# ON THE ADDITIVITY OF EQUIVARIANT COMMUTATIVE OPERADS

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Abstract. We define the category of $G$-operads and the hierarchy of $\mathcal{N}-\infty$-*operads*, which are suboperads of the terminal $G$-operad $\mathrm{Comm}_G$ containing $\mathbb{E}_\infty$. We exhibit an isomorphism between the category of $\mathcal{N}^\infty$ operads and the poset of *indexing systems*, which are nice subcategories of $\underline{\mathbb{F}}_G$; this witnesses $\mathcal{N}^\infty$ operads as describing *commutative multiplication with restriction and some multiplicative transfers*. Indeed, their algebras in Cartesian categories are incomplete Mackey functors and their algebras in Mackey functors are incomplete Tambara functors.

After this, we discuss some in-progress research. Namely, we construct a *Boardman-Vogt tensor product* of $G$-operads, and prove that (nonempty) tensor products of $\mathcal{N}^\infty$ operads correspond with joins of indexing systems, i.e. there is an $\mathcal{N}^\infty(I \cup J)$-monoidal equivalence

$$\mathbf{Alg}^{\mathcal{N}^\infty(I)}\mathbf{Alg}^{\mathcal{N}^\infty(J)}\mathcal{C} \simeq \mathbf{Alg}^{\mathcal{N}^\infty(I\cup J)}\mathcal{C}$$

for all $\mathcal{N}^\infty(I \cup J)$-monoidal categories $\mathcal{C}$, allowing $G$-commutative structures to be constructed "one transfer at a time."

**Foreword.** The following are notes prepared for a casual talk in the zygotop seminar concerning research which is currently in-progress cite . Though I will attempt to confine these notes to their own proofs, citations to the literature, and well-marked conjecture, the reader should read with the understanding that they are particularly error-prone.

## 1. Introduction

In [Dre71], the concept of a *Mackey functor* was introduced; this structure was described as consisting of functors $M_I : \mathcal{O}_G \to \mathbf{Mod}_R$ and $M_R : \mathcal{O}_G^{\mathrm{op}} \to \mathbf{Mod}_R$ which agree on $\mathcal{O}_G^{\simeq}$ and satisfying the *double coset formula*

$$R_J^H I_K^H = \prod_{x\in[J\backslash H/K]} I^J_{J\cap xKx^{-1}} \cdot \mathrm{conj}_X R_{x^{-1}Jx\cap K}$$

for all $J, K \subset H$, where $R_J^K := M_R(G/J \to G/K)$ and similar for $I$. The ur-example of this is the assignment $H \mapsto \mathbf{Rep}_H(R)$ with covariant functoriality $\mathrm{Ind}$ and contravariant functoriality $\mathrm{Res}$. This was repackaged and generalized into the modern definition of the *category of $\mathcal{C}$-valued $G$-Mackey functors*

$$\mathcal{M}_G(\mathcal{C}) := \mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_G), \mathcal{C}),$$

where $\mathbb{F}_G$ denotes the category of finite $G$-sets.

In parallel, the concept of *transfer maps in group cohomology* was being developed in **[Evens]**, later lifted to genuine equivariant cohomology in **[Greenlees]**, and finally developed as a functor

$$N_H^G : \mathrm{Sp}_H \to \mathrm{Sp}_G$$

in [HHR16], which played a crucial role in the solution to the Kervaire invariant one problem. These were noted in [HH16] to satisfy the conditions of a *Symmetric monoidal Mackey functor*, a notion they distinguished from their notion of *$G$-symmetric monoidal categories* due to coherence issues.

In the broad program announced in [Bar+16], the correct notion of *$G$-symmetric monoidal $G$-$\infty$-categories* (henceforth $G$-symmetric monoidal categories) was introduced:

**Definition 1.1.** Let $\mathcal{C}$ have finite products. Then, the category of $G$-commutative monoids in $\mathcal{C}$ is

$$\mathrm{CMon}_G(\mathcal{C}) := \mathcal{M}_G(\mathcal{C}).$$

The category of $G$-symmetric monoidal categories is $\mathrm{CMon}_G(\mathbf{Cat})$.

We similarly define the *category of small $G$-categories* as

$$\mathbf{Cat}_G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Cat}) \simeq \mathbf{Cat}^{\mathrm{cocart}}_{/\mathcal{O}_G^{\mathrm{op}}},$$

where the equivalence is the *straightening-unstraightening construction* of [HTT]. This has an adjunction

$$\mathrm{Tot} : \mathbf{Cat}_G \rightleftarrows \mathbf{Cat} : \mathrm{CoFr}^G$$

where Tot takes the total category of a cocartesian fibration and $\mathrm{CoFr}^G(\mathcal{C})$ is classified by functor categories

$$\mathrm{CoFr}^G(\mathcal{C})_H := \mathrm{Fun}(\mathcal{O}_H^{\mathrm{op}}, \mathcal{C})$$

with functoriality dictated by pullback. In particular, the *G-category of small G-categories* $\underline{\mathbf{Cat}}_G := \mathrm{CoFr}^G(\mathcal{C})$ has $G$-fixed points given by **Cat**.

Let $\underline{\mathbb{F}}_{G,*} := \mathrm{CoFr}^G(\mathbb{F}_*)$. We may understand an object in $\underline{\mathbb{F}}_{G,*}$ as a map of finite $G$-sets $S \to U$ where $U$ is an orbit (recognizing $S$ as induced from $H$ if $U \simeq G/H$), and a map $\begin{pmatrix} S \\ \downarrow \\ U \end{pmatrix} \to \begin{pmatrix} S' \\ \downarrow \\ U' \end{pmatrix}$ as a span

$$\begin{tikzcd} S \arrow[d] & T \arrow[l] \arrow[r] \arrow[d] & S' \arrow[d] \\ U & V \arrow[l] \arrow[r, equal] & U' \end{tikzcd}$$

whose associated map $T \to S \times_U U'$ is a summand inclusion. We say that such maps are *active* if they are forwards, and *inert* if they are backwards.

**Definition 1.2.** A *G-operad* is a functor $\pi : \mathcal{O}^\otimes \to \mathrm{Tot}\underline{\mathbb{F}}_{G,*}$ such that

(1) $\mathcal{O}^\otimes$ has $\pi$-cocartesian lifts for inert morphisms with specified domains,
(2) the $\pi$-cocartesian lifts induce equivalences on the categories of colors

$$\mathcal{O}^\otimes_{(S\to U)} \simeq \prod_{W \in \mathrm{Orb}(S)} \mathcal{O}^\otimes_{(W=W)}.$$

(3) For any morphism $\psi : \begin{pmatrix} S \\ \downarrow \\ U \end{pmatrix} \to \begin{pmatrix} S' \\ \downarrow \\ U' \end{pmatrix}$ in $\mathrm{Tot}\underline{\mathbb{F}}_{G,*}$, pair $(x,y) \in \mathcal{O}_{(S\to U)} \times \mathcal{O}_{(S'\to U')}$ and collection of cocartesian edges $\{y \to y_W \mid W \in \mathrm{Orb}(S)\}$ lying over the inert morphisms $S \hookleftarrow W = W$, the induced map

$$\mathrm{Map}^{\psi}_{\mathcal{O}^\otimes}(x,y) \to \prod_{W \in \mathrm{Orb}(S)} \mathrm{Map}^{\psi|_W}_{\mathcal{O}^\otimes}(x, y_W)$$

is an equivalence.

Morphisms of $G$-operads are morphisms over $\mathrm{Tot}\underline{\mathbb{F}}_{G,*}$ preserving cocartesian lifts for inert morphisms.

This is a straightforward, but heavy, generalization of the $\infty$-operads of [HA] to the equivariant world, and we suggest the interested reader consult [BHS22] for a less heavy variant or [NS22] for the original. In particular, postcomposition along the inclusion functor $\mathbb{F} \hookrightarrow \mathrm{Tot}\underline{\mathbb{F}}_{G,*}$ induces a fully faithful functor $\mathrm{Op} \hookrightarrow \mathrm{Op}_G$.

An early observation about genuine equivariant homotopy coherent algebraic structures is that the structure of transfers *does not come canonically* from an $\mathbb{E}_\infty$-structure; that is, $\mathbb{E}_\infty \in \mathrm{Op}_G$ is not terminal. The failure of $\mathbb{E}_\infty$ to be terminal is parameterized by the category of $N^\infty$*-operads*:

**Definition 1.3.** Write $\mathrm{Comm}_G^\otimes := \left(\mathrm{Tot}\underline{\mathbb{F}}_{G,*} = \mathrm{Tot}\underline{\mathbb{F}}_{G,*}\right)$ for the terminal $G$-operad. A $G$-operad $\mathcal{O}^\otimes$ is *subterminal* if the unique morphism $\mathcal{O}^\otimes \to \mathrm{Comm}_G^\otimes$ is a monomorphism, i.e. $\mathcal{O}_U^\otimes \simeq *$ for all $U$ and $\mathrm{Map}^\psi_{\mathcal{O}}(x,y) \in \{*, \emptyset\}$ for all $\psi : \pi(x) \to \pi(y)$.

An $N^\infty$ *operad* is a subterminal $G$-operad $\mathcal{O}^\otimes$ admitting a map $\mathbb{E}_\infty \to \mathcal{O}^\otimes$.

Write $\mathcal{N}_G^\infty \subset \mathrm{Op}_G$ for the full subcategory consisting of $N^\infty$-operads, and write $\widehat{\mathcal{N}}_G^\infty := \mathcal{N}_G^\infty \cup \left\{\mathcal{O}^{otimes}_{\mathrm{triv}}\right\}$ The following proposition is an easy exercise in category theory:

**Proposition 1.4.** *The category $\widehat{\mathcal{N}}_G^\infty$ is a poset, i.e. all of its mapping spaces are contractible or empty.*

In ref , we endow $\mathrm{Op}_G$ with a *Boardman-Vogt symmetric monoidal structure*, satisfying the universal property that

$$\mathbf{Alg}^{\mathcal{O}\otimes\mathcal{P}}(\mathcal{C}) \simeq \mathbf{Alg}^{\mathcal{O}}\mathbf{Alg}^{\mathcal{P}}(\mathcal{C}).$$

We would like to characterize the tensor products of these, but to do so, we need a candidate, which are called *indexing systems*.

**Definition 1.5.** An *indexing system* is a core-containing subcategory $\mathcal{O}_G^{\simeq} \hookrightarrow I \hookrightarrow \mathcal{O}_G$ which is closed under base change, i.e. for any

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\alpha'"] & V \arrow[d, "\alpha"] \\ U' \arrow[r] & V' \end{tikzcd}$$

with $U \hookrightarrow V' \times_{U'} V$ a summand inclusion and $\alpha \in I$, we have $\alpha' \in I$. The poset of indexing systems under inclusion is denoted $\mathrm{Ind}-\mathrm{Sys}_G$, and the poset of indexing systems with an added initial object is denoted $\widehat{\mathrm{Ind}-\mathrm{Sys}}_G$.

Given an indexing system, there is a corresponding full subcategory of $\underline{\mathbb{F}}_{G,*}$ which happens to have the structure of a $G$-operad. We call this functor $\mathcal{N}^\infty(-) : \widehat{\mathrm{Ind}-\mathrm{Sys}}_G \to \mathrm{Op}_G$, with value on $\varnothing$ given by $\mathcal{O}^{\otimes}_{\mathrm{triv}}$.

**Theorem A.** *The functor $\mathcal{N}^\infty(-) : \widehat{\mathrm{Ind}-\mathrm{Sys}}_G \to \mathrm{Op}_G$ is fully faithful with image $\widehat{\mathcal{N}}_G^\infty$. Furthermore, this functor is symmetric monoidal for the cocartesian structure on $\widehat{\mathrm{Ind}-\mathrm{Sys}}_G$ and the BV tensor product on $\mathrm{Op}_G$; this supplies a canonical equivalence*

$$\mathbf{Alg}^{\mathcal{N}^\infty(I)}\mathbf{Alg}^{\mathcal{N}^\infty(J)}\mathcal{C} \simeq \mathbf{Alg}^{\mathcal{N}^\infty(I\cup J)}\mathcal{C}$$

*for all indexing systems $I, J$.*

*Remark.* One may worry about the comparison between models for $G$-operads, as our notion of $N^\infty$-operads is ostensibly embedded deep within the world of $G$-$\infty$-operads, which are not known to be equivalent to the $\infty$-category presented by the graph model structure or by genuine $G$ operads. However, by ref, all notions of $N^\infty$ operads coincide .

## 2. The Boardman-Vogt tensor product

### 2.1. The $G$-symmetric monoidal envelope.

### 2.2. The internal hom on $G$-operads.

### 2.3. Day convolution and the categorical Fourier transform.

## 3. Commutative operads

### 3.1. Tensor products of subterminal operads.

### 3.2. Synthesis.