A crash course on foundations of equivariant homotopy theory

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This document is meant to serve as lecture notes for a standalone talk at the Harvard zygotop seminar. Audience members and readers are assumed to have background at the level of a first or second year graduate student in homotopy theory; in particular, familiarity of the language of Joyal and Lurie's ∞-categories is assumed, as well as many of the basic results laid out in [Lur09] and [Lur17].

As astutely observed in the intro to Arun Debray's notes from Andrew Blumberg's session of UT Austin M392C, equivariant homotopy theory is poorly served by textbooks [Deb17]. I would personally reccomend Debray's notes for a good introduction to the theory, though they do not cover spectral Mackey functors.

Andrew Blumberg, as captured in Arun Debray's lecture notes for UT Austin M392C [Deb17].

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Contents

1 Unstable equivariant homotopy

For the duration of this section, fix G a compact Lie group. Let $\mathbf{Top}_G := \mathbf{Top}^{BG}$ denote the category of topological spaces with G-action.

Unstable equivariant homotopy is the study of $\mathbf{Top}_G$ up to homotopy, i.e. of suitable (simplicial) localizations of $\mathbf{Top}_G$. In this section, we briefly probe two such notions: that of *Borel equivariant homotopy*, and that of *Bredon equivariant homotopy*. The former is often referred to as *naïve equivariant homotopy theory* or *homotopy theory of spaces with G-action*, and the latter *genuine equivariant homotopy*, though I will avoid such morally loaded terminology.

Following this probe, we state *Elmendorf's theorem*, which realizes Bredon equivariant homotopy theory as the homotopy theory of presheafs on the category of orbits.

1.1 Borel equivariant homotopy

Definition 1.1. A *Borel weak equivalence* is a map of topological G-spaces $f : X \to Y$ whose underlying map of topological spaces is a weak equivalence.

It is not too hard to see that the simplicial localization $\mathbf{Top}_G\left[\mathrm{Borel}-\mathrm{weq}^{-1}\right]$ is equivalent to the presheaf category

$$\mathcal{S}^{BG} \simeq \mathbf{Top}_G\left[\mathrm{Borel}-\mathrm{weq}^{-1}\right]$$

As a consequence, we only remember the *homotopical* versions of invariants of a G-space in Borel equivariant homotopy theory; among these are the *homotopy fixed points* and *homotopy orbits*

$$X^{hG} = \lim_G X,$$
$$X_{hG} = \mathrm{colim}_G X,$$

where (co)limits are understood to be taken in the ∞-category $\mathcal{S}^{BG}$. Every invariant one could want of a space then immediately become invariants of a Borel equivariant space, by applying them to either X^{hG} or X_{hG}.

In this realm, we have spectral sequences

$$E^2_{s,t} \simeq H_s(BG; H_t(X)) \implies H_{s+t}\left(X_{hG}\right),$$
$$E_2^{s,t} \simeq H^{-s}(BG; \pi_t(X)) \implies \pi_{s+t}\left(X^{hG}\right);$$

The former is just the Serre spectral sequence. The latter is referred to as the *homotopy fixed point spectral sequence*, and can be gotten via the postnikov tower spectral sequence(?) for X^{hG}.

There are reasons to want something finer than this; we sometimes want to track the information of the point-set fixed points of a G-action. For instance, it's easy to see that every element of $\mathcal{S}^{BG}$ with contractible underlying space is *Borel-equivariantly contractible*, i.e. equivalent to the terminal object, which is modeled by $*$. However, it's not hard to come up with interesting group actions on contractible spaces whose point set fixed points are nontrivial.

In settings where we care to track this sort of data, we will use *Bredon equivariant homotopy theory*.

1.2 Bredon equivariant homotopy

Definition 1.2. A *Bredon weak equivalence* is a map of topological G-spaces $f : X \to Y$ whose point-set fixed point map $f^H : X^H \to Y^H$ is a homotopy equivalence for each closed subgroup $H \le G$.

In particular, Bredon weak equivalences are Borel weak equivalences. We denote by $\mathcal{S}_G$ the simplicial localization $\mathbf{Top}_G\left[\mathrm{Bredon} - \mathrm{WEQ}^{-1}\right]$; it is immediately evident that $\mathcal{S}^{BG}$ can be presented as a localization of $\mathcal{S}_G$ at the class of maps whose underlying map of spaces is an equivalence.

This supports some useful analogs to classical unstable homotopy theory. For instance, this is the right setting for CW complexes and whitehead's theorem. We first define cell complexes, following [Bre67].

Definition 1.3. A G-CW complex is an element of $\mathbf{Top}_G$ acquired by iterated pushouts along (trivially) shifted G-spheres; that is, a *d-dimensional G-CW complex* is a point-set pushout of G-spaces

$$\begin{tikzcd} P \times S^{d-1} \arrow[r, hook] \arrow[d] & P \times D^d \arrow[d] \\ X_{d-1} \arrow[r, "\iota"] \arrow[ur, phantom, "\ulcorner", very near end] & X_d \end{tikzcd}$$

where P is a discrete G-set, G acts trivially on S^{d-1} and D^d, and X_{d-1} is a $(d-1)$-dimensional G-CW complex. The map ι is said to be the $d-1$ skeleton map. A *G-CW complex* is a sequential colimit of d-dimensional CW complexes along skeleton maps.

We can define G-homotopy equivalences a number of ways; the most efficient way is as a map $f : X \to Y$ inducing weak equivalences

$$\mathrm{Map}(Z, X)^G \xrightarrow{\sim} \mathrm{Map}(Z, Y)^G.$$

Since $X^H \simeq \mathrm{Map}(G/H, X)^G$ for all $H \le G$, this is evidently included in the class of Bredon weak equivalences. The following result of Matsumoto is an equivariant analog to Whitehead's theorem:

Theorem 1.4 ([Mat71])**.** *A map $f : X \to Y$ of G-CW complexes is a Bredon weak equivalence if and only if it is a G-homotopy equivalence.*

Bredon homotopy also supports several other useful classical theorems, many of which are included in [May96]:

- Bredon weak equivalences support a G-CW approximation theorem;
- Bredon weak equivalences support a G-cellular approximation theorem;
- Bredon weak equivalences support a G-manifold triangularization theorem; in particular, every compact smooth G-manifold has the structure of a finite G-CW complex.

As a corollary to the last of these facts, we have the following:

Corollary 1.5. *Let X be a G-space gotten by iterated pushouts along inclusions $\{S(V) \hookrightarrow D(V)\}_{V \in \mathbf{Rep}_{\mathbb{R}}(G)}$, where $S(V)$ and $D(V)$ are the unit sphere and disk G-spaces of a real G representation V. Then, X is a G-CW complex.*

Proof. The corollary follows by triangularizing the inclusion of smooth G-manifolds $S(V) \hookrightarrow D(V)$; using the associated relative G-CW complex structure, we may refine an attachment of a $D(V)$-cell to an iterated attachment of G-CW cells. $\square$

For the rest of this section, we assume that G is finite. Note that the category $\mathbf{Set}^{BG}$ is freely generated under coproducts by the subcategory $\mathcal{O}_G \subset \mathbf{Set}^{BG}$ of *transitive G-sets*, each of which are of the form G/H for some $H \leq G$; hence we may view (Bredon) G-spaces as cell complexes for shifts of $\mathcal{O}_G$ by trivial representation spheres.

We now have a $\mathcal{O}_G$-worth of objects constituting *points*; in the same way that we used the *Eilenberg-Steenrod axioms* to uniquely determine (co)homology theories based on the value of a point, Bredon define a notion of *reduced equivariant (co)homology* uniquely extended from a functor $\mathcal{O}_G^{\mathrm{op}} \to \mathbf{Ab}$ via stability and excision. We skip details on this, as we'll recover this notion of cohomology as a special case of Mackey functor cohomology soon. First, we take a detour into the combinatorics of it all.

1.3 Elmendorf's theorem

The *corepresentability of fixed points* formula

$$X^H \simeq \mathrm{Map}(G/H, X)^G$$

yields functoriality of fixed points in the Bredon equivariant category; in particular, this yields an evident functor

$$(-)^{(-)} : \mathcal{S}_G \to \mathrm{Fun}\left(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}\right)$$

Elmendorf's theorem asserts that this summarizes the entirety of unstable equivariant homotopy:

Theorem 1.6 ([Elm83])**.** *The functor constructed above yields an equivalence $\mathcal{S}_G \xrightarrow{\sim} \mathrm{Fun}\left(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}\right)$.*

2 Mackey functors

2.1 Discrete Mackey functors as coefficients for equivariant cohomology

A good reference for this section is Megan Shulman's thesis [Shu14]. We begin by recalling the definition of the *Burnside category*

Definition 2.1. The *Burnside category* is the category of spans $\mathrm{A}(G) := \mathrm{Span}(\mathbf{Fin}_G)$

Due to some categorical nonsense (namely, that $\mathbf{Fin}_G$ is *disjunctive*), $\mathrm{A}(G)$ is semiadditive, with direct sums given by coproducts in $\mathbf{Fin}_G$. This allows us to define Mackey functors:

Definition 2.2. The *category of Mackey functors* is the product-preserving functor category

$$\mathcal{M}_G := \mathrm{Fun}_\times\left(\mathrm{A}(G), \mathbf{Ab}\right)$$

2.2 Spectral Mackey functors

3 Stable equivariant homotopy

3.1 A handwave towards orthogonal spectra, and Guillou-May

[BH21]

3.2 Homotopy, (co)homology

3.3 Geometric fixed points

3.4 Slices

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