THE BOGOMOLOV-TIAN-TODOROV THEOREM

1. Introduction

The goal of this document is to give an exposition of the Bogomolov-Tian-Todorov theorem in characteristic zero, and indicate the necessary modifications for the result to hold in characteristic p. Our discussion begins classically, but we shall take a more sophisticated perspective in later sections. Using this modern point of view, we then discuss a generalization to Hochschild homology. Most of this document is written from the perspective of an algebraic topologist; in particular, Section 4 unapologetically relies heavily on homotopical machinery. (The most accessible portion of this document is Section 3.1, which we included for the sake of concreteness.)

1.1. **The statement.** We begin by defining the class of objects under study.

Definition 1.1. A smooth connected projective variety X over a field k (of any characteristic) is said to be weakly Calabi-Yau if its canonical sheaf ω_X is trivial. It is said to be Calabi-Yau if $\mathrm{H}^i(X;\mathcal{O}_X)=0$ for $0<i<\dim(X)$.

The result we shall discuss is the following.

Theorem 1.2 (Bogomolov-Tian-Todorov). *The deformation theory of a Calabi-Yau variety over a field of characteristic zero is unobstructed, and the deformation space is smooth.*

The first proof is analytic, and is a rephrasing of the original proofs of Bogomolov, Tian, and Todorov (see [Bog78, Tia87, Tod89]). Our perspective will nonetheless be tainted by algebra. The second proof is algebraic, and is originally due to Ran and Kawamata (see [Ran92, Kaw92]).

In the course of these proofs, we will see that the unobstructedness and smoothness of the deformations of a Calabi-Yau variety X are consequences of the degeneration of the Hodge-de Rham spectral sequence at the E_1-page. The noncommutative analogue of this result is the degeneration of the Tate spectral sequence (going from Hochschild homology to periodic cyclic homology). Motivated by this observation, we then prove a noncommutative analogue of Theorem 1.2. Namely, we show (see Proposition 4.14):

Proposition 1.3. *Let $\mathcal{C}$ be a weakly n-Calabi-Yau k-linear stable ∞-category. If k is of characteristic p, assume that $\mathcal{C}$ lifts to $\mathrm{W}_2(k)$, and that $\mathrm{HH}_*(\mathcal{C})$ is concentrated in dimensions $>-p$ and $<p$. Then $\mathrm{HH}_*(\mathcal{C})$ is quasi-isomorphic to a formal abelian dg-Lie algebra.*

This implies Theorem 1.2, as well as a generalization of Schröer's characteristic p statement (see [Sch03]).

2. The analytic proof of Theorem 1.2

2.1. **Beginning the proof.** Our goal in this section is to present a proof of the following slight variation of Theorem 1.2.

Theorem 2.1. *Let X be a compact Kähler manifold with trivial canonical bundle. Then the universal local family of deformations of X is smooth.*

The proof of Theorem 2.1 will occupy the remainder of this section. We begin with an analytic interpretation of the results of Section 3.1.

Construction 2.2. Let $J \in C^{\infty}(T_X \otimes T_X^*)$ be the complex structure on X, so that J determines a decomposition $T_X \otimes \mathbf{C} = T_X^{1,0} \oplus T_X^{0,1}$. Let J_t be a family of complex structures on X, depending smoothly on t. Then there is an induced decomposition $T_X \otimes \mathbf{C} = T_{X_t}^{1,0} \oplus T_{X_t}^{0,1}$, which (as is usual when one has two direct sum decompositions of a vector space) defines an element $\alpha_t \in \mathrm{Hom}(T_X^{0,1}, T_X^{1,0})$; in particular, $\alpha_t \in \mathrm{Hom}(T_X^{0,1}, T_X^{1,0})$ uniquely determines an *almost* complex structure on X.

In order for a given $\alpha_t \in \mathrm{Hom}(T_X^{0,1}, T_X^{1,0})$ to define a complex structure on X, it must satisfy the integrability condition $[T_{X_t}^{0,1}, T_{X_t}^{0,1}] \subseteq T_{X_t}^{0,1}$.

Lemma 2.3. *The integrability condition $[T_{X_t}^{0,1}, T_{X_t}^{0,1}] \subseteq T_{X_t}^{0,1}$ is satisfied if and only if α_t satisfies the Maurer-Cartan equation*

$$\overline{\partial}\alpha_t + [\alpha_t, \alpha_t] = 0.$$

Proof. The vector fields of type $(0,1)$ for $J_t = J + \alpha_t$ are of the form $v + \alpha_t(v)$ for $v \in T_X^{0,1}$, so we need $[v + \alpha_t(v), w + \alpha_t(w)] \in T_{X_t}^{0,1}$ for all $v, w \in T_X^{0,1}$. Note that

$$[v + \alpha_t(v), w + \alpha_t(w)] = [v, w] + [v, \alpha_t(w)] - [w, \alpha_t(v)] + [\alpha_t(v), \alpha_t(w)].$$

Now, $[\alpha_t(v), \alpha_t(w)] \in T_X^{0,1}$ because J is integrable, and therefore it suffices to show that

$$[v, \alpha_t(w)] - [w, \alpha_t(v)] = -[\alpha_t(v), \alpha_t(w)].$$

In turn, it suffices to consider the case $v = \overline{\partial}_{z_i}$ and $w = \overline{\partial}_{z_j}$, corresponding to local holomorphic coordinates z_i. In this case, $[v, \alpha_t(w)] = \overline{\partial}_{z_i}(\alpha_t(\overline{\partial}_{z_j}))$, and so the desired condition translates (and is equivalent) to

$$\overline{\partial}_{z_i}(\alpha_t(\overline{\partial}_{z_j})) - \overline{\partial}_{z_j}(\alpha_t(\overline{\partial}_{z_i})) = -[\alpha_t(\overline{\partial}_{z_i}), \alpha_t(\overline{\partial}_{z_j})].$$

This is precisely the Maurer-Cartan equation. □

Corollary 2.4. *The integrability condition $[T_{X_t}^{0,1}, T_{X_t}^{0,1}] \subseteq T_{X_t}^{0,1}$ is satisfied if for each $\alpha_1 \in \mathrm{H}^1(X; T_X)$, there is a formal power series $\sum_{i \geq 1} \alpha_i t^i$ solving the Maurer-Cartan equation.*

Proof. Consider the power series expansion of α_t, i.e., write $\alpha_t = \sum_{i \geq 1} \alpha_i t^i$. Then the Maurer-Cartan equation translates into a family of equations

$$\overline{\partial}\alpha_k = -\sum_{1 \leq i \leq k-1} [\alpha_i, \alpha_{k-i}]. \tag{2.1}$$

In particular, we have $\overline{\partial}\alpha_1 = 0$, and thus α_1 defines a holomorphic 1-form, i.e., an element of $\mathrm{H}^1(X; T_X)$. This is precisely the class in $\mathrm{H}^1(X; T_X)$ corresponding to the deformation X_t under Proposition 3.8. The statement of the corollary follows. □

Example 2.5. Equation (2.1) gives an identity $\overline{\partial}\alpha_2 = -[\alpha_1, \alpha_1]$, so (at the very least) we need $[\alpha_1, \alpha_1] \in \mathrm{H}^2(X; T_X)$ to vanish. This is exactly the obstruction class of Proposition 3.9.

Remark 2.6. Of course, one way to prove that (2.1) admits solutions is to just show that $\mathrm{H}^2(X; T_X) = 0$; this, however, is not true for Calabi-Yau varieties. One might then try to show that the symmetric bilinear form $\mathrm{H}^1(X; T_X) \otimes \mathrm{H}^1(X; T_X) \to \mathrm{H}^2(X; T_X)$ is zero — but this only helps us with showing that $[\alpha_1, \alpha_1]$ vanishes.

2.2. **Differential graded Lie algebras.** To solve (2.1) and resolve the issues raised in Remark 2.6, we recall some of the theory of differential graded Lie algebras (which we will just abbreviate to dg-Lie algebra), and its connection to deformation theory.

Definition 2.7. A dg-Lie algebra $\mathfrak{g}$ is a Lie algebra in dg-k-modules. Explicitly, it is a dg-k-module $(\mathfrak{g}_*, d)$ and a graded antisymmetric bracket $[,] : \mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ satisfying the graded Jacobi identity, such that

$$d[g_1, g_2] = [dg_1, g_2] + (-1)^i [g_1, dg_2].$$

The motivation, in this context, for introducing dg-Lie algebras is the following example.

Example 2.8. Let X be a smooth k-variety, and let $\mathfrak{ks}(X)$ denote the dg-Lie algebra given by $\mathfrak{ks}(X)_n = \mathrm{H}^0(X; \Omega_X^{0,n}(T_X))$. The derivation d is the operator $\overline{\partial}$, and the Lie bracket is the bracket on holomorphic vector fields. The dg-Lie algebra $\mathfrak{ks}(X)$ is known as the Kodaira-Spencer dg-Lie algebra.

Before proceeding, we need the following definition.

Definition 2.9. Let $\mathfrak{g}$ be a nilpotent dg-Lie algebra, so that $\mathfrak{g}_0$ is a nilpotent Lie algebra. Then there is an action of $\exp(\mathfrak{g}_0)$ on $\mathfrak{g}_1$, given by

$$e^g \cdot h = h + \sum_{n \geq 0} \frac{([g, -])^n}{(n+1)!}([g, h] - dg).$$

Construction 2.10. Let $\mathfrak{g}$ be a dg-Lie algebra. Define a functor $\mathrm{Def}_{\mathfrak{g}} : \mathrm{Art}_k \to \mathrm{Set}$ as follows. Let

$$\mathrm{MC}_{\mathfrak{g}}(A, \mathfrak{m}_A) = \left\{ x \in \mathfrak{g}_1 \otimes \mathfrak{m}_A | dx + \frac{1}{2}[x, x] = 0 \right\}$$

denote the set of Maurer-Cartan elements of $\mathfrak{m}_A \otimes \mathfrak{g}_*$. There is an action of $\mathfrak{g}_0 \otimes \mathfrak{m}_A$ on $\mathrm{MC}_{\mathfrak{g}}(A, \mathfrak{m}_A)$, obtained from Definition 2.9 via the observation that $\mathfrak{m}_A$ is nilpotent (viewed as a cdga in degree zero). Finally, one defines $\mathrm{Def}_{\mathfrak{g}}(A, \mathfrak{m}_A) = \mathrm{MC}_{\mathfrak{g}}(A, \mathfrak{m}_A)/\mathfrak{g}_0 \otimes \mathfrak{m}_A$.

Remark 2.11. The factor of $1/2$ appearing in Construction 2.10 is not very relevant for our purposes; it stems from viewing the differential on a holomorphic vector field in the Kodaira-Spencer dg-Lie algebra as a bracket of $\overline{\partial}$ with the holomorphic vector field.

We shall not spend much time exploring Construction 2.10, but we note some explicit calculations.

Proposition 2.12. *There is an isomorphism of k-vector spaces* $\mathrm{Def}_{\mathfrak{g}}(k[\epsilon]/\epsilon^2) \cong \mathrm{H}^1(\mathfrak{g})$.

Proof. An element $x \in \mathfrak{g}_1 \otimes (\epsilon)$ lives in $\mathrm{MC}_{\mathfrak{g}}(k[\epsilon]/\epsilon^2)$ if and only if $dx + \frac{1}{2}[x, x] = 0$. Write $x = g \otimes h\epsilon$; then, we need

$$-dg \otimes h\epsilon = d(g \otimes h\epsilon) = -\frac{1}{2}[g \otimes h\epsilon, g \otimes h\epsilon] = -\frac{1}{2}[g, g] \otimes h^2 \epsilon^2 = 0.$$

Therefore, $\mathrm{MC}_{\mathfrak{g}}(k[\epsilon]/\epsilon^2)$ is isomorphic to the 1-cycles in $\mathfrak{g}$. To conclude, it suffices to understand the action of $\mathfrak{g}_0 \otimes (\epsilon)$. But if $g \in \mathfrak{g}_0 \otimes (\epsilon)$ and $h \in \mathfrak{g}_1 \otimes (\epsilon)$, then $[g, h] = 0$, and so

$$e^g \cdot h = h + \sum_{n \geq 0} \frac{([g, -])^n}{(n+1)!}([g, h] - dg) = h + dg.$$

This implies the desired claim. $\square$

Proposition 2.13. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be quasi-isomorphic dg-Lie algebras. Then* $\mathrm{Def}_{\mathfrak{g}_1}$ *is isomorphic to* $\mathrm{Def}_{\mathfrak{g}_2}$.

Remark 2.14. This proposition can be made more precise (and more general). Work of Pridham, Lurie, and Gaitsgory-Rozenblyum (see [Pri10, Lur11, GR17]) gives an equivalence between dg-Lie algebras up to quasi-isomorphism and formal moduli problems over fields of characteristic zero. We shall not say more about this here, but we can briefly describe the correspondence as follows. There is an equivalence between associative (i.e., $\mathbf{E}_1$-) algebra objects in formal moduli problems and connected formal moduli problems (given by looping and delooping). The category of $\mathbf{E}_1$-algebras (i.e., group objects) in formal moduli problems is equivalent to the category of (derived) formal groups over k, which in turn (because of the characteristic zero assumption) is equivalent to the category of dg-Lie algebras. The composite of these equivalences is the Lurie-Pridham-Gaitsgory-Rozenblyum equivalence.

More precisely, a formal moduli problem X is sent to the dg-Lie algebra given by the tangent space at the identity of ΩX, i.e., $T_{X,x}[-1]$ (where $x \in X(k)$ is a basepoint), and a dg-Lie algebra $\mathfrak{g}$ is sent to $B\exp(\mathfrak{g})$. If $\mathfrak{g}$ is nilpotent, then $B\exp(\mathfrak{g}) = \mathrm{Spf}(k)/\exp(\mathfrak{g}) = \mathrm{Spf}(k^{\mathfrak{g}})$. The homotopy invariants $k^{\mathfrak{g}}$ is, of course, the Chevalley-Eilenberg complex $C^*(\mathfrak{g})$. Therefore, the moduli problem associated to $\mathfrak{g}$ is the functor $\mathrm{Art}_k \to \mathrm{Set}$ given by $A \mapsto \mathrm{Map}_k(C^*(\mathfrak{g}), A)$. The Chevalley-Eilenberg complex is adjoint to the Koszul duality functor $\mathbf{D} : (\mathrm{Art}_k)^{\mathrm{op}} \to \mathrm{Lie}_k$, and so $\mathrm{Map}_k(C^*(\mathfrak{g}), A) \simeq \mathrm{Map}_{\mathrm{Lie}_k}(\mathbf{D}(A), \mathfrak{g})$. It is not at all clear how any of this relates to the Maurer-Cartan story from this abstract perspective, but, as we said above, we shall not address this here.

The following is the result relevant to solving (2.1).

Proposition 2.15. *Let $\mathfrak{g}$ be a formal dg-Lie algebra (i.e., $\mathfrak{g}$ is quasi-isomorphic to $\mathrm{H}^*(\mathfrak{g})$). Then $\varprojlim_n \mathrm{Def}_{\mathfrak{g}}(k[\epsilon]/\epsilon^n) \to \mathrm{Def}_{\mathfrak{g}}(k[\epsilon]/\epsilon^2)$ maps isomorphically onto the image of $\mathrm{Def}_{\mathfrak{g}}(k[\epsilon]/\epsilon^3) \to \mathrm{Def}_{\mathfrak{g}}(k[\epsilon]/\epsilon^2) \cong \mathrm{H}^1(\mathfrak{g})$. The image of this map consists of those $g \in \mathrm{H}^1(\mathfrak{g})$ such that $[g,g] = 0$.*

Proof. Let $x \in \mathrm{Def}_{\mathfrak{g}}(k[\epsilon]/\epsilon^2)$, so that $x = g \otimes \epsilon$ for some $g \in \mathfrak{g}_1$ (which we will just write as $g\epsilon$). Then x lifts to $\mathrm{Def}_{\mathfrak{g}}(k[\epsilon]/\epsilon^3)$ if and only if there is an $h \in \mathfrak{g}_1$ such that $g\epsilon + h\epsilon^2$ satisfies the Maurer-Cartan equation. If $\mathfrak{g}$ is formal, then it is isomorphic to the graded Lie algebra $\mathrm{H}^*(\mathfrak{g})$, and so the Maurer-Cartan equation asserts that $[x,x] = 0$. In particular, we find that $g\epsilon + h\epsilon^2$ satisfies the Maurer-Cartan equation if and only if

$$[g\epsilon + h\epsilon^2, g\epsilon + h\epsilon^2] \equiv \epsilon^2[g,g] \pmod{\epsilon^3}$$

vanishes, i.e., if and only if $[g,g] = 0$. This clearly implies that $g\epsilon \in \mathrm{Def}_{\mathfrak{g}}(k[\epsilon]/\epsilon^n)$ for every $n \geq 3$. □

Remark 2.16. It is not true that the map $\mathrm{Def}_{\mathfrak{g}}(k[\epsilon]/\epsilon^n) \to \mathrm{Def}_{\mathfrak{g}}(k[\epsilon]/\epsilon^3)$ is an isomorphism if $\mathfrak{g}$ is a dg-Lie algebra.

2.3. Returning to the analytic proof.

In this section, we complete the proof of Theorem 2.1 using Proposition 2.15. It follows from this proposition and Proposition 2.13 that (2.1) can be solved for any $\alpha_1 \in \mathrm{H}^1(X; T_X) = \mathrm{H}^1(\mathfrak{ks}(X))$ if $\mathfrak{ks}(X)$ is quasi-isomorphic to a formal dg-Lie algebra with trivial bracket. To show this, we will introduce the notion of a dg-BV algebra.

Definition 2.17. Let (A, d) be a (always commutative) dg-algebra over a field k. A Batalin-Vilkovisky algebra structure on A (often just shortened so that A is called a dg-BV algebra) is a map $\Delta : A \to A$ of degree -1 which squares to zero such that Δ is an operator of order 2, i.e.,

$$\Delta(abc) = \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|-1)|b|}b\Delta(ac) - (\Delta a)bc - (-1)^{|a|}a(\Delta b)c - (-1)^{|a|+|b|}ab(\Delta c).$$

Definition 2.18. A Gerstenhaber algebra is a dg-algebra A with a Lie bracket (known as the Gerstenhaber bracket) which satisfies the Poisson identity

$$[a, bc] = [a,b]c + (-1)^{|b||c|}[a,c]b.$$

Lemma 2.19. *Any dg BV-algebra (A, Δ) defines a Gerstenhaber algebra via*

$$[a,b] = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}\Delta(a)b - a\Delta(b).$$

Theorem 2.20. *Let $\mathfrak{g}$ be a dg-Lie algebra. Suppose $\mathfrak{g}$ admits the structure of a dg BV-algebra via an operator Δ, such that the induced Lie bracket (via Lemma 2.19) is the Lie bracket of $\mathfrak{g}$. Suppose that $\Delta d + d\Delta = 0$, and the ∂-$\overline{\partial}$-lemma is satisfied, i.e.,*

$$\mathrm{im}(d) \cap \ker(\Delta) = \ker(d) \cap \mathrm{im}(\Delta) = \mathrm{im}(\Delta \circ d).$$

Let $\mathfrak{g}'$ denote the Lie algebra $(\ker(\Delta), [,], d)$, and let $\mathfrak{g}''$ denote the Lie algebra $(\ker(\Delta)/\mathrm{im}(\Delta), 0, 0)$. Then there are quasi-isomorphisms of dg-Lie algebras

$$\mathfrak{g}'' \xleftarrow{\sim} \mathfrak{g}' \xrightarrow{\sim} \mathfrak{g}.$$

In particular, $\mathfrak{g}$ is quasi-isomorphic to a formal dg-Lie algebra with trivial bracket.

We refer the reader to Theorem 4.11 for a generalization of Theorem 2.20. Before we prove Theorem 2.20, let us finish the proof of Theorem 2.1.

Proof of Theorem 2.1. As mentioned above, Proposition 2.15 and Proposition 2.13 show that (2.1) can be solved for any $\alpha_1 \in \mathrm{H}^1(X; T_X) = \mathrm{H}^1(\mathfrak{ks}(X))$ if $\mathfrak{ks}(X)$ is quasi-isomorphic to a formal dg-Lie algebra with trivial bracket. By Theorem 2.20, it suffices to show that there is an operator Δ satisfying the conditions of the theorem.

Since X has trivial canonical bundle, there is a trivializing section $\omega \in \mathrm{H}^0(X; \omega_X)$, also known as the volume form. Let $n = \dim(X)$. Contracting with ω defines an isomorphism $\eta : \bigwedge^p T_X \cong \Omega_X^{n-p}$. The dg-Lie algebra $\mathfrak{ks}(X)$ sits inside a larger (graded) dg-Lie algebra $\mathfrak{ks}(X)^\bullet$, given by $\mathfrak{ks}(X)^{p,q} = \Omega^{0,q}\left(\bigwedge^p T_X\right)$. Define the operator $\Delta : \mathfrak{ks}(X)^{p,q} \to \mathfrak{ks}(X)^{p-1,q}$ by the composite

$$\mathfrak{ks}(X)^{p,q} = \Omega^{0,q}\left(\bigwedge^p T_X\right) \xrightarrow{\eta} \Omega^{n-p,q}(X) \xrightarrow{\partial} \Omega^{n-p+1,q}(X) \xrightarrow{\eta^{-1}} \Omega^{0,q}\left(\bigwedge^{p-1} T_X\right) \cong \mathfrak{ks}(X)^{p-1,q}.$$

The Tian-Todorov theorem (whose proof we shall omit) is the statement that the Lie bracket on $\mathfrak{ks}(X)$ is exactly the Gerstenhaber bracket. The other condition of Theorem 2.20 is the ∂-$\overline{\partial}$-lemma of Kähler geometry. $\square$

Remark 2.21. The proof of Theorem 2.1 provided above shows that the Kodaira-Spencer dg-Lie algebra $\mathfrak{ks}(X)$ is quasi-isomorphic to a formal dg-Lie algebra with trivial bracket; as such, an immediate consequence is the degeneration of the Hodge-de Rham spectral sequence for a compact Kähler manifold with trivial canonical bundle.

Proof of Theorem 2.20. We first show that $\mathfrak{g}' \to \mathfrak{g}''$ is a morphism of dg-Lie algebras. Namely, we need to show that d vanishes on $\ker(\Delta)/\mathrm{im}(\Delta)$, and that the bracket vanishes on $\ker(\Delta)/\mathrm{im}(\Delta)$. For the latter claim, it suffices to show that the bracket of two elements of $\ker(\Delta)$ is in the image of Δ. This follows from the hypothesis that the Lie bracket on $\mathfrak{g}$ is the Gerstenhaber bracket. To show that d vanishes on $\ker(\Delta)/\mathrm{im}(\Delta)$, it suffices to show that for any $x \in \ker(\Delta)$, the element dx is in the image of Δ. Since $\Delta dx = -d\Delta x$, we have $dx \in \mathrm{im}(d) \cap \ker(\Delta)$; therefore, the ∂-$\overline{\partial}$-lemma assumption implies that dx is also in the image of Δ.

We now show that there are quasi-isomorphisms as indicated in the theorem. We first show that the map $\mathfrak{g}' \to \mathfrak{g}$ is a quasi-isomorphism. Injectivity on cohomology is obvious, so we prove surjectivity. Suppose an element in $\mathrm{H}^*(\mathfrak{g})$ is represented by some $x \in \mathfrak{g}$ with $dx = 0$. Then $\Delta x \in \ker(d) \cap \mathrm{im}(\Delta) = \mathrm{im}(\Delta \circ d)$, so $\Delta x = d\Delta y$ for some y. It follows that $x + dy$ is killed by d, is cohomologous to x, and is killed by Δ; therefore, $x + dy \in \mathrm{H}^*(\mathfrak{g}')$, as desired.

It remains to show that the map $\mathfrak{g}' \to \mathfrak{g}''$ is a quasi-isomorphism. We first prove injectivity on cohomology. Suppose a class in $\mathrm{H}^*(\mathfrak{g}')$, represented by some $x \in \mathfrak{g}'$ such that $dx = 0$, is killed in $\mathrm{H}^*(\mathfrak{g}') = \mathfrak{g}'$. We need to show that $x = dy$ for some y. Since the class of x maps to zero in $\mathfrak{g}'$, we know that $x = \Delta z$ for some z. Thus $x \in \ker(d) \cap \mathrm{im}(\Delta)$, and hence $x \in \mathrm{im}(d)$ by the ∂-$\overline{\partial}$-lemma, as desired. Surjectivity on cohomology is equally easy. $\square$

We shall return to Theorem 2.20 in Section 4, where we prove a generalization (see Theorem 4.11).

3. The algebraic proof of Theorem 1.2

3.1. **Principles of deformation theory.** In this section, we discuss some general principles of deformation theory. Although our discussion will be in the algebraic setting, the results are true more generally.

Definition 3.1. Let Art_k denote the category of local Artinian k-algebras $(A, \mathfrak{m}_A)$ whose residue field is k. Let X be a k-scheme. A deformation of X over $(A, \mathfrak{m}_A) \in \mathrm{Art}_k$ is an A-scheme X_A such that $X_A \times_A A/\mathfrak{m}_A \cong X$. An isomorphism of deformations is an isomorphism of A-schemes which reduces to the identity on the special fiber (i.e., on X). We denote the collection of isomorphism classes of deformations over $(A, \mathfrak{m}_A)$ by $\mathrm{Def}_X(A)$. This defines a functor $\mathrm{Def}_X : \mathrm{Art}_k \to \mathrm{Set}$.

Remark 3.2. The deformations of X over $(A, \mathfrak{m}_A)$ *a priori* form a groupoid, but we will only consider the underlying discrete groupoid. This is problematic in general, but see Proposition 3.7.

Theorem 3.3 (Schlessinger). *A functor $F : \mathrm{Art}_k \to \mathrm{Set}$ is pro-representable by a formal power series ring in n variables over k if the following conditions are satisfied:*

(a) *$F(k) = *$.*
(b) *$F(k[\epsilon]/\epsilon^2)$ is n-dimensional over k.*
(c) *F is formally smooth, i.e., if $A \to A'$ is a surjective morphism, then $F(A) \to F(A')$ is surjective.*
(d) *If $A \to B \leftarrow C$ is a pair of surjections, then $F(A \times_B C) \to F(A) \times_{F(B)} F(C)$ is a bijection.*

Remark 3.4. A slight modification of Theorem 3.3 proves pro-representability of F (not necessarily by a power-series ring) in the weaker situation where only the first and second conditions hold, and the final condition is modified to the statement that if $A \to B \leftarrow C$ is a pair of surjections, then $F(A \times_B C) \to F(A) \times_{F(B)} F(C)$ is a surjection.

An even weaker statement can be obtained by somewhat weaker conditions on F. Namely, we can replace the target of F with the category of groupoids, and write $\mathrm{Art}_{\mathrm{W}}$ to denote the category of local Artin rings with residue field k (these are $\mathrm{W}(k)$-algebras), so F is a functor $\mathrm{Art}_{\mathrm{W}} \to \mathrm{Gpd}$. Remark 3.5 shows that $\mathcal{C}_{k[\epsilon]/\epsilon^2}$ acquires the structure of a k-vector space. The weaker version of Theorem 3.3 can then be stated as follows. Suppose $F : \mathrm{Art}_{\mathrm{W}} \to \mathrm{Gpd}$ is a functor such that:

(a) (a) and (b) of Theorem 3.3 are satisfied;
(b) if $A \to B \leftarrow C$ is a pair of maps with $A \to B$ surjective, then $F(A \times_B C) \to F(A) \times_{F(B)} F(C)$ is a surjection (on the collection of objects up to isomorphism);
(c) in the above setting, the map $F(k[\epsilon]/\epsilon^2 \times_k C) \to F(k[\epsilon]/\epsilon^2) \times F(C)$ is a bijection (on the collection of objects up to isomorphism).

(In SGA 7, a functor satisfying this condition is called a semihomogeneous cofibered groupoid with finite-dimensional tangent space.) Then, there is a pro-object R in $\mathrm{Art}_{\mathrm{W}}$, and an object $\mathcal{X} \in F(R)$, such that the map $\phi : \mathrm{Hom}(R, -) \to F$, which sends a map $R \to A$ to the object $\mathcal{X} \otimes_R A \in F(A)$, is smooth, and induces a bijection on tangent spaces.

Remark 3.5. To make sense of the second condition, we need to know that the set $F(k[\epsilon]/\epsilon^2)$ is a k-vector space. Indeed, k acts on $k[\epsilon]/\epsilon^2$ via a map $k \to \mathrm{End}(k[\epsilon]/\epsilon^2)$; this induces scalar multiplication on $F(k[\epsilon]/\epsilon^2)$. Now, $k[\epsilon]/\epsilon^2$ is a group algebra in Art_k, and so there is a map $F(k[\epsilon]/\epsilon^2 \times_k k[\epsilon]/\epsilon^2) \to F(k[\epsilon]/\epsilon^2)$. The third property of Theorem 3.3 implies that the source is actually $F(k[\epsilon]/\epsilon^2) \times F(k[\epsilon]/\epsilon^2)$. There is therefore a map $F(k[\epsilon]/\epsilon^2) \times F(k[\epsilon]/\epsilon^2) \to F(k[\epsilon]/\epsilon^2)$; this is the desired addition on $F(k[\epsilon]/\epsilon^2)$.

Remark 3.6. When applied to the functor Def_X, the final condition in Theorem 3.3 expresses the statement that a deformation over $A \times_B C$ is equivalent to the datum of a pair of deformations over A and C, and an isomorphism of their restrictions to B.

Proof of Theorem 3.3. We must show that there is a bijection $\mathrm{Hom}_k(k[[x_1,\cdots,x_n]],A) \xrightarrow{\sim} F(A)$. The proof proceeds by induction on the length of A. If A is of length 1, then $A = k$, and the bijection follows from the fact that these are just one-element sets. Suppose now that A has length > 1, and let $x \in A$ be an element killed by $\mathfrak{m}_A$. The k-algebra A/x is of length one less than that of A, so there is a bijection $\mathrm{Hom}_k(k[[x_1,\cdots,x_n]],A/x) \simeq F(A/x)$. There is an isomorphism $A \times_{A/x} A \simeq k[\epsilon]/\epsilon^2 \times_k A$, and hence A is a $k[\epsilon]/\epsilon^2$-torsor over A/x.

It follows that there is a bijection

$$F(A) \times F(k[\epsilon]/\epsilon^2) \simeq F(A \times_k k[\epsilon]/\epsilon^2) \simeq F(A \times_{A/x} A) \simeq F(A) \times_{F(A/x)} F(A),$$

which exhibits $F(A)$ as a $F(k[\epsilon]/\epsilon^2)$-torsor over $F(A/x)$. Similarly, there is a bijection between $\mathrm{Hom}_k(k[[x_1,\cdots,x_n]],A) \times \mathrm{Hom}_k(k[[x_1,\cdots,x_n]],k[\epsilon]/\epsilon^2)$ and $\mathrm{Hom}_k(k[[x_1,\cdots,x_n]],A) \times_{\mathrm{Hom}_k(k[[x_1,\cdots,x_n]],A/x)} \mathrm{Hom}_k(k[[x_1,\cdots,x_n]],A)$, exhibiting $\mathrm{Hom}_k(k[[x_1,\cdots,x_n]],A)$ is a $\mathrm{Hom}_k(k[[x_1,\cdots,x_n]],k[\epsilon]/\epsilon^2)$-torsor over $\mathrm{Hom}_k(k[[x_1,\cdots,x_n]],A/x)$. By the inductive hypothesis, we are reduced to showing that $\mathrm{Hom}_k(k[[x_1,\cdots,x_n]],k[\epsilon]/\epsilon^2) \simeq F(k[\epsilon]/\epsilon^2)$. But $\mathrm{Hom}_k(k[[x_1,\cdots,x_n]],k[\epsilon]/\epsilon^2) \simeq k^n$, and so we win using the second hypothesis in Theorem 3.3. □

Theorem 3.3 is rather formal, and we would like to understand it better in the special case of the functor Def_X. In order to even apply Theorem 3.3, we need to resolve the issue raised in Remark 3.2. The discussion that follows is a special case of Kodaira-Spencer theory. One can write these proofs in a "coordinate-free" way, but we have chosen not to do so for concreteness.

Proposition 3.7. *Let $\pi : X \to \mathrm{Spec}(k)$ be a smooth scheme. Let X' be a deformation of X to $k[\epsilon]/\epsilon^2$. There is a bijection between $\mathrm{H}^0(X; T_X)$ and the set of isomorphisms of X' which restrict to the identity on X.*

More generally, if $(A, \mathfrak{m}_A) \in \mathrm{Art}_k$ and X' is a deformation of X, then there is a bijection between $\mathrm{Hom}(\Omega^1_X, \pi^*\mathfrak{m}_A)$ and automorphisms of X' which are the identity on X.

Proof. Let $\pi : X' \to \mathrm{Spec}(k[\epsilon]/\epsilon^2)$ denote the structure morphism. Let $f : \mathcal{O}_{X'} \to \mathcal{O}_{X'}$ be an automorphism restricting to the identity on X; then, $f - \mathrm{id} : \mathcal{O}_{X'} \to \mathcal{O}_{x'}$ kills the ideal sheaf $\pi^*(\epsilon)$ of X, and so descends to a map $f - \mathrm{id} : \mathcal{O}_X \to \pi^*(\epsilon)$. One now checks that this map is a derivation. Explicitly: we can reduce to the affine case, where $X = \mathrm{Spec}(A)$, and $X' = \mathrm{Spec}(B)$ with $B/\epsilon \cong A$. Any automorphism $f : B \to B$ which restricts to the identity on A must therefore be of the form $f(a + \epsilon a') = a + \epsilon(a' + g(a))$ for some additive $g : A \to A$. It follows that

$$a_1 a_2 + \epsilon g(a_1 a_2) = f(a_1 a_2) = f(a_1) f(a_2) = (a_1 + \epsilon g(a_1))(a_2 + \epsilon g(a_2)) = a_1 a_2 + \epsilon(g(a_1) a_2 + a_1 g(a_2)),$$

and hence g is a derivation. Similarly, any derivation g gives an automorphism of X'. □

Proposition 3.8. *There is an isomorphism $\mathrm{Def}_X(k[\epsilon]/\epsilon^2) \cong \mathrm{H}^1(X; T_X)$ of k-vector spaces.*

More generally, if $(A, \mathfrak{m}_A) \in \mathrm{Art}_k$, then there is a bijection between $\mathrm{Ext}^1_X(\Omega^1_X, \pi^*\mathfrak{m}_A)$ and deformations of X to $(A, \mathfrak{m}_A)$.

Proof. We first show that if $\mathrm{Spec}(R)$ is a smooth k-scheme, then there is a unique deformation of R to $k[\epsilon]/\epsilon^2$. To see this, write $R = k[x_1,\cdots,x_n]/(f_1,\cdots,f_m)$. Lift the f_i's to $g_i \in k[\epsilon, x_1,\cdots,x_n]/\epsilon^2$, and let $R' = k[\epsilon, x_1,\cdots,x_n]/(\epsilon^2, g_1,\cdots,g_m)$. This is a deformation of R to $k[\epsilon]/\epsilon^2$ (so we get existence). If R' and R'' are two deformations, then we need to define an isomorphism $R' \cong R''$ of $k[\epsilon]/\epsilon^2$-algebras. To see this, note that there is a commutative diagram

$$\begin{tikzcd} k[\epsilon]/\epsilon^2 \arrow[r] \arrow[d] & R'' \arrow[d] \\ R' \arrow[r] \arrow[ur, dashed] & R, \end{tikzcd}$$

and the dotted map lifts by formal smoothness. This map is an isomorphism.

For the general case, let us cover X by affine opens $\{U_i\}$, and let $\{U_i'\}$ denote the associated deformations. Since X is smooth, it is a local complete intersection, so we can choose the U_i to be complete intersections. Therefore, U_i' is a trivial deformation of U_i (by the preceding discussion), and we choose an isomorphism $\phi_i : U_i \times_k k[\epsilon]/\epsilon^2 \cong U_i'$. There is an induced automorphism $\psi_{ij} = \phi_i^{-1}\phi_j$ of $U_{ij} \times_k k[\epsilon]/\epsilon^2$. Proposition 3.7 tells us that ψ_{ij} defines a derivation $\mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$, and hence a section $\psi_{ij} \in \mathrm{H}^0(U_{ij}; T_{U_{ij}})$. The sum $\psi_{ij} + \psi_{jk} + \psi_{ki}$ vanishes, because it corresponds to the automorphism

$$\phi_i^{-1}\phi_j\phi_j^{-1}\phi_k\phi_k^{-1}\phi_i = \mathrm{id}_{U_{ijk}}.$$

It follows that ψ_{ij} is a Cech 1-cocycle, and hence a class in $\mathrm{H}^1(X; T_X)$. To show that this class is well-defined, we need to show that any other choice of isomorphism ϕ_i changes ψ_{ij} by a 1-coboundary. To see this, note that if ϕ_i' is another choice of isomorphism $U_i \times_k k[\epsilon]/\epsilon^2 \cong U_i'$, then $\phi_i^{-1}\phi_i'$ is an automorphism of $U_i \times_k k[\epsilon]/\epsilon^2$, and hence comes from a section $s_i \in \mathrm{H}^0(U_i; T_X)$. In particular, $\psi_{ij}' = \psi_{ij} + s_i - s_j$, as desired. □

Proposition 3.9. *There is a canonical class* $\mathrm{ob} \in \mathrm{H}^2(X; T_X)$ *which obstructs the existence of a deformation of X to $k[\epsilon]/\epsilon^2$.*

More generally, if $(A, \mathfrak{m}_A) \in \mathrm{Art}_k$, then there is a class $\mathrm{ob} \in \mathrm{Ext}^2_X(\Omega^1_X, \pi^*\mathfrak{m}_A)$ *which obstructs deformations of X to $(A, \mathfrak{m}_A)$.*

Proof. In the proof of Proposition 3.8, we showed that there is a unique deformation of a smooth affine to $k[\epsilon]/\epsilon^2$. Again, let us cover X by affine opens $\{U_i\}$, and let $\{U_i'\}$ denote the associated deformations. Since X is smooth, it is a local complete intersection, so we can choose the U_i to be complete intersections. Then there is an isomorphism $\psi_{ij} : U_i'|_{U_{ij}} \cong U_j'|_{U_{ij}}$, since U_{ij} is a smooth affine, and these have *unique* deformations. In order for these to glue, they must satisfy the cocycle condition. Therefore, $\psi_{ij}\psi_{jk}^{-1}\psi_{ki}$ gives an automorphism of $U_k'|_{ijk}$; this must be the identity for a deformation of X to exist. It is not hard to see that this class defines a 2-cocycle, so it defines a class in $\mathrm{H}^0(U_{ijk}; T_{U_{ijk}})$ by Proposition 3.7. Up to 2-coboundaries, this class is independent of choices, and so we get a well-defined element $\mathrm{ob} \in \mathrm{H}^2(X; T_X)$. □

Remark 3.10. If X is not necessarily smooth, then one has to work with the cotangent complex (rather than just the Kähler differentials/tangent bundle). Namely, there is a canonical bijection between $\mathrm{Ext}^0_X(L_X, \pi^*\mathfrak{m}_A)$ and automorphisms of a deformation of X to $(A, \mathfrak{m}_A)$ which restrict to the identity on X. There is a canonical bijection between $\mathrm{Ext}^1_X(L_X, \pi^*\mathfrak{m}_A)$ and deformations of X to $(A, \mathfrak{m}_A)$. There is a moduli-theoretic interpretation for the higher cohomology of the cotangent complex, too, involving the language of derived algebraic geometry.

Remark 3.11. The obstruction class of Proposition 3.9 may be given the following interpretation. Suppose $\pi : X_n \to \mathrm{Spec}(k[\epsilon]/\epsilon^{n+1})$ is a deformation of X to $k[\epsilon]/\epsilon^{n+1}$. There is an exact sequence

$$0 \to T_X \to T_{X_n/k[\epsilon]/\epsilon^{n+1}} \to T_{X_{n-1}/k[\epsilon]/\epsilon^n} \to 0$$

induces a long exact sequence in cohomology, and, in particular, a boundary map $\mathrm{H}^1(X_{n-1}; T_{X_{n-1}/k[\epsilon]/\epsilon^n}) \to \mathrm{H}^2(X; T_X)$. By Proposition 3.8, the deformation X_n defines a class $\alpha \in \mathrm{H}^1(X_{n-1}; T_{X_{n-1}/k[\epsilon]/\epsilon^n})$, and its image under the boundary map is the class $\mathrm{ob} \in \mathrm{H}^2(X; T_X)$. This classifies the obstruction of extending X_n to a deformation $X_{n+1} \to \mathrm{Spec}(k[\epsilon]/\epsilon^{n+2})$. (If ob vanishes, then the long exact sequence in cohomology tells us that α lifts to $\mathrm{H}^1(X_n; T_{X_n/k[\epsilon]/\epsilon^{n+1}})$, and hence represents a deformation of X_n to X_{n+1}.)

Remark 3.12. Proposition 3.7 tells us that if $\mathrm{H}^0(X;T_X)=0$, then Def_X naturally lands in sets. Note that if X is Calabi-Yau over $\mathbf{C}$, then there is an isomorphism $\mathrm{H}^0(X;T_X)\cong \mathrm{H}^n(X;\Omega^1_X)^\vee$. There is an abstract isomorphism $\mathrm{H}^n(X;\Omega^1_X)\cong \mathrm{H}^1(X;\omega_X)$, which is isomorphic to $\mathrm{H}^1(X;\mathcal{O}_X)=0$ since the canonical bundle is zero (and the vanishing of cohomology for Calabi-Yau varieties).

3.2. **The algebraic proof.** The proof of Theorem 1.2 provided in this section is an argument due originally to Ran and Kawamata (see [Ran92, Kaw92]). (We learnt it from notes of Daniel Litt; see [LL17].) We first state the following result, and defer its proof.

Theorem 3.13 (T^1-lifting)**.** *Let k be a field of characteristic zero, and let R be a complete local Noetherian k-algebra with residue field k. Then $R\cong k[[x_1,\cdots,x_n]]$ if and only if for each $(A,\mathfrak{m}_A)\in \mathrm{Art}_k$, and each surjection $M\to N$ of A-modules, the induced map $\mathrm{Map}_k(R,A\oplus M)\to \mathrm{Map}_k(R,A\oplus N)$ is a surjection, where $A\oplus M$ and $A\oplus N$ denote the square-zero extensions of A by M and N, respectively.*

Before proving Theorem 1.2, we need the following result, which we will not prove.

Theorem 3.14 (Deligne-Illusie, [DI87])**.** *Let S be a scheme over a field of characteristic zero, and let $f:X\to S$ be a smooth proper morphism. Then $R^q f_*\Omega^p_{X/S}$ is locally free, and its formation commutes with base-change.*

Remark 3.15. An important consequence/result related to Theorem 3.14 is the degeneration of the Hodge-de Rham spectral sequence at the E_1-page for compact Kähler manifolds. The discussion in Section 2, and, in particular, the ∂-$\overline{\partial}$-lemma, is sufficient to prove Theorem 3.14 (see also Remark 2.21). Indeed, the Hodge-de Rham spectral sequence has E_1-page $E_1^{p,q}=\mathrm{H}^p(X;\Omega^q_X)$, and converges to $\mathrm{H}^{p+q}_{\mathrm{dR}}(X;\mathbf{C})$. To show that the spectral sequence degenerates, recall how the differentials work: if $x\in \mathrm{H}^p(X;\Omega^q_X)$, then pick a $\overline{\partial}$-closed form α_1 representing x. The class $\partial\alpha_1$ is $\overline{\partial}$-closed, and represents a class in $\mathrm{H}^p(X;\Omega^{q+1}_X)$; this is $d_1(x)$. If this is zero, then $\partial\alpha_1=\overline{\partial}\alpha_2$ for some α_2, and so $\partial\alpha_2$ is $\overline{\partial}$-closed. The class represented by $\partial\alpha_2$ is $d_2(x)$. Inductively, if $d_i(x)=0$ for $i\leq n$, then $\partial\alpha_{n-1}=\overline{\partial}\alpha_n$, and $d_{n+1}(x)=\partial\alpha_n$. If we start off by choosing α_1 harmonic (which can be done by the ∂-$\overline{\partial}$-lemma), then $\alpha_2=0$ is a perfectly valid choice. This shows that $d_1(x)=0$, and moving inductively in this manner shows that all differentials vanish.

Proof of Theorem 1.2. We first show that Def_X is pro-representable. To show this, we first claim that for any smooth k-scheme X, the following are equivalent:

(a) the functor Def_X is pro-representable;
(b) for every surjection $f:B\to A$ in Art_k such that $\mathfrak{m}_B\ker(f)=0$, and every deformation X_B of X to B, any automorphism of X_A fixing X extends to an automorphism of X_B fixing X;
(c) in the setting of (b), the exact sequence
$$\mathrm{H}^1(X;T_X)\otimes\ker(f)\to \mathrm{Def}_X(B)\to\mathrm{Def}_X(A)\to \mathrm{H}^2(X;T_X)\otimes\ker(f)$$
is exact on the left.

We first show that (c) is equivalent to (a). It suffices to show that Def_X satisfies the conditions of Theorem 3.3 (in particular, the modification indicated in Remark 3.4). To show that the map $\mathrm{Def}_X(A\times_B C)\to \mathrm{Def}_X(A)\times_{\mathrm{Def}_X(B)}\mathrm{Def}_X(C)$ is surjective, it suffices to observe that (c) yields a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(X;T_X)\otimes\ker(f) \arrow[r] \arrow[d, equal] & \mathrm{Def}_X(A\times_B C) \arrow[r] \arrow[d] & \mathrm{Def}_X(C) \arrow[r] \arrow[d] & \mathrm{H}^2(X;T_X)\otimes\ker(f) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{H}^1(X;T_X)\otimes\ker(f) \arrow[r] & \mathrm{Def}_X(A) \arrow[r] & \mathrm{Def}_X(B) \arrow[r] & \mathrm{H}^2(X;T_X)\otimes\ker(f).
\end{tikzcd}$$

To show that (b) is equivalent to (c), note that for any deformation X_B of X to B, there is an exact sequence

$$\mathrm{Aut}^0(X_B) \to \mathrm{Aut}^0(X_A) \to \mathrm{H}^1(X;T_X)\otimes \ker(f) \to \mathrm{Def}_X(B) \to \mathrm{Def}_X(A) \to \mathrm{H}^2(X;T_X)\otimes \ker(f),$$

and so the obstruction to the exactness on the left of the sequence in (c) is precisely the surjectivity of $\mathrm{Aut}^0(X_B) \to \mathrm{Aut}^0(X_A)$. This is the condition in (b).

Utilizing the above criterion and Proposition 3.7, we conclude that Def_X is pro-representable for any smooth scheme X such that $\mathrm{H}^0(X;T_X) = 0$. Having obtained pro-representability, we can now use Theorem 3.13 to prove Theorem 1.2. Namely, we must show that $\mathrm{Def}_X(A\oplus M) \to \mathrm{Def}_X(A\oplus N)$ is surjective for every $A \in \mathrm{Art}_k$ and surjection $M \to N$ of A-modules. It suffices to show that for any deformation $\pi : X_A \to \mathrm{Spec}(A)$ of X, and any lift of X_A to $A\oplus N$, there is a lift of X_A to $A\oplus M$. Since the deformations of X_A to $A\oplus N$ form a torsor for $\mathrm{H}^1(X_A;T_{X_A/A}\otimes \pi^*N)$, it suffices to show that the map $\mathrm{H}^1(X_A;T_{X_A/A}\otimes \pi^*M) \to \mathrm{H}^1(X_A;T_{X_A/A}\otimes \pi^*N)$ is surjective.

Observe that $\mathrm{H}^1(R\Gamma(T_{X_A/A}\otimes \pi^*M)) \cong \mathrm{H}^1(R\Gamma(T_{X_A/A})\otimes^{\mathbf{L}} M)$. If each $\mathrm{H}^i(X_A;T_{X_A/A})$ is locally free, then the Tor/Künneth spectral sequence degenerates, and we would find that $\mathrm{H}^1(R\Gamma(T_{X_A/A})\otimes^{\mathbf{L}} M) \cong \mathrm{H}^1(X_A;T_{X_A/A})\otimes M$. This certainly finishes the proof of surjectivity, because the map $\mathrm{H}^1(X_A;T_{X_A/A}\otimes \pi^*M) \to \mathrm{H}^1(X_A;T_{X_A/A}\otimes \pi^*N)$ is then just the one induced by tensoring the surjection $M \to N$ with $\mathrm{H}^1(X_A;T_{X_A/A})$.

To show that each $\mathrm{H}^i(X_A;T_{X_A/A})$ is locally free, observe that Serre duality gives an isomorphism $\mathrm{H}^i(X_A;T_{X_A/A}) \cong \mathrm{H}^{n-i}(X_A;\Omega^1_{X_A/A}\otimes \omega_{X_A/A})^\vee$. First, observe that $\omega_{X_A/A}$ is trivial, i.e., X_A is itself weakly Calabi-Yau. (Indeed, Theorem 3.14 implies that $\mathrm{H}^0(X_A;\omega_{X_A/A})$ is isomorphic to $\mathrm{H}^0(X;\omega_X)\otimes_k A \cong A$, and is thus trivializable.) Therefore, $\mathrm{H}^{n-i}(X_A;\Omega^1_{X_A/A}\otimes \omega_{X_A/A})^\vee \cong \mathrm{H}^{n-i}(X_A;\Omega^1_{X_A/A})^\vee$. Theorem 3.14 implies that $\mathrm{H}^{n-i}(X_A;\Omega^1_{X_A/A})$ is locally free, and so its dual is also locally free, as desired. □

Proof of Theorem 3.13. Choose generators $f_1,\cdots,f_n$ for $\mathfrak{m}_R/\mathfrak{m}_R^2$; these define a map $f : k[[x_1,\cdots,x_n]] \to R$, whose kernel we will denote by J. We need to show that $J = 0$. It suffices to show that $J \subseteq (x_1,\cdots,x_n)^m$ for all m. For notational simplicity, let us just write $\mathfrak{m}$ to denote $(x_1,\cdots,x_n)$. Note that $J \subseteq \mathfrak{m}^2$; this starts off the induction. The existence of the inclusion $J \subseteq \mathfrak{m}^m$ is equivalent to the quotient map $k[[x_1,\cdots,x_n]] \to k[[x_1,\cdots,x_n]]/\mathfrak{m}^m$ factoring through f.

Define a map

$$\phi : k[[x_1,\cdots,x_n]] \to k[[x_1,\cdots,x_n]]/\mathfrak{m}^m \oplus \bigoplus_{1\le i\le n} k[[x_1,\cdots,x_n]]/\mathfrak{m}^{m-1} t_i,$$

where the t_i are just module generators (to be thought of as dx_i), as follows: a power series f is sent to $f + \sum_i t_i \partial_{x_i} f$. Since $J \subseteq \mathfrak{m}^m$, the derivative ∂_{x_i} sends I to $\mathfrak{m}^{m-1}$. In particular, ϕ kills J, and hence factors through R. The hypothesis of Theorem 3.13, applied with $A = k[[x_1,\cdots,x_n]]/\mathfrak{m}^m$, shows that ϕ lifts to a map

$$R \to k[[x_1,\cdots,x_n]]/\mathfrak{m}^m \oplus \bigoplus_{1\le i\le n} k[[x_1,\cdots,x_n]]/\mathfrak{m}^m t_i.$$

This implies that $\partial_{x_i} f \in \mathfrak{m}^m$ for all $f \in I$. Since we are in characteristic zero, this implies that $I \subseteq \mathfrak{m}^{m+1}$. □

Remark 3.16. The proof of Theorem 3.13 suggests exactly where it fails in characteristic p. Namely, consider $R = k[x]/x^p$; then the induced map $\mathrm{Map}_k(R, A\oplus M) \to \mathrm{Map}_k(R, A\oplus N)$ is always a surjection for any surjection $M \to N$.

3.3. **Theorem 1.2 in characteristic p.** In this section, we shall recall (mostly without proof) results of Schröer proving the characteristic p analogue of Theorem 3.13; see [Sch03]. Let k be a field of characteristic p, and let $\mathrm{W} = \mathrm{W}(k)$. Let $\mathrm{W}_m = \mathrm{W}/p^m$.

Definition 3.17. Let $F : \mathrm{Art}_{\mathrm{W}} \to \mathrm{Gpd}$ be a functor satisfying the conditions in the second paragraph of Remark 3.4 (i.e., is a semihomogeneous cofibered groupoid with finite-dimensional tangent space). Say that F admits a deformation theory if there is a k-vector space T^2 such that for each surjection $B \to A$ with kernel I isomorphic to k, there is a map $\mathrm{ob} : F(A) \to T^2 \otimes_k I$ such that $X_A \in F(A)$ extends to an object in $F(B)$ if and only if $\mathrm{ob}(X_A) = 0$.

If $A \in \mathrm{Art}_{\mathrm{W}}$ and $X \in F(A)$, let $T_F(X/A)$ denote the underlying set of the lax fiber product $F(A[\epsilon]/\epsilon^2) \times_{F(A)} \{X\}$. Concretely, $T_F(X/A)$ is the set of isomorphism classes of objects $Y \in F(A[\epsilon]/\epsilon^2)$ along with a morphism $Y \times_{A[\epsilon]/\epsilon^2} A \to X$ in $F(A)$.

Let $\mathrm{Art}_{\mathrm{W}}^{\wedge}$ denote the category of pro-objects in $\mathrm{Art}_{\mathrm{W}}$ of the form $\{A/\mathfrak{m}_A^n\}$; this allows us to view W itself as an object of $\mathrm{Art}_{\mathrm{W}}^{\wedge}$. We denote by $F^{\wedge} : \mathrm{Art}_{\mathrm{W}}^{\wedge} \to \mathrm{Gpd}$ the induced functor.

Say that F is smooth if the ring R of Remark 3.4 is a formally smooth W-algebra.

Schröer's analogue of Theorem 3.13 states the following.

Theorem 3.18 (T^1-lifting in characteristic p)**.** *Let $F : \mathrm{Art}_{\mathrm{W}} \to \mathrm{Gpd}$ be a functor satisfying the conditions in the second paragraph of Remark 3.4 (i.e., is a semihomogeneous cofibered groupoid with finite-dimensional tangent space). Suppose:*

(a) *F admits a deformation theory;*
(b) *there is a formal object $\mathfrak{X}' \in F^{\wedge}(\mathrm{W})$.*
(c) *for each $X \in \mathrm{W}_m$ and $Y \in \mathrm{W}_{m+1}$ equipped with a map $Y \times_{\mathrm{W}_{m+1}} \mathrm{W}_m \to X$, the map $T_F(Y/\mathrm{W}_{m+1}) \to T_F(X/\mathrm{W}_m)$ is surjective.*
(d) *for each $Y \in \mathrm{W}_{m,d}$ and $X \in \mathrm{W}_{m,d-1}$ equipped with a map $Y \times_{\mathrm{W}_{m,d}} \mathrm{W}_{m,d-1} \to X$, the map $T_F(Y/\mathrm{W}_{m,d}) \to T_F(X/\mathrm{W}_{m,d-1})$ is surjective.*

Then F is smooth.

We will not prove Theorem 3.18 here. To state the analogue of Theorem 1.2 is characteristic p, we introduce some notation: let $\mathrm{W}_{m,d}$ denote the pd-W-algebra $(\mathrm{W}/u^m)\langle t\rangle/(\gamma_d(t), \gamma_{d+1}(t), \cdots)$. If k is a perfect field, then given any pd-$\mathrm{W}(k)$-algebra A, we may form the crystalline cohomology groups $\mathrm{H}^*_{\mathrm{cris}}(X; A)$. Recall that this is the global sections of the structure sheaf on the crystalline site $\mathrm{cris}(X/A)$, whose objects are pd-thickenings $U \hookrightarrow V$ (over A) of (Zariski) opens $U \subseteq X$; the structure sheaf simply assigns the coordinate ring of V to a thickening $U \hookrightarrow V$.

Theorem 3.19 (Schröer)**.** *Let k be a perfect field of characteristic $p > 0$, and let X be a weakly Calabi-Yau k-scheme such that $\dim(X) \leq p$. Suppose:*

- *there is a smooth proper formal deformation $\mathfrak{X}' \to \mathrm{Spf}(\mathrm{W}(k))$ of X.*
- *$\mathrm{H}^i_{\mathrm{cris}}(X; \mathrm{W}_{m,d})$ is a free $\mathrm{W}_{m,d}$-module for all $m, d > 0$.*

Then the versal deformation $\mathfrak{X} \to \mathrm{Spf}(R)$ of X has a smooth base.

Proof. Consider the functor $\mathrm{Def}_X : \mathrm{Art}_{\mathrm{W}} \to \mathrm{Gpd}$ sending $A \in \mathrm{Art}_{\mathrm{W}}$ to the groupoid of smooth proper schemes X_A over A, along with an isomorphism $\phi : X_A \times_A A/\mathfrak{m}_A \xrightarrow{\sim} X$. We shall check that each of the conditions of Theorem 3.18 are satisfied. To check condition (a), we must show that Def_X admits a deformation theory; this follows from Proposition 3.9: one just sets $T^2 = \mathrm{H}^2(X; T_X)$. Condition (b) is our assumption that X lifts to a smooth proper formal scheme $\mathfrak{X}'$ over $\mathrm{W}(k)$. The most interesting conditions to check, of course, are conditions (c) and (d).

By Proposition 3.8, we know that $T_F(X/A) = \mathrm{Ext}^1(\Omega^1_{X_A/A}, \mathcal{O}_{X_A})$, and so to prove the desired surjectivity in (c) and (d), we can argue as in the algebraic proof of Theorem 1.2. For simplicity, let us denote

$A = \mathrm{W}_{m,d}$ (the same argument works for $A = \mathrm{W}_m$). We first claim that the cohomology of each $\Omega^i_{X_A/A}$ is a free A-module. To see this, recall that $\mathrm{H}^i_{\mathrm{cris}}(X;A) \cong \mathrm{H}^i_{\mathrm{dR}}(X_A)$, so our assumption on the crystalline cohomology of X shows that $\mathrm{H}^i_{\mathrm{dR}}(X_A)$ is a free A-module. By our assumption on the dimension of X, the Hodge-de Rham spectral sequence for $\mathrm{H}^*_{\mathrm{dR}}(X)$ degenerates at the E_1-page (by Deligne-Illusie; see [DI87]). It follows that if $\ell = \mathrm{length(A)}$, then

$$\ell \sum_{i+j=n} \dim \mathrm{H}^i(X;\Omega^j_X) = \ell \dim \mathrm{H}^n_{\mathrm{dR}}(X) = \dim \mathrm{H}^n_{\mathrm{dR}}(X_A) \leq \sum_{i+j=n} \dim \mathrm{H}^i(X_A;\Omega^j_{X_A/A}).$$

This inequality is in fact an equality, because we know that $\dim \mathrm{H}^i(X_A;\Omega^j_{X_A/A}) \leq \ell \dim \mathrm{H}^i(X;\Omega^j_X)$. This therefore forces $\dim \mathrm{H}^i(X_A;\Omega^j_{X_A/A}) = \ell \dim \mathrm{H}^i(X;\Omega^j_X)$, and hence the cohomology of each $\Omega^j_{X_A/A}$ is a free A-module.

Having obtained this, the same argument as in the algebraic proof of Theorem 1.2 can now be applied. Indeed, we observe that the canonical bundle of X_A over A is trivial. To show this, observe that the cohomological freeness of $\Omega^j_{X_A/A}$ implies that $\mathrm{H}^0(X_A;\omega_{X_A/A})$ is isomorphic to $\mathrm{H}^0(X;\omega_X) \otimes_k A$, hence is trivializable. The triviality of the canonical bundle in turn implies that $(\Omega^1_{X/A})^\vee \cong (\Omega^1_{X_A/A})^\vee \otimes \omega_{X_A/A} \cong \Omega^{n-1}_{X_A/A}$, where $n = \dim(X)$. Therefore, $\mathrm{Ext}^1(\Omega^1_{X_A/A}, \mathcal{O}_{X_A}) \cong \mathrm{H}^1(X_A;\Omega^{n-1}_{X_A/A})$. Using the cohomological freeness of $\Omega^j_{X_A/A}$ again, we finally conclude that the map $T_F(Y/\mathrm{W}_{m,d+1}) \to T_F(X/\mathrm{W}_{m,d})$ is the tensoring of the surjection $W_{m,d+1} \to W_{m,d} = A$ with $\mathrm{H}^1(X_A;\Omega^{n-1}_{X_A/A})$, and hence is itself surjective, as desired. □

As we explained in Section 2.2, the smoothness of the deformation space of X and the unobstructedness of deformations may also be understood in terms of the formality and abelianness of the Kodaira-Spencer dg-Lie algebra $\mathfrak{ks}(X) = \mathrm{H}^0(X;\Omega^{0,*}(T_X))$. (This, in turn, implies degeneration of the Hodge-de Rham spectral sequence for Calabi-Yau varieties, as mentioned in Remark 2.21.) It is therefore natural to wonder whether the hypotheses of Theorem 3.19 imply the formality and abelianness of $\mathfrak{ks}(X)$ in characteristic p. We will show that this is indeed the case, and that the formality and abelianness of $\mathfrak{ks}(X)$ in characteristic p implies formality and abelianness in characteristic zero (just as Deligne and Illusie showed that degeneration of the Hodge-de Rham spectral sequence in characteristic p implies degeneration in characteristic zero). The arguments used to prove this result are quite general, and are applicable in the generality of noncommutative geometry; we will therefore prove the claimed result in Section 4.

4. Noncommutative analogues

Theorem 1.2 and Theorem 3.19 show that if k is a perfect field, and X is a smooth proper k-scheme whose canonical bundle is trivial such that $\dim(X) \leq p$ if $\mathrm{char}(k) = p > 0$ and X lifts to $\mathrm{W}(k)$, then the deformation theory of X is unobstructed, and the deformation space is smooth. Ultimately, both boil down to the degeneration of the Hodge-de Rham spectral sequence for X.

There is a noncommutative analogue of the degeneration of the Hodge-de Rham spectral sequence: if $\mathcal{C}$ is a k-linear smooth and proper stable ∞-category, then Kaledin and Mathew have shown (see [Kal08, Kal17, Mat17]) that the Tate spectral sequence for $\mathrm{HP}(\mathcal{C})$ degenerates for arbitrary $\mathcal{C}$ if k is of characteristic zero, and in characteristic p if $\mathrm{HH}_*(\mathcal{C})$ is concentrated in dimensions $\geq -p$ and $\leq p$ and $\mathcal{C}$ lifts to $\mathrm{W}_2(k)$. If $\mathcal{C} = \mathrm{QCoh}(X)$ (and the dimension condition is satisfied in characteristic p), then $\mathrm{HH}_*(\mathcal{C})$ is isomorphic to the Dolbeaut cohomology of X, and $\mathrm{HP}_*(\mathcal{C})$ is a 2-periodic version of the de Rham cohomology of X; the Tate spectral sequence is a 2-periodic version of the Hodge-de Rham spectral sequence. In light of this, it is natural to ask if there is a noncommutative analogue of Theorems 1.2 and 3.19. Our goal in this section is to show that this is indeed the case.

4.1. **Hodge-de Rham/homotopy fixed points degeneration and formality.** The goal of this section is to prove a generalization of Theorem 2.20. The main result of this section is Theorem 4.11, originally discussed (in characteristic zero) in a paper of Katzarkov-Kontsevich-Pantev (see [KKP08]). We will let k denote a perfect field of arbitrary characteristic.

Recollection 4.1. Let M be a k-module spectrum with an S^1-action. Then $\pi_* M$ acquires the structure of a dg-k-module, via the differential coming from the isomorphism $\mathrm{H}^*(\mathrm{S}^1; k) \cong k[\epsilon]/\epsilon^2$.

Definition 4.2. Let $\mathfrak{g}$ be a dg-Lie algebra over k. Say that $\mathfrak{g}$ has an S^1-source if there is a Lie algebra M in projective k-module spectrum, such that $\pi_* M \cong \mathfrak{g}$, and M is equipped with an S^1-action which induces the differential on $\mathfrak{g}$; the module M is called the S^1-source of $\mathfrak{g}$.

Remark 4.3. If k is of characteristic zero, then the rational formality of S^1 implies that every dg-Lie algebra $\mathfrak{g}$ has an S^1-source (given by $\mathfrak{g}$ itself).

Remark 4.4. Suppose $\mathfrak{g}$ is a dg-Lie algebra over k with S^1-source M. Then $M^{h\mathrm{S}^1}$ is also Lie algebra object in $k^{h\mathrm{S}^1}$-module spectra. Since $\pi_* k^{h\mathrm{S}^1} \cong k[[\beta]]$ with β the Bott element in degree 2, we see that $\pi_* M^{h\mathrm{S}^1}$ is a dg-Lie algebra over $k[[\beta]]$.

Proposition 4.5. *Let $\mathfrak{g}$ be a dg-Lie algebra over k with S^1-source M. Then $\mathfrak{g}$ is quasi-isomorphic to a formal dg-Lie algebra if the S^1-homotopy fixed points spectral sequence for $M^{h\mathrm{S}^1}$ degenerates. The converse is true in characteristic zero.*

Proof. In fact, the S^1-homotopy fixed points spectral sequence for $M^{h\mathrm{S}^1}$ degenerates if and only if M has trivial S^1-action. Indeed, M is a projective k-module, and hence is a summand of a free k-module spectrum; the degeneration of the S^1-homotopy fixed points spectral sequence implies that M is S^1-equivariantly a summand of a free k-module spectrum with trivial S^1-action, and hence itself has trivial S^1-action. The converse in characteristic zero is an easy consequence of the rational formality of S^1. □

Construction 4.6. Suppose $\mathfrak{g}$ is a dg-Lie algebra over k with S^1-source M such that the S^1-homotopy fixed points spectral sequence degenerates. (For simplicity, we will just refer to this degeneration as HFPSS degeneration.) Then $\pi_*(M^{h\mathrm{S}^1}) \cong \mathfrak{g}[[\beta]]$, where β is the Bott class in degree 2. Remark 4.4 tells us that $\mathfrak{g}[[\beta]]$ is therefore a dg-Lie algebra over $k[[\beta]]$. Let D denote the differential on $\mathfrak{g}[[\beta]]$; then D modulo β is the differential on $\mathfrak{g}$. Since the S^1-homotopy fixed points spectral sequence degenerates, Proposition 4.5 implies that $D = \beta\widetilde{\Delta}$ for some operator $\widetilde{\Delta}$ on $\mathfrak{g}[[\beta]]$ of degree -1 which squares to zero.

Warning 4.7. In the setting of Construction 4.6, the action of $\widetilde{\Delta}$ on β need not be trivial. In other words, $\widetilde{\Delta}$ need not be base-changed from an operator of degree -1 on $\mathfrak{g}$.

Definition 4.8. Let $\mathfrak{g}$ be a dg-Lie algebra over k. Suppose $\mathfrak{g}$ admits the structure of a dg BV-algebra via an operator Δ, such that the induced Lie bracket (via Lemma 2.19) is the Lie bracket of $\mathfrak{g}$, and $d\Delta + \Delta d = 0$. We call this structure a dg-BV-Lie algebra, and denote it by $(\mathfrak{g}, \Delta)$.

If $\mathfrak{g}$ is a dg-Lie algebra over k with S^1-source M with HFPSS degeneration, then Construction 4.6 gives an operator $\widetilde{\Delta}$ on $\mathfrak{g}[[\beta]]$; say that $\mathfrak{g}$ is a dg-BV-Lie algebra with S^1-source M with HFPSS degeneration if $\widetilde{\Delta}$ is base-changed from an operator Δ on $\mathfrak{g}$ such that $(\mathfrak{g}, \Delta)$ is a dg-BV-Lie algebra.

Proposition 4.9. *Let $\mathfrak{g}$ be a dg-BV-Lie algebra with S^1-source M with HFPSS degeneration. If k is of characteristic p, suppose that $\mathfrak{g}$ is concentrated in degrees $> -p$ and $< p$. Then $\mathfrak{g}((\beta)) := \mathfrak{g}[[\beta]] \otimes_{k[[\beta]]} k((\beta))$ is quasi-isomorphic to an abelian dg-Lie algebra.*

Proof. Let f denote the operator on $\mathfrak{g}((\beta))$ given by

$$f(x) = \beta\Delta(x) + [x, x].$$

It suffices to show that there is some automorphism of $\mathfrak{g}((\beta))$ such that $f(x) = \beta\Delta$. Consider the automorphism ϕ defined by

$$\phi(x) = \beta(e^{x/\beta} - 1) = \sum_{n\geq 1} \frac{x^n}{n!\beta^{n-1}},$$

with inverse given by

$$\phi^{-1}(x) = \beta \log\left(\frac{x}{\beta} + 1\right) = \sum_{n\geq 1} (-1)^{n-1} \frac{x^n}{n\beta^{n-1}}.$$

These series are well-defined in characteristic zero, and our assumption that $\mathfrak{g}$ is concentrated in degrees $> -p$ and $< p$ kicks in when k is of characteristic p to ensure that ϕ and ϕ^{-1} remain well-defined. In these coordinates, we have

$$f(\phi(x)) = (\beta\Delta(x) + [x,x])e^{x/\beta} = \beta(\beta\Delta(x\beta^{-1}) + \beta[x\beta^{-1}, x\beta^{-1}])e^{x/\beta} = \beta\Delta(\phi(x)),$$

as desired. □

Remark 4.10. The automorphism ϕ is essentially the inverse of the Bernoulli series.

Theorem 4.11. *Let $\mathfrak{g}$ be a dg-BV-Lie algebra with S^1-source M with HFPSS degeneration. If k is of characteristic p, suppose that $\mathfrak{g}$ is concentrated in degrees $> -p$ and $< p$. Then $\mathfrak{g}$ is quasi-isomorphic to a formal abelian dg-Lie algebra.*

Proof. Proposition 4.5 shows that $\mathfrak{g}$ is quasi-isomorphic to a formal dg-Lie algebra. It remains to show that $\mathfrak{g}$ is quasi-isomorphic to an abelian dg-Lie algebra. Since $\mathfrak{g}$ is a dg-BV-Lie algebra with HFPSS degeneration, this is equivalent to showing that $\mathfrak{g}[[\beta]]$ is quasi-isomorphic to an abelian dg-Lie algebra. This is a consequence of Proposition 4.9, which shows that $\mathfrak{g}((\beta))$ is quasi-isomorphic to an abelian dg-Lie algebra. □

4.2. **Hochschild homology.** In this section, we study the Hochschild homology of Calabi-Yau categories.

Definition 4.12. Let $\mathcal{C}$ be a fully dualizable object of the $(\infty, 2)$-category of cocomplete k-linear stable ∞-categories, and assume that $\mathcal{C}$ is compactly generated. The ∞-category $\mathcal{C}$ then admits a Serre functor $\mathrm{S} : \mathcal{C} \to \mathcal{C}$, characterized by natural equivalences $\mathrm{Hom}_{\mathcal{C}}(X,Y)^\vee \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Y, \mathrm{S}(X))$ for all $X, Y \in \mathcal{C}^\omega$. A weak n-Calabi-Yau structure on a smooth and proper ∞-category $\mathcal{C}$ which admits a Serre functor S is the choice of an equivalence between S and the functor $\Sigma^n : \mathcal{C} \to \mathcal{C}$. We will just say that $\mathcal{C}$ is weakly n-Calabi-Yau if it admits a weak n-Calabi-Yau structure.

Example 4.13. Let $\mathcal{C} = \mathrm{QCoh}(X)$. Then $\mathcal{C}$ admits a Serre functor, given by tensoring with the dualizing complex ω_X. The category $\mathcal{C}$ is weakly n-Calabi-Yau if and only if X is an n-dimensional weak Calabi-Yau scheme.

The main result of this section is the following.

Proposition 4.14. *Let $\mathcal{C}$ be a weakly n-Calabi-Yau k-linear stable ∞-category. If k is of characteristic p, assume that $\mathcal{C}$ lifts to $\mathrm{W}_2(k)$, and that $\mathrm{HH}_*(\mathcal{C})$ is concentrated in dimensions $> -p$ and $< p$. Then $\mathrm{HH}_*(\mathcal{C})$ admits the structure of a dg-BV-Lie algebra with S^1-source $\mathrm{HH}(\mathcal{C})$ with HFPSS degeneration, and hence (by Theorem 4.11) is quasi-isomorphic to a formal abelian dg-Lie algebra.*

Proof sketch. As mentioned above, the conditions on $\mathcal{C}$ imply (by Mathew and Kaledin) that the HFPSS for the canonical S^1-action on $\mathrm{HH}(\mathcal{C})$ degenerates. To prove the desired result, it therefore suffices to show that if $\mathcal{C}$ is weakly Calabi-Yau, then $\mathrm{HH}_*(\mathcal{C})$ admits the structure of a dg BV-algebra via an operator Δ satisfying $\Delta d + d\Delta = 0$ (and just define the Lie bracket via Lemma 2.19). Since $\mathcal{C}$ admits a Serre functor S, and one can view $\mathrm{HH}(\mathcal{C})$ as $\mathrm{Map}_{\mathrm{End}_k(\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{S})$, we see that if $\mathcal{C}$ is weakly n-Calabi-Yau, then $\mathrm{HH}^*(\mathcal{C}) \simeq \mathrm{HH}_{*-n}(\mathcal{C})$, in *homological* grading. The S^1-action on the Hochschild cohomology of $\mathcal{C}$

defines an operator of cohomological degree 1, and hence of homological degree −1. Under the above equivalence of Hochschild homology and cohomology, this translates to the desired operator Δ. □

Remark 4.15. One can prove a strengthening of Proposition 4.14 to prove the formality of $\mathrm{HH}(\mathcal{C})$ itself. This implies that the deformation theory of a weakly n-Calabi-Yau k-linear stable ∞-category satisfying the conditions of the proposition is unobstructed and smooth.

Remark 4.16. Proposition 4.14 implies Theorem 1.2: as discussed in Section 2.3, it suffices to show that $\mathfrak{ks}(X)$ is quasi-isomorphic to a formal abelian dg-Lie algebra, which in turn follows from the formality and abelianness of the graded dg-Lie algebra $\mathfrak{ks}(X)^\bullet$ with

$$\mathfrak{ks}(X)^{p,q} = \mathrm{H}^0\left(X;\Omega^{0,q}\left(\bigwedge^p T_X\right)\right) \cong \mathrm{H}^0(X;\Omega_X^{n-p,q}) \cong \mathrm{H}^q(X;\Omega_X^{n-p}).$$

(Note that the first isomorphism used the isomorphism $\bigwedge^p T_X \cong \Omega_X^{n-p}$ provided by choosing a volume form $\omega \in \mathrm{H}^0(X;\omega_X)$ trivializing ω_X.) Since the Hochschild-Kostant-Rosenberg (HKR) spectral sequence degenerates in characteristic zero, there is an isomorphism $\mathrm{HH}_*(X) \cong \mathrm{H}^*(X;\Omega_X^*)$, which respects the BV-operator Δ. The formality and abelianness of $\mathfrak{ks}(X)^\bullet$ now follows from Proposition 4.14.

Similarly, Proposition 4.14 implies (a generalization of) Theorem 3.19. If X is a Calabi-Yau variety over a perfect field k of characteristic $p > 0$, such that $\dim(X) < p$ and X lifts to $\mathrm{W}_2(k)$, then $\mathcal{C} = \mathrm{Perf}(X)$ satisfies the hypotheses of Proposition 4.14, and therefore $\mathrm{HH}_*(X)$ is formal. The assumption that $\dim(X) < p$ also implies that the HKR spectral sequence for X degenerates at the E_2-page, and therefore $\mathrm{HH}_*(X) \cong \mathfrak{ks}(X)^\bullet$. As discussed above, the formality and abelianness of $\mathfrak{ks}(X)^\bullet$ implies the smoothness and unobstructedness of deformations of X.

Remark 4.17. We remarked in the previous section that the formality and abelianness of $\mathfrak{ks}(X)$ in characteristic p implies formality and abelianness in characteristic zero. This is true in the sense that the HFPSS degeneration for $\mathrm{HH}(\mathcal{C})$ in characteristic zero can be proved as a consequence of the HFPSS degeneration in characteristic p.

References

[Bog78] F. Bogomolov. Hamiltonian Kählerian manifolds. *Dokl. Akad. Nauk SSSR*, 243(5):1101–1104, 1978. (Cited on page 1.)

[DI87] P. Deligne and L. Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987. (Cited on pages 9 and 12.)

[GR17] D. Gaitsgory and N. Rozenblyum. *A study in derived algebraic geometry. Vol. II. Deformations, Lie theory and formal geometry*, volume 221 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017. (Cited on page 4.)

[Kal08] D. Kaledin. Non-commutative Hodge-to-de Rham degeneration via the method of Deligne-Illusie. *Pure Appl. Math. Q.*, 4(3, Special Issue: In honor of Fedor Bogomolov. Part 2):785–875, 2008. (Cited on page 12.)

[Kal17] D. Kaledin. Spectral sequences for cyclic homology. In *Algebra, geometry, and physics in the 21st century*, volume 324 of *Progr. Math.*, pages 99–129. Birkhäuser/Springer, Cham, 2017. (Cited on page 12.)

[Kaw92] Y. Kawamata. Unobstructed deformations. *J. Algebraic Geom.*, 1(2):183–190, 1992. (Cited on pages 1 and 9.)

[KKP08] L. Katzarkov, M. Kontsevich, and T. Pantev. Hodge theoretic aspects of mirror symmetry. In *From Hodge theory to integrability and TQFT tt*-geometry*, volume 78 of *Proc. Sympos. Pure Math.*, pages 87–174. Amer. Math. Soc., Providence, RI, 2008. (Cited on page 13.)

[LL17] D. Litt and H. Liu. Notes for Topics in AG: Deformation Theory. `http://math.columbia.edu/~hliu/classes/f17-algebraic-geometry.pdf`, 2017. (Cited on page 9.)

[Lur11] J. Lurie. Derived Algebraic Geometry X: Formal Moduli Problems. `https://www.math.ias.edu/~lurie/papers/DAG-X.pdf`, 2011. (Cited on page 4.)

[Mat17] A. Mathew. Kaledin's degeneration theorem and topological Hochschild homology. `https://arxiv.org/abs/1710.09045`, 2017. (Cited on page 12.)

[Pri10] J. Pridham. Unifying derived deformation theories. *Adv. Math.*, 224(3):772–826, 2010. (Cited on page 4.)

[Ran92] Z. Ran. Lifting of cohomology and unobstructedness of certain holomorphic maps. *Bull. Amer. Math. Soc. (N.S.)*, 26(1):113–117, 1992. (Cited on pages 1 and 9.)

[Sch03] S. Schröer. The T^1-lifting theorem in positive characteristic. *J. Algebraic Geom.*, 12(4):699–714, 2003. (Cited on pages 1 and 11.)

[Tia87] G. Tian. Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric. In *Mathematical aspects of string theory (San Diego, Calif., 1986)*, volume 1 of *Adv. Ser. Math. Phys.*, pages 629–646. World Sci. Publishing, Singapore, 1987. (Cited on page 1.)

[Tod89] A. Todorov. The Weil-Petersson geometry of the moduli space of $\mathrm{SU}(n \geq 3)$ (Calabi-Yau) manifolds. I. *Comm. Math. Phys.*, 126(2):325–346, 1989. (Cited on page 1.)

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