := \mathrm{End}(V)$. (For ρ to be a homomorphism, we need $\rho([a,b]) = [\rho(a), \rho(b)] = \rho(a)\rho(b) - \rho(b)\rho(a)$ for all $a, b \in \mathfrak{g}$. Because ρ is k-linear, it suffices to check for a, b in the basis of $\mathfrak{g}$.)

Let's do some examples.

Example 3.4. For any Lie algebra $\mathfrak{g}$, we can choose any V, and let $\rho = 0$. This is called the trivial representation. We can also choose $V = \mathfrak{g}$, and define $\rho(a)(v) = [a, v]$. This is called the adjoint representation of $\mathfrak{g}$. This requires a little bit of checking, and it turns out that you have to use the Jacobi identity.

Example 3.5 (Optional)**.** If G is a Lie group, any C^∞-group homomorphism $G \to \mathrm{GL}(V)$ induces a Lie algebra representation $\mathfrak{g} \to \mathfrak{gl}(V)$. If $\mathfrak{g}$ is induced from a Lie group G, you can obtain the two representations mentioned above. (For example, for the trivial representation of $\mathfrak{g}$, you can just choose the trivial representation of G. The adjoint representation is a little trickier, but that's left as an exercise.)

We want to link this to representations of associative algebras. What associative k-algebra does a representation of a Lie algebra correspond to? The answer is what's called the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$.

Definition 3.6. Suppose you have a basis $(x_i)_{i \in I}$ of $\mathfrak{g}$. The bracket is determined by what it's doing on the basis elements. You can just write out $[x_i, x_j]$, which is an element of $\mathfrak{g}$, so it can be written as $\sum_{k \in I} c_{ij}^k x_k$ with $c_{ij}^k \in k$. (This is basically a three-dimensional table that satisfies some rules to ensure that the Jacobi identity, etc. holds (it'd be interesting to write this out explicitly).) Define $\mathcal{U}(\mathfrak{g}) = k\langle x_i | i \in I\rangle / \langle x_i x_j - x_j x_i - \sum_{k \in I} c_{ij}^k x_k | i, j \in I\rangle$.

Remark 3.7. You do this because you want the representations of $\mathcal{U}(\mathfrak{g})$ to be representations of $\mathfrak{g}$. *A priori*, the x_is can be arbitrary endomorphisms, but since we impose the relations, it's almost a tautology that representations of $\mathcal{U}(\mathfrak{g})$ are representations of $\mathfrak{g}$.

3.1. **Tensor products.**

Definition 3.8. Let M, N, P be abelian groups. Say that a map $f : M \times N \to P$ is *bi-additive* if $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$ and $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$ for all $m_1, m_2, m \in M$ and $n, n_1, n_2 \in N$.

Definition 3.9. A tensor product $M \otimes N$ of M and N is the universal target of a bi-additive map from $M \times N$. This means that $M \otimes N$ is an abelian group, with a bi-additive map $M \times N \to M \otimes N$ that sends $(m, n) \mapsto m \otimes n$, such that any other bi-additive map $M \times N \to P$ splits as $M \times N \to M \otimes N \to P$ for a unique map $M \otimes N \to P$.

Proposition 3.10. *If $M \otimes N$ exists, it's unique up to isomorphism.*

Proof. Trivial, basic category theory. Poonen gave a proof, but it's exactly what you'd expect it to be. □

We still don't know that it exists, though.

Proposition 3.11. *The tensor product exists!*

Proof. It's the obvious construction. Define $M \otimes N := \{$free abelian group generated by $(m, n) | m \in M,\ n \in N\}/\{$the relations $(m_1 + m_2, n) - (m_1, n) - (m_2, n),\ (m, n_1 + n_2) - (m, n_1) - (m, n_2)$ for all $m_1, m_2, m \in M,\ n, n_1, n_2 \in N\}$. This clearly works. □

Warning 3.12. Not all elements of $M \otimes N$ look like $m \otimes n$; rather, they're of the form $\sum_{i \in I} m_i \otimes n_i$ where I is a finite set.

We can upgrade this; instead of considering just abelian groups, we can consider modules.

Definition 3.13. Let A be a ring. Let M be a right A-module and N a left A-module. A *middle linear map* is a bi-additive map $f : M \times N \to P$ such that $f(ma, n) = f(m, an)$ for all $m \in M$ and $n \in N$. Define $M \otimes_A N$ is the universal target of a middle linear map from $M \times N$.

Definition 3.14. Let A, B be rings. An (A, B)-bimodule is a abelian group M with $A \times M \to M$ making M a left A-module and $M \times B \to M$ making M a right B-module that respect each that other, i.e., $(am)b = a(mb)$ for all $a \in A$, $m \in M$, and $b \in B$.

Remark 3.15. Suppose M is an (A, B)-bimodule, and another left B-module N, you can tensor them together to get $M \otimes_B N$, which is a left A-module. The multiplication is defined by $a(m \otimes n) := (am) \otimes n$. More generally, suppose M is an (A, B)-bimodule, and another (B, C)-bimodule N, you can tensor them together to get $M \otimes_B N$, which is an (A, C)-bimodule in the obvious way.

Example 3.16. As a special case, if A is commutative, then left A-modules are exactly right A-modules and vice-versa, so they are also the same thing as (A, A)-bimodules. We just write A-modules. In this case, if M and N are A-modules, then $M \otimes_A N$ is another A-module.

Example 3.17. As an important case, if k is a field, consider two k-vector spaces V, W with bases $(v_i)_{i\in I}$ and $(w_j)_{j\in J}$. Then $V \otimes_k W$ is the vector space spanned by $v_i \otimes w_j$ for $(i, j) \in I \times J$. For example, if $\dim V = 2$ and $\dim W = 3$, then $\dim V \otimes_k W = 6$. We'll prove this next time.

4. More on tensor products and semisimple representations

Clarifying what we said last time: Ado's theorem is true over any field. Let's consider the tensor product of vector spaces. We want to prove the following result, stated last time.

Proposition 4.1. *If k is a field, consider two k-vector spaces V, W with bases $(v_i)_{i\in I}$ and $(w_j)_{j\in J}$. Then $V \otimes_k W$ is the vector space spanned by $v_i \otimes w_j$ for $(i, j) \in I \times J$.*

Proof. Let T be the vector space spanned by $v_i \otimes w_j$ for $(i, j) \in I \times J$. Define $V \times W \to T$ by requiring $(v_i, w_j) \mapsto v_i \otimes w_j$. and extend by k-linearly. We need to prove that for any other middle linear map $V \times W \to P$, then there is a unique homomorphism $T \to P$ such that the following diagram commutes.

$$\begin{tikzcd} V \times W \arrow[r] \arrow[dr] & T \arrow[d, dashed, "g"] \\ & P \end{tikzcd}$$

Let's first prove uniqueness. If g exists, then consider $g(\sum c_{ij} v_i \otimes w_j)$. Because g is supposed to be a homomorphism, this is $\sum g(c_{ij} v_i \otimes w_j)$. This should be able to be rewritten as $\sum f(c_{ij} v_i, w_j)$ by commutativity. This specifies g completely, because if g exists, then it must be specified by this formula.

Existence follows, because this formula defines a homomorphism g because f is biadditive. We only need to check that this makes the diagram commutes. Because of biadditivity, it suffices to check on $(cv_i, dw_j) \in V \times W$. The map $V \times W \to T \to P$ takes $(cv_i, dw_j) \mapsto (cd)v_i \otimes w_j \mapsto f(cdv_i, w_j)$. The diagonal map $f : V \times W \to P$ takes $(cv_i, dw_j) \mapsto f(cv_i, dw_j)$. But because of middle linearity, $f(cv_i, dw_j) = f(cdv_i, w_j)$. $\square$

You can also take the tensor product of multiple vector spaces. We write $V^{\otimes n} = \bigotimes_n V$.

Let's talk about maps between tensor products. Given abelian group homomorphisms $f : M \to M'$ and $g : N \to N'$, you can tensor them to get $f \otimes g : M \otimes N \to M' \otimes N'$ defined by $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$. This is induced by the bi-additive pairing $M \times N \to M' \otimes N'$ that takes $(m, n) \mapsto f(m) \otimes g(n)$. More generally, if you're given $f : M_A \to M'_A$ (where the $_A$ means that they're right A-modules) and $g :_A N \to_A N'$, then you get $f \otimes g : M \otimes N \to M' \otimes N'$. Recall:

Definition 4.2. If V is a vector space, the dual vector space is $V^* = \mathrm{Hom}_k(V, k)$. If $e_1, \cdots, e_n$ is a basis of V, then there's a dual basis $e^1, \cdots, e^N$, defined by $e^j(e_i) = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$. So $\dim V^* = \dim V$.

Definition 4.3. An element of $V^{\otimes n} \otimes (V^*)^{\otimes m}$ is called a tensor of type (m, n), and a typical element looks like $\sum_{i_1, \cdots, i_n} c^{i_1, \cdots, i_n}_{j_1, \cdots, j_m} e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e^{j_1} \otimes \cdots \otimes e^{j_m}$ where $c^{i_1, \cdots, i_n}_{j_1, \cdots, j_m} \in k$.

Definition 4.4. The tensor square $T^2V := V^{\otimes 2}$. If V has basis $e_1, \cdots, e_N$, then the basis of T^2V is $e_i \otimes e_j$ whose dimension is N^2. There's also what's called the symmetric square, written $S^2V = \mathrm{Sym}^2 V := V \otimes V/(k\text{-span of } v \otimes w - w \otimes v,\ v, w \in V)$. The basis is written $e_i e_j$ where $i \leq j$, and the dimension is $n(n+1)/2$. Lastly, there's the alternating square $\bigwedge^2 V = \Lambda^2 V := V \otimes V/(k\text{-span of } v \otimes v,\ v \in V)$. The basis is written $e^i \wedge e_j$ where $i < j$, so the dimension is $N(N-1)/2$.

Example 4.5. If $V = k \cdot x + k \cdot y$, then $\mathrm{Sym}^2 V = k \cdot x^2 + k \cdot xy + k \cdot y^2 = \{\text{quadratic forms in } x, y\}$.

Definition 4.6. You can do all of this to get the dth tensor power, the dth symmetric power, and the dth alternating power. The dimension of the dth tensor power is N^d. The dth symmetric power is $\mathrm{Sym}^d V = V^{\otimes d}/\langle \sigma(t) - t,\ \sigma \in S_d,\ t \in V^{\otimes d}\rangle$, whose dimension is, by the stars and bars argument (namely, you let $*|**||*$ correspond to $e_1e_2e_2e_4$), the number of stars and bars you can make with d stars and $N-1$ bars, which is $\binom{N+d-1}{N-1}$. The dth alternating power, also called the exterior power, is $\Lambda^d V := V^{\otimes d}/\langle e_1 \otimes e_2 \otimes \cdots \otimes e_d | e_i = e_j \text{ for some } i \neq j\rangle$. The dimension of this vector space is $\binom{N}{d}$, because you only need to choose d indices.

Remark 4.7. If $t \in T^dV$ and $t' \in T^eV$, then you get $t \otimes t' \in T^{d+e}V$. The *tensor algebra* is $TV := \bigoplus_{d \geq 0} T^dV$. This is an associative k-algebra. Note that $T^0V = k$. If $V = \langle x_1, \cdots, x_n\rangle$, then $TV = k\langle x_1, \cdots, x_n\rangle$, and T^dV has as basis words of length d.

Remark 4.8. You can also have $\mathrm{Sym} V = \bigoplus \mathrm{Sym}_{d \geq 0} \mathrm{Sym}^d V = TV/\langle v \otimes w - w \otimes v | v, w \in V\rangle$ where $\langle v \otimes w - w \otimes v | v, w \in V\rangle$ is a two-sided ideal (so it takes care of the tensors with more terms). If $V = \langle x_1, \cdots, x_n\rangle$, then $\mathrm{Sym} V = k[x_1, \cdots, x_n]$. Lastly, you can also have $\Lambda V = \bigoplus_{d \geq 0} \Lambda^d V = TV/\langle v \otimes v | v \in V\rangle$. This is actually a finite-dimensional algebra, whose dimension is $\sum_{0 \leq d \leq N} \binom{N}{d} = 2^N$. (When $V = \langle x_1, \cdots, x_n\rangle$, the alternating algebra ΛV is $\Lambda_k(x_1, \cdots, x_n)$, the exterior algebra.)

Definition 4.9. If V is a Lie algebra, then its universal enveloping algebra is:

$$\mathcal{U}(V) := TV/\langle v \otimes w - w \otimes v - [v, w] | v, w \in V \rangle.$$

This doesn't decompose as a sum like the above examples.

We've postponed section 2.14 of the textbook.

4.1. **Back to representation theory.** Recall that irreducibility is stronger than indecomposability, and that an irreducible representation is also called simple.

Proposition 4.10. *If V is finite-dimensional, then $V \simeq \bigoplus V_i$ for some indecomposable V_i.*

Proof. Ridiculously easy. □

If V is indecomposable, this does not imply V is irreducible, because it might have a subrepresentation $W \neq 0, V$ such that $W^\perp$ (called the complement of W) isn't a subrepresentation.

Definition 4.11. Say that V is semisimple (also called completely reducible) if it is a direct sum of irreducible representations.

Example 4.12. Let $A = \mathrm{M}_n(k)$, and let $V = k^n$ be the space of column vectors. This is called the standard representation of A. Let $Y = \mathrm{M}_n(k)$ itself, called the regular representation. Now, V is irreducible, because the A-module generated by any $v \in V$ is the whole of V itself. Is Y irreducible? Clearly not, since $Y = \bigoplus Y_i$ where Y_i consists of those matrices whose ith column is nonzero, and all other columns are zero. But $Y_i \simeq k^n$ as an A-representation, so Y is semisimple. Note that $Y = \left(\bigotimes^n k\right) \oplus V \simeq \bigoplus^n V$. This is essentially 3.1.2 of the book.

5. Subrepresentations of semisimple representations

Next week, we have a guest lecturer, the celebrity Pavel Etingof. There's no office hours next week. Remember that the next PSet is due Monday.

Last time, we did this. If $A = \mathrm{M}_n(k)$, $V = k^n$, and $Y = A$ (called the regular representation). Then we showed that $Y \simeq \bigoplus^n V \simeq N \otimes_k V$ where $N = \bigoplus^n k$. We'll write $nV := \bigoplus^n V$ - you're *not* multiplying by n. This decomposition isn't canonical, as we'll see. Similarly, for any associative k-algebra A, if V is an irreducible representation of dimension n and $Y = \mathrm{End}(V)$, then $Y \simeq nV$. This is just the result above, except without bases.

Recall that a semisimple A-representation is one that is a direct sum of irreducible representations. What are the subrepresentations of semisimple representations? Let's do an example.

Example 5.1. Let $k = \overline{k}$ be algebraically closed. Let A be an associative k-algebra, and let V, W be irreducible representations such that $V \not\simeq W$. What are the irreducible subrepresentations of $V \oplus W$? Two obvious things are V and W.

Lemma 5.2. *They're the only subrepresentations.*

Proof. To see this, if P is another irreducible subrepresentation, then the projection to V is either 0 or an isomorphism (this is essentially Schur's lemma), and similarly for W. They can't both be isomorphisms because $V \not\simeq W$. Therefore one of the projections is 0, so $P \subseteq V$ or $P \subseteq W$. Since V, W are irreducible, either $P = V$ or $P = W$. □

Example 5.3. What are the irreducible subrepresentations of $V \oplus V$. Clearly there's V. But this isn't the only subrepresentation. Let $V \to V \oplus V$ - there are multiple such maps, because you can send $v \mapsto (c_1 v, c_2 v)$ for any $(c_1, c_2) \in k^2/(0,0)$. The only subrepresentations must be graphs of endomorphisms, but by Schur's lemma, $\mathrm{End}_A V = k$. So, {irreps of $V \oplus V$} $\leftrightarrow$ {1-dimensional subspaces of $k \oplus k$}.

Let's go to the general case. A typical semisimple representation is of the form $\bigoplus X_1 \oplus \bigoplus X_2 \oplus \bigoplus X_3 \cdots = \bigoplus_{\text{irreps } X} N_i \otimes X_i$ where N_i is a k-vector space of dimension given by the number of copies of X_i. This is called the *isotypic decomoposition of a semisimple representation*.

Definition 5.4. Given V, X representations, define the evaluation map $\mathrm{Hom}_A(X, V) \otimes X \to V$ via $g \otimes x \mapsto g(x)$.

Proposition 5.5. *Let $k = \overline{k}$, and let A be an associative k-algebra. If V is a finite-dimensional semisimple A-representation. Then the map*

$$\bigoplus_{\text{irreducible } X \text{ up to isomorphism}} \mathrm{Hom}_A(X, V) \otimes_k X \to V$$

is an isomorphism of representations.

Proof. If this is true for V_1, V_2, then this is true for V_1 and V_2. Therefore, it suffices to consider irreducible V. Then $\mathrm{Hom}_A(X, V) = \begin{cases} \mathrm{End}_A(V) = k & \text{if } X \simeq V \\ 0 & \text{if } X \not\simeq V \end{cases}$ by Schur's lemma. So the map $\bigoplus_X \mathrm{Hom}_A(X, V) \otimes_k X \to V$ is a direct sum $0 \oplus \cdots \oplus 0 \oplus (k \otimes_k V) \oplus 0 \cdots \oplus 0 \to V$ - but this is an isomorphism! □

This is functorial in V, i.e., if $V = \bigoplus_X N_X \otimes X$ and $V' = \bigoplus_X N'_X \otimes X$, then:

{homomorphisms of representations $V \to V'$} $\leftrightarrow$ {tuple $(f_X)_X$ of k-linear maps $f_X : N_X \to N'_X$}

How does that work? Given f, get $N_X = \mathrm{Hom}_A(X, V) \to \mathrm{Hom}_A(X, V') = N'_X$ given by $g \mapsto f \circ g$ for each X, which gives the required map. This is why people like (irreducible, at least) semisimple representations, because every question reduces to looking at vector spaces. How do you tell if f is injective? All you have to do is check that f_X is injective for all X, and similarly for surjectivity, and hence for isomorphisms.

Corollary 5.6. *The subrepresentations of $V = \bigoplus N_X \otimes X$ are $\bigoplus M_X \otimes X$ where $M_X \subseteq N_X$ is a subspace for each X. The quotients are $\bigoplus Q_X \otimes X$ where Q_X is a quotient of N_X for each X. Everything's kinda really easy.*

In all of this, we used Schur's lemma. But there's also a variant.

Remark 5.7. If $k \neq \overline{k}$ or V isn't necessarily finite-dimensional, then $D_X := \operatorname{End}_A X$, where X is irreducible, is a division algebra (because $X \to X$ is either 0 or an isomorphism, but then the inverse is a homomorphism - so every nonzero element has an inverse, and this is exactly what a division algebra is!), and $N_X := \operatorname{Hom}_A(X, V)$ is a right D_X-module. Then:

$$\bigoplus_{\text{irreducible } X \text{ up to isomorphism}} \operatorname{Hom}_A(X, V) \otimes_{D_X} X \to V$$

is an isomorphism of representations.

Lemma 5.8. *Let $k = \overline{k}$, A an associative k-algebra, and V a finite-dimensional irreducible representations. If $v_1, \cdots, v_n \in V$ are linearly independent, and if $w_1, \cdots, w_n \in V$ are arbitrary vectors, then there exists some $a \in A$ such that $av_i = w_i$ for all i.*

Proof. Consider the A-representation homomorphism $A \to V^n$ sending $a \mapsto (av_1, \cdots, av_n)$. You can check that this is a homomorphism of representations. But $V^n = k^n \otimes_k V$. If this map isn't surjective, then what's the image? It's image will itself be a subrepresentation of V^n, but V^n is semisimple, so subrepresentations of V^n are contained in $H \otimes V$ where H is a hyperplane, i.e., subspace of k^n where H is defined by $c_1 x_1 + \cdots + c_n x_n = 0$ for some $c_1, \cdots, c_n \in k$ not all 0. Then $H \otimes V = \{(z_1, \cdots, z_n) \in V^n | c_1 z_! + \cdots + c_n z^n = 0\}$. Since the image of the map is contained in this thing, then $c_1(av_1) + \cdots + c_n(av_n) = 0$ for all $a \in A$. Associative algebras have a 1, so if $a = 1$, then you have $c_1 v_1 + \cdots + c_n v_n = 0$. This is a contradiction because of linear independence. This finishes the proof since the map is surjective. □

Theorem 5.9 (Density theorem)**.** *Let $k = \overline{k}$, A an associative k-algebra, and V a finite-dimensional representation of A.*

(1) *If V is irreducible, then the k-algebra homomorphism $A \to \operatorname{End} V$ is surjective.*

(2) *If $V = \bigoplus V_i$ where the V_is are irreducible and pairwise nonisomorphic, then $A \to \prod \mathrm{End}(V_i)$ is surjective.*

Proof. For the first part, let $T \in \mathrm{End}\, V$. Let $v_1, \cdots, v_n$ be a basis of V. By our lemma, there is $a \in A$ such that $av_i = Tv_i$ for all i. Then a maps to T, so we're done.

For the second part, let $d_i = \dim V_i$. What does $\mathrm{End}\, V_i$ look like? This is just $d_i \times d_i$-matrices. So $\mathrm{End}\, V_i \simeq d_i V_i$ as A-representations. If you take $\prod \mathrm{End}\, V_i$, this is isomorphic[1] to $\bigoplus d_i V_i$ because the direct sum and direct product are the same in vector spaces. The image of $A \to \bigoplus d_i V_i$ is isomorphic to $\bigoplus e_i V_i$ where $e_i \leq d_i$. By the first part of the lemma, $A \to \mathrm{End}\, V_i = d_i V_i$ is surjective. This doesn't work if $e_i < d_i$, so $e_i = d_i$, and therefore the image of $A \to \prod \mathrm{End}\, V_i$ had to be the whole thing, i.e., it's surjective. ◻

Next thing we'll do is classify all the representations of a matrix algebra.

Proposition 5.10. *Let $A = \mathrm{M}_n(k)$. Let $V = k^n$ be the standard representation.*

(1) *Any other finite-dimensional representation W of A is isomorphic to mV for some $m \geq 0$.*
(2) *The irreducible subrepresentations are just isomorphic to k^n.*

Proof.
(1) Let $w_1, \cdots, w_r$ be basis of W. Then $rA \to W$ by sending the ith basis element of rA maps to w_i. This is automatically surjective. Thus W is a quotient of rA. But as a representation, $rA \simeq r(nV) = rnV$ because of what we saw before. The quotients are just smaller number of copies of V added together, i.e., $W \simeq nV$ for some $m \leq rn$.
(2) Easy once you have the first part.

◻

Suppose that A_1, A_2 are associative k-algebras with representations V_1, V_2 respectively. Then $V_1 \times V_2$ is a representation of $A_1 \times A_2$ with $(a_1, a_2) \cdot (v_1, v_2) := (a_1 v_1, a_2 v_2)$.

Proposition 5.11. *Every representation of $A_1 \times A_2$ arises this way.*

Proof. Let V be a representation of $A_1 \times A_2$. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$, both in $A_1 \times A_2$. These are independent idempotents. So their product is 0 and their sum is 1. Let $V_1 = e_1 V$ and $V_2 = e_2 V$. These are subreps of V. Moreover, they intersect trivially because $e_1 e_2 = 0$, and they generate the whole thing. For the

[1] Rant about direct products and direct sums: you *cannot* take the direct sum of rings! If you could do $R \oplus M$, then the map $R \to R \oplus M$ isn't a ring homomorphism because it sends $1 \mapsto (1, 0)$, which isn't the identity in $R \oplus M$. Since $R \oplus M$ is defined that way, this doesn't work.

latter claim, if $v \in V$, then $e_1 v + e_2 v = v$. Therefore, $V = V_1 \oplus V_2$. The action of $A_1 \times A_2$ on V_i factors through A_i, i.e., $(a_1, a_2) \cdot e_1 v = (a_1, a_2)(1, 0)v = (a_1, 0)v$, so it depends only on a_1. Therefore V_1 is an A_1-representation, and similarly for V_2. □

Corollary 5.12. *The irreps of $A_1 \times A_2$ are $V_1 \times \{0\}$ and $\{0\} \times V_2$ where V_1 is an irreducible A_1-representation and V_2 is an irreducible A_2-representation.*

Theorem 5.13. *Let $A = M_{d_1}(k) \times \cdots \times M_{d_r}(k)$. Then the irreducible A-representations are $V_1 = k^{d_1}, \cdots, V_r = k^{d_r}$, all nonisomorphic. Every finite-dimensional A-representation is a direct sum of copies of $V_1, V_2, \cdots, V_r$. So they are all semisimple representations, so we say that A is a semisimple k-algebra.*

We have a few more minutes, so we'll end with a definition.

Definition 5.14. Let A be an associative k-algebra. Let V be a representation of A. A filtration of V is a sequence of subrepresentations $0 = V_0 \subseteq \cdots \subseteq V_n = V$. The quotients of the filtration are V_i/V_{i-1} for each $i = 1, \cdots, n$.

Proposition 5.15. *If V is finite-dimensional, then there is a filtration whose quotients are irreducible.*

Proof. Induct on dimension of V. If $\dim V = 0$, there are no quotients! If V is irreducible, then just take a one-step filtration. Otherwise, $0 \subset W \subset V$. Choose a filtration of W with irreducible quotients. Then look at V/W. Then you get a filtration with irreducible quotients. But submodules between W and V correspond to submodules of V/W by one of the isomorphism theorems, which creates a filtration of V with irreducible quotients. □

6. Guest Lecture: Pavel Etingof

Let A be a finite-dimensional k-algebra. Later we'll assume that k is algebraically closed, but not at the moment.

Definition 6.1. The radical of A, $\mathrm{Rad}(A)$, is the set of all $x \in A$ which act by 0 in all irreducible representations of A.

Proposition 6.2. *It's a two-sided ideal.*

Proof. Trivial. □

Proposition 6.3 (Slightly less trivial)**.**

1. *Let I be a nilpotent (two-sided) ideal in A, i.e., $I^n = \{x^n | x \in I\} = 0$ for some n. Then $I \subseteq \mathrm{Rad}(A)$.*
2. $\mathrm{Rad}(A)$ *is itself a nilpotent ideal, i.e., it's the largest nilpotent ideal in A.*

Proof.

1. Suppose V is an irreducible representation of A, and $v \in V$. Consider $Iv \subseteq V$. This is a subrepresentation of V because I is an ideal. If this subrepresentation isn't zero, then it must be everything, because V is irreducible, i.e., there is $x \in I$ such that $xv = v$. But then $x^n \neq 0$ for every n since $x^n v = v$. This is a contradiction because I is nilpotent. Thus $Iv = 0$, so $I \subseteq \operatorname{Rad}(A)$.
2. Let $0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$ be a (finite!) filtration of A with irreducible A_i/A_{i-1}. We showed before that this exists. Here, we're viewing A as the regular representation. If $x \in \operatorname{Rad}(A)$, then x acts by zero on A_i/A_{i-1}, so $xA_i \subseteq A_{i-1}$. This means that x^n acts by zero where n is the length of this chain, i.e., $x^n \cdot 1 = x^n = 0$.

□

Example 6.4. Suppose $A = k[x]/(x^n)$. Then $\operatorname{Rad}(A) = (x)$. The only irreducible module is k with x acting by 0.

Now let's assume that $k = \overline{k}$.

Theorem 6.5. *A finite-dimensional (unital) algebra A has only finitely many irreducible representations $V_1, \cdots, V_r$ up to isomorphism, all of which are finite-dimensional, and $A/\operatorname{Rad}(A) \simeq \bigoplus_{i=1}^r \operatorname{End} V_i$ as A-algebras.*

Proof. Let V is an irreducible representation, and let $v \neq 0 \in V$. Then $0 \neq Av \subseteq V$, so $Av = V$ because V is irreducible. But then $\dim V \leq \dim A < \infty$. All the irreps are finite-dimensional. Let $V_1, \cdots, V_r$ be pairwise nonisomorphic irreducible representations of A. Then we have $\rho = \bigoplus_{i=1}^r \rho_i : A \to \bigoplus_{i=1}^r \operatorname{End} V_i$, which is surjective by the density theorem. This means that $r \leq \sum_{i=1}^r \dim \operatorname{End} V_i = \sum_{i=1}^r (\dim V_i)^2 \leq \dim A$, which is finite dimensional, i.e., there are finitely many irreps. What remains to prove is the isomorphism. Assume $V_1, \cdots, V_r$ are all irreps up to isomorphism. Then $\ker \rho = \operatorname{Rad}(A)$. The isomorphism theorems give a canonical isomorphism $A/\operatorname{Rad}(A) \simeq \bigoplus_{i=1}^r \operatorname{End} V_i$. □

Corollary 6.6.

$$\sum_{i=1}^r (\dim V_i)^2 \leq \dim A$$

Proof. Already done in the proof above, but for the record:

$$\sum_{i=1}^r \dim \operatorname{End} V_i = \sum_{i=1}^r (\dim V_i)^2 = \dim A - \dim \operatorname{Rad}(A) \leq \dim A$$

□

Example 6.7. Let A be the algebra of upper triangular $n \times n$-matrices. For all $i = 1, \cdots, n$, we have irreducible, pairwise nonisomorphic, 1-dimensional representation of A, sending $A \ni x \mapsto x_{ii}$. Note that $(xy)_{ii} = x_{ii}y_{ii}$ because the matrices are upper triangular. Let $I = \{x | x_{ii} = 0 \text{ for all } i\}$, which is precisely all strictly upper triangular matrices. But $I \subseteq \mathrm{Rad}(A)$ since it's nilpotent, but also $\mathrm{Rad}(A) \subseteq I$ since for all $x \in \mathrm{Rad}(A)$, $x_{ii} = 0$, so $I = \mathrm{Rad}(A)$. This means that $V_1, \cdots, V_n$ are the only irreducible representations, and $\mathrm{Rad}(A)$ is strictly upper triangular matrices. The same thing is true for lower triangular $n \times n$-matrices.

Example 6.8. Let's consider a generalization. Let A be the collection of block matrices with blocks of size $m_1, \cdots, m_r$, i.e., "upper triangular matrices made out of matrices". Similarly, we have irreducible representations $V_i = k^{m_i}$, where $x \mapsto x_{ii}$ (which is a matrix, not a number). Then $\mathrm{Rad}(A) = \{\text{strictly block upper triangular matrices}\}$. This is a generalization of the previous example.

Definition 6.9. A finite-dimensional algebra is said to be semisimple if $\mathrm{Rad}(A) = 0$. By default, for the rest of today, k is assumed to be algebraically closed.

Proposition 6.10. *The following conditions are equivalent (TFAE - this is like LOL, mathematicians were the first to invent this language) for a finite-dimensional algebra A.*

(1) *A is semisimple.*
(2) $\sum_{i=1}^r (\dim V_i)^2 = \dim A$ *where* $V_1, \cdots, V_r$ *are irreps of A.*
(3) $A \simeq \bigoplus_{i=1}^r \mathrm{M}_{d_i}(k)$ *for some* r, d_i.
(4) *Any finite-dimensional representation of A is completely reducible, i.e., isomorphic to a direct sum of irreps.*
(5) *A viewed as the regular representation is compltely reducible.*

Proof. We saw before that $\sum_{i=1}^r (\dim V_i)^2 = \dim A - \dim \mathrm{Rad}(A)$. But $\mathrm{Rad}(A) = 0$, so $\sum_{i=1}^r (\dim V_i)^2 = \dim A$. Conversely, if $\sum_{i=1}^r (\dim V_i)^2 = \dim A$, then $\dim \mathrm{Rad}(A) = 0$, so (1) is equivalent to (2). Also, by the previous Theorem, $A \simeq \bigoplus \mathrm{End}(V_i) \simeq \bigoplus \mathrm{M}_{d_i}(k)$, so (1) implies (3). In addition, (3) implies (1) because $\mathrm{Rad}(\bigoplus \mathrm{M}_{d_i}(k)) = 0$, which is immediately clear.

(3) implies (4) because of classifications of direct sums of matrix algebras, which we did before. Clearly (4) implies (5). It remains to prove that (5) implies (3). To prove this, we have a direct sum decomposition $A = \bigoplus_{i=1}^r d_i V_i$. Consider $\mathrm{End}_A(A)$. But $\mathrm{End}_A(A) = A^{op}$ (exercise, but it's not hard to prove), and $\mathrm{End}_A(\bigoplus_{i=1}^r d_i V_i) = \bigoplus_{i=1}^r \mathrm{M}_{d_i}(k)$ because $\mathrm{Hom}_A(V_i, V_i) = k$ by Schur's lemma and $\mathrm{Hom}_A(V_i, V_j) = 0$ if $i \neq j$. We have an isomorphism $\mathrm{M}_d(k) \to \mathrm{M}_d(k)^{op}$ by

sending $B \mapsto B^T$. So $A = \bigoplus_{i=1}^r \mathrm{M}_{d_i}(k)$, and we're done! This is a proof modulo this exercise[2]. $\square$

6.1. **Characters of representations.**

Definition 6.11. Let V be a finite-dimensional representation of A, with $\rho : A \to \mathrm{End}(V)$. The character of V is the linear function $\chi_V : A \to k$ given by $\chi_V(a) = \operatorname{Tr}\rho(a)$. Clearly this is an invariant of representations up to isomorphism.

Example 6.12. If $W \subseteq V$, then $\chi_V = \chi_W + \chi_{V/W}$. Exercise! This is one of the drawbacks of characters.

Theorem 6.13.
(1) *Characters of distinct irreps of A are linearly independent.*
(2) *Let $[A, A]$ be the space of $[a, b] = ab - ba$ where $a, b \in A$. This is a subspace of A. It's not any kind of ideal - it's just a subspace. If A is semisimple, characters of irreps of A form a basis in $(A/[A, A])^* \subseteq A^*$.*

Remark 6.14. χ_V vanishes on $[A, A]$ since $\operatorname{Tr}(\rho(ab-ba)) = \operatorname{Tr}(\rho(a)\rho(b)-\rho(b)\rho(a)) = 0$. Therefore χ_V is a linear function $A/[A, A] \to k$.

Proof, hopefully right.
(1) Suppose $V_1, \cdots, V_r$ are distinct irreps of A. Then theres a surjective (by the density theorem) map $\rho : A \to \bigoplus_{i=1}^r \operatorname{End} V_i$. Let $\chi_1, \cdots, \chi_r$ be characters of $V_1, \cdots, V_r$. They're linearly independent since they are linear functionals on $\operatorname{End} V_1, \cdots, \operatorname{End} V_r$. That is, if $\sum_{i=1}^r \lambda_i \chi_i = 0$, then $\sum_{i=1} \lambda_i \operatorname{Tr}\rho_i(a) = 0$ for any a. These ρ_i can be chosen independently of each other because of the density theorem. So for any $M_1, \cdots, M_r$ where $M_i \in \operatorname{End} V_i$, we have $\sum \lambda_i \operatorname{Tr} M_i = 0$, and so all $\lambda_i = 0$.
(2) $A = \bigoplus_{i=1}^r \mathrm{M}_{d_i}(k)$. We want to compute $[A, A]$. This is $\bigoplus_{i=1}^r [\mathrm{M}_{d_i}(k), \mathrm{M}_{d_i}(k)]$. So what's $[\mathrm{M}_{d_i}(k), \mathrm{M}_{d_i}(k)]$? The claim is that this is precisely $\mathfrak{sl}_{d_i}(k)$, i.e., matrices of trace 0. It's easy because one inclusion is clear since any element in the commutator must have trace zero. Now $\mathfrak{sl}_d(k)$ has basis $E_{ij}, i \neq j$ and $E_{ii} - E_{i+1,i+1}$. Well, $E_{ij} = [E_{ik}, E_{kj}]$, and $E_{ii} - E_{i+1,i+1} = [E_{i,i+1}, E_{i+1,i}]$. This proves the reverse inclusion. This whole thing means that $[A, A] = \bigoplus_{i=1}^r \mathfrak{sl}_{d_i}(k)$, i.e., $\dim A/[A, A] = r$. But $\chi_1, \cdots, \chi_r$ are linearly independent by (1) in $(A/[A, A])^*$, so they are a basis because $(A/[A, A])^*$ has dimension r.

$\square$

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[2]Was this assigned as an exercise? No. Alright, let's do the exercise then. Suppose A is any unital algebra. and $\phi \in \operatorname{End}_A(A)$. Then we want to show that there exists a unique $b \in A$ such that $\phi(a) = ab$. Just let $b = \phi(1)$. Consider $\phi_b \circ \phi_c(a) = acb = \phi_{cb}$, so that's why we get the opposite of A.

Theorem 6.15 (Jordan-Holder theorem)**.** *Let V be a finite-dimensional representation of A (any algebra), and let $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ and $0 = V'_0 \subset V'_1 \subset \cdots \subset V'_m = V$ be two filtrations of V such that $W_i = V_i/V_{i-1}$ and $W'_i V'_i/V'_{i-1}$ are irreducible. Then $n = m$, and there is a permutation σ of $1, \cdots, n$ such that $W_i \simeq W'_{\sigma(i)}$.*

First proof, we'll have an error. $\chi_V = \sum_{i=1}^{n} \chi_{W_i} = \sum_{i=1}^{m} \chi_{W'_i}$. Since characters of irreps are linearly independent, the number of times each irrep occurs among W_i and W'_i is the same. □

This isn't exactly right, because it only works over fields of characteristic 0.

Second proof, correct and more general. Let's induct in the dimension of V. The base case is clear. If $W_1 = W'_1$ (I mean literally equal). Well, then, we are done, since this reduces to the statement for $V/W_1 = V/W'_1$. This is the induction assumption. So WLOG we can assume that $W_1 \neq W'_1$. Because they're irreducible, $W_1 \cap W'_1 = 0$. We have an inclusion $W_1 \oplus W'_1 \hookrightarrow V$. Let $U = V/(W_1 \oplus W'_1)$. Then $\dim U < \dim V$. Fix a filtration of U as $0 = U_0 \subset U_1 \subset \cdots \subset U_p = U$ with $U_i/U_{i-1} = Z_i$ irreducible. We can always do this. Then we have a filtration of V/W_1 with successive quotients $W'_1, Z_1, \cdots, Z_p$, and another one of V/W'_1 with successive quotients $W_1, Z_1, \cdots, Z_p$. But we already have another filtration of V/W_1 given by $W_2, W_3, \cdots, W_n$. Similarly, V/W'_1 already has a filtration with successive quotients $W'_2, \cdots, W'_m$. We can use the induction hypothesis to get that $W'_1, Z_1, \cdots, Z_p$ is isomorphic to $W_2, \cdots, W_n$ up to permuation, and similarly $W_1, Z_1, \cdots, Z_p$ is isomorphic to $W'_2, \cdots, W'_m$ up to isomorphic. But now, $W_1, W_2, \cdots, W_n = W_1, W'_1, Z_1, \cdots, Z_p$ and $W'_1, W'_2, \cdots, W'_m = W'_1, W_1, Z_1, \cdots, Z_p$, so $W_1, W_2, \cdots, W_n = W'_1, W'_2, \cdots, W'_m$, so $m = n$, and the lists are the same up to permutation of representations. □

Remark 6.16. This works in any characteristic, and over a field that isn't necessarily algebraically closed. It even works for rings, i.e., this holds for modules over rings. Also works for groups.

Next week, we'll do the Krull-Schmidt theorem.

7. Krull-Schmidt, etc.

Came to class a bit late.

Theorem 7.1. *Let A be an algebra over $k = \overline{k}$ (this assumption isn't necessary, there'll be an exercise asking how to do the proof without this). Any finite-dimensional representation V of A uniquely decomposes as a direct sum of indecomposable representation (up to isomorphism and order of summands).*

This is a rather nontrivial theorem.

Proof. Existence is clear. We only need to prove uniqueness. We're going to do this by induction in $\dim V$. The base of induction is clear (for $\dim V = 1$, $V \simeq k$). Let $V = V_1 \oplus \cdots \oplus V_n$ and $V = V_1' \oplus \cdots \oplus V_m'$ be two decompositions into indecomposables. Now we have maps $i_s : V_s \hookrightarrow V$, $p_s : V \to V_s$, $i_s' : V_s' \hookrightarrow V$, $p_s' : V \to V_s'$. What are the properties of these things? We have the obvious property that $p_s i_s = \mathrm{id}_{V_s}$, and $\sum_s i_s p_s = \mathrm{id}_V$ because every vector is the sum of its componennts in the direct sum, and similarly for i_s' and p_s'. Define a map $\theta_s : V_1 \xrightarrow{i_1} V \xrightarrow{p_s'} V_s' \xrightarrow{i_s'} V \xrightarrow{p_1} V_1$, i.e., $\theta_s = p_1 i' s p' s i_1$. If we consider $\sum_{i=1}^n \theta_s$, this is $\sum_{i=1}^n p_1 i_s' p_s' i_1 = p_1 i_1 = \mathrm{id}_{V_1} = 1$.

Lemma 7.2. *Let W be a finite-dimensional indecomposable representation of A. Then:*

(1) *If $\theta : W \to W$ is an endomorphism of representation, then θ is either nilpotent or an isomorphism.*
(2) *If $\theta_1, \cdots, \theta_n : W \to W$ are nilpotent endomorphisms, then $\sum_{i=1}^n \theta_s$ is nilpotent.*

Remark 7.3. This is like Schur's lemma, because indecomposable is weaker than irreducibility.

Proof. We'll only do this for algebraically closed fields.

(1) Over an algebraically closed field, we can write $W = \bigoplus_\lambda W(\lambda)$. The $W(\lambda)$ are generalized eigenspaces. Now, each of these is a subrepresentation, so only one of them can be nonzero. This means that $W = W(\lambda)$ for some λ. If $\lambda = 0$, then θ is nilpotent, because all eigenvalues are zero. If $\lambda \neq 0$, then θ is an isomorphism.

    Exercise 7.4. Extend this to any field k. If you have any linear operator, then the space splits uniquely into the direct sum of the spaces where the operator is nilpotent, and where it's an isomorphism.

(2) We'll induct on n. The base case is clear, so we'll do the induction step. Suppose θ isn't nilpotent. Then by part 1, θ is an isomorphism. So $\sum_{i=1}^n \theta^{-1}\theta_s = 1$. So $1 - \theta^{-1}\theta_n = \theta^{-1}\theta_1 + \cdots + \theta^{-1}\theta_{n-1}$. Now, $\theta^{-1}\theta_n$ isn't an isomoprhism, since θ^n is not. So by the previous part, $\theta^{-1}\theta_n$ is nilpotent (so all of its eigenvalues are 0). Then $1 - \theta^{-1}\theta_n$ has only eigenvalue 1. But $\theta^{-1}\theta_1 + \cdots + \theta^{-1}\theta_{n-1}$ are all nilpotent, and hence not isomorphisms, so by the previous part, they're nilpotent. This is a contradiction with the induction assumption.

□

Let's return to the proof of the theorem. By the lemma, because $\sum_{s=1}^n \theta_s = 1$, there is some s such that θ_s isn't nilpotent, i.e., an isomorphism. WLOG we can assume that $s = 1$ (otherwise relabel). Recall that $\theta_1 : V_1 \xrightarrow{i_1} V \xrightarrow{p_1'} V_s' \xrightarrow{i_1'} V \xrightarrow{p_1} V_1$. Define $f = p_1' i_1$ and $g = p_1 i_1'$. Then $g \circ f = \theta_1$. So $V_1' = \operatorname{Im} f \oplus \ker g$ by basic linear algebra, and this is a decomposition of subrepresentations because f and g are maps of representations. Now, V_1' is indecomposable, so $\operatorname{Im} f = 0$ or $\ker g = 0$. But $\operatorname{Im} f = 0$ implies that $f = 0$, which isn't possible. This means that $\ker g = 0$, and so f, g are isomorphisms because g is a map between spaces of the same dimension.

Let $B = \bigoplus_{j=2}^n V_i$ and $B' = \bigoplus_{j=2}^m V_j'$. We have an obvious embedding $h : B \hookrightarrow V \to B'$, the latter of which is a projection along V_1'. We claim that h is an isomorphism. Now, $\dim B = \dim B'$ because $V_1 \simeq V_1'$, so it suffices to show that $\ker h = 0$. Let $v \in \ker h \subseteq V$. This means that $v \in V_1'$ because V_1' is collapsed to zero. But also, $p_1 i_1' v = 0$, so $gv = 0$ by definition. Because g is an isomorphism, $v = 0$. This means that $\ker h = 0$, and thus h is an isomorphism. Now the theorem follows from the induction assumption because $\dim B < \dim V$, and there exists a permutation $\sigma : [m] \to [n]$ such that $\sigma(1) = 1$ and $V_j' \simeq V_{\sigma(j)}$. □

Remark 7.5. The theorem holds in any abelian category where objects have finite length. In particular, it's true over rings. This isn't true if modules have infinite length. For example, let $A = C(\mathbf{R}/\mathbf{Z})$ be the algebra of real-values continuous functions on the circle, i.e., continuous functions that are periodic with period 1. Let $M = \{f : \mathbf{R} \to \mathbf{R} | f(x+1) = -f(x)\}$. The claim is that M is not isomorphic to A, and $M \oplus M \simeq A \oplus A$, and A, M are both indecomposable. Left as an exercise. This means that the Krull-Schmidt theorem fails in this case.

7.1. **Representations of tensor products.** Suppose A, B are algebras over some k. Then $A \otimes_k B =: A \otimes B$ is also an algebra where $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$. Left as an exercise to check if this is right.

Exercise 7.6. $\mathrm{M}_n(k) \otimes \mathrm{M}_m(k) \simeq \mathrm{M}_{mn}(k)$. A good way to think about this is that if V, W are finite-dimensional vector spaces, then $\operatorname{End} V \otimes \operatorname{End} W \simeq \operatorname{End}(V \otimes W)$ by $a \otimes b \mapsto a \otimes b$. This is completely canonical.

Theorem 7.7. *Let k be algebraically closed..*

(1) *If V is an irreducible finite-dimensional representation of A and W an irreducible finite-dimensional representation B, then $V \otimes W$ is an irreducible representation of $A \otimes B$.*

(2) *For any irreducible finite-dimensional representation M of $A \otimes B$, there exist unique up to isomorphism V (irrep of A) and W (irrep of B) such that $M \simeq V \otimes W$.*

Remark 7.8. Part 2 fails if the representations aren't finite-dimensional. For example, take the Weyl algebra $\mathbf{C}\langle x, \partial\rangle/(\partial x - x\partial - 1)$. Let $A = B = A_1$. Then $A \otimes B = A_2 = \mathbf{C}\langle x_1, x_2, \partial_1, \partial_2\rangle$. Part 1 also fails in infinite-dimensional cases. For example, a commutative algebra A is an irreducible A-module iff A is a field (trivial). Let $A = B = \mathbf{C}(x)$. Let $V = W = A = B$. Then $A \otimes B = \mathbf{C}(x) \otimes \mathbf{C}(y)$ is rational functions with denominator $p(x)q(y)$. This is not a field because it doesn't contain $1/(x-y)$ because $x - y$ can't be written as the product of two functions in only x and only y. This is not irreducible as an irreducible $A \otimes B$-module.

Remark 7.9. How important is it that k be algebraically closed? Let $A = B = \mathbf{C}$ be a $\mathbf{R}$-algebra. Let $V = W = \mathbf{C}$. Then V, W are obviously irreducible. What is $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$? We claim that this is precisely $\mathbf{C} \oplus \mathbf{C}$. This isn't a field anymore, so $V \otimes W$ is not irreducible.

Proof of theorem. (1) There's $\alpha : A \to \operatorname{End} V$ that's surjective by the density theorem. Similarly, the map $\beta : B \to \operatorname{End} W$ is also surjective. So $\alpha \otimes \beta : A \otimes B \to \operatorname{End} V \otimes \operatorname{End} W$ is surjective. But $\operatorname{End} V \otimes \operatorname{End} W \simeq \operatorname{End} V \otimes W$. This means that $V \otimes W$ is irreducible. We did use algebraically closed here because the density theorem needs k to be algebraically closed.

(2) Let A' be the image of A in $\operatorname{End} M$, and similarly for B'. Let M be a finite-dimensional irrep of $A \otimes B$. Then there's a factorization $A \otimes B \to A' \otimes B' \to \operatorname{End} M$, and A', B' are finite-dimensional since M is also finite-dimensional. WLOG we can replace A, B with A', B', so we've reduced to the finite-dimensional case.

Lemma 7.10. $\operatorname{Rad}(A \otimes B) = \operatorname{Rad}(A) \otimes B + A \otimes \operatorname{Rad}(B)$.

Proof. Let $J = \operatorname{Rad}(A) \otimes B + A \otimes \operatorname{Rad}(B)$. Then J is nilpotent because $\operatorname{Rad}(A)^n = 0$ and $\operatorname{Rad}(B)^m = 0$, so $J^{n+m} = 0$ by the binomial theorem. In fact, it's true that $J^{n+m-1} = 0$. This means that $J \subseteq \operatorname{Rad}(A \otimes B)$. If we consider $A \otimes B/J$, this is $A/\operatorname{Rad}(A) \otimes B/\operatorname{Rad}(B)$ by linear algebra[3], and this is an algebra isomorphism. This is pretty easy to see. Now, $A/\operatorname{Rad}(A)$ and $B/\operatorname{Rad}(B)$ are semisimple. This means that $A \otimes B/J$ is semisimple, so $\operatorname{Rad}(A \otimes B) \subseteq J$ (because semisimple algebras don't have any nonzero nilpotent ideals, and the image of $\operatorname{Rad}(A \otimes B)$ in $A \otimes B/J$ is nilpotent), and thus $J = \operatorname{Rad}(A \otimes B)$. ◻

[3]If you have two spaces $U \subseteq X$ and $V \subseteq Y$, then $(X \otimes Y)/(U \otimes Y + X \otimes V) \simeq X/U \oplus Y/V$.

Now we can finish our proof. Suppose M is an irrep of $A \otimes B$. Then the action of $A \otimes B$ on M factors through $A \otimes B/\operatorname{Rad}(A \otimes B) \simeq A/\operatorname{Rad}(A) \otimes B/\operatorname{Rad}(B)$, and $A/\operatorname{Rad}(A) \simeq \bigoplus_i \operatorname{End}(V_i)$ and $B/\operatorname{Rad}(A) \simeq \bigoplus_j \operatorname{End}(W_j)$. This means that $A \otimes B/\operatorname{Rad}(A \otimes B) \simeq \bigoplus_{i,j} \operatorname{End}(V_i \otimes W_j)$. So $M \simeq V_i \otimes W_j$ for some unique i, j.

□

7.2. **Representations of finite groups.** Recall that a representation of G is a map $\rho : G \to \mathrm{GL}(V)$ where V is a vector space over k. This is the same thing as a representation of the group algebra $k[G]$, i.e., a map $\rho : k[G] \to \operatorname{End}(V)$.

Theorem 7.11 (Maschke's theorem)**.** *Let $|G| \neq 0$ in $k = \overline{k}$. Then:*

(1) *$k[G]$ is semisimple.*
(2) *We have an algebra isomorphism $\psi : k[G] \to \bigoplus_i \operatorname{End} V_i$ defined via $\psi(g) = \bigoplus_i g|_{V_i}$ where the V_i are the distinct irreps of G. In particular, this is an isomorphism of representations of G. So the regular representation $k[G]$ on G is isomorphic to $\bigoplus_i (\dim V_i) V_i$, i.e., each irrep occurs as many times as its dimension. This means that $|G| = \sum_i (\dim V_i)^2$. This is called the sum of squares formula. This is in $\mathbf{Z}$, not just in the field.*

Proof. (2) follows from (1) by the theory of finite-dimensional algebras because a semisimple thing is isomorphic to a direct sum of endomorphisms of irreps. We automatically obtain the sum of squares thing as well. We only have to prove part 1, i.e., show that every representation is completely reducible, i.e., we need to show that if $W \subseteq V$ is a subrep, then there is $W' \subseteq V$ such that $V \simeq W \oplus W'$.

Let $\widehat{W}$ be any linear complement of W in V. Let $p : V \to V$ be defined via $p|_W = \mathrm{id}_W$ and $p|_{\widehat{W}} = 0$. The point is that $\widehat{W}$ isn't a subrepresentation in general. You want to make this a subrepresentation. Define $\overline{p} = \frac{1}{|G|} \sum_{g \in G} \rho(g) p \rho(g)^{-1} : V \to V$. You prove that $\overline{p}$ is a projection to W along $W' \subseteq V$, and $W' = \ker \overline{p}$. Then W' is a subrepresentation, so that $V = W \oplus W'$. Poonen will probably go into more detail next time as we're short on time. □

8. Group representations

Definition 8.1. Let V, W be representations of a group G. Then G acts on $\operatorname{Hom}_k(V, W)$ by carrying some $F : V \to W$ via:

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d, "g"] & W \arrow[d, "g"] \\ V \arrow[r, "{}^g F"] & W \end{tikzcd}$$

More precisely, $({}^gF)(v) = gF(g^{-1}v)$. This makes the diagram commute. Check that this makes $\mathrm{Hom}_k(V, W)$ into another representation of G.

Proposition 8.2. *F is a homomorphism of G-representations if and only if ${}^gF = F$ for all $g \in G$.*

Proof. F should respect the action of g. □

Let W be a subspace of V.

Definition 8.3. A complement of W in V is another subspace W' such that $V = W \oplus W'$.

Definition 8.4. A projection $p : V \to W$ is a linear map such that $p|_V = \mathrm{id}_W$.

Proposition 8.5. *Projections $p : V \to W$ are in bijection with complements of W in V*

given by $p \mapsto \ker p$.

There is a variant of this. Let k be a field and let A be an associative k-algebra. Let W be a subrepresentation of an A-representation V.

Proposition 8.6. *Projections $p : V \to W$ that are homomorphisms of representations are in bijection with A-representation complements of W in V.*

Corollary 8.7. *Let $A = kG$. Then G-invariant projections $V \to W$ are in bijection with G-representation complements of W in V.*

Proposition 8.8. *For a short exact sequence $0 \to W \xrightarrow{i} V \xrightarrow{s} W \to 0$ of A-representations, the following are equivalent.*

1. *There exists an A-representation complement of W in V, i.e., $V = W \oplus W'$ for some other A-subrepresentation $W' \subseteq V$.*
2. *There is a map $\sigma : Q \to V$ so that $s \circ \sigma = \mathrm{id}_Q$.*
3. *There is a map $p : V \to W$ such that $p \circ i = \mathrm{id}_W$.*

Proof. First exercise in this week's pset. □

Recall:

Theorem 8.9. *The following are equivalent over k (not necessarily algebraically closed):*

1. $\mathrm{Rad}(A) = 0$*, i.e., A is semisimple.*
2. *every subrepresentation of a finite-dimensional A-representation has a complement.*
3. *every short exact sequence of finite-dimensional A-representations splits.*

(4) *every finite-dimensional A-representations is completely reducible.*
(5) *the regular representation is completely reducible.*
(6) $A \simeq \prod_{i=1}^{r} \mathrm{M}_{d_i}(D_i)$ *where D_i is a finite dimensional division algebra over k.*

If k is algebraically closed, then the following additional conditions are equivalent to those above.

(7) $A \simeq \prod_{i=1}^{r} \mathrm{M}_{d_i}(k)$ *because there are no finite dimensional division algebras over an algebraically closed field. (This is because if there was such a thing, then there'd be a finite extension of k but this does not exist owing to the algebraic closure of k.)*
(8) $\dim A = \sum_{i=1}^{r} (\dim V_i)^2$ *where the V_i are irreducible representations.*

We want to show:

Theorem 8.10. *Let k be a field and let G be a finite group. If* $\mathrm{char}(k) \not| \#G$ *(automatic when the characteristic of k is zero), then kG is semisimple.*

Proof. We'll check condition two of the above. Let W be a subrepresentation of a finite-dimensional G-representation V. We need to find a G-representation complement of W. But complements are the same thing as projections. Equivalently, we therefore need to find a G-invariant projection $V \to W$.

We can find a vector space complement because you can take a basis of W and extend to a basis of V, and let W' be the span of the other things. This gives a vector space projection $p : V \to W$. It doesn't respect the G-action. How do you make this respect the G-action?

Try to define: $q := \sum_{g \in G} ({}^g p)$. This is G-invariant (because if you multiply by $h \in G$ it just permutes the elements of the sum), but it's not a projection: if $w \in W$, then $q(w) = \sum_{g \in G} gp(g^{-1}w)$. Now $g^{-1}w \in W$, but p is a projection so it acts trivially, so this gives $\sum_{g \in G} gg^{-1}w = \#G \cdot w$. It's pretty clear how to fix this. So let's try $r := \frac{1}{\#G} \sum_{g \in G}^{g} p$, then this is a G-invariant projection onto W. Then $\ker r$ is the complement of W. $\square$

Corollary 8.11. *All of the other properties hold. As a corollary, if $k = \overline{k}$ and* $\mathrm{char}(k)$ *does not divide $\#G$, if $V_1, \cdots, V_r$ are all irreducible representations of G up to isomorphism, then* $\#G = \sum (\dim V_i)^2$.

Proposition 8.12 (converse to maschke's theorem)**.** *If k is a field and G is a finite group, if the characteristic of k does divide the order of G, then kG is not semisimple.*

Proof. Start with the regular representation kG and the trivial representation k. Then there's the map $kG \to k$ sending $g \mapsto 1$. This is surjective, so you can look at the kernel, which is called I and called the augmentaion ideal. This is $I = \{\sum a_g g | \sum a_g = 0\}$. Think of this as another G-representation. This sits in a short exact sequence $0 \to I \to kG \to k \to 0$. We will show that this does not split, which means that kG can't be semisimple.

Suppose kG is semisimple, so that the ses above splits. then there is a $\sigma : k \to kG$ that's a section of $kG \to k$. This σ should be a G-homomorphism. σ is determined by $\sigma(1) =: \sum b_g g$. To say that σ is a G-homomorphism means that for all $h \in G$, you have $h \sum b_g g = \sum b_g g$ because h acts trivially on k. Because h is arbitrary, this means that $b_g = b_{g'} =: b$ for all $g, g' \in G$. So $1 \mapsto b \sum_{g \in G} g$, and thus $\sum_{g \in G} b = \#Gb = 1$ because $g \mapsto 1$ under $kG \to k$, so $\#G \neq 0$ in k, contradiction. □

8.1. **Characters.**

Definition 8.13. Let V be a finite dimensional G-representation given by $\rho : G \to \mathrm{GL}(V)$. Define the character of V via $\chi_V(g) := \mathrm{Tr}\, \rho(g)$ for all $g \in G$.

Also, any function $f : G \to k$ is called a class function if f is constant on conjugacy classes, i.e., such that $f(ghg^{-1}) = f(h)$ for all $g, h \in G$.

Proposition 8.14. *For any finite-dimensional G-representation V, the character $\chi_V(g)$ is a class function.*

Proof. $\rho(ghg^{-1}) = \rho(g)\rho(h)\rho(g)^{-1}$ is conjugate to $\rho(h)$ in $\mathrm{GL}(V)$. But two conjugate matrices have the same trace, so $\chi_V(ghg^{-1}) = \chi_V(h)$. □

Theorem 8.15. *Let $k = \overline{k}$, and let G be a finite group such that the characteristic of k doesn't divide $\#G$. Then $\{\chi_V | \text{irreps } V\}$ is a basis of $\{\text{class functions}\}$.*

Proof. Let $A = kG$. Then you can define $[A, A]$ to be the k-span of $xy - yx$ where $x, y \in A$; in fact it suffices to do this on the basis of A. So if $A = kG$, then this is the same as the k-span of $gh - hg$. Let $h = jg^{-1}$. Then this is the k-span of $\{gjg^{-1} - j | g, j \in G\}$. This means that $A/[A, A]$ is the k-span of conjugacy classes of G. So $(A/[A, A])^*$ is exactly the space of class functions on G.

On the other hand, by writing $A \simeq \prod \mathrm{M}_{d_i}(k)$, so you can compute $A/[A, A]$ factor by factor, and we shows that $\{\chi_V | \text{irreps } V\}$ form a basis for $(A/[A, A])^*$. □

Corollary 8.16. *If $k = \overline{k}$, and G is a finite group with the usual Maschke condition, then $\#\{\text{irreps of } G\}/\simeq = \#\{\text{conjugacy classes of } G\}$.*

Proof. A basis for class functions is basically conjugacy classes. □

Example 8.17. If $k = \overline{k}$ and G is finite abelian with the Maschke condition, then conjugacy classes of G are exactly the elements of G. We also showed that all irreducible representations are one-dimensional. So the number of irreps is $\#G$. Then $kG \simeq \prod_{i=1}^{\#G} k$. This is like the first homework where we considered $k = \mathbf{C}$ and $G = \mathbf{Z}/n\mathbf{Z}$.

Example 8.18. Let $k = \mathbf{C}$ and $G = \mathbf{Z}/3\mathbf{Z}$. We can write down the characters in a table. here $\omega := e^{2\pi i/3}$.

	0	1	2
χ_1	1	1	1
χ_ω	1	ω	ω^2
χ_{ω^2}	1	ω^2	ω

Because $\mathrm{GL}_1(k) = k^\times$. I don't know how to livetex tables fast.

Given a group G, recall that $[G,G] := \langle ghg^{-1}h^{-1}\rangle$, which is a normal subgroup of G. The abelianization of G is $G^{ab} = G/[G,G]$. It's clearly abelian. This has a universal property that if A is another abelian group that pretends to be an abelianization so that there's a map $G \to A$, there is a unique map $G^{ab} \to A$.

Now, one dimensional representations of a group G are homomorphisms $G \to \mathrm{GL}_1(k) = k^*$. But these are exactly homomorphisms $G^{ab} \to k^\times$ because of the universal property. If $k = \overline{k}$ and G is finite, then $\#\{\text{one-dimensional reps}\} = \#G^{ab}$.

Example 8.19. Let $k = \mathbf{C}$, and let $G = A_4$ be the group of rotation (no reflections) preserving symmetries of a regular tetrahedron. What's the order of G? It's $12 = 4!/2$. Let's write down the elements of G. Well:

$$G = \{1, \text{three elements of order } 2, \text{eight elements of order } 3\}$$

The elements of order two are $(12)(34), (14)(23), (13)(24)$, and the elements of order 3 look like $(123), (234), (341), (412)$ and their inverses. The conjugacy classes are as follows. $[1], [(12)(34), (14)(23), (13)(24)], [(123), (234), (341), (412)]$, [inverses]. This means that $[G,G] = \{1, \text{three elements of order } 2\}$, and the last two conjugacy clases are the nontrivial cosets of $[G,G]$. So $G/[G,G]$ is cyclic of order three, so it's definitely cyclic of order three. This means that there are three irreducible one-dimensional representations.

There's only going to be only one other irreducible representation, because there are only four other conjugacy classes. What is the dimension of this last one? This is exactly $1^2 + 1^2 + 1^2 + a^2 = 12$, so $a = 3$. This three-dimensional representation is the whole of $\mathbf{R}^3$ itself and hence also on $\mathbf{C}^3$! You have a character table like the following.

	1	(12)(34)	(123)	(132) = $(123)^{-1}$
χ_1	1	1	1	1
χ_ω	1	1	ω	ω^2
χ_{ω^2}	1	1	ω^2	ω
χ_V	3	-1	0	0

These are independent, so V is irreducible. This is the full characterization.

9. More on characters

Office hours Friday, October 7, has moved to 3-4 pm. No office hours on Monday (holiday), and no class on Tuesday (holiday). Homework 5 is due Monday October 17 at 11:59 pm.

Recall the character table of A_4:

	1	(12)(34)	(123)	(132) = $(123)^{-1}$
χ_1	1	1	1	1
χ_ω	1	1	ω	ω^2
χ_{ω^2}	1	1	ω^2	ω
χ_V	3	-1	0	0

The $\chi_1, \chi_\omega, \chi_{\omega^2}, \chi_V$ are characters of irreducible representations, and $1, (12)(34), (123), (132)$ are representatives on conjugacy classes. We figured out the character table last time. We knew that the dimension of the last thing is 3. How do we know it's irreducible? It's because if it was, then the character of V would be a linear combination of the other characters, but this is not true. You can do this for any group. This is enough because each character is a class function, i.e., is determined by its value on the conjugacy classes.

9.1. **Dual representations.**

Recall that if G acts on V and W, then G acts on $\mathrm{Hom}(V, W) := \mathrm{Hom}_k(V, W)$. As a special case, if V is finite-dimensional and $W = k$ with the trivial action.

(1) Then G acts on $\mathrm{Hom}(V, k) = V^*$. This is called the *dual representation*/contragredient representation of V. This acts via $({}^g f)(v) := f(g^{-1}v)$. You'd normally have to multiply by g but the action of G on k is trivial anyway.

(2) Another way to think about it is to consider the evaluation $V^* \times V \to k$. This pairing should respect the G-action. What does this mean? If you do $\langle {}^g f, gv\rangle = \langle f, v\rangle$, i.e., ${}^g f(gv) = f(v)$, which is the same as saying ${}^g f(w) = f(g^{-1}w)$, so you get exactly the same thing.

(3) Yet another way: Given $\rho_V : G \to \mathrm{GL}(V)$, we get $\rho_{V^*} : G \to \mathrm{GL}(V^*)$ satisfying $\langle \rho_{V^*}(g) f, \rho_V(g) v\rangle = \langle f, v\rangle$. By definition, $\langle \rho_{V^*}(g) f, \rho_V(g) v\rangle = \langle \rho_V(g)^* \rho_{V^*}(g) f, v\rangle$, where $\rho_V(g)^*$ is the "transpose" (makes sense after you

choose a basis of V). But this means that $\rho_V(g)^*\rho_{V^*}(g) = 1$, and thus $\rho_{V^*}(g) = (\rho_V(g)^*)^{-1} = (\rho_V(g)^{-1})^* = \rho_V(g^{-1})^*$.

(4) Another way is in terms of characters. We just take the transpose of the above equation to get $\chi_{V^*}(g) = \chi_V(g^{-1})$ because transpose doesn't change the trace.

(5) This is a really useful one, so even if you're tired, I still have to tell you. If $k = \mathbf{C}$ and G is finite, then $\chi_{V^*}(g) = \overline{\chi_V(g)}$, i.e., the complex conjugate of $\chi_V(g)$.

Proof. Let $\rho_V(g)$ have eigenvalues $\lambda_1, \cdots, \lambda_n$ counted with multiplicity. Then $\rho_V(g^{-1})$ has eigenvalues $\lambda_1^{-1}, \cdots, \lambda_n^{-1}$. If $\#G = m$, then $\rho_V(g)^m = \rho_V(g^m) = \rho_V(1)$, so $\lambda_1, \cdots, \lambda_n$ are nth roots of unity where n is the order of g. The inverse of an nth root of unity is the complex conjugate, so $\chi_V(g) = \lambda_1 + \cdots + \lambda_n$, and thus $\chi_{V^*}(g) = \overline{\lambda_1} + \cdots + \overline{\lambda_n} = \overline{\chi_V(g)}$. □

9.2. **Tensor products.**

(1) If G acts on V and W, then G acts on $V \otimes W := V \otimes_k W$. This acts via $g(v \otimes w) := gv \otimes gw$. You can check that this is well defined.

(2) The analogue of number 3 above is $\rho_{V\otimes W}(g) = \rho_V(g) \otimes \rho_W(g)$.

(3) The analogue of number 4 is $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$.

Proof. If $A \in \mathrm{M}_m(k)$ has eigenvalues $\lambda_1, \cdots, \lambda_m$ and $B \in \mathrm{M}_n(k)$ has eigenvalues $\mu_1, \cdots, \mu_n$, then $A \otimes B$ has eigenvalues $\{\lambda_i \otimes \mu_j\}_{1\le i\le m, 1\le j\le n}$. If you add up all these eigenvalues, it'll be the product of $\lambda_1 + \cdots + \lambda_m$ and $\mu_1 + \cdots + \mu_n$. □

9.3. **Hom spaces.**

Proposition 9.1. *If* $\dim W < \infty$*, then* $\mathrm{Hom}_k(V, W) \simeq V^* \otimes_k W$.

Proof. Let's define a map in the opposite direction. Take $f \otimes w \mapsto (v \mapsto f(v)w)$ where $f \in V^*$ and $w \in W$. This is a bilinear map, so it's a well-defined map from the tensor product. If you fix V, and if you know that it is an isomorphism for W_1 and W_2, then it's an isomorphism for $W_1 \oplus W_2$. This lets you reduce W into direct summands, i.e., $\bigoplus_{i=1}^{\dim W} k$. But this is clearly an isomorphism for $W = k$, so we're done. □

Corollary 9.2. *If V, W are finite-dimensional representations of G over $k = \mathbf{C}$, then $\chi_{\mathrm{Hom}(V,W)}(g) = \overline{\chi_V(g)}\chi_W(g)$ for all $g \in G$.*

Proof. Just use the facts above. □

9.4. **Orthogonality of characters.** Let G be a finite group. Define a Hermitian[4] inner product on {functions $G \to \mathbf{C}$} by the following formula:

$$(f_1, f_2) := \frac{1}{\#G} \sum_{g\in G} f_1(g)\overline{f_2(g)}$$

"sesquilinear"

This pairing is *positive definite*, meaning that if $f \neq 0$, then $(f, f) = \frac{1}{\#G} \sum_{g\in G} f(g)\overline{f}(g) > 0$ because you're adding up a bunch of positive things.

Theorem 9.3. *If V and W are finite-dimensional representations of G over $k = \mathbf{C}$, then $(\chi_V, \chi_W) = \dim \mathrm{Hom}_G(W, V)$.*

This is going to take a little while, so we're going to prove a bunch of lemmas.

Lemma 9.4. *Let $p := \frac{1}{\#G} \sum_{g\in G} g \in \mathbf{C}G$. If X is an irrep of G, then p acts on X as* $\begin{cases} 1 & X \simeq \mathbf{C} \text{ (trivial rep)} \\ 0 & X \not\simeq \mathbf{C} \end{cases}$.

Proof. If $h \in G$, then $hp = p$ in $\mathbf{C}G$.

(1) If $X \simeq \mathbf{C}$, then each g acts as 1 so $\sum_g g$ is multiplication by $\#G$, so p acts as 1.
(2) If $X \not\simeq cc$, and $x \in X$, then $hpx = px$. So px is going to be fixed by every element of the group. So either $px = 0$, or px spans a copy of the trivial representation in X. This is a problem because X is irreducible. This means that $px = 0$.

□

Take the trace of this and get:

Corollary 9.5. *If X is irreducible, then $\frac{1}{\#G} \sum_{g\in G} \chi_X(g) = \begin{cases} 1 & X \simeq \mathbf{C} \\ 0 & X \not\simeq \mathbf{C} \end{cases}$.*

Corollary 9.6. *For any finite-dimensional representation of V, if $V^G := \{v \in V | gv = v \text{ for all } g \in G\}$ (the fixed subspace of V = the isotypic component consisting of the direct sum of copies of the trivial representation of G in V), then $\frac{1}{\#G} \sum_{g\in G} \chi_V(g) = \dim V^G$ because V is a direct sum of irreducible, so each irreducible gives one term in this sum, but because of the above corollary, this simply counts the number of copies of the trivial representation, which is exactly the dimension of V^G.*

[4] This means $(f_1, f_2) = \overline{(f_2, f_1)}$.

Proof of theorem.

$$\begin{aligned}(\chi_V, \chi_W) &= \frac{1}{\#G}\sum_{g\in G}\chi_V(g)\overline{\chi_W(g)}\\ &= \frac{1}{\#G}\sum_{g\in G}\chi_{V\otimes W^*}(g)\\ &= \frac{1}{\#G}\sum_{g\in G}\chi_{\mathrm{Hom}(W,V)}(g)\end{aligned}$$

This is the same thing, by the above corollary, as $\dim(\mathrm{Hom}(W, V)^G)$. A homomorphism fixed by the group is exactly a homomorphism respecting the G-action, proving the theorem. □

Corollary 9.7. $\dim \mathrm{Hom}_G(V, W) = \dim \mathrm{Hom}_G(W, V)$ *because* (χ_V, χ_W) *is a real number and* $(\chi_V, \chi_W) = \dim \mathrm{Hom}_G(W, V)$.

Corollary 9.8. *The* χ_V *for irreducible* V *form an* orthonormal *basis for* {*class functions*}.

Proof. If V, W are irreducible, then $(\chi_V, \chi_W) = \dim \mathrm{Hom}_G(V, W) = \begin{cases}1 & V = W\\ 0 & V \neq W\end{cases}$ by Schur's lemma. So different characters have zero inner product. □

(Can you replace complex conjugation with a Galois action? This might be interesting to work out.)

Corollary 9.9. *If* $V = \bigoplus_X n_X X$ *where* $n_X \in \mathbf{Z}_{\geq 0}$, *then* $(\chi_V, \chi_V) = \sum n_X^2$.

Corollary 9.10 (Incredibly useful)**.** *So* V *is irreducible if and only if* $(\chi_V, \chi_V) = 1$.

This also shows us that the character of the 3-dimensional representation of A_4 is actually irreducible.

9.5. **Orthogonality relations II.** I missed a little here because I went to the bathroom. Recall that $M \in \mathrm{M}_n(\mathbf{C})$ is unitary means that M preserves the standard Hermitian inner product $\sum_{i=1}^n x_i\overline{y}_i$ on $\mathbf{C}^n$, i.e., M has orthonormal columns, i.e., $M^*M = I$ where M^* is the conjugate transpose, i.e., $MM^* = 1$, i.e., M has orthonormal rows. I want to apply this to the character table.

Given $g \in G$, let $Z_g := \{h \in G | gh = hg\}$ is called the centralizer of $g \in G$. Then $\#Z_g \times \#$conjugacy class of $g = \#G$. This is because the conjugacy class of g is the orbit of the conjugation action of G on g (given by $h : (g \mapsto hgh^{-1})$) and Z_g is the stabilizer for this action, i.e., such that the elements of G such that $hgh^{-1} = g$. This is the orbit-stabilizer formula.

Lemma 9.11. *The matrix M indexed by irrep characters χ and representatives g of conjugacy classes whose (χ, g)-entry is $\frac{\chi(g)}{\sqrt{\#Z_g}}$ is unitary.*

Proof. Check inner products of rows.

$$\begin{aligned}
\sum_{\text{representatives } g} \frac{\chi(g)}{\sqrt{\#Z_g}} \frac{\overline{\chi'(g)}}{\sqrt{\#Z_g}} &= \frac{1}{\#G} \sum_{\text{representatives } g} (\#\text{conjugacy class of } g)\, \chi(g)\overline{\chi'(g)} \\
&= \frac{1}{\#G} \sum_{g \in G} \chi(g)\overline{\chi'(g)} \\
&= (\chi, \chi') \\
&= \begin{cases} 1 & \chi = \chi' \\ 0 & \chi \neq \chi' \end{cases}
\end{aligned}$$

The second equality holds because you're each representative of the conjugacy class is added #conjugacy class of g times, and characters are constant on conjugacy classes. This finishes the proof. □

Corollary 9.12. *For $g, h \in G$:*

$$\sum_{\textit{irreducible } \chi} \chi(g)\overline{\chi(h)} = \begin{cases} \#Z_g & \textit{if } g \textit{ is conjugate to } h \\ 0 & \textit{else} \end{cases}$$

Proof. This says that the columns are orthonormal. □

The textbook has another proof.

9.6. **Unitary representations over C.** Unitary representations over **C** are finite-dimensional **C**-vector spaces equipped with:

- A positive-definite Hermitian inner product.
- A homomorphism $\rho : G \to \mathrm{GL}(V)$ such that $\rho(g)$ is unitary (with respect to the inner product on V) for every $g \in G$, i.e., $(\rho(g)v, \rho(g)w) = (v, w)$ for all $g \in G$ and $v, w \in V$.

We'll show next time that every representation can be "upgraded".

10. Unitary representations

Definition 10.1. Let V be a **C**-vector space. A sesquilinear[5] inner product/form on V is a biadditive pairing $(,) : V \times V \to \mathbf{C}$ such that $(\lambda v, w) = \lambda(v, w)$, and

[5] "Sesqui-" means $1\frac{1}{2}$

$(v, \lambda w) = \overline{\lambda}(v, w)$ for all $\lambda \in \mathbf{C}$ and $v, w \in V$. A Hermitian form on V is a biadditive pairing $(,) : V \times V \to \mathbf{C}$ such that $(\lambda v, w) = \lambda(v, w)$, and $(w, v) = \overline{(v, w)}$.

Exercise 10.2. A Hermitian form is sesquilinear.

Definition 10.3. Let V be a $\mathbf{C}$-vector space. Let $\overline{V} := V$ with the same addition but with scalar multiplication $\lambda \cdot v := \overline{\lambda} v$.

Proposition 10.4. *There's a bijection* $\{$*sesquilinear forms*$\} \to \mathrm{Hom}_{\mathbf{C}}(\overline{V}, W^*)$ *by sending* $(,) \mapsto (w \mapsto (v, w))$ *where* $w \in \overline{V}$ *and* $(w \mapsto (v, w)) \in V^*$.

Proof. Easy. □

If V is a representation of G over $\mathbf{C}$, then so is $\overline{V}$, using the same G-action.

Exercise 10.5. If V is irreducible, then so is $\overline{V}$.

Corollary 10.6. *There's a bijection* $\{G$*-invariant sesquilinear forms*$\} \to \mathrm{Hom}_G(\overline{V}, W^*)$ *by sending* $(,) \mapsto (w \mapsto (v, w))$ *where* $w \in \overline{V}$ *and* $(w \mapsto (v, w)) \in V^*$. *In addition, there's a bijection between the set of* $\{G$*-invariant nondegenerate sesquilinear forms*$\}$ *and* $\{$*isomorphisms* $\overline{V} \to V^*\}$.

Recall from last time:

Definition 10.7. Unitary representations of G are finite-dimensional $\mathbf{C}$-vector spaces V equipped with:

(1) A positive-definite Hermitian form $(,) : V \times V \to \mathbf{C}$, i.e., $(v, v) > 0$ for all $v \neq 0$.
(2) A homomorphism $\rho : G \to \mathrm{GL}(V)$ such that $\rho(g)$ is unitary for all g, i.e., $(\rho(g)v, \rho(g)w) = (v, w)$.

Theorem 10.8. *Let G be a finite group.*

(1) *Every finite-dimensional representation V of G over* $\mathbf{C}$ *can be equipped with a positive definite Hermitian form making it a unitary representation.*
(2) *If V is irreducible, then the Hermitian form is unique up to scaling.*

Proof. (1) Let $B(v, w)$ be any positive definite Hermitian form on V. For example, identify $V = \mathbf{C}^n$ after choosing a basis, and use $B(\mathbf{x}, \mathbf{y}) = \sum x_i \overline{y}_i$. We can make this G-invariant. We know that G acts on $\{$Hermitian forms on $V\}$. Define:

$$(v, w) := \frac{1}{\#G} \sum_{g \in G} B(gv, gw)$$

Then G preserves this new pairing, and it's still a positive definite Hermitian form.

(2) Assume V is irreducible. Suppose this isn't unique, so you have $(,)_1$ and $(,)_2$ two G-invariant positive definite Hermitian forms on V. Positive definite maps are nondegenerate, so these correspond to isomorphisms of representations $\phi_1, \phi_2 : \overline{V} \to V^*$. Then $\phi_1^{-1} \circ \phi_2 \in \mathrm{Hom}_G(\overline{V}, \overline{V}) = \mathbf{C}$ because $\overline{V}$ is irreducible by the exercise above. This means that one of these isomorphisms is just a multiple of the other one, i.e., $\phi_2 = \lambda \phi_1$, so one of the pairings is a complex multiple of the other, i.e., $(,)_2 = \lambda(,)_1$. Choose $v \neq 0$. Then $(v, v)_2 = \lambda(v, v)_1$. But $(v, v)_2$ and $(v, v)_1$ are both positive real numbers, so λ is also a positive real number.

□

New proof of Maschke's theorem over $\mathbf{C}$. Let G be a finite group. Let W be a subrepresentation of a finite-dimensional $\mathbf{C}$-representation V. We need to prove that there is a G-representation complement of W in V. So upgrade V to a unitary representation. Define $W^\perp := \{v \in V | (v, w) = 0 \text{ for all } w \in W\}$. Since $(,)$ is positive definite, $V = W \oplus W^\perp$, and since G preserves W and $(,)$, it preserves $W^\perp$ as well. This is the G-representation complement we want. □

Remark 10.9. This still uses averaging, because we have to use it to define the inner product.

10.1. **Orthogonality III: matrix entries.** Equip $\mathbf{C}^n$ with the usual Hermitian form, i.e., $(v, w) := \sum v_i \overline{w}_i$. The standard basis $e_1, \cdots, e_n$ is an orthonormal basis of $\mathbf{C}^n$. If $A = (a_{ij}) \in \mathrm{M}_n(\mathbf{C})$, then Ae_i picks out the ith column vector, and (Ae_i, e_j) picks out the jth coordinate of the ith column, i.e., $(Ae_i, e_j) = a_{ji}$. This is a matrix entry.

A more coordinate-free way to talk about this is the following. Let V be a finite-dimensional vector space with a positive-definite Hermitian inner product $(,)$. Choose an orthonormal basis of V with respect to $(,)$. Let $A \in \mathrm{End}(V)$. Then (Av_i, v_j) is called a *matrix entry*.

Suppose in addition that we have $\rho : G \to \mathrm{End}\, V$ respecting $(,)$ that is an irreducible unitary representation where G is a finite group. For each (i, j), we get a "matrix entry" function $t_{ij}^V : G \to \mathbf{C}$ given by $g \mapsto (\rho(g)v_i, v_j)$ for each g. It's not a homomorphism or anything, it's just a function.

Proposition 10.10. (1) *If V, W are nonisomorphic irreducible representations, then:*

$$\langle t_{ij}^V, t_{IJ}^W \rangle := \frac{1}{\#G} \sum_{g \in G} t_{ij}^V(g) \overline{t_{IJ}^W(g)} = 0$$

(2) *And:*

$$\langle t^V_{ij}, t^V_{IJ}\rangle = \begin{cases} \frac{1}{\dim V} & \textit{if } (i,j) = (I,J) \\ 0 & \textit{else} \end{cases}$$

(3) *The* t^V_{ij} *as* V, i, j *vary form an orthogonal (not quite orthonormal) basis of the space of functions* $G \to \mathbf{C}$.

Proof. (1) Let $\{v_i\}$ be an orthonormal basis for V, and $\{w_i\}$ be an orthonormal basis for W. Also, $\overline{W}$ is an irrep with $\overline{(,)_W}$, and it has orthonormal basis $\{\overline{w}_i\}$ (this is actually the same as $\{w_i\}$). Now we're ready to do this massive computation. By definition,

$$\begin{aligned} \langle t^V_{ij}, t^W_{IJ}\rangle &= \frac{1}{\#G}\sum_{g\in G} t^V_{ij}(g)\overline{t^W_{IJ}(g)} \\ &= \frac{1}{\#G}\sum_{g\in G} (gv_i, v_j)\overline{(gw_I, w_J)} \\ &= \frac{1}{\#G}\sum_{g\in G} (g(v_i \otimes \overline{w}_I), v_i \otimes w_j)_{V\otimes\overline{W}} \\ &= (P(v_i \otimes \overline{w}_I), v_J \otimes \overline{w}_J)_{V\otimes\overline{W}} \end{aligned}$$

(Note that gv is $\rho(g)v$.) Where $P := \frac{1}{\#G}\sum_{g\in G} g$ is the averaging operator defined before. Now, $V\otimes\overline{W} \sim V\otimes W^* \simeq \operatorname{Hom}(W,V)$, and P projects $V\otimes\overline{W}$ onto $(V\otimes\overline{W})^G \simeq \operatorname{Hom}(W,V)^G = \operatorname{Hom}_G(W,V) = \begin{cases} \mathbf{C} & V = W \\ 0 & \text{else} \end{cases}$ by Schur's lemma. This is great, because if $W \neq V$, then P maps everything to zero, so $\langle t^V_{ij}, t^W_{IJ}\rangle = (P(v_i\otimes\overline{w}_I), v_J \otimes \overline{w}_J)_{V\otimes\overline{W}} = 0$ if $V \neq W$.

(2) Now suppose $V = W$, and let $n = \dim V$. There's a bilinear map $V \times \overline{V} \to \mathbf{C}$ taking $v \otimes w \mapsto (v, w)$. It obviously respects the G-action because G acts trivially on $\mathbf{C}$, so the map is a G-homomorphism. This is a surjection of representations. If you view $\mathbf{C}$ as being a subrepresentation of $V \otimes \overline{V}$, then this *is* P.

More precisely, consider the exact sequence $0 \to K \to V\otimes\overline{V} \to \mathbf{C} \to 0$ where K is the kernel of this pairing, which by definition is $K = \{\sum a_{k\ell} v_k \otimes \overline{v}_\ell : \sum_{k=1}^n a_{kk} = 0\}$ because $(v_k, v_\ell) = 0$ if $k \neq \ell$ and 1 if $k = \ell$. This is a G-subrepresentation of $V \otimes \overline{V}$. This exact sequence splits because by Maschke's theorem, $\mathbf{C}G$ is semisimple. There's a splitting $s : \mathbf{C} \to V \otimes \overline{V}$ where $1 \mapsto \frac{1}{n}\sum_{k=1}^n v_k \otimes \overline{v_k}$. It's easy to see that this is in

$K^\perp$. So $V \otimes \overline{V} \simeq K \oplus s(\mathbf{C})$. Then P is the projection onto $s(\mathbf{C}) = K^\perp$. So $P(v_i \otimes \overline{v}_j) = \frac{(v_i, v_j)}{n} \sum_{k=1}^n v_k \otimes \overline{v_k}$. Because the $\{v_i\}$ are an orthonormal basis, $(v_i, v_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. Therefore:

$$\begin{aligned} \langle t_{ij}^V, t_{IJ}^V \rangle &= (P(v_i \otimes \overline{v}_I), v_J \otimes \overline{v}_J)_{V \otimes \overline{V}} \\ &= \left(\frac{(v_i, v_I)}{n} \sum_{k=1}^n v_k \otimes \overline{v_k}, v_j \otimes \overline{v}_J \right) \\ &= \begin{cases} 1/n & (i, j) = (I, J) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Because $(v_j, \overline{v}_J)$ vanishes most of the time, and when $j = J$, it matches up exactly with one of the terms of the bizarre-looking sum $\sum_{k=1}^n v_k \otimes \overline{v_k}$.

(3) Clearly (1), (2) $\Rightarrow$orthogonality. Why is it a basis? The number of t_{ij}^Vs is $\sum_V (\dim V)(\dim V)$ where each $\dim V$ gives the number of is (resp. js). But $\sum_V (\dim V)(\dim V) = \#G = \dim_{\mathbf{C}}\{\text{functions } G \to \mathbf{C}\}$. In particular, they're lienarly independent.

□

"You all look like you're glad it's over."

10.2. **Representations over R.**

Theorem 10.11 (Frobenius' theorem (1877))**.** *The only finite-dimensional division algebras D over* $\mathbf{R}$ *are* $\mathbf{R}, \mathbf{C}, \mathbb{H}$.

We'll give a new proof of this. Every finite-dimensional field extension of $\mathbf{R}$ lies between $\mathbf{R}$ and $\overline{\mathbf{R}} = \mathbf{C}$. There's not much room because $\mathbf{C}$ is a degree 2 extension of $\mathbf{R}$, so the only field extensions of $\mathbf{R}$ are $\mathbf{R}$ and $\mathbf{C}$. If $d \in D \setminus \mathbf{R}$, then $\mathbf{R}[d] \subset D$ is a commutative domain of finite dimension over $\mathbf{R}$. Hence it's a field, so $\mathbf{R}[d] \simeq \mathbf{C}$. Let me stop here.

11. Representations over R

Theorem 11.1 (Frobenius' theorem (1877))**.** *The only finite-dimensional (associatice) division algebras D over* $\mathbf{R}$ *are* $\mathbf{R}, \mathbf{C}, \mathbb{H}$.

Proof. If $d \in D \setminus \mathbf{R}$, then $\mathbf{R}[d]$ is a copy of $\mathbf{C}$ in $\mathbb{H}$. Suppose that $D \neq \mathbf{R}$, and choose an element so that you fix a copy of $\mathbf{C}$ in D. Let $i = \sqrt{-1}$ in this copy of $\mathbf{C}$. View D as a left $\mathbf{C}$-vector space. The map $D \to D$ sending $x \mapsto ixi^{-1}$ is a $\mathbf{C}$-linear automorphism of order 2. This is like a representation of $\mathbf{Z}/2\mathbf{Z}$, so you have $D =$

$D^+ \oplus D^-$, where $D^+ = \{x | ixi^{-1} = x\}$. This is exactly $\{x$ that commute with $i\}$. This contains $\mathbf{C}$, so if $x \in D^+$, then $\mathbf{C}[x]$ is a finite field extension of $\mathbf{C}$, so $\mathbf{C}[x] = \mathbf{C}$, i.e., $x \in \mathbf{C}$.

Suppose $D \neq \mathbf{C}$. Then $D^- \neq 0$ by the above argument. Choose nonzero $j \in D^-$. The $\mathbf{C}$-linear map $D^- \to D^+$ that takes $x \mapsto xj$ (this lands in D^+, as you can check) is injective because j is a unit. This means that $\dim_{\mathbf{C}} D^- \leq \dim_{\mathbf{C}} D^+ = 1$, so $\dim_{\mathbf{C}} D^- \leq 1$. We know that it's not zero, so $\dim_{\mathbf{C}} D^- = 1$. We already know an element! So $D^- = \mathbf{C}j$.

(1) We have $j^2 \in D^+ = \mathbf{C}$.
(2) $\mathbf{R}[j]$ is another copy of $\mathbf{C}$, so $j^2 \in \mathbf{R} + \mathbf{R}j$.

Now, $\mathbf{C}$ and $\mathbf{R} + \mathbf{R}j$ are different two-dimensional copies because one of them contains j which isn't in $\mathbf{C}$. This means that $\mathbf{C} \cap (\mathbf{R} + \mathbf{R}j) = \mathbf{R}$. Because j^2 is in both of these, we know that $j^2 \in \mathbf{R}$. We know it's also nonzero because $D^- \neq 0$. We want $j^2 = -1$. If $j^2 > 0$, then $j^2 = r^2$ for some real number, so $(j + r)(j - r) = 0$. Because this is a division algebra, this would mean that $j = \pm r$, but this is ridiculous. Therefore $j^2 < 0$. Scale j to assume that $j^2 = -1$. Now it's really starting to look like the quaternions!

Then $D = \mathbf{C} \oplus \mathbf{C}j = \mathbf{R} + \mathbf{R}i \oplus \mathbf{R} + \mathbf{R}j$ with $i^2 = -1$ and $j^2 = -1$. Also, $ij = -ji$ because i and j anticommute. So $D = \mathbb{H}$. □

What is $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$? Define $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ via $a \otimes b \mapsto (ab, a\overline{b})$. You can check that this is an isomorphism. What is $\mathbb{H} \otimes_{\mathbf{R}} \mathbf{C}$? We claim that this is $\mathrm{M}_2(\mathbf{C})$. You can embed $\mathbb{H}$ into End($\mathbb{H}$ as a right $\mathbf{C}$-vector space). You view $\mathbb{H}$ as $1\mathbf{C} + \mathbf{C}j$. But this is just $\mathrm{M}_2(\mathbf{C})$. You can make this explicit: $1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $i \mapsto \begin{pmatrix} i & \\ & -i \end{pmatrix}$, $j \mapsto \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, and $ij \mapsto \begin{pmatrix} & -i \\ i & \end{pmatrix}$. This means that we can now write:

D	$D \otimes_{\mathbf{R}} \mathbf{C}$
$\mathbf{R}$	$\mathbf{C}$
$\mathbf{C}$	$\mathbf{C} \times \mathbf{C}$
$\mathbb{H}$	$\mathrm{M}_2(\mathbf{C})$

Suppose you're given a $\mathbf{R}$-vector space W. You can complexify to get a $\mathbf{C}$-vector space $W_{\mathbf{C}} := W \otimes_{\mathbf{R}} \mathbf{C}$. You have $W \hookrightarrow W_{\mathbf{C}}$, by sending $w \mapsto w \otimes 1$. Then an $\mathbf{R}$-basis of W is a $\mathbf{C}$-basis of $W_{\mathbf{C}}$. It's the same basis, just viewed as a complex vector space. For the rest of this lecture, we'll let W be a $\mathbf{R}$-vector space, and V is a complex vector space. If V is a $\mathbf{C}$-vector space, you can "restrict scalars" to get ${}_{\mathbf{R}}V$, which is V viewed as an $\mathbf{R}$-vector space of twice the dimension. This is what you get when you only remember to multiply by real numbers. The two operations described here are *not* inverses!

How does $({}_{\mathbf{R}}V)_{\mathbf{C}}$ relate to V? This is of twice the dimension, and it turns out that there is a canonical isomorphism $({}_{\mathbf{R}}V)_{\mathbf{C}} \simeq V \oplus \overline{V}$, by $v \otimes \lambda \mapsto (\lambda v, \overline{\lambda} v)$. That this is an isomorphism follows from the following argument. The map is well-defined, and is $\mathbf{R}$-bilinear. If $\dim V = n$, then both sides have dimension $2n$. We therefore only need to check injectivity. A general element of $({}_{\mathbf{R}}V)_{\mathbf{C}}$ looks like $v \otimes 1 + w \otimes i$, which mapsto $(0, 0)$. Then $v + iw = 0$ and $v + \overline{i}w = v - iw = 0$. So $v = 0 = w$. So the kernel of this map is zero, and it's injective.

What if these things are representations of some finite group? Suppose W is a $\mathbf{R}G$-module. Then $\chi_{W_{\mathbf{C}}} = \chi_W$ because they're the same basis, so the action of G is just when you literally view the same matrix but with entries in a different field (namely, $\mathbf{C}$), that contains $\mathbf{R}$! If V is a $\mathbf{C}G$-module. Then $\chi_{{}_{\mathbf{R}}V} = \chi_V + \chi_{\overline{V}} = \chi_V + \overline{\chi_V}$ because $({}_{\mathbf{R}}V)_{\mathbf{C}} \simeq V \oplus \overline{V}$, and then you use the previous thing about χ_W.

If I give you $W \otimes_{\mathbf{R}} \mathbf{C}$, can you recover W? No. You need to additionally know how $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ acts on this. For example, $\mathbf{R}^n$ consists of vectors in $\mathbf{C}^n$ fixed by coordinate-wise complex conjugation.

Definition 11.2. Let V be a $\mathbf{C}$-space. Complex conjugation $J : V \to V$ is $\mathbf{C}$-*antilinear*, i.e., J is additive, and $J(\lambda v) = \overline{\lambda} J(v)$ for all $v \in V$ and $\lambda \in \mathbf{C}$. (This is equivalent to saying that J is a $\mathbf{C}$-linear map $V \to \overline{V}$).

Proposition 11.3. *There is an equivalence of categories between* $\mathbf{R}$*-vector spaces and* $\mathbf{C}$*-vector spaces V with a* $\mathbf{C}$*-antilinear automorphism J with $J^2 = 1_V$. We'll only prove the bijection.*

Proof. If W is a $\mathbf{R}$-vector space, let $W \mapsto W_{\mathbf{C}}$ equipped with $W_{\mathbf{C}} \to W_{\mathbf{C}}$ by $w \otimes \lambda \mapsto w \otimes \overline{\lambda}$. To go the other way, you take $(V, J) \mapsto V^{J=1} := \{v \in V | J(v) = v\}$. □

You can think of $\{1_V, J\}$ as defining an action of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ on V that is compatible with the action of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ on $\mathbf{C}$. There is a generalization of this to the case when you replace $\mathbf{C}/\mathbf{R}$ with any Galois extension, where an action of $\mathrm{Gal}(K/k)$ on V that is compatible with the action of $\mathrm{Gal}(K/k)$ on K. This is called descent theory. It's related to something called Hilbert 90, but we won't talk about this.

Now, $\mathrm{M}_{m,n}(\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} = \mathrm{M}_{m,n}(\mathbf{C})$.

Lemma 11.4. *If W, X are* $\mathbf{R}$*-vector space, then* $\mathrm{Hom}_{\mathbf{R}}(W, X) \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathrm{Hom}_{\mathbf{C}}(W_{\mathbf{C}}, X_{\mathbf{C}})$. *In particular, if $W = X$, then* $\mathrm{End}_{\mathbf{R}}(W) \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathrm{End}_{\mathbf{C}}(W_{\mathbf{C}})$.

"Today is like lemma day." We're almost ready to do the classification.

If $g \in G$ acts on V with eigenvalues $\lambda_1, \cdots, \lambda_n$ listed with multiplicity (roots of the characteristic polynomial). Then g acts on $V \otimes V$ with eigenvalues $\lambda_i \lambda_j$ (you can see why this is true if the action of g is diagonalizable, but this is also true in general) for all (i, j). How does g acts on $\mathrm{Sym}^2(V)$? So some of these duplicate $\lambda_i\lambda_j$ are removed, i.e., g acts on $\mathrm{Sym}^2(V)$ with eigenvalues $\lambda_i\lambda_j$ for all $i \le j$. Also, g acts on $\Lambda^2 V$ with eigenvalues $\lambda_i\lambda_j$ for all $i < j$. Also, g^2 acts on V with eigenvalues λ_i^2 for $i = 1, 2, \cdots, n$. There are going to be certain relations between these.

Corollary 11.5. *It's easy to see that* $\chi_{\mathrm{Sym}^2 V} + \chi_{\Lambda^2 V} = \chi_{V\otimes V}$.

Corollary 11.6. *Also,* $\chi_{\mathrm{Sym}^2 V}(g) - \chi_{\Lambda^2 V}(g) = \chi_V(g^2)$.

Now we can classify representations over $\mathbf{R}$.

Let G be a finite group, W an irreducible $\mathbf{R}G$-module, and $D := \mathrm{End}_G(W)$, which is a finite-dimensional division algebra over $\mathbf{R}$ by Schur's lemma. By Frobenius, $D = \begin{cases} \mathbf{R} \\ \mathbf{C} \\ \mathbb{H} \end{cases}$. Let $n = \dim_D W$ (which makes sense because W is a right D-vector space, which is in general different from $\dim_{\mathbf{R}} W$), and V be an irreducible subrepresentation of $W_{\mathbf{C}}$. We have an enormous table[6]:

$D := \mathrm{End}_G(W)$	$\mathrm{End}_G(W_{\mathbf{C}})$	Decomp. of $W_{\mathbf{C}}$	Decomp. of ${}_{\mathbf{R}}W$	$\dim_{\mathbf{R}} W$	$\dim_{\mathbf{C}} V$
$\mathbf{R}$	$\mathbf{C} = \mathbf{R} \otimes_{\mathbf{R}} \mathbf{C}$	V	$W \oplus W$	n	n
$\mathbf{C}$	$\mathbf{C} \times \mathbf{C}$	$V \oplus V'$ by above	W	$2n$	n
$\mathbb{H}$	$\mathrm{M}_2(\mathbf{C})$	$V \oplus V$ by above	W	$4n$	$2n$

Let me explain why:

(1) The second column follows because because $\mathrm{End}_{\mathbf{R}}(W) \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathrm{End}_{\mathbf{C}}(W_{\mathbf{C}})$.
(2) The third column is because if $W_{\mathbf{C}} = \bigoplus n_i V_i$, then $\mathrm{End}_G(W_{\mathbf{C}}) = \prod \mathrm{M}_{n_i}(\mathbf{C})$. Also, V is said to be of αth time where $\alpha \in \{\text{real, complex, quaternionic}\}$. In the third column, also, view W as ${}_{\mathbf{R}}U$ for some U vector space U. Then $W_{\mathbf{C}} = U \oplus \overline{U}$. Now, $W_{\mathbf{C}} = V \oplus V'$, but decomposition into irreducibles is unique, so $V' = \overline{V}$.
(3) The second-to-last column is because $\dim_{\mathbf{R}} V = n \times \dim_{\mathbf{R}} D$.

Definition 11.7. A representation V of G over $\mathbf{C}$ is *realizable over* $\mathbf{R}$ if $V = W_{\mathbf{C}}$ for some W of G over $\mathbf{R}$.

Are the things we have above realizable? In the real case, this is clearly realizable. In the $\mathbf{C}$ and $\mathbb{H}$ cases: if V were $X_{\mathbf{C}}$, then $0 \subset X_{\mathbf{C}} \subset W_{\mathbf{C}}$ (strict inequality). By

[6] One preliminary computation: We know that $\chi_{\mathbf{R}V} = \chi_V + \chi_{\overline{V}}$. This is $\chi_{\mathbf{R}V} = \begin{cases} 2\chi_W & \text{real case} \\ \chi_{W_{\mathbf{C}}} = \chi_W & \text{complex case} \\ 2\chi_V = \chi_{W_{\mathbf{C}}} = \chi_W & \end{cases}$.

Noether-Deuring (the exercise that we did), then $0 \subseteq X \subseteq W$. But W is irreducible, so such an X can't exist. Therefore W is not realizable in the $\mathbf{C}$ and $\mathbb{H}$ cases.

Is $V \simeq \overline{V}$? I.e., is χ_V real valued? In the real case, $\chi_V = \chi_W$, since $V = W_{\mathbf{C}}$, so it's clearly yes here. For the complex case, $V \not\simeq \overline{V}$. In the complex case, $\chi_V = \frac{1}{2}\chi_W$, so χ_V is real valued.

Now, $\mathrm{Hom}(V, V^*) \simeq V^* \otimes V^* = (V \otimes V)^* = (\mathrm{Sym}^2 V)^* \oplus (\Lambda^2 V)^*$ by our copmutations above. Now, an element of $(V \otimes V)^*$ is just a bilinear form $V \times V \to \mathbf{C}$, which decomposes as the direct sum of {symmetric bilinear forms} and {skew-symmetric/alternating bilinear forms} (everything here is $\mathbf{C}$-bilinear, not sesquilinear or something). This means that $\mathrm{Hom}_G(V, V^*) \simeq$ {G-invariant symmetric bilinear forms}$\oplus$ {G-invariant skew-symmetric bilinear forms}. What is $\dim \mathrm{Hom}_G(V, V^*)$? If $V \not\simeq V^*$, then this is 0 and if they are isomorphic, then it's 1 by Schur's lemma. What happens on the other side? Therefore there's at most one G-invariant bilinear form up to scalar multiplication, and it's either skew-symmetric or not.

Is $V \simeq V^*$? It's yes in the real and quaternionic cases, and no in the complex case. This is the same as what we got when we asked about realizability. Why? Fix a G-invariant positive definite Hermitian form $(,)$ on V. This defined $\overline{V} \simeq V^*$, so realizability is the same as asking if $V \simeq V^*$.

What about whether the G-invariant bilinear form is symmetric or not? Suppose $\mathrm{Hom}_G(V, V^*)$ is one-dimensional. Let $B(,)$ be a nonzero G-invariant bilinear form. This is unique up to $\mathbf{C}^\times$. For each $w \in V$, then $(-, w)$ defines a $\mathbf{C}$-linear functional on V. But also, $(-, w) = B(-, J(w))$ for a unique $J(w) \in V$. Then $J : V \to V$ is $\mathbf{C}$-antilinear by Hermitian-ity, and is unique up to $\mathbf{C}^\times$. What happens if you take J^2? We know that J^2 is $\mathbf{C}$-linear and hence is multiplication by some $r \in \mathbf{C}$. Replacing J by cJ changes r to $c\bar{c}r$ (to see this, if $J(J(v)) = rv$, then $cJ(cJ(v)) = c\bar{c}J(J(v)) = c\bar{c}rv$).

If B is symmetric, then $(J(v), J(v)) = B(J(v), J^2)$ by definition of J. But this is $B(J(v), rv) = rB(J(v), v)$. Since it's symmetric, this is $rB(v, J(v)) = r(v, v)$. This is true for all v. So $r \in \mathbf{R}_{>0}$. If B is skew-symmetric, then you do the same thing and find that $r \in \mathbf{R}_{<0}$. This means that you can scale if and only if B is symmetric. Therefore the G-invariant bilinear form in the real case is symmetric, and is skew-symmetric in the quaternionic case.

Said some stuff about Frobenius-Schur indicator defined by $\mathrm{FS}(v) = \frac{1}{\#G}\sum_g \chi_V(g^2)$, which turns out to be 1 in the real case, 0 in the complex case, and -1 in the quaternionic case. We'll prove this next time. Also, not all this stuff is in the textbook.

12. Frobenius-Schur indicator, and $\dim V | \#G$ if V is irreducible (Guest lecture, Pavel Etingof)

Definition 12.1. Let G be a finite group, and let V be an irreducible representation of G over $\mathbf{C}$. The Frobenius-Schur indicator of V is $\mathrm{FS}(V) = \frac{1}{\#G} \sum_{g \in G} \chi_V(g^2)$.

Proposition 12.2. *The following is true.*

$$\mathrm{FS}(V) = \begin{cases} 1 & V \text{ real} \\ -1 & V \text{ quaternionic} \\ 0 & V \text{ complex.} \end{cases}$$

Proof. Last time we showed that $\chi_V(g^2) = \chi_{\mathrm{Sym}^2 V}(g) - \chi_{\Lambda^2 V}(g)$. Therefore:

$$\frac{1}{\#G} \sum_{g \in G} \chi_V(g^2) = \frac{1}{\#G} \sum_{g \in G} \chi_{\mathrm{Sym}^2 V}(g) - \frac{1}{\#G} \sum_{g \in G} \chi_{\Lambda^2 V}(g)$$
$$= (\chi_{\mathrm{Sym}^2 V}, \chi_1) - (\chi_{\Lambda^2 V}, \chi_1) = \dim(\mathrm{Sym}^2 V)^G - \dim(\Lambda^2 V)^G$$

But now, $(\mathrm{Sym}^2 V)^G \oplus (\Lambda^2 V)^G \simeq (V \otimes V)^G \simeq \mathrm{Hom}_G(V^*, V)$, and now $\dim(\mathrm{Hom}_G(V^*, V)) \leq 1$. If V is real, then $(\mathrm{Sym}^2 V)^G \simeq \mathbf{C}$, $(\Lambda^2 V)^G = 0$, so that $\mathrm{FS}(V) = 1$. If V is quaternionic, then $(\mathrm{Sym}^2 V)^G = 0$, $(\Lambda^2 V)^G = 1$, so that $\mathrm{FS}(V) = 0$. If V is complex, then $(\mathrm{Sym}^2 V)^G = (\Lambda^2 V)^G = 0$, so $\mathrm{FS}(V) = 0$. □

Theorem 12.3 (Frobenius-Schur)**.** *The number of involutions of G (elements of order ≤ 2) equals $\sum_{irrep\ V} \dim V \cdot \mathrm{FS}(V)$.*

Proof. Compute the trace of $B := \frac{1}{\#G} \sum_{g \in G} g^2 \in \mathbf{C}G$ in two ways.

(1) $\mathrm{Tr}\,|_{\mathbf{C}G}(g^2) = \begin{cases} 0 & g^2 = 1 \\ \#G & g^2 = 1 \end{cases}$. Therefore $\mathrm{Tr}(B)$ is the number of involutions.

(2) By Maschke's theorem, $\mathbf{C}G = \bigoplus_{\mathrm{irrep}\ V} \dim V \cdot V$, so $\mathrm{Tr}\,|_{\mathbf{C}G}(B) = \sum_{\mathrm{irrep}\ V} \dim V \cdot \mathrm{Tr}_V(B)$. But now $\mathrm{Tr}_V(B) = \frac{1}{\#G} \sum_{g \in G} \mathrm{Tr}_V(g^2) = \mathrm{FS}(V)$, so $\mathrm{Tr}\,|_{\mathbf{C}G}(B) = \sum_{\mathrm{irrep}\ V} \dim V \cdot \mathrm{FS}(V)$.

□

Corollary 12.4. *All irreducible representations V of G are defined over $\mathbf{R}$ (so that $V = \widetilde{V} \otimes_{\mathbf{R}} \mathbf{C}$, where $\widetilde{V}$ is a real representation) if and only if the number of involutions of G is $\sum_{irrep\ V} \dim V$. One example of a group satisfying this is S_n.*

We're going to have to use algebraic number theory.

12.1. **Algebraic numbers and algebraic integers.** This is a well-known topic, so I'll be rather brief. There are two definitions.

Definition 12.5. Let $z \in \mathbf{C}$. This is called an algebraic number (resp. algebraic integer) if it satisfies a monic polynomial equation $p(z) = 0$ were $p(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ with rational (resp. integer) coefficients.

Definition 12.6. $z \in \mathbf{C}$ is an algebraic number (resp. algebraic integer) if it is an eigenvalue of a rational (resp. integer) matrix.

Proposition 12.7. *These definitions are equivalent.*

Proof. The second definition implies the first because you can just take the characteristic polynomial of the matrix. Conversely, you use the "companion matrix" of the polynomial p, which has the form:

$$\begin{pmatrix} 1 & 0 & \cdots & -a_n \\ 0 & 1 & \cdots & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_1 \end{pmatrix}$$

□

Theorem 12.8. *Let* $\mathbf{A}$ *denote the set of algebraic integers, and* $\overline{\mathbf{Q}}$ *the set of algebraic numbers. Then* $\mathbf{A}$ *is a field, and* $\overline{\mathbf{Q}}$ *is a field that's the algebraic closure of* $\mathbf{Q}$.

Proof. If $\alpha, \beta \in \mathbf{A}$ or $\overline{\mathbf{Q}}$, then α is an eigenvalue of A, and β is an eigenvalue of B, so $Av = \alpha v$ and $Bw = \beta w$ where $v, w \neq 0$. Then $(A\otimes 1 + 1\otimes B)(v\otimes w) = (\alpha \pm \beta)(v \otimes w)$, so that $\alpha \pm \beta \in \mathbf{A}$ or $\overline{\mathbf{Q}}$. Similarly, for the product, $(A \otimes B)(v \otimes w) = \alpha\beta(v \otimes w)$, so $\alpha\beta \in \mathbf{A}$ or $\overline{\mathbf{Q}}$. This proves that these are subrings of $\mathbf{C}$.

Also, if $\alpha \neq 0 \in \overline{\mathbf{Q}}$, satisfying $p(\alpha) = 0$, then $\frac{1}{\alpha}$ satisfies $\widetilde{p}\left(\frac{1}{\alpha}\right) = 0$, where $\widetilde{p}(x) = x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \cdots + \frac{a_1}{a_n} x + \frac{1}{a_n}$. This means that $\overline{Q}$ is a field. It's also the algebraic closure of $\mathbf{Q}$ by the fundamental theorem of algebra. □

Proposition 12.9.

$$\mathbf{A} \cap \mathbf{Q} = \mathbf{Z}$$

Proof. Suppose $\alpha = \frac{p}{q}$ in lowest terms. Then $\alpha^n + a_1\alpha^{n-1} + \cdots + a_n = 0$ for $a_i \in \mathbf{Z}$. The denominator of α^n is a^n, and the denominator of the other terms divides q^{n-1}, so $q = 1$, because if $q > 1$, then you'd have $0 = q^n\alpha^n + q^n a_1 \alpha^{n-1} + \cdots + q^n a_n$. Now, $q^n a_1 \alpha^{n-1} + \cdots + q^n a_n$ is divisible by q. Also $q^n a_1 \alpha^{n-1} = q a_1 p^{n-1}$ and $q \nmid p^n$, so we get a contradiction (just rearrange the equation $0 = q^n\alpha^n + q^n a_1\alpha^{n-1} + \cdots + q^n a_n$). □

If $\alpha \in \overline{\mathbf{Q}}$, we can consider the minimial polynomial of α, which is a monic polynomial $p(x)$ of minimial degree such that $p(\alpha) = 0$. This is unique, becuase if you had two such polynomials, you could take their difference and divide by some coefficient to get a polynomial of smaller degree that α is a root of, which is a contradiction to the minimality of the degree. Also, any other polynomial $q(x)$ with rational coefficients such that $q(\alpha) = 0$ is divisible by $p_\alpha(x)$.

Definition 12.10. The roots of $p_\alpha(x)$ are called the algebraic conjugates of α. They are the roots of any rational polynomial $q(x)$ such that $q(\alpha) = 0$.

Clearly an algebraic conjugate to an algebraic integer is also an algebraic integer.

Proposition 12.11. *If α is an algebraic integer, then p_α has integer coefficients.*

Proof. We know that $p_\alpha(x) = (x - \alpha) \cdots (x - \alpha_m)$ where the $\alpha_i \in \mathbf{A}$ are algebraic conjugates of α. By the Vieta theorem, coefficients of p_α are in $\mathbf{A}$. But they're also in $\mathbf{Q}$, and hence in $\mathbf{Z}$. □

Lemma 12.12. *If $\alpha_1, \cdots, \alpha_m \in \overline{\mathbf{Q}}$, then all algebraic conjugates of $\alpha_1 + \cdots + \alpha_m$ are of the form $\alpha'_1 + \cdots + \alpha'_m$ where the α'_i are algebraic conjugates to α_i.*

Remark 12.13. This does not say that if α'_i are algebraic conjugates to α_i, then $\alpha'_1 + \cdots + \alpha'_m$ is conjugate to $\alpha_1 + \cdots + \alpha_m$. For example, consider $\sqrt{2} + (-\sqrt{2}) = 0$. Then you can replace $-\sqrt{2}$ with its conjugate, $\sqrt{2}$, but $2\sqrt{2}$ is clearly not conjugate to 0.

Proof. The α_i are eigenvalues of A_i with rational entries, so $z = \alpha_1 + \cdots + \alpha_m$ is an eigenvalue of $A := A_1 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes A_2 \otimes \cdots \otimes 1 + 1 \otimes 1 \otimes \cdots \otimes A_m$. Therefore any conjugate of z' is an eigenvalue of A. But any eigenvalue of A is a sum of eigenvalues of $A_1, \cdots, A_m$, i.e., is of the form $\alpha'_1 + \cdots + \alpha'_m$ where the α'_i are eigenvalues of A_i. We can pick an A_i of minimal size, so that the characteristic polynomial of A_i is p_{α_i}, so that the α'_i are all algebraic conjugates of α_i. □

12.2. **Frobenius divisibility.**

Theorem 12.14 (Frobenius divisibility)**.** *If G is a finite group and V an irreducible representation of G over* $\mathbf{C}$*, then* $\dim V | \#G$*.*

Proof. Let $C \subseteq G$ be a conjugacy class, let $g \in C$, and V an irrep of G. Define $\lambda = \frac{\#C}{\dim V}\chi_V(g)$. Note that $\chi_V(g)$ are algebraic integers because it is the sum of eigenvalues of g on V, which are roots of unity.

Proposition 12.15. *λ is an algebraic integer.*

Proof. Let $p = \sum_{h\in C} h \in \mathbf{Z}G$. This is a central element of $\mathbf{C}G$, because it's a conjugacy class (so stuff gets permuted). By Schur's lemma, it acts by a scalar $\widetilde{\lambda}$ on V. It'll turn out that $\widetilde{\lambda} = \lambda$. We'll compute $\mathrm{Tr}_V(p)$, which is $\sum_{h\in C}\mathrm{Tr}_V(h) = \#C \cdot \chi_V(g)$, but also $\mathrm{Tr}_V(p) = \widetilde{\lambda}\cdot \dim V$. Therefore $\widetilde{\lambda} = \frac{\#C}{\dim V}\chi_V(g) = \lambda$. We are not done yet. But $\mathbf{Z}G$ is a finitely generated $\mathbf{Z}$-module, so every element $q \in \mathbf{Z}G$ satisfies a monic polynomial equation $p(q) = 0$ with integer coefficients (just take the integer matrix of the action of q whose characteristic polynomial is what you want). Therefore $p(\lambda) = 0$, i.e., $\lambda \in \mathbf{A}$. □

Let $C_1, \cdots, C_n$ be the conjugacy classes of G, so $G = \coprod_{i=1}^n C_i$. Let g_{C_i} be a representative of C_i, so that we have as above $\lambda_i = \frac{\#C_i}{\dim V}\chi_V(g_{C_i})$. We will consider $\sum_{i=1}^n \lambda_i\overline{\chi_V(g_{C_i})}$. This belongs to $\mathbf{A}$ because $\lambda_i \in \mathbf{A}$, and $\overline{\chi_V(g_{C_i})} \in \mathbf{A}$ because $\chi_V(g_{C_i})$ is the sum of eigenvalues of g_{C_i} on V, which are roots of unity because $g_{C_i}^{m_i} = 1$. Therefore:

$$\begin{aligned}\sum_{i=1}^n \lambda_i\overline{\chi_V(g_{C_i})} &= \sum_{i=1}^n \frac{\#C_i}{\dim V}\chi_V(g_{C_i})\overline{\chi_V(g_{C_i})} = \frac{1}{\dim V}\sum_{i=1}^n\sum_{g\in C_i}\chi_V(g)\overline{\chi_V(g)} \\ &= \frac{1}{\dim V}\sum_{g\in G}\chi_V(g)\overline{\chi_V(g)} = \frac{1}{\dim V}(\chi_V,\chi_V)\times \#G = \frac{\#G}{\dim V}\end{aligned}$$

Therefore $\frac{\#G}{\dim V} \in \mathbf{A}\cap\mathbf{Q} = \mathbf{Z}$, so that $\dim V | \#G$. □

"It's cute because we didn't use algebraic integers in the statement."

12.3. **Burnside's theorem.** This is an even more striking application, because it doesn't even use representations in the statement!

Definition 12.16. Let G be a finite group. It's said to be *solvable* if there is a chain of normal subgroups $\{e\} = G_1 \le G_2 \le G_3 \le \cdots \le G_n = G$ such that G_i/G_{i-1} is abelian for all i.

Why is it called solvable? Comes from Galois theory.

Theorem 12.17. *Any group of order p^aq^b where p, q are primes is solvable.*

Remark 12.18.
(1) Any group of order p^a is nilpotent (easy), and in particular solvable.
(2) This isn't true for three primes, for example A_5 with $\#A_5 = 60 = 2^2\times 3\times 5$.
(3) There's a longer proof of Burnside's theorem purely using group theory.

We'll prove an independent theorem first, that'll be used to prove Burnside's theorem.

Theorem 12.19. *Let $C \subseteq G$ be a conjugacy class and V an irrep of G over $\mathbf{C}$, such that $\gcd(\#C, \dim V) = 1$. Then either $\chi_V(g) = 0$ or g acts on V acts as a scalar.*

Proof.

Lemma 12.20. *If $\varepsilon_1, \cdots, \varepsilon_n$ are roots of unity such that $a := \frac{1}{n}\sum_{i=1}^n \varepsilon_i$ is an algebraic integer, then either $a = 0$ or $\varepsilon_i = \varepsilon_j$ for all i, j, so that $a = \varepsilon_i$ for any i.*

Proof. Suppose the ε_i are not all the same. Then $|\varepsilon_1 + \cdots + \varepsilon_n| < |\varepsilon_1| + \cdots + |\varepsilon_n|$ (strict inequality). But this is n, so that $|a| < 1$. If a' is an algebraic conjugate of a, then $a' = \frac{1}{n}(\varepsilon_1' + \cdots + \varepsilon_n')$ where ε_i' are some algebraic conjugates of ε_i, i.e., they are roots of unity. So $|a'| \leq 1$. Thus $b := \prod_{a' \text{ conjugate to } a} a'$ satisfies $|b| < 1$. But b is an integer, because it's the last coefficient of the minimal polynomial, so $b = 0$, and thus $a = 0$. This completes the proof of this lemma. □

Let $\dim V = n$. Let $\epsilon_1, \cdots, \epsilon_n$ be eigenvalues of g on V (roots of unity). We know from before that $\frac{\#C}{n}\chi_V(g) \in \mathbf{A}$. We also know that $\gcd(\#C, n) = 1$, so there are $a, b \in \mathbf{Z}$ such that $a\#C + bn = 1$. So $\frac{a\#C+bn}{n}\chi_V(g)$ is equal to $\chi_V(g)/n$, and also is $a\frac{\#C}{n}\chi_V(g) + b\chi_V(g)$. Now, $\frac{\#C}{n}\chi_V(g)$ is an algebraic integer by our previous result, so $a\frac{\#C}{n}\chi_V(g) + b\chi_V(g) \in \mathbf{A}$. Thus $\frac{\chi_V(g)}{n} \in \mathbf{A}$. But this is $\frac{1}{n}(\epsilon_1 + \cdots + \epsilon_n)$. By the lemma, this is either zero, so that $\chi_V(g) = 0$, or $\epsilon_1 = \cdots = \epsilon_n =: \epsilon$, which means that g acts by a scalar ($\epsilon - \mathrm{id}$) because g is diagonalizable, being an element with finite order. □

We'll stop here.

13. Burnside's theorem

On the way to Burnside's theorem...

Theorem 13.1 (Theorem 5.4.4)**.** *If V is an irreducible $\mathbf{C}$-representation of G, C a conjugacy class such that $\gcd(\#C, \dim V) = 1$, then for all $g \in G$, $\chi_V(g) = 0$ or g acts as a scalar on V.*

Theorem 13.2 (Theorem 5.4.6)**.** *Let G be a finite group such that there's C a conjugacy class of size p^k for some prime p and $k \geq 1$. Then G has a normal subgroup H with $1 \subset H \subset G$ (strict inclusions).*

(We're going to be writing $\overline{\mathbf{Z}}$ for $\mathbf{A}$, it seems.)

Proof. Let $g \neq 1 \in C$. Then $\sum_{\text{irrep } V} \chi_V(g)\overline{\chi_V(1)} = 0$. One of these terms is just going to be one, from the trivial representation. Thus $\sum_{\text{irrep } V} \chi_V(g)\overline{\chi_V(1)} = 1 + \sum_{\text{irrep } V \neq \mathbf{C}} \chi_V(g) \dim V = 0$ in $\overline{\mathbf{Z}}$, because we know that $\chi_V(1) = \dim V$.

The idea is to look at this equation "mod p". If $\chi_V(g)\dim V \in p\overline{\mathbf{Z}}$ for all $V \not\simeq \mathbf{C}$, then $1 \in p\overline{\mathbf{Z}}$. This means that $1/p \in \overline{Z}$. But it's also in $\mathbf{Q}$, so $1/p \in \mathbf{Z}$. Contradiction. Thus we may pick V such that $\chi_V(g)\dim V \notin p\overline{\mathbf{Z}}$. Then $\chi_V(g) \neq 0$. But also $\chi_V(g) \in \overline{\mathbf{Z}}$, which means that $p \nmid \dim V$.

By the above theorem (Theorem 5.4.4), since $\gcd(p^k, \dim V) = 1$, either $\chi_V(g) = 0$ or g acts as a scalar. But $\chi_V(g) \neq 0$, which means that g acts as a scalar. Also, any $a \in C$ acts as a conjugate of the matrix of g, so any other $a \in C$ acts via a scalar - in fact, the same scalar! (Another way to see this, which I think is more simple, is that you can apply the same argument because $\chi_V(g) = \chi_V(a)$.)

Let $H := \langle ab^{-1} | a, b \in \mathbf{C}\rangle$. This is automatically a normal subgroup of G because everything is preserved under conjugation. How do we know that it's not the whole group? How doo the element of H act on V? Any element of H must act as the identity because in ab^{-1} they give the *same* matrix! But G does not act trivially, so $H \neq G$. It's left to prove that this isn't trivial. But this is obvious since $\#C > 1$. □

"This is the official acting learning day, I think this is what the president said."

Theorem 13.3. *Any group G of order $p^a q^b$ is solvable. Here the p, q are suppose to be primes and $a, b \in \mathbf{Z}_{\geq 0}$.*

Note that the smallest nonsolvable group is A_5, and $\#A_5 = 2^2 \times 3 \times 5$. This is one step on the way to classifying finite simple groups but it's not clear that it's completely complete. You can actually write down a character table before you figure out the group itself. Anyway, I'm getting sidetracked.

Proof. By induction on $\#G$. Let Z be the center of G, so Z is solvable (I'm thinking of the definition where the quotients are cyclic, not the ones where they're just abelian – but they're actually equivalent by the structure theorem of finite abelian groups).

Case I: $Z \neq 1$. Then G/Z is smaller than G, so this is solvable by the inductive hypothesis. You have $1 \leq Z \leq G$, all the quotients of which are solvable. So G is solvable.

Case II: $Z = 1$. We'll partition the group into conjugacy classes. $\#G = 1 + \#C_2 + \cdots + \#C_n$ where 1 is the identity and the C_i are other conjugacy classes. No other elements are in the center, so none of the $\#C_i$ are of size 1. Also, $\#C_i | \#G$ by the orbit-stabilizer theorem because the orbit is the conjugacy class and the stabilizer is the conjugacy class. Thus each $\#C_i$ is a power of p times a power of q.

If any of the $\#C_i$ is p^k or q^k for some $k \geq 1$, then the above theorem (Theorem 5.4.6) says that there exists a normal subgroup H such that $1 \subset H \subset G$ (strict inclusion). When that happens, you're done anyway because H and G/H are solvable by the induction hypothesis. The same argument for Case I shows that G is solvable.

If they're not prime powers, then $pq|\#C_i$ for all $i \geq 2$. Ok so then $\#G \equiv 1 \bmod p$, so $p|\#G$. Also, $\#G \equiv 1 \bmod q$ so $q \not| \#G_i$. But this is weird because $\#G = p^a q^b$. Thus $G = \{1\}$, so G is solvable. □

"It's a pretty sneaky proof."

13.1. **Representations of products.** Fix a field k and let G, H be finite groups. Let $V_1, \cdots, V_r$ be the irreducible k-representations of G and $W_1, \cdots, W_s$ the irreps of H. Then $\{V_i \otimes W_j | 1 \leq i \leq r, 1 \leq j \leq s\}$ are the distinct k-representations of $G \times H$. Also, $\chi_{V_i \otimes W_j}(g, h) = \chi_{V_i}(g)\chi_{W_j}(h)$. That's all I want to say, there's not a whole lot more to say.

13.2. **Virtual representations.** Fix k and G. Then {virtual representations of G} is defined to be {free abelian group generated by the irreps over k} where the operation is $\oplus$.

Example 13.4. If V_1, V_2 are distinct irreps of G, then $2V_1 + 3V_2$ is a representation but $-2V_1 + 3V_2$ can't be written as a direct sum of V_1, V_2 so this is a virtual representation.

Definition 13.5. If $V = \sum n_i V_i$ is a virtual representation, then $\chi_V := \sum n_i \chi_{V_i}$.

Lemma 13.6. *Let V be a virtual representation over $\mathbf{C}$. If $(\chi_V, \chi_V) = 1$ and $\chi_V(1) > 0$, then V is an irreducible representation.*

Proof. If $V = \sum n_i V_i$, then $(\chi_V, \chi_V) = \sum n_i^2$ because the V_i are orthogonal. This is pretty hard for this to be equal to 1. So one of the $n_i = \pm 1$ and all the other ones are 0. Thus $V = \pm 1 V_i$. Finally, $\chi_{V_i}(1) = \dim V_i > 0$, so $V = V_i$, otherwise there'd be a contradiction. □

13.3. **Induced and restricted representations.** Fix a field k. Let $H \leq G$. You have maps {reps of H} $\leftrightarrow$ {reps of G}. If you have a representation V of G, you just restrict to H's action to get $\mathrm{Res}_H^G V$. If W is a representation of H, you can get a representation of G, called $\mathrm{Ind}_H^G W$.

Definition 13.7. $\mathrm{Res}_H^G V := V$ with $h \in H$ acting as it did on V (forgets the action of elements outside H). The homomorphism is just the restriction $\rho|_H$ where $\rho : G \to \mathrm{GL}(V)$.

Induction is much more complicated. $\mathrm{Ind}_H^G W := \{f : G \to W | f(hx) = hf(x) \forall x \in G, h \in H\}$ with G-action given by $({}^g f)(x) := f(xg)$.

Remark 13.8. You can imagine that $\mathrm{Ind}_H^G W = \mathrm{Map}_H(G, W) \simeq \mathrm{Hom}_{kH}(kG, W)$ where the latter thing takes f to the k-linear extension of f.

Warning 13.9. Some books use a different definition of the induction representation, where they define it as $\mathrm{Ind}_H^G W := kG \otimes_{kH} W$. The definitions are equivalent if $(G : H) < \infty$. It's kinda like the relation between the direct sum and the direct product. You're going to prove this in your homework.

Remark 13.10. If $G = \coprod_{p\in P} Hp$ where P is the set of coset representatives, then:

$$\mathrm{Ind}_H^G W = \mathrm{Map}_H\left(\coprod_{p\in P} Hp, W\right) = \mathrm{Map}(P, W) = \prod_{p\in P} W$$

The last thing follows because of H-equivariance.

Corollary 13.11. *If* $(G : H) < \infty$*, then* $\dim \mathrm{Ind}_H^G W = (G : H) \dim W$.

Example 13.12. Let G be a finite group. Then $\mathrm{Ind}_{\{1\}}^G k = \mathrm{Map}(G, k)$. Let $\delta_1(X) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$. Then $\{{}^g\delta_1 | g \in G\}$ (turns out that ${}^g\delta_1 = \delta_{g^{-1}}$) is a basis for $\mathrm{Map}(G, k)$. The group G just acts by permuting the ${}^g\delta_1$. So $\mathrm{Ind}_{\{1\}}^G k = kG$, the regular representation.

Example 13.13. Assume $(G : H) < \infty$. We'll say that $H\backslash G = \{\text{cosets } Hg\}$ and likewise for G/H. These are finite because of my assumption. If k is the trivial representation of H, then $\mathrm{Ind}_H^G k = \mathrm{Map}_H(G, k)$. But H acts trivially on k, so any H-equivariant map will be constant on cosets of H. But which cosets? Right cosets, i.e., $H\backslash G$. Thus $\mathrm{Ind}_H^G k = \mathrm{Map}(H\backslash G, k)$. I claim that this is $k[G/H]$ (the k-vector space with basis G/H). The way this works is by taking a basis element of $k[G/H]$, say gH, to $\delta_{Hg^{-1}}$ which is a basis vector of $\mathrm{Map}(H\backslash G, k)$. Because G acts on G/H, we know that G acts on $k[G/H]$. Thus $\mathrm{Ind}_H^G W = k[G/H]$. This is the permutation representation associated to the left G-set G/H.

13.4. **Characters of induced representations.** Let $(G : H) < \infty$. Let W be a representation of H. Let $V := \mathrm{Ind}_H^G W$. What is χ_V in terms of χ_W?

Proposition 13.14 (Frobenius' formula)**.** *Let R be any set of right coset representatives for $H \subseteq G$. Then:*

$$\chi_V(g) = \sum_{r\in R} \chi_W(rgr^{-1})$$

This looks like nonsense because rgr^{-1} need not be in H. The convention is that if $\chi_W(g) = 0$ if $g \notin H$. I just ignore the terms where it doesn't make sense.

We know that $G = \coprod_{r\in R} Hr$. Take inverses of all the elements, so you get $G = \coprod_r r^{-1}H$. The k-span of G is $kG = \bigoplus_r r^{-1} \otimes kH$. I'll just not write the tensor anymore.

Proof. By homework, $V \simeq kG \otimes_{kH} W \simeq \bigoplus_{r\in R} r^{-1}kH \otimes_{kH} W = \bigoplus_r r^{-1}W$ (there's an implicit tensor in the last thing). Now how does G act on this? Maybe I really should save this for next time. □

14. Frobenius reciprocity

I don't know what happened in the beginning because I came rather late. Looks like we did Young diagrams? Some result that was written in the board before I came was:

Corollary 14.1. *If G is finite and* char(k) *doesn't divide $\#H$, then* $\chi_V(g) = \frac{1}{\#H}\sum_{g\in G}\chi_W(rgr^{-1})$.

Let me continue with what's actually happening right now.

Theorem 14.2 (Frobenius reciprocity)**.** *Let $H \leq G$. If V is a representation of G and W a representation of H, then* $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \simeq \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$ *given by* $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \ni \alpha \mapsto [v \mapsto (\alpha v)(1)] \in \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$ *and* $\mathrm{Hom}_H(\mathrm{Res}_H^G V, W) \ni \beta \mapsto [v \mapsto (g \mapsto \beta(gv))] \in \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$.

Proof. ... (I don't know why this was written on the board.) □

Corollary 14.3 (Take dimension in Frobenius reciprocity)**.** *If $k = \mathbf{C}$, then* $(\chi_V, \chi_{\mathrm{Ind}_H^G W}) = (\chi_{\mathrm{Res}_H^G V}, W)$. *And for irreducible V and W, the multiplicity of V in* $\mathrm{Ind}_H^G W$ *= the multiplicity of W in* $\mathrm{Res}_H^G V$.

14.1. Representations for S_n.

Definition 14.4. A *partition* of n is a sequence of positive integers (called *parts*) $\lambda = (\lambda_1, \cdots, \lambda_p)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ and $\sum_{i=1}^p \lambda_i = n$. We write $\lambda > \mu$ by the lexicographical ordering, i.e., if the first $\lambda_i - \mu_i$ that is nonzero is positive.

You can obtain a Young diagram from $\lambda = (\lambda_1, \cdots, \lambda_p)$, which is where you have λ_1 squares on the top row, λ_2 squares on the next, and so on. A Young tableau is when you fill in a Young diagram with integers $\leq n$, each only once.

Example 14.5. Let T_λ be the Young tableau with $1, \cdots, n$ in order (where by "in order" I mean you read $1, \cdots, n$ in the same way as you'd read a book).

Each $g \in G := S_n$ acts on {Young tableaux}.

Given λ, you can define the *row subgroup*, denoted P_λ, defined as the collection of $g \in S_n$ that maps each number in T_λ to a number in the same row. What is

$\#P_\lambda$? It's exactly $\lambda_1! \cdots \lambda_p!$. There's also the column subgroup, Q_λ, defined as the collection of $g \in S_n$ that maps each number in T_λ to a number in the same column. It's clear that $Q_\lambda \cap P_\lambda = \{1\}$.

Definition 14.6. Young projectors are defined as $a_\lambda = \frac{1}{\#P_\lambda} \sum_{g \in P_\lambda} g$ and $b_\lambda = \frac{1}{\#Q_\lambda} \sum_{g \in Q_\lambda} (-1)^g g$, where $(-1)^g = \mathrm{sgn}(g)$. Let $c_\lambda := a_\lambda b_\lambda \in \mathbf{C}G$.

What is the coefficient of 1 in c_λ? It's $\frac{1}{\#P_\lambda \#Q_\lambda}$, because the only contribution to c_λ comes from the identity (if there was another g then you'd have to multiply with g^{-1} but $Q_\lambda \cap P_\lambda = \{1\}$).

14.2. **Specht modules.** There's one for each partition, defined as $V_\lambda := (\mathbf{C}G)c_\lambda$. It's a left $\mathbf{C}G$-module, i.e., a representation of S_n.

Theorem 14.7. *Fix n. The V_λ for all parititions λ of n form all the irreducible representations of S_n, each occurring once up to isomorphism.*

It's going to take us a while to prove this. There'll be a bunch of lemmas.

Lemma 14.8. *If $p \in P_\lambda$, then $a_\lambda p = a_\lambda$, and $pa_\lambda = a_\lambda$. If $q \in Q_\lambda$, then $b_\lambda q = (-1)^q b_\lambda$, and $qb_\lambda = (-1)^q b_\lambda$.*

Proof. Too trivial to bother writing down. □

14.2.1. *A game.* We're going to play a game. (It looks like the Young diagrams on the board whose purpose I didn't understand are going to be used for this game). If I give you two different Young tableaux associated to a partition, can you permute the rows of one of them, and permute the columns of the other one, to get the same new Young tableau? With the one that we've been given, the answer is yes, obviously. But if you have two elements that in the same row in one Young tableau and in the same column in the other one, you can't do it!

In terms of group theory, this means the following. Let Q'_λ be the collection of elements of S_n that map each (illegible) of T' to a number in the same column. This is exactly $gQ_\lambda g^{-1}$. You fall into two cases. Note that these are the only two cases, i.e., the only obstruction is if you have two elements that in the same row in one Young tableau and in the same column in the other one. This is because if the obstruction didn't exist, you could always solve the problem (this is Case I)!

Case I, i.e., when the game is solvable. There is $p \in P_\lambda$ and $q \in Q_\lambda$ such that $p = (gq^{-1}g^{-1})g$ where $(gq^{-1}g^{-1})$ is an element of Q'_λ. This means I can first apply g to go up and then come back down in Q'_λ. This means that $pq = g$.

Case II, i.e., when the game isn't solvable. There exists a transposition $p \in P^\lambda$, and it's also in Q'_λ. Thus $p = gqg^{-1}$, where p is a transposition in P^λ, and q is a transposition in Q_λ.

Using this, I can prove some group theory lemmas.

14.2.2. *Continuing onwards.*

Lemma 14.9. *For $g \in G$:*

$$a_\lambda g b_\lambda = \begin{cases} (-1)^q c_\lambda & g = pq \text{ with } p \in P_\lambda, q \in Q_\lambda \\ 0 & g \notin P_\lambda Q_\lambda \end{cases}$$

Proof. If $g = pq$, then:

$$a_\lambda g b_\lambda = a_\lambda pq b_\lambda = a_\lambda (-1)^q b_\lambda = (-1)^q c_\lambda$$

If $g \neq pq$, we're in case II as above. So there exist transpositions $p \in P_\lambda$ and $q \in Q_\lambda$ such that $p = gqg^{-1}$, i.e., $pg = gq$. Then:

$$\begin{aligned} a_\lambda g b_\lambda &= (a_\lambda p) g b_\lambda \\ &= a_\lambda g (q b_\lambda) \\ &= a_\lambda g (-1) b_\lambda \\ &= -a_\lambda g b_\lambda \end{aligned}$$

This is because q is a transposition, $(-1)^q = -1$. So, $a_\lambda g b_\lambda = 0$. □

Corollary 14.10. *We see that $a_\lambda(\mathbf{C}G)b_\lambda \subseteq \mathbf{C}c_\lambda$.*

Corollary 14.11. *We also see that $c_\lambda(\mathbf{C}G)c_\lambda \subseteq \mathbf{C}c_\lambda$ because the left hand side is $a_\lambda(b_\lambda \mathbf{C}G a_\lambda)b_\lambda$.*

Corollary 14.12. *c_λ^2 is in $\mathbf{C}c_\lambda$.*

Corollary 14.13. *It turns out that:*

$$c_\lambda^2 = \frac{n!}{\#P_\lambda \# Q_\lambda \dim V_\lambda} c_\lambda$$

Proof. Write $c_\lambda^2 = \mu c_\lambda$ for some $\mu \in \mathbf{C}$. Then $c_\lambda|_{\mathbf{C}G}$ has matrix whose trace is $(\dim V_\lambda)\mu$. On the other hand, $\operatorname{Tr} c_\lambda|_{\mathbf{C}G} = n! \times (\text{coeff of } 1) = \frac{n_\lambda}{\#P_\lambda \# Q_\lambda}$. You now solve for μ, and substitute. □

Lemma 14.14. *Let T be a Young tableau of shape λ and T' of shape μ. If $\lambda > \mu$, then there exists distinct i, j in the same row of T and the same column of T'.*

Proof. If the first row of T is greater than T', then by the Pigeonhole principle, such (i, j) must exist. WLOG, permute the columns of T' so that the first row entries of T land in the first row of T'. But then you just use the Pigeonhole principle anyway. You just think about this at home. Find some pigeons, and work through it. □

Let's apply this lemma to the case when $T = T_\lambda$ and $T' = gT_\mu$. There exists a transposition in $P_\lambda \cap Q'_\mu$. Thus if $\lambda > \mu$, then $a_\lambda x b_\mu = 0$ for all $x \in \mathbf{C}G$.

15. Representations of S_n

Last time: let $G = S_n$, $a_\lambda := \frac{1}{\#P_\lambda}\sum_{g\in P_\lambda} g$, $b_\lambda := \frac{1}{\#Q_\lambda}\sum_{g\in Q_\lambda}(-1)^g g$, $c_\lambda := a_\lambda b_\lambda$, and $V_\lambda := (\mathbf{C}G)c_\lambda$. The goal is to show:

THEOREM 15.1. *For fixed n, the V_λ for partitions λ of n are all the irreducible representations of S_n over $\mathbf{C}$, each occurring once up to isomorphism.*

So far, we have shown that $c_\lambda^2 = \frac{n!}{\#P_\lambda \#Q_\lambda \dim V_\lambda} c_\lambda$, and $c_\lambda(\mathbf{C}G)c_\mu = \begin{cases} \mathbf{C}c_\lambda & \text{if } \lambda = \mu \\ 0 & \text{else} \end{cases}$. This follows from the fact that if $\lambda > \mu$, then $a_\lambda x b_\mu = 0$ for all $x \in \mathbf{C}G$.

Lemma 15.2. *Let A be any ring, and e an idempotent. Then $1-e$ is also an idempotent. Let M be an A-module. Under $M \simeq \mathrm{Hom}_A(A, M)$ sending $m \mapsto (a \mapsto am)$, the decompositions $M \simeq eM \oplus (1-e)M \simeq \mathrm{Hom}_A(A, M) \simeq \mathrm{Hom}(Ae, M) \oplus \mathrm{Hom}(A(1-e), M)$ gives isomorphisms $eM \simeq \mathrm{Hom}(Ae, M)$ and $(1-e)M \simeq \mathrm{Hom}(A(1-e), M)$.*

Proof. For any $m \in M$, $(a \mapsto am) \in \mathrm{Hom}(Ae, M)$ if and only if $(a \mapsto am)$ kills $A(1-e)$ if and only if $(1-e)m = 0$ if and only if $m = em$ if and only if $m \in eM$. The same argument for $\mathrm{Hom}(A(1-e), M)$. □

Now we prove our desired result.

Proof of THEOREM. For $\lambda \geq \mu$, we know that:

$$\begin{aligned} \mathrm{Hom}_G(V_\lambda, V_\mu) &= \mathrm{Hom}_G((\mathbf{C}G)c_\lambda, (\mathbf{C}G)c_\mu) \\ &= c_\lambda(\mathbf{C}G)c_\mu \end{aligned}$$

The second line comes when we take $M = (\mathbf{C}G)c_\mu$ and $e = c_\lambda$ and invoke the above lemma (it doesn't matter that c_λ is not exactly an idempotent, you're multiplying by all elements of $\mathbf{C}G$ anyway). Taking dimension gives:

$$\dim \mathrm{Hom}_G(V_\lambda, V_\mu) = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{else} \end{cases}$$

This top line means that V_λ is irreducible. The bottom line means that $V_\lambda \not\simeq V_\mu$. We now need to show that these are all the irreducibles. We know that the number of irreducible representations is the number of conjugacy classes in S_n. These are just the number of cycle types, which is just the number of partitions of n. And look how many irreducibles we constructed! This means we have the right number, i.e., the V_λ are all the irreps. □

What are the dimensions of each V_λ? We are going to actually compute the whole character!

15.1. **Some induced representations.** Let $U_\lambda := \mathrm{Ind}_{P_\lambda}^{S_n}\mathbf{C}$. What is this? We proved that this would be $\mathbf{C}[S_n/P_\lambda] = \{\sum_{g\in G} a_g g | a_g \in \mathbf{C} \text{ depends only on } gP_\lambda\} \simeq (\mathbf{C}S_n)a_\lambda$, where the latter thing is by definition. How does U_λ decompose? This is the dimension of the homs. What is $\mathrm{Hom}_G(U_\lambda, V_\mu) = \mathrm{Hom}_G((\mathbf{C}G)a_\lambda, (\mathbf{C}G)c_\mu)$. We can apply the lemma. Well, a_λ is an idempotent (which you can check; the constant factor $\frac{1}{\#P_\lambda}$ is what *makes* it an idempotent). Thus this becomes $a_\lambda(\mathbf{C}G)c_\mu = \begin{cases} \mathbf{C}c_\lambda & \text{if } \mu = \lambda \\ 0 & \mu < \lambda \end{cases}$. We proved the top line last time. We don't know what happens in the case $\lambda > \mu$. At least we can say that $U_\lambda = \bigoplus_{\mu\geq\lambda} k_{\mu\lambda} V_\mu$ where $k_{\lambda\lambda} = 1$ and $k_{\mu\lambda}$ are nonnegative integers, called Kostka numbers.

15.1.1. *Preliminaries.* Let $x_1, \cdots, x_N$ be indeterminates. The capital N means that this is very large. Some people just take this to be ∞ but you should think of this as very large. For $m \geq 0$, define the power sum $H_m(x) := \sum_{i=1}^N x_i^m$. If $\lambda = (\lambda_1, \cdots, \lambda_p)$ and $N \geq p$, then write $x^\lambda := \prod_{i=0}^N x_i^{\lambda_i}$ where $\lambda_k := 0$ if $k > p$.

We know that cycle types are the conjugacy classes in S_n. We're going to represent cycle types by a sequence $i = (i_1, i_2, \cdots)$ of positive integers where the $i_m = \#$ of m-cycles. For example, the identity would be $(n, 0, 0, \cdots)$. This gives a conjugacy class C_i of S_n. If $c \in C_i$, what's the centralizer Z_c? If $c = (123)(456)(78)$, then powers of $(123), (456), (78)$ are in this. But $(14)(25)(36)$ is also in the centralizer, as long as you keep the order. Then Z_c is the semidirect product $\prod_{m\geq 1} S_{i_m} \rtimes (\mathbf{Z}/m\mathbf{Z})^{i_m}$ where the $(\mathbf{Z}/m\mathbf{Z})^{i_m}$ comes from rotating each m-cycle, and the S_{i_m} permutes the m-cycles. In particular, $\#Z_m = \prod_{m\geq 1} i_m! m^{i_m}$. You can work out the sizes of the conjugacy classes? By orbit-stabilizer, $\#C_i = \dfrac{\#S_n}{\#Z_c} = \dfrac{n!}{\prod_{m\geq 1} i_m! m^{i_m}}$. Those are all the preliminaries.

15.2. **Character of U_λ.**

Proposition 15.3. *If $c \in C_i$ and if $[x^\lambda]$ is the coefficient of x^λ in $\prod_{m\geq 1} H_m(x)^{i_m}$, then:*

$$\chi_{U_\lambda}(c) = [x^\lambda] \prod_{m\geq 1} H_m(x)^{i_m}$$

Proof. Sometimes we'll just write $\chi_{U_\lambda}(C_i)$. We know from the character of an induced representation that:

$$\chi_{U_\lambda}(c) = \frac{1}{\#P_\lambda} \sum_{g\in S_n} \chi_{\mathbf{C}}(gcg^{-1})$$

But $\chi_{\mathbf{C}}(gcg^{-1}) = \begin{cases} 1 & \text{if } gcg^{-1} \in P_\lambda \\ 0 & \text{else} \end{cases}$. This sum is just:

$$\begin{aligned}\chi_{U_\lambda}(c) &= \frac{1}{\#P_\lambda}\sum_{g\in S_n}\chi_{\mathbf{C}}(gcg^{-1})\\ &= \frac{1}{\#P_\lambda}\sum_{p\in P_\lambda\cap C_i}\sum_{g \text{ s.t. } gcg^{-1}=p} 1\end{aligned}$$

The conjugation orbit of c are in bijection with the cosets of Z_c. So this is going to be:

$$\chi_{U_\lambda}(c) = \frac{\#(P_\lambda\cap C_i)\#Z_c}{\#P_\lambda}$$

We're not done yet. What is $\#(P_\lambda \cap C_i)$? This is the number of elements of P_λ of cycle type i. Remember that P_λ are the permutations that preserve the rows. Because $P_\lambda = S_{\lambda_1}\times\cdots S_{\lambda_p}$. Thus:

$$\#(P_\lambda\cap C_i) = \sum_r\prod_{j=1}^p \#\text{elements of } S_{\lambda_j} \text{ of cycle type } (r_{jm})_{m\geq 1}$$

Where r_{jm} := # of m-cycles in the jth row for an element of P_λ. Fix j. Then $\sum_m mr_{jm} = \lambda_j$. Also, $\sum_j r_{jm} = i_m$. I'm just looking at all the ways you can take cycles and distribute them among the rows. What is this number "#elements of S_{λ_j} of cycle type $(r_{jm})_{m\geq 1}$"? This is $\frac{\lambda_j!}{\prod_{m\geq 1} r_{jm}!m^{r_{jm}}}$ by our formula $\#C_i = \frac{\#S_n}{\#Z_c} = \frac{n!}{\prod_{m\geq 1} i_m!m^{i_m}}$. The conclusion is that:

$$\begin{aligned}\chi_{U_\lambda}(C_i) &= \frac{\#Z_c\#(P_\lambda\cap C_i)}{\#P_\lambda}\\ &= \frac{\left(\sum_r\prod_{m\geq 1} i_m!m^{i_m}\right)\left(\frac{\lambda_j!}{\prod_{m\geq 1} r_{jm}!m^{r_{jm}}}\right)}{\prod_j\lambda_j!}\\ &= \sum_r\prod_{m\geq 1}\frac{i_m!}{\prod_j r_{jm}!}\\ &= \sum_r\prod_{m\geq 1}[x_1^{mr_{1m}}x_2^{mr_{2m}}\cdots](x_1^m+\cdots+x_N^m)^{i_m}\\ &= [x^\lambda]\prod_{m\geq 1}(x_1^m+\cdots+x_N^m)^{i_m}\end{aligned}$$

The proof is, it's done. $\square$

"I guess you have to like combinatorics if you're going to like today's lecture". Define $U_{\text{any permuatation of }\lambda} := U_\lambda$, and say that $U_\lambda := 0$ if any λ_j is negative. The nice thing about the formula for the character is that it's still true even if any λ_i is negative. Define the Vandermonde determinant:

$$\begin{aligned}\Delta(x) &= \prod_{1\le i<j<N}(x_i - x_j)\\ &= \det\begin{pmatrix} x_1^{N-1} & x_1^{N-2} & \cdots & 1\\ x_2^{N-1} & x_2^{N-2} & \cdots & 1\\ \vdots & \vdots & \ddots & \vdots\\ x_N^{N-1} & x_N^{N-2} & \cdots & 1\end{pmatrix}\\ &= \sum_{\sigma\in S_n}(-1)^\sigma x^{\sigma(\rho)}\end{aligned}$$

Where $\rho = (N-1, N-2, \cdots, 0) \in \mathbf{Z}^N_{\ge 0}$.

Theorem 15.4. *The character of the V_λ is:*

$$\begin{aligned}\chi_{V_\lambda}(C_i) &= [x^{\lambda+\rho}]\Delta(x)\prod_{m\ge 1}H_m(x)^{i_m}\\ &= [x^\lambda]\prod_{i<j}\left(1-\frac{x_j}{x_i}\right)\prod_{m\ge 1}H_m(x)^{i_m}\end{aligned}$$

Where the latter equation is obtained from the first by dividing out(?).

Proof. The RHS defines a class function that we'll call θ_λ. It suffices to show that $\theta_\lambda = \chi_{V_\lambda}$ plus a $\mathbf{Z}$-linear combination of χ_{V_λ} for $\lambda < \mu$. Then we'll show that $(\theta_\chi, \theta_\chi) = 1$. But $(\theta_\chi, \theta_\chi) = (\chi_{V_\lambda}, \chi_{V_\lambda})$ + sum of squares of multiplicities $=$ 1 + sum of squares of multiplicities, so that sum of squares of multiplicities $= 0$, and then we're done.

Let's prove the first statement. We have:

$$\theta_\lambda(C_i) = \sum_{\sigma\in S_n}(-1)^\sigma[x^{\lambda+\rho-\sigma(\rho)}]\prod_m H_m(x)^{i_m} \tag{1}$$

$$\theta_\lambda = \sum_{\sigma\in S_n}(-1)^\sigma\chi_{U_{\lambda+\rho-\sigma(\rho)}} \tag{2}$$

The first line is because this keeps track of all the ways you can obtain $x^{\lambda+\rho}$. The second line follows from our computation of the character of U_λ. We know that $\sigma(\rho) \le \rho$ in the lexicographic ordering, with equality if and only if $\sigma = 1$. So $\lambda+\rho-\sigma(\rho) \ge \lambda+\rho-\rho = \lambda$ with equality if and only if $\sigma = 1$. Thus the equation becomes:

$$\begin{aligned}\theta_\lambda &= \sum_{\sigma\in S_n}(-1)^\sigma \chi_{U_{\lambda+\rho-\sigma(\rho)}}\\ &= \chi_{U_\lambda} + \mathbf{Z}\text{-linear combination of higher } (\mu > \lambda)\, \chi_{U_\lambda}\\ &= (\chi_{V_\lambda} + \text{higher}) + \text{higher}\end{aligned}$$

That's exactly what we wanted.

Now we'll compute the inner product. This is going to be the messiest thing we've done all year.

$$\begin{aligned}(\theta_\lambda,\theta_\lambda) &= \frac{1}{n!}\sum_{\text{cycle types, i.e., } i=(i_1,i_2,\cdots)} \#C_i\theta_\lambda(C_i)^2\\ &= \frac{1}{n!}\sum_i \frac{n!}{\prod_m m^{i_m} i_m!}[x^{\lambda+\rho}y^{\lambda+\rho}]\Delta(x)\Delta(y)\prod_{m\geq 1} H_m(x)H_m(y)\\ &= [x^{\lambda+\rho}y^{\lambda+\rho}]\Delta(x)\Delta(y)\sum_i\prod_m \frac{\left(\frac{(\sum_j x_j^m)(\sum_k y_k^m)}{m}\right)^{i_m}}{i_m!}\end{aligned}$$

where the third line is expanding out the second line. We can replace the sum in the third line with the sum over *all* sequences $(i_1, i_2, \cdots)$ in $\mathbf{Z}_{\geq 0}$ eventually 0 (only the i with $\sum m i_m = n$ contribute, though). Now, we can bring the product out to get:

$$\begin{aligned}\sum_i\prod_m \frac{\left(\frac{(\sum_j x_j^m)(\sum_k y_k^m)}{m}\right)^{i_m}}{i_m!} &= \prod_{m\geq 1}\sum_{i_m=0}^\infty \frac{\left(\sum_{j,k=1}^N \frac{(x_j y_k)^m}{m}\right)^{i_m}}{i_m!}\\ &= \prod_{m\geq 1}\exp\left(\sum_{j,k}\frac{(x_j y_k)^m}{m}\right)\\ &= \prod_{j,k}\exp\left(-\ln(1-x_j y_k)\right)\\ &= \prod_{j,k}\frac{1}{1-x_j y_k}\end{aligned}$$

So that:

$$\begin{aligned}
(\theta_\lambda, \theta_\lambda) &= [x^{\lambda+\rho} y^{\lambda+\rho}] \Delta(x)\Delta(y) \sum_i \prod_m \frac{\left(\frac{(\sum_j x_j^m)(\sum_k y_k^m)}{m}\right)^{i_m}}{i_m!} \\
&= [x^{\lambda+\rho} y^{\lambda+\rho}] \Delta(x)\Delta(y) \prod \frac{1}{1 - x_j y_k} \\
&= [x^{\lambda+\rho} y^{\lambda+\rho}] \det \left(\frac{1}{1 - x_j y_k} \right)_{1 \le j,k \le N} \\
&= [x^{\lambda+\rho} y^{\lambda+\rho}] \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^N \frac{1}{1 - x_j y_{\sigma(j)}}
\end{aligned}$$

If $x^{\lambda+\rho} y^{\lambda+\rho}$ appears here, then the exponent of x_j is the exponent of $y_{\sigma(j)}$. But this is the exponent of y_j because we need the same exponents in $x^{\lambda+\rho} y^{\lambda+\rho}$. But $\lambda + \rho$ has strictly decreasing because λ is already weakly decreasing and ρ is strictly decreasing. Thus this isn't possible unless $j = \sigma(j)$ for all j. This means that:

$$\begin{aligned}
(\theta_\lambda, \theta_\lambda) &= [x^{\lambda+\rho} y^{\lambda+\rho}] \prod_{j=1}^N \frac{1}{1 - x_j y_j} \\
&= 1
\end{aligned}$$

So we're done. ☐

16. More on representations of S_n, Lie algebra representations, Schur-Weyl duality

16.1. **The hook-length formula.** Last time: Let $\rho = (N-1, N-2, \cdots, 0)$. Define the Vandermonde determinant:

$$\begin{aligned}
\Delta(x) &= \prod_{1 \le i < j < N} (x_i - x_j) \\
&= \det \begin{pmatrix} x_1^{N-1} & x_1^{N-2} & \cdots & 1 \\ x_2^{N-1} & x_2^{N-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_N^{N-1} & x_N^{N-2} & \cdots & 1 \end{pmatrix} \\
&= \sum_{\sigma \in S_n} (-1)^\sigma x^{\sigma(\rho)}
\end{aligned}$$

And let $H_m(x) = x_1^m + \cdots + x_N^m$.

Theorem 16.1. *If $c \in S_n$ has i_m m-cycles, then*

$$\chi_{V_\lambda}(c) = [x^{\lambda+\rho}]\Delta(x)\prod_{m\geq 1} H_m(x)^{i_m}$$

Today we're going to figure out the dimension of V_λ.

Consider a typical partition and consider its Young diagram. For every square S, you count the number of squares below it, the number of squares on its right, and the square itself. We write h_S for this number.

Theorem 16.2 (Hook-length formula)**.** *It's called the hook-length formula for reasons that'll be obvious once I write out the formula.*

$$\dim V_\lambda = \frac{n!}{\prod_{\text{squares } S \in \text{Young diagram of } \lambda} h_S}$$

Proof. Write $\lambda + \rho = (\ell_1, \ell_2, \cdots, \ell_N)$, so that $\ell_j = \lambda_j + \rho_j = \lambda_j + N - j$. Define $P_d(z) := z(z-1)\cdots(z-d+1)$. This clearly satisfies: $\frac{P_j(\ell)}{\ell!} = \frac{1}{(\ell-j)!}$. Then:

$$\begin{aligned}
\dim V_\lambda &= \chi_{V_\lambda}(1)\\
&= [x^{\lambda+\rho}]\Delta(x)(x_1+\cdots+x_N)^n\\
&= [x^{\lambda+\rho}]\left(\sum_{s\in S_n}(-1)^s\prod_{j=1} x_j^{N-s(j)}\right)(x_1+\cdots+x_N)^n\\
&= \sum_{s\in S_n}(-1)^s\left[\prod_j x_j^{\ell_j-(N-s(j))}\right](x_1+\cdots+x_N)^n\\
&= \sum_{s\in S_n}(-1)^s\frac{n!}{\prod_j(\ell_j-N+s(j))!}\\
&= \frac{n!}{\ell_1!\cdots\ell_N!}\sum_{s\in S_n}(-1)^s\prod_j P_{N-s(j)}(\ell_j)\\
&= \frac{n!}{\prod_{k=1}^N \ell_k!}\prod_{i<j}(\ell_i-\ell_j)\\
&= \frac{n!}{\prod_{i=1}^N \frac{\ell_i(\ell_i-1)(\ell_i-2)\cdots 1}{\prod_{j=i+1}^N(\ell_i-\ell_j)}}
\end{aligned}$$

∎

Because 1 has n 1-cycles. Also, to find the coefficient of $x^{\lambda+\rho}$ in that thing, you just need to find the sum of the coefficients that add up. You use the multinomial theorem to expand. In the fifth line, you drop the term if any $\ell_j - N + s(j)$ is negative. And we jump to the seventh line because $\sum_{s\in S_n}(-1)^s \prod_j P_{N-s(j)}(\ell_j)$ is a determinant, namely $\det\begin{pmatrix} P_{N-1}(x_1) & P_{N-2}(x_1) & \cdots & 1 \\ P_{N-1}(x_2) & P_{N-2}(x_2) & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ P_{N-1}(x_N) & P_{N-2}(x_N) & \cdots & 1 \end{pmatrix}$. But because this is just the consequence of adding rows in $\begin{pmatrix} x_1^{N-1} & x_1^{N-2} & \cdots & 1 \\ x_2^{N-1} & x_2^{N-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_N^{N-1} & x_N^{N-2} & \cdots & 1 \end{pmatrix}$, this determinant is the same as $\Delta(x)$.

It turns out that if you cancel out $\prod_{i=1}^{N} \frac{\ell_i(\ell_i-1)(\ell_i-2)\cdots 1}{\prod_{j=i+1}^{n}(\ell_i-\ell_j)}$, what's left are the λ_i hook lengths from the ith row. This is left as an exercise.

16.2. Tensor products and duals of Lie algebra representations.

Definition 16.3. Let $\mathfrak{g}$ be a Lie algebra. Let $\mathfrak{p}_V : \mathfrak{g} \to \operatorname{End} V$ and $\mathfrak{p}_W : \mathfrak{g} \to \operatorname{End} W$ be two representations. Define $\mathfrak{p}_{V\otimes W}(x) := \mathfrak{p}_V(x) \otimes 1 + 1 \otimes \mathfrak{p}_W(x) \in \operatorname{End}(V \otimes W)$. Also, $\mathfrak{p}_{V^*} = -\mathfrak{p}_V(x)^* \in \operatorname{End} V^*$ (where * denotes the transpose). These are again representations of $\mathfrak{g}$.

Where does this come from? Suppose you start with Lie group representations $\rho_V : G \to \mathrm{GL}(V)$ and $\rho_W : G \to \mathrm{GL}(W)$. We know how to tensor these. We just let $\rho_V \otimes \rho_W : G \to \mathrm{GL}(V \otimes W)$ by sending $g \mapsto \rho_V(g) \otimes \rho_W(g)$. We now consider the associated Lie algebra representation. What you do is take the derivative of these homomorphisms at the identity. This gives $\mathfrak{p}_V : T_eG = \mathfrak{g} \to T_1(\mathrm{GL}(V)) = \mathfrak{gl}(V) = \operatorname{End}(V)$ and $\mathfrak{p}_W : \mathfrak{g} \to \mathfrak{gl}(W) = \operatorname{End}(W)$. So, consider $d(\rho_V \otimes \rho_W)|_{g=1} : \mathfrak{g} \to \mathfrak{gl}(V \otimes W) = \operatorname{End}(V \otimes W) = \operatorname{End}(V) \otimes \operatorname{End}(W)$. You use the product rule to evaluate this. And, well, what is it? It's going to be $d_{\rho_V}|_{g=1} \otimes \rho_W|_{g=1} + \rho_V|_{g=1} \otimes d\rho_W|_{g=1}$. This is the map that sends a tangent vector x to $\mathfrak{p}_V(x) \otimes 1 + 1 \otimes \mathfrak{p}_W(x)$ because ρ_V, ρ_W are homomorphisms. The proof for $\mathfrak{p}_{V^*}$ is similar.

16.3. Schur-Weyl duality. Before I talk about Schur-Weyl duality, I'll need something called the double centralizer theorem.

Theorem 16.4 (Double centralizer theorem)**.** *Let k be an algebraically closed field. Let E be a finite-dimensional k-vector space. Let A be a semisimple subalgebra of* $\operatorname{End} E$*. You can view E as an A-module. Define $B := \operatorname{End}_A E = \{b \in$* $\operatorname{End} E$ *that commute with all $a \in A\}$. So it's like a centralizer. Then:*

(1) $\operatorname{End}_B E = A$.
(2) *B is semisimple.*
(3) *As a representation for $A \otimes_k B$, $E = \bigoplus_{i \in I} V_i \otimes W_i$ where the V_i are all the irreducibles of A and similarly for the W_i. There's actually a natural bijection between irreps of A and irreps of B. I'll tell you how to pair them up in the proof.*

Proof. For the second part, let $(V_i)_{i \in I}$ be all the irreps of A. We assumed that A is semisimple, so $A \simeq \prod_{i \in I} \operatorname{End} V_i$. As an A-module, $V_i \subseteq \operatorname{End} V_i$. But $\operatorname{End} V_i \subseteq A \subseteq \operatorname{End} E \simeq E^{\dim E}$. So every V_i occurs in E. The isotypic decomposition of E as a semisimple A-module is $E = \bigoplus_{i \in I} V_i \otimes_k \operatorname{Hom}_A(V_i, E)$ by Schur's lemma. Call $\operatorname{Hom}_A(V_i, E) =: W_i$. Since V_i occurs in E, we know that $W_i \neq 0$. Then $\operatorname{End}_A E = \prod_{i \in I} \operatorname{End} W_i$ by a homework assignment you did. This is what we defined B to be. This is good, because that's about as semisimple as it gets! It's a product of matrix algebras.

Since $B = \prod_{i \in I} \operatorname{End} W_i$, the W_i are all the irreducibles of B. I guess that finishes the last part.

For the first part, we know that the isotypic decomposition of E as a B-module is $E = \bigoplus_i V_i \otimes W_i$ where the V_i are thought of as vector spaces that index the irreducibles W_i. This implies that $\operatorname{End}_B E = \prod_{i \in I} \operatorname{End} V_i$. But this is just A. □

The book says this all in one sentence. "The book gets more and more sketchy as you go on." We're going to use this result in a special case. Let $E = V^{\otimes n}$ for some finite-dimensional $\mathbf{C}$-vector space V. Then S_n acts on E by permuting the tensors. So E is a representation of S_n. So we get $\mathbf{C}S_n \to \operatorname{End} E$.

Let's do an example first, to see why it's an interesting thing. Suppose $n = 2$. Well, S_2 acts on $V^{\otimes 2}$. But S_2 isn't a very exciting group. You can try to decompose this into its isotypic components, and you get $V^{\otimes 2} \simeq \operatorname{Sym}^2 V \oplus \Lambda^2 V$ because S_2 acts trivially on $\operatorname{Sym}^2 V$ and S_2 acts as the sign representation. The summands are also representations of $\mathrm{GL}(V)$. This gives you way of constructing representations of $\mathrm{GL}(V)$. Now we're going to do this for all n.

In the general case, let $A = \operatorname{Im}(\mathbf{C}S_n \to \operatorname{End} E)$. Since $\mathbf{C}S_n$ is semisimple by Maschke's theorem, its quotient A is semisimple. On the other hand, there's also an action of $\mathrm{GL}(V)$. You can either use the Lie group $\mathrm{GL}(V)$, or you can think of it as a Lie algebra representation. More precisely, on the other hand, the Lie algebra $\mathfrak{gl}(V)$ acts on V and hence on $V^{\otimes n}$ by our tensor products of representations of Lie algebras action constructed above. Explicitly, each $x \in \mathfrak{gl}(V)$ acts as $x \otimes 1 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes 1 \otimes \cdots \otimes x =: \Delta_n(x)$ on $V^{\otimes n}$. You can think of $\Delta_n(x) \in \operatorname{End} E$.

Now we have two things that are acting. A is acting on E, and so is $\mathfrak{gl}(V)$. The algebras in the double centralizer theorem are supposed to be associative k-algebras, so $\mathfrak{gl}(V)$ doesn't really satisfy the hypotheses. But we just pass to the universal enveloping algebra. More precisely:

Proposition 16.5. $\mathrm{End}_A E$ *is* $\mathrm{Im}(\mathcal{U}(\mathfrak{gl}(V)) \to \mathrm{End}\, E)$ *where* $\mathcal{U}(\mathfrak{gl}(V))$ *is the universal enveloping algebra generated by elements* $x \in \mathfrak{g}$.

Proof. We know that $\mathrm{End}\, E = \mathrm{End}(V^{\otimes n}) = (\mathrm{End}\, V)^{\otimes n}$. To get the ones that commute with A, I just want S_n-homomorphisms. I need to take S_n-invariants, to get $\mathrm{End}_A E = \mathrm{End}_{S_n} E = [(\mathrm{End}\, V)^{\otimes n}]^{S_n}$. This is the left hand side. Now, what's the right hand side? I think I'm going to have to erase the double centralizer theorem even though that's what we're trying to apply. You can probably remember what that means. Anyway, the right hand side is $\mathrm{Im}(\mathcal{U}(\mathfrak{gl}(V)) \to \mathrm{End}\, E)$, which is the $\mathbf{C}$-algebra generated by $\{\Delta_n(x) | x \in \mathfrak{gl}(V) = \mathrm{End}\, V\}$. This is equal to $[(\mathrm{End}\, V)^{\otimes n}]^{S_n}$ by a special case of the following more general lemma.

Lemma 16.6. *For any* $\mathbf{C}$*-algebra* X*, the elements* $\Delta_n(x)$ *generate* $(X^{\otimes n})^{S_n}$ *as a* $\mathbf{C}$*-algebra.*

Proof. I'll show for $n = 2$. You'll get how the general case goes. Let Y be the $\mathbf{C}$-subalgebra generated by $\Delta_2(x)$. What can we find in Y? Well, Y contains $\dfrac{\Delta_2(x)^2 - \Delta_2(x^2)}{2} = \dfrac{(x \otimes 1 + 1 \otimes x)^2 - (x^2 \otimes 1 - 1 \otimes x^2)}{2} = x \otimes x$ for all $x \in X$. And also Y also contains $(x+y) \otimes (x+y) - x \otimes x - y \otimes y = x \otimes y + y \otimes x$ for all $x, y \in X$. But these span $(X^{\otimes 2})^{S_2}$. □

□

The double centralizer theorem now implies Schur-Weyl duality for S_n and $\mathfrak{gl}(V)$.

Theorem 16.7 (Schur-Weyl duality for S_n and $\mathfrak{gl}(V)$)**.**

(1) *The subalgebras* $A := \mathrm{Im}(\mathbf{C}S_n \to \mathrm{End}\, V^{\otimes n})$ *and* $B := \mathrm{Im}(\mathcal{U}(\mathfrak{gl}(V)) \to \mathrm{End}\, V^{\otimes n})$ *are centralizers of each other in* $\mathrm{End}\, V^{\otimes n}$.
(2) *A and B are semisimple.*
(3) *As* $(A \otimes B)$*-modules,*

$$V^{\otimes n} = \bigoplus_{\text{partitions } \lambda \text{ of } n} V_\lambda \otimes L_\lambda$$

Where the L_λ *are the distinct irreducible representations of* $\mathfrak{gl}(V)$ *or* 0.

Proof. Only the last part doesn't directly follow. Why do you have 0s? The double centralizer theorem says that $V^{\otimes n} = \bigoplus_{i \in I} V_i \otimes W_i$ where the V_i are irreps of A and similarly for B. But remember that A is a quotient of $\mathbf{C}S_n$. So the irreducibles of A are a subset of the irreps of S_n. Similarly, W_i are all the irreps of B, which are a subset of irreps of $\mathcal{U}(\mathfrak{gl}(n))$, i.e., irreps of $\mathfrak{gl}(n)$. So we're done. □

The $n = 2$ case is what we said before. We'll talk later about when you use the Lie group $\mathrm{GL}(V)$.

17. Schur-Weyl duality

Friday is a holiday!
Recall we proved:

Theorem 17.1 (Schur-Weyl duality for S_n and $\mathfrak{gl}(V)$)**.**
(1) *The subalgebras* $A := \mathrm{Im}(\mathbf{C}S_n \to \mathrm{End}\, V^{\otimes n})$ *and* $B := \mathrm{Im}(\mathcal{U}(\mathfrak{gl}(V)) \to \mathrm{End}\, V^{\otimes n})$ *are centralizers of each other in* $\mathrm{End}\, V^{\otimes n}$.
(2) *A and B are semisimple.*
(3) *As $(A \otimes B)$-modules,*

$$V^{\otimes n} = \bigoplus_{\text{partitions } \lambda \text{ of } n} V_\lambda \otimes L_\lambda$$

Where the L_λ are the distinct irreducible representations of $\mathfrak{gl}(V)$ *or* 0.

We want to do this for the Lie group $\mathrm{GL}(V)$ as well. We have:

Proposition 17.2. *Let* $B' = \mathrm{Im}(\mathbf{C}[\mathrm{GL}(V)] \to \mathrm{End}\, V^{\otimes n})$. *Then* $B' = B$.

Proof. Well B' is the span of $\{g^{\otimes n} | g \in \mathrm{GL}(V)\}$ and B is the span of $\{g^{\otimes n} | g \in \mathrm{End}(V)\}$. Since $\mathrm{GL}(V)$ is dense in $\mathrm{End}\, V$, B' is dense in B. Thus $B' = B$, because they're both subspaces in a finite-dimensional complex vector space. □

We therefore obtain:

Corollary 17.3 (Schur-Weyl duality for S_n and the Lie group $\mathrm{GL}(V)$)**.** *As $(S_n \times \mathrm{GL}(V))$-representations:*

$$V^{\otimes n} \simeq \bigoplus_{\text{partitions } \lambda \text{ of } n} V_\lambda \otimes L_\lambda$$

Where the L_λ are the distinct irreducible representations of $\mathrm{GL}(V)$ *or* 0.

17.1. **Schur polynomials.** Let $N \geq 0$. Let $\lambda = (\lambda_1, \cdots, \lambda_p)$ be a partition of n with $p \leq N$. Recall that we had the Vandermonde determinant:

$$\begin{aligned}
\Delta(x) &= \prod_{1 \leq i < j < N} (x_i - x_j) \\
&= \det \begin{pmatrix} x_1^{N-1} & x_1^{N-2} & \cdots & 1 \\ x_2^{N-1} & x_2^{N-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_N^{N-1} & x_N^{N-2} & \cdots & 1 \end{pmatrix} \\
&= \sum_{\sigma \in S_n} (-1)^\sigma x^{\sigma(\rho)}
\end{aligned}$$

Say that $\lambda_k = 0$ if $k > p$. Define:

$$\begin{aligned}
D_\lambda(x) &= \det \begin{pmatrix} x_1^{\lambda_1 + N-1} & x_1^{\lambda_2 N-2} & \cdots & x_1^{\lambda_N} \\ x_2^{\lambda_1 N-1} & x_2^{\lambda_2 N-2} & \cdots & x_2^{\lambda_N} \\ \vdots & \vdots & \ddots & \vdots \\ x_N^{\lambda_1 N-1} & x_N^{\lambda_2 N-2} & \cdots & x_N^{\lambda_N} \end{pmatrix} \\
&= \det(x_i^{\lambda_j + N - j})
\end{aligned}$$

Example 17.4. A dumb example, but $D_0(x) = \Delta(x)$.

Just like $\Delta(x)$, this determinant $D_\lambda(x)$ is antisymmetric, i.e., it changes sign if any x_i, x_j are interchanged with $i \neq j$. Thus $(x_i - x_j) | D_\lambda$ for all $i < j$. In particular, $\Delta | D_\lambda$.

Definition 17.5. The Schur polynomial $S_\lambda(x) := \frac{D_\lambda(x)}{D_0(x)} = \frac{D_\lambda(x)}{\Delta(x)}$. This is a symmetric function in $x_1, \cdots, x_N$ because if you interchange variables, then both the numerator and denominator change sign.

Let's do some computations. Notice that:

$$D_\lambda(1, z, z^2, \cdots, z^{N-1}) = \prod_{1 \leq i < j \leq N} (z^{\lambda_i + N - i} - z^{\lambda_j + N - j})$$

It's a Vandermonde determinant where you've replaced the determinant by something else. Therefore, we find that:

$$S_\lambda(1, z, z^2, \cdots, z^{N-1}) = \prod_{i<j} \frac{z^{\lambda_i - i} - z^{\lambda_j - j}}{z^{-i} - z^{-j}}$$

And, in particular, you can use L'Hopital's rule (bet you'd never thought that you'd use this again!):

$$\lim_{z\to 1} S_\lambda(1, z, z^2, \cdots, z^{N-1}) = S_\lambda(1, 1, \cdots, 1) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

We're going to use this compute the characters of L_λ. Let us prove the following preliminary result.

Lemma 17.6.

$$\Delta(x) \prod_{m\geq 1} (x_1^m + \cdots + x_N^m)^{i_m} = \sum_{\text{partitions } \lambda \text{ of } n \text{ such that } p\leq N} \chi_\lambda(C_i) D_\lambda(x)$$

Proof. Compare coefficients of x^ℓ for all $\ell = (\ell_1, \cdots, \ell_N)$. Both sides are antisymmetric, so it's enough to consider the case $\ell_1 > \ell_2 > \cdots > \ell_N$ (strict inequality because if they're equal then the coefficient is zero by antisymmetricity). Both sides of this equation are also homogeneous polynomials of degree $(N-1) + (N-2) + \cdots + 0 + n$, and the degree of $D_\lambda(x)$ is $(\lambda_1 + N - 1) + (\lambda_2 + N - 2) + \cdots + (\lambda_N) = \lambda_1 + \cdots + \lambda_N + (N-1) + (N-2) + \cdots + 0 = (N-1) + (N-2) + \cdots + 0 + n$. Without loss of generality, therefore $\ell_1 + \cdots + \ell_N = (N-1) + (N-2) + \cdots + 0 + n$. Let $\rho = (N-1, N-2, \cdots, 0)$. Then $\ell = \mu + \rho$ where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$, and thus $\mu_1 + \cdots + \mu_N = n$.

The coefficient of x^ℓ on the left hand side is $\chi_{V_\mu}(C_i)$. And the coefficient on the RHS is as follows. The only monomial in $D_\lambda(x)$ with strictly decreasing exponents are the terms where you take the diagonal term in the determinant, i.e., the one corresponding to the identity permutation. It's just $x^{\lambda+\rho}$. Thus the coefficient of x^ℓ in the RHS is $[x^{\lambda+\rho}] \sum \chi_{V_\lambda}(C_i) x^{\lambda+\rho} = \chi_{V_\mu}(C_i)$. □

By the way, this whole proof was a single line in the book.

Fix $n \in \mathbf{Z}_{\geq 0}$ and V be a $\mathbf{C}$-vector space. These determine V_λ and L_λ (as a representation of $\mathrm{GL}(V)$). Let $N = \dim V$. Define $S_\lambda(x)$ as before.

Theorem 17.7 (Weyl character formula)**.** *Let $\lambda = (\lambda_1, \cdots, \lambda_p)$. Then:*

(1) *$L_\lambda \neq 0$ if and only if $p < N$.*
(2) *If $p \leq N$, then "the character of L_λ is S_λ". More precisely, if $g \in \mathrm{GL}(V)$ has eigenvalues $x_1, \cdots, x_n$ (listed with multiplicity). Then:*

$$\chi_{L_\lambda}(g) = S_\lambda(x_1, \cdots, x_N)$$

And thus:

$$\dim L_\lambda = S_\lambda(1, 1, \cdots, 1) = \prod_{1\leq i<j\leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

Proof. Let $g \in \mathrm{GL}(V)$, ,and let $s \in S_n$. We'll compute the trace of $g \cdot s$ on each side of the Schur-Weyl formula(?). We know that $V^{\otimes n} \simeq \bigoplus_{\text{partitions } \lambda \text{ of } n} V_\lambda \otimes L_\lambda$. The left hand side:

$$\begin{aligned}
\mathrm{Tr}(g^{\otimes n} s|_{V^{\otimes n}}) &= \prod_{m \geq 1} \mathrm{Tr}(g^m)^{i_m} \\
&= \prod_{m \geq 1} (x_1^m + \cdots + x_N^m)^{i_m} \\
&= \sum_{\text{partitions } \lambda \text{ of } n \text{ such that } p \leq N} \chi_\lambda(C_i) S_\lambda(x)
\end{aligned}$$

The second line is homework. Actually, I don't think Etingof did this either, he said something like "This is an easy step". So for homework, you're going to complete this "easy step". The last equality comes from the lemma The right hand side:

$$\chi_{\bigoplus V_\lambda \otimes L_\lambda}(g^{\otimes n} s) = \sum_{\text{partitions } \lambda \text{ of } n} \chi_{V_\lambda}(s) \chi_{L_\lambda}(g)$$

These match for all i, and the columns of the character table are independent. So the only way you can have a linear combination of columns equaling a linear combination of columns means that the coefficients match up. So because you're considering partitions λ such that $p \leq N$, some of these are going be zero, which proves part 1. In particular:

$$\chi_{L_\lambda}(g) = \begin{cases} S_\lambda(x) & \text{if } p \leq N \\ 0 & \text{if } p > N \end{cases}$$

If $p \leq N$, then $\dim L_\lambda = \chi_{L_\lambda}(1) = S_\lambda(1, 1, \cdots, 1) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i} > 0$. Thus we get what we want. □

Are the L_λ are the finite-dimensional irreps of $\mathrm{GL}(V)$? The answer is no (to be explained sometime soon, hopefully). What are the other ones? We're gonna need some notation. When I write $\det V := \Lambda^N V$ where $N = \dim V$. Then $\dim \det V = \binom{N}{N} = 1$. The reason for this notation is (Scott Kominers explained this to me a few days ago, it's a rather cute thing) because if $g \in \mathrm{End}\, V$ induces an endomorphism of $\det V$, and this is multiplication by some scalar $\det g$. And this scalar is the determinant of g. Also, write 1^N for $(1, \cdots, 1)$.

Proposition 17.8. *We find that:*

$$L_{\lambda + 1^N} \simeq L_\lambda \oplus \det V$$

Proof. Note that the tensor of an irreducible with a 1-dimensional representation is always irreducible and it appears in $V^{\otimes n} \otimes V^{\otimes N} = V^{\otimes n+N}$. Thus by Jordan-Holder, $L_\lambda \otimes \det V \simeq L_\mu$ for some partition μ of $n + N$. We'll compare characters. We find that:

$$\chi_{L_\lambda \otimes \det V}(g) = S_\lambda(x) x_1 \cdots x_N = S_\mu(x) = \chi_{L_\mu}(g)$$

Both are ratios, so you can find that $D_\lambda(x) x_1 \cdots x_N = D_\mu(x)$. Compare the lexicographically largest monomial on both sides by expanding out the determinant to show that $\lambda + 1^N = \mu$. □

Definition 17.9. A function $f : \mathrm{GL}(V) \to \mathbf{C}$ is algebraic if $f(g)$ is a polynomial in the entries of g and g^{-1}, i.e., $f \in \mathbf{C}[g_{11}, \cdots, g_{nn}, \frac{1}{\det g}] = \mathcal{O}(\mathrm{GL}(V))$, which is the affine coordinate ring of $\mathrm{GL}(V)$.

Definition 17.10. A function $f : \mathrm{GL}(V) \to W \simeq \mathbf{C}^r$ is algebraic if each coordinate function is algebraic.

Definition 17.11. A representation Y of $\mathrm{GL}(V)$ is algebraic if and only if $\mathrm{GL}(V) \xrightarrow{\rho} \mathrm{GL}(Y) \subseteq \mathrm{End}(Y)$ is algebraic.

The direct sum, tensor product, dual representations, subrepresentations, quotients of algebraic reps are algebraic. Let $(\det V)^{-r} := ((\det V)^*)^{\times r}$. The character of L_λ is a polynomial $S_\lambda(x_1, \cdots, x_N)$. The character of $L_\lambda \otimes (\det V)^{-r}$ is $S_\lambda(x_1, \cdots, x_N)(x_1 \cdots x_N)^{-r}$, which isn't a polynomial if $r \gg 0$. Poonen wrote $r \gg 1$, and told a story about what Serre said. Don't worry about it. It just means that r is very big. So $L_\lambda \otimes (\det V)^{-r}$ is not necessarily one of the L_μ. We'll write $L_\lambda \otimes (\det V)^{-r} =: L_{\lambda - r \cdot 1^N}$. Now we have L_λ defined for any decreasing sequence of integers (not necessarily positive). Call λ the highest weight of L_λ.

Theorem 17.12. *Every finite-dimensional algebraic representation of* $\mathrm{GL}(V)$ *is a drect sum of irreducible representation, each of which is L_λ for a unique λ.*

Proof. If ϕ is an algebraic map from $\mathrm{End}(V)$ or $\mathrm{GL}(V)$ to some $\mathbf{C}$-vector space X, and $h \in \mathrm{GL}(V)$, define $h\phi$ via $(h\phi)(g) = \phi(gh)$. Let Y be a finite-dimensional algebraic representation of $\mathrm{GL}(V)$. Let Y_{triv} be Y with the trivial $\mathrm{GL}(V)$-action. We have $\mathrm{GL}(V) \times Y \to Y$, and if you fix $y \in Y$, you can view it as a map $\mathrm{GL}(V) \to Y$. In particular, $Y \hookrightarrow \mathrm{AlgMaps}(\mathrm{GL}(V), Y_{\mathrm{triv}})$ as representations (check this at home). And $\mathrm{AlgMaps}(\mathrm{GL}(V), Y_{\mathrm{triv}}) = \mathrm{AlgMaps}(\mathrm{GL}(V), \mathbf{C}) \otimes Y_{\mathrm{triv}} = \mathcal{O}(\mathrm{GL}(V)) \otimes Y_{\mathrm{triv}} \simeq \mathcal{O}(\mathrm{GL}(V)) \oplus \cdots \oplus \mathcal{O}(\mathrm{GL}(V))$. I need to show that $\mathcal{O}(\mathrm{GL}(V))$ is a direct sum of L_λs.

Let $\mathcal{O}(\mathrm{GL}(V))_{d,r} := \{\frac{q}{\det^r} | q \in \mathbf{C}[\{g_{ij}\}]$ homogeneous of degree $d\}$. Then $\bigoplus_{d,r} \mathcal{O}(\mathrm{GL}(V))_{d,r}$ surjects onto $\mathcal{O}(\mathrm{GL}(V))$. And:

$$\mathcal{O}(\mathrm{GL}(V)) \simeq \mathrm{Sym}^d((\mathrm{End}\, V)^*) \otimes (\det V)^{-r}$$

Where $\mathrm{End}(V)^*$ are the linear polynomials in the g_{ij}. And, $V \otimes V^*_{\mathrm{triv}} \simeq \mathrm{End}(V)^*$ via $v \otimes f \mapsto [\alpha \mapsto f(\alpha v)]$. Since V is a direct sum of L_λs (it's $L_{(1)}$), so are $E := (\mathrm{End}\, V)^* \simeq V \oplus \cdots \oplus V$, and thus $E^{\otimes d}$ will also be decomposed, and hence $\mathrm{Sym}^d(E)$, and thus $O(\mathrm{GL}(V))_{d,r}$ are direct sums of the L_λ, and so $O(\mathrm{GL}(V))$ is a direct sum, so Y will also be a direct sum. Alright, so we're done. □

18. The Peter-Weyl theorem, and representations of $\mathrm{GL}_2(\mathbf{F}_q)$ over $\mathbf{C}$

Recall:

Theorem 18.1. *Recall that $f \in \mathbf{C}[g_{11}, \cdots, g_{nn}, \frac{1}{\det g}] = O(\mathrm{GL}(V)) =: O$. Every finite-dimensional algebraic representation of $\mathrm{GL}(V)$ is a drect sum of irreducible representation, each of which is L_λ for a unique λ with $\lambda_i \in \mathbf{Z}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$.*

There's also the following theorem.

Theorem 18.2 (Peter-Weyl theorem for algebraic representations of $\mathrm{GL}(V)$)**.** *Let $\mathrm{GL}(V) \times \mathrm{GL}(V)$ act on O via $(g, h) \cdot \phi \mapsto x \mapsto \phi(g^{-1}xh)$. Then:*

$$O \simeq \bigoplus_\lambda L_\lambda^* \otimes L_\lambda$$

as $\mathrm{GL}(V) \times \mathrm{GL}(V)$-representations.

Proof. Consider O as a representation of the second $\mathrm{GL}(V)$. We know that O is a direct sum of copies of the L_λ. We just have to figure out how many. And we know how to do this, because $O = \bigoplus_\lambda \mathrm{Hom}_{\mathrm{GL}(V)}(L_\lambda, O) \otimes L_\lambda$.

In general, if X is any irreducible representation of $\mathrm{GL}(V)$, then there is an isomorphism $X^* \to \mathrm{Hom}_{\mathrm{GL}(V)}(X, O)$ by sending $f \mapsto [x \mapsto (g \mapsto f(gx))]$. You can go the other way $\mathrm{Hom}_{\mathrm{GL}(V)}(X, O) \to X^*$ by sending $h \mapsto [x \mapsto h(x)(1)]$. This is not only an isomorphism of vector spaces, but actually as a representation. So we are done. □

Remark 18.3. Explicitly, the isomorphism is given by $\bigoplus_\lambda L_\lambda^* \otimes L_\lambda \to O$ given by $\sum_\lambda f_\lambda \otimes x_\lambda \mapsto [g \mapsto f_\lambda(gx_\lambda)]$. This is a $\mathrm{GL}(V) \times \mathrm{GL}(V)$-representation isomorphism, as you should check.

18.1. $\mathbf{C}$-representations of $\mathrm{GL}_2(\mathbf{F}_q)$.

Given $g \in \mathrm{GL}_2(\mathbf{F}_q)$, let $P_g(x)$ be its characteristic polynomial.

Lemma 18.4. *Non-scalar conjugacy classes in $\mathrm{GL}_2(\mathbf{F}_q)$ are in bijection with polynomials of the form $x^2 - ax + b$ with $a \in \mathbf{F}_q$, $b \in \mathbf{F}_q^\times$ by sending $[g] \mapsto P_g$.*

The map in the other direction sends $x^2 - ax + b$ *to* $\begin{pmatrix} 0 & -b \\ 1 & a \end{pmatrix}$*, which is the matrix of multiplication of* x *on* $\mathbf{F}_q[x]/(x^2 - ax + b)$*.*

Proof. The composition in one direction is obvious. For the other direction, let g be a nonscalar 2×2-matrix with characteristic polynomial $x^2 - ax + b$. Then g isn't scalar, so there is some vector v that isn't an eigenvector. So v, gv are independent, and hence they're a basis. Then the matrix of g with respect to this basis is $\begin{pmatrix} 0 & ? \\ 1 & ? \end{pmatrix}$. And if you want to get the characteristic polynomial $x^2 - ax + b$, you must have $\begin{pmatrix} 0 & -b \\ 1 & a \end{pmatrix}$. Thus g is conjugate to this matrix, so we are done. □

Note that there is an embedding $i : \mathbf{F}_{q^2} \to \mathrm{End}_{\mathbf{F}_q}(\mathbf{F}_{q^2}) \simeq \mathrm{M}_2(\mathbf{F}_q)$ given by $\alpha \mapsto [x \mapsto \alpha x]$. Taking unit groups gives $\mathbf{F}_{q^2}^\times \to \mathrm{GL}_2(\mathbf{F}_q)$. If $\alpha \in \mathbf{F}_{q^2}^\times \setminus \mathbf{F}_q^\times$, then $i(\alpha)$ has characteristic polynomial equal to the minimal polynomial of $\alpha/\mathbf{F}_q$.

So, we get (where, in the third line, $a, b \in \mathbf{F}_q^\times$ and $a \neq b$):

$P_g(x)$	g is called	examples of g	# conj. class	$(Z_g, \#Z_g)$[a]
$(x-a)^2$	*scalar* if $g = \begin{pmatrix} a & \\ & a \end{pmatrix}$	$\begin{pmatrix} a & \\ & a \end{pmatrix}$	$(q-1)$	$(\mathrm{GL}_2(\mathbf{F}_q), (q^2-1)(q^2-q))$
	parabolic otherwise	$\begin{pmatrix} a & 1 \\ & a \end{pmatrix}$	$(q-1)$	$\left(\left\{\begin{pmatrix} c & d \\ & c \end{pmatrix}\right\}, q(q-1)\right)$
$(x-a)(x-b)$	*hyperbolic*	$\begin{pmatrix} a & \\ & b \end{pmatrix}$	$\binom{q-1}{2}$[b]	$\left(\left\{\begin{pmatrix} c & \\ & d \end{pmatrix}\right\}, (q-1)^2\right)$
irred$/\mathbf{F}_q$	elliptic	$i(\alpha)$	$\frac{q^2-q}{2}$[c]	$(i(\mathbf{F}_{q^2}^\times), q^2-1)$

[a]the centralizer.
[b]The number of conjugacy classes of hyperbolic elements is $\frac{(q-1)(q-2)}{2}$ because it's determined by a, b.
[c]The number of irreducible polynomials over $\mathbf{F}_q$ is determined by some $\mathbf{F}_{q^2} \setminus \mathbf{F}_q$ along with its Galois conjugate, which gives us $\frac{q^2-q}{2}$.

Counting this up shows that the number of conjugacy classes is $q^2 - 1$ which is the number of irreps we want.

Remark 18.5. The size of each conjugacy class going from top to bottom is $1, q^2 - 1, q(q+1), q(q-1)$.

We want to find irreps that are as small as possible, and since the dimension of the induced representation is $(\mathrm{GL}_2(\mathbf{F}_q) : H)$ times the dimension, we want to find H that has small index, i.e., it's big in $\mathrm{GL}_2(\mathbf{F}_q)$. Let's now find subgroups of $\mathrm{GL}_2(\mathbf{F}_q)$.

name	subgroups	order	index
duh	$\mathrm{SL}_2(\mathbf{F}_q)$	$q(q^2-1)$	$q-1$
Borel subgroup	$B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$	$q(q-1)^2$	$q+1$
Non-split Cartan subgroup	$i(\mathbf{F}_{q^2}^\times)$	q^2-1	q^2-q
	$H = \{\begin{pmatrix} a & b \\ & a \end{pmatrix}\}$	$q(q-1)$	q^2-1

Let's consider one-dimensional irreps first. What is the abelianization of $\mathrm{GL}_2(\mathbf{F}_q)$?

Lemma 18.6. *The commutator* $[\mathrm{GL}_2(\mathbf{F}_q), \mathrm{GL}_2(\mathbf{F}_q)] = \mathrm{SL}_2(\mathbf{F}_q)$.

Proof. One direction is that $\det(ghg^{-1}h^{-1}) = 1$. For the other direction, you can show that $[\mathrm{GL}_2(\mathbf{F}_q), \mathrm{GL}_2(\mathbf{F}_q)]$ contains $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$, where $a \in \mathbf{F}_q^\times$, and these generate $\mathrm{SL}_2(\mathbf{F}_q)$. □

Corollary 18.7. *We find that* $\mathrm{GL}_2(\mathbf{F}_q)^{ab} = \mathrm{GL}_2(\mathbf{F}_q)/[\mathrm{GL}_2(\mathbf{F}_q), \mathrm{GL}_2(\mathbf{F}_q)] \simeq \mathbf{F}_q^\times$.

So, the one-dimensional representations of $\mathrm{GL}_2(\mathbf{F}_q)$ are given as follows:

Proposition 18.8. *For every* $\xi : \mathbf{F}_q^\times \to \mathbf{C}^\times$*, define* $\mathbf{C}_\xi$ *as* $\mathbf{C}$ *with* $\rho(g) = \xi(\det(g))$*. This is one-dimensional and irreducible. There are* $\#\operatorname{Hom}(\mathbf{F}_q^\times, \mathbf{C}^\times) = (q-1)$ *such representations.*

The next kind of reps are called *principal series representations*. You can check that $B^\times$ is $\mathbf{F}_q^\times \times \mathbf{F}_q^\times$ (for q large enough?). Given $\lambda_1, \lambda_2 : \mathbf{F}_q^\times \to \mathbf{C}^\times$, define $B \to \mathbf{C}^\times$ via $\begin{pmatrix} a & b \\ & c \end{pmatrix} \mapsto \lambda_1(a)\lambda_2(c)$. This defines a 1-dimensional representation $\mathbf{C}_{\lambda_1,\lambda_2}$ of B. Define $V_{\lambda_1,\lambda_2} := \operatorname{Ind}_B^{\mathrm{GL}_2(\mathbf{F}_q)} \mathbf{C}_{\lambda_1,\lambda_2}$.

Proposition 18.9.
1. *If* $\lambda_1 \neq \lambda_2$*, then* V_{λ_1,λ_2} *is irreducible of dimension* $(\mathrm{GL}_2(\mathbf{F}_q) : B) \times 1 = q+1$.
2. *If* $\lambda_1 = \lambda_2 =: \lambda$*, then* $V_{\lambda,\lambda}$ *is* $\mathbf{C}_\lambda \oplus W_\lambda$ *where* $\mathbf{C}_\lambda$ *is as before, and* W_λ *is irreducible of dimension* q.
3. *These new irreducible representations are all distinct except that* $V_{\lambda_1,\lambda_2} \simeq V_{\lambda_2,\lambda_1}$.

Sketch of proof. First, $\mathbf{C}_\lambda$ appears in $V_{\lambda,\lambda}$, since $(\mathbf{C}_\lambda, V_{\lambda,\lambda}) = (\mathbf{C}_\lambda, \operatorname{Ind}_B^{\mathrm{GL}_2(\mathbf{F}_q)} \mathbf{C}_{\lambda,\lambda}) = (\operatorname{Res}_B^{\mathrm{GL}_2(\mathbf{F}_q)} \mathbf{C}_\lambda, \mathbf{C}_{\lambda,\lambda})$. The first one sends $\begin{pmatrix} a & b \\ & c \end{pmatrix} \mapsto \lambda(ac)$, and the second maps $\begin{pmatrix} * & * \\ & * \end{pmatrix} \mapsto \lambda(a)\lambda(c)$. And look, they're the same. So $(\operatorname{Res}_B^{\mathrm{GL}_2(\mathbf{F}_q)} \mathbf{C}_\lambda, \mathbf{C}_{\lambda,\lambda}) = 1$.

Thus $W_\lambda = V_{\lambda,\lambda} - \mathbf{C}_\lambda$ (thought of as a virtual representation) is an an actual representation.

Now, compute characters and check that $(V_{\lambda_1,\lambda_2}, V_{\lambda_1,\lambda_2}) = \begin{cases} 1 & \lambda_1 \neq \lambda_2 \\ 2 & \lambda_1 \neq \lambda_2 \end{cases}$, because $V_{\lambda,\lambda} = \mathbf{C}_\lambda \oplus W_\lambda$. This also shows that W_λ are also irreducible. Then check which irreducible characters are equal. □

We don't have quite enough yet.

Then we have the complementary series representations. The book obtains these by tensoring the known reps and studying how they decompose. But we are going to induce everything up. Let's identify $\mathbf{F}_{q^2}^\times$ with $i(\mathbf{F}_{q^2}^\times)$.

(1) Let $\theta : \mathbf{F}_{q^2}^\times \to \mathbf{C}^\times$. Define $K_\theta := \mathrm{Ind}_{\mathbf{F}_{q^2}^\times}^{\mathrm{GL}_2(\mathbf{F}_q)} \mathbf{C}_\theta$ of dimension $q^2 - q$. We can guess that this is not irreducible, it's rather huge.
(2) Fix a nontrivial $\rho : \mathbf{F}_q \to \mathbf{C}^\times$ where $\mathbf{F}_q$ is viewed as an additive group. Given $\eta : \mathbf{F}_q^\times : \mathbf{C}^\times$, define $H \to \mathbf{C}^\times$ by sending $\begin{pmatrix} a & b \\ & a \end{pmatrix} = \begin{pmatrix} a & \\ & a \end{pmatrix}\begin{pmatrix} 1 & b/a \\ & 1 \end{pmatrix}$ to $\eta(a)\rho(b/a)$. Let $J_\eta = \mathrm{Ind}_H^{\mathrm{GL}_2(\mathbf{F}_q)}[H \to \mathbf{C}^\times]$, of dimension $q^2 - 1$.
(3) There is the Frobenius endomorphism $\mathbf{F}_{q^2}^\times \to \mathbf{F}_{q^2}^\times$ via $x \mapsto x^q$. It's the nontrivial Galois automorphism of this field. When $\theta \neq \theta^q$ (i.e., it's not equal to its Galois conjugate), define $D_\theta := J_{\theta|_{\mathbf{F}_q^\times}} - K_\theta$. We know that it's a virtual representation of $\mathrm{GL}_2(\mathbf{F}_q)$ of dimension $q - 1$. At least it has positive dimension, so there's some hope that it's an actual representation.

Lemma 18.10. (1) *D_θ is an actual representation, and it's irreducible.*
(2) *$D_\theta \simeq D_{\theta'}$ if and only if $\theta' = \theta$ or $\theta' = \theta^q$.*

Sketch of proof. Compute characters, and check that $(D_\theta, D_\theta) = 1$, showing by a lemma we showed a while ago that D_θ is an actual representation, and that it's irreducible. The second statement also follows the characters. □

Corollary 18.11. *The number of distinct D_θ is $(q^2-1)-(q-1)$ where the (q^2-1) comes from the number of θ and the $(q-1)$ comes from the number of θ such that $\theta^{q-1} = 1$. And you divide by 2 because the θ are identified with their conjugate, i.e., θ and θ^q. So the number of distinct D_θ is $\frac{q^2-q}{2}$.*

Now you can count and find that you have $q^2 - 1$ irreducible representations, which was the number of conjugacy classes in $\mathrm{GL}_2(\mathbf{F}_q)$. So we are done. Maybe

I'll write down for you the character table.

Conj. class	$\begin{pmatrix} a & \\ & a \end{pmatrix}$	$\begin{pmatrix} a & 1 \\ & a \end{pmatrix}$	$\begin{pmatrix} a & \\ & b \end{pmatrix}$	$i(\alpha)$
$\mathbf{C}_\xi$	$\xi(a^2)$	$\xi(a^2)$	$\xi(ab)$	$\xi(\text{norm of } \alpha) = \xi(\alpha^{q+1})$[a]
V_{λ_1,λ_2}	$(q+1)\lambda_1(a)\lambda_2(a)$	$\lambda_1(a)\lambda_2(a)$	$\lambda_1(a)\lambda_2(a) + \lambda_1(b)\lambda_2(a)$	0[b]
W_λ	$q\lambda(a^2)$	0	$\lambda(ab)$	$-\lambda(\alpha^{q+1})$[c]
D_θ	$(q-1)\theta(a)$	$-\theta(a)$	0	$-\theta(a) - \theta(\alpha^q)$

[a]Because $N_{\mathbf{F}_{q^2}/\mathbf{F}_q}(\alpha) = \alpha \cdot \text{Galois conjugate}(\alpha) = \alpha \cdot \alpha^q$

[b]eigenvalues don't have values in $\mathbf{F}_q$ or something

[c]subtract

19. Artin's theorem, Dynkin diagrams

Let $\mathrm{Rep}(G)$ be the collection of virtual representations of G over $\mathbf{C}$, i.e., the collection of $\mathbf{Z}$-linear combinations of irreps. This is isomorphic to $\mathbf{Z}^{\#\text{conjugacy classes}}$. Then $\mathbf{Q} \otimes \mathrm{Rep}(G)$ is the collection of $\mathbf{Q}$-linear combinations of irreps, and similarly for $\mathbf{C} \otimes \mathrm{Rep}(G)$. If $H \subseteq G$ is a subgroup, you get two maps $\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ and $\mathrm{Res}_H^G : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$.

Theorem 19.1. *Let G be a finite group. Let $\mathcal{H}$ be a collection of subgroups of G, closed under conjugation (i.e., if $H \in \mathcal{H}$ and $g \in G$, then $gHg^{-1} \in \mathcal{H}$). Then the following are equivalent.*

1. *G is the union of the $H \in \mathcal{H}$.*
2. *Every finite-dimensional representation of G is a $\mathbf{Q}$-linear combination of $\{\mathrm{Ind}_H^G V | H \in \mathcal{H}, V \text{ an irrep of } H\}$.*

Proof. G is the union of the $H \in \mathcal{H}$ if and only if the following statement is true: a class function $f : G \to \mathbf{C}$ vanishes on every $H \in \mathcal{H}$, then $f = 0$. But this happens if and only if $\mathbf{C} \otimes \mathrm{Rep}(G) \xrightarrow{\mathbf{C}\otimes\bigoplus_{H\in\mathcal{H}} \mathrm{Res}_H^G} \bigoplus_{H\in\mathcal{H}} \mathbf{C} \otimes \mathrm{Rep}(H)$ is injective. By Frobenius reciprocity, this happens if and only if $\bigoplus_{H\in\mathcal{H}} \mathbf{C}\otimes\mathrm{Rep}(H) \xrightarrow{\mathbf{C}\otimes\bigoplus_{H\in\mathcal{H}} \mathrm{Ind}_H^G} \mathbf{C} \otimes \mathrm{Rep}(G)$ is surjective. Now this is a statement about the rank of a matrix – and this isn't dependent on the field itself. So this happens if and only if $\bigoplus_{H\in\mathcal{H}} \mathbf{Q} \otimes \mathrm{Rep}(H) \xrightarrow{\mathbf{C}\otimes\bigoplus_{H\in\mathcal{H}} \mathrm{Ind}_H^G} \mathbf{Q} \otimes \mathrm{Rep}(G)$ is surjective. This is equivalent to the second statement. This finishes the proof. □

Remark 19.2. Note that $\bigoplus_{H\in\mathcal{H}} \mathbf{C} \otimes \mathrm{Rep}(H) \xrightarrow{\mathbf{C}\otimes\bigoplus_{H\in\mathcal{H}} \mathrm{Ind}_H^G} \mathbf{C} \otimes \mathrm{Rep}(G)$ is surjective if and only if $\bigoplus_{H\in\mathcal{H}} k \otimes \mathrm{Rep}(H) \xrightarrow{\mathbf{C}\otimes\bigoplus_{H\in\mathcal{H}} \mathrm{Ind}_H^G} k \otimes \mathrm{Rep}(G)$ is surjective where $k \subseteq \mathbf{C}$ is a field of characteristic zero. But then you can say things like every representation of G is a $\mathbf{Q}(\sqrt{2})$-linear combination of $\{\mathrm{Ind}_H^G V$ such that $H \in \mathcal{H}, V$ an irrep of $H\}$. And this is weaker statement.

In particular, if $\mathcal{H}$ is the collection of cyclic subgroups of G, then the first condition is satisfied, so that we obtain:

Corollary 19.3. *Every (complex) representation of G is a $\mathbf{Q}$-linear combination of $\{\mathrm{Ind}_H^G V$ such that H a cyclic subgroup of G, V an irrep of $H\}$. Note that V is one-dimensional, so the representations are "easy" to build.*

I'm going to state without proof the following result.

Theorem 19.4 (Brauer's theorem)**.** *Every finite-dimensional representation of G is a $\mathbf{Z}$-linear combination of $\{\mathrm{Ind}_H^G V$ such that $H \leq G, V$ one-dimensional representations of $H\}$. You can actually choose the H to be the "p-elementary subgroups", that we won't talk about.*

Proof. Serre's book. □

Remark 19.5. This is used in number theory to prove that Artin L-functions are meromorphic. Maybe I'll talk about this in the end of the semester. It's actually conjectured that if you take the Artin L-function associated to a representation that isn't the trivial representation, then it's holomorphic. But I won't go into this now.

19.1. **Dynkin diagrams, and then quiver reps.** It's going to be impossible to livetex this.

Definition 19.6. Let Γ be a (finite undirected connected) graph (where we allow multiple edges between two vertices), with no self loops. Let r_{ij} be the number of edges between vertices i and j. For all i, j, we know that $r_{ij} \in \mathbf{Z}_{\geq 0}$. Clearly the data of all the r_{ij} gives the data of the graph. Note that $r_{ii} = 0$ and $r_{ij} = r_{ji}$. Define the adjacency matrix $R_\Gamma = (r_{ij}) \in \mathrm{M}_{|V(\Gamma)|}(\mathbf{Z})$ where $V(\Gamma)$ is the set of vertices of Γ.

Definition 19.7. Define the Cartan matrix as $A = A_\Gamma := 2I - R_\Gamma$.

These are both symmetric matrices with entries in $\mathbf{Z}$. For $x, y \in \mathbf{R}^n$, define $B(x, y) = B_\Gamma(x, y) = x^T Ay$. This is a symmetric bilinear form. You can consider $B(x, x)$, which is a quadratic form.

Example 19.8. Suppose Γ has three vertices (i, j, k), with $r_{ij} = 2$, $r_{jk} = 1$, and $r_{ik} = 0$. Then $R_\Gamma = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. This means that $B(x, x) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 4x_1x_2 - 2x_2x_3$.

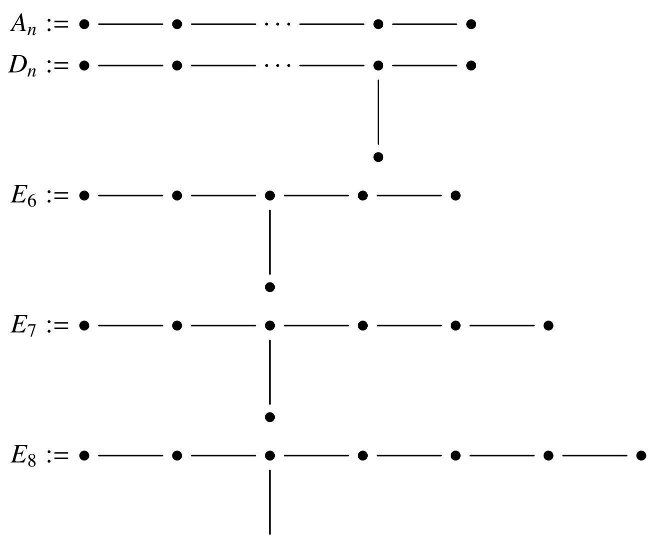
Definition 19.9. Say that Γ is a (simply laced) Dynkin diagram if A is positive definite, i.e., $B(x, y)$ is positive definite, i.e., $B(x, x) > 0$ for $x \neq 0$.

Remark 19.10. Given a matrix A, you can test if it's positive definite by the following result: a matrix if positive definite iff all principal minors are positive definite, i.e., det(the upper left $m \times m$-minor of A) > 0 for $m = 1, 2, \cdots, |V(\Gamma)|$.

Example 19.11. Our example with $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ is not positive definite because $\det \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = 0$.

Definition 19.12. Say that Γ is an affine (simply laced) Dynkin diagram if A is positive semidefinite, i.e., $B(x, y)$ is positive semidefinite and not positive definite, i.e., $B(x, x) \geq 0$ for $x \neq 0$ and equality holds for some nonzero x. This is equivalent to saying that det(the upper left $m \times m$-minor of A) ≥ 0 for $m = 1, 2, \cdots, |V(\Gamma)|$, with equality for some m.

Theorem 19.13. Γ *is a Dynkin diagram if and only if* Γ *is isomorphic to one of the following.*



The last three are called exceptional Dynkin diagrams. See p. 154 of the textbook.

Proof. Your homework. □

Let $k = \overline{k}$. Let Q be a connected finite quiver. Let $\Gamma_Q = \Gamma$ be the graph obtained by forgetting the arrow directions. We want to classify the representations of the quiver Q.

Definition 19.14. Q is of finite type if it has only finitely many indecomposable representations over k (up to isomorphism of course).

Theorem 19.15 (Gabriel's theorem)**.** *Q is of finite type if and only if Γ is a Dynkin diagram.*

Proof. Q being of finite type implies Γ is a Dynkin diagram is homework. The other direction we'll do later. In fact, when Γ is a Dynkin diagram, we'll classify all indecomposables. $\square$

Example 19.16. Suppose Q is $\bullet \to \bullet$. Then a rep of Q is $V \xrightarrow{A} W$. But this is the direct sum of $\ker A \to 0$, $V/\ker A \to \operatorname{Im} A$, and $0 \to W/\operatorname{Im} A$. The first is a multiple of the irrep $\mathbf{C} \to 0$, the second is a multiple of $\mathbf{C} \xrightarrow{\sim} \mathbf{C}$, and the third is a multiple of $0 \to \mathbf{C}$. Thus Q is of finite type, and this is exactly what you want from Gabriel's theorem since $\Gamma_Q = A_2$.

Example 19.17. Suppose Q is of the form:

$$\begin{tikzcd}
& \bullet \arrow[d] & \\
\bullet \arrow[r] & \bullet & \bullet \arrow[l] \\
& \bullet \arrow[u] &
\end{tikzcd}$$

This is not a Dynkin diagram. I'm going to show you that Q is not of finite type. Let $\dim W = 2$ and $\dim V_1 = \cdots = \dim V_4 = 1$ where the V_i are subspaces of W. Then I can consider:

$$\begin{tikzcd}
& V_2 \arrow[d, hook] & \\
V_1 \arrow[r, hook] & W & V_3 \arrow[l, hook'] \\
& V_4 \arrow[u, hook] &
\end{tikzcd}$$

There is an identification of W with k^2, which essentially amount to choosing a basis, that is unique up to an overall scalar such that $V_1 = k \cdot (1, 0)$, $V_2 = k \cdot (0, 1)$,

$V_3 = k \cdot (1,1)$, and $V_4 = k \cdot (1,\lambda)$ for some $\lambda \in \{0,1\}$. For each $\lambda \in \{0,1\}$, this gives an indecomposable representation X_λ of Q, and the isomorphism type of X_λ determines λ.

19.2. **Roots.** Forget quivers for now, and consider a Dynkin diagram Γ. Thus $B(x,x)$ is positive definite. If $x \in \mathbf{Z}^n$, then $B(x,x) = x^T(2I - R_\Gamma)x \in 2\mathbf{Z}_{\geq 0}$ because R_Γ is symmetric and you can pair things up. The smallest it could be is 2.

Definition 19.18. A root is a vector $x \in \mathbf{Z}^n$ such that $B(x,x) = 2$. A simple root is a standard basis vector $\alpha_i := (0, \cdots, 1, \cdots, 0)$ for some i.

Exercise 19.19. α_i really is a root. This is easy because the diagonal entries of R_Γ are zero, and if you didn't have R_Γ, then $x^T(2I - R_\Gamma)x = 2\sum_i^n x_i^2$.

Lemma 19.20. *If $\alpha = \sum k_i\alpha_i$ is a root, then either $k_i \geq 0$ for all i or $k_i \leq 0$ for all i. That is, you can't have positive and negative coefficients.*

Proof. Suppose not; say $k_i > 0$ and $k_j < 0$ where j is as close to i as possible. Suppose i and j are vertices of Γ. Remove the edge next to i (connecting i to some i') that's closer to j. This gives two graphs Γ_1, Γ_2. Let $\beta = \sum_{m\in\Gamma_1} k_m\alpha_m$ and $\gamma = \sum_{m\in\Gamma_2} k_m\alpha_m$. Since $k_i > 0$, $\beta \neq 0$, so $B(\beta,\beta) \geq 2$ and $B(\gamma,\gamma) \geq 2$. Also, $B(\beta,\gamma) = B(k_i\alpha_i, k_{i'}\alpha_{i'}) = -k_ik_{i'} \geq 0$ because $k_i, k_{i'}$ don't have the same sign. Thus $2 = B(\alpha,\alpha) = B(\beta,\beta) + B(\gamma,\gamma) + 2B(\beta,\gamma\gamma) \geq 2 + 2 + 0 = 4$. This is a contradiction. $\square$

Remark 19.21. I do not understand this proof at all.

20. Dynkin diagrams, roots

I came ridiculously late. We recalled a few things about roots.

Let $\Gamma = A_2$. Define $\left(\mathbf{R}^2 \text{ with } B\right) \hookrightarrow \left(\mathbf{R}^3 \text{ with standard inner product}\right)$. Then $\alpha_1 \mapsto e_1 - e_2$ and $\alpha_2 \mapsto e_2 - e_3$. Check that $B(\alpha_1,\alpha_1) = B(\alpha_2,\alpha_2) = 2$ and $B(\alpha_1,\alpha_2) = -1$.

For $\Gamma = D_n$, you can do a similar thing, by sending $\alpha_i \mapsto e_i - e_{i+1}$ for $1 \leq i \leq n-1$, and $\alpha_n = e_{n-1} + e_n$. And then $\mathbf{Z}^n \leftrightarrow \{x \in \mathbf{Z}^n | \sum x_i \text{ is even}\}$, $\{\text{roots}\} \leftrightarrow \{\pm e_i \pm e_j | 1 \leq i < j \leq n\}$, and $\{\text{simple roots}\} \leftrightarrow \{e_i \pm e_j | 1 \leq i < j \leq n\}$.

Lemma 20.1. *Let α be a root. The reflection in the hyperplane $\alpha^\perp$ is given by $s_\alpha(x) = x - B(x,\alpha)\alpha$.*

Proof. If $x \in \alpha^\perp$, then $s_\alpha(x) = x - 0\alpha = x$. If $x \in \alpha$, then $s_\alpha(\alpha) = \alpha - 2\alpha = -\alpha$. $\square$

Definition 20.2. Define $s_i = s_{\alpha_i}$ for $i = 1, \cdots, n$ where the α_i are the simple roots.

Definition 20.3. The Weyl Group $W = W(\Gamma) :=$ subgroup of $O(\mathbf{R}^n, B)$ generated by $s_1, \cdots, s_n$.

Lemma 20.4. *W is finite.*

Proof. There is a homomorphism $W \to \mathrm{Aut}\{\text{roots}\}$. And the number of roots is finite (because you have lattice vectors on a bounded region). This homomorphism is claimed to be injective, because the roots span $\mathbf{R}^n$ (an element of the Weyl group is determined by how it acts on the roots). □

Lemma 20.5. *Let $v \in \mathbf{R}^n$. If $s_1 \cdots s_m$ fixes v (the composition of distinct reflections), then each s_i fixes v.*

Proof. Applying s_i changes at most the ith coordinate. So this means that each s_i fixes v. □

Definition 20.6. The Coxeter element $c := s_1 \cdots s_n \in W \subseteq O(\mathbf{R}^n, B) \subseteq \mathrm{GL}_n(\mathbf{R})$.

Implicit in this definition is that I fixed an ordering of the vertices in the Dynkin diagram, because that's what gave us the roots (because W is generally noncommutative).

Corollary 20.7. *If $cv = v$, then $v = 0$.*

Proof. Suppose $cv = v$, then each s_i fixes v. On the other hand, $s_i v = v - B(v, \alpha_i)\alpha_i$. Thus $B(v, \alpha_i) = 0$ for all i. But the α_i form a basis and B is nondegenerate because it's positive definite. This tells you that $v = 0$. □

Corollary 20.8. *The matrix $c - 1$ is invertible.*

Corollary 20.9. *There exists $M \in \mathbf{Z}_{>0}$ such that $c^{M-1} + \cdots + c + 1 = 0$ in $\mathrm{M}_n(\mathbf{R})$.*

Proof. Since W is a finite group, $c^M = 1$ for some $M \in \mathbf{Z}_{>0}$. Then $(c-1)(c^{M-1} + \cdots + 1) = 0$, but $(c-1)$ is invertible, so you're done. □

Lemma 20.10 (Lemma A)**.** *Let $\beta \in \mathbf{R}^N$ be nonzero. Then there exists $N \in \mathbf{Z}_{\geq 0}$ such that $c^N\beta$ has at least one negative coordinate.*

Proof. Well, $c^{M-1}\beta + \cdots + c\beta + \beta = 0$ in $\mathbf{R}^n$. If all of them had nonnegative coordinates, then all have to be zero if they add up to zero. That's ridiculous, because $\beta \neq 0$. □

Lemma 20.11 (Lemma B)**.** *Every positive root α is of the form $s_n s_{n-1} \cdots s_1 s_n s_{n-1} \cdots s_{q+1} s_q \alpha_i$ (possibly wrapping around multiple times, where we stop at some q) for some i.*

Proof. This is equivalent to saying that $\alpha_i = s_q \cdots s_{n-1} s_n \alpha$ for some q. In the sequence $r_0 = \alpha$, $r_1 := s_n\alpha$, ..., $r_i = s_{n-i+1} s_{n-i+2} \cdots s_n \alpha$, we'll eventually get $s_1 \cdots s_n \alpha = c\alpha$, and then $c^2\alpha$, etc. Every r_i is a root because every reflection preserves length, and hence roots. Eventually, some r_j has a negative coordinate.

This follows from Lemma A. Choose the smallest such j. I had to apply some reflection s_i to get r_j from r_{j-1}. By minimality of j, r_{j-1} is a positive root. How can this happen? Well, on the other hand, what does s_i do to a vector? It changes only one coordinate. So s_i changes the ith coordinate. And what this means for us is that all the other ith coordinates of r_{j-1} are zero. Thus $r_{j-1} \in \mathbf{R}_{>0} \cdot \alpha_i$. It's also a root, so we know its length, which is 2. This means that $r_{j-1} = \alpha_i$. □

Now let's go back to quivers.

20.1. **Quivers.** Let Q be a finite quiver (no self-loops). Let Γ be the associated undirected graph, labeled by $1, \cdots, n$. If V is a representation of Q, you can define $\dim(V) := (\dim V_1, \cdots, \dim V_n) \in \mathbf{Z}^n$.

Example 20.12. Let i be one vertex of Q. Let $k_{(i)}$ be the representation with k at vertex i and zero everywhere else. Then $\dim k_{(i)} = (0, \cdots, 1, \cdots, 0) = \alpha_i$.

Theorem 20.13 (Gabriel's theorem (improved version))**.**
(1) *If Γ is a Dynkin diagram, then $\{$indecomposables of $Q\}/iso$ is in bijection with $\{$positive roots$\}$ given by sending $V \mapsto \dim(V)$.*
(2) *If Γ is not a Dynkin diagram, then $\{$indecomposables of $Q\}/iso$ is infinite.*

Proof. Second part is homework.

The strategy of the proof is as follows. Define operations on indecomposable representations that correspond to simple reflections on vectors in $\mathbf{Z}^n$. We can't finish the proof today, but we'll start.

Fix a quiver Q and a vertex i. Assume that Γ is a Dynkin diagram. Suppose that i is a *sink*. This means that all arrows to i point inward. I can obtain $\overline{Q_i}$ by flipping all arrows that go into i while leaving everything else alone. Suppose V is a representation of this quiver. I can combine all the incoming maps into one, to get $\phi : \bigoplus_{j \to i} V_j \to V_i$.

Definition 20.14. V is *surjective at i* if ϕ is surjective.

In general, write $V_i = \operatorname{Im}\phi \oplus Z$ where Z is some vector space complement of $\operatorname{Im}\phi \subseteq V_i$. Then V is the direct sum of the representation where I replace V_i by $\operatorname{Im}\phi$ and leave everything else unchanged, and the representation where I replace V_i by Z and change everything else to 0. The first subrepresentation is surjective at i, and the second is the direct sum of $\dim Z$ copies of $k_{(i)}$. As a corollary, if V is indecomposable, then either V is surjective at i or $V \simeq k_{(i)}$.

Now I'm ready to tell you what the analogue of simple reflections are. Define the reflection functor (also known as the Coxeter functor) as follows. Let $\operatorname{Rep}(Q)$ be the category of all representations of Q. The reflection functor is a functor

$\operatorname{Rep} Q \xrightarrow{F_i^+} \operatorname{Rep} \overline{Q_i}$ by sending V (where i is a sink) to the representation of $\overline{Q_i}$ given by leaving everything unchanged, except where you replace V_i with $\ker \phi$. What are the maps in this representation? Most of the maps are the same, and the only thing that changes are the maps at i. The maps $\ker \phi \to V_j$ are defined as follows. $\ker \phi \hookrightarrow \bigoplus_{j \to i} V_j \xrightarrow{\mathrm{pr}_{V_j}} V_j$. Now i is no longer a sink, it's a source.

I'll stop here, but we'll continue next week. □

21. Finishing Gabriel's theorem, and compact groups

Proof, continued. We note that the quiver representation $k_{(i)}$ is sent via F_i^+ to the zero representation of $\overline{Q_i}$.

Proposition 21.1. *Suppose that V is surjective at i. If F_i^+ maps V to W, then s_i maps* $\dim(V)$ *to* $\dim(W)$.

Proof. Most of the dimensions stay the same, and the only place where the dimension could possibly change is at the vertex i. This is also what s_i does. Recall that $s_i(x) = x - B(x, \alpha_i)\alpha_i$. This means that we just have to look at position i. The ith coordinate of $\dim(W)$ is $\dim W_i = \dim \ker \phi$. It is important that $\ker \phi$ is surjective, so that $\dim \ker \phi = \left(\sum_{j \to i} \dim V_j\right) - \dim \operatorname{Im} \phi = \left(\sum_{j \to i} \dim V_j\right) - \dim V_i$. What we really want to know if $\dim W_i - \dim V_i$? This is going to be $\left(\sum_{j \to i} \dim V_j\right) - 2 \dim V_i$. This has entries exactly the same as $A_\Gamma = 2I - R$ and $a_{ij} = 2\delta_{ij} - r_{ij}$, so this becomes $-\sum_j a_{ij} \dim V_j = -B(\dim(V), \alpha_i)$. The last equality is what you get when you expand the definition. What we conclude is that $\dim W - \dim V = -B(\dim(V), \alpha_i)\alpha_i$, so we're done. □

Suppose instead that i is a "source" – exactly what you get when you reverse the arrows in the definition of a sink. Let ψ be the map $V_i \to \bigoplus_{j \leftarrow i} V_j$. We can say that V is injective at i is ψ is injective. Recall that we showed that if V is indecomposable, then either V is surjective at i or $V \simeq k_{(i)}$. You have an analogue of this for these representations as well. You also get a functor $\operatorname{Rep} Q \xrightarrow{F_i^-} \operatorname{Rep} \overline{Q_i}$ by sending V where i is a source to the representation of $\overline{Q_i}$ given by leaving everything unchanged, except where you replace V_i with $\operatorname{coker} \phi$. What are the maps in this representation? Most of the maps are the same, and the only thing that changes are the maps at i. You send $V_j \hookrightarrow \bigoplus_{j \leftarrow i} V_j \to \operatorname{coker} \phi$.

There's an analogous proposition to what we just did above. I'll just skip it because the statement and proof are pretty much the same. Now I'm out of boards. But I want everything. Let me just erase the board with the proposition.

Here's the question. Suppose I reflect twice. What happens then to the representation?

Proposition 21.2. *Let i be a sink. Let V be a representation of Q that's surjective at i. Then:*

(1) *$F_i^+ V$ is injective at i.*
(2) *$F_i^- F_i^+ V \simeq V$.*
(3) *If $V \neq 0$, then $F_i^+ V \neq 0$.*

Proof. Recall that $\phi : \bigoplus_{j \to i} V_j \to V_i$ is surjective, so that you can fit this into an exact sequence:

$$0 \to K = \ker \phi \xrightarrow{\psi} \bigoplus_{j \to i} V_i \xrightarrow{\phi} V_i \to 0$$

Now I can draw what's going to happen. It is obvious that $F_i^+ V$ is injective at i. Now, if I reflect a second time, I get coker(ψ). But this isomorphic to V_i. So we are done with the second part as well. Also the last part is obvious because if $V \neq 0$ and $F_i^+ V = 0$, then $F_i^- F_i^+ V = 0$, so $V \simeq 0$, contradiction. □

There's an analogue:

Proposition 21.3. *Let i be a source. Let V be a representation of Q that's injective at i. Then:*

(1) *$F_i^- V$ is surjective at i.*
(2) *$F_i^+ F_i^- V \simeq V$.*
(3) *If $V \neq 0$, then $F_i^- V \neq 0$.*

Another proposition.

Proposition 21.4. *Let i be a sink, and let V be an indecomposable representation. If $V \simeq k_{(i)}$, then $F_i^+ V = 0$. If V is surjective at i, we'll prove that $F_i^+ V$ is indecomposable. If $V \simeq k_{(i)}$, then* $\dim V = \alpha_i$*. If V is surjective at i, then s_i* $\dim(V)$ *has nonnegative coordinates.*

Let's call the statements about $V \simeq k_{(i)}$ the "red box" statements and the statements about V when it's surjective at i the "blue box" statement. Then: the statement in each box are equivalent to each other, and exactly one box is true for V.

Proof. If $V \simeq k_{(i)}$, then $F_i^+ V = 0$ and $F_i^+ V$ is not indecomposable. Also $\dim V = \alpha_i$, so $s_i \dim V = -\alpha_i$. Note that not all coordinates are nonnegative.

Suppose V is surjective at i. Then $F_i^+ V$ is nonzero, and $s_i \dim(V) = \dim(F_i^+ V)$, which obviously has nonnegative coordinates. I still have to prove that $F_i^+ V$ is indecomposable. Suppose it wasn't indecomposable, so that $F_i^+ V = W_1 \oplus W_2$.

Then F_i^- yields V on the LHS. But F_i^- preserves direct sums, as you should show, so $F_i^- W_1 \oplus F_i^- W_2$. Since $F_i^+ V$ is injective at i, W_1, W_2 are also injective at i, which implies by the above proposition, that $F_i^- W_1$ and $F_i^- W_2$ are nonzero. This contradicts indecomposability of V. □

Lemma 21.5. *Suppose* Γ *is Dynkin. One can label the vertices of* Q *as* $1, 2, \cdots, n$ *so that whenever* $i \to j$*,* $i < j$*.*

Proof. Start at some vertex of Q. Since there are no cycles, follow the arrows until you get stuck. If you didn't get stuck, then you'd have a cycle. So following arrows eventually leads to a sink, which we'll label n. Remove that vertex n, and repeat what's left, and label the sink you end up hitting $(n-1)$. □

Remark 21.6. This holds for any finite tree.

Now we're really ready to prove Gabriel's theorem. Fix such a labeling of the quiver. Let V be an indecomposable rep of Q, and define $V^{(0)} := V$. We have its dimension vector $\dim V$. Define $V^{(1)} := F_n^+ V$, which is a representation of $\overline{Q_n}$. After I do this, n is no longer a sink. This means that in $\overline{Q_n}$, $(n-1)$ is a sink. So we can reflect to get $V^{(2)} := F_{n-1}^+ V^{(1)}$ which is a representation of $\overline{(\overline{Q_n})_{n-1}}$. This is rather horrible notation. You can continue to reflect until you hit $V^{(n)} := F_1^+ V_{(n-1)}$, which is a representation of Q. And it's a representation of Q because each edge is flipped twice – and each edge has two vertices, which appears twice in our process of iterating F_k^+. Now, our representation $V^{(n)}$ might be different from V – in fact, it often will.

We can still consider $F_n^+ V^{(n)}$, etc. We can continue to do this as long we end up in the "red box", i.e., until $V^{(q)}$ is $k_{(\ell)}$. Reflecting once more gives you zero. On the roots side, doing all these reflections gives us $\dim V^{(q)} = \alpha_\ell$. We *know* that we end up with some such $V^{(q)} \simeq k_{(\ell)}$ because of Lemma A that we proved last time (use CTRL+F here). What we've therefore found is that $\alpha_\ell = s_{\ell-1} \cdots s_{n-1} s_n \dim(V)$. So $\dim V = s_n s_{n-1} \cdots s_{\ell-1} \alpha_\ell$. Since reflections preserve length, it follows that $\dim V$ is a positive root.

We've shown one direction. Suppose α is a positive root. Then $\alpha = s_n s_{n-1} \cdots s_1 s_n s_{n-1} \cdots s_{q+1} s_q \alpha_i$. You get an indecomposable via $F_n^- F_{n-1}^- \cdots F_{q+1}^- F_q^- k_{(i)}$ for every α. Because we showed that $F_i^- F_i^+ V \simeq V$, we are done (the map is surjective, as we've just shown, and $F_i^- F_i^+ V \simeq V$ implies injectivity). □

For the rest of the semester, we're going to study the representation theory of compact groups.

21.1. **Representations of compact groups.** I'm going to presume knowledge of topology, functional analysis, and measure theory. If you don't know anything about this, talk to me. I refer you to Serre, another one by Sepanski (about compact Lie groups), and another one by Deitman and Echterhoff.

Definition 21.7. A topological group is a group G with a (Hausdorff) topology such that $G \times G \to G$ and $G \xrightarrow{\_^{-1}} G$ are continuous.

Remark 21.8. Does anyone know of an example where $G \times G \to G$ is continuous but $G \to G$ (inverse) isn't? I don't.

Definition 21.9. A compact group is a topological group such that G is compact.

I was gonna give a bunch of examples, but I guess I'll do that next time.

22. Compact groups

Serre's book has a two-page summary of the basics of compact groups. Sepanski covers all of that in full detail, plus a little more. Recall:

Definition 22.1. A topological group is a group G with a (Hausdorff) topology such that $G \times G \to G$ and $G \xrightarrow{\_^{-1}} G$ are continuous.

Example 22.2. Recall I asked for an example where $G \times G \to G$ is continuous but $G \to G$ (inverse) isn't. One example of this is $\mathbf{A}_k^\times \subseteq \mathbf{A}_k$ where $\mathbf{A}_k^\times$ is the idele group and $\mathbf{A}_k$ is the adele ring, where $\mathbf{A}_k^\times$ is endowed with the subspace topology.

Definition 22.3. A compact group is a topological group such that G is compact.

If $z = a + bi + cj + dk$, define $\bar{z} = a - bi - cj - dk$. Define $\mathrm{M}_n(\mathbb{H}) :=$ End($\mathbb{H}^n$ as a right $\mathbb{H}$-module), so the matrices are acting on the left on $\mathbb{H}^n$. Define $\mathrm{GL}_n(\mathbb{H}) :=$ Aut($\mathbb{H}^n$ as a right $\mathbb{H}$-module). For $v, w \in \mathbf{R}^n, \mathbf{C}^n, \mathbb{H}^n$, define:

$$(v, w) := \begin{cases} \sum v_i w_i & \text{inner product} \\ \sum \bar{v}_i w_i & \text{Hermitian inner product} \\ \sum \bar{v}_i w_i & \text{symplectic/Hermitian/quaternionic inner product} \end{cases}$$

A unit vector is v such that $(v, v) = 1$, and {unit vectors} is a sphere, hence compact. If $A \in \mathrm{M}_n(\mathbf{R}, \mathbf{C}, \mathbb{H})$, then define the adjoint A^* as the unique matrix such that $(Av, w) = (v, A^* w)$ for all v, w.

Example 22.4 (Of compact groups)**.**
(1) Compact Lie groups (Sepanski, §1.1.4):
   (a) Under addition, $\mathbf{R}/\mathbf{Z}$, which is isomorphic to S^1 via the exponential map. Also, you have tori $(\mathbf{R}/\mathbf{Z})^n$.

   (b) Orthogonal group $O(n) := \{g \in GL_n(\mathbf{R}) : (gv, gw) = (v, w)\} = \{g \in GL_n(\mathbf{R}) : g^* = g^{-1}\}$ of matrices that preserve length.
   (c) Unitary group $U(n) := \{g \in GL_n(\mathbf{C}) : (gv, gw) = (v, w)\} = \{g \in GL_n(\mathbf{C}) : g^* = g^{-1}\}$.
   (d) Compact symplectic group $Sp(n) := \{g \in GL_n(\mathbb{H}) : (gv, gw) = (v, w)\} = \{g \in GL_n(\mathbb{H}) : g^* = g^{-1}\}$. This is not the same thing as the value of the algebraic group Sp_{2n} on $\mathbf{R}$ or $\mathbf{C}$. There is some relation, but we won't worry about that.
   (e) These are all compact groups because $g^*g = 1$ imples that each column of g is a unit vector. Therefore $\{g|g^*g = 1\}$ is a closed subset of a product of n spheres, hence it's compact.
   (f) Special orthogonal group $SO(n) := \ker(O(n) \xrightarrow{\det} \{\pm 1\})$. This is the connected component of the identity in $O(n)$, because it can be thought of as having two components, namely of matrices whose determinant is ± 1.
   (g) Special unitary group $SU(n) := \ker(U(n) \xrightarrow{\det} S^1)$.
   (h) There's no special compact symplectic real group. These are all real Lie groups, called the classical compact Lie groups. There are others, associated with G_2, F_4, E_6, E_7, E_8 that we won't talk about. Also finite covers, products, quotients by finite central subgroups. These are actually all the compact Lie groups. (There's some overlap – for example, $U(1) = S^1$ – there are a couple of other little accidents.)
(2) Compact groups that aren't manifolds that also exist. Any finite group with the discrete topology works.
(3) Profinite groups:

   Definition 22.5. A profinite group is an inverse limit $G = \varprojlim G_i$ of finite groups, where WLOG the maps $G_j \to G_i$ are surjective.

   Some examples are:
   (a) The p-adics $\mathbf{Z}_p := \varprojlim \mathbf{Z}/p^k\mathbf{Z}$ where $\mathbf{Z}/p^k\mathbf{Z} \to \mathbf{Z}/p^{k-1}\mathbf{Z}$ is reduction modulo p. An element of $\mathbf{Z}_p$ is giving one element of each group that are compatible, i.e., an element of $\mathbf{Z}_p$ is $(a_i)_{i>0}$ such that $a_i \in \mathbf{Z}/p^i\mathbf{Z}$ such that $a_j \bmod p^i = a_i$ in $\mathbf{Z}/p^i\mathbf{Z}$ for all $j \geq i$. This is like an integer in base p but you have an infinite number of digits.
   (b) The profinite completion $\widehat{\mathbf{Z}} := \varprojlim \mathbf{Z}/n\mathbf{Z}$ with respect to maps $\mathbf{Z}/N\mathbf{Z} \to \mathbf{Z}/M\mathbf{Z}$ where $M|N$. You're taking the projective limit here over some partially ordered set. It follows from the Chinese remainder theorem that $\widehat{\mathbf{Z}} \simeq \prod_p \mathbf{Z}_p$.

(c) $\mathrm{GL}_2(\mathbf{Z}_p) := \{\text{matrices whose determinant is in } \mathbf{Z}_p^\times\}$. This is the same as $\varprojlim \mathrm{GL}_2(\mathbf{Z}/p^n\mathbf{Z})$. This holds for GL_n as well.

(d) For any field k, if k_s is a separable closure, then $k_s = \bigcup_{\text{f. Galois ext. } k_s \supseteq L/k} L.$

Then $\mathrm{Gal}(k_s/k) = \varprojlim \mathrm{Gal}(L/k)$ with restriction maps $\mathrm{Gal}(M/k) \to \mathrm{Gal}(L/k)$ by $\alpha \mapsto \alpha|_L$ whenever $k \subseteq L \subseteq M \subseteq k_s$.

What's the topology? For finite groups, the topology is the stupidest one. There are two equivalent descriptions.

(a) Give each G_i the discrete topology. Give $\prod G_i$ the product topology – so far, this is compact (by Tychonoff). Then $\varprojlim G_i \subseteq \prod G_i$ is a closed subset of $\prod G_i$ (because it's cut out by equations), hence is compact in the subspace topology. (By the way, if you haven't studied topology just come to office hours.)

(b) Let $\pi_i : G \to G_i$ be the projection. Then the collection $\{\pi_i^{-1}(g_i) | i \in \mathcal{I}, g_i \in G_i\}$ is a basis of open sets for the topology on G. In fact, profinite groups are totally disconnected because points are closed in each G_i.

Remark 22.6. An open subgroup of topological group is closed, because if $H \subseteq G$ is an open subgroup, then the cosets of H are also open (because translating is continuous) – but its complement is the union of all its cosets, so H must be closed. If G is compact, then the cosets of H (along with H) form a cover, hence there are finitely many cosets.

Let $C(G) := \{\text{continuous functions } G \to \mathbf{C}\}$.

Theorem 22.7 (There exists a unique Haar measure)**.** *Let G be a compact group. Then there exists a unique linear functional $C(G) \to \mathbf{C}$ sending $f \mapsto \int_G f$ (also written $\int_G f(g)dg$) such that:*

(1) *If $f \geq 0$ (real-valued with nonnegative values), then $\int_G f \geq 0$.*

(2) *"$\int_G$ is right invariant": for any $f \in C(G)$ and $h \in G$:*

$$\int_G f(g)dg = \int_G f(gh)dg$$

i.e., if ${}^h f$ is defined as ${}^h f(g) = f(gh)$, then $\int_G f = \int_G^h f$.

(3) *"G has volume 1". This means that $\int_G 1 = 1$.*

Remark 22.8. Haar is the German word for hair. It's also the name of a mathematician. I don't know why I just told you that.

I'm not going to prove this, there's not much time left in the semester. If you know measure theory, then the Riesz representation theorem says that there exists a measure, called the Haar measure.

Proof. Deitmar, App. B. □

Lemma 22.9. *$\int_G$ is also left invariant.*

Proof. If $j \in G$, then $f \mapsto \int_G f(jg)dg$ is a new integration (i.e. it satisfies the three conditions), hence by uniqueness it equals $\int_G f(g)dg$. □

There's a generalization with locally compact groups. Then sometimes this lemma can fail.

Example 22.10.
(1) $G = \mathbf{R}/\mathbf{Z}$, where dg comes from the Lebesgue measure on $\mathbf{R}$.
(2) If G is a finite group, each element has $1/\#G$. Thus $\int_G f = \frac{1}{\#G}\sum_{g\in G} f(g)$. In general, this integral is like an averaging operator.
(3) If G is a profinite group, and if $H \subseteq G$ is an open subgroup, then $(G : H)$ is finite, and thus each coset of H (including H itself) has measure $1/(G : H)$. If f is constant on each coset, i.e., constant mod H, then $\int_G f = \frac{1}{(G:H)}\sum_{g\in G/H} f(g)$.

You can define:

$$L^2(G) := \left\{ f : G \to \mathbf{C} : \int_G |f|^2 < \infty \right\} / \{f : f \text{ is 0 outside a measure zero subset}\}$$

This contains $C(G)$. If $f, g \in L^2(G)$, you can define $(f, g) := \int_G f\overline{g}$.

22.1. **Finite-dimensional representations of compact groups.** References are Serre, Chapter 4, and Sepanski, Chapter 2. Let G be a compact group.

Definition 22.11. A finite-dimensional representation of G is a finite-dimensional complex vector space V equipped with a continuous homomorphism $G \to \mathrm{GL}(V)$, i.e., with a continuous action $G \times V \to V$.

Example 22.12. $G = \mathbf{R}/\mathbf{Z}$. The irreducible finite-dimensional representations are 1-dimensional, indexed by $n \in \mathbf{Z}$: these are just $\chi_n : \mathbf{R}/\mathbf{Z} \to \mathbf{C}^\times$ via $t \mapsto e^{2\pi int}$. The claim is that these are all of them.

The great thing is that pretty much everything you know about representations of finite groups is true for representations of compact groups. The proofs are different, though. Replacing $\frac{1}{\#G}\sum_{g\in G}$ by $\int_G$ extends the following theorems from representations of finite G to compact G (the proofs are also exactly the same):

(1) Finite-dimensional $\mathbf{C}$-representations of G are semisimple.
(2) On any finite-dimensional representation V, there exists a G-invariant Hermitian form (unique up to a positive scalar if V is irreducible).
(3) Schur: for any finite-dimensional irreps, $\mathrm{Hom}_G(V, W) \simeq \begin{cases} \mathbf{C} & V \simeq W \\ 0 & \text{else} \end{cases}$.
(4) For $V = \bigoplus m_i V_i$ where $V_1, \cdots, V_r$ are finite-dimensional irreps of G, and $W = \bigoplus n_i V_i$, where $m_i, n_i \in \mathbf{Z}_{\geq 0}$, then $(\chi_V, \chi_W) = \sum m_i n_i$ (we called this the orthogonality of characters, I).
(5) In particular, $(\chi_V, \chi_V) = 1$ if and only if V is irreducible (and $\chi_V(1) > 0$ if V is only virtual *a priori*).
(6) Orthogonality of matrix entries (called orthogonality III, I think).
(7) Finite-dimensional representations of $G_1 \times G_2$ are exactly tensors of finite-dimensional representations of G_1 and G_2.
(8) Definitions of the dual, complex conjugate representations exist, and $V^* \simeq \overline{V}$. Also, tensor powers, direct sums, Homs, symmetric powers, exterior powers, virtual representations, Res_H^G (if $H \subseteq G$ is a *closed* subgroup, so that it's also compact), and $\mathrm{Ind}_H^G V$ (for now, if H is a closed subgroup of finite index, i.e., H is an open subgroup of G).
(9) The character formula for Ind_H^G works, as does Frobenius reciprocity.
(10) Results on finite-dimensional real representations translate, and the Frobenius-Schur indicator is now $\int_G \chi_V(g^2) dg$.

There, that was a review of the whole course. Some things fail, though. (number of irreps = number of conjugacy classes, and $\mathbf{C}G \simeq \prod \mathrm{End}\, V_i$ for finite G) Also, what about infinite-dimensional representations? We'll talk about Hilbert spaces. Read Deitmar, Chapter 2 if you haven't seen Hilbert spaces before. See office hours on website, they've shifted. Some questions: we know that most things pass on to compact groups, but how about:

(1) the set of irreducible characters is an orthonormal basis for class functions (and the number of irreps is the number of conjugacy classes)? The answer turns out to be OK if it is interpreted as the Hilbert space $L^2(G)^G$ (fixed points of the conjugation action of G on $L^2(G)$). If you do this for finite groups, you would get the class functions.
(2) $\mathbf{C}G \simeq \prod \mathrm{End}\, V_i$ over all the irreps V_i of G (and $\#G = \sum (\dim V_i)^2$)? The answer for this is the Peter-Weyl theorem for compact groups, which says that $L^2(G) \simeq \widehat{\bigoplus} V_i^* \otimes V_i$ as $G \times G$-representations over all irreps V_i. As one big representation of G, $L^2(G) \simeq \widehat{\bigoplus} \dim(V_i) V_i$.

(3) What about infinite-dimensional **C**-representations? If you think of Hilbert spaces on which G acts and unitary representations, then each infinite-dimensional representation is a $\widehat{\bigoplus}$ of finite-dimensional irreps.

22.2. **General definitions for infinite-dimensional representations.** See Sepanski §3.2.

Definition 22.13. A **C**-vector space V with a (Hausdorff) topology is a *topological vector space* if $V \times V \xrightarrow{+} V$ and $\mathbf{C} \times V \to V$ are continuous where **C** has the Euclidean topology.

If we didn't assume Hausdorff, we could just quotient out by the closure of 0, and this'll give something that's Hausdorff.

From now on, G is a topological group.

Definition 22.14. A representation of G is a topological vector space V with an action of G such that $G \times V \to V$ is continouous. A G-homomorphism between representations V and W of G is a continuous map $\phi : V \to W$ such that $\phi(gv) = g\phi(v)$ for all $g \in G$ and $v \in V$.

Remark 22.15. You could define a representation as $G \to \mathrm{GL}(V)$, but you have to do something bizarre for the topology on $\mathrm{GL}(V)$.

Definition 22.16. A subrepresentation of a representation V is a closed subspace $W \subseteq V$ such that W is a G-representation.

Definition 22.17. V is irreducible if it's nonzero, and there's no nontrivial subrepresentation.

This is actually a little too general for what we want. A Banach space is a topological vector space whose topology comes from a metric, that's actually a norm, such that it's complete. We will consider *Hilbert spaces*:

Definition 22.18. A Hilbert space is a **C**-vector space V equipped with a Hermitian form $(-,-) : V \times V \to \mathbf{C}$ such that V is complete with respect to the associated norm $||v|| := \sqrt{(v,v)}$. You can check that this makes a Hilbert space into a metric space.

It's the best kind of infinite-dimensional space.

Example 22.19. $L^2(G)$ along with our norm defined before. More generally, $L^2(X,\mu)$ for any measure space (X,μ). The nontrivial part is proving completeness.

Definition 22.20. $(v_i)_{i\in I}$ is an *orthonormal basis* of V if $(v_i, v_j) = \delta_{ij}$, and if every $x \in V$ is uniquely expressible as $\sum_{i\in I} c_i v_i$ for $c_i \in \mathbf{C}$ such that $\sum_{i\in I} |c_i|^2 < \infty$ (possibly infinite).

Strictly speaking it's not a basis if you forget the topology.

Theorem 22.21. *Every Hilbert space has an orthonormal basis*

Example 22.22. It will turn out that $(e^{2\pi int})_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R}/\mathbf{Z})$; i.e., every $f(t) \in L^2(\mathbf{R}/\mathbf{Z})$ is uniquely expressible as $\sum_{n\in\mathbf{Z}} a_n e^{2\pi int}$ for $a_n \in \mathbf{C}$ such that $\sum |a_n|^2 < \infty$. This is the main theorem of Fourier analysis, so it's not that easy – but it'll follow from what we do later regarding compact groups.

Definition 22.23. Let $(V_i)_{i\in I}$ be Hilbert spaces. Let $\widehat{\bigoplus} V_i$ be the completion (as a metric space) of $\bigoplus V_i$, i.e., $\widehat{\bigoplus} V_i = \{\sum_{i\in I} v_i : \sum |v_i|^2 < \infty\}$, with $(\sum v_i, \sum w_i) = \sum (v_i, w_i)$. You can check that this works.

22.3. **Operators.** See Deitmar-Echterhoff, Chapter 5.

Definition 22.24. Let $T : V \to W$ be a continuous $\mathbf{C}$-linear map of Hilbert spaces. Then there exists a unique $T^* : W \to V$ such that $(Tv, w) = (v, T^* w)$. Say that $T : V \to V$ is self-adjoint if $T^* = T$. It's called normal if $TT^* = T^*T$.

Definition 22.25. Let V be a Hilbert space, and let $T : V \to V$ be a linear operator ($\mathbf{C}$-linear map). The operator norm $\|T\| := \sup_{v\in V, v\neq 0} \dfrac{|Tv|}{|v|} \in \mathbf{R}_{\geq 0} \cup \{\infty\}$.

Definition 22.26. Using this, we have (in decreasing order of inclusions):

(1) Bounded operators, which means that $\|T\| < \infty$. This is actually equivalent to the condition that T is continuous.
(2) Compact operators, which means that $\overline{T(\text{unit ball})}$ is compact – the unit ball in a Hilbert space is not compact, so it squashes things. This is equivalent to saying that T is a limit of (a Cauchy sequence of) finite-rank operators where the limit is taken with respect to the operator norm.
(3) Finite-rank operators, which says that $T(V)$ is finite-dimensional. (So the compact operators are the closure of finite-rank operators in the space of all bounded operators.)

Example 22.27. Say V has an orthonormal basis (v_i) and $Tv_i = \alpha_i v_i$ for some $\alpha_i \in \mathbf{C}$. Then T is bounded if the α_i are bounded, i.e., there exists C such that $|\alpha_i| \leq C$ for all i. Also, T is compact if $\alpha_i \to 0$. Clearly T is finite-rank if all but finitely many α_i are zero.

Recall:

Theorem 22.28 (Spectral theorem for normal operators)**.** *Suppose $V = \mathbf{C}^n$. Let $T : V \to V$ be a normal operator. For each eigenvalue λ, let $V_\lambda = \ker(T - \lambda I)$ be the λ-eigenspace of T on V. Then:*

(1) *$V \simeq \bigoplus V_\lambda$. So V has an orthonormal basis consisting of eigenvectors.*
(2) *Let $E_\lambda \in \operatorname{End} V$ be the orthogonal projection onto V_λ. Then $T = \sum_{\text{eigenvalues } \lambda} \lambda E_\lambda$. With respect to a suitable orthonormal basis, there'll be a diagonal block in the matrix of E_λ that's completely the identity, and all other diagonal blocks will be zero. So if $S \subseteq \{\text{eigenvalues}\}$, then $E(S) := \sum_{\lambda \in S} E_\lambda$ is another orthogonal projection.*
(3) *If $U \in \operatorname{End}(V)$ commutes with T, then U commutes with every E_λ, i.e., U can have any matrix on diagonal blocks of the appropriate size.*

Now for Hilbert spaces:

Theorem 22.29 (Spectral theorem for normal compact operators, Deitmar-Echterhoff, 5.2)**.** *Suppose V be a Hilbert space. Let $T : V \to V$ be a normal compact operator. For each eigenvalue λ, let $V_\lambda = \ker(T - \lambda I)$ be the λ-eigenspace of T on V. Then:*

(1) *$\{\text{eigenvalues}\}$ is either finite or is a countable sequence converging to 0.*
(2) *$V \simeq \widehat{\bigoplus} V_\lambda$, and each eigenspace is finite-dimensional for each nonzero λ. So V has an orthonormal basis consisting of eigenvectors.*
(3) *Let $E_\lambda \in \operatorname{End} V$ be the orthogonal projection onto V_λ. Then $T = \sum_{\text{eigenvalues } \lambda} \lambda E_\lambda$. With respect to a suitable orthonormal basis, there'll be a diagonal block in the matrix of E_λ that's completely the identity, and all other diagonal blocks will be zero. So if $S \subseteq \{\text{eigenvalues}\}$, then $E(S) := \sum_{\lambda \in S} E_\lambda$ is another orthogonal projection.*
(4) *If U is a bounded operator that commutes with T, then U commutes with every E_λ, i.e., U can have any matrix on diagonal blocks of the appropriate size.*

There is a more general version that works for bounded operators:

Example 22.30. Suppose $V = \{(a_n)_{n \in \mathbf{Z}} : a_n \in \mathbf{C}, \sum |a_n|^2 < \infty\}$. This is a Hilbert space, and has an orthonormal basis from the characteristic functions. Let $T : V \to V$ be the shift operator (this is bounded – it preserves the norm). Then T has no nonzero eigenvectors.

Theorem 22.31 (Spectral theorem for normal bounded operators)**.** *Suppose V be a Hilbert space. Let $T : V \to V$ be a normal bounded operator. Then:*

(1) *There exists a projection-valued measure $E : \mathcal{B}(\mathbf{C}) \to \{$orthonormal projection $V \to V\}$ (where $\mathcal{B}(\mathbf{C})$ is the Borel σ-algebra on $\mathbf{C}$), such that $T = \int_{\mathbf{C}} \lambda dE := \lim \sum_{\text{finite}} \lambda_i E(\Delta_i)$ for $\mathbf{C} = \coprod \Delta_i$ and $\lambda_i \in \Delta_i$ where the Δ_i are Borel sets.*

(2) *If a bounded operator $\mathcal{U} : V \to V$ commutes with T, then $\mathcal{U}$ commutes with $E(\Delta)$ for every Borel set $\Delta \subseteq \mathbf{C}$.*

This is the fanciest spectral theorem. We're going to use this to study representations. We're going to upgrade Schur's lemma – if you recall the proof, we know that $T : V \to V$ would map $V_\lambda \to V_\lambda$ – which makes $V = V_\lambda$, so $T = \lambda I$. This won't work anymore! There aren't any eigenspaces. We have to be more clever.

23. Unitary representations and Schur's lemma

Definition 23.1. A representation of a topological group G on a Hilbert space V is *unitary* if $(,)$ is G-invariant, i.e., $(gv, gw) = (v, w)$.

Remark 23.2. If a compact group G acts on a Hilbert space V and $(,)$ is not G-invariant, we can replace it by its average over G to make it G-invariant.

23.1. Schur's lemma.

Lemma 23.3. *Let G be a topological group and let V, W be irreducible unitary representations (implicit is that they're Hilbert spaces). Then:*

$$\mathrm{Hom}_G(V, W) = \begin{cases} \mathbf{C} & \text{if } V = W \\ 0 & \text{if } V \neq W \end{cases}$$

Proof. Let $T \in \mathrm{Hom}_G(V, W)$. Suppose $T \neq 0$. Then:

(1) T is injective, because $\ker T$ is a closed subspace of V, but V is irreducible,
(2) Also, T^* is injective because $T^* : W^* = W \to V^* = V$. Same argument works.
(3) If $S \in \mathrm{Hom}_G(V, V)$ is normal, then $S \in \mathbf{C}$.

   Proof. By the spectral theorem for normal bounded operators, $S = \lim \sum \lambda_i E(\Delta_i)$. For all $g \in G$, we know that $g|_V$ commutes with S because S is G-invariant. Thus $g|_V$ commutes with each $E(\Delta)$. Thus $E(\Delta)$ is also a G-invariant map, i.e., $E(\Delta) \in \mathrm{Hom}_G(V, V)$. This is weird because $E(\Delta)$ is a projection onto some G-invariant subspace. Thus, $E(\Delta)$ is 0 or 1. This means that $\sum \lambda_i E(\Delta_i) \in \mathbf{C}$. And $S = \lim \sum \lambda_i E(\Delta_i)$, so $S \in \mathbf{C}$. □

(4) We didn't know in advance that T was normal. Then $T^*T = c$ for some $c \in \mathbf{C}^\times$.

Proof. Apply (3) to $T^*T : V \to V$. This is a normal operator, because $(T^*T)^* = T^*(T^*)^* = T^*T$. This scalar is nonzero because T^*T is injective. □

(5) $TT^* = d$ for some $d \in \mathbf{C}^\times$ by the same argument.
(6) $T : V \to W$ is an isomorphism.

Proof. T has left inverse $c^{-1}T^*$, and has right inverse $d^{-1}T^*$. So it's invertible and hence is an isomorphism. □

We're done with the second part (when $V \not\simeq W$) by contrapositive.

Now suppose $V = W$. Write $T = A + iB$ where $A^* = A$ and $B^* = B$ (namely let $A = (T + T^*)/2$ and $B = (T - T^*)/2i$). By (3), $A, B \in \mathbf{C}$, so we're done. □

23.2. **Ingredients for the next proof.**

(1) We defined $\int_G f$ for continuous $f : G \to \mathbf{C}$. We can also define $\int_G F(g)dg$ for continuous $F : G \to \{\text{bounded operators } V \to V\}$ where the latter thing has topology coming from the metric defined via the operator norm. How do you actually do this? You write $\int_G F(g)dg = \lim \sum F(g_i)\mu(\Delta_i)$ where $\mu(\Delta_i)$ is the volume of Δ_i, i.e., $\int_{\Delta_i} 1$ where $G = \coprod_{\text{finite}} \Delta_i$ such that Δ_i are Borel subsets and $g_i \in \Delta_i$.
(2)

Definition 23.4. $T : V \to V$ is *positive* if $(Tv, v) \geq 0$ for all v and $(Tv, v) > 0$ for some v.

Example 23.5. If $p : V \to V$ is an orthogonal projection onto a nonzero closed subspace W then p is positive. To see this, write $V = W \oplus W^\perp$. Then $(p(w + w'), w + w') = (w, w + w') = (w, w) \geq 0$ because this is a positive-definite inner product.

Theorem 23.6. *Let G be a compact group. Let V be a unitary representation of G. Then $V \simeq \widehat{\bigoplus} V_i$ for some finite-dimensional irreps V_i.*

Proof.

Lemma 23.7 (Key lemma)**.** *If $V \neq 0$, then there exists a nonzero finite-dimensional irreducible subrepresentation W.*

Proof. Let X be any finite-dimensional subspace. It's probably not G-invariant. Let $p : V \to V$ be the orthogonal projection (hence self-adjoint). If $g \in G$, then define ${}^g p$ as before, so it's an orthogonal projection onto gX. Let $Q = \int_G {}^g p$. Thus Q is self-adjoint, G-invariant, and compact. The image of Q is my G-invariant subspace, but this isn't exactly right. It could be zero – or infinite-dimensional!

Claim 23.8. $Q \neq 0$.

Proof. To see this, $(Qv, v) = \int_G ({}^g pv, v) \geq 0$ for all v. Also, if v is such that $(pv, v) > 0$ (exists because p is positive), then because $({}^g pv, v) > 0$ when $g = 1$, by continuity, there exists $\epsilon > 0$ and an open neighborhood U of 1 such that $({}^g pv, v) \geq \epsilon$. Thus $(Qv, v) = \int_G ({}^g pv, v) \geq \epsilon \cdot \mathrm{vol}(U)$. This is > 0 by homework. This finishes the proof of the claim. □

Now apply the spectral theorem for compact operators. This implies that Q has a nonzero eigenvalue λ and the λ-eigenspace W_λ of Q is finite-dimensional. Since Q is G-invariant, so if W_λ. Thus W_λ is a finite-dimensional subrepresentation. Choose any irrep $W \subset W_\lambda$, so we're done. □

Corollary 23.9. *Any irreducible unitary representation of a compact group G must be finite-dimensional.*

Now we'll finish the proof of the theorem. This is similar to the proof that any vector space has a basis (Zorn's lemma). Consider a collection $\{V_i\}_{i \in I}$ of pairwise orthogonal finite-dimensional irreducible representations that you've obtained by peeling off things from V. Any collection of collections ordered by inclusion has an upper bound (the union). That's exactly what you need to apply Zorn's lemma, which in this case says that there exists a maximal collection $\{V_i\}_{i \in I}$. Let $W = \widehat{\bigoplus_{i \in I}} W_i \subset V$.

Claim 23.10. $W = V$.

Proof. If not, then $V = W \oplus W^\perp$ where $W^\perp \neq 0$. The key lemma tells you that $W^\perp$ has a finite-dimensional irreducible subrepresentations, which contradicts maximality. □

□

Let X be a finite-dimensional irreducible representation of G.

Definition 23.11. The X-isotypic omponent of V (call it V_X) is the closure of the sum of all subrepresentations of V isomorphic to X.

Proposition 23.12. *If $V = \widehat{\bigoplus} V_i$ with V_i irreducible, then $V_X = \widehat{\bigoplus_{V_i \simeq X}} V_i$.*

Proof. Clearly V_X contains each V_i that's isomorphic to X, and V_X is closed, so $V_X \supseteq \widehat{\bigoplus}_{V_i \simeq X} V_i$. For the other inclusion, we have $V = \left(\widehat{\bigoplus}_{i:V_i\simeq X} V_i\right) \oplus \left(\widehat{\bigoplus}_{j:V_j\not\simeq X} V_j\right)$. Where could I have a copy of X in $\widehat{\bigoplus}_{j:V_j\not\simeq X} V_j$? Each copy of X in V must project to zero in each $V_j \not\simeq X$. This is because of Schur's lemma between nonisomorphic representations. Thus X projects to zero in $\widehat{\bigoplus}_{j:V_j\not\simeq X} V_j$, so $X \subseteq \widehat{\bigoplus}_{i:V_i\simeq X} V_i$, i.e., $\widehat{\bigoplus}_{i:V_i\simeq X} V_i \supseteq V_X$. □

Corollary 23.13. $V \simeq \widehat{\bigoplus}_X V_X$.

Proposition 23.14. $\mathrm{Hom}_G(X, V) \otimes X \simeq V_X$ *by the evaluation map.*

24. Towards Peter-Weyl

Proposition 24.1. $\mathrm{Hom}_G(X, V) \otimes X \simeq V_X$ *by the evaluation map where* $\mathrm{Hom}_G(X, V)$ *is (for now) just a complex vector space.*

Proof. Write $V = \widehat{\bigoplus}_{i\in I} X \oplus \widehat{\bigoplus}_{j\in J, V_j\not\simeq X} V_j$. There are projections $\pi_i : \widehat{\bigoplus}_{i\in I} X \to X$ and $\pi_j : \widehat{\bigoplus}_{j\in J,V_j\not\simeq X} V_j \to V_j$. Any vector $v \in V$ is determined by $\pi_i(v)$ and $\pi_j(v)$. So $\phi \in \mathrm{Hom}_G(X, V)$ is determined by the composites $X \xrightarrow{\phi} V \xrightarrow{\pi_i} X$ and $X \xrightarrow{\phi} V \xrightarrow{\pi_j} V_j$. Maps in the latter collection must be zero, and maps in the former collection is multiplication by some scalar $c_i \in \mathbf{C}$. Thus it suffices to specify how ϕ maps to V_X. Conversely, $(c_i)_{i\in I}$ gives a well-defined map $X \xrightarrow{x\mapsto(c_i x)_{i\in I}} V_X \subseteq V$ if and only if $\sum_{i\in I} |c_i|^2 < \infty$. Thus $\mathrm{Hom}_G(X, V) \simeq \widehat{\bigoplus}_{i\in I} \mathbf{C}$. Then $\mathrm{Hom}_G(X, V) \otimes X \simeq \widehat{\bigoplus}_{i\in I} \mathbf{C} \widehat{\otimes} X \simeq \widehat{\bigoplus}_{i\in I} \mathbf{C} \otimes X$ where the latter isomorphism holds since X is finite-dimensional[7]. Then this is exactly $\widehat{\bigoplus}_{i\in I} X = V_X$ as desired. □

Corollary 24.2 (Isotypic decomposition)**.** $V \simeq \widehat{\bigoplus}_X \mathrm{Hom}_G(X, V) \otimes X$.

Remark 24.3. For $S, T \in \mathrm{Hom}_G(X, V)$, define $(S, T) := S^*T \in \mathrm{Hom}_G(X, X) = \mathbf{C}$. You can check that it's a Hermitian inner product. If $\phi \in \mathrm{Hom}_G(X, V)$ corresponds to $(c_i)_{i\in I}$ under $\mathrm{Hom}_G(X, V) \simeq \widehat{\bigoplus}_{i\in I} \mathbf{C}$, then $(\phi, \phi) = \sum_{i\in I} |c_i|^2 < \infty$. Thus the isomorphisms in the above proof are isometric, i.e., they preserve the Hermitian forms.

[7]so you can reduce to the case $X = \mathbf{C}$, and then you just have $\widehat{\bigoplus}_{i\in I} \mathbf{C} \widehat{\otimes} \mathbf{C}$ which is just $\widehat{\bigoplus}_{i\in I} \mathbf{C}$, so that $\widehat{\bigoplus}_{i\in I} \mathbf{C} \widehat{\otimes} X$ is just a finite direct sum of Hilbert spaces – which is already complete.

24.1. **Getting to Peter-Weyl.** Peter-Weyl is about the decomposition of $L^2(G)$, which is analogous to the decomposition of $\mathbf{C}G$. Another one'll come up, so we'll start numbering from (2).

(2) $C(G) := \{\text{continuous } f : G \to \mathbf{C}\}$. This is a Banach space with respect to $|f|_\infty := \sup_{x\in G} |f(x)|$; you know this is finite because G is compact (always assumed throughout the lecture). In particular it's complete with respect to the $|\cdot|_\infty$ norm. The topology on $C(G)$ is defined via the $|\cdot|_\infty$-norm.

(3) This sits inside:

$$L^2(G) := \{f : G \to \mathbf{C} \text{ such that } \int |f|^2 < \infty\}/\text{functions} = 0 \text{ outside a measure zero subset}$$

This is a Hilbert space with respect to the $|\cdot|_2$-norm, i.e., $|f|_2 = \sqrt{(f,f)} = \sqrt{\int |f|^2}$ where $(f,g) := \int f\overline{g}$. In fact, this is complete with respect to this norm, so that $L^2(G)$ is the completion of $C(G)$ with respect to the $|\cdot|_2$-norm $|\cdot|_2$. The topology on $L^2(G)$ is defined via the $|\cdot|_2$-norm.

I want to make these into representations. Let $G\times G$ act on $C(G)$ as follows: an element $(g,h) \in C(G)$ maps $f(x) \in C(G)$ to $f(g^{-1}xh) = {}^{(g,h)}f$.

Proposition 24.4. *This makes $C(G)$ a continuous representation of $G\times G$.*

Proof. It is a $\mathbf{C}$-linear left action (easy – "I am just too lazy to do it"). Continuity? We need: $\sup_{x\in G} \left|f_1(g_1^{-1}xh_1) - f_2(g_2^{-1}xh_2)\right|$ is small when g_1 is close to g_2, h_1 is close to h_2, and f_1 is close to f_2. The idea is that $f_1(g_1^{-1}xh_1) \doteq f_1(g_2^{-1}xh_1)$ (there's a dot on the equality sign, watch out for that below) where the $|\cdot|_\infty$-error is uniformly small as x varies because f_1 is uniformly continuous (it's uniformly continuous because continuity = uniform continuity on a compact space), so $f_1(g_1^{-1}xh_1) \doteq f_1(g_2^{-1}xh_1) \doteq f_1(g_2^{-1}xh_2) \doteq f_2(g_2^{-1}xh_2)$ where the last $\doteq$ holds because the error is uniformly bounded by $|f_1 - f_2|_\infty$. You can make this formal through ϵ-δ stuff. "I'm sliding the details under the carpet – but we don't have a carpet and you might be able to see through the details." $\square$

Proposition 24.5. *The same definition of the action on $C(G)$ also makes $L^2(G)$ a continuous unitary representation of $G\times G$.*

Proof. It is a $\mathbf{C}$-linear left action (easy). Now use the $|\cdot|_2$-norm to measure the error. Given (g_1,h_1,f_1) close to (g_2,h_2,f_2), choose $f\in C(G)$ such that $|f-f_2|_2$ is small (possible since $C(G)$ is $|\cdot|_2$-dense in $L^2(G)$). Then ${}^{(g_1,h_1)}f_1 \doteq {}^{(g_1,h_1)}f_2$ because $|{}^{(g_1,h_1)}f_2 - {}^{(g_1,h_1)}f_1| = |f_2 - f_1|$, which we can assume is arbitrarily small. Now, ${}^{(g_1,h_1)}f_2 \doteq {}^{(g_1,h_1)}f$ because $|{}^{(g_1,h_1)}f_2 - f| = |f_2 - f|$, which we can also assume is arbitrarily small. Next, ${}^{(g_1,h_1)}f \doteq {}^{(g_2,h_2)}f$ because $|\cdot|_2 \le |\cdot|_\infty$ which is small (use our argument for $C(G)$). And by similar arguments we have the chain ${}^{(g_2,h_2)}f \doteq {}^{(g_2,h_2)}f_2$, as desired. Now we have to prove unitarity, which comes because of the left-right invariance of the integral (Haar measure). □

Lemma 24.6. *View $L^2(G)$ as a representation of the second G (so $f(x) \mapsto {}^g f := f(xg) = {}^{(1,g)}f$). This representation is faithful.*

Proof. Let $g \neq 1 \in G$. Since G is Hausdorff, there exist disjoint open neighborhoods $U \ni 1$ and $V \ni g$. Then 1_U (characteristic function) satisfies $(1_U, 1_V) =$ measure of $U \cap V = 0$. On the other hand, $(1_U, {}^g 1_V) =$ measure of $U \cap Vg^{-1} > 0$ by your homework. So $1_V \neq {}^g 1_V$. The difference is definitely not null, so $1^V \neq {}^g 1_V$ in $L^2(G)$. □

Define:

(1) $\mathrm{MC}(G)$ = matrix coefficients. A *matrix coefficient* of G is a composition $\rho : G \to \mathrm{GL}(V) \xrightarrow{L|_{\mathrm{GL}(V)}} \mathbf{C}$ where (V, ρ) is a finite-dimensional continuous representation of G and $L : \mathrm{End}(V) \to \mathbf{C}$ is a $\mathbf{C}$-linear functional, and the set of all these things is $\mathrm{MC}(G)$.

Example 24.7. $G \xrightarrow{\rho} \mathrm{GL}(V) \xrightarrow{\text{take } (i,j) \text{ entry}} \mathbf{C}$ sends $g \mapsto \rho(g)_{(i,j)}$, which is why it's called matrix coefficients. Anything in $\mathrm{MC}(G)$ is continuous.

Proposition 24.8.
(1) *$\mathrm{MC}(G)$ is an associative $\mathbf{C}$-subalgebra of $C(G)$ – in particular, it has a* 1.
(2) *If $f \in \mathrm{MC}(G)$ then the pointwise complex conjugate $\overline{f} \in \mathrm{MC}(G)$.*
(3) *$\mathrm{MC}(G)$ separates points of G, i.e., if $g_1, g_2 \in G$ are distinct, then there exists $f \in \mathrm{MC}(G)$ such that $f(g_1) \neq f(g_2)$.*

Proof.
(1) 1 comes from the trivial representation $\mathbf{C}$ and $L = \mathrm{id}$. If f_1 comes from (V_1, L_1) and f_2 comes from (V_2, L_2), you can get $f_1 + f_2$ via $(V_1 \oplus V_2, L_1 + L_2)$ and $f_1 f_2$ via $(V_1 \otimes V_2, L_1 \otimes L_2)$. If $c \in \mathbf{C}$ and f comes from (V, L) then cf comes from (V, cL).
(2) If f comes from (V, L) then $\overline{f}$ comes from $(\overline{V}, \overline{L})$.
(3) Write $L^2(G) = \widehat{\bigoplus} V_\alpha$ for some finite-dimensional irreducible representations (V_α, ρ_α) since all irreps of a compact group are finite-dimensional. Given $g_1 \neq g_2 \in G$, then $g_1 g_2^{-1}$ acts nontrivially on $L^2(G)$. So it acts nontrivially on some V_α. This means that g_1 and g_2 act differently, i.e. have different $n \times n$-matrices on V_α. Then there exists some (i, j) such that $\rho_\alpha(g_1)_{ij} \neq \rho_\alpha(g_2)_{ij}$ so that the matrix coefficient $\rho_\alpha(g)_{ij}$ separates g_1 and g_2.

□

Who cares about these three properties? The Stone-Weierstrass theorem gives:

Corollary 24.9 (Peter-Weyl theorem). $\mathrm{MC}(G)$ *is dense in* $C(G)$ *with respect to the* $|\cdot|_\infty$*-norm.*

There are about four different Peter-Weyl theorems, and this is one of them. We're going to prove the thing that *I* call the Peter-Weyl theorem (namely that $L^2(G) \simeq \widehat{\bigoplus}_X X^* \otimes X = \widehat{\bigoplus}_X \operatorname{End} X$) next time.

25. Peter-Weyl

Last day! Do fill out the online evaluations. I'm hoping we can beat all the other classes, so get moving!

Theorem 25.1 (Peter-Weyl). *There are about four different Peter-Weyl theorems, so we'll just group them all together.*

(1) *Every irreducible unitary representation of G on a Hilbert space is finite-dimensional. We proved this by averaging projections, and then using the spectral theorem for compact operators.*
(2) *Every unitary representation decomposes as $\widehat{\bigoplus}_{i\in I} V_i$ for some finite-dimensional irreducibles V_i, which we proved by peeling off representations and using Zorn's lemma.*
(3) *The space* $\mathrm{MC}(G) \subseteq C(G) \subseteq L^2(G)$ *of matrix coefficients is:*
   (a) *dense in $C(G)$ with respect to $|\cdot|_\infty$. We proved this last time by checking the conditions of the Stone-Weierstrass theorem.*
   (b) *dense in $L^2(G)$ with respect to $|\cdot|_2$. We haven't proved this yet.*
(4) *There's a decomposition*

$$L^2(G) \simeq \widehat{\bigoplus_{\textit{irreps } X}} X^* \otimes X$$

as unitary representations of $G \times G$, where $X^* \otimes X$ is equipped with $\frac{1}{\dim X}(-,-)_{\mathrm{HS}}$.
(5) *The matrix entries* $\sqrt{\dim X}\rho_X(g)_{ij}$ *form an orthonormal basis for the Hilbert space $L^2(G)$.*
(6) *For each irrep X of G, let $\chi_X : G \to \mathbf{C}$ be its character. Then:*
   (a) $\operatorname{span}\{\chi_X : X \textit{ irrep}\}$ *is dense in the space of continuous class functions.*
   (b) *The χ_X form an orthonormal basis of the L^2 class functions.*

The challenge is to finish all this in the next hour and eight minutes. Fortunately we've done a lot of the hard work already.

Proof. 3a) implies that:

Corollary 25.2. $\mathrm{MC}(G)$ *is dense in* $C(G)$ *with respect to* $|\cdot|_2$.

Proof. $|\cdot|_2 \le |\cdot|_\infty$. □

Corollary 25.3 (3b)**.** $\mathrm{MC}(G)$ *is dense in* $L^2(G)$ *with respect to* $|\cdot|_2$.

Proof. $\mathrm{MC}(G) \xrightarrow{\text{dense}} C(G) \xrightarrow{\text{dense}} L^2(G)$. □

Example 25.4. By your homework, $\mathrm{MC}(\mathbf{R}/\mathbf{Z}) = \left\{ \sum_{n \in S \subsetneq \mathbf{Z}} a_n e^{2\pi i n t} \right\}$ where S is finite. These you could call the trigonometric polynomials, and thus 3a) is saying that every continuous periodic function $f : \mathbf{R}/\mathbf{Z} \to \mathbf{C}$ can be uniformly approximated by trigonometric polynomials and 3b) is saying that every $f \in L^2(\mathbf{R}/\mathbf{Z})$ is a $|\cdot|_2$-limit of trigonometric polynomials. This is the main theorem of Fourier analysis.

Now, there's a couple of things to explain here. Recall that G acts on X^* as follows: if $\lambda \in X^* = \mathrm{Hom}(X, \mathbf{C})$, then $({}^g\lambda)(x) = \lambda(g^{-1}x)$. The inverse is just to make it a left action. Thus $G \times G$ acts on $X^* \otimes X$.

Proposition 25.5. *The map from the ordinary direct sum* $\bigoplus_{\mathit{irreps}\, X} X^* \otimes X \to C(G) \subseteq L^2(G)$ *defined as* $\lambda \otimes v \mapsto (x \mapsto \lambda(xv))$ *is an injective* $G \times G$*-map whose image is* $\mathrm{MC}(G)$.

Proof. $\underline{G \times G\text{-map:}}$ We just check and find that it commutes:

$$\begin{tikzcd} \lambda \otimes v \arrow[r, mapsto] \arrow[d, mapsto, "{(g,h)}"] & (x \mapsto \lambda(xv)) \arrow[d, mapsto, "{(g,h)}"] \\ ({}^g\lambda) \otimes (hv) \arrow[r, mapsto] & (x \mapsto \lambda(g^{-1}xhv)) \end{tikzcd}$$

It's just unraveling the definition. It works.

Image: If $v_1, \cdots, v_n$ is a basis of X and $\lambda_1, \cdots, \lambda_n$ is the dual basis of X^*, then $\lambda_i \otimes v_j$ is a basis of $X^* \otimes X$. Then $\lambda_j \otimes v_i \mapsto (g \mapsto \rho_X(g)_{ji})$. This is because $\rho_X(g)_{ji} = \lambda_j(gv_i)$. As X and i, j vary, these things span all the matrix coefficients.

Injective: The functions $\rho_X(g)_{ij}$ *are* linearly independent (in fact they're orthogonal and nonzero)! We did this for finite group, but the same proof works here. Thus the map is injective. □

The next thing to do is define the Hilbert-Schmidt inner product.

Definition 25.6. Let X be a finite-dimensional Hilbert space. The *Hilbert-Schmidt inner product* on $\mathrm{End}(X)$ is $(S,T)_{\mathrm{HS}} := \mathrm{Tr}(S^*T) \in \mathbf{C}$. You can check that it's positive-definite and so on.

Example 25.7. $(1_X, 1_X)_{\mathrm{HS}} = \dim X$. Also, let E_{ij} be the matrix that has a 1 in (i, j)th entry and is zero elsewhere. Then $(E_{ij}, E_{ij})_{\mathrm{HS}} = 1$ with a 1 in the (j, j)-position. In fact these things form an orthonormal basis.

Since $\mathrm{End}(X) = X^* \otimes X$, we have a Hermitian form on $X^* \otimes X$.

Proof of 4) and 5). We already have the map $\bigoplus_{\text{irreps } X} X^* \otimes X \to L^2(G)$ that's a map of representations. We claim that this respects inner products. This is the orthonormal theorem III or something. It also has dense image. So it induces an isometry $\widehat{\bigoplus}_{\text{irreps } X} X^* \otimes X \to L^2(G)$. This is 4).

For 5), we already kinda did this because if you identify X with $\mathbf{C}^n$ by choosing a basis, then $\mathrm{End}(X) = \mathrm{M}_n(\mathbf{C})$. You have $E_{ij} \in \mathrm{M}_n(\mathbf{C})$ that corresponds to $E_{ij} := \lambda_i \otimes v_j \in \mathrm{End}(X)$. Then E_{ij} are not orthonormal in $X^* \otimes X$ because we rescaled by $1/\dim(X)$. So we add in a factor to find that $\{\sqrt{\dim X} E_{ij}\}_{X,i,j}$ is an orthonormal basis of $\widehat{\bigoplus}_{\text{irreps } X} X^* \otimes X$, and these map to the matrix coefficients $\{\sqrt{\dim X}\rho_X(g)_{ij}\}_{X,i,j} \in L^2(G)$, and hence because $\widehat{\bigoplus}_{\text{irreps } X} X^* \otimes X \to L^2(G)$ is an isometry, it follows that $\{\sqrt{\dim X}\rho_X(g)_{ij}\}_{X,i,j}$ forms an orthonormal basis of $L^2(G)$. $\square$

As a corollary:

$$L^2(G) \simeq \bigoplus_{\text{irreps } X} (\dim X) X$$

as representations of the "second G", just like for finite groups.

Let's prove 6b).

Definition 25.8. Define a "conjugation" action of G on $L^2(G)$ by $G \xrightarrow{\text{diagonal}} G \times G \to \mathrm{Aut}(L^2(G))$. Explicitly, $g \mapsto (f(x) \mapsto f(g^{-1}xg))$.

Then $L^2(G)^G$ is the invariant subspace under this action, and it's the collection of L^2-class functions. This is the G-invariants of the left-hand-side of 4).

On the right hand side, let's choose a basis $e_1, \cdots, e_n$ of X. Let $e_1^*, \cdots, e_n^*$ be the dual basis of X^*. Then $(X^* \otimes X)^G = (\mathrm{End}\, X)^G = \mathbf{C}$ by Schur's lemma. This isomorphism can be made very explicit. $1 \in \mathbf{C}$ is the orthonormal basis of $\mathbf{C} \subseteq$

End(X) wtih respect to $\frac{1}{\dim X}(-,-)_{\mathrm{HS}}$. This corresponds to $\sum_{i=1}^n e_i^* \otimes e_i \in (X^* \otimes X)^G$. So now, in our isomorphism:

$$L^2(G) \simeq \widehat{\bigoplus_{\text{irreps } X}} X^* \otimes X$$

we know the G-invariants!

We know that $\sum_{i=1}^n e_i^* \otimes e_i$ maps to some orthonormal basis for the class functions $L^2(G)^G$. Our isomorphism sends $\sum_{i=1}^n e_i^* \otimes e_i \mapsto (g \mapsto \sum_{i=1}^n e_i^*(ge_i) = \sum_{i=1}^n \rho_X(g)_{ii} = \chi_X(g))$. Thus after taking G-invariants we get:

$$L^2(G)^G \simeq \left(\widehat{\bigoplus_{\text{irreps } X}} X^* \otimes X \right)^G = \operatorname{span}\{\chi_X\}_{\text{irreps } X}$$

Part a) is the same idea. I'll skip that since we don't have a whole lot of time. □

As a corollary, we have the following kinda useless description.

Corollary 25.9. *Every compact group is a closed subgroup of a product of unitary groups: $G \subseteq \prod_{i \in I} U(n_i)$ where $n_i \geq 0$.*

An annoying thing is that I'm not telling you how big this product is. It's not a Lie group or anything; it could be a *huge* product. For example, an uncountable product of circles (using Tychonoff we know it's compact).

Proof. The action of G on each irrep X defines a homomorphism $G \to U(\dim X) \subseteq \mathrm{GL}_{\dim X}(\mathbf{C}) = \mathrm{GL}(X)$. Then the product homomorphism $\phi : G \to \prod_{\text{irreps } X} U(\dim X)$ is injective because if $g \in \ker \phi$, then g acts trivially on every X. Thus because:

$$L^2(G) \simeq \bigoplus_{\text{irreps } X} (\dim X) X$$

we know that g acts trivially on $L^2(G)$. *But* we showed that this is a faithful representation of G, and hence $g = 1$. Do you know why G is closed? This is because G is compact, its image $\phi(G)$ is compact. It's compact in a compact Hausdorff space, and hence $\phi(G)$ is closed. I'm not quite done. I should really also argue that the topology on G is what you get from the subspace topology on $\phi(G)$.

One way is to show is that closed sets correspond under the isomorphism, i.e., closed subsets of G correspond under ϕ to the closed subsets of $\phi(G)$. To see this, if $C \subseteq G$ is closed, then $\phi(C)$ is closed because C is compact, hence $\phi(C)$ is compact, hence closed. In the other direction, if $C \subseteq G$ is closed, then $\phi^{-1}(C)$ is closed in G because ϕ is continuous. □

With that deep fact, I'll end this class. Everybody applauds.

Well maybe I won't end this because there's a lot more to say, especially for Lie groups. You classified irreps of sl_2 and this can be generalized. But we have only three minutes. You can take the other classes that have "Lie" in the title.