ON TENSOR PRODUCTS WITH EQUIVARIANT COMMUTATIVE OPERADS

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Abstract. We affirm and generalize a conjecture of Blumberg and Hill: unital weak $\mathcal{N}_\infty$-operads are closed under ∞-categorical Boardman-Vogt tensor products and the resulting tensor products correspond with *joins of weak indexing systems*; in particular, we acquire a natural G-symmetric monoidal equivalence

$$\underline{\mathrm{CAlg}}^{\otimes}_I \underline{\mathrm{CAlg}}^{\otimes}_J \mathcal{C} \simeq \underline{\mathrm{CAlg}}^{\otimes}_{I \vee J} \mathcal{C}.$$

We accomplish this by showing that $\mathcal{N}^{\otimes}_{I\infty}$ is $\overset{\mathrm{BV}}{\otimes}$-idempotent and $\mathcal{O}^{\otimes}$ is local for the corresponding smashing localization if and only if $\mathcal{O}$-monoid G-spaces satisfy I-indexed Wirthmüller isomorphisms.

Ultimately, we accomplish this by advancing the equivariant higher algebra of *cartesian and cocartesian I-symmetric monoidal ∞-categories*. Additionally, we acquire a number of structural results concerning G-operads, including a canonical lift of $\otimes$ to a presentably symmetric monoidal structure and a general disintegration & assembly procedure for computing tensor products of non-reduced unital G-operads. All such results are proved in the generality of atomic orbital ∞-categories.

We also achieve the expected corollaries for (iterated) Real topological Hochschild and cyclic homology and construct a natural I-symmetric monoidal structure on right modules over an $\mathcal{N}_{I\infty}$-algebra.

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Introduction

We're concerned with the relationship between homotopy-coherent interchange and equivariant commutative algebras, incarnated via $\mathcal{N}_{I\infty}$-*algebras* (henceforth I-*commutative algebras*) in the sense of [BH15; Ste25]. In particular, in [Ste25] we constructed a natural "pointwise" G-symmetric monoidal structure $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ on $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. We hope to answer the following questions, where $\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{C}) \coloneqq \underline{\mathrm{Alg}}_{\mathcal{N}_{I\infty}}^{\otimes}(\mathcal{C})$.

Questions. Let $\mathcal{O}^{\otimes}$ be a unital G-operad and $I, J \subset \mathbb{F}_G$ a pair of unital weak indexing categories.

- (I) When is the forgetful natural transformation $\mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{CAlg}}_I^{\otimes}(-) \implies \mathrm{CAlg}_I(-)$ an equivalence?
- (II) When is the forgetful natural transformation $\mathrm{CAlg}_I\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(-) \implies \mathrm{Alg}_{\mathcal{O}}(-)$ an equivalence?
- (III) What is the (unique) G-operad $\mathcal{O}^{\otimes}$ with natural equivalence $\mathrm{CAlg}_I\underline{\mathrm{CAlg}}_J^{\otimes}(-) \simeq \mathrm{Alg}_{\mathcal{O}}(-)$? ◁

Each of the left hand sides of these proposed equivalences are corepresented by *Boardman-Vogt tensor products* of G-operads, so these Questions (I) and (II) are equivalent to the question of when distinguished maps $\mathcal{N}_{I\infty}^{\otimes} \to \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ and $\mathcal{O}^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes}$ are equivalences; moreover Question (III) asks the value of the tensor product $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes}$. In this form, [BH15, Conj 6.27] conjectured an answer.

Conjecture (Blumberg-Hill)**.** If I and J are *indexing categories* then $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{I \vee J\infty}^{\otimes}$. ◁

We begin by completely characterizing G-operad algebras in (co)cartesian I-symmetric monoidal ∞-categories: cocartesian I-symmetric monoidal structures are characterized by the property that their G-objects canonically lift to $\mathcal{O}$-algebras for any reduced I-operad $\mathcal{O}^{\otimes}$, and cartesian I-symmetric monoidal structures are characterized by an $\mathcal{O}$-*monoid* formula generalizing [HA, Prop 2.4.2.5]. Using this, we show that Question (I) is true precisely when $\mathcal{O}^{\otimes}$ is *reduced* and I-commutative algebras admit underlying $\mathcal{O}$-algebra structures.

We conclude that the unique map $\mathbb{E}_0^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$ witnesses $\mathcal{N}_{I\infty}^{\otimes}$ as an idempotent algebra in $\mathrm{Op}_G^{\mathrm{uni}}$; Question (II) asks to classify the associated smashing localization. Indeed, we show that the equivalence holds whenever $\mathcal{O}$-algebra G-spaces satisfy I-indexed Wirthmüller isomorphisms.

Since $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{S}_G)$-ambidextrous arities form a weak indexing category, we find that the intersection of the $\mathcal{N}_{I\infty}^{\otimes}$- and $\mathcal{N}_{J\infty}^{\otimes}$-smashing local categories is the $\mathcal{N}_{I\vee J\infty}^{\otimes}$-smashing local category, constructing an equivalence $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{I\vee J\infty}^{\otimes}$ in full generality. This answers Question (III) by affirming the evident unital extension of Blumberg-Hill's conjecture, constructing a (unique) natural equivalence

$$\mathrm{CAlg}_I\underline{\mathrm{CAlg}}_J^{\otimes}(\mathcal{C}) \simeq \mathrm{CAlg}_{I\vee J}(\mathcal{C}).$$

This is the third part of an ongoing project to develop the parameterized and equivariant higher algebra predicted in [BDGNS16; NS22] into simply usable foundations for equivariant homotopy theory and K-theory [Ste24; Ste25]; as such, we spend the last third of the paper fleshing out higher algebraic corollaries.

These corollaries fall into two classes: the first class gives $\mathrm{Comm}_G^{\otimes} \in \mathrm{Op}_G$ a unique idempotent algebra structure, which determines a unique compatible idempotent algebra structure on its G-symmetric monoidal envelope, enabling *symmetric monoidality* of the equivariant equifibered perspective of [BHS22; BS24b; HK24]. From this, we lift Op_G with the Boardman-Vogt tensor product to a canonical presentably symmetric monoidal G-∞-category; as an application, we develop equivariant operadic disintegration and assembly, and the associated distributivity of $\overset{\mathrm{BV}}{\otimes}$ allows us to compute tensor products of unital G-operads whose underlying G-∞-categories are G-spaces in terms of tensor products of reduced G-operads.

The second class simply applies Questions (I) and (II): by answering Question (II) for $\mathcal{N}_{I\infty}^{\otimes} \simeq \mathbb{E}_{\infty}^{\otimes}$, we get an $\mathcal{O}$-symmetric monoidal structure on left modules over an $\mathcal{O}$-algebra; for instance, specializing to $\mathcal{O}^{\otimes} = \mathcal{N}_{J\infty}^{\otimes}$ confirms a hypothesis of Hill [Hil17, Rmk 3.15].

By answering Question (I) for $\mathcal{O}^{\otimes} \simeq \mathbb{E}_V^{\otimes}$, we acquire an I-commutative algebra structure on (lax) I-symmetric monoidal $\mathbb{E}_V$-algebra invariants of I-commutative algebras; for instance, this constructs an I-commutative algebra structure on Real topological Hochschild homology and Real topological cyclic homology of an I-commutative algebra whenever I-commutative algebras have underlying $\mathbb{E}_\sigma$-algebras.

We now move to a more careful account of the background, motivation, and main results of this paper.

Background and motivation. Let $\mathcal{C}$ be a 1-category with finite products. Recall that a *commutative monoid in* $\mathcal{C}$ is the data

$$A \in \mathrm{Ob}(\mathcal{C}), \qquad \text{multiplication } \mu\colon A \times A \to A, \text{ and} \qquad \text{unit } \eta\colon * \to A,$$

subject to the usual unitality, associativity, and commutativity assumptions; more generally, if $(\mathcal{C},\otimes,1)$ is a symmetric monoidal 1-category, a *commutative algebra in* $\mathcal{C}$ is the data of

$$R \in \mathrm{Ob}(\mathcal{C}), \qquad \text{multiplication } \mu\colon R\otimes R \to R, \text{ and } \qquad \text{unit } \eta\colon 1 \to R,$$

satisfying analogous conditions. When $\mathcal{C} = \mathrm{Set}$, this recovers the traditional theory of commutative monoids, and when $\mathcal{C} = \mathrm{Mod}_k$ with the tensor product of k-modules, this recovers the traditional theory of commutative k-algebras. These have been the subject of a great deal of homotopy theory in three guises:

(i) We may define the $(2,1)$-category $\mathsf{Span}(\mathbb{F})$ to have objects the finite sets, morphisms from X to Y the spans of finite sets $X \leftarrow R \rightarrow Y$, 2-cells the isomorphisms of spans

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] \arrow[d, dashed, "\sim"] & \\ X & R' \arrow[l] \arrow[r] & Y, \end{tikzcd}$$

and composition the pullback of spans

$$\begin{tikzcd} & & R_{XZ} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & R_{XY} \arrow[dl] \arrow[dr] & & R_{YZ} \arrow[dl] \arrow[dr] & \\ X & & Y & & Z. \end{tikzcd}$$

If $\mathcal{C}$ is an ∞-category, then we define the ∞-*category of commutative monoids in* $\mathcal{C}$ as the $\mathcal{C}$-valued models of the associated Lawvere theory; that is, we define the product-preserving functor category

$$\mathrm{CMon}(\mathcal{C}) \coloneqq \mathrm{Fun}^{\times}(\mathsf{Span}(\mathbb{F}),\mathcal{C}),$$

noting that products in $\mathsf{Span}(\mathbb{F})$ correspond with disjoint unions of finite sets. Indeed, if $\mathcal{C}$ is a 1-category and A a commutative monoid in $\mathcal{C}$, we flesh this out with the dictionary

$$\begin{aligned} ([2] = [2] \to [1]) &\qquad\longmapsto\qquad \mu\colon A^{\times 2} \to A; \\ (\varnothing = \varnothing \to [1]) &\qquad\longmapsto\qquad \eta\colon * \simeq A^{\times 0} \to A; \\ ([1] \leftarrow [2] = [2]) &\qquad\longmapsto\qquad \Delta\colon A \to A^{\times 2} \\ ([1] \leftarrow \varnothing = \varnothing) &\qquad\longmapsto\qquad !\colon A \to A^{\times 0} \simeq *. \end{aligned}$$

Unitality, associativity, and commutativity are conveniently packaged by functoriality. This turns out to be equivalent to Graeme Segal's *special* Γ *spaces* [Seg74] when $\mathcal{C} = \mathcal{S}$, and for general $\mathcal{C}$, it recovers the anologously defined theory in $\mathcal{C}$ (see [BHS22, Ex 3.1.6, Prop 3.1.16, Prop 5.2.14]).

(ii) We say that a pointed ∞-category is *semiadditive* if it has finite products and coproducts and for all finite sets S, the "identity matrix" natural transformation $\coprod_{s\in S}(-) \implies \prod_{s\in S}(-)$ is an equivalence. The full subcategory $\mathrm{Pr}^{L,\oplus} \subset \mathrm{Pr}^{L}$ of *semiadditive presentable* ∞-*categories* possesses a localization functor $L_{\oplus}\colon \mathrm{Pr}^{L} \to \mathrm{Pr}^{L,\oplus}$, which we study.

(iii) Let Op denote the ∞-category of operads.[1] Then, there is a terminal operad $\mathrm{Comm}^{\otimes} \simeq \mathbb{E}_{\infty}^{\otimes}$; given $\mathcal{C}$ a symmetric monoidal ∞-category, we may form the ∞-category of *commutative algebra objects*

$$\mathrm{CAlg}(\mathcal{C}) \coloneqq \mathrm{Alg}_{\mathrm{Comm}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_{\infty}}(\mathcal{C}).$$

We study this and its specialization to the cartesian symmetric monoidal structure.

These three perspectives each present the same ∞-category, i.e. [Cra11; GGN15] show that

$$\mathrm{CMon}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C}^{\times}) \simeq L_{\oplus}\mathcal{C}. \tag{1}$$

As a result, translating between these perspectives has proved invaluable; for instance, [GGN15] uses Perspectives (ii) and (iii) to construct an essentially unique symmetric monoidal structure on $\mathrm{CMon}(\mathcal{C})$ and [CHLL24a] uses Perspectives (i) and (iii) to model commutative algebras in $\mathrm{CMon}(\mathcal{C})^{\otimes}$ as models for the Lawvere theory of *commutative semirings*.

Crucially, Perspectives (i) and (iii) may be used to construct homotopical lifts of the *Eckmann-Hilton argument*; for instance, in [SY19], it is shown that for *any* reduced operad $\mathcal{O}^{\otimes}$, the forgetful functors

$$\mathrm{CAlgAlg}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C}) \leftarrow \mathrm{Alg}_{\mathcal{O}}\mathrm{CAlg}^{\otimes}(\mathcal{C}),$$

[1] This is unambiguous [HM23], but we will tend to model these as ∞-operads in the sense of [HA].

are equivalences for the "pointwise" symmetric monoidal structure on algebras. Such an equivalence may be exhibited by recognizing the far left and far right side each as algebras over the *Boardman-Vogt tensor product* $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes}$ and each arrow as pullback along the canonical map

$$\mathrm{Comm}^{\otimes} \simeq \mathrm{triv}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes} \xrightarrow{\mathrm{can} \otimes \mathrm{id}} \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes};$$

that Eq. (1) consists of equivalences reduces to the well-known fact that $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes} \in \mathrm{Op}$ is terminal, which one can quickly prove via Perspectives (ii) and (iii).

This result is used ubiquitously to replace (lax) symmetric monoidal functors $\mathrm{Alg}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathcal{C}^{\otimes}$ with (lax) symmetric monodial endofunctors

$$\mathrm{CAlg}^{\otimes}(\mathcal{C}) \simeq \mathrm{CAlg}^{\otimes}\mathrm{Alg}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathrm{CAlg}^{\otimes}(\mathcal{C});$$

for instance, this underlies the symmetric monoidal structure on left-modules [HA] and the multiplicative structure on factorization homology [HA, Thm 5.5.3.2], TC [NS18, § IV.2], and algebraic K-theory [Bar15].

This paper concerns the analogs of Perspectives (i) to (iii) in the equivariant theory of algebra stemming from Hill-Hopkins-Ravanel's use of *norms of G-spectra* on the Kervarire invariant one problem, as well as the resulting theory of *indexed tensor products and (co)products* (see [BDGNS16; HH16; NS22]).

For the rest of this introduction, fix G a finite group. In G-equivariant homotopy theory, the point is replaced with elements of the *orbit category* $\mathcal{O}_G \subset \mathrm{Set}_G$, whose objects are homogeneous G-sets $[G/H]$; indeed, Elmendorf's theorem [Elm83] realizes G-spaces as coefficient systems $\mathcal{S}_G \simeq \mathrm{Fun}\left(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}\right)$.[2] In G-equivariant higher category theory, ∞-categories are thus replaced with G-∞-categories

$$\mathrm{Cat}_G := \mathrm{Fun}\left(\mathcal{O}_G^{\mathrm{op}}, \mathrm{Cat}\right).$$

In G-equivariant higher algebra, following Perspective (i), we may form the effective Burnside 2-category $\mathrm{Span}(\mathbb{F}_G)$ whose objects are finite G-sets, whose morphisms are spans, whose 2-cells are isomorphisms of spans, and whose composition is pullback; the following central definition is the heart of this subject.

Definition. The *∞-category of G-commutative monoids in $\mathcal{C}$* is the product-preserving functor ∞-category

$$\mathrm{CMon}_G(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_G), \mathcal{C});$$

the *∞-category of small G-symmetric monoidal ∞-categories* is

$$\mathrm{Cat}_G^{\otimes} := \mathrm{CMon}_G(\mathrm{Cat}). \qquad \triangleleft$$

These are a homotopical lift of Dress' *semi-Mackey functors* [Dre71] (c.f. [Lin76]). Indeed, given $\mathcal{C}^{\otimes} \in \mathrm{Cat}_G^{\otimes}$ a G-symmetric monoidal ∞-category, pullback along the product-preserving functor

$$\iota_H \colon \mathrm{Span}(\mathbb{F}) \xrightarrow{* \mapsto G/H} \mathrm{Span}(\mathbb{F}_G)$$

constructs a symmetric monoidal ∞-category $\mathcal{C}_H^{\otimes} := \iota_H^* \mathcal{C}^{\otimes}$ whose underlying ∞-category $\mathcal{C}_H$ is the value of $\mathcal{C}^{\otimes}$ on the orbit $[G/H]$. For all subgroups $K \subset H \subset G$, the covariant and contravariant functoriality of $\mathcal{C}^{\otimes}$ then yield symmetric monoidal *restriction* and *norm* functors

$$\mathrm{Res}_K^H \colon \mathcal{C}_H^{\otimes} \to \mathcal{C}_K^{\otimes},$$
$$N_K^H \colon \mathcal{C}_K^{\otimes} \to \mathcal{C}_H^{\otimes},$$

which satisfy a form of Mackey's *double coset formula*.

Example ([BH21; CHLL24b])**.** There is a presentably G-symmetric monoidal ∞-category $\underline{\mathrm{Sp}}_G^{\otimes}$ with:

- H-value given by the symmetric monoidal ∞-category $\left(\underline{\mathrm{Sp}}_G^{\otimes}\right)_H \simeq \mathrm{Sp}_H^{\otimes}$ of *genuine H-spectra*,
- restriction functors $\mathrm{Res}_K^H \colon \mathrm{Sp}_H^{\otimes} \to \mathrm{Sp}_K^{\otimes}$ given by the usual restriction functors, and
- norm functors $N_K^H \colon \mathrm{Sp}_K^{\otimes} \to \mathrm{Sp}_H^{\otimes}$ given by the *HHR norm* of [HHR16].

In fact, this structure is completely determined by its unit object $\mathbb{S}_G \in \mathrm{Sp}_G^{\otimes}$. $\triangleleft$

[2] Maps $[G/K] \to [G/H]$ may equivalently be presented as elements of g such that $gKg^{-1} \subset H$, modulo K; see e.g. [Die09] for details.

Fix $\mathcal{C}^\otimes \in \mathrm{Cat}_G^\otimes$. If $H \subset G$ is a subgroup and $S \in \mathbb{F}_H$ a finite H-set, we may form the induced G-set $\mathrm{Ind}_H^G S \to [G/H]$, and the covariant and contravariant functoriality then yield an *S-indexed tensor product* and *S-indexed diagonal*

$$\bigotimes_K^S : \mathcal{C}_S \to \mathcal{C}_H, \qquad\qquad \Delta^S : \mathcal{C}_H \to \mathcal{C}_S.$$

where $\mathcal{C}_S := \prod_{[H/K]\in\mathrm{Orb}(S)} \mathcal{C}_K$. Note that N_H^K is the $[H/K]$-indexed tensor product and Res_K^H the $[H/K]$-indexed diagonal. As explained in [Ste25], the "orbit collapse" factorization $\mathrm{Ind}_H^G S \to \coprod_{[H/K]\in\mathrm{Orb}(S)}[G/H] \to [G/H]$ yields natural equivalences

$$\bigotimes_K^S X_K \simeq \bigotimes_{[H/K]\in\mathrm{Orb}(S)} N_K^H X_K, \qquad\qquad \Delta^S(X) = \left(\mathrm{Res}_K^H X\right)_{[H/K]\in\mathrm{Orb}(S)},$$

so we may often reduce arguments about S-indexed tensor products to to binary tensor products and norms. Similarly, we define the *S-indexed tensor power*

$$X_H^{\otimes S} := \bigotimes_K^S \left(\Delta^S X_H\right) \simeq \bigotimes_K^S \mathrm{Res}_H^K X_H \simeq \bigotimes_{[H/K]\in\mathrm{Orb}(S)} N_K^H \mathrm{Res}_K^H X_H.$$

If it exists, the pointwise left-adjoint to Δ^S is the *indexed coproduct*

$$\coprod_K^S X_K \simeq \coprod_{[H/K]\in\mathrm{Orb}(S)} \mathrm{Ind}_K^H S,$$

where Ind_K^H is the left adjoint to the restriction map $\mathcal{C}_H \to \mathcal{C}_K$. The *indexed products* are defined analogously.

Given $H \subset G$ a subgroup, we say that $\mathcal{C}$ is *H-pointed* if $\mathcal{C}_K$ is pointed for all $(K) \subset (H)$. Given $S \in \mathbb{F}_H$, we say that S is *$\mathcal{C}$-ambidextrous* if $\mathcal{C}$ is H-pointed, $\mathcal{C}$ admits S-indexed products and coproducts, and the *Wirthmüller* natural transformation

$$W_S : \coprod_K^S(-) \implies \prod_K^S(-)$$

(called the *norm* in [Nar16, § 5]) is an equivalence. We say that $\mathcal{C}$ is *G-semiadditive* if S is $\mathcal{C}$-ambidextrous for all $S \in \mathbb{F}_H$ and $H \subset G$. More generally, if $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_G$ is a weak indexing system corresponding with the weak indexing category $I \subset \mathbb{F}_G$ (see [Ste24] or our review in Section 1.2), we say that $\mathcal{C}$ is *I-semiadditive* if S is $\mathcal{C}$-ambidextrous whenever $S \in \mathbb{F}_{I,H}$.

In this level of generality, Perspectives (i) and (ii) are known to present equivalent ∞-categories of I-commutative monoids; indeed, the *semiadditive closure* theorem of [CLL24, Thm B] demonstrates that $\mathrm{Pr}_G^{L,I-\oplus} \subset \mathrm{Pr}_G^L$ is a smashing localization implemented by

$$L_{I-\oplus}(\mathcal{C}) \simeq \underline{\mathrm{CMon}}_I(\mathcal{C}) := \underline{\mathrm{Fun}}_G^\times\left(\mathrm{Span}_I(\underline{\mathbb{F}}_G), \mathcal{C}\right),$$

and in particular, when $\mathcal{C}$ is a G-∞-category of coefficient systems

$$\underline{\mathrm{Coeff}}^G(\mathcal{D})_H := \mathrm{Fun}\left(\mathcal{O}_H^{\mathrm{op}}, \mathcal{D}\right),$$

[CLL24, Thm C] yields the formula

$$\underline{\mathrm{CMon}}_I\left(\underline{\mathrm{Coeff}}^G(\mathcal{D})\right)_H \simeq \mathrm{Fun}^\times(\mathrm{Span}_I(\mathbb{F}_H), \mathcal{D}),$$

where $\mathrm{Span}_I(\mathbb{F}_H) \subset \mathrm{Span}(\mathbb{F}_H)$ is the wide subcategory of spans whose forward maps lie in the restriction of I to $\mathbb{F}_H$. Thus, we set the notation $\mathrm{CMon}_I(\mathcal{D}) := \underline{\mathrm{CMon}}_I\left(\underline{\mathrm{Coeff}}^G(\mathcal{D})\right)_G \simeq \mathrm{Fun}^\times(\mathrm{Span}_I(\mathbb{F}_G), \mathcal{D})$ and make the following definition.

Definition. For I is a weak indexing category, the *∞-category of small I-symmetric monoidal ∞-categories* is

$$\mathrm{Cat}_I^\otimes := \mathrm{Fun}^\times(\mathrm{Span}_I(\mathbb{F}_G), \mathrm{Cat}). \qquad \triangleleft$$

Following through on Perspective (iii), algebraic objects $X_\bullet$ in a G-symmetric monoidal ∞-category should possess collections of *S-ary operations* $X_H^{\otimes S} \to X_H$ subject to various coherences, controlled by a theory of *genuine equivariant operads*; we use Nardin-Shah's ∞-category Op_G, whose objects we call *G-operads*. Given $\mathcal{O}^\otimes \in \mathrm{Op}_G$ a G-operad, $K \subset H \subset G$ a pair of subgroups, $S \in \mathbb{F}_H$ a finite H-set, and T_i a finite K_i-set for all orbits $[H/K_i] \subset S$, in [Ste25] we constructed a *space of S-ary operations* $\mathcal{O}(S)$, *operadic composition maps*

$$\gamma\colon \mathcal{O}(S) \otimes \bigotimes_{[H/K_i]\in \mathrm{Orb}(S)} \mathcal{O}(T_i) \to \mathcal{O}\left(\coprod_{[H/K_i]\in\mathrm{Orb}(S)} \mathrm{Ind}_{K_i}^H T_i\right), \tag{2}$$

operadic restriction maps

$$\mathrm{Res}\colon \mathcal{O}(S) \to \mathcal{O}\left(\mathrm{Res}_K^H S\right), \tag{3}$$

and *equivariant symmetric group action*

$$\rho\colon \mathrm{Aut}_H(S) \times \mathcal{O}(S) \to \mathcal{O}(S). \tag{4}$$

We made the following simplifying definition.

Definition. A $\mathcal{O}^\otimes \in \mathrm{Op}_G$ *has one color* if $\mathcal{O}(*_H) = *$ for all $H \subset G$; these span a full subcategory $\mathrm{Op}_G^{\mathrm{oc}} \subset \mathrm{Op}_G$. ◁

We showed in [Ste25, Thm A] that Eqs. (3) and (4) lift to a monadic functor $\mathrm{Op}_G^{\mathrm{oc}} \to \mathrm{Fun}(\mathrm{Tot}\Sigma_G, \mathcal{S})$, i.e. one color G-operads are monadic over G-symmetric sequences; in particular, $(S \mapsto \mathcal{O}(S) \mid H \subset G, S \in \mathbb{F}_H)$ are jointly conservative.

When $\mathcal{O}^\otimes$ has one color, an $\mathcal{O}$-algebra in a G-symmetric monoidal ∞-category $\mathcal{C}^\otimes$ can intuitively be viewed as a tuple $\left(X_H \in \mathcal{C}_H^{BW_G(H)}\right)_{G/H \in \mathcal{O}_G}$ with $X_K \simeq \mathrm{Res}_K^H X_H$ for all $K \subset H \subset G$, together with $\mathcal{O}(S)$-actions

$$\mu_S\colon \mathcal{O}(S) \to \mathrm{Map}_{\mathcal{C}_H}\left(X_H^{\otimes S}, X_H\right) \tag{5}$$

for all $H \subset G$ and $S \in \mathbb{F}_H$, homotopy-coherently compatible with Eqs. (2) to (4).[3] We are concerned with the following examples.

Example. There exists a terminal G-operad $\mathrm{Comm}_G^\otimes$, which is characterized up to (unique) equivalence by the property that $\mathrm{Comm}_G(S)$ is contractible for all $S \in \mathbb{F}_H$; its algebras are endowed with contractible spaces of maps $X_H^{\otimes S} \to X_H$ for all $S \in \mathbb{F}_H$, as well as coherent homotopies witnessing their compatibility. We call these *G-commutative algberas*.

On one hand, we saw in [Ste25, § 2.7] that Comm_G-algebras present a homotopical lift of Hill-Hopkins' *G-commutative monoids* [HH16, § 4], though we prefer to reserve this name for the Cartesian case, following the convention of [HA]. On the other hand, our model agrees with that of [CHLL24b], so the recent *homotopical Tambara functor theorem* of Cnossen, Lenz, and Linskens [CHLL24b, Thm B] presents G-commutative algebra objects in $\underline{\mathrm{Sp}}_G^\otimes$ (i.e. *G-commutative ring spectra*) as a form of *homotopical G-Tambara functors*.

Additionally, the recent rectification theorem of Lenz, Linskens, and Pützstück [LLP25] establishes G-commutative ring spectra as a Dwyer-Kan localization of strict commutative algberas in symmetric (or orthogonal) G-spectra at the weak equivalences transferred from a "positive stable" model structure. ◁

Example. Let V be a real orthogonal G-representation. There is a *little V-disks G-operad* $\mathbb{E}_V^\otimes$ whose structure spaces are *spaces of equivariant configurations*:

$$\mathbb{E}_V(S) \simeq \mathrm{Conf}_S^H(V)$$

(see [Hil22; Hor19]). This is modelled by the *Steiner graph G-operad*, so e.g. pointed G-spaces of the form $X = \Omega^V Y := \mathrm{Map}_*\left(S^V, Y\right)$ lift to $\mathbb{E}_V$-spaces by composition of loops [GM11; HHKWZ24]; moreover, many $\mathbb{E}_V$-ring spectra may be constructed as Thom G-spectra of V-fold loop maps [HHKWZ24]. ◁

[3] Here, $W_G(H) = N_G(H)/H$ is the *Weyl group* of $H \subset G$, i.e. the automorphism group of the homogeneous G-set $[G/H]$. The restriction-compatible data specified above may be more familiarly referenced as a *G-object*; it's canonically extended from a choice $X_G \in \mathcal{C}_G$.

Example. Given $I \subset \mathbb{F}_G$ a weak indexing category, in [Ste25] we constructed a *weak* $\mathcal{N}_\infty$ *G-operad* $\mathcal{N}_{I\infty}^{\otimes}$ which is characterized up to (unique) equivalence by its structure spaces

$$\mathcal{N}_{I\infty}(S) \simeq \begin{cases} * & S \in \underline{\mathbb{F}}_I \\ \varnothing & S \notin \underline{\mathbb{F}}_I \end{cases} \tag{6}$$

These recover the $\mathcal{N}_\infty$-operads of [BH15] when I is an indexing category, i.e. $\mathcal{N}_{I\infty}(n \cdot *_G) \simeq *$ for $n \in \mathbb{N}$; in general, they are identified as the sub-terminal objects of Op_G [Ste25, Thm C]. ◁

For instance, we verify in Corollary 3.15 that the condition $V \oplus V \simeq V$ for an orthogonal G-representation V implies that $\mathbb{E}_V$ is a weak $\mathcal{N}_\infty$-operad, which is an $\mathcal{N}_\infty$-operad precisely when V^G is positive-dimensional; In particular, $\mathrm{Comm}_G^{\otimes} \simeq \mathbb{E}_{\infty\rho}^{\otimes} \simeq \mathcal{N}_{\mathbb{F}_{\mathcal{T}}\infty}^{\otimes}$. Moreover, $\mathbb{E}_\infty^{\otimes}$ presents the initial $\mathcal{N}_\infty$-operad, and its algebras are *naive* commutative algebra objects [Ste25, § 3.3]:

$$\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C}_G).$$

If I is an indexing category, the structure of an $\mathcal{N}_{I\infty}$-ring spectrum is intuitively viewed as commutative ring structures on each spectrum X_H, connected by multiplicative I-indexed norms, suitably compatible with the restriction and (additive) transfer structures inherent to G-spectra. We refer to $\mathcal{N}_{I\infty}$-algebras in general as *I-commutative algebras* and $\mathcal{N}_{I\infty}$-ring spectra as *I-commutative ring spectra*, writing

$$\mathrm{CAlg}_I(\mathcal{C}) := \mathrm{Alg}_{\mathcal{N}_{I\infty}}(\mathcal{C}).$$

In this paper, we are primarily concerned with homotopy coherently interchanging $\mathcal{O}$– and $\mathcal{P}$-algebra structures, which are implemented as algebras over *Boardman-Vogt tensor product* $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ of [Ste25]; in particular, we are concerned with computing $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes}$, which corepresents pairs of interchanging I- and J-commutative algebra structures.

To start, in [Ste25, § 2.6] we characterized $I \mapsto \mathcal{N}_{I\infty}^{\otimes}$ as right adjoint to the *arity support* construction

$$A\mathcal{O} := \left\{ T \to S \;\middle|\; \prod_{[H/K] \in \mathrm{Orb}(S)} \mathcal{O}(T \times_S [H/K]) \neq \varnothing \right\} \subset \mathbb{F}_G;$$

when $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ have one object, we will show that $A(\mathcal{O} \otimes \mathcal{P}) = A\mathcal{O} \vee A\mathcal{P}$, the latter denoting the join in the poset of weak indexing category. This constructs a unique pairing $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \to \mathcal{N}_{I \vee J\infty}^{\otimes}$.

Intuitively, given an algebra with $I \vee J$-indexed norms, we may separate these into I- and J-indexed norms together with coherent homotopies witnessing interchange between the two. Now, the transfer system for $I \vee J$ consists of those inclusions $K \subset H$ which can be factored as

$$K \subset K_{I1} \subset K_{J1} \subset K_{I2} \subset \cdots \subset K_{Jn} \subset H$$

where $K_{I\ell} \subset K_{J\ell}$ is in I and $K_{J\ell} \subset K_{I(\ell+1)}$ is in J (see [Rub21, Prop 3.1]); intuition suggests that we may combine interchanging I- and J-commutative algebra structures to construct an $I \vee J$-commutative algebra structure. Indeed, Blumberg and Hill conjectured that there is an equivalence $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{I \vee J\infty}^{\otimes}$ [BH15, Conj 6.27]; the main theorem of this paper confirms their conjecture in Op_G, as well as characterizing exactly how far we may weaken I and J.

Summary of main results. Recall that a weak indexing category $I \subset \mathbb{F}_G$ is *almost essentially unital* if whenever a non-isomorphism $T \sqcup T' \to S$ lies in I, the factor map $T \to S$ lies in I, and *almost-unital* if additionally $*_G \in I$. We begin with a rigidity result for (co)cartesian I-symmetric ∞-categories under almost-unitality.

Theorem A. *When I is almost-unital, there are fully faithful embeddings $(-)^{I-\sqcup}$ and $(-)^{I-\times}$ making the following commute:*

$$\begin{tikzcd}
\mathrm{Cat}_I^{\sqcup} \arrow[r, hook, "(-)^{I-\sqcup}"] \arrow[dr, "U"'] & \mathrm{Cat}_I^{\otimes} \arrow[d, "U"] & \mathrm{Cat}_I^{\times} \arrow[l, hook', "(-)^{I-\times}"'] \arrow[dl, "U"] \\
& \mathrm{Cat}_G &
\end{tikzcd}$$

The essential image of $(-)^{I-\sqcup}$ is spanned by the I-symmetric monoidal ∞-categories whose I-indexed tensor products are indexed coproducts, and $(-)^{I-\times}$ by those whose I-indexed tensor products are indexed products.

Remark. After this introduction, we replace $\mathcal{O}_G$ with an atomic orbital ∞-category $\mathcal{T}$; we prove Theorem A as well as the other theorems in this introduction in this setting, greatly generalizing the stated results at the cost of ease of exposition. ◁

We refer to I-symmetric monoidal ∞-categories of the form $\mathcal{C}^{I-\times}$ as *cartesian*, and $\mathcal{C}^{I-\sqcup}$ *cocartesian*.

Remark. Given I-symmetric monoidal ∞-categories $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ and an I-product-preserving functor $F\colon \mathcal{C} \to \mathcal{D}$ between their underlying G-∞-categories, we may define the ∞-*category of I-symmetric monoidal lifts*

$$\begin{tikzcd} \mathrm{Fun}_G^{\otimes,F}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}_G^{\otimes}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes}) \arrow[d] \\ \{F\} \arrow[r] & \mathrm{Fun}_G(\mathcal{C},\mathcal{D}) \end{tikzcd}$$

To interpret Theorem A as a rigidity theorem, note that it directly implies that whenever $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are cartesian (resp. cocartesian), the core space $\mathrm{Fun}_G^{\otimes,F}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})^{\simeq}$ is contractible if F is I-product preserving (I-coproduct preserving) and empty otherwise. Moreover, we confirm this fact without taking cores in Proposition A.21 and Corollary A.23. ◁

To state our remaining theorems, we need the following definition.

Definition. An I-operad $\mathcal{O}^{\otimes}$ is *unital* if the unique map $f\colon \mathcal{O}^{\otimes} \to \mathcal{N}_{I\infty}$ induces an equivalence

$$\mathcal{O}(\varnothing_H) \simeq \mathcal{N}_{I\infty}(\varnothing_H)$$

for all $H \subset G$ (c.f. Eq. (6)); an I-operad is *reduced* if additionally f induces an equivalence

$$\mathcal{O}(*_H) \simeq \mathcal{N}_{I\infty}(*_H).$$

A G-operad $\mathcal{O}^{\otimes}$ is *almost essentially unital* (resp *almost essentially reduced*) if it's unital (reduced) as an $A\mathcal{O}$-operad and $A\mathcal{O}$ is almost essentially unital. ◁

Algebraically, we identify cartesian I-commutative algebras with I-commutative monoids and cocartesian (unital) I-commutative algebras with G-objects, identifying Perspectives (i) to (iii).

Theorem B. *If I is almost-unital, $\mathcal{C}^{\otimes}$ is a cartesian I-symmetric monoidal ∞-category, and $\mathcal{O}^{\otimes}$ is an I-operad, then the forgetful functor*

$$U\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \longrightarrow \mathrm{Fun}_G(\mathrm{Tot}_G\mathcal{O}^{\otimes},\mathcal{C})$$

is fully faithful with image spanned by the G-functors $\mathrm{Tot}_G\mathcal{O}^{\otimes} \to \mathcal{C}$ sending S-indexed tuples to S-indexed products; in particular, this specializes to an equivalence

$$\mathrm{CAlg}_I\left(\mathcal{C}^{I-\times}\right) \xrightarrow{\sim} \underline{\mathrm{Fun}}_G^{I-\oplus}\left(\underline{\mathbb{F}}_{I,*},\mathcal{C}\right).$$

In particular, in the case of coefficient systems, we acquire an equivalence

$$\mathrm{Alg}_{\mathcal{O}}\left(\underline{\mathrm{Coeff}}^G\mathcal{D}^{I-\times}\right) \simeq \mathrm{Seg}_{\mathrm{TotTot}_G\mathcal{O}}(\mathcal{D}) \simeq \mathrm{Seg}_{\mathrm{Tot}\mathcal{O}}(\mathcal{D}),$$

where $\mathrm{Seg}_{(-)}(-)$ refers to Segal objects *in the sense of [CH21]. Hence there is an additional equivalence*

$$\mathrm{CAlg}_I\left(\underline{\mathrm{Coeff}}^G\mathcal{D}^{I-\times}\right) \simeq \mathrm{CMon}_I(\mathcal{D}).$$

Moreover, for all unital I-operads $\mathcal{O}^{\otimes}$, the forgetful functor yields an equivalence

$$\mathrm{Alg}_{\mathcal{O}}\left(\mathcal{C}^{I-\sqcup}\right) \xrightarrow{\sim} \mathrm{Fun}_G(U\mathcal{O},\mathcal{C}).$$

References. This is Propositions 1.51 and 1.62 and Corollaries 1.52 to 1.54. □

In this theorem, $\mathrm{TotTot}_G\mathcal{O}$ is the total ∞-category of the fibration over $\underline{\mathbb{F}}_{G,*}$ and $\mathrm{Tot}\mathcal{O}$ is the total ∞-category of the fibration over $\mathrm{Span}(\mathbb{F}_G)$.

Remark. The composed equivalence $\underline{\mathrm{Fun}}_G^{I-\oplus}\left(\underline{\mathbb{F}}_{I,*},\underline{\mathrm{Coeff}}^G\mathcal{D}\right) \simeq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_G),\mathcal{D})$ is not new; indeed, it was claimed for the complete weak indexing system as far back as [Nar16], it was proved in greater generality than this article in [CLL24], and we verified in [Ste25, § A] that it also follows from [BHS22], as well as the more general comparison between the two Segal object models for cartesian algebras. The new content is the identification of these notions with G-operad algebras.

Moreover, in the case that $\mathcal{D} = \mathcal{S}$ and that I is an indexing category, this is a direct analog to [Mar24, Thm A] in the ∞-categorical setting; the reader should interpret this relationship as a lift of Pavlov-Scholbach's comparison result [PS18, Thm 1.3] for a particularly nice choice of G-operad and value category. ◃

In Section 1.4 we verify that $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ is cartesian when $\mathcal{C}$ is. Following this, in Section 2.1 we show that I-indexed tensor products in $\underline{\mathrm{CAlg}}^{\otimes}_{I}\mathcal{C}$ are indexed coproducts (i.e. its underlying I-symmetric monoidal ∞-category is *cocartesian*) and that this completely characterizes $\mathcal{N}^{\otimes}_{I\infty}$. The heart of our strategy uses the explicit monadic description of [Ste25, § 2.4] to reduce to the case of G-spaces $\mathcal{C}^{\otimes} \simeq \underline{\mathcal{S}}_G^{G-\times}$; in this case, we may easily see that the cartesian I-symmetric monoidal ∞-category $\underline{\mathrm{CAlg}}^{\otimes}_{I}(\underline{\mathcal{S}}_G^{G-\times}) \simeq \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_G)^{I-\times}$ is cocartesian, as its underlying G-∞-category is I-semiadditive by [CLL24, Thm B-C]. We conclude the following.

Theorem C. *Let $\mathcal{O}^{\otimes}$ be an almost essentially reduced G-operad. Then, the following conditions are equivalent.*

(a) The G-∞-category $\underline{\mathrm{Alg}}_{\mathcal{O}}\underline{\mathcal{S}}_G$ is $A\mathcal{O}$-semiadditive.

(b) The unique map $\mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{A\mathcal{O}\infty}$ is an equivalence.

Moreover, for all almost essentially unital weak indexing categories I and I-symmetric monoidal ∞-categories $\mathcal{C}^{\otimes}$, the I-symmetric monoidal ∞-category $\underline{\mathrm{CAlg}}^{\otimes}_{I}\mathcal{C}$ is cocartesian.

Theorems B and C together with conservativity of $\underline{\mathrm{Alg}}_{(-)}(\underline{\mathcal{S}}_G)$ (as in [Ste25, § 2.4]) yields the following.

Corollary D. *$\mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty}$ is a weak $\mathcal{N}_{\infty}$-operad if and only if I is almost essentially unital. In this case, if $\mathcal{O}^{\otimes}$ is a reduced I-operad, then the unique map $\mathcal{O}^{\otimes} \otimes \mathcal{N}^{\otimes}_{I\infty} \to \mathcal{N}^{\otimes}_{I\infty}$ is an equivalence.*

In particular, whenever I is almost unital, there exists a map $\mathrm{triv}^{\otimes}_G \to \mathcal{N}^{\otimes}_{I\infty}$ witnessing $\mathcal{N}^{\otimes}_{I\infty}$ as an idempotent object in Op_G. We verified in [Ste25, Thm D] that $\mathrm{Env}\colon \mathrm{Op}_G \to \mathrm{Cat}^{\otimes}_G$ is compatible with the unit and tensor products under the mode symmetric monoidal structure on $\mathrm{Cat}^{\otimes}_G$; this yields a $\circledast$-idempotent algebra structure on $\underline{\mathbb{F}}_G^{G-\sqcup} = \mathrm{Env}(\mathrm{Comm}_G) \in \mathrm{Cat}^{\otimes}_G$, and hence a symmetric monoidal structure on $\underline{\mathrm{Cat}}^{\otimes}_{G,/\underline{\mathbb{F}}_G^{G-\sqcup}}$. We acquire an equivariantization of a modification of [BS24a, Thm E].

Corollary E. *There exists a unique symmetric monoidal structure $\underline{\mathrm{Op}}^{\otimes}_G$ on $\underline{\mathrm{Op}}_G$ attaining a (necessarily unique) symmetric monoidal structure on the fully faithful G-functor*

$$\mathrm{Env}^{/\underline{\mathbb{F}}_G^{G-\sqcup}}\colon \underline{\mathrm{Op}}^{\otimes}_G \longrightarrow \underline{\mathrm{Cat}}^{\otimes}_{G,/\underline{\mathbb{F}}_G^{G-\sqcup}}$$

of [BHS22; NS22] with respect to $\circledast$; the tensor product of this structure is $\overset{\mathrm{BV}}{\otimes}$.

Idempotent objects correspond with smashing localizations, i.e. they classify particular properties [HA, § 4.8.2]; in Theorem 2.6, we conclude that the smashing localization corresponding with $\mathcal{N}^{\otimes}_{I\infty} \in \mathrm{Op}^{\mathrm{red}}_J$ classifies the property of *having I-indexed Wirthmüller isomorphisms*

$$\begin{aligned} \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \simeq \mathcal{O}^{\otimes} \quad &\iff \quad \forall \mathcal{C}^{\otimes} \in \mathrm{Cat}^{\otimes}_J,\ \ \forall S \in \mathbb{F}_{I,V},\ \ \forall (X_U)_S \in \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})_S \quad W_S\colon \coprod_U^S X_U \xrightarrow{\sim} \bigotimes_U^S X_U \\ &\iff \quad \underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_G) \text{ is } I\text{-semiadditive.} \end{aligned}$$

Recall that tensor products of idempotents algebras are idempotent algebras, classifying the intersection of the associated smashing localizations [CSY20, Prop 5.1.8]; conveniently, indexed semiadditivity is classified by a weak indexing category [Ste25, § 1.2], so $\underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_G)$ is $I \vee J$-semiadditive if and only if it is I-semiadditive and J-semiadditive. This allows us to affirm Blumberg and Hill's conjecture with respect to $\overset{\mathrm{BV}}{\otimes}$.

Theorem F. $\mathcal{N}^{\otimes}_{(-)\infty}\colon \mathrm{wIndex}_G \to \mathrm{Op}_G$ *restricts to a fully faithful symmetric monoidal G-right adjoint*

$$\begin{tikzcd} \underline{\mathrm{wIndex}}_G^{aE\mathrm{uni}} \arrow[rr, bend right, "\mathcal{N}^{\otimes}_{(-)\infty}"'] \arrow[rr, phantom, "\perp"] & & \underline{\mathrm{Op}}_G^{aE\mathrm{uni}} \arrow[ll, bend right, "A"'] \end{tikzcd}$$

Furthermore, the resulting tensor product of weak $\mathcal{N}_\infty$-operads is computed by the Borelified join

$$\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}^{\otimes}_{\mathrm{Bor}^G_{cI\cap cJ}(I \vee J)\infty}.$$

Hence when I, J are almost-unital weak indexing categories and $\mathcal{C}^\otimes$ is an $I \vee J$-symmetric monoidal ∞-category, there is a canonical equivalence of $I \vee J$-symmetric monoidal ∞-categories

$$\underline{\mathrm{CAlg}}_I^{\otimes} \underline{\mathrm{CAlg}}_J^{\otimes}(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_{I\vee J}^{\otimes}(\mathcal{C}).$$

For instance, using [CHLL24b, Thm 4.3.6] to identify I-Tambara functors in an ∞-category $\mathcal{C}$ with I-commutative algebras in Mackey functors, this confirms that $I \vee J$-Tambara functors are equivalent to I-commutative algebras in J-Tambara functors with respect to the box product.

Remark. The reader interested in computing tensor products of G-operads may benefit from reading the combinatorial characterization of joins of weak indexing systems in terms of *closures* in [Ste24, § 2.3]; there, we prove that the join of weak indexing systems $\mathbb{F}_I \vee \mathbb{F}_J$ is computed by closing the union $\mathbb{F}_I \cup \mathbb{F}_J$ under iterated I and J-indexed coproducts. ◃

Relationship to the literature. There are three main bodies of literature which present results in homotopy-coherently equivariant algebra: the model categorical, the atomic orbital, and the global. We now attempt to give a bit of a Rosetta stone to connect our definitions to the model categorical and global settings.

We established in [Ste24] that our weak indexing categories specialize to Blumberg-Hill's indexing categories [BH18] in the case $\mathcal{T} = \mathcal{O}_G$ and $n \cdot *_G \to *_G$ lies in I for all $n \in \mathbb{N}$, and our weak indexing systems to the indexing systems of [BH15] when $n \cdot *_G \in \mathbb{F}_{I,G}$ for all $n \in \mathbb{N}$; moreover, this was shown to be compatible with Bonventre's nerve in [Bon19; Ste25], which is intertwines with the underlying G-symmetric sequence and restricting to an equivalence on at-most-one-color G-operads with 0-truncated structure spaces, showing that our weak $\mathcal{N}_\infty$-operads specialize to those of [BH18; BP21; GW18; Rub21]. Additionally, we saw in [Ste24] that our weak indexing categories specialize the *weakly extensive span pairs* of [CHLL24b] to the case that the larger category is $\mathbb{F}_{\mathcal{T}}$.

We saw in [Ste25] that our algebras agree with Blumberg-Hill's in the discrete setting, and combining Corollary 1.54 with the main result of [Mar24] identifies a Dwyer-Kan localization of the latter with the former in the case $\underline{\mathcal{S}}_G^{G-\times}$. Of course, the recent results of Lenz, Linskens, and Pützstück [LLP25, Thm A] have established *rectification* to G-commutative ring spectra, establishing them as presented by Hammock localization of a right-transferred structure on commutative algebras, $\mathrm{CAlg}(\mathrm{Sp}_G^\Sigma)$, with respect to the positive stable model structure on symmetric G-spectra (or equivalently, on orthogonal G-spectra). The author is not aware of any study into rectification for the incomplete case.

Our I-symmetric monoidal ∞-categories and I-operads generalize [NS22] and specialize [LLP25] by [Ste25, § A] and by definition, respectively. Moreover, there is a homotopical *operadic nerve* construction mapping the (equivalent) settings of [BH15; BP21; Per18] to Op_I and $\mathrm{Cat}_I^\otimes$ by [Bon19; Ste25].

The author is not aware of a comparison result between $\overset{\mathrm{BV}}{\otimes}$ and the point-set Boardman-Vogt tensor product appearing in [BH15]; moreover, the "derived" tensor product appearing in [Rub21] is only defined on an ∞-category which is equivalent to Index_G, so it's not clear that it makes sense to ask for a comparison to $\overset{\mathrm{BV}}{\otimes}$ other than confirming that $\overset{\mathrm{BV}}{\otimes}$ confirms Blumberg-Hill's conjecture, as demanded by the results of [Rub21, Thm A] (after which Rubin explicitly claimed that the conjecture remained open).

For (co)cartesian I-symmetric monoidal ∞-categories, we show in Section 1.4 that our definitions generalize [NS22] and agree with [CHLL24b] when both are defined; in particular, we recover a non-$(\infty,2)$-categorical reproof of the identification theorem of the two, as conjectured in [CHLL24b] and verified in [CLR25]. The author is not aware of a definition to be compared with in the model-categorical setting, but comparisons with such constructions will be as easy as verifying that those structures have indexed tensor products which present (derived) indexed products.

Some versions of our results on cocartesianness and algebras are proved independently in the literature; though it is not clear that Nardin-Shah's $\mathcal{T}$-operad of algebras [NS22] agrees with ours, they confirmed that their version of $\underline{\mathrm{CAlg}}_{\mathcal{T}}^{\otimes}(\mathcal{C})$ is cocartesian and claimed that $\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{C})$ is I-cocartesian when I is an indexing category. Moreover, it is shown in [LLP25, Prop 2.26] that $\overline{(-)^{I-\sqcup}}$ *admits* a left adjoint, but this left adjoint is not computed therein (whereas we confirm it to be U in the almost-unital case as Corollary 2.16).

Additionally, after the first preprint version of this article appeared on the arxiv, the author learned of independent accounts of some aspects the cartesian and cocartesian structure in [Hil24a, § A] and [Yan25,

§ 4.1], respectively. In particular, the former recovers Corollary 1.54 and the latter Proposition 1.62 in the case $I = \mathcal{T} = \mathcal{O}_G$.

Notation and conventions. We assume that the reader is familiar with the technology of higher category theory and higher algebra as developed in [HTT] and [HA, § 2-3], though we encourage the reader to engage with such technologies via a "big picture" perspective akin to that of [Gep19, § 1-2] and [Hau23, § 1-3]. In particular, our treatment is almost entirely model agnostic–we only pierce the veil in Section A.1 and use quasicategorical language in order to verify that a few functors are exponentiable.

We additionally assume that the reader is familiar with *parameterized* higher category theory over an ∞-category as developed in [Sha22; Sha23]; the material reviewed in the prequel [Ste25, § 1] will be enough. In particular,

- $\mathcal{T}$ will always be an atomic orbital ∞-category in the sense of [NS22], $\mathbb{F}_{\mathcal{T}}$ its corresponding ∞-category of finite $\mathcal{T}$-sets, and $\underline{\mathbb{F}}_{\mathcal{T}}$ its corresponding $\mathcal{T}$-1-category of finite $\mathcal{T}$-sets.
- $\mathcal{F} \subset \mathcal{T}$ will always be a $\mathcal{T}$-family in the sense of [Ste24].
- $I \subset \mathbb{F}_{\mathcal{T}}$ and $\underline{\mathbb{F}}_I$ will always be a weak indexing category and corresponding weak indexing system in the sense of [Ste24]. $c(I)$ will be its color family and $v(I)$ its unit family.
- Cat will always be the ∞-category of small ∞-categories, $\mathrm{Cat}_{\mathcal{T}}$ of small $\mathcal{T}$-∞-categories, and $\mathrm{Cat}_I^{\otimes}$ of small I-symmetric monoidal ∞-categories. All ∞-categories will be assumed to be small unless otherwise mentioned.
- $\mathcal{T}$-operad will always mean $\mathcal{T}$-∞-operad in the sense of [NS22] and $\mathrm{Op}_{\mathcal{T}}$ the ∞-category of $\mathcal{T}$-operads.

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1. I-symmetric monoidal categories and I-operads

We begin in Section 1.1 by recalling results of [CLL24; Nar16; NS22; Ste24; Ste25] concerning the theory of I-commutative monoids and I-symmetric monoidal ∞-categories. Moving on, in Section 1.2 we recall results of [NS22; Ste25] concerning $\mathcal{T}$-operads; in either case, all reviewed information was used in the preceding article [Ste25]. We then go on in Section 1.3 to begin to carefully study the interactions of restriction, arity-borelification, arity-support, and Boardman-Vogt tensor products. We finish the section in Section 1.4, where we develop a number of foundational results on (co)cartesian I-symmetric monoidal ∞-categories, ultimately elaborating on the technical minutiae of Section A.

1.1. Recollections on I-commutative monoids and I-symmetric monoidal ∞-categories.

For the rest of this paper, we fix $\mathcal{T}$ an atomic orbital ∞-category.

1.1.1. *Weak indexing systems and semiadditivity.* We will use the following machinery of [Ste24].

Definition 1.1. A *$\mathcal{T}$-weak indexing category* is a subcategory $I \subset \mathbb{F}_{\mathcal{T}}$ satisfying the following conditions:

(IC-a) (restrictions) I is stable under arbitrary pullbacks in $\mathbb{F}_{\mathcal{T}}$, and
(IC-b) (segal condition) $T \to S$ and $T' \to S$ are both in I if and only if $T \sqcup T' \to S \sqcup S'$ is in I.

A *$\mathcal{T}$-weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ satisfying the following conditions:

(IS-a) whenever the V-value $\mathbb{F}_{I,V} := (\underline{\mathbb{F}}_I)_V$ is nonempty, we have $*_V \in \underline{\mathbb{F}}_{I,V}$, and
(IS-b) $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is closed under $\underline{\mathbb{F}}_I$-indexed coproducts. ◁

We say that a $\mathcal{T}$-weak indexing system $\underline{\mathbb{F}}_I$:

(i) has one color if for all $V \in \mathcal{T}$, we have $\mathbb{F}_{I,V} \neq \varnothing$,
(ii) is almost essentially unital (or aE-unital) if whenever $\underline{\mathbb{F}}_I$ has a non-contractible V-set, $\varnothing_V \in \mathbb{F}_{I,V}$,

(iii) is almost-unital (or a-unital) it's almost essentially unital and has one color,
(iv) is unital if $\varnothing_V \in \mathbb{F}_{I,V}$ for all $V \in \mathcal{T}$, and
(v) is an *indexing system* if the subcategory $\underline{\mathbb{F}}_{I,V} \subset \mathbb{F}_V$ is closed under finite coproducts for all $V \in \mathcal{T}$.

These occupy embedded sub-posets

$$\mathrm{Index}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}.$$

Given a weak indexing category $I \subset \mathbb{F}_{\mathcal{T}}$, we denote the *$I$-admissible V-sets* by

$$\mathbb{F}_{I,V} := \left\{ S \in \mathbb{F}_{I,V} \mid \mathrm{Ind}_V^{\mathcal{T}} S \to V \in I \right\} \subset \mathbb{F}_V;$$

Condition (IC-a) guarantees that these assemble into a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$, which contains all of the information of I by Condition (IC-b). In [Ste24, Thm A] we proved the following and expressed the conditions of Definition 1.1 in the language of weak indexing categories.

Proposition 1.2 (Generalized [BH18, Thm 1.4])**.** *The assignment $I \mapsto \underline{\mathbb{F}}_I$ implements an equivalence between the posets of $\mathcal{T}$-weak indexing categories and $\mathcal{T}$-weak indexing systems.*

Intuitively, Condition (IS-a) corresponds with identity arrows in I and Condition (IS-b) with composition. We will need the following invariants of weak indexing systems.

Construction 1.3. Given $\underline{\mathbb{F}}_I$ a weak indexing system, we define the *color and unit families*

$$c(I) := \{ V \in \mathcal{T} \mid *_V \in \mathbb{F}_{I,V} \} \subset \mathcal{T};$$
$$v(I) := \{ V \in \mathcal{T} \mid \varnothing_V \in \mathbb{F}_{I,V} \} \subset \mathcal{T}.$$

Indeed, we saw that these are families in [Ste25]. ◁

One reason to study this is *indexed semiadditivity* in the sense of the following definitions of Nardin.

Definition 1.4. A $\mathcal{T}$-∞-category $\mathcal{C}$ is said to be *V-pointed* if $\mathcal{C}_U$ is a pointed ∞-category for all $U \to V$. Given $S \in \mathbb{F}_V$ is a finite V-set, $\mathcal{C}$ V-pointed admitting S-indexed products and coproducts, and $(X_U)_S \in \mathcal{C}_S$ an S-tuple, we define the *S-indexed Wirthmuller map* $W_{S,(X_U)} \colon \coprod_U^S X_U \to \prod_U^S X_U$ to extend the following maps via the universal property for S-indexed coproducts:

$$W_{S,(X_U),W} \colon X_W \simeq X_W \times \prod_{U'}^{\mathrm{Res}_W^V S - W} *_{U'} \xrightarrow{(\mathrm{id};!)} X_W \times \prod_{U'}^{\mathrm{Res}_U^V S - W} \mathrm{Res}_{U'}^V X_{o(U')} \simeq \mathrm{Res}_W^V \prod_U^S X_U$$

where $o(U') \in \mathrm{Orb}(S)$ is the orbit whose restriction contains U'. We say that S is *$\mathcal{C}$-ambidextrous* if $\mathcal{C}$ is V-pointed and $W_{S,(X_U)}$ is an equivalence for all $(X_U) \in \mathcal{C}_S$; given $\underline{\mathbb{F}}_I$ a weak indexing system, we say that $\mathcal{C}$ is *I-semiadditive* if S is $\mathcal{C}$-ambidextrous for all $S \in \underline{\mathbb{F}}_I$. ◁

Remark 1.5. The map $W_{S,(X_U),W}$ is determined via the universal property for S-indexed products by its projections $W_{S,(X_U),W,W'} \colon X_W \to \mathrm{Res}_W^V \mathrm{CoInd}_{W'}^V X_{W'}$, which are zero when $W \neq W'$, and otherwise they are the map induced under functoriality of products by the dashed arrow

$$\begin{tikzcd}
\mathrm{Ind}_W^V *_W \arrow[drr, equal, bend left] \arrow[dr] \arrow[ddr, equal, bend right] & & \\
& \mathrm{Ind}_W^V \mathrm{Res}_W^V \mathrm{Ind}_W^V *_W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ind}_W^V *_W \arrow[d] \\
& \mathrm{Ind}_W^V *_W \arrow[r] & V
\end{tikzcd}$$

In particular, they match the norms constructed in [Nar16]. ◁

In [Ste25] we proved that the collection of $\mathcal{C}$-ambidextrous finite V-sets form a weak indexing system and concluded the following important observation.

Proposition 1.6 ([Ste25, § 1.2])**.** *Let $\vee$ denote the join in* $\mathrm{wIndexCat}_{\mathcal{T}}$*. Then, $\mathcal{C}$ is I-semiadditive and J-semiadditive if and only if $\mathcal{C}$ is $I \vee J$-semiadditive.*

1.1.2. *I-commutative monoids.* In [Bar14], the notion of *adequate* triple was defined, consisting of triples $(\mathcal{C},\mathcal{C}_b,\mathcal{C}_f)$ with $\mathcal{C}_f,\mathcal{C}_B \subset \mathcal{C}$ a pair of wide subcategories satisfying pullback-stability and distributivity conditions; if I is a weak indexing category, then $(\mathbb{F}_{c(I)},\mathbb{F}_{c(I)},I)$ is an adequate triple.

Adequate triples form a full subcategory $\mathrm{Trip}^{\mathrm{Adeq}} \subset \mathrm{Fun}(\bullet \to \bullet \leftarrow \bullet, \mathrm{Cat})$; [Bar14] constructed a functor

$$\mathrm{Span}_{-,-}(-)\colon \mathrm{Trip}^{\mathrm{Adeq}} \to \mathrm{Cat},$$

called the *effective Burnside category.* In the case that $c(I)$ is a 1-category (e.g. $\mathcal{T}$ has a terminal object, see [NS22, Prop 2.5.1]), $\mathbb{F}_{c(I)}$ is a 1-category, so the effective Burnside category

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) := \mathrm{Span}_{\mathbb{F}_{c(I)},I}(\mathbb{F}_{c(I)})$$

is a $(2,1)$-category with objects agreeing with $\mathbb{F}_{c(I)}$, morphisms the spans $X \leftarrow R \xrightarrow{f} Y$ with f in I, 2-cells the isomorphisms of spans, and composition of morphisms computed by pullbacks in $\mathbb{F}_{c(I)}$ (which are guaranteed to be morphisms in $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ by pullback-stability of I).

Much of the technical work of [Bar14; BGS20] has been extended by [HHLN23], so we generally refer the reader there. At any rate, we recall this in order to define *homotopical incomplete semi-Mackey functors for I*, which we call *I-commutative monoids*.

Definition 1.7. If $\mathcal{C}$ is an ∞-category with finite products, then an *I-commutative monoid in $\mathcal{C}$* is a product-preserving functor $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathcal{C}$. More generally, if $\mathcal{D}$ is a $\mathcal{T}$-∞-category with I-indexed products, then an *I-commutative monoid in $\mathcal{D}$* is an I-product-preserving $\mathcal{T}$-functor $\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}}) \to \mathcal{D}$. We write

$$\begin{aligned}
\underline{\mathrm{CMon}}(\mathcal{D}) &:= \underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\times}(\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}}),\mathcal{D})\\
\mathrm{CMon}(\mathcal{D}) &:= \Gamma^{\mathcal{T}}\,\underline{\mathrm{CMon}}(\mathcal{D})\\
\underline{\mathrm{CMon}}(\mathcal{C}) &:= \underline{\mathrm{CMon}}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}\,\mathcal{C}\right)\\
\mathrm{CMon}(\mathcal{C}) &:= \mathrm{CMon}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}\,\mathcal{C}\right).
\end{aligned}$$ ◁

An important result of Cnossen-Lenz-Linskens resolves the notational clash.

Proposition 1.8 ([CLL24, Thm C])**.** *When $\mathcal{C}$ is an ∞-category, restriction furnishes an equivalence*

$$\mathrm{CMon}(\mathcal{C}) \simeq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}),\mathcal{C}),$$

and more generally, we have $\underline{\mathrm{CMon}}(\mathcal{C})_V \simeq \mathrm{Fun}^{\times}_V(\mathrm{Span}_I(\mathbb{F}_V),\mathcal{C})$ with restriction given by pullback along $\mathrm{Span}_I(\mathbb{F}_V) \to \mathrm{Span}_I(\mathbb{F}_W)$.

Let I be a one-object weak indexing category and let $\mathrm{Cat}_{\mathcal{T}}^{I-\times} \subset \mathrm{Cat}_{\mathcal{T}}$ be the (non-full) subcategory whose objects are $\mathcal{T}$-∞-categories admitting I-indexed products and functors preserving I-indexed products. Let $\mathrm{Cat}_I^{I-\oplus} \subset \mathrm{Cat}_{\mathcal{T}}^{I-\times}$ be the full subcategory spanned by I-semiadditive $\mathcal{T}$-∞-categories. The following result is fundamental in the theory of equivariant semiadditivity and equivariant higher algebra.

Theorem 1.9 ([CLL24, Thm B])**.** *The inclusion $\mathrm{Cat}_{\mathcal{T}}^{I-\oplus} \subset \mathrm{Cat}_{\mathcal{T}}^{I-\times}$ has left adjoint $\underline{\mathrm{CMon}}(-)$.*

1.1.3. *I-symmetric monoidal ∞-categories.* The following definition is central to equivariant higher algebra.

Definition 1.10. The *∞-category of small I-symmetric monoidal ∞-categories* is $\mathrm{Cat}_I^{\otimes} := \mathrm{CMon}_I(\mathrm{Cat})$. ◁

We refer to maps in $\mathrm{Cat}_I^{\otimes}$ as *I-symmetric monoidal functors*. An important lemma is the following.

Lemma 1.11 ([CH21, Cor 8.2])**.** *If $\mathcal{C}$ is an ∞-category and I a one-object weak indexing category, then the* ***underlying coefficient system*** *functor $\mathrm{CMon}_I(\mathcal{C}) \to \mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ is conservative; in particular, if a I-symmetric monoidal functor's underlying $\mathcal{T}$-functor is an equivalence, then it is a $\mathcal{T}$-symmetric monoidal equivalence.*

Now, these are defined for the following notation's sense.

Notation 1.12. Given an I-symmetric monoidal ∞-category and an I-map $\mathrm{Ind}_V^{\mathcal{T}} S \to V$ with $V \in \mathcal{T}$, we denote the covariant functorialty of $\mathcal{C}^{\otimes}$ by $\bigotimes_U^S\colon \mathcal{C}_S \to \mathcal{C}_V$ and the contravariant functoriality by $\Delta^S\colon \mathcal{C}_V \to \mathcal{C}_S$. ◁

We will also need *presentability*. In his thesis, Nardin defined a $\mathcal{T}$-symmetric monoidal ∞-category $\underline{\mathrm{Pr}}_{\mathcal{T}}^{L,\otimes}$ of *presentable $\mathcal{T}$-∞-categories*, whose S-ary tensor products are characterized by mapping ∞-categories

$$\mathrm{Fun}^L_{\mathcal{T}}\left(\bigotimes_U^S \mathcal{C}_U, \mathcal{D}\right) := \mathrm{Fun}^{S-\partial}_{\mathcal{T}}\left(\prod_U^S \mathcal{C}_U, \mathcal{D}\right)$$

where $\mathrm{Fun}^{S-\partial}_{\mathcal{T}}$ consists of the "S-distributive $\mathcal{T}$-functors." We will not care too much about the details of this in general, and instead shunt the interested the reader to [QS22b, Def 5.14]. Nevertheless, we care about the following case.

Example 1.13. $\mathrm{Fun}_V^{2*_V-\partial}(\mathcal{C}\times\mathcal{D},\mathcal{E}) \subset \mathrm{Fun}(\mathcal{C}\times\mathcal{D},\mathcal{E})$ is the full subcategory of $\mathcal{T}$-functors whose fibers $\mathcal{C}\times\{D\},\{C\}\times\mathcal{D}\to\mathcal{E}$ all strongly preserve $\mathcal{T}$-colimits. ◁

Now, we make the following definition.

Definition 1.14. A *distributive I-symmetric monoidal ∞-category* is an I-symmetric monoidal ∞-category $\mathcal{C}^\otimes$ whose S-tensor functors $\otimes_U^S\colon \mathcal{C}_S \to \mathcal{C}_V$ are S-distributive. A *presentably I-symmetric monoidal ∞-category* is a distributive I-symmetric monoidal ∞-category whose underlying $c(I)$-∞-category is presentable. ◁

We will also need the following construction.

Construction 1.15. The "opposite category" construction $\mathrm{op}\colon \mathrm{Cat}\to\mathrm{Cat}$ is an equivalence, so in particular it is product-preserving. Hence postcomposition with $(-)^{\mathrm{op}}$ yields *fiberwise opposite* functor

$$(-)^{\mathrm{vop}}\colon \mathrm{Cat}_I^\otimes \to \mathrm{Cat}_I^\otimes.$$

Note that the underlying category $(\mathcal{C}^\otimes)^{\mathrm{vop}}$ is the traditional *vertical opposite $\mathcal{T}$-∞-category* $\mathcal{C}^{\mathrm{vop}}$. ◁

1.2. Recollections on $\mathcal{T}$-operads.

1.2.1. *I-operads.* In [Ste25], we made the following definition.

Definition 1.16. An *I-operad* is a functor $\pi\colon \mathrm{Tot}\mathcal{O}^\otimes \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ satisfying the following conditions.

(a) Tot'$\mathcal{O}^\otimes$ has π-cocartesian lifts for backwards maps in $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$;
(b) (Segal condition for colors) writing $\mathcal{O}_S \simeq \pi^{-1}(S)$, for every $S\in\mathbb{F}_{\mathcal{T}}$, cocartesian transport along the π-cocartesian lifts lying over the inclusions $(S\leftarrow U = U \mid U\in\mathrm{Orb}(S))$ together induce an equivalence
$$\mathcal{O}_S \simeq \prod_{U\in\mathrm{Orb}(S)} \mathcal{O}_U;$$
(c) (Segal condition for multimorphisms) for every map of orbits $T\to S$ in I and pair of objects $(\mathbf{C},\mathbf{D})\in\mathcal{O}_T\times\mathcal{O}_U$, postcomposition with the π-cocartesian lifts $\mathbf{D}\to D_U$ lying over the inclusions $(S\leftarrow U=U\mid U\in\mathrm{Orb}(S))$ induces an equivalence
$$\mathrm{Map}^{T\to S}_{\mathrm{Tot}\mathcal{O}^\otimes}(\mathbf{C},\mathbf{D}) \simeq \prod_{U\in\mathrm{Orb}(S)} \mathrm{Map}^{T\leftarrow T_U\to U}_{\mathrm{Tot}\mathcal{O}^\otimes}(\mathbf{C},D_U).$$
where $T_U := T\times_S U$.

The ∞-category of I-operads is defined to be a localizing subcategory

$$\begin{tikzcd} \mathrm{Op}_I \arrow[r, hook, bend right] & \mathrm{Cat}^{\mathrm{int-cocart}}_{/\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})} \arrow[l, bend right, "L_{\mathrm{Op}_I}"'] \end{tikzcd};$$

that is, a morphism of I-operads is a functor $\mathrm{Tot}\mathcal{O}^\otimes\to\mathrm{Tot}\mathcal{P}^\otimes$ over $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$sending $\pi_{\mathcal{O}}$-cocartesian morphisms to $\pi_{\mathcal{P}}$-cocartesian morphisms. We also call these *$\mathcal{O}$-algebras in $\mathcal{P}$* and we let

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) := \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}(\mathcal{O}^\otimes,\mathcal{P}^\otimes) \subset \mathrm{Fun}_{/\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}(\mathcal{O}^\otimes,\mathcal{P}^\otimes)$$

be the full subcategory spanned by $\mathcal{O}$-algebras in $\mathcal{P}$. ◁

This doesn't obviously recover the notion of [NS22]. To discuss the comparison, we temporarily assume the reader is familiar with *fibrous patterns* in the sense of [BHS22] (which are essentially *weak Segal fibrations* in the sense of [CH21]).

Construction 1.17. Let $\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T}}^{\vee}$ be the cartesian unstraightening of the functor $V \mapsto \mathbb{F}_V$, so that its objects are $\mathcal{T}$-arrows $S \to U$ with $V \in \mathcal{T}$ and its morphisms $f\colon (T \to V) \longrightarrow (S \to U)$ are commutative diagrams between arrows, i.e.

$$\begin{tikzcd} T \arrow[r, dashed, "f^\circ"] \arrow[rr, bend left, "f_s"] \arrow[dr] & S \times_U V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d] \\ & V \arrow[r, "f_t"] & U \end{tikzcd}$$

We say that f is *s.i.* if f° is a summand inclusion and I*-tdeg* if f_t is an identity arrow and f_s lies in I. Then, we define the algebraic pattern

$$\mathrm{Tot}\underline{\mathbb{F}}_{I,*} \coloneqq \mathrm{Span}_{s.i.,I-tdeg}\left(\mathrm{Tot}\mathbb{F}_{\mathcal{T}}^{\vee}\right).$$

The map of triples $(\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T}}^{\vee}, s.i., I-tdeg) \to (\mathbb{F}_{\mathcal{T}}, all, I)$ induces a Segal morphism $s\colon \mathrm{Tot}\underline{\mathbb{F}}_{I,*} \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$. ◃

We recover Nardin-Shah's notion of $\mathcal{T}$-∞-operads by the following result.

Proposition 1.18 ([BHS22; Ste25])**.** *The ∞-category $\mathrm{Op}_{\mathcal{T},\infty}$ of [NS22] is equivalent to fibrous $\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}$-patterns, and s induces an equivalence*

$$\mathrm{Op}_I \simeq \mathrm{Fbrs}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})) \xrightarrow{\sim} \mathrm{Fbrs}\left(\mathrm{Tot}\underline{\mathbb{F}}_{I,*}\right).$$

In particular, we get a composite functor

$$\mathrm{Tot}_{\mathcal{T}}\colon \mathrm{Op}_I \hookrightarrow \mathrm{Cat}^{\mathrm{int-cocart}}_{/\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})} \xrightarrow{s^*} \mathrm{Cat}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T},I}} \xrightarrow{U} \mathrm{Cat}_{\mathcal{T}}$$

The following observation about this composite functor is key. We greatly strengthen it in Section A.6.

Observation 1.19. $\mathrm{Tot}_{\mathcal{T}}\colon \mathrm{Op}_I \to \mathrm{Cat}_{\mathcal{T}}$ is conservative, since each of the component arrows are conservative. ◃

Now, it follows by unwinding definitions that a cocartesian fibration $\pi\colon \mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is an I-operad if and only if its unstraightening $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Cat}$ is an I-symmetric monoidal category. [BHS22] and [NS22] thus independently construct an adjunction

$$\begin{tikzcd} \mathrm{Op}_I \arrow[r, bend left, "\mathrm{Env}_I"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_I^{\otimes} \arrow[l, bend left, "U"] \end{tikzcd}.$$

Now, Op_I has a terminal object $\mathcal{N}_{I\infty}^{\otimes}$, and in [Ste25] we computed $\mathrm{Env}_I\mathcal{N}_{I\infty}^{\otimes} \simeq \underline{\mathbb{F}}_I^{I-\sqcup}$, i.e. it is the weak indexing system for I with indexed tensor products given by indexed coproducts; [BHS22, Prop 4.21] then verifies that the *sliced* left adjoint $\mathrm{Env}_I^{/\underline{\mathbb{F}}_I^{I-\sqcup}}\colon \mathrm{Op}_I \to \mathrm{Cat}^{\otimes}_{I,/\underline{\mathbb{F}}_I^{I-\sqcup}}$ is fully faithful and identifies its image, i.e. Op_I is a colocalizing subcategory of I-symmetric monoidal ∞-categories over $\underline{\mathbb{F}}_I^{I-\sqcup}$ consisting of the *equifibrations*.

Now, the following construction is very occasionally important.

Construction 1.20. Given I a weak indexing category and $V \in \mathcal{T}$, define the $\mathcal{T}_{/V}$-weak indexing category I_V to consist of those maps over V which lie in I. Define

$$\mathrm{Span}_I(\mathbb{F}_V) \coloneqq \mathrm{Span}_{I_V}(\mathbb{F}_V).$$

This is evidently functorial under *unslicing* functors; in particular, pullback along $\mathrm{Span}_I(\mathbb{F}_V) \to \mathrm{Span}_I(\mathbb{F}_W)$ yields a functor

$$\mathrm{Res}_V^W\colon \mathrm{Op}_{I_W} \to \mathrm{Op}_{I_V}.$$

We refer to the associated $\mathcal{T}$-∞-category as $\underline{\mathrm{Op}}_I$. ◃

1.2.2. *The underlying $\mathcal{T}$-symmetric sequence.*

Definition 1.21. The *underlying $\mathcal{T}$-∞-category* $U\mathcal{O}$ of an I-operad $\mathcal{O}^{\otimes}$ is the straightening of the pullback

$$\begin{tikzcd} \mathrm{Tot}U\mathcal{O} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{O}^{\otimes} \arrow[d] \\ \mathcal{T}^{\mathrm{op}} \arrow[r] & \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \end{tikzcd}$$

A $\mathcal{T}$-operad *has at most one color* if each value $U\mathcal{O}_V$ is either empty or contractible, *has at least one color* if $U\mathcal{O}_V$ is nonempty for each V, has *has one color* if $U\mathcal{O} \simeq *_{\mathcal{T}}$. These occupy full subcategories

$$\mathrm{Op}_I^{\mathrm{oc}} \subset \mathrm{Op}_I^{\leq \mathrm{oc}}, \mathrm{Op}_I^{\geq \mathrm{oc}} \subset \mathrm{Op}_I. \qquad \triangleleft$$

In [Ste25, § 2.3] we defined an *underlying $\mathcal{T}$-symmetric sequence* functor and proved the following.

Theorem 1.22 ([Ste25, Thm A])**.** *The underlying $\mathcal{T}$-symmetric sequence functor* $\mathrm{sseq}\colon \mathrm{Op}_{\mathcal{T}}^{\leq oc} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ *is monadic; in particular, it is conservative.*

The V-objects in $\underline{\Sigma}_{\mathcal{T}} \simeq \underline{\mathbb{F}}_{\mathcal{T}}^{\simeq}$ are finite V-sets; given $S \in \Sigma_V \simeq \mathbb{F}_V^{\simeq}$, writing $\mathcal{O}(S)$ for $\mathrm{sseq}\,\mathcal{O}^{\otimes}(S)$, we remember this as saying that at-most-one-color $\mathcal{T}$-operads are identified conservatively by their S-ary structure spaces. Using this, we defined the full subcategory of *$\mathcal{T}$-d-operads* as those with $(d-1)$-truncated structure spaces:

$$\mathrm{Op}_{\mathcal{T},d} := \left\{ \mathcal{O}^{\otimes} \mid \forall S,\, \mathcal{O}(S) \in \mathcal{S}_{\leq (d-1)} \right\} \subset \mathrm{Op}_{\mathcal{T}}$$

In [Ste25, § 2.5], we verified the following.

Proposition 1.23 ([Ste25])**.** *The inclusion $\mathrm{Op}_{\mathcal{T},d} \subset \mathrm{Op}_{\mathcal{T}}$ has a left adjoint $h_d\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T},d}$, and given $\mathcal{P}^{\otimes} \in \mathrm{Op}_{\mathcal{T},d}$, the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ is a d-category; moreover, if $\mathcal{P}^{\otimes} \in \mathrm{Op}_{\mathcal{T},0}$, then $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ is either empty or contractible. In particular, $\mathrm{Op}_{\mathcal{T},d}$ is a $(d+1)$-category and $\mathrm{Op}_{\mathcal{T},0}$ is a poset.*

We call $h_d\mathcal{O}^{\otimes}$ the *homotopy $\mathcal{T}$-d-operad of $\mathcal{O}^{\otimes}$*. We went on to compute the *free $\mathcal{O}$-algebra monad*; for algebras in a cartesian structure on coefficient systems in a cocomplete cartesian closed ∞-category $\mathcal{C}$, this sends $X \in \mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ to the coefficient system $T_{\mathcal{O}}X$ with

$$(T_{\mathcal{O}}X)^V \simeq \coprod_{S \in \mathbb{F}_V} \left(\mathrm{Fr}_{\mathcal{C}}\,\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S)} X^U \right)_{h\mathrm{Aut}_V(S)},$$

where $\mathrm{Fr}_{\mathcal{C}}\colon \mathcal{S} \to \mathcal{C}$ is the unique symmetric monoidal left adjoint. In particular, given $S \in \mathbb{F}_V$, in [Ste25] we found a natural splitting $\mathrm{Fr}_{\mathcal{C}}\,\mathcal{O}(S) \oplus J \simeq (T_{\mathcal{O}}S)^V$. A multiple-color version of this argument yielded the following.

Proposition 1.24 ([Ste25, § 2.4])**.** *A map of $\mathcal{T}$-operads $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is an h_d-equivalence if and only if:*

(a) the underlying $\mathcal{T}$-functor $U\varphi\colon U\mathcal{O} \to U\mathcal{P}$ is essentially surjective, and

(b) the pullback functor $\varphi^*\colon \mathrm{Alg}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\leq(d-1)}) \to \mathrm{Alg}_{\mathcal{O}}\left(\underline{\mathcal{S}}_{\mathcal{T},\leq(d-1)}\right)$ is an equivalence.

In particular, φ is an equivalence if and only if it is U-essentially surjective and induces an equivalence on algebras in $\underline{\mathcal{S}}_{\mathcal{T}}$.

1.2.3. *Rudiments of weak $\mathcal{N}_\infty$-operads.* In [Ste25, § 2.2], we constructed a family of $\mathcal{T}$-operads:

Proposition 1.25 ([Ste25])**.** *Let $I \subset J \subset \mathbb{F}_{\mathcal{T}}$ be pullback-stable subcategories. Then, $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}_J(\mathbb{F}_{\mathcal{T}})$ presents a J-operad if and only if I is a weak indexing category.*

These are called weak $\mathcal{N}_\infty$-operads; in the case that I is an indexing category, these are called $\mathcal{N}_\infty$-operads. To state their universal property, we defined the *arity support* subcategory

$$A\mathcal{O} := \left\{ T \to S \;\middle|\; \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T \times_S U) \neq \varnothing \right\} \subset \mathbb{F}_{\mathcal{T}},$$

Theorem 1.26 ([Ste25, § 2.6])**.** *The arity support of a $\mathcal{T}$-operad is a weak indexing category, and the associated essential surjection admits a fully faithful right adjoint*

$$\begin{tikzcd} \mathrm{Op}_{\mathcal{T}} \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathrm{wIndexCat}_{\mathcal{T}} \arrow[l, bend left, "\mathcal{N}_{(-)\infty}"] \end{tikzcd}$$

The essential image of $\mathcal{N}_{(-)\infty}$ is spanned by $\mathcal{T}$-operads $\mathcal{O}^{\otimes}$ satisfying the following equivalent conditions.

(a) $\mathcal{O}^{\otimes}$ is a weak $\mathcal{N}_\infty$-operad.

(b) $\mathcal{O}^\otimes$ *is a* $\mathcal{T}$*-0-operad.*
(c) The map of $\mathcal{T}$*-operads* $\mathcal{O}^\otimes \to \mathrm{Comm}_{\mathcal{T}}^\otimes$ *is a monomorphism.*

In particular, this isolates the weak $\mathcal{N}_\infty$-operads as those possessing a fully faithful unslicing functor

$$\mathrm{Op}_{\mathcal{T},/\mathcal{N}_{I\infty}} \hookrightarrow \mathrm{Op}_{\mathcal{T}}.$$

This yields functors for change of weak indexing systems, which we use to generically specialize to $I = \mathcal{T}$.

Proposition 1.27 ([Ste25])**.** *Postcomposition along* $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}_J(\mathbb{F}_{\mathcal{T}})$ *yields a fully faithful embedding*

$$\mathrm{Op}_I \simeq \mathrm{Op}_{J,/\mathcal{N}^\otimes_{I\infty}} \hookrightarrow \mathrm{Op}_J.$$

We denote the associated push-pull adjunction as

$$\begin{tikzcd} \mathrm{Op}_I \arrow[r, bend left, "E_I^J"] \arrow[r, phantom, "\perp"] & \mathrm{Op}_J; \arrow[l, bend left, "\mathrm{Bor}_I^J"] \end{tikzcd} \tag{7}$$

it follows from Theorem 1.26 that $A\mathrm{Bor}_I^J\mathcal{O} \simeq I \cap A\mathcal{O}$ and $AE_I^J\mathcal{O} \simeq A\mathcal{O}$. We have several examples:

Example 1.28. Let I^{triv} be the initial one-color weak indexing category. The corresponding weak $\mathcal{N}_\infty$-operad $\mathrm{triv}_{\mathcal{T}}^\otimes := \mathcal{N}^\otimes_{I^{\mathrm{triv}}\infty}$ is called the *trivial* $\mathcal{T}$*-operad*, and it is characterized by its algebras [NS22; Ste25]

$$\underline{\mathrm{CAlg}}_{I^{\mathrm{triv}}}(\mathcal{O}) \simeq U\mathcal{O};$$

in particular, the restriction of the underlying $\mathcal{T}$-∞-category construction to I^{triv}-operads yields an equivalence $\mathrm{Op}_{I^{\mathrm{triv}}} \xrightarrow{\sim} \mathrm{Cat}_{\mathcal{T}}$ and $E^J_{I^{\mathrm{triv}}}$ is compatible with U [NS22]; that is, Eq. (7) takes the form of an adjunction

$$\begin{tikzcd} \mathrm{Cat}_{\mathcal{T}} \arrow[r, bend left, "\mathrm{triv}(-)^\otimes"] \arrow[r, phantom, "\perp"] & \mathrm{Op}_{\mathcal{T}}, \arrow[l, bend left, "U"] \end{tikzcd}$$

i.e. given a $\mathcal{T}$-∞-category $\mathcal{C}$, we acquire a $\mathcal{T}$-operad characterized by its algebras [NS22]

$$\mathrm{Alg}_{\mathrm{triv}(\mathcal{C})}(\mathcal{O}) \simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}, U\mathcal{O});$$

this formula allows for an alternative construction for $\mathrm{triv}(\mathcal{C})$ is as the operadic localization [Ste25]

$$\mathrm{triv}(\mathcal{C})^\otimes \simeq L_{\mathrm{Op}_{\mathcal{T}}}(\mathcal{C} \to \mathcal{T}^{\mathrm{op}} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})). \qquad \triangleleft$$

Example 1.29. Given $\mathcal{F} \subset \mathcal{T}$ a family, define

$$\mathbb{F}^0_{\mathcal{T},\mathcal{F},V} := \begin{cases} \{\varnothing_V, *_V\} & V \in \mathcal{F} \\ \{*_V\} & V \notin \mathcal{F}. \end{cases}$$

Let $I^0_{\mathcal{T},\mathcal{F}}$ be the associated weak indexing category; this is the initial one-color weak indexing category with $v(I) \supset \mathcal{F}$. We define $\mathbb{E}^\otimes_{0,\mathcal{F}} := \mathcal{N}^\otimes_{I^0_{\mathcal{T},\mathcal{F}}\infty}$; in particular, we write $\mathbb{E}_0^\otimes := \mathbb{E}^\otimes_{0,\mathcal{T}}$.

It was shown in [NS22, Thm 5.2.10] (and re-shown in [Ste25, § 3.3]) that there is an equivalence

$$\Gamma^{\mathcal{T}} \underline{\mathrm{Alg}}^\otimes_{\mathbb{E}_0}(\mathcal{C}) \simeq \left(\Gamma^{\mathcal{T}}\mathcal{C}\right)^\otimes_{1/}. \qquad \triangleleft$$

Example 1.30. Let $\underline{\mathbb{F}}^\infty$ be the initial indexing system, i.e.

$$\mathbb{F}^\infty_V := \{n \cdot *_V \mid n \in \mathbb{N}\}.$$

Let I^∞ be its indexing category. We write $\mathbb{E}^\otimes_\infty := \mathcal{N}^\otimes_{I^\infty\infty}$. In [Ste25, § 3.3] we constructed an equivalence

$$\Gamma^{\mathcal{T}} \underline{\mathrm{Alg}}^\otimes_{\mathbb{E}_\infty}(\mathcal{C}) \simeq \mathrm{CAlg}^\otimes\left(\Gamma^{\mathcal{T}}\mathcal{C}\right). \qquad \triangleleft$$

Example 1.31. The terminal $\mathcal{T}$-operad $\mathrm{Comm}^\otimes_{\mathcal{T}} = (\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) = \mathrm{Span}(\mathbb{F}_{\mathcal{T}}))$ is the $\mathcal{N}_\infty$-operad for the terminal indexing category $\mathbb{F}_{\mathcal{T}} = \mathbb{F}_{\mathcal{T}}$. $\triangleleft$

1.2.4. *The Boardman-Vogt tensor product.* In [Ste25, Thm D], in the case that $\mathcal{T}$ has a terminal object, we equipped $\mathrm{Op}_{\mathcal{T}}$ with a closed *Boardman-Vogt tensor product*

$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} := L_{\mathrm{Op}}\left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \longrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \times \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \xrightarrow{\wedge} \mathrm{Span}(\mathbb{F}_{\mathcal{T}})\right),$$

Its internal hom is denoted $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P})$; its underlying $\mathcal{T}$-∞-category is denoted $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P})$, and it has values

$$\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P})_V \simeq \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{O}}(\mathrm{Res}^{\mathcal{T}}_V \mathcal{P}).$$

We verified several properties in [Ste25]; for instance, $\underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C})$ is an I-symmetric monoidal ∞-category when $\mathcal{C}$ is, functorially for I-symmetric monoidal maps in $\mathcal{C}^{\otimes}$ and $\mathcal{T}$-operad maps in $\mathcal{P}^{\otimes}$. We interpret $\mathcal{O} \otimes \mathcal{P}$-algebras as *homotopy-coherently interchanging pairs of $\mathcal{O}$-algebra and $\mathcal{P}$-algebra structures* via the following.

Recollection 1.32. Suppose $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal 1-category and $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are one-color $\mathcal{T}$-operads. We saw in [Ste25] that an $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$-algebra structure on a $\mathcal{T}$-object $X \in \Gamma^{\mathcal{T}} \mathcal{C}$ is equivalently viewed as a pair of $\mathcal{O}$-algebra and $\mathcal{P}$-algebra structures subject to the *interchange relation* that, for all $\mu_S \in \mathcal{O}(S)$ and $\mu_T \in \mathcal{P}(T)$, the following diagram commutes.

$$\begin{tikzcd}
\bigotimes_U^S X_V^{\otimes \mathrm{Res}^V_U T} \simeq X_V^{\otimes S \times T} \simeq \bigotimes_W^T X_V^{\otimes \mathrm{Res}^V_W S} \arrow[r, "(\mathrm{Res}^V_W \mu_S)"] \arrow[d, "(\mathrm{Res}^V_U \mu_T)"'] & X_V^{\otimes T} \arrow[d, "\mu_T"] \\
X_V^{\otimes S} \arrow[r, "\mu_S"] & X_V
\end{tikzcd}$$

A morphism of $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$-algebras is simply a morphism of $\mathcal{T}$-objects which is simultaneously an $\mathcal{O}$-algebra map and a $\mathcal{P}$-algebra map. ◁

The following proposition exhibited a key role played by $\mathrm{triv}^{\otimes}_{\mathcal{T}}$.

Proposition 1.33 ([Ste25, Thm D.(3)])**.** $\mathrm{triv}^{\otimes}_{\mathcal{T}}$ *is the* $\overset{\mathrm{BV}}{\otimes}$*-unit; hence there is an equivalence of $\mathcal{T}$-operads*

$$\underline{\mathrm{Alg}}_{\mathrm{triv}_{\mathcal{T}}}(\mathcal{O}) \simeq \mathcal{O}^{\otimes}$$

We also saw that $\overset{\mathrm{BV}}{\otimes}$ is compatible with the *Mode* (i.e. Day coonvolution or box product) structure.

Proposition 1.34 ([Ste25, Thm D.(7)])**.** *The $\mathcal{T}$-symmetric monoidal envelope intertwines the mode symmetric monoidal structure on* $\mathrm{Cat}^{\otimes}_{\mathcal{T}}$ *with Boardman-Vogt tensor products, i.e.*

$$\mathrm{Env}\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right) \simeq \mathrm{Env}\left(\mathcal{O}^{\otimes}\right) \circledast \mathrm{Env}\left(\mathcal{P}^{\otimes}\right).$$

Furthermore, $\mathrm{Env}\left(\mathrm{triv}^{\otimes}_{\mathcal{T}}\right)$ *is the $\circledast$-unit.*

To use all of these results, for the remainder of Sections 1 and 2 we will make the following assumption. In Corollary 3.1, we will establish Assumption (b) in full generality, so the results of Sections 1 and 2 will apply for arbitrary $\mathcal{T}$ after that point.

Assumption 1.35. We assume one of the following things is true.

(a) $\mathcal{T}$ has a terminal object.
(b) $\overset{\mathrm{BV}}{\otimes} \colon \underline{\mathrm{Op}}_{\mathcal{T}} \times \underline{\mathrm{Op}}_{\mathcal{T}} \to \underline{\mathrm{Op}}_{\mathcal{T}}$ is a $\mathcal{T}$-bifunctor whose restriction $\mathrm{Op}_V \times \mathrm{Op}_V \to \mathrm{Op}_V$ is $\overset{\mathrm{BV}}{\otimes}$ over $\mathcal{T}_{/V}$. ◁

1.3. **Restriction and arity-borelification.** We now expand on Res^V_U and Bor^J_I.

1.3.1. *Operadic restriction and (co)induction.* Recall from [Ste25, § 2.3] that the underlying $\mathcal{T}$-symmetric sequence forms a $\mathcal{T}$-functor $\underline{\mathrm{sseq}} \colon \underline{\mathrm{Op}}^{\mathrm{red}}_{\mathcal{T}} \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \underline{\mathcal{S}}_{\mathcal{T}})$; in particular, restriction of $\underline{V}$-operads lies over restriction of $\underline{V}$-symmetric sequences. This upgrades Theorem 1.26 to an adjunction of $\mathcal{T}$-∞-categories.

Proposition 1.36. $\mathrm{Res}_V^W \mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{N}_{\mathrm{Res}_V^W I\infty}^{\otimes}$*; more generally,* $A \dashv \mathcal{N}_{(-)\infty}^{\otimes}$ *lifts to a* $\mathcal{T}$*-adjunction*

$$\begin{tikzcd} \underline{\mathrm{Op}}_{\mathcal{T}} \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{wIndex}}_{\mathcal{T}} \arrow[l, bend left, "\mathcal{N}_{(-)\infty}^{\otimes}"] \end{tikzcd}$$

Proof. Restriction compatiblility of the underlying symmetric sequence implies that $\mathrm{Res}_V^W A\mathcal{O} = A\,\mathrm{Res}_V^W \mathcal{O}$, lifting A to a $\mathcal{T}$-functor $\underline{\mathrm{Op}}_{\mathcal{T}} \to \underline{\mathrm{wIndex}}_{\mathcal{T}}$ whose V-value is $A : \mathrm{Op}_V \to \mathrm{wIndex}_V$. The right adjoints $\mathcal{N}_{(-)\infty}^{\otimes}$ uniquely lift to a right $\mathcal{T}$-adjoint to $\mathcal{N}_{(-)\infty}^{\otimes}$ by [HA, Prop 7.3.2.6], completing the proposition. $\square$

Since A is a $\mathcal{T}$-left adjoint, it is compatible with $\mathcal{T}$-colimits. Applying this for indexed coproducts, we immediately acquire the following convenient properties of A.

Corollary 1.37. *If* $\mathcal{O}, \mathcal{P}$ *are* $\mathcal{T}$*-operads, then we have*

$$A(\mathcal{O} \sqcup \mathcal{P}) = A\mathcal{O} \vee A\mathcal{P}.$$

If $\mathcal{Q}$ *is a* V*-operad, then we have*

$$A\mathrm{Ind}_V^W \mathcal{Q} = \mathrm{Ind}_V^W A\mathcal{Q}.$$

We may use an analogous argument to that of [BHS22, Lem 4.1.13] to show that $\underline{\mathrm{Op}}_{\mathcal{T}}$ strongly admits $\mathcal{T}$-limits; since the fully faithful $\mathcal{T}$-functor $\underline{\mathrm{Op}}_{\mathcal{T}} \to \underline{\mathrm{Cat}}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$ possesses pointwise left adjoints (given by L_{Fbrs}), it possesses a $\mathcal{T}$-left adjoint; in particular, we may compute $\mathcal{T}$-limits of $\mathcal{T}$-operads in $\underline{\mathrm{Cat}}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$. Then, an analogous argument using [BHS22, Prop 2.3.7] constructs $\mathcal{T}$-limits in $\underline{\mathrm{Cat}}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$ in $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathrm{Span}(\mathbb{F}_{\mathcal{T}}), \underline{\mathrm{Cat}}_{\mathcal{T}})_{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$, which strongly admits $\mathcal{T}$-limits, as its a slice $\mathcal{T}$-∞-category of a functor $\mathcal{T}$-∞-category into a $\mathcal{T}$-∞-category which strongly admits $\mathcal{T}$-limits. In particular, this constructs a right adjoint to $\mathrm{Res}_V^W : \mathrm{Op}_W \to \mathrm{Op}_V$, which we call CoInd_V^W.

Proposition 1.38. *If* $\mathcal{O}^{\otimes}$ *is a* V*-*d*-operad, then* $\mathrm{CoInd}_V^W \mathcal{O}^{\otimes}$ *is a* W*-*d*-operad.*

Proof. This follows simply by taking right adjoints within the following diagram

$$\begin{tikzcd} \mathrm{Op}_W \arrow[r, "\mathrm{Res}_V^W"] \arrow[d] & \mathrm{Op}_V \arrow[d] \\ \mathrm{Op}_{W,d} \arrow[r, "\mathrm{Res}_V^W"] & \mathrm{Op}_{V,d} \end{tikzcd} \tag{8}$$

$\square$

Corollary 1.39. *There exist equivalences*

$$\mathrm{sseq}\,\mathrm{CoInd}_V^W \mathcal{O}^{\otimes} \simeq \mathrm{CoInd}_V^W \mathrm{sseq}\,\mathcal{O}^{\otimes};$$
$$A\mathrm{CoInd}_V^W \mathcal{O} = \mathrm{CoInd}_V^W A\mathcal{O}.$$

Proof. The first statement equivalence by noting that $\mathrm{Fr}\,\mathrm{Res}_V^W = \iota_V^{W*}\mathrm{Fr}$ and taking right adjoints. The second follows by taking right adjoints of Eq. (8) in the case $d = 0$. $\square$

We care about $\mathrm{CoInd}_V^W \mathcal{O}^{\otimes}$ because it is a structure borne by *norms of algebras* as follows.

Construction 1.40. Let I be a W-weak indexing category containing the map $V \to W$, let $\mathcal{P}^{\otimes} \to \mathrm{CoInd}_V^W \mathcal{O}^{\otimes}$ be a functor of one-object I-operads, and let $\mathcal{C}^{\otimes}$ be a I-symmetric monoidal ∞-category. Then, the adjunct map $\varphi\colon \mathrm{Res}_V^W \mathcal{P} \to \mathcal{O}^{\otimes}$ participates in a commutative diagram of symmetric monoidal functors

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Res}_V^W \mathcal{C}) \arrow[r, "\varphi^*"] \arrow[d, "U_V"] & \mathrm{Alg}_{\mathrm{Res}_V^W \mathcal{P}}(\mathrm{Res}_V^W \mathcal{C}) \arrow[r, "N_V^W"] \arrow[d, "U_V"] & \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \arrow[d, "U_W"] \\ \mathcal{C}_V \arrow[r, equal] & \mathcal{C}_V \arrow[r, "N_V^W"] & \mathcal{C}_W \end{tikzcd}$$

Intuitively, $\mathrm{CoInd}_V^W \mathcal{O}^{\otimes}$ bears the *universal* natural structure on $N_V^W X$ for all $X \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. ◁

1.3.2. *Color and arity Borelification.* Let $\mathcal{F} \subset \mathcal{T}$ be a $\mathcal{T}$-family. There is a terminal $\mathcal{F}$-colored weak indexing category $\mathbb{F}_{\mathcal{F}}$; we refer to $\mathbb{F}_{\mathcal{F}}$-Borelification as *$\mathcal{F}$-Borelification* and write $\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} := \mathrm{Bor}_{\mathbb{F}_{\mathcal{F}}}^{\mathcal{T}}$. Note that

$$\mathrm{Op}_{\mathcal{F}} \simeq \mathrm{Op}_{I_{\mathcal{F}}}.$$

Let $\mathrm{triv}_{\mathcal{F}}^{\otimes} := \mathrm{triv}(*_{\mathcal{F}})^{\otimes}$; this is the weak $\mathcal{N}_\infty$-operad for the initial $\mathcal{F}$-colored weak indexing category $I_{\mathcal{F}}^{\mathrm{triv}} = \mathbb{F}_{\mathcal{F}}^{\simeq} = E_{\mathcal{F}}^{\mathcal{T}} I^{\mathrm{triv}}$, and in particular, there is an equivalence

$$\mathrm{triv}_{\mathcal{F}}^{\otimes} \simeq E_{\mathcal{F}}^{\mathcal{T}} \mathrm{triv}_{\mathcal{F}}^{\otimes}. \tag{9}$$

This is our first nontrivial example of a $\overset{\mathrm{BV}}{\otimes}$-idempotent $\mathcal{T}$-operad.

Proposition 1.41 (Color-borelification)**.** *Given $\mathcal{F} \in \mathrm{Fam}_{\mathcal{T}}$ a $\mathcal{T}$-family, there is a natural equivalence*

$$\mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathcal{O}) \simeq \Gamma^{\mathcal{F}} \mathcal{O};$$

hence there is a natural equivalence

$$\mathrm{triv}_{\mathcal{F}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes} \simeq E_{\mathcal{F}}^{\mathcal{T}} \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{O}^{\otimes}.$$

Proof. The first statement follows by using Eq. (9) and Proposition 1.33 to construct equivalences

$$\mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathcal{O}) \simeq \mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}}(\mathcal{O})) \simeq \Gamma^{\mathcal{F}} \mathcal{O}.$$

The second statement then follows by Yoneda's lemma, noting that

$$\begin{aligned}
\mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}} \otimes \mathcal{O}}(\mathcal{P}) &\simeq \mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}}} \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{P}) \\
&\simeq \Gamma^{\mathcal{F}} \mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \\
&\simeq \mathrm{Alg}_{\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{O}}(\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{P}) \\
&\simeq \mathrm{Alg}_{E_{\mathcal{F}}^{\mathcal{T}} \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{O}}(\mathcal{P}).
\end{aligned}$$

□

Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$, we define the color family $c(\mathcal{O}) := c(A\mathcal{O}) = \{V \mid \mathcal{O}_V \neq \varnothing\}$.

Remark 1.42. Proposition 1.41 exhibits $\mathrm{Im} E_{\mathcal{F}}^{\mathcal{T}} = \left\{\mathcal{O}^{\otimes} \in \mathrm{Op}^{\mathcal{T}} \mid c(\mathcal{O}) \subset \mathcal{F}\right\}$ as a $\otimes$-*ideal*, i.e. if $c(\mathcal{O}) \subset \mathcal{F}$, and $\mathcal{P}^{\otimes}$ is arbitrary, then $c\left(\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{P}\right) \subset \mathcal{F}$. ◁

This is important in part because it reduced $\overset{\mathrm{BV}}{\otimes}$ computations to the at-least one color case.

Observation 1.43. There is a string of natural equivalences

$$\begin{aligned}
\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} &\simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{c\mathcal{O}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{c\mathcal{P}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}, \\
&\simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{c\mathcal{O} \cap c\mathcal{P}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}, \\
&\simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{c\mathcal{O} \cap c\mathcal{P}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{c\mathcal{O} \cap c\mathcal{P}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}, \\
&\simeq E_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}} \mathrm{Bor}_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}} \left(\mathcal{O}^{\otimes}\right) \overset{\mathrm{BV}}{\otimes} E_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}} \mathrm{Bor}_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}} \left(\mathcal{P}^{\otimes}\right), \\
&\simeq E_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}} \left(\mathrm{Bor}_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}} \left(\mathcal{O}^{\otimes}\right) \overset{\mathrm{BV}}{\otimes} \mathrm{Bor}_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}} \left(\mathcal{P}^{\otimes}\right)\right).
\end{aligned}$$

Moreover, the $c\mathcal{O} \cap c\mathcal{P}$-operads $\mathrm{Bor}_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}}(\mathcal{O}^{\otimes})$ and $\mathrm{Bor}_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}}(\mathcal{P}^{\otimes})$ both have at least one color. ◁

Having done this, we may compute arity-supports of arbitrary tensor products of $\mathcal{T}$-operads.

Proposition 1.44. *Suppose $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are $\mathcal{T}$-operads. Then,*

$$A\left(\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{P}\right) = E_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}} \mathrm{Bor}_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}} (A\mathcal{O} \vee A\mathcal{P}).$$

Proof. By Observation 1.43, we have equivalences

$$A\left(\mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes}\right) \simeq E_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}} A\left(\mathrm{Bor}_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}} \left(\mathcal{O}^{\otimes}\right) \overset{\mathrm{BV}}{\otimes} \mathrm{Bor}_{c\mathcal{O} \cap c\mathcal{P}}^{\mathcal{T}} \left(\mathcal{P}^{\otimes}\right)\right),$$

so it suffices to prove the proposition in the case that $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ have at least one color.

In this case, first note that there exist maps

$$\mathcal{O}^{\otimes} \otimes \mathrm{triv}_{\mathcal{T}}^{\otimes}, \mathrm{triv}_{\mathcal{T}}^{\otimes} \otimes \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes},$$

and applying A together with the universal property for joins yields an inequality

$$A\mathcal{O} \vee A\mathcal{P} \leq A(\mathcal{O} \vee \mathcal{P}).$$

To provide the inequality in the other direction, by Proposition 1.36, in the case of Assumption (b) we may pass to a restriction and assume that $\mathcal{T}$ has a terminal object; in this case, there exists a composite map

$$\mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes} \to \mathcal{N}^{\otimes}_{A\mathcal{O}\infty} \otimes \mathcal{N}^{\otimes}_{A\mathcal{P}\infty} \to \mathcal{N}^{\otimes}_{A\mathcal{O}\vee A\mathcal{P}\infty} \otimes \mathcal{N}^{\otimes}_{A\mathcal{O}\vee A\mathcal{P}\infty} \to \mathcal{N}^{\otimes}_{A\mathcal{O}\vee A\mathcal{P}\infty},$$

whose last map is presented by the bifunctor

$$\begin{tikzcd} \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \times \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \arrow[r, "\wedge"] \arrow[d] & \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \arrow[d] \\ \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \times \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \arrow[r, "\wedge"] & \mathrm{Span}(\mathbb{F}_{\mathcal{T}}); \end{tikzcd}$$

here, the top map is defined canonically by the fact that weak indexing categories $I \subset \mathbb{F}_{\mathcal{T}}$ are closed under cartesian products [Ste24]. Applying A to this map yields $A(\mathcal{O} \vee \mathcal{P}) \leq A\mathcal{O} \vee A\mathcal{P}$, as desired. $\square$

We immediately acquire the following corollary.

Corollary 1.45. $\mathrm{Op}_I \subset \mathrm{Op}_{\mathcal{T}}$ *is closed under binary tensor products; if I has one color, then* $\mathrm{triv}^{\otimes}_{\mathcal{T}} \in \mathrm{Op}_I$.

1.4. **(Co)cartesian I-symmetric monoidal ∞-categories.** Fix I an almost-unital weak indexing system. In this section, we characterize *cartesian and cocartesian* I-symmetric monoidal ∞-categories, in part as examples of interest and in part as universal construction.

We defer the minutiae of these to Section A, where we construct ∞-categories $\mathcal{C}^{I-\times}, \mathcal{C}^{I-\sqcup}$ over $\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}$, verifying that $\mathcal{C}^{I-\times}$ is an I-symmetric monoidal ∞-category precisely when $\mathcal{C}$ has I-indexed products, that $\mathcal{C}^{I-\sqcup}$ is always an I-operad, and that that $\mathcal{C}^{I-\sqcup}$ is an I-symmetric monoidal ∞-category precisely when $\mathcal{C}$ has I-indexed coproducts. Most of this abuts to unenlightening technicalities about parameterized higher category theory, which we defer to Section A, summarizing the outcomes as they become relevant.

1.4.1. *(Co)cartesian rigidity.* Denote by $\mathrm{Cat}^{I-\sqcup}_I, \mathrm{Cat}^{I-\times}_I \subset \mathrm{Cat}_{\mathcal{T}}$ the replete subcategories with objects given by $\mathcal{T}$-∞-categories attaining I-indexed coproducts (resp. products) and with morphisms given by $\mathcal{T}$-functors which preserve I-indexed coproducts (products). In Section A, we prove the following.

Theorem A'. *There are fully faithful embeddings $(-)^{I-\sqcup}, (-)^{I-\times}$ making the following commute:*

$$\begin{tikzcd} \mathrm{Cat}^{I-\sqcup}_I \arrow[r, hook, "(-)^{I-\sqcup}"] \arrow[dr, "U"'] & \mathrm{Cat}^{\otimes}_I \arrow[d, "U"] & \mathrm{Cat}^{I-\times}_I \arrow[l, hook', "(-)^{I-\times}"'] \arrow[dl, "U"] \\ & \mathrm{Cat}_{\mathcal{T}} & \end{tikzcd}$$

The image of $(-)^{I-\sqcup}$ is spanned by the I-symmetric monoidal ∞-categories whose I-admissible indexed tensor functors $\otimes^S : \mathcal{C}_S \to \mathcal{C}_V$ are left adjoint to the indexed diagonal $\Delta^S : \mathcal{C}_V \to \mathcal{C}_S$ (i.e. whose indexed tensor products are are indexed coproducts), and the image of $(-)^{I-\times}$ is spanned by those whose I-admissible indexed tensor functors $\otimes^S$ are right adjoint to Δ^S.

We call I-symmetric monoidal ∞-categories of the form $\mathcal{C}^{I-\sqcup}$ *cocartesian*, and $\mathcal{C}^{I-\times}$ *cartesian*.

Philosophical remark 1.46. In higher category theory, a fundamental rigidity result is that of *adjoints*; by [HTT, Prop 5.2.6.2] the full subcategory of $\mathrm{Fun}(\mathcal{D},\mathcal{C})^{\mathrm{op}}$ spanned by functors right adoint to a fixed functor $L : \mathcal{C} \to \mathcal{D}$ is contractible. Moreover, adjointness itself as a property requires only finitely data to test per-object, separately (c.f. [HTT, Prop 5.2.2.9, 5.2.2.12]).

It is this rigidity which we are leveraging in Theorem A': in essence, the coherent data witnessing I-symmetric monoidality of a functor $\mathcal{C}^{I-\times} \to \mathcal{D}^{I-\times}$ is constructed (up to contractible ambiguity) by the *property* that the underlying $\mathcal{T}$-functor $\mathcal{C} \to \mathcal{D}$ is compatible with the adjunctions $\Delta^S \dashv \bigotimes^S$. ◃

Many similar definitions have been made in the literature. Luckily, they all agree.

Remark 1.47. [NS22] constructed a pair of structures $\mathcal{C}^{\coprod}, \mathcal{C}^{\prod}$ which, after unwinding definitions, satisfy the conditions of Theorem A' in the case $I = \mathcal{T}$. In particular, there are unique I-symmetric monoidal equivalences $\mathrm{Bor}_I^{\mathcal{T}} \mathcal{C}^{\coprod} \simeq \mathcal{C}^{I-\sqcup}$ and $\mathrm{Bor}_I^{\mathcal{T}} \mathcal{C}^{\prod} \simeq \mathcal{C}^{I-\times}$ lying over the identity whenever $\mathcal{C}$ admits finite indexed (co)products.

Moreover, [CHLL24b] introduced another structure, specifically a *cartesian I-symmetric monoidal structure* on $\underline{\mathbb{F}}_{\mathcal{T}}$, and conjectured it to be equivalent to Nardin-Shah's construction; this conjecture was recently verified in [CLR25] by verifying universality of the unfurling construction used in Nardin-Shah's construction. We acquire an independent proof of this result without serious $(\infty,2)$-category theory: Cnossen-Haugseng-Lenz-Linsken's construction satisfies the condition of Theorem A', so there is a unique pair of I-symmetric monoidal equivalences $\mathrm{Bor}_I^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{T}}^{\prod} \simeq \underline{\mathbb{F}}_{\mathcal{T}}^{I-\times} \simeq \underline{\mathbb{F}}_{\mathcal{T},\times}$ lying over the identity.

Moreover, after drafts of this article were made public, [LLP25, Def 2.15] constructed another structure, which specializes to a *cocartesian I-symmetric monoidal structure* $\mathcal{C}^{I-\coprod}$; by [LLP25, Lem 2.16(i)], in the case that $\mathcal{C}$ has I-indexed coproducts, $\mathcal{C}^{I-\coprod}$ satisfies the conditions of Theorem A', so there is a unique I-symmetric monoidal equivalence $\mathcal{C}^{I-\coprod} \simeq \mathcal{C}^{I-\sqcup}$ lying over the identity. ◁

We prove the following in Section A as a precursor to Theorem A', though it also follows from it.

Proposition 1.48. *There is a unique equivalence $\left(\mathcal{C}^{I-\times}\right)^{\mathrm{vop}} \simeq (\mathcal{C}^{\mathrm{vop}})^{I-\sqcup}$ lying over the identity.*

Before characterizing the algebras in $\mathcal{C}^{I-\sqcup}$ and $\mathcal{C}^{I-\times}$, we point out that they are often presentable.

Proposition 1.49. *Suppose $\mathcal{C}$ is a presentable ∞-category with I-indexed products and coproducts.*

(1) $\underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{C}^{I-\sqcup}$ is presentably I-symmetric monoidal.

(2) If finite products in $\mathcal{C}$ commute with colimits separately in each variable (i.e. it is Cartesian closed), then $\underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{C}^{I-\times}$ is presentably I-symmetric monoidal,

Proof. It follows from Hilman's characterization of parameterized presentability [Hil24b, Thm 6.1.2] that $\underline{\mathrm{Coeff}}^{\mathcal{T}}$ is presentable, so we're tasked with proving that the $\mathcal{T}$-symmetric monoidal structures are distributive. The first case is just commutativity of colimits with colimits, and the second is [NS22, Prop 3.2.5]. ☐

1.4.2. *$\mathcal{O}$-monoids.* We will identify algebras in $\mathcal{C}^{I-\times}$ with the following.

Definition 1.50. Fix $\mathcal{O}^{\otimes}$ an I-operad and $\mathcal{C}$ a $\mathcal{T}$-∞-category. Then, an *$\mathcal{O}$-monoid in $\mathcal{C}$* is a $\mathcal{T}$-functor $M\colon \mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathcal{C}$ satisfying the condition that, for each orbit $V \in \mathcal{T}$, each finite V-set $S \in \mathbb{F}_V$, and each S-tuple $X = (X_U) \in \mathcal{O}_S$, the canonical maps $M(X) \to \mathrm{CoInd}_U^V M(X_U)$ realize $M(X)$ as the indexed product

$$M(X) \simeq \prod_U^S M(X_U).$$ ◁

Indeed, we prove the following equivariant lift of [HA, Prop 2.4.2.5] as Proposition A.16.

Proposition 1.51. *Given $\mathcal{O}^{\otimes}$ an I-operad and $\mathcal{C}$ a $\mathcal{T}$-∞-category with I-indexed products, the forgetful functor*

$$\mathrm{Alg}_{\mathcal{O}}\left(\mathcal{C}^{I-\times}\right) \longrightarrow \mathrm{Fun}_{\mathcal{T}}\left(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}, \mathcal{C}\right)$$

is fully faithful with image spanned by the $\mathcal{O}$-monoids.

Corollary 1.52. *Given $\mathcal{O}^{\otimes}$ an I-operad and $\mathcal{D}$ an ∞-category with finite products, the forgetful functor*

$$\mathrm{Alg}_{\mathcal{O}}\left(\underline{\mathrm{Coeff}}^{G}(\mathcal{D})^{I-\times}\right) \longrightarrow \mathrm{Fun}\left(\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes}, \mathcal{D}\right)$$

is fully faithful with image spanned by $\mathrm{Seg}_{\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes}}(\mathcal{D})$.

Proof. After Proposition 1.51, it suffices to characterize the image of $\mathcal{O}$-monoids under the equivalence

$$\mathrm{Fun}(\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes}, \mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{T}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}, \underline{\mathrm{Coeff}}_G(\mathcal{D})).$$

By [Nar17, Ex 1.17], given a finite V-set $S \in \mathbb{F}_V$ and writing $\mathrm{Tot}S \simeq \coprod_{U \in \mathrm{Orb}(S)} \mathcal{T}_{/U}$ for the total ∞-category of the associated V-category, the above identification turns S-indexed products into right Kan extensions:

$$\begin{tikzcd}
\mathrm{Fun}_{\mathcal{T}}(S, \underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{D})) \arrow[r, "\prod^S"] \arrow[d, phantom, "\wr"] & \mathrm{Coeff}^{\mathcal{T}}(\mathcal{D}) \arrow[d, phantom, "\wr"] \\
\mathrm{Fun}(\mathrm{Tot}S, \mathcal{D}) \arrow[r, "\mathrm{RKE}"] & \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{D})
\end{tikzcd}$$

Thus the image of $\mathrm{Mon}_{\mathcal{O}}(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D})$ is those functors $\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathcal{D}$ whose image of an object $((X_U), S) \in \mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes}$ is right Kan extended along elementary maps, which is exactly the relevant Segal condition. ☐

Corollary 1.53. *Given $\mathcal{O}^{\otimes}$ an I-operad and $\mathcal{D}$ an ∞-category with finite products, the forgetful functor*

$$\mathrm{Alg}_{\mathcal{O}}(\underline{\mathrm{Coeff}}^{G}(\mathcal{D})^{I-\times}) \longrightarrow \mathrm{Fun}(\mathrm{Tot}\mathcal{O}^{\otimes}, \mathcal{D})$$

is fully faithful with image spanned by $\mathrm{Seg}_{\mathrm{Tot}\mathcal{O}^{\otimes}}(\mathcal{D})$.

Proof. Apply Corollary 1.52 and the equivalence $\mathrm{Seg}_{\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes}}(\mathcal{C}) \simeq \mathrm{Seg}_{\mathrm{Tot}\mathcal{O}^{\otimes}}(\mathcal{C})$ constructed in [Ste25, § A]. ☐

We finally identify Perspectives (i) to (iii) from the introduction.

Corollary 1.54 ("CMon = CAlg")**.** *There is a canonical equivalence $\underline{\mathrm{CMon}}_I(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_I(\mathcal{C}^{I-\times})$ over $\mathcal{C}$.*

Proof. Our proof is similar to that of [Nar16, Thm 6.5]; there is a pullback square over $\mathcal{C}$

$$\begin{tikzcd}
\mathrm{CMon}_I(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{CAlg}_I(\mathcal{C}^{I-\times}) \arrow[d] & \simeq & \mathrm{Fun}^{I-\times}(\mathrm{Span}(\underline{\mathbb{F}}_{\mathcal{T}}), \mathcal{C}) \arrow[d] \\
\mathrm{Fun}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}})) \arrow[r] & \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{CAlg}}_I(\mathcal{C}^{I-\times})\right) & \simeq & \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Fun}}^{I-\times}(\mathrm{Span}(\underline{\mathbb{F}}_{\mathcal{T}}), \underline{\mathcal{S}}_{\mathcal{T}}))
\end{tikzcd}$$

so it suffices to prove this in the case $\mathcal{C} = \underline{\mathcal{S}}_{\mathcal{T}}$. There, we simply compose equivalences as follows

$$\mathrm{CMon}_I(\underline{\mathcal{S}}_{\mathcal{T}}) \xrightarrow{1.8} \mathrm{CMon}_I(\mathcal{S}) \xrightarrow{1.53} \mathrm{CAlg}_I(\underline{\mathcal{S}}_{\mathcal{T}}^{I-\times})$$

☐

Remark 1.55. As with much of the rest of this subsection, Corollary 1.54 possesses an alternative strategy where both are shown to furnish the I-semiadditive closure, the latter using [CLL24, Thm B]. The above argument was chosen for brevity, as its requisite parts are also needed elsewhere. ◁

Remark 1.56. In the case $\mathcal{C} \simeq \underline{\mathcal{S}}_G$, and I is an indexing category, the analogous result was recently proved in [Mar24] for a Dwyer-Kan localization of algebras over the corresponding *graph G-operads*. To the knowledge of the author, this is one of the first concrete higher-categorical indications that the genuine operadic nerve of [Bon19] may induce equivalences between ∞-categories of algebras. ◁

1.4.3. *$\mathcal{F}$-unitality.* We now study I-operadic unitality, beginning with the following definition.

Definition 1.57. We say that an I-operad $\mathcal{O}^{\otimes}$ is *unital* if $\mathcal{O}(\varnothing_V; C) = *$ for all $C \in \mathcal{O}_V$ with $V \in v(I)$, and *reduced* if also $\mathcal{O}(*_V) = *$ for all $V \in c(I)$. More generally if $\mathcal{F} \subset v(I)$ is a family, we say that $\mathcal{O}^{\otimes}$ is *$\mathcal{F}$-unital* if $\mathcal{O}(\varnothing_V; C) \simeq *$ for all $C \in \mathcal{O}_V$ and $V \in \mathcal{F}$ and *$\mathcal{F}$-reduced* if also $\mathcal{O}(*_V) = *$ for all $V \in \mathcal{F}$; equivalently, $\mathcal{O}^{\otimes}$ is $\mathcal{F}$-unital (resp. $\mathcal{F}$-reduced) if and only if $\mathrm{Bor}^I_{I \cap \mathbb{F}_{\mathcal{F}}}\mathcal{O}^{\otimes}$ is unital (reduced). ◁

An under-appreciated case of unitality is the (equivariantly) *symmetric monoidal case.*

Observation 1.58. If $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal ∞-category with unit $v(I)$-object $1_{\bullet}$ and $X \in \mathcal{C}_V$, then the Segal conditon for multimorphisms constructs an equivalence

$$\mathrm{Map}_{\mathcal{O}^{\otimes}}(\varnothing_V, X) \simeq \mathrm{Map}_{\mathcal{C}_V}(1_V, X);$$

hence $\mathcal{C}^{\otimes}$ is $\mathcal{F}$-unital if and only if $1_{\bullet} \in \Gamma^{\mathcal{F}}\mathcal{C}$ is initial. In particular, if $\mathrm{Bor}^I_{I \cap \mathbb{F}_{\mathcal{F}}}\mathcal{C}^{\otimes}$ is cartesian, then $\mathcal{C}^{\otimes}$ is $\mathcal{F}$-unital if and only if $U\mathcal{C}$ is $\mathcal{F}$-pointed. ◁

In fact, we may reverse this, characterizing unitality in terms of the I-symmetric monoidal envelope.

Observation 1.59. We may identify an object $(T, \mathbf{C}) \in \mathrm{Env}_I(\mathcal{O})_V$ with an I-admissible finite V-set T and an S-color $\mathbf{C}$; the space of maps $(S, \mathbf{B}) \to (T, \mathbf{C})$ lying over a fixed map $\psi \colon T \to S$ is precisely the multimorphism space $\mathrm{Mul}^{\psi}_{\mathcal{O}}(\mathbf{C}; \mathbf{B})$. In particular, letting $\varnothing_V$ denote the unique $\varnothing_V$-color of $\mathcal{O}$ for $V \in \mathcal{F}$, we find that

$$\mathrm{Map}_{\mathrm{Env}_I(\mathcal{O})}(\varnothing_V, (S, \mathbf{C})) \simeq \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(\varnothing_V; C_U).$$

Products of spaces are contractible if and only if their factors are contractible; hence $\mathcal{O}^{\otimes}$ is $\mathcal{F}$-unital if and only if $\mathrm{Env}_I(\mathcal{O})^{\otimes}$ is $\mathcal{F}$-unital. ◁

We can identify this via an algebraic mapping-in property as follows.

Lemma 1.60. *If $\mathcal{C}^\otimes$ is an I-symmetric monoidal ∞-category, then $\mathcal{C}^\otimes$ is $\mathcal{F}$-unital if and only if the forgetful $\mathcal{T}$-functor $U\colon \underline{\mathrm{Alg}}_{\mathbb{E}_{0,\mathcal{F}}}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*

Proof. The forward implication follows from the computation Lemma A.11 in the case $I^0_{\mathcal{F}}$, so assume U is an equivalence. Then, for all $V \in \mathcal{F}$, $\mathcal{C}_V^{1_V/} \simeq \mathrm{Alg}_{\mathbb{E}_{0,\mathcal{F}}}(\mathcal{C})_V \to \mathcal{C}$ is an equivalence, so $1_V \in \mathcal{C}_V$ is initial. Thus Observation 1.58 implies the lemma. $\square$

We can replace $\mathbb{E}^\otimes_{0,\mathcal{F}}$ with an *arbitrary* $\mathcal{F}$-unital I-operad, retaining the above property.

Lemma 1.61 (Incomplete [NS22, Thm 5.2.11])**.** *If $\mathcal{O}^\otimes$ is an $\mathcal{F}$-unital I-operad and $\mathcal{C}^\otimes$ is an I-symmetric monoidal ∞-category, then $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})$ is $\mathcal{F}$-unital.*

Proof. Using the same trick as Lemma 1.60, we prove this when $\mathcal{F} = v(I) = \mathcal{T}$. Then, in light of Observation 1.58, this is simply [NS22, § 5.2.11]. $\square$

1.4.4. *$\mathcal{O}$-comonoids, indexed semiadditivity.* We prove the following fundamental fact as Lemma A.11.

Proposition 1.62. *Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category and $\mathcal{O}^\otimes$ a unital I-operad. The forgetful functor is an equivalence*

$$\underline{\mathrm{Alg}}_{\mathcal{O}}\left(\mathcal{C}^{I-\sqcup}\right) \xrightarrow{\sim} \underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{O}, \mathcal{C}).$$

We use this to construct another recognition result for cocartesian I-symmetric monoidal ∞-categories.

Construction 1.63. Let $\mathcal{C}^\otimes$ be an I-symmetric monoidal ∞-category satisfying the property that $1_\bullet \in \Gamma^{v(I)}\mathcal{C}$ is initial and let $(X_U) \in \mathcal{C}_S$ be an S-tuple for some $S \in \underline{\mathbb{F}}_I$. The $\otimes$-*Wirthmüller map for* (X_U) is the map

$$W_{S,(X_U)}\colon \coprod_U^S X_U \longrightarrow \bigotimes_U^S X_U$$

classified by the summand maps

$$\begin{tikzcd} X_W \arrow[r, "W_{S,(X_U),U'}"] \arrow[d, phantom, "\wr"] & \mathrm{Res}^V_U \bigotimes^S_W X_W. \arrow[d, phantom, "\wr"] \\ X_W \otimes \bigotimes_{U'}^{\mathrm{Res}^V_U S - W} 1_{U'} \arrow[r, "(\mathrm{id};!)"] & X_W \otimes \bigotimes_{U'}^{\mathrm{Res}^V_U S - W} \mathrm{Res}^{o(U')}_{U'} X_{o(U')} \end{tikzcd}$$

where $o(U') \in \mathrm{Orb}(S)$ is the orbit whose restriction contains U'. 1_U exists and is initial by almost-unitality. Dually, if $\mathcal{C}^\otimes$ is an I-symmetric monoidal ∞-category such that $1_\bullet \in \Gamma^{v(I)}\mathcal{C}$ the $\otimes$-*co-Wirthmüller map* $W^{co}_{S,(X_U)}\colon \bigotimes_U^S X_U \to \prod_U^S X_U$ is the Wirtmüller map for $(X_U)_S$ in the fiberwise opposite $\mathcal{C}^{\otimes,\mathrm{vop}}$. $\triangleleft$

Lemma 1.64. *Let $\mathcal{C}^\otimes$ be an I-symmetric monodial ∞-category.*

(1) $\mathcal{C}^\otimes$ is cocartesian if and only if $1_\bullet \in \Gamma^{v(I)}\mathcal{C}$ is initial and $W_{S,(X_U)}$ is an equivalence for all $S \in \underline{\mathbb{F}}_I$ and $(X_U) \in \mathcal{C}_S$;

(2) $\mathcal{C}^\otimes$ is cartesian if and only if $1_\bullet \in \Gamma^{v(I)}\mathcal{C}$ is terminal and $W^{co}_{S,(X_U)}$ is an equivalence for all $S \in \underline{\mathbb{F}}_I$ and $(X_U) \in \mathcal{C}_S$.

Proof. (1) is Observation A.24, and part (2) follows directly from part (1) under Proposition 1.48. $\square$

Observation 1.65. When $\mathcal{C}^\otimes$ is cartesian, the assumption that $1_\bullet$ is initial is precisely the assumption that $\mathcal{C}$ is $v(I)$-pointed; moreover, unwinding definitions, $W_{S,(X_U)}$ matches the Wirthmüller map of Definition 1.4. $\triangleleft$

Finally, we find the indexed semiadditivity of [CLL24; Nar16] within equivariant higher algebra.

Corollary 1.66 (Equivariant [GGN15, Prop 2.3])**.** *Suppose $\mathcal{C}$ is a $\mathcal{T}$-∞-category with I-indexed products. Then, the following conditions are equivalent.*

(a) $\mathcal{C}$ is I-semiadditive.
(b) There exists an I-symmetric monoidal equivalence $\mathcal{C}^{I-\times} \simeq \mathcal{C}^{I-\sqcup}$ lying over the identity.
(c) The forgetful $\mathcal{T}$-functor $\underline{\mathrm{CMon}}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence.

Proof. We have proved this redundantly. (a) $\iff$ (b) is Lemma 1.64 and Observation 1.65. (b) $\implies$ (c) is Corollary 1.54 and Proposition 1.62. (c) $\iff$ (a) is [CLL24, Cor 7.8]. $\square$

1.4.5. *Pointwise indexed tensor products of $\mathcal{O}$-monoids.* Last, we characterize $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}^{I-\times})$ categorically.

Lemma 1.67. *Fix $F\colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ an I-symmetric monoidal functor.*

(1) If F is I-coproduct preserving then there is a homotopy $FW_{S,(X_U)} \sim W_{S,F(X_U)}$.
(2) If F is I-product preserving then there is a homotopy $FW^{co}_{S,(X_U)} \sim W^{co}_{S,F(X_U)}$.
(3) If F is I-coproduct-preserving and conservative and $\mathcal{C}^{\otimes}$ is cocartesian then $\mathcal{D}^{\otimes}$ is cocartesian.
(4) If F is I-product preserving and conservative and $\mathcal{C}^{\otimes}$ is cartesian then $\mathcal{D}^{\otimes}$ is cartesian.
(5) If F is a fiberwise-monadic right $\mathcal{T}$-adjoint and $\mathcal{C}^{\otimes}$ is cartesian then $\mathcal{D}^{\otimes}$ is cartesian.

Proof. For (1), since F is I-coproduct preserving, we're tasked with constructing a homotopy making the following diagram commute

$$\begin{tikzcd} F\left(X_U \otimes \bigotimes_W^{\mathrm{Res}_U^V S-U} 1_W\right) \arrow[r, "(\mathrm{id};!)"] \arrow[d, phantom, "\wr"] & F\left(X_U \otimes \bigotimes_W^{\mathrm{Res}_U^V S-U} X_W\right) \arrow[d, phantom, "\wr"] \\ FX_U \otimes \bigotimes_W^{\mathrm{Res}_U^V S-U} 1_U \arrow[r, "(\mathrm{id};!)"] & FX_U \otimes \bigotimes_W^{\mathrm{Res}_U^V S-U} FX_W \end{tikzcd}$$

In fact, there is a contractible space of such choices. (2) follows by applying fiberwise-opposites.

For (3), applying (1) and Lemma 1.64 shows that $FW_{S,(X_U)}$ is an equivalence for all $S \in \underline{\mathbb{F}}_I$ and $(X_U) \in \mathcal{D}_S$, so conservativity implies that $W_{S,(X_U)}$ is an equivalence and Lemma 1.64 concludes that $\mathcal{D}^{\otimes}$ is cocartesian. (4) follows similarly from (2). (5) is directly implied by (4), since fiberwise monadicity implies conservativity and right $\mathcal{T}$-adjoints are I-product preserving. □

Applying (5) of Lemma 1.67 to $U\colon \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}\left(\mathcal{C}^{I-\times}\right) \to \mathcal{C}^{I-\times}$ immediately yields the following.

Corollary 1.68. *If $\mathcal{C}$ has I-indexed products, then $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}\left(\mathcal{C}^{I-\times}\right)$ is a cartesian I-symmetric monoidal ∞-category.*

2. I-commutative algebras

Philosophical remark 2.1. On one hand, it follows from Proposition 1.24 that I-operads are determined conservatively by their theories of *algebras in I-symmetric monoidal categories*; indeed, it suffices to characterize their algebras in the universal case $\underline{\mathcal{S}}_{\mathcal{T}}^{I-\times}$.

On the other hand, the right adjoint $\mathrm{Cat}_I^{\otimes} \to \mathrm{Op}_I$ is full on cores, since automorphisms in the slice category $\mathrm{Cat}_{/\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}$ automatically preserve cocartesian morphisms. Hence the associated map of spaces

$$\begin{tikzcd} \mathrm{Cat}_I^{\otimes,\simeq} \arrow[r] & \mathrm{Op}_I^{\simeq} \arrow[r] & \mathrm{Fun}(\mathrm{Op}_I, \mathrm{Cat})^{\simeq} \\ \mathcal{C}^{\otimes} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[rr, mapsto] & & \mathrm{Alg}_{(-)}(\mathcal{C}) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

is a summand inclusion. That is, an I-symmetric monoidal ∞-category is determined (functorially on equivalences) by its categories of $\mathcal{O}$-algebras for each I-operad $\mathcal{O}$. ◁

Following along these lines and using Proposition 1.51, we will generally characterize algebraic theories in *arbitrary* settings by reducing to the universal case of $\underline{\mathcal{S}}_{\mathcal{T}}^{\mathcal{T}-\times}$, which we study using category theoretic means. Indeed, in Sections 2.1 and 2.2 we use this to bootstrap I-semiadditivity of $\underline{\mathrm{CMon}}_I(\mathcal{C})$ to I-cocartesianness of $\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{C})$ for $\mathcal{C}^{\otimes}$ an arbitrary I-operad. Using work from Section A, we use this to conclude generalizations of Theorem C and Corollary D, answering Questions (I) and (II).

We take this to its logical extreme in Section 2.2, using this to completely characterize the smashing localizations associated with $\otimes$-idempotent weak $\mathcal{N}_\infty$-operads. As promised in the introduction, we use this classification to prove a generalization of Theorem F, answering Question (III). Following this, in Section 2.3 we show that our results are sharp; if I is not almost essentially unital, then $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ fails to be connected, so $\mathcal{N}_{I\infty}^{\otimes}$ is (abstractly) idempotent under $\overset{\mathrm{BV}}{\otimes}$ if and only if I is almost essentially unital.

2.1. **Indexed tensor products of I-commutative algebras.** Fix I an almost-unital weak indexing system. In Proposition 1.62, we showed that every object in a cocartesian I-symmetric monoidal structure bears a canonical I-commutative algebra algebra structure, i.e. $\underline{\mathrm{CAlg}}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence. In this subsection, we demonstrate the converse, i.e. we show the following.

Theorem 2.2 (Indexed tensor products of I-commutative algebras)**.** *The following are equivalent for $\mathcal{C}^\otimes \in \mathrm{Cat}_I^\otimes$.*

(a) $\mathcal{C}^\otimes$ is cocartesian.
(b) For all unital I-operads $\mathcal{O}^\otimes$, the forgetful functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{T}}(U\mathcal{O}, \mathcal{C})$ is an equivalence.
(c) The forgetful $\mathcal{T}$-functor $\underline{\mathrm{CAlg}}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence.
(d) There exists an I-symmetric monoidal ∞-category $\mathcal{D}^\otimes$ and an equivalence $\underline{\mathrm{CAlg}}_I^\otimes(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}^\otimes$.

The implications (a) $\implies$ (b), (c) are simply Proposition 1.62. For the implication (b) $\implies$ (a), note that Lemma 1.61 states that $\mathcal{C}^\otimes$ is unital; hence Yoneda's lemma applied to $\mathrm{Op}_I^{\mathrm{uni}}$ constructs an I-operad equivalence $\mathcal{C}^\otimes \simeq \mathcal{C}^{I-\sqcup}$, which is an I-symmetric monoidal equivalence by Philosophical remark 2.1. The implication (c) $\implies$ (d) follows by neglect of assumptions. To summarize, we've arrived at the implications

$$\begin{tikzcd} & & \text{(d)} \\ \text{(b)} \arrow[r, Leftrightarrow] & \text{(a)} \arrow[r, Rightarrow] & \text{(c)} \arrow[ul, Rightarrow] \end{tikzcd} \tag{10}$$

Our workhorse lemma for closing the gap is the following.

Lemma 2.3. *The following are equivalent for $\mathcal{P}^\otimes \in \mathrm{Op}_{\mathcal{T}}$:*

(e) The $\mathcal{T}$-∞-category $\underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive.
(f) For all $\mathcal{O}^\otimes \in \mathrm{Op}_I^{\mathrm{uni}}$, the forgetful functor

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}_{\mathcal{P}}^\otimes(\underline{\mathcal{S}}_{\mathcal{T}}) \longrightarrow \mathrm{Alg}_{\mathrm{triv}(\mathrm{U}\mathcal{O})}\underline{\mathrm{Alg}}_{\mathcal{P}}^\otimes(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Fun}_{\mathcal{T}}(U\mathcal{O}, \underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}))$$

is an equivalence.
(g) For all $\mathcal{O}^\otimes \in \mathrm{Op}_I^{\mathrm{uni}}$, the map $\mathrm{triv}(\mathcal{O})^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes \to \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes$ is an equivalence.
(h) For all $\mathcal{O}^\otimes \in \mathrm{Op}_I^{\mathrm{uni}}$ and $\mathcal{Q}^\otimes \in \mathrm{Op}_I$, the forgetful $\mathcal{T}$-operad map

$$\underline{\mathrm{Alg}}_{\mathcal{O}\otimes\mathcal{P}}^\otimes(\mathcal{Q}) \simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}_{\mathcal{P}}^\otimes(\mathcal{Q}) \longrightarrow \underline{\mathrm{Alg}}_{\mathrm{triv}(U\mathcal{O})}^\otimes \underline{\mathrm{Alg}}_{\mathcal{P}}^\otimes(\mathcal{Q})$$

is an equivalence

Proof. Since Corollary 1.68 shows that $\mathrm{Bor}_I^{\mathcal{T}} \underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\underline{\mathcal{S}}_{\mathcal{T}})$ is cartesian, Corollary 1.66 identifies the bi-implication (e) $\iff$ (f) with (a) $\iff$ (b) applied to $\mathrm{Bor}_I^{\mathcal{T}} \underline{\mathrm{Alg}}_{\mathcal{P}}^\otimes(\underline{\mathcal{S}}_{\mathcal{T}})$. (f) $\implies$ (g) follows from Proposition 1.24, and the implications (g) $\implies$ (h) $\implies$ (f) are obvious. □

Proof of Theorem 2.2. After the implications illustrated in Eq. (10), it suffices to show for all $\mathcal{D}^\otimes \in \mathrm{Cat}_I^\otimes$ that $\underline{\mathrm{CAlg}}_I(\mathcal{D})$ satisfies (b), i.e. (d) $\implies$ (b); by Lemma 2.3, it suffices to prove that $\underline{\mathrm{CAlg}}_I(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive. But in fact, Corollary 1.54 constructs an equivalence $\underline{\mathrm{CAlg}}_I(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}})$ and the latter is I-semiadditive by Cnossen-Lenz-Linsken's result, Theorem 1.9. □

Rephrasing things somewhat, we've arrived at the following theorem.

Theorem C'. *Let $\mathcal{O}^\otimes$ be an almost essentially reduced $\mathcal{T}$-operad. Then, the following properties are equivalenent.*

(a) The $\mathcal{T}$-∞-category $\underline{\mathrm{Alg}}_{\mathcal{O}}\underline{\mathcal{S}}_{\mathcal{T}}$ is $A\mathcal{O}$-semiadditive.
(b) The unique map $\mathcal{O}^\otimes \to \mathcal{N}_{A\mathcal{O}\infty}^\otimes$ is an equivalence.

Furthermore, for any almost essentially unital weak indexing system I and I-symmetric monoidal ∞-category $\mathcal{C}^\otimes$, the I-symmetric monoidal ∞-category $\underline{\mathrm{CAlg}}_I^\otimes\mathcal{C}$ is cocartesian.

Proof. By Lemma 1.64, Corollary 1.68, and Theorem 2.2, Condition (a) is equivalent to the forgetful $\mathcal{T}$-functor

$$\underline{\mathrm{CAlg}}_{A\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \underline{\mathrm{Alg}}_{\mathcal{O}}\underline{\mathrm{CAlg}}_{A\mathcal{O}}^\otimes(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \underline{\mathrm{CAlg}}_{A\mathcal{O}}\underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\underline{\mathcal{S}}_{\mathcal{T}}) \longrightarrow \underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$$

being an equivalence, which is equivalent to Condition (b) by Proposition 1.24. The remaining statement follows immediately from the implication (d) $\implies$ (a) of Theorem 2.2. □

This implies that $\mathcal{N}_{I\infty}^{\otimes} \in \mathrm{Op}_I^{\mathrm{red}}$ is $\overset{\mathrm{BV}}{\otimes}$-absorptive.

Corollary 2.4. *Let $\mathcal{O}^{\otimes}$ be an almost-reduced I-operad. The map $F\colon \mathcal{N}_{I\infty}^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes}$ is an equivalence.*

Proof. By Theorem 2.2, the forgetful map

$$F^*\colon \mathrm{Alg}_{\mathcal{O}\otimes\mathcal{N}_{I\infty}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{N}_{I\infty}}(\underline{\mathcal{S}}_{\mathcal{T}}) \to \underline{\mathrm{Alg}}_{\mathcal{N}_{I\infty}}(\underline{\mathcal{S}}_{\mathcal{T}})$$

is an equivalence. The statement then follows from Proposition 1.24. □

Remark 2.5. At this point, we may answer Question (I) of the introduction; if $\mathcal{O}^{\otimes}$ is almost-essentially unital and $\mathcal{N}_{I\infty}^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes}$ is an equivalence, then Proposition 1.44 implies that $\mathcal{O}^{\otimes}$ is an almost-reduced I-operad; in particular, the assumptions of Corollary D are necessary and sufficient. ◁

2.2. **The smashing localization for $\mathcal{N}_{I\infty}^{\otimes}$ and Blumberg-Hill's conjecture.** In view of Corollary 2.4, when I is an almost-unital weak indexing category the unique map $\mathrm{triv}_{\mathcal{T}}^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$ induces an equivalence

$$\mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{\mathcal{T}}^{\otimes} \xrightarrow{\sim} \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes},$$

i.e. it uniquely witnesses $\mathcal{N}_{I\infty}^{\otimes}$ as an idempotent object in the sense of [HA, Def 4.8.2.1]. To conclude Theorem F, we will characterize the smashing localization classified by $\mathcal{N}_{I\infty}^{\otimes}$-modules.[4]

2.2.1. *The smashing localization classified by $\mathcal{N}_{I\infty}^{\otimes}$.* $(-)\overset{\mathrm{BV}}{\otimes}\mathcal{N}_{I\infty}^{\otimes}$ classifies algebraic Wirthmüller isomorphisms.

Theorem 2.6. *Let I be an almost essentially unital weak indexing system. Then, a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ possesses an equivalence $\mathcal{P}^{\otimes}\overset{\mathrm{BV}}{\otimes}\mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{P}^{\otimes}$ if and only if the following conditions are satisfied:*

- *(a) $c(\mathcal{P}) = c(I)$, and*
- *(b) $\underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive.*

By the arity support computation of Proposition 1.44, Theorem 2.6 is equivalent to the following.

Proposition 2.7. *Let I be an almost-unital weak indexing system. Then, an at-least one color $\mathcal{T}$-operad $\mathcal{P}^{\otimes}$ satisfies $\mathcal{P}^{\otimes}\overset{\mathrm{BV}}{\otimes}\mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{P}^{\otimes}$ if and only if $\underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive.*

Proof. Just as in the proof of Lemma 2.3, note that Theorem 2.2 implies that the conditions of Lemma 2.3 are equivalent to the additional condition

- (j) The forgetful $\mathcal{T}$-functor $\underline{\mathrm{Alg}}_{\mathcal{N}_{I\infty}\otimes\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \underline{\mathrm{CAlg}}_I\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \to \underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is an equivalence.

But this is equivalent to the desired equivalence by Proposition 1.24 □

2.2.2. *The proof of Blumberg-Hill's conjecture.* We start by answering Question (III).

Proposition 2.8. *When I and J are almost-unital, there is an equivalence $\mathcal{N}_{I\infty}^{\otimes}\overset{\mathrm{BV}}{\otimes}\mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{I\vee J\infty}^{\otimes}$.*

Proof. By [CSY20, Prop 5.1.8], $\mathcal{N}_{I\infty}^{\otimes}\overset{\mathrm{BV}}{\otimes}\mathcal{N}_{J\infty}^{\otimes} \in \mathrm{Op}_{\mathcal{T}} \simeq \mathrm{Op}_{\mathcal{T},\mathrm{triv}_{\mathcal{T}}^{\otimes}/}$ is an idempotent object classifying the conjunction of the properties which are classified by $\mathcal{N}_{I\infty}^{\otimes}$ and $\mathcal{N}_{J\infty}^{\otimes}$; that is, $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ is fixed by $(-)\overset{\mathrm{BV}}{\otimes}\mathcal{N}_{I\infty}^{\otimes}\overset{\mathrm{BV}}{\otimes}\mathcal{N}_{J\infty}^{\otimes}$ if and only if $\underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive and J-semiadditive. Proposition 1.6, identifies this with the property that $\underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is $I \vee J$-semiadditive , i.e. $\mathcal{O}^{\otimes}$ is fixed by $(-)\overset{\mathrm{BV}}{\otimes}\mathcal{N}_{I\vee J}^{\otimes}$. Thus, we have

$$\mathcal{N}_{I\vee J\infty}^{\otimes} \simeq \mathcal{N}_{I\vee J\infty}^{\otimes}\overset{\mathrm{BV}}{\otimes}\mathcal{N}_{I\infty}^{\otimes}\overset{\mathrm{BV}}{\otimes}\mathcal{N}_{J\infty}^{\overset{\mathrm{BV}}{\otimes}} \simeq \mathcal{N}_{I\infty}^{\otimes}\overset{\mathrm{BV}}{\otimes}\mathcal{N}_{J\infty}.$$ □

We may now conclude the full theorem, which we restate in the atomic orbital case.

[4] The identification between idempotent objects and smashing localizations is stated in [CSY20; HA] under the unnecessary specification that the constructions live in a given *symmetric monoidal* ∞-category, as this leads to canonical lifts of idempotent objects to idempotent algebras. In fact, their arguments for the identification only make use of the underlying $\mathbb{A}_2$-structure and the *existence* of a braiding $A \otimes B \simeq B \otimes A$, separately for each pair. Even before Section 3.1, $(\mathrm{Op}_{\mathcal{T}}, \overset{\mathrm{BV}}{\otimes}, \mathrm{triv}_{\mathcal{T}}^{\otimes})$ with the braiding determined by symmetry of the universal property for $\overset{\mathrm{BV}}{\otimes}$ is certainly such a structure.

Theorem F'. $\mathcal{N}^{\otimes}_{(-)\infty}\colon \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T}}$ *restricts to a fully faithful symmetric monoidal $\mathcal{T}$-right adjoint*

$$\begin{tikzcd} \underline{\mathrm{wIndex}}_{\mathcal{T}}^{aE\mathrm{uni},\otimes} \arrow[rr, bend right, "\mathcal{N}^{\otimes}_{(-)\infty}"'] \arrow[rr, phantom, "\perp"] & & \underline{\mathrm{Op}}_{\mathcal{T}}^{aE\mathrm{uni},\otimes}. \arrow[ll, bend right, "A"'] \end{tikzcd}$$

Furthermore, the resulting tensor product of weak $\mathcal{N}_\infty$-operads is computed by the Borelified join

$$\mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{J\infty} \simeq \mathcal{N}^{\otimes}_{\mathrm{Bor}^{\mathcal{T}}_{cI\cap cJ}(I\vee J)\infty}. \tag{11}$$

Hence whenever I, J are almost-unital weak indexing categories and $\mathcal{C}^{\otimes}$ is an $I \vee J$-symmetric monoidal ∞-category, there is a canonical equivalence of $I \vee J$-symmetric monoidal ∞-categories

$$\underline{\mathrm{CAlg}}^{\otimes}_{I}\underline{\mathrm{CAlg}}^{\otimes}_{J}(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_{I\vee J}(\mathcal{C}). \tag{12}$$

Proof of Theorem F'. The $\mathcal{T}$-adjunction is precisely Proposition 1.36, and Eqs. (11) and (12) will follow from symmetric monoidality of $\mathcal{N}^{\otimes}_{(-)\infty}$ and the support computation of Proposition 1.44.

We're left with proving that almost essentially unital weak $\mathcal{N}_\infty$-operads are closed under $\overset{\mathrm{BV}}{\otimes}$, i.e. the unique map $\varphi\colon \mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{J\infty} \to \mathcal{N}^{\otimes}_{I\vee J\infty}$ is an equivalence. By Observation 1.43, it suffices to prove that $\mathrm{Bor}^{\mathcal{T}}_{cI\cap cJ}(\varphi)$ is an equivalence, i.e. we may assume that I and J are almost-unital; this is Proposition 2.8. $\square$

2.2.3. $\mathcal{N}^{\otimes}_{I\infty}$ *classifies I-cocartesianness.* We now study a variant of Theorem 2.6, motivated by the following.

Observation 2.9. The computation of [HA, § 2.3.1] and resulting theory may be stated simply: the operad $\mathbb{E}_0^{\otimes}$ is an idempotent object in $\mathrm{Op}^{\otimes}$ under the unique map $\mathrm{triv}^{\otimes} \to \mathbb{E}_0^{\otimes}$, and the corresponding smashing localization classifies unitality. In particular, a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is a $\mathbb{E}_0^{\otimes}$-module with respect to $\overset{\mathrm{BV}}{\otimes}$ if and only if the unit object $1_{\mathcal{C}} \in \mathcal{C}^{\otimes}$ is initial, i.e. for all $X \in \mathcal{C}$, the unique map $X^{\sqcup\varnothing} \to X^{\otimes\varnothing}$ is an equivalence; that is, $(-)\overset{\mathrm{BV}}{\otimes}\mathbb{E}_0^{\otimes}$ classifies I_0-*cocartesianness* in the symmetric monoidal case. ◁

We say that a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ is I-*cocartesian* if the identity on $U\mathcal{O}$ is adjunct to an equivalence $\mathrm{Bor}^{\mathcal{T}}_{I}\mathcal{O}^{\otimes} \xrightarrow{\sim} U\mathcal{O}^{I-\sqcup}$. We begin with an I-operadic variant of Theorem 2.2 and Proposition 2.7.

Proposition 2.10. *Given I an almost-unital weak indexing category, the following are equivalent for $\mathcal{P}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$.*

- *(a) $\mathcal{P}^{\otimes}$ is I-cocartesian.*
- *(b) For all unital I-operads $\mathcal{O}^{\otimes}$, the forgetful functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(I\mathcal{O}, \mathcal{P})$ is an equivalence.*
- *(c) The forgetful $\mathcal{T}$-operad map $\underline{\mathrm{CAlg}}^{\otimes}_{I}(\mathcal{P}) \to \mathcal{P}^{\otimes}$ is an equivalence.*
- *(d) There exists a $\mathcal{T}$-operad $\mathcal{Q}^{\otimes}$ and an equivalence of $\mathcal{T}$-operads $\underline{\mathrm{CAlg}}^{\otimes}_{I}(\mathcal{Q}) \xrightarrow{\sim} \mathcal{P}^{\otimes}$.*
- *(k) The canonical map $\mathcal{P}^{\otimes} \to \mathcal{P}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty}$ is an equivalence.*

Proof. The proof of the equivalence between Conditions (a) to (d) is identical to Theorem 2.2, so we omit it. Now, by a standard two-out-of-three argument, $\mathcal{P}^{\otimes}$ is local for the smashing localization associated with $\mathcal{N}^{\otimes}_{I\infty}$ if and only if pullback along the localization map $\mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty}$ induces an equivalence $\mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{CAlg}}^{\otimes}_{I}(\mathcal{P})^{\simeq} \simeq \mathrm{Alg}_{\mathcal{O}\otimes\mathcal{N}_{I\infty}}(\mathcal{P})^{\simeq} \xrightarrow{\sim} \mathrm{Alg}_{\mathcal{O}}(\mathcal{P})^{\simeq}$ for all $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$, which is equivalent to the condition that $\underline{\mathrm{CAlg}}^{\otimes}_{I}(\mathcal{P}) \to \mathcal{P}^{\otimes}$ is an equivalence by Yoneda's lemma. $\square$

Applying Proposition 1.44 to Proposition 2.10 yields a variant of Theorem 2.6, answering Question (II).

Proposition 2.11. *Let I be an almost essentially unital weak indexing category and $\mathcal{O}^{\otimes}$ a $\mathcal{T}$-operad. Then, $\mathcal{O}^{\otimes}$ admits an (essentially unique) $\mathcal{N}^{\otimes}_{I\infty}$-module structure if and only if the following conditions hold:*

- *(a) $c(\mathcal{O}) = c(I)$, and*
- *(b) $\mathcal{O}^{\otimes}$ is I-cocartesian.*

Remark 2.12. If $J \subset I$ and $\mathcal{O}^{\otimes}$ is I-cocartesian, then $\mathcal{O}^{\otimes}$ is J-cocartesian. In particular, applying this to $I \cap \mathbb{E}^{\otimes}_{0,v(I)} \subset I$ shows when I is unital that the conditions of Proposition 2.11 implies that $\mathcal{O}^{\otimes}$ is unital. ◁

Given a related pair of weak indexing categories $I \subset J$, let $\mathrm{Op}_J^{I-\mathrm{cocart}} \subset \mathrm{Op}_J$ be the full subcategory of I-cocartesian J-operads. We find that $\mathrm{Op}_J^{I-\mathrm{cocart}}$ is absorptive under $\overset{\mathrm{BV}}{\otimes}$ and the internal hom.

Corollary 2.13. *Suppose $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are at-least-one-color J-operads such that either $\mathcal{O}^{\otimes}$ or $\mathcal{P}^{\otimes}$ are I-cocartesian. Then, $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ and $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P})$ are I-cocartesian.*

Proof. I-cocartesianness of $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ follows from Proposition 2.11. If $\mathcal{P}^{\otimes}$ is I-cocartesian, then Proposition 2.10 constructs equivalences

$$\mathrm{CAlg}_I \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) \simeq \mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{CAlg}}^{\otimes}_I(\mathcal{P}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{P}),$$

so the result follows from another application of Proposition 2.10. If $\mathcal{O}^{\otimes}$ is I-cocartesian, the result follows from two more applications of Proposition 2.10. as we acquire equivalences

$$\underline{\mathrm{CAlg}}_I \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) \simeq \underline{\mathrm{Alg}}_{\mathcal{N}_{I\infty} \otimes \mathcal{O}}(\mathcal{P}) \simeq \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P}). \tag{13}$$ □

Finally, we vastly generalize the results of [HA, § 2.3.1].

Corollary 2.14. *Suppose $I \subset J$ is a related pair of almost-unital weak indexing categories. Then, $\mathrm{Op}_J^{I-\mathrm{cocart}} \subset \mathrm{Op}_J$ is a smashing localization and a cosmashing colocalization:*

$$\begin{tikzcd} \mathrm{Op}_J^{I-\mathrm{cocart}} \arrow[r, hook] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathrm{Op}_J \arrow[l, bend right=40, "\mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} (-)"'] \arrow[l, bend left=40, "\underline{\mathrm{CAlg}}^{\otimes}_I(-)"] \end{tikzcd}$$

Proof. Proposition 2.11 exhibits the top adjunction and Theorem 2.2 shows that $\underline{\mathrm{CAlg}}^{\otimes}_I : \mathrm{Op}_J \to \mathrm{Op}_J$ factors through $\mathrm{Op}_J^{I-\mathrm{cocart}} \subset \mathrm{Op}_J$. Moreover, applying $\Gamma^{\mathcal{T}}(-)^{\simeq}$ to Eq. (13) yields a natural equivalence

$$\mathrm{Map}_{\mathrm{Op}_J}\left(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}\right) \simeq \mathrm{Map}_{\mathrm{Op}_J^{I-\mathrm{cocart}}}\left(\mathcal{O}^{\otimes}, \underline{\mathrm{CAlg}}^{\otimes}_I(\mathcal{P})\right)$$

for all J-operads $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ such that $\mathcal{O}^{\otimes}$ is I-cocartesian, yielding the bottom adjunction. □

2.2.4. *(Co)localization to unital I-operads.* The specialization of Theorem 2.2 and Proposition 2.11 to $I \cap I_0$ is quite useful, so we state it explicitly here.

Corollary 2.15. *Given I an almost-unital weak indexing system and $\mathcal{O}^{\otimes} \in \mathrm{Op}_I$, the following are equivalent:*

(a) For all unital I_0-operads $\mathcal{P}^{\otimes}$, the forgetful I-operad map $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{O}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}(U\mathcal{P})}(\mathcal{O}^{\otimes})$ is an equivalence.
(b) The forgetful I-operad map $\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{0,v(I)}}(\mathcal{O}) \to \mathcal{O}^{\otimes}$ is an equivalence.
(c) $\mathcal{O}^{\otimes}$ is unital.
(d) There exists an equivalence $\mathcal{O}^{\otimes} \simeq \mathbb{E}^{\otimes}_{0,v(I)} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes}$.
(e) For all I-operads $\mathcal{C}^{\otimes}$, the forgetful I-operad map $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}} \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{0,v(I)}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ is an equivalence.
(f) For all I-operads $\mathcal{C}^{\otimes}$, the I-operad $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ is unital.
(g) The $\mathcal{T}$-∞-category $\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{S})$ is $v(I)$-pointed.

In particular, $\mathrm{Op}_I^{\mathrm{uni}} \subset \mathrm{Op}_I$ is a smashing localization and cosmashing colocalization:

$$\begin{tikzcd} \mathrm{Op}_I^{\mathrm{uni}} \arrow[r, hook] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathrm{Op}_I \arrow[l, bend right=40, "\mathbb{E}^{\otimes}_{0,v(I)} \overset{\mathrm{BV}}{\otimes} (-)"'] \arrow[l, bend left=40, "\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{0,v(I)}}(-)"] \end{tikzcd}$$

The double adjunction is Corollary 2.14 and the following diagram shows how to recover the corollary.

$$\begin{tikzcd} (c) \arrow[r, Leftrightarrow, "2.11"] \arrow[d, Rightarrow, "2.10"'] & (d) \arrow[r, Leftrightarrow, "\text{Yoneda}"] & (e) \arrow[r, Rightarrow, "2.10"] & (f) \arrow[d, Rightarrow, "1.58"] \\ (a) \arrow[r, Rightarrow, "\text{obvious}"'] & (b) \arrow[ul, Rightarrow, "1.60"] & & (g) \arrow[ull, Rightarrow, "2.3"] \end{tikzcd}$$

2.2.5. *The underlying $\mathcal{T}$-∞-category.* We get an immediate corollary from Theorem A' and Theorem 2.2.

Corollary 2.16. *Suppose I is almost-unital. Then, $U_{\mathrm{uni}}\colon \underline{\mathrm{Op}}_I^{\mathrm{uni}} \to \underline{\mathrm{Cat}}_{\mathcal{T}}$ is left $\mathcal{T}$-adjoint to $(-)^{I-\sqcup}$.*

Warning 2.17. Corollary 2.16 shows that no nontrivial $\mathcal{T}$-colimit of one-color $\mathcal{T}$-operads has one color; in particular, no one-color $\mathcal{T}$-operads are the result of a nontrivial induction. ◁

We use this to compute the $\mathcal{T}$-∞-category underlying Boardman-Vogt tensor products.

Proposition 2.18. *The underlying category functor $U|_{\mathrm{uni}}\colon \mathrm{Op}_I^{\mathrm{uni}} \to \mathrm{Cat}_{\mathcal{T}}$ sends*

$$U\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right) \simeq U(\mathcal{O}^{\otimes}) \times U(\mathcal{P}^{\otimes}).$$

Proof. Corollaries 2.13 and 2.16 yield a string of natural equivalences

$$\begin{aligned}
\mathrm{Fun}_{\mathcal{T}}\left(U\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right), \mathcal{C}\right) &\simeq \mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}\left(\mathcal{C}^{I-\sqcup}\right)\\
&\simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}\left(\mathcal{C}^{I-\sqcup}\right)\\
&\simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Fun}}_{\mathcal{T}}\left(U(\mathcal{P}^{\otimes}), \mathcal{C}\right)^{I-\sqcup}\\
&\simeq \mathrm{Fun}_{\mathcal{T}}\left(U(\mathcal{O}^{\otimes}), \underline{\mathrm{Fun}}_{\mathcal{T}}\left(U(\mathcal{P}^{\otimes}), \mathcal{C}\right)\right)\\
&\simeq \mathrm{Fun}_{\mathcal{T}}\left((U(\mathcal{O}^{\otimes}) \times U(\mathcal{P}^{\otimes}), \mathcal{C}\right),
\end{aligned}$$

so the result follows by Yoneda's lemma. □

Applying Observation 1.43 and Propositions 1.44 and 2.18, we acquire the following.

Corollary 2.19. *The full subcategories $\mathrm{Op}_{\mathcal{T}}^{\mathrm{red}} \subset \mathrm{Op}_{\mathcal{T}}^{a\,\mathrm{red}} \subset \mathrm{Op}_{\mathcal{T}}^{aE\,\mathrm{red}} \subset \mathrm{Op}_{\mathcal{T}}$ are closed under $\overset{\mathrm{BV}}{\otimes}$.*

Finally, this gives us a formula for the I-Borelification of tensor products with $\mathcal{N}_{I\infty}^{\otimes}$.

Corollary 2.20. *Given a $\mathcal{T}$-operad $\mathcal{C}^{\otimes}$ and an almost essentially unital weak indexing system I, there exists an equivalence $\mathrm{Bor}_I^{\mathcal{T}}\left(\mathcal{C}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}\right) \simeq U\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{0,v(I)}^{\otimes}\right)^{I-\sqcup}$. In particular, if $\mathrm{Bor}_I^{\mathcal{T}}\mathcal{O}^{\otimes}$ is a unital I-operad, then*

$$\mathrm{Bor}_I^{\mathcal{T}}\left(\mathcal{C}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}\right) \simeq U\mathcal{C}^{I-\sqcup}.$$

Proof. In sight of Observation 1.43 we may assume I is almost-unital. Proposition 2.18 shows that these have the same underlying $\mathcal{T}$-category and Corollary 2.14 shows that the left-hand size is cocartesian, so Theorem A' yields the corollary. □

We see in the next subsection that this behaves poorly outside of the almost essentially unital setting.

2.3. **Failure of the nonunital equivariant Eckmann-Hilton argument.** We say that a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ with at most one color is *n-connected* if the nonempty structure spaces $\mathcal{O}(S)$ are each n-connected. We write the full subcategory of n-connected $\mathcal{T}$-operads as

$$\mathrm{Op}_{\mathcal{T},\geq n+1}^{\leq \mathrm{oc}} \subset \mathrm{Op}_{\mathcal{T}}^{\leq \mathrm{oc}}.$$

By Proposition 1.24, this is equivalent to the condition that the forgetful functor $\mathrm{CAlg}_{A\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is an equivalence for all $\mathcal{T}$-symmetric monoidal $(n+1)$-categories , which itself is equivalent to the same condition in the case $\mathcal{C} \simeq \mathcal{S}_{\leq n}$. We first observe compatibility with $\overset{\mathrm{BV}}{\otimes}$ in the almost-essentially reduced setting.

Corollary 2.21. $\mathrm{Op}_{\mathcal{T},\geq n+1}^{aE\,\mathrm{red}}$ *is closed under $\overset{\mathrm{BV}}{\otimes}$.*

Proof. By Proposition 1.44 and Theorem F', $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes} \in \mathrm{Op}_{\mathcal{T},\geq n+1}^{aE\,\mathrm{red}}$ participate in a string of natural equivalences

$$\begin{aligned}
\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\leq n}) &\simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\leq n})\\
&\simeq \mathrm{CAlg}_{A\mathcal{O}}\underline{\mathrm{CAlg}}_{A\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\leq n})\\
&\simeq \mathrm{CAlg}_{A\mathcal{O}\vee A\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\leq n})\\
&\simeq \mathrm{CAlg}_{A(\mathcal{O}\otimes\mathcal{P})}(\underline{\mathcal{S}}_{\mathcal{T},\leq n}),
\end{aligned}$$

induced by the unique map $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \to \mathcal{N}_{A(\mathcal{O}\otimes\mathcal{P})}^{\otimes}$, so Proposition 1.24, implies that $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ is n-connected. Corollary 2.19 implies that $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ is almost essentially reduced. □

Remark 2.22. The unit object $\mathrm{triv}_{\mathcal{T}}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$ is n-connected for all n, so n-connected $\mathcal{T}$-operads are closed under k-fold tensor products for all $k \in \mathbb{N}$. ◁

The example $\mathrm{triv}_{\mathcal{T}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes}$ demonstrates that this is the best we can say without further assumptions on the $\mathcal{T}$-operads in question; the author hopes to return to this question in forthcoming work, constructing analogues to [SY19]. For the time being, we demonstrate that Corollary 2.21 dramatically fails without the *almost essentially unital* assumption, exhibiting a failure of the nonunital Eckmann-Hilton argument.

Observation 2.23. Fix I a weak indexing system. By Proposition 1.44, there is a contractible space of diagrams of the following form:

$$\mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{c(I)}^{\otimes} \xrightarrow{\mathrm{id} \otimes !} \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes} \rightarrow \mathcal{N}_{I\infty}^{\otimes};$$

furthermore, the composite $\mathcal{N}_{I\infty}^{\otimes} \rightarrow \mathcal{N}_{I\infty}^{\otimes}$ is homotopic to the identity since $\mathcal{N}_{I\infty}^{\otimes}$ has contractible endomorphism space. In particular, this implies that there is a unique natural *split diagonal* diagram

$$\begin{tikzcd} & \mathrm{CAlg}_I \underline{\mathrm{CAlg}}_I^{\otimes}(-) \arrow[dr, "U"] & \\ \mathrm{CAlg}_I(-) \arrow[ur, "\delta"] \arrow[rr, equal] & & \mathrm{CAlg}_I(-) \end{tikzcd}$$

δ takes a structure to two interchanging copies of itself, and U simply forgets one of the structures. ◁

A weak ∞-categorical form of the *Eckmann-Hilton argument* for I-commutative algebras would state that the functor U is an equivalence, or equivalently, δ is an equivalence, i.e. $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ is ∞-connected; the specialization to $(n+1)$-categories is that $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ is n-connected. Unfortunately, this does not hold for all $I \in \mathrm{wIndex}_{\mathcal{T}}$. The following simple counterexample was pointed out to the author by Piotr Pstrągowski.

Example 2.24. Let R be a nonzero commutative ring and let $\mathrm{Comm}_{nu}^{\otimes}$ be the weak $\mathcal{N}_{\infty}$-$*$-operad associated with the $*$-weak indexing system $\mathbb{F}^{nu} = \mathbb{F} - \{\varnothing\}$. Then, the Abelian group underlying R supports a $\mathrm{Comm}_{nu}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}_{nu}^{\otimes}$ structure given by the two multiplications

$$\mu(r,s) = rs, \qquad\qquad \mu_0(r,s) = 0,$$

which are easily seen to satisfy interchange but be distinct. This lies outside of the essential image of

$$\delta\colon \mathrm{Alg}_{\mathrm{Comm}_{nu}}(\mathbf{Ab}) \longrightarrow \mathrm{Alg}_{\mathrm{Comm}_{nu}} \underline{\mathrm{Alg}}_{\mathrm{Comm}_{nu}}(\mathbf{Ab}),$$

so δ is not an equivalence; by Proposition 1.24, this implies that $\mathrm{Comm}_{nu}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}_{nu}^{\otimes}$ is not connected. ◁

In the positive direction, [SY19] yields a classification of $\overset{\mathrm{BV}}{\otimes}$-idempotent algebras in *reduced* ∞-operads. In fact, Example 2.24 shows that the associated unitality assumption only misses one example among nonequivariant one-color weak $\mathcal{N}_{\infty}$-operads.

Corollary 2.25. *A weak $\mathcal{N}_{\infty}$-$*$-operad $\mathcal{O}^{\otimes}$ possesses a map $\mathrm{triv}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ inducing an equivalence*

$$\mathcal{O}^{\otimes} \xrightarrow{\sim} \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes}$$

if and only if $\mathcal{O}^{\otimes}$ is equivalent to $\mathrm{triv}^{\otimes}$, $\mathbb{E}_0^{\otimes}$, or $\mathbb{E}_{\infty}^{\otimes}$.

Proof. [SY19, Cor 5.3.4] covers the reduced case, so it suffices to assume that $\mathcal{O}(\varnothing) = \varnothing$ and show that $\mathcal{O}^{\otimes} \simeq \mathrm{triv}^{\otimes}$. Note that $\mathrm{Comm}_{nu}^{\otimes}$ is the terminal nonunital $\mathcal{N}_{\infty}$-$*$-operad, i.e. there exists a map $\mathcal{O}^{\otimes} \rightarrow \mathrm{Comm}_{nu}^{\otimes}$, yielding a diagram

$$\begin{tikzcd} \mathcal{O}^{\otimes} \otimes \mathcal{O}^{\otimes} \arrow[r, "\varphi"] & \mathrm{Comm}_{nu}^{\otimes} \otimes \mathrm{Comm}_{nu}^{\otimes} \\ \mathcal{O}^{\otimes} \arrow[u] \arrow[r] & \mathrm{Comm}_{nu}^{\otimes} \arrow[u] \end{tikzcd}$$

Pulling back Example 2.24, we find that if $\mathcal{O}(n) = *$ for any $n \neq 1$, then $\varphi^* R \in \mathrm{Alg}_{\mathcal{O} \otimes \mathcal{O}}(\mathbf{Ab})$ is not in the image of the diagonal; contrapositively, $\mathcal{O}(n) = \varnothing$ when $n \neq 1$, i.e. it's equivalent to $\mathrm{triv}^{\otimes}$. □

We saw in [Ste24] that $\left\{\mathrm{triv}^{\otimes} \rightarrow \mathbb{E}_0^{\otimes} \rightarrow \mathbb{E}_{\infty}^{\otimes}\right\} = \mathrm{Op}_*^{a\mathrm{uni,\ weak}\text{-}\mathcal{N}_{\infty}}$. In this section, we introduce an equivariant analogue to this argument in order to prove the following proposition.

Proposition 2.26. *Suppose $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ is connected. Then, I almost essentially unital.*

By combining Proposition 2.26 and Corollary 2.4, we conclude the remaining part of Corollary D.

Corollary 2.27. *$\mathcal{N}_{I\infty}^{\otimes} \otimes \mathcal{N}_{I\infty}^{\otimes}$ is a weak $\mathcal{N}_\infty$-operad if and only if I is almost essentially unital; in particular, there exists a (necessarily unique) map $\mathrm{triv}^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$ inducing an equivalence $\mathcal{N}_{I\infty}^{\otimes} \xrightarrow{\sim} \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ if and only if I is almost-unital.*

To show Proposition 2.26, we pass to a universal case. First, the weak indexing system.

Recollection 2.28. In [Ste24], we computed the terminal weak indexing system with unit family $\mathcal{F}$ to be

$$\mathbb{F}_{\mathcal{F}^{\perp}-nu,V} = \begin{cases} \mathbb{F}_V & V \in \mathcal{F}; \\ \mathbb{F}_V - \{S \mid \forall U \in \mathrm{Orb}(S),\, U \in \mathcal{F}\} & V \notin \mathcal{F}; \end{cases}$$

in particular, $\underline{\mathbb{F}}_I$ fails to be almost essentially unital if and only if there is some non-contractible W-set in $\mathbb{F}_{v(I)^{\perp}-nu,W} \cap \mathbb{F}_{I,W}$ for some $W \in v(I)^{\perp}$. We refer to the associated weak indexing category as $I_{\mathcal{F}^{\perp}-nu}$; note that $I_{\mathcal{F}^{\perp}-nu} \subset \mathbb{F}_{\mathcal{T}}$ is the wide subcategory of maps $T \to S$ such that either $S, T \in \mathbb{F}_{\mathcal{F}}$ or $S, T \in \mathbb{F}_{\mathcal{F}^{\perp}}$. ◁

Now, we construct a family of problematic $\mathcal{N}_{I_{\mathcal{F}^{\perp}-nu}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I_{\mathcal{F}^{\perp}-nu}}^{\otimes}$-algebras.

Construction 2.29. Let M be a $\mathcal{T}$-commutative monoid in pointed sets. We define a new functor

$$M^0 \colon h_1 \mathrm{Span}_{I_{\mathcal{F}^{\perp}-nu}}(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Set}_*$$

which agrees with M on objects, backwards maps, forwards maps lying in $\mathbb{F}_{\mathcal{F}}$, but whose forward maps lying in $\mathbb{F}_{\mathcal{F}^{\perp}}$ are zero. This is evidently functorial on backwards and forward maps, and the restriction to backwards maps is product-preserving. We're left with verifying the double coset formula that, given a cartesian square as on the left such that $S, T, R \in \mathcal{T}$ and $f, f' \in I_{\mathcal{F}^{\perp}-nu}$, the right square commutes, where $(-)_*$ denotes covariant functoriality and $(-)^*$ contravariant.

$$\begin{tikzcd} & R \times_S T \arrow[dl, "g'"'] \arrow[dr, "f'"] \arrow[dd, phantom, "\lrcorner"] & \\ R \arrow[dr, "f"'] & & T \arrow[dl, "g"] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} & M^0(R \times_S T) \arrow[dr, "f'_*"] & \\ M^0(R) \arrow[ur, "g'^*"] \arrow[dr, "f_*"'] & & \widetilde{M}^0(T) \\ & M^0(R) \arrow[ur, "g^*"'] & \end{tikzcd}$$

The assertions that $f, f' \in I_{\mathcal{F}^{\perp}-nu}$ and that $\mathcal{F}$ is a family together imply that $T \in \mathcal{F}$ if and only if the entire diagram lives in $\mathbb{F}_{\mathcal{F}}$, and $T \in \mathcal{F}^{\perp}$ if and only if the entire diagram lives in $\mathbb{F}_{\mathcal{F}^{\perp}}$. In the former case, the right diagram commutes by the double coset formula for M, and in the latter case it commutes as each composite map is zero. ◁

Lemma 2.30. *For all $\mathcal{T}$-commutative algebras M, the $I_{\mathcal{F}^{\perp}-nu}$-commutative algebras M and M_0 interchange.*

Proof. Note a diagram of $\mathcal{T}$-coefficient systems in a 1-category commutes if and only if the V-fixed point diagram commutes for all $V \in \mathcal{T}$; the V-fixed points of the diagram in Recollection 1.32 in our case correspond with the diagram

$$\begin{tikzcd} \left(X^T\right)^S \arrow[d, "(\mathrm{tr}_T)_S"] & \simeq & X^{S \times T} & \simeq & \left(X^S\right)^T \arrow[r, "(\mathrm{tr}_S^0)_T"] & X^T \arrow[d, "\mathrm{tr}_T"] \\ X^S \arrow[rrrrr, "\mathrm{tr}_S^0"] & & & & & X \end{tikzcd}$$

where tr_* is the indexed multiplication in M and tr_*^0 is the indexed multiplication in M^0; when $V \in \mathcal{F}^{\perp}$, this commutes as each of the composites factor through a zero map.

Moreover, note that $\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} I_{\mathcal{F}^{\perp}-nu}$ is unital, so Corollary 2.4 implies that $\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} I_{\mathcal{F}^{\perp}-nu}$-algebras interchange with themselves; in particular, the interchange relation of Recollection 1.32 for M with itself implies the same relation for M and M_0 whenever $V \in \mathcal{F}$. □

We are left with constructing a highly nontrivial $\mathcal{T}$-commutative algebra; we choose a universal one.

Construction 2.31. Since the "isomorphism classes of objects" functor $\pi_0 \colon \mathrm{Cat} \to \mathrm{Set}$ preserves limits, pushforward along it lifts to a functor

$$\underline{\pi}_0 \colon \mathrm{Cat}_{\mathcal{T}}^{\otimes} \simeq \mathrm{CMon}_{\mathcal{T}}(\mathrm{Cat}) \to \mathrm{CMon}_{\mathcal{T}}(\mathrm{Set});$$

the *effective Burnside $\mathcal{T}$-commutative monoid* is $\underline{A}_{\mathcal{T}} \coloneqq \underline{\pi}_0 \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$. We denote its image under the maps

$$\mathrm{CMon}_{\mathcal{T}}(\mathrm{Set}) \simeq \mathrm{CMon}_{\mathcal{T}}(\mathrm{Set}_*) \to \mathrm{CMon}_{I_{\mathcal{F}^{\perp}-nu}}(\mathrm{Set}_*)$$

implied by Corollaries 1.54 and 2.4 by $\widetilde{\underline{A}}_{\mathcal{T}}$. ◁

Lemma 2.32. *The S-indexed multiplication in $\widetilde{\underline{A}}_{\mathcal{T}}$ and $\widetilde{\underline{A}}_{\mathcal{T}}^0$ are distinct for all $S \in \mathbb{F}_{\mathcal{F}^{\perp}-nu,V} - \{*_V\}$ and $V \in \mathcal{F}^{\perp}$.*

Proof. It suffices to prove that, for all $S \neq \varnothing_V \in \mathbb{F}_V$, the S-ary multiplication of $\underline{A}_{\mathcal{T}}$ takes some element to another element other than the unit; unraveling definitions, this is equivalent to the property that some nonempty V-set can be expressed as an S-indexed coproduct. S provides such an example. □

We now exhibit failure of the non-almost-essentially-unital 1-categorical Eckmann-Hilton argument.

Proof of Proposition 2.26. Note that

$$\begin{aligned}\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes} \text{ is connected} &\iff h_1 \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{N}_{A(\mathcal{N}_{I\infty} \otimes \mathcal{N}_{I\infty})}^{\otimes} \simeq \mathcal{N}_{I\infty}^{\otimes} \\ &\implies \mathrm{CAlg}_I(\mathrm{Set}_*) \longrightarrow \mathrm{Alg}_{\mathcal{N}_{I\infty} \otimes \mathcal{N}_{I\infty}}(\mathrm{Set}_*) \text{ is essentially surjective.}\end{aligned}$$

Furthermore, Lemmas 2.30 and 2.32 construct an $\mathcal{N}_{v(I)^{\perp}-nu\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{v(I)^{\perp}-nu\infty}^{\otimes}$-algebra A satisfying the condition that its two individual structure maps $A(S) \to A(*_V)$ differ whenever $V \in v(I)^{\perp}$ and $S \neq *_V$. Since I is not almost essentially unital, it must admit some noncontractible $S \in \mathbb{F}_{I,V}$ for $V \in v(I)^{\perp}$, so the pullback $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ structure on A has two distinct underlying I-algebra structures, implying it is outside of this essential image. The contrapositive shows that $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ is not connected. □

Remark 2.33. Using the above argument, one can show that if $\mathcal{O}^{\otimes}$ is a idempotent object in $\mathcal{T}$-operads, then its nullary spaces $\mathcal{O}(\varnothing_V)$ are nonempty. If additionally $\mathcal{O}(\varnothing_V)$ are assumed to be contractible (i.e. $\mathcal{O}^{\otimes}$ is almost-unital), then Proposition 2.18 shows that the underlying fixed point catgeories $\mathcal{O}_V$ are all idempotent algebras, i.e. they are contractible. Hence $\mathcal{O}^{\otimes}$ will be shown to be almost-reduced. In forthcoming work, we will develop an equivariant lift of [SY19], which would imply that every idempotent almost-unital $\mathcal{T}$-operad is a weak $\mathcal{N}_{\infty}$-operad. ◁

3. Corollaries in higher algebra

We now indulge in a number of corollaries. We begin in Section 3.1 by making the equivariant equifibered perspective symmetric monoidal, establishing that $\overset{\mathrm{BV}}{\otimes}$ is $\mathcal{T}$-bifunctorial and distributive, fulfilling Assumption (b) in full generality. We make use of the distributivity in Section 3.2, where we apply the disintegration and assembly procedure of Section B to compute tensor products of $\mathcal{T}$-space colored $\mathcal{T}$-operads. Then, in Section 3.3 we show how Corollary D constructs an I-symmetric monoidal structure on right-modules over an I-commutative algebra.

Moving on from this, in Section 3.4 we spell out the cases of *equivariant Dunn additivity* that follows from Corollary D and Theorem F; in Section 3.5 we show that equivariant factorization homology is G-symmetric monoidal and spell out how to infinitely iterate equivariant factorization homology and TCR.

3.1. Coherences and restrictions of equivariant Boardman-Vogt tensor products.

We would like to construct coherences for $\overset{\mathrm{BV}}{\otimes}$ using the argument of [BS24a], but it is currently not known whether Env_I is monic in Cat, so we must modify their argument: we use that the *sliced* envelope fully-faithful [BHS22, Prop 4.2.1].

Now, we proved in Corollary 2.27 that $\mathrm{Comm}_{\mathcal{T}}^{\otimes}$ bears a unique structure as an idempotent object; in particular, Proposition 1.34 shows that Env is compatible with finitary tensor products, so it induces a $\circledast$-idempotent object structure on $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \in \mathrm{Cat}_{\mathcal{T}}^{\otimes}$. By [HA, Prop 4.8.2.9] this underlies a unique $\mathbb{E}_{\infty}$-algebra under the mode structure. Then, Corollary C.8 constructs a symmetric monoidal $\mathcal{T}$-∞-category structure on the $\mathcal{T}$-overcategory $\underline{\mathrm{Cat}}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$ whose underlying tensor functor has value

$$\mathcal{C} \circledast \mathcal{D} \xrightarrow{\pi_{\mathcal{C}} \circledast \pi_{\mathcal{D}}} \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \circledast \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \xrightarrow{\sim} \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}.$$

and whose unit is

$$\mathrm{Env}\left(\mathrm{triv}_{\mathcal{T}}^{\otimes}\right) \xrightarrow{\eta} \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}.$$

We acquire existence and uniqueness of the Boardman-Vogt symmetric monoidal structure.

Corollary E'. *$\underline{\mathrm{Op}}_{\mathcal{T}} \subset \underline{\mathrm{Cat}}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$ is a symmetric monoidal subcategory under $\circledast$, with unit corresponding with $\mathrm{triv}^{\otimes}_{\mathcal{T}}$ and tensor bifunctor corresponding with $\overset{\mathrm{BV}}{\otimes}$. Hence there exists a unique symmetric monoidal $\mathcal{T}$-∞-category $\underline{\mathrm{Op}}^{\otimes}_{\mathcal{T}}$ and symmetric monoidal $\mathcal{T}$-functor*

$$\underline{\mathrm{Op}}^{\otimes}_{\mathcal{T}} \to \underline{\mathrm{Cat}}^{\otimes,\circledast}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$$

lifting the sliced $\mathcal{T}$-symmetric monoidal envelope.

Proof. We're tasked with proving that the image of $\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}(-)$ contains the unit and is closed under binary tensor products. The unit is Proposition 1.34, and by construction we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Env}\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right) \arrow[r, "\mathrm{Env}(\pi_{\mathcal{O}} \overset{\mathrm{BV}}{\otimes} \pi_{\mathcal{P}})"] \arrow[d, phantom, "\wr"] & \mathrm{Env}\left(\mathrm{Comm}^{\otimes}_{\mathcal{T}} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes}_{\mathcal{T}}\right) \arrow[d, phantom, "\wr"] & \mathrm{Env}\left(\mathrm{Comm}^{\otimes}_{\mathcal{T}}\right) \arrow[l, "\mathrm{Env}(\mathrm{id}\otimes\eta)"', "\sim"] \arrow[d, phantom, "\wr"] \\
\mathrm{Env}\left(\mathcal{O}^{\otimes}\right) \circledast \mathrm{Env}\left(\mathcal{P}^{\otimes}\right) \arrow[r, "\pi_{\mathrm{Env}(\mathcal{O}^{\otimes})} \circledast \pi_{\mathrm{Env}(\mathcal{P}^{\otimes})}"'] & \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \circledast \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} & \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \arrow[l, "\sim", "\mathrm{id}\circledast\eta"']
\end{tikzcd}$$

Inverting $\mathrm{Env}(\mathrm{id}\otimes\eta)$ and $\mathrm{id}\circledast\eta$ yields the desired equivalence

$$\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right) \simeq \mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}\left(\mathcal{O}^{\otimes}\right) \circledast \mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}\left(\mathcal{P}^{\otimes}\right). \qquad \square$$

Corollary 3.1. *There exists a G-bifunctor $\overset{\mathrm{BV}}{\otimes} : \underline{\mathrm{Op}}_{\mathcal{T}} \times \underline{\mathrm{Op}}_{\mathcal{T}} \longrightarrow \underline{\mathrm{Op}}_{\mathcal{T}}$ whose V-value $\mathrm{Op}_V \times \mathrm{Op}_V \to \mathrm{Op}_V$ is*

$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \simeq L_{\mathrm{Op}_V}\left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \longrightarrow \mathrm{Span}(\mathbb{F}_V) \times \mathrm{Span}(\mathbb{F}_V) \xrightarrow{\wedge} \mathrm{Span}(\mathbb{F}_V)\right).$$

This recovers the correct construction in the nonequivariant case.

Corollary 3.2. *When $\mathcal{T} = *$, there is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Op}^{\otimes}_{*} \simeq \mathrm{Op}^{\otimes},$$

where the latter is the Boardman-Vogt symmetric monoidal ∞-category of [BS24a].

Proof. In [Ste25] we supplied an equivalence $\mathrm{Op}_* \simeq \mathrm{Op}$, so it suffices to upgrade this to a symmetric monoidal equivalence. In fact, the unslicing functor $\mathrm{Cat}^{\otimes}_{\infty,/\mathbb{F}^{\sqcup}} \to \mathrm{Cat}^{\otimes}_{\infty}$ bears a symmetric monoidal structure (see Corollary C.6), so Corollary E' constructs a symmetric monoidal structure on the composite $\mathrm{Op}^{\otimes}_{*} \to \mathrm{Cat}^{\otimes}_{\infty}$. Thus [BS24a, Thm E] constructs a symmetric monoidal equivalence extending the equivalence $\mathrm{Op}_* \simeq \mathrm{Op}$. $\square$

Corollary 3.3. *Let I be a one color weak indexing system and $n \in \mathbb{N} \cup \{\infty\}$. Then, $\mathrm{Op}_I \subset \mathrm{Op}_{\mathcal{T}}$ is a symmetric monoidal subcategory, $\mathrm{Op}_I^{\mathrm{uni}} \subset \mathrm{Op}_I$ is a smashing localization, and the following are symmetric monoidal full subcategory inclusions:*

$$\begin{array}{ccccccc}
\mathrm{Op}^{aE\,\mathrm{red}}_{I,\geq n} & \subset & \mathrm{Op}^{aE\,\mathrm{red}}_{I} & \subset & \mathrm{Op}^{aE\mathrm{uni}}_{I} & \subset & \mathrm{Op}_I \\
\mathrm{Op}^{\mathrm{red}}_{I,\geq n} & \subset & \mathrm{Op}^{\mathrm{red}}_{I} & \subset & \mathrm{Op}^{\mathrm{uni}}_{I} & &
\end{array}$$

Proof. The first statement follows from Proposition 1.44 and the second from Corollary 2.15. $\mathrm{triv}^{\otimes}_{\mathcal{T}}$ and $\mathbb{E}^{\otimes}_{0,v(I)}$ are ∞-connected, so in particular, the symmetric monoidal units are compatible with each of the above subcategory inclusions. We're left with verifying that each subcategory inclusion is closed under tensor products; the lefthand inclusions both follow from Corollary 2.21, the middle inclusions both follow from Proposition 2.18, and the righthand inclusion is Corollary 2.19. $\square$

We finish the subsection by confirming a convenient structural result.

Corollary 3.4. *$\underline{\mathrm{Op}}^{\otimes}_{I}$, and $\underline{\mathrm{Op}}^{I-\mathrm{cocart},\otimes}_{I}$ are presentably symmetric monoidal $\mathcal{T}$-∞-categories.*

To see this, $\underline{\mathrm{Op}}^{\otimes}_{I}$ is presentable by the localizing inclusion $\underline{\mathrm{Op}}_I \subset \underline{\mathrm{Cat}}_{\mathcal{T},/\underline{\mathbb{F}}_{I}^{I-\sqcup}}$, and it is distributive by the tensor-hom $\mathcal{T}$-adjunction $(-)\overset{\mathrm{BV}}{\otimes}\mathcal{O}^{\otimes} \dashv \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(-)$. The remaining case follows from the following easy lemma.

Lemma 3.5. *If $L\colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ is a smashing $\mathcal{T}$-localization and $\mathcal{D}^{\otimes}$ is a presentably symmetric monoidal $\mathcal{T}$-∞-category, then $\mathcal{C}^{\otimes}$ is a presentably symmetric monoidal $\mathcal{T}$-∞-category.*

Proof. It's clear that $\mathcal{C}$ is a presentable $\mathcal{T}$-∞-category, so we're left with verifying that $-\otimes C\colon \mathcal{C}\to\mathcal{C}$ possesses a right $\mathcal{T}$-adjoint; by the usual argument, it suffices to show this on fixed points, so we may assume $\mathcal{T}=*$.

We claim that $-\otimes C \dashv \hom_{\mathcal{D}}(C,-)$, the latter denoting the $\mathcal{D}$-internal hom. It suffices to verify that $\hom_{\mathcal{D}}(C,D)$ is L-local for all $D\in\mathcal{D}$. We apply the standard argument: if $f\colon X\to Y$ is an L-equivalence, then $C\otimes f\sim C\otimes Lf\colon C\otimes X\to C\otimes Y$ is an equivalence, so the horizontal arrows in the following are equivalences.

$$\begin{tikzcd}
\mathrm{Map}(Y,\hom(C,D)) \arrow[r, "f^*"] \arrow[d, phantom, "\wr"] & \mathrm{Map}(X,\hom(C,D)) \arrow[d, phantom, "\wr"] \\
\mathrm{Map}(Y\otimes C,D) \arrow[r, "(C\otimes f)^*"] & \mathrm{Map}(X\otimes C,D)
\end{tikzcd}$$

The fact that the top arrow is an equivalence is the desired locality. □

3.2. **Disintegration and equivariant Boardman-Vogt tensor products.**

We show the following generalization of the main results of [HA, § 2.3.3-2.3.4] in Section B.

Theorem 3.6 (Disintegration and assembly). *Let X be a $\mathcal{T}$-space. Taking fibers yields an equivalence*

$$\underline{\mathrm{Op}}_{I,/X^{I-\sqcup}} \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(X,\underline{\mathrm{Op}}_I).$$

The counit of this specifies a natural equivalence

$$\underline{\mathrm{colim}}_{x\in X}\left(\mathrm{Res}^{\mathcal{T}}_{\mathrm{stab}(x)}\mathcal{O}^{\otimes}\times_{\mathrm{Res}^{\mathcal{T}}_{\mathrm{stab}(x)}X^{I-\sqcup}}\mathcal{N}^{\otimes}_{I_V\infty}\right)\xrightarrow{\sim}\mathcal{O}^{\otimes}.$$

Here, $\underline{\mathrm{colim}}_{x\in X}$ refers to a $\mathcal{T}$-colimit of an X-indexed diagram. Given $x\in X^V$ we've written $\mathrm{stab}(x):=V$. Given $\mathcal{O}^{\otimes}$ a $\mathcal{T}$-operad, I a one-color $\mathcal{T}$-weak indexing category, and $x\in\mathcal{O}_V$ a V object, we define the *reduced endomorphism I_V-operad of x* to be the pullback

$$\begin{tikzcd}
\mathrm{End}^{I,\mathrm{red}}_x(\mathcal{O}) \arrow[r, "\iota_x"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Res}^{\mathcal{T}}_V\mathcal{O}^{\otimes} \arrow[d, "\eta"] \\
\mathcal{N}^{\otimes}_{I_V\infty} \arrow[r, "\{x\}"'] & \mathrm{Res}^{\mathcal{T}}_V U\mathcal{O}^{I-\sqcup}
\end{tikzcd}$$

In the case $I=\mathcal{T}$, we simply write $\mathrm{End}^{\mathrm{red}}_x(\mathcal{O}):=\mathrm{End}^{\mathcal{T},\mathrm{red}}_x(\mathcal{O})$.

Remark 3.7. If $\mathcal{O}^{\otimes}$ is unital (resp. almost-unital) then $\mathrm{End}^{\mathrm{red}}_x(\mathcal{O})$ is reduced (almost-reduced). ◁

We acquire the following from Theorem 3.6.

Corollary 3.8. *Suppose $\mathcal{O}^{\otimes}$ is a $\mathcal{T}$-operad whose underlying $\mathcal{T}$-∞-category $U\mathcal{O}$ is a $\mathcal{T}$-space and I is a one-color weak indexing system. Then, the inclusion maps ι_x assemble to a $\mathcal{T}$-colimit diagram in I-operads:*

$$\underline{\mathrm{colim}}_{x\in U\mathcal{O}}\mathrm{End}^{I,\mathrm{red}}_x(\mathcal{O})\xrightarrow{\sim}\mathrm{Bor}^{\mathcal{T}}_I\mathcal{O}.$$

In essence, this says that an at-least one color $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ whose underlying $\mathcal{T}$-∞-category is a $\mathcal{T}$-space disintegrates into a $U\mathcal{O}$-local system of one color $\mathcal{T}$-operads, and the $U\mathcal{O}$-indexed colimit $\mathcal{T}$-operad (i.e. the Grothendieck construction) assembles $\mathcal{O}^{\otimes}$ from this local system. In particular, $\mathcal{O}$-algebras are $U\mathcal{O}$-indexed systems of $\mathrm{End}^{\mathrm{red}}_x(\mathcal{O})$-algebras:

$$\begin{aligned}\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) &\simeq \underline{\mathrm{Alg}}_{\underline{\mathrm{colim}}_{x\in U\mathcal{O}}\mathrm{End}^{\mathrm{red}}_x(\mathcal{O})}(\mathcal{C})\\ &\simeq \underline{\lim}_{x\in U\mathcal{O}}\underline{\mathrm{Alg}}_{\mathrm{End}^{\mathrm{red}}_x(\mathcal{O})}(\mathcal{C}).\end{aligned}$$

The corresponding picture for $\mathcal{O}^{\otimes}\overset{\mathrm{BV}}{\otimes}\mathcal{P}^{\otimes}$-algebras is $U\mathcal{O}\times U\mathcal{P}$-local systems of $\mathrm{End}^{\mathrm{red}}_x(\mathcal{O})\overset{\mathrm{BV}}{\otimes}\mathrm{End}^{\mathrm{red}}_y(\mathcal{P})$-algebras: that is, we can compute tensor products of at-least one color $\mathcal{T}$-operads in terms of one color $\mathcal{T}$-operads, as long as they are $\mathcal{T}$-space colored.

Corollary 3.9 (Disintegration of tensor products). *Suppose $\mathcal{O}^{\otimes},\mathcal{P}^{\otimes},\mathcal{Q}^{\otimes}$ are at-least one colored $\mathcal{T}$-operads whose underlying $\mathcal{T}$-∞-categories are $\mathcal{T}$-spaces and $\varphi\colon\mathcal{O}^{\otimes}\overset{\mathrm{BV}}{\otimes}\mathcal{P}^{\otimes}\to\mathcal{Q}^{\otimes}$ is a map such that*

(a) the underlying map of $\mathcal{T}$-spaces $U\varphi\colon U\mathcal{O}\times U\mathcal{P}\to U\mathcal{Q}$ is an equivalence, and
(b) for all pairs $(x,y)\in U\mathcal{O}\times U\mathcal{P}$, φ pulls back to an equivalence

$$\varphi_{(x,y)}\colon\ \mathrm{End}^{\mathrm{red}}_x(\mathcal{O})\overset{\mathrm{BV}}{\otimes}\mathrm{End}^{\mathrm{red}}_y(\mathcal{P})\to\mathrm{End}^{\mathrm{red}}_{(x,y)}(\mathcal{Q}).$$

Then φ is an equivalence.

Proof. Corollaries 3.4 and 3.8 construct equivalences of arrows

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \arrow[d, "\varphi"] \arrow[r, phantom, "\simeq"] & \underline{\mathrm{colim}}_{x \in U\mathcal{O}} \mathrm{End}^{\mathrm{red}}_x(\mathcal{O}) \overset{\mathrm{BV}}{\otimes} \underline{\mathrm{colim}}_{y \in U\mathcal{P}} \mathrm{End}^{\mathrm{red}}_y(\mathcal{P}) \arrow[d] \arrow[r, phantom, "\simeq"] & \underline{\mathrm{colim}}_{(x,y) \in U\mathcal{O} \times U\mathcal{P}} \mathrm{End}^{\mathrm{red}}_x(\mathcal{O}) \overset{\mathrm{BV}}{\otimes} \mathrm{End}^{\mathrm{red}}_y(\mathcal{P}) \arrow[d, "\sim"] \\
\mathcal{Q}^{\otimes} \arrow[r, phantom, "\simeq"] & \underline{\mathrm{colim}}_{(x,y) \in U\mathcal{O} \times U\mathcal{P}} \mathrm{End}^{\mathrm{red}}_{(x,y)}(\mathcal{Q}) \arrow[r, equal] & \underline{\mathrm{colim}}_{(x,y) \in U\mathcal{O} \times U\mathcal{P}} \mathrm{End}^{\mathrm{red}}_{(x,y)}(\mathcal{Q})
\end{tikzcd}$$

The right vertical arrow is an equivalence by assumption, so φ is an equivalence by two out of three. $\square$

We will make crucial use of this in forthcoming work concerning variants of $\mathbb{E}_V^{\otimes}$ with tangential structure.

3.3. **Norms of right-modules over I-commutative algebras.** Let I be an indexing category, $\mathcal{C}^{\otimes}$ an I-symmetric monoidal ∞-category, $t\colon W \to V$ an I-admissible transfer, A an I_V-commutative algebra, and M a right module over the associative algebra underlying $\mathrm{Res}_W^V A$. Then, we may define the *A-module norm of M* by the base-changed A-module

$$_A N_W^V M \coloneqq A \otimes_{N_W^V \mathrm{Res}_W^V A} N_W^V M;$$

that is, the normed multiplication recognizes A as an $N_W^V \mathrm{Res}_W^V A$-module, and the A-module norm of M is the free A-module on the normed $N_W^V \mathrm{Res}_W^V A$-algebra of M; see [Yan23] for a detailed account in the C_p-equivariant case.

In this subsection, we use the equivalence $\mathcal{N}_{I\infty}^{\otimes} \simeq \mathbb{E}_1^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ to lift this to an I-symmetric monoidal structure, yielding coherent functoriality and a coherent double coset formula for A-module norms. To do this, we begin by bootstrapping G-symmetric monoidality of the right module construction from the non-equivariant case.

Observation 3.10. Fix $\mathcal{O}^{\otimes}$ a $\mathcal{T}$-operad. By [HA, Rmk 4.8.3.8], functors $F\colon \mathrm{TotTot}_{\mathcal{T}} \mathcal{O}^{\otimes} \to \mathrm{Cat}^{\mathrm{Alg}}$ are data

$$\begin{tikzcd}
 & \mathcal{C}^{\otimes} \arrow[d, "\pi_F"] \\
\mathbb{E}_1^{\otimes} \times \mathrm{TotTot}_{\mathcal{T}} \mathcal{O}^{\otimes} \arrow[ur, "A_F"] \arrow[r, equal] & \mathbb{E}_1^{\otimes} \times \mathrm{TotTot}_{\mathcal{T}} \mathcal{O}^{\otimes}
\end{tikzcd}$$

such that π is a cocartesian fibration whose fibers $\mathcal{C}_V^{\otimes} \to \mathbb{E}_1^{\otimes}$ are the unstraightenings of small monoidal ∞-categories and such that the composite arrows $\mathbb{E}_1^{\otimes} \times \{O\} \hookrightarrow \mathcal{C}^{\otimes}$ are associative algebras. Moreover, unwinding definitions and applying Lemma A.18, the condition that F corresponds with an $\mathcal{O}$-monoid $\mathrm{Tot}_{\mathcal{T}} \mathcal{O}^{\otimes} \to \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathrm{Cat}^{\mathrm{Alg}}$ corresponds with the condition that each of the fibers $\mathcal{C}_n^{\otimes} \to \mathrm{TotTot}_{\mathcal{T}} \mathcal{O}^{\otimes}$ is an $\mathcal{O}$-monoidal ∞-category. $\triangleleft$

Given $\mathcal{C}^{\otimes} \in \mathrm{Cat}^{\otimes}_{\mathbb{E}_1 \otimes \mathcal{O}}$ and $A \in \mathrm{Alg}_{\mathbb{E}_1 \otimes \mathcal{O}}(\mathcal{C})$, we acquire a functorial diagram

$$\begin{tikzcd}
\mathrm{Tot}_{\mathcal{T}} \mathcal{O}^{\otimes} \arrow[dr, dashed] \arrow[drr, "{(A,\mathcal{C}^{\otimes})}"] \arrow[drrrr, "\mathrm{RMod}_A(\mathcal{C})^{\otimes}"] \arrow[dddr, equal] & & & & \\
 & \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathrm{Cat}^{\mathrm{Alg}} \times_{\underline{\mathrm{Coeff}}^{\mathcal{T}} \mathrm{Alg}(\mathrm{Cat})} \mathrm{Tot}_{\mathcal{T}} \mathcal{O}^{\otimes} \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\lrcorner"] & \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathrm{Cat}^{\mathrm{Alg}} \arrow[r, "\Theta"] \arrow[d, "U"] & \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathrm{Cat}^{\mathrm{Mod}} \arrow[r, "Y"] & \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathrm{Cat} \\
 & & \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathrm{Alg}(\mathrm{Cat}) \arrow[d, phantom, "\wr"] & & \\
 & \mathrm{Tot}_{\mathcal{T}} \mathcal{O}^{\otimes} \arrow[r, "\mathcal{C}^{\otimes}"] & \underline{\mathrm{Alg}}(\underline{\mathrm{Coeff}}^{\mathcal{T}} \mathrm{Cat}) & &
\end{tikzcd}$$

Θ and Y are product preserving functors, so $\mathrm{RMod}_A(\mathcal{C})^{\otimes}$ is an $\mathcal{O}$-monoid in Cat, i.e. an $\mathcal{O}$-monoidal ∞-category. Unwinding definitions, this proves the following proposition.

Proposition 3.11. *Let $\mathcal{O}^\otimes$ be a $\mathcal{T}$-operad and let $\mathcal{C}^\otimes$ an $\mathbb{E}_1 \otimes \mathcal{O}$-monoidal ∞-category. There is a lift*

$$\begin{tikzcd} & & \mathrm{Cat}_{\mathcal{O}}^{\otimes} \arrow[d, "\Gamma^{\mathcal{T}}"] \\ \mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathcal{C}) \arrow[r] \arrow[urr, bend left, "\mathrm{RMod}^{\otimes}_{(-)}(\mathcal{C})"] & \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \arrow[r, "\mathrm{RMod}_{(-)}(\mathcal{C})"] & \mathrm{Cat}, \end{tikzcd}$$

natural separately in $\mathcal{O}^\otimes$ and $\mathcal{C}^\otimes$; that is, left modules over $\mathbb{E}_1 \otimes \mathcal{O}$-algebras bear a natural $\mathcal{O}$-algebra structure.

We immediately acquire the following corollary, confirming a hypothesis of [Hil17, Rmk 3.15].

Corollary 3.12. *Let $\mathcal{O}^\otimes$ be a $\mathcal{T}$-operad whose underlying I^∞-operad is $\mathbb{E}_\infty$ and $\mathcal{C}^\otimes$ an $\mathcal{O}$-monoidal ∞-category. There is a lift*

$$\begin{tikzcd} & & \mathrm{Cat}_{\mathcal{O}}^{\otimes} \arrow[d] \\ \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r] \arrow[urr, bend left, "\mathrm{RMod}^{\otimes}_{(-)}(\mathcal{C})"] & \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \arrow[r, "\mathrm{RMod}_{(-)}(\mathcal{C})"] & \mathrm{Cat} \end{tikzcd}$$

natural separately in $\mathcal{O}^\otimes$ and $\mathcal{C}^\otimes$. In particular, if I is an indexing category, the ∞-category of right-modules over an I-commutative algebra admits a natural I-symmetric monoidal structure.

3.4. **Equivariant infinitary Dunn additivity.** In [Bon19], a *genuine operadic nerve* 1-categorical functor was constructed between a model of graph-G operads and a model of G-operads. In [Ste25], we lifted this to a conservative functor of ∞-categories $N^\otimes \colon g\mathrm{Op}_G \to \mathrm{Op}_G$. Given V an orthogonal G-representation, we define

$$\mathbb{E}_V^\otimes \coloneqq N^\otimes D_V,$$

where D_V is the *little V-disks graph G-operad* of [GM17], whose n-ary $G \times \Sigma_n$ space is the configuration space

$$D_V(n) \simeq \mathrm{Conf}_n(V)$$

by [GM17, Lem 1.2]. The resulting unital G-operad $\mathbb{E}_V$ was studied in [Hor19], who showed for instance that

$$\mathbb{E}_V(S) \simeq \mathrm{Conf}_S^H(V) \coloneqq \operatorname*{colim}_{\substack{W \subset V \\ \text{fin.dim}}} \mathrm{Conf}_S^H(W),$$

in view of the fact that the assignment $\mathcal{O} \mapsto \mathcal{O}(S)$ preserves sifted colimits [Ste25, § 2.3]; here, $\mathrm{Conf}_S^H(W)$ is the space of H-equivariant configurations of S into W under the compact open topology.

A weak form of the following easy claim appears to be folklore.

Proposition 3.13. *Let G be a topological group, $H \subset G$ a closed subgroup, $S \in \mathbb{F}_H$ a finite H-set admitting an configuration $\iota : S \hookrightarrow W$, and V, W orthogonal G-representations whose associated map*

$$\mathrm{Conf}_S^H(V) \hookrightarrow \mathrm{Conf}_S^H(V \oplus W)$$

is an equivalence. Then, $\mathrm{Conf}_S^H(V)$ is contractible.

Proof. Linear interpolation to ι yields a deformation of $\mathrm{Map}^H(S, V \oplus W)$ onto $\{\iota\}$. The path of a point beginning in the subspace $\mathrm{Conf}_S^H(V) \subset \mathrm{Conf}_S^H(V \oplus W)$ consisting of configurations with zero projection to W lands within $\mathrm{Conf}_S^H(V \oplus W)$ at all times; composing this deformation after the deformation retract $\mathrm{Conf}_S^H(V \oplus W) \xrightarrow{\sim} \mathrm{Conf}_H^S(V)$ yields a deformation retract of $\mathrm{Conf}_S^H(V \oplus W)$ onto $\{\iota\}$, so it is contractible.[5] By the equivalence $\mathrm{Conf}_S^H(V) \simeq \mathrm{Conf}_S^H(V \oplus W)$, the space $\mathrm{Conf}_S^H(V)$ is contractible as well. □

Remark 3.14. This argument only produces *contractibility*, whereas the nonequivariant argument using Fadell and Neuwirth's fibration [FN62] sharply characterizes *n-connectivity* of $\mathrm{Conf}_k(\mathbb{R}^n)$, and hence of $\mathbb{E}_k^\otimes$; the author will equivariantize this in forthcoming work. ◁

[5] Said explicitly, let $h \colon [0,1] \to \mathrm{Conf}_S^H(V \oplus W)$ be the deformation retract onto those configurations with zero projection to W. Then, our deformation retract h' onto $\iota(w)$ is computed by

$$h'(t) = \begin{cases} h(2t) & t \leq \frac{1}{2}, \\ (2-2t) \cdot h(1) + (2t-1)\iota & t \geq \frac{1}{2}. \end{cases}$$

The second is an *isotopy* since $h(1)$ and ι are pointwise-linearly independent embeddings.

We say that V is a *weak universe* if it is a direct sum of infinitely many copies of a collection of irreducible orthogonal G-representations; equivalently, there is an equivalence $V \simeq V \oplus V$. Given V an orthogonal G-representation, we let $AV := A\mathbb{E}_V$, i.e. AV corresponds with the weak indexing system $\underline{\mathbb{F}}^V = \underline{\mathbb{F}}_{AV}$ of finite H-sets admitting an embedding into V. The following corollary follows immediately from Proposition 3.13.

Corollary 3.15. *If there exists an equivalence $\mathbb{E}_V^{\otimes} \simeq \mathbb{E}_{V\oplus W}^{\otimes}$, then the canonical map $\mathrm{Bor}_{AW}^G \mathbb{E}_V^{\otimes} \to \mathcal{N}_{AW}^{\otimes}$ is an equivalence; in particular, if V is a weak universe, then there is a unique equivalence*

$$\mathbb{E}_V^{\otimes} \xrightarrow{\sim} \mathcal{N}_{AV}^{\otimes}.$$

Observation 3.16. If V is a *universe* (i.e. it is a weak universe admitting a positive-dimensional fixed point locus), then it admits embeddings of all finite sets with trivial G-action; in this case, $\mathbb{E}_V^{\otimes}$ is not just a weak $\mathcal{N}_\infty$-operad, but an $\mathcal{N}_\infty$-operad. ◁

Much study has been dedicated to the less general setting of *universes*; for instance, Rubin has given a complete and simple characterization of those indexing systems (equivalently, transfer systems) occurring as the arity-support of an $\mathbb{E}_V$-operad in [Rub19] for G abelian.

An inclusion $V \subset W$ yields a map of graph G-operads $D_V \to D_W$, hence a map $\mathbb{E}_V^{\otimes} \to \mathbb{E}_W^{\otimes}$. This yields a map of weak indexing systems $\underline{\mathbb{F}}^V \to \underline{\mathbb{F}}^W$; in [Ste24] we showed that this is additive, i.e.

$$\underline{\mathbb{F}}^V \vee \underline{\mathbb{F}}^W = \underline{\mathbb{F}}^{V\oplus W}. \tag{14}$$

Corollary 3.17 (Equivariant infinitary Dunn additivity)**.** *Let G be a finite group and V, W real orthogonal G-representations satisfying at least one of the following conditions:*

(a) V, W are weak G-universes, or
(b) the functoriality map $\mathbb{E}_V^{\otimes} \to \mathbb{E}_{V\oplus W}^{\otimes}$ is an equivalence.

Then there is a canonical equivalence

$$\mathbb{E}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} \to \mathbb{E}_{V\oplus W}^{\otimes};$$

equivalently, for any G-symmetric monoidal category $\mathcal{C}$, there are canonical equivalences[6]

$$\mathrm{Alg}_{\mathbb{E}_V} \underline{\mathrm{Alg}}_{\mathbb{E}_W}^{\otimes}(\mathcal{C}) \leftarrow \mathrm{Alg}_{\mathbb{E}_{V\oplus W}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_W} \underline{\mathrm{Alg}}_{\mathbb{E}_V}^{\otimes}(\mathcal{C}).$$

Proof. Given Corollary 3.15, case (a) follows from Theorem F and Eq. (14) and case (b) follows from Corollary D. □

Remark 3.18. In [Szc24], an ostensibly similar result to Corollary 3.17 is proved: given D_V the *little Disks graph G-operad*, Szczesny constructs a non-homotopical Boardman-Vogt tensor product $\otimes$ and a canonical map $D_V \otimes D_W \to D_{V\oplus W}$, which he shows to be a weak equivalence of graph G-operads in [Szc24, Thm 7.1]. Neither this result nor Corollary 3.17 imply each other.

On one hand, Szczesny's result concerns a tensor product with no known homotopical properties, so it is incomparable with results concerning ∞-categories of algebras defined by homotopy-coherent universal properties. On the other hand, while Corollary 3.17 is homotopical, it only concerns cases where at least one of the representations induces I-symmetric monoidal ∞-categories of algebras whose indexed tensor products are indexed coproducts; this property will not be satisfied for any nontrivial indexed tensor products in the finite-dimensional case, so the range of representations in Szczesny's result is significantly larger. The author will address the general case in forthcoming work. ◁

3.5. Norms on Real topological Hochschild and cyclic homology.

3.5.1. *Factorization homology in general.* In classical algebra, there is a well-known tensor products of functors $F, G : \mathcal{C} \to \mathcal{D}$ using monoidal structure of $\mathcal{D}$: the *pointwise tensor product* sets $F \otimes G(C) := F(C) \otimes G(C)$. We will use a lift of this due to Nardin-Shah.

Theorem 3.19 ([NS22, Thm 3.3.1, Thm. 3.3.3])**.** *Let $\mathcal{K}$ be a $\mathcal{T}$-∞-category and $\mathcal{C}^{\otimes}$ a $\mathcal{T}$-operad. Then, there exists a unique (functorial) I-operad structure $\mathrm{Fun}_{\mathcal{T}}(\mathcal{K}, \mathcal{C})^{\otimes-\mathrm{ptws}}$ on $\mathrm{Fun}_{\mathcal{T}}(\mathcal{K}, \mathcal{C})$ satisfying the universal property*

$$\mathrm{Alg}_{\mathcal{O}}(\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \mathcal{C})^{\otimes-\mathrm{ptws}}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}))$$

[6] What we mean by "canonical" depends on the case; for case (a), there is a contractible space of equivalences, and for case (b), this equivalence comes from inverting arrows of the zigzag $\mathbb{E}_{V\oplus W}^{\otimes} \simeq \mathbb{E}_{V\oplus W}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_G^{\otimes} \xrightarrow{\mathrm{id}\otimes !} \mathbb{E}_{V\oplus W}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} \xleftarrow{\iota\otimes \mathrm{id}} \mathbb{E}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes}$.

for $\mathcal{O} \in \mathrm{Op}_I$. Furthermore, when $\mathcal{C}^\otimes$ is I-symmetric monoidal, $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}$ is I-symmetric monoidal and satisfies the universal property

$$\mathrm{Fun}_{\mathcal{T}}^{I-\otimes}\left(\mathcal{D}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}\right) \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{K}, \underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})\right).$$

If additionally, S is I-admissible, then the S-indexed tensor product of $(F_U) \in \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})_S^{\otimes-\mathrm{ptws}}$ has value

$$\begin{tikzcd} \mathcal{D}_V \arrow[r, "\Delta^S"] \arrow[rrr, bend right, "\bigotimes_U^S F_U"'] & \mathcal{D}_S \arrow[r, "(F_U)"] & \mathcal{C}_S \arrow[r, "\bigotimes^S"] & \mathcal{C}_V \end{tikzcd}$$

The following proposition is easy, so we omit its proof.

Proposition 3.20. *There exists a natural equivalence $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}} \simeq \underline{\mathrm{Alg}}^\otimes_{\mathrm{triv}(\mathcal{K})}(\mathcal{C})$.*

Lemma 3.21. *Let $\mathcal{C}$ be a $\mathcal{T}$-symmetric monoidal category, $\mathcal{O}^\otimes$ a $\mathcal{T}$-operad, and $\mathcal{K}$ a $\mathcal{T}$-category.*

1. *There is a natural $\mathcal{T}$-symmetric monoidal functor $U\colon \underline{\mathrm{Alg}}^\otimes_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathrm{Env}(\mathcal{O}),\mathcal{C})^{\otimes-\mathrm{ptws}}$.*
2. *There is a lift of coevaluation of $\mathcal{T}$-functors,*

$$\begin{tikzcd} & \underline{\mathrm{Fun}}^\otimes_G\left(\underline{\mathrm{Fun}}_G(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}},\mathcal{C}\right). \arrow[d, "U"] \\ \mathcal{K} \arrow[ur, "\mathrm{coev}"] \arrow[r, "\mathrm{coev}"'] & \underline{\mathrm{Fun}}_G\left(\underline{\mathrm{Fun}}_G(\mathcal{K},\mathcal{C}),\mathcal{C}\right). \end{tikzcd}$$

natural in the sense that varying $\mathcal{K}$ forms a $\mathcal{T}$-functor $\underline{\mathrm{Cat}}_{\mathcal{T}} \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathrm{Infl}_e^{\mathcal{T}}\partial\Delta^1, \underline{\mathrm{Cat}}_{\mathcal{T}})$.

Proof. (1) is pullback along the $\mathcal{T}$-operad map $\mathrm{triv}(\mathrm{Env}(\mathcal{O}))^\otimes \to \mathcal{O}^\otimes$ adjunct to the structure functor $\mathrm{Env}(\mathcal{O}) \to \mathcal{O}$. For (2), note that $\mathrm{triv}(-)^\otimes\colon \underline{\mathrm{Cat}}_{\mathcal{T}} \to \underline{\mathrm{Op}}_{\mathcal{T}}$ is left adjoint to U; in particular, the $\mathcal{T}$-colimit of the constant $\mathcal{T}$-functor $\chi_{\mathcal{K}}\colon \mathcal{K} \to \underline{\mathrm{Cat}}_{\mathcal{T}}$ valued at $*$ is $\mathcal{K}$, which yields a $\mathcal{T}$-colimit expression

$$\mathrm{triv}(\mathcal{K})^\otimes \simeq \mathrm{colim}_{V\to\mathcal{K}} \mathrm{triv}_V^\otimes,$$

which on maps out yields a $\mathcal{T}$-limit expression *of $\mathcal{T}$-symmetric monoidal ∞-categories*

$$\underline{\mathrm{Fun}}_G(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}} \simeq \lim_{V\to\mathcal{K}} \mathrm{Res}_V^{\mathcal{T}}\mathcal{C}^\otimes.$$

In particular, we're tasked with constructing a lift

$$\begin{tikzcd} & \underline{\mathrm{colim}}_{\mathcal{K}}\underline{\mathrm{Fun}}^\otimes_G(\mathcal{C}^\otimes,\mathcal{C}^\otimes) \arrow[d, "U"'] \\ \underline{\mathrm{colim}}_{\mathcal{K}} * \arrow[ur, "\mathrm{coev}"] \arrow[r, "\mathrm{coev}"'] & \underline{\mathrm{colim}}_{\mathcal{K}}\underline{\mathrm{Fun}}_G(\mathcal{C}^\otimes,\mathcal{C}^\otimes) \end{tikzcd}$$

by taking a $\mathcal{T}$-colimit, it suffices to define the diagram in the case $\mathcal{K} = *_{\mathcal{T}}$ compatible with restriction. We choose the diagonal and top functor each picking out the identity. □

We use this to construct a G-symmetric monoidal lift for *genuine equivariant factorization homology.*

Corollary 3.22. *Given $\mathcal{C}^\otimes$ a distributive G-symmetric monoidal ∞-category, genuine equivariant factorization homology assembles to a G-functor*

$$\int\colon \underline{\mathrm{Mfld}}^{V-fr,G} \to \underline{\mathrm{Fun}}^\otimes_G\left(\underline{\mathrm{Alg}}^\otimes_{\mathbb{E}_V}(\mathcal{C}),\mathcal{C}\right),$$

natural with respect to G-sifted G-colimit preserving G-symmetric monoidal functors in $\mathcal{C}$. In particular, evaluating at M yields a commutative diagram of G-symmetric monoidal functors

$$\begin{tikzcd} \underline{\mathrm{CAlg}}^\otimes_{AV}(\mathcal{C}) \arrow[r, "\int_M"] \arrow[d, "U"] & \underline{\mathrm{CAlg}}^\otimes_{AV}(\mathcal{C}) \arrow[d, "U"] \\ \underline{\mathrm{Alg}}^\otimes_{\mathbb{E}_V}(\mathcal{C}) \arrow[r, "\int_M"] & \mathcal{C}^\otimes \end{tikzcd}$$

Proof. In the notation of [Hor19], let $\iota^{\otimes} : \underline{\mathrm{Disk}}^{G,V-fr,\sqcup} \to \underline{\mathrm{Mfld}}^{G,V-fr,\sqcup}$ be the G-symmetric monoidal inclusion of V-framed G-disks into V-framed G-manifolds. By [Hor19, Prop. 4.1.4], $\int_M$ may be presented as the G-value of a composition

$$\int_M : \underline{\mathrm{Alg}}_{\mathbb{E}_V}(\mathcal{C}) \simeq \underline{\mathrm{Fun}}_G^{\otimes}\left(\underline{\mathrm{Disk}}^{G,V-fr}, \mathcal{C}\right) \xrightarrow{U} \underline{\mathrm{Fun}}_G\left(\underline{\mathrm{Disk}}^{G,V-fr}, \mathcal{C}\right) \xrightarrow{\iota_!} \underline{\mathrm{Fun}}_G\left(\underline{\mathrm{Mfld}}^{G,V-fr}, \mathcal{C}\right) \xrightarrow{\mathrm{ev}_M} \mathcal{C}.$$

To construct the desired G-functor $\int_M$, it suffices to construct G-symmetric monoidal lifts of U and $\iota_!$; then, we have a G-symmetric monoidal functor

$$\begin{tikzcd}
\underline{\mathrm{Mfld}}^{V-fr,G} \arrow[rr, "\int"] \arrow[d, "\mathrm{coev}"] & & \underline{\mathrm{Fun}}_G^{\otimes}\left(\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C}), \mathcal{C}\right) \arrow[d, phantom, "\wr"] \\
\underline{\mathrm{Fun}}_G^{\otimes}\left(\underline{\mathrm{Fun}}_G\left(\underline{\mathrm{Mfld}}^{G,V-fr}, \mathcal{C}\right), \mathcal{C}\right) \arrow[r, "U^*"] & \underline{\mathrm{Fun}}_G^{\otimes}\left(\underline{\mathrm{Fun}}_G\left(\underline{\mathrm{Disk}}^{G,V-fr}, \mathcal{C}\right), \mathcal{C}\right) \arrow[r, "(\iota_!)^*"] & \underline{\mathrm{Fun}}_G^{\otimes}\left(\underline{\mathrm{Fun}}_G^{\otimes}\left(\underline{\mathrm{Disk}}^{G,V-fr}, \mathcal{C}\right), \mathcal{C}\right)
\end{tikzcd}$$

whose value on M is $\int_M$. Indeed, U is given by Lemma 3.21.

For $\iota_!$, we use the G-symmetric monoidality of G-operadic left Kan extension argued in Corollary D.3, noting that G-siftedness of the relevant slice G-category $\underline{\mathrm{Disk}}^{G,V-fr}_{/M}$ follows from [Hor19, Lem. 5.2.7]. □

3.5.2. *Multiplication and norms on* THR. We specialize Corollary 3.22 to $(V, M) = (\sigma, S^{\sigma})$.

Corollary 3.23. *Real topological Hochschild homology lifts to a C_2-symmetric monoidal functor*

$$\mathrm{THR} : \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_\sigma}(\mathcal{C}) \to \mathcal{C};$$

/ in particular, THR *lifts to a C_2-symmetric monoidal endofunctor*

$$\mathrm{THR} : \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{V+\infty\sigma}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{V+\infty\sigma}}(\mathcal{C}).$$

Given $A \in \mathrm{Alg}_{\mathbb{E}_{V+\infty\sigma}}(\mathcal{C})$, there is an equivalence

$$\mathrm{THR}(A) \simeq \underline{\mathrm{colim}}_{S^{\sigma}} A,$$

with colimit taken in $\mathrm{Alg}_{\mathbb{E}_{V+\infty\sigma}}(\mathcal{C})$, naturally in A.

Proof. The last sentence is the only part which does not follow immediately from combining Horev's facorization homology formula [Hor19, Rmk 7.1.2] with Corollaries 3.17 and 3.22. It suffices to show the colimit property for $\mathcal{O} \simeq \mathcal{O} \otimes \mathbb{E}_\sigma$-algebras whenever $\mathrm{Bor}^{\mathcal{T}}_{A\sigma}\mathcal{O}^{\otimes} \simeq \mathcal{N}^{\otimes}_{A\sigma\infty}$, which holds for $\mathbb{E}_{V+\infty\sigma}$ by Proposition 3.13. In any case, naturality of the dihedral bar construction together with the the Wirthmüller maps of Construction 1.63 yields a diagram

$$\begin{tikzcd}
\vdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A^{\sqcup \mu_3} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & A^{\sqcup \mu_2} \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d] & A \arrow[r] \arrow[d] & \mathrm{colim}_{S^{\sigma}} A \arrow[d, "\varphi"] \\
\vdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A^{\otimes \mu_3} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A^{\otimes \mu_2} \arrow[r, shift left=1] \arrow[r, shift right=1] & A \arrow[r] & \mathrm{THR}(A);
\end{tikzcd}$$

where μ_n is the n-element "dihedral" C_2-set, i.e. the unique σ-admissible C_2-set of size n, and each row is a geometric realization diagram. When the domain category is $A\sigma$-semiadditive, the vertical maps between the bar constructions are equivalences, so φ is an equivalence. The result then follows by $A\sigma$-semiadditivity of $\mathcal{O}$-algebras, as in Theorem F'. □

Remark 3.24. The computation $\mathrm{THR}(A) = \mathrm{colim}_{S^{\sigma}} A$ when A is pulled back from a C_2-commutative algebra is not new; indeed, it appears as [QS19, Rmk 5.4]. In fact, the ambiguity induced by the potential discrepancy between our construction $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ and that of [NS22, Thm 5.3.4] vanishes for the I-symmetric monoidal structure on $\mathrm{CAlg}_I(\mathcal{C})$ by applying Theorem A' in view of the fact that each are cocartesian [NS22, Thm 5.3.9]. The new element of this identification is that the operation on C_2-commutative algebras is induced canonically from the operation on $\mathbb{E}_\sigma$-algebras and that the colimit formulas need only an $\mathbb{E}_{\infty\sigma}$-algebra structure. ◃

Remark 3.25. In the above, we only needed $\mathcal{C}$ to be an $\mathbb{E}_\sigma$-monoidal ∞-category; however, to easily understand $\mathcal{O}$-algebras, one ought to assume that $\mathcal{C}$ is an $\mathcal{O}$-monoidal ∞.category. ◃

Now, we can construct a circle action on THR.

Construction 3.26. Define $\underline{\mathrm{Diff}}^{V-fr}(M) \subset \underline{\mathrm{Emb}}^{V-fr}(M,M)$ to be the topological subspace of diffeomorphisms and embeddings of M with conjugation G-action, considered as a (grouplike) $\mathbb{E}_1$-G-space. Precomposing the functoriality of Corollary 3.22 along the action $B\underline{\mathrm{Diff}}^{V-fr} \to B\underline{\mathrm{Aut}}_{\underline{\mathrm{Mfld}}^{V-fr}}(M) \subset \underline{\mathrm{Mfld}}^{G,V-fr}$ yields a $\underline{\mathrm{Diff}}^{V-fr}(M)$-action on $\int_M(-)$ through G-symmetric monoidal natural transformations, where $B\underline{\mathrm{Diff}}^{V-fr}(M)$ is the unique connected G-space with $\Omega B\underline{\mathrm{Diff}}^{V-fr}(M) \simeq \underline{\mathrm{Diff}}^{V-fr}(M)$ as $\mathbb{E}_1$-G-spaces (see [HA, § 5.2.6]). In particular, this yields a natural lift

$$\begin{tikzcd} & \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})^{B\underline{\mathrm{Diff}}^{V-fr}(M)} \arrow[d, "U"] \\ \underline{\mathrm{Alg}}_{\mathcal{O}\otimes\mathbb{E}_V}(\mathcal{C}) \arrow[ur, dashed] \arrow[r, "\int_M"'] & \mathcal{C}. \end{tikzcd}$$

Applying the left-action $S^\sigma \to \underline{\mathrm{Diff}}^{\sigma-fr}(S^\sigma)$ yields a lift

$$\begin{tikzcd} & \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})^{BS^\sigma} \arrow[d, "U"] \\ \underline{\mathrm{Alg}}_{\mathcal{O}\otimes\mathbb{E}_\sigma}(\mathcal{C}) \arrow[ur, dashed] \arrow[r, "\mathrm{THR}"'] & \mathcal{C} \end{tikzcd}$$

We refer to this as the S^σ *action on* THR. ◁

In fact, this type of action has been seen in previous work.

Remark 3.27. For $\psi\colon G \to G/N$ a surjective topological group homomorphism, the N*-free* G*-family* is

$$B^\psi_{G/N}N = \left\{[G/H] \;\middle|\; \mathrm{Res}^G_N([G/H]) \text{ is a free } N\text{-set}\right\} \subset \mathcal{O}_G^{\mathrm{op}};$$

[QS19, Lem 4.14] recognizes the "quotient by N" forgetful functor

$$B^\psi_{G/N}N \to \mathcal{O}^{\mathrm{op}}_{G/N},$$

as the unstraightening of a G/N-space, which we also refer to as $B^\psi_{G/N}N$. In particular, for an arbitrary Abelian group A, they define $B^t_{C_2}A$ with respect to the semidirect product extension $A \to A \rtimes C_2 \to C_2$ for the C_2-action on A by inversion. In the case that $A = \mathbb{T}$ is the circle group, this is given by the usual extension $\mathbb{T} = \mathrm{SO}(2) \to O(2) \to C_2$.

We may explicitly identify $B^t_{C_2}\mathbb{T}$ by hand; it's well-known that there are exactly two conjugacy classes of subgroups $(H) \subset O(2)$ whose homogeneous spaces $[O(2)/H]$ are $\mathbb{T}$-free: the reflections and the trivial subgroup. The normalizer of a subgroup generated by a reflection is a dihedral group of order 4, so

$$\mathrm{Aut}_{O(2)}([O(2)/C_2]) \simeq W_{O(2)}(C_2) \simeq D_4/C_2 \simeq C_2.$$

In particular, we may picture the unstraightening of $B^t_{C_2}\mathbb{T}$ via the diagram

$$\begin{tikzcd} B^t_{C_2}\mathbb{T} \arrow[d] \arrow[r, leftrightsquigarrow, "\text{Unstraightening}"] & {[O(2)/C_2]} \arrow[loop left, "C_2"] \arrow[d, Rightarrow] & {[O(2)/e]} \arrow[l] \arrow[loop right, "O(2)"] \\ *_{C_2} \arrow[r, leftrightsquigarrow] & {[C_2/C_2]} & {[C_2/e]} \arrow[l] \arrow[loop right, "C_2"] \end{tikzcd}$$

In particular, taking fibers, we find that $\left(\Omega B^t_{C_2}\mathbb{T}\right)^e = \mathbb{T}$, $\left(\Omega B^t_{C_2}S^1\right)^{C_2} = C_2$, and the restriction map $C_2 \to \mathbb{T}$ is the unique nontrivial such homomorphism. This evidently agrees with the circle group structure on S^σ, inducing an equivalence $BS^\sigma \simeq B^t_{C_2}\mathbb{T}$, so we really have acquired a natural lift

$$\begin{tikzcd} & \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})^{h_{C_2}\mathbb{T}} \arrow[d, "U"] \\ \underline{\mathrm{Alg}}_{\mathcal{O}\otimes\mathbb{E}_\sigma}(\mathcal{C}) \arrow[ur, dashed] \arrow[r, "\mathrm{THR}"'] & \mathcal{C} \end{tikzcd}$$

where we write $\mathcal{C}^{h_{C_2}\mathbb{T}} \simeq \underline{\mathrm{Fun}}_{C_2}\left(B^t_{C_2}\mathbb{T}, \mathcal{C}\right)$. ◁

The following reformulation of the equivalence $\mathrm{THR}(A) \simeq \mathrm{colim}_{S^\sigma} A$ may be familiar.

Observation 3.28. Let Γ be a grouplike $\mathbb{E}_1$-$\mathcal{T}$-space. Evaluation and left Kan extension yield an adjunction

$$-\otimes\Gamma\colon \Gamma^{\mathcal{T}}\mathcal{C} \rightleftarrows \mathrm{Fun}_{\mathcal{T}}(B\Gamma, \mathcal{C})\colon U;$$

unwinding the left Kan extension formula along $*_{\mathcal{T}} \to B\Gamma$ shows that the $\mathcal{T}$-object underlying $-\otimes\Gamma$ is the constant $\Gamma = \Omega B\Gamma$-indexed $\mathcal{T}$-colimit functor; in particular, Corollary 3.23 understands $\mathrm{THR}(A)$ to be the free $\mathbb{E}_{V+\infty\sigma}$-object on A with C_2-equivariant S^σ-action. This free action agrees with that of Construction 3.26, and in particular this identifies our action with that of [QS22a, § 5]. ◁

3.5.3. *Multiplication and norms on* TCR. Having produced a C_2-symmetric monoidal construction

$$\mathrm{THR}\colon \mathrm{Alg}_{\mathbb{E}_\sigma}\left(\mathrm{Sp}_{C_2}\right) \to \underline{\mathrm{Sp}}^{\otimes}_{B_{C_2}\mathbb{T}} \coloneqq \underline{\mathrm{Fun}}_{C_2}\left(B^t_{C_2}\mathbb{T}, \underline{\mathrm{Sp}}^{\otimes}_{C_2}\right)^{\otimes-\mathrm{ptws}}$$

which lifts to Quigley-Shah's construction, we're poised to become the "future work" indicated in [QS19, Warning 0.12] by constructing a lax symmetric monoidal p-typical (Borel) Real topological cyclic homology functor which lifts to Quigley-Shah's construction. Now, consider the C_2-space map $i\colon B^t_{C_2}\mu_{p^\infty} \to B^t_{C_2}\mathbb{T}$ defined by the following colimit

$$\begin{tikzcd} B^t_{C_2}e \arrow[r, hook] \arrow[drrrrr, bend right] & \cdots \arrow[r, hook] & B^t_{C_2}\mu_{p^n} \arrow[r, hook] \arrow[drrr, bend right] & B^t_{C_2}\mu_{p^{n+1}} \arrow[r, hook] \arrow[drr, bend right] & \cdots \arrow[r, hook] & B^t_{C_2}\mu_{p^\infty} \arrow[d, "i"] \\ & & & & & B^t_{C_2}\mathbb{T} \end{tikzcd}$$

Postcomposing with i^* yields a C_2-symmetric monoidal functor $\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_\sigma}(\mathrm{Sp}_{C_2}) \to \underline{\mathrm{Sp}}^{\otimes}_{B_{C_2}\mu_{p^\infty}}$.

Following Nikoulaus and Scholze [NS18, Cons IV.2.1] Quigley-Shah defined the *C_2-operad of (Borel) Real p-cyclotomic spectra* by the pullback C_2-operad

$$\begin{tikzcd} \underline{\mathbb{R}\mathrm{CycSp}}^{\otimes}_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Fun}}_{C_2}\left(\mathrm{Infl}^{C_2}_e\Delta^1, \underline{\mathrm{Sp}}^{\otimes}_{C_2}\right)^{\otimes-\mathrm{ptws}} \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ \underline{\mathrm{Sp}}^{\otimes}_{B^t_{C_2}\mu_{p^\infty}} \arrow[r, "\left(U, (-)^{t_{C_2}\mu_p}\right)"'] & \underline{\mathrm{Sp}}^{\otimes}_{C_2} \times \underline{\mathrm{Sp}}^{\otimes}_{C_2} \end{tikzcd}$$

where $(-)^{t_{C_2}\mu_p}$ has the lax C_2-symmetric monoidal structure of [QS19, Setup 2.1]; this is a *lax equalizer of C_2-operads*. Pullback-stability of cocartesian fibrations guarantees that $\underline{\mathbb{R}\mathrm{CycSp}}^{\otimes} \to \underline{\mathrm{Sp}}^{\otimes}_{C_2}$ is a cocartesian fibration of C_2-operads, and in particular, $\underline{\mathbb{R}\mathrm{CycSp}}^{\otimes}$ is a C_2-symmetric monoidal ∞-category.

In [QS22a, Const 5.5], Quigley-Shah define a lax C_2-symmetric monoidally natural *dihedral Tate diagonal* functor $\Delta_p\colon \underline{\mathrm{Sp}}^{\otimes}_{C_2} \to \underline{\mathrm{Fun}}_{C_2}\left(\mathrm{Infl}^{C_2}_e\Delta^1, \underline{\mathrm{Sp}}^{\otimes}_{C_2}\right)^{\otimes-\mathrm{ptws}}$ whose composites are $\mathrm{ev}_1 \circ \Delta_p A \sim \left(A^{\otimes\mu_p}\right)^{t_{C_2}\mu_p}$ and $\mathrm{ev}_0 \sim \mathrm{id}$.

Remark 3.29. Note that the full subcategory $\mathbb{F}^\sigma_{C_2} = \{\mu_n \mid n \in \mathbb{N}\} \subset \mathbb{F}_{C_2}$ of σ-admissible C_2-sets participates in a weak indexing system with $\mathbb{F}^\sigma_e = \mathbb{F}$; in particular these together are closed under indexed coproducts. One can easily identify the underlying set of output by the *double coset formula* for indexed coproducts, which in this case simply shows that $\left|\coprod^S_U X_U\right| = \sum_{U\in\mathrm{Orb}(S)} |U|\cdot|X_U|$; since $\mathrm{Res}^{C_2}_e\colon \mathbb{F}^\sigma_{C_2} \to \mathbb{F}^\sigma_e = \mathbb{F}$ is injective, the dihedral Tate diagonal of an indexed tensor power has signature $X^{\otimes\mu_n} \longrightarrow \left((X^{\otimes\mu_n})^{\otimes\mu_p}\right)^{t_{C_2}\mu_p} \simeq \left(X^{\otimes\mu_{np}}\right)^{t_{C_2}\mu_p}$. ◁

Construction 3.30 (Dihedral Frobenius)**.** Naturality of the dihedral Tate diagonal, commutativity of C_2-colimits with C_2-colimits, and the coassembly map for C_2-limits yields a natural diagram

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X^{\otimes\mu_3} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\Delta_p(X^{\otimes 3})"] & X^{\otimes\mu_2} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\Delta_p(X^{\otimes 2})"] & X \arrow[r] \arrow[d, "\Delta_p(X)"] & \mathrm{THR}(X) \arrow[d] \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \left(X^{\otimes\mu_{3p}}\right)^{t_{C_2} mu_p} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \left(X^{\otimes\mu_{2p}}\right)^{t_{C_2}\mu_p} \arrow[r, shift left] \arrow[r, shift right] & \left(X^{\otimes\mu_p}\right)^{t_{C_2}\mu_p} \arrow[r] & \operatorname{colim}\left(X^{\otimes\mu_p}\right)^{t_{C_2}\mu_p} \arrow[d] \\
\Big(\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X^{\otimes\mu_{3p}} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & X^{\otimes\mu_{2p}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X^{\otimes\mu_p}\Big)^{t_{C_2}\mu_p} \arrow[r] \arrow[d] & \left(\operatorname{colim}\left(X^{\otimes\mu_p}\right)\right)^{t_{C_2}\mu_p} \arrow[d] \\
\Big(\cdots \underset{\rightarrow}{\overset{\rightarrow}{\vdots}} & X^{\otimes\mu_{3p}} \underset{\rightarrow}{\overset{\rightarrow}{\vdots}} \cdots \underset{\rightarrow}{\overset{\rightarrow}{\vdots}} & X^{\otimes\mu_{2p}} \underset{\rightarrow}{\overset{\rightarrow}{\vdots}} \cdots \underset{\rightarrow}{\overset{\rightarrow}{\vdots}} & X^{\otimes\mu_p} \underset{\rightarrow}{\overset{\rightarrow}{\vdots}} \cdots \rightrightarrows X\Big)^{t_{C_2}\mu_p} \arrow[r] & \mathrm{THR}(X)^{t_{C_2}\mu_p}
\end{tikzcd}$$

We acquire a (lax C_2-symmetric monoidally-) natural composite map $\varphi_p \colon \mathrm{THR}(X) \to \mathrm{THR}(X)^{t_{C_2}\mu_p}$, yielding a lax C_2-symmetric monoidal functor

$$\begin{tikzcd}
\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_\sigma}\left(\underline{\mathrm{Sp}}_{C_2}\right) \arrow[drr, bend left, "\varphi_p"] \arrow[dr, dashed, "\widetilde{\mathrm{THR}}"] \arrow[ddr, bend right, "\mathrm{THR}"'] & & \\
& \underline{\mathbb{R}\mathrm{CycSp}}^{\otimes}_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Fun}}_{C_2}\left(\mathrm{Infl}_e^{C_2}\Delta^1, \underline{\mathrm{Sp}}^{\otimes}_{C_2}\right)^{\otimes-\mathrm{ptws}} \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
& \underline{\mathrm{Sp}}^{\otimes}_{B^t_{C_2}\mu_{p^\infty}} \arrow[r, "\left(U,(-)^{t_{C_2}\mu_p}\right)"'] & \underline{\mathrm{Sp}}^{\otimes}_{C_2} \times \underline{\mathrm{Sp}}^{\otimes}_{C_2}
\end{tikzcd}$$

◁

Now, in [QS19, § 2.3] they also define a lax C_2-symmetric monoidal functor $\mathrm{TCR}(-,p) \colon \underline{\mathbb{R}\mathrm{CycSp}}^{\otimes}_p \to \underline{\mathrm{Sp}}^{\otimes}_{C_2}$. From this, we may conclude the following.

Corollary 3.31. *Given $\mathcal{O}^{\otimes}$ a C_2-operad, there is a commutative diagram of lax C_2-symmetric monoidal functors*

$$\begin{tikzcd}
\underline{\mathrm{CAlg}}^{\otimes}_{C_2}\left(\underline{\mathrm{Sp}}_{C_2}\right) \arrow[r, "\mathrm{THR}"] \arrow[d] & \underline{\mathrm{CAlg}}^{\otimes}_{C_2}\left(\underline{\mathbb{R}\mathrm{CycSp}}_p\right) \arrow[r, "\mathrm{TCR}(-{,}p)"] \arrow[d] & \underline{\mathrm{CAlg}}_{C_2}\left(\underline{\mathrm{Sp}}_{C_2}\right) \arrow[d] \\
\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_\sigma\otimes\mathcal{O}}\left(\underline{\mathrm{Sp}}_{C_2}\right) \arrow[r, dashed] \arrow[d] & \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}\left(\underline{\mathbb{R}\mathrm{CycSp}}_p\right) \arrow[r] \arrow[d] & \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}\left(\underline{\mathrm{Sp}}_{C_2}\right) \arrow[d] \\
\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_\sigma}\left(\underline{\mathrm{Sp}}_{C_2}\right) \arrow[r, dashed] \arrow[dr, "\mathrm{THR}"'] & \underline{\mathbb{R}\mathrm{CycSp}}^{\otimes}_p \arrow[r, "\mathrm{TCR}(-{,}p)"] \arrow[d] & \underline{\mathrm{Sp}}^{\otimes}_{C_2} \\
& \underline{\mathrm{Sp}}^{\otimes}_{C_2} &
\end{tikzcd}$$

whose top row recovers the constructions of [QS19]. In particular, if I is a C_2-weak indexing system containing the dihedral C_2-sets, then Quigley-Shah's p-typical Real topological cyclic homology is lifted from a lax C_2-symmetric monoidal endofunctor of I-commutative ring spectra.

Remark 3.32. The conditions of Corollaries 3.23 and 3.31 were varied for the sake of diversity of examples; they can be interchanged, and each can be weakened to construct (lax) C_2-symmetric monoidal endofunctors of $\mathcal{O}$-algebras whenever $\mathrm{Bor}_{A\sigma}^{C_2}\mathcal{O}^{\otimes} \simeq \mathbb{E}^{\otimes}_{\sigma\infty} \simeq \mathcal{N}^{\otimes}_{A\sigma\infty}$. ◁

Remark 3.33. For simplicity, we made the p-*typical* construction; however, Real cyclotomic spectra and their Real topological cyclic homology were constructed integrally in [QS22a], and we may lift our constructions to a lax C_2-symmetric monoidal Real cyclotomic structure as follows.

$$\begin{tikzcd}
\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_\sigma}\left(\underline{\mathrm{Sp}}_{C_2}\right) \arrow[dr, dashed, "\widetilde{\mathrm{THR}}"] \arrow[ddr, bend right, "\mathrm{THR}"'] \arrow[drr, bend left, "(\varphi_p)"] & & \\
& \underline{\mathbb{R}\mathrm{CycSp}}^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod\limits_{p \text{ prime}} \underline{\mathrm{Fun}}_{C_2}\left(\mathrm{Infl}_e^{C_2}\Delta^1, \underline{\mathrm{Sp}}^{\otimes}_{B^t_{C_2}\mathbb{T}}\right)^{\otimes-\mathrm{ptws}} \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
& \underline{\mathrm{Sp}}^{\otimes}_{B^t_{C_2}\mathbb{T}} \arrow[r, "\left(U, (-)^{tC_2\mu_p}\right)_p"'] & \prod\limits_{p \text{ prime}} \underline{\mathrm{Sp}}_{B^t_{C_2}\mathbb{T}} \times \underline{\mathrm{Sp}}_{B^t_{C_2}\mathbb{T}}
\end{tikzcd}$$

The lax C_2-symmetric monoidal *Real topological Cyclic homology* functor may be defined to be the composite

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_\sigma}\left(\underline{\mathrm{Sp}}_{C_2}\right) \xrightarrow{\widetilde{\mathrm{THR}}} \underline{\mathbb{R}\mathrm{CycSp}}^{\otimes} \xrightarrow{\mathrm{TCR}} \underline{\mathrm{Sp}}^{\otimes}_{C_2}. \qquad \triangleleft$$

3.5.4. *Speculations on genuine equivariant* THH *for other groups.* Merling posed the following question.

Question 3.34 ([AimPL, Prob 1.6])**.** Is it possible to build a version of THH for G-ring spectra which is G-symmetric monoidal? $\triangleleft$

On its face, Question 3.34 receives a positive answer by setting $\mathcal{C} = \underline{\mathrm{Sp}}^{\otimes}_G$ in the following.

Corollary 3.35. *Given $\mathcal{C}^{\otimes}$ a G-symmetric monoidal ∞-category, there is a G-symmetric monoidal functor*

$$\int_{S^1} : \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_1}(\mathcal{C}) \to \mathcal{C}^{\otimes}$$

whose H-value functor $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}_H) \to \mathcal{C}_H$ is THH.

Proof. Specialize Corollary 3.22 to $M = S^1$. □

This is possibly unsatisfying; it certainly recovers the notion of [MQS24], but it recovers neither twisted nor Real THH. We can give a more general result recovering the latter as follows.

Corollary 3.36. *Let $\mathcal{C}^{\otimes}$ be a G-symmetric monoidal ∞-category, V be an orthogonal G-representation, and $S \subset V$ an embedded G-set. Then, there exists a G-symmetric monoidal functor*

$$\mathrm{THH}_{S,V} : \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C}) \to \mathcal{C}^{\otimes},$$

natural in G-symmetric monoidal strongly G-colimit preserving G-functors in $\mathcal{C}$, satisfying

$$\mathrm{THH}_{S,V}(A) \simeq A \otimes_{A^{\otimes S}} A.$$

Proof. Define the invariant closed topological subspace

$$S^{\succ(S,V)} := \mathbb{R}S \cup \{\infty\} \subset S^V.$$

Giving S^V the standard metric, pick some small $\varepsilon > 0$ and let $\tau S^{\succ(S,V)} \subset S^V$ be the open ball around $S^{\succ(S,V)}$ of radius ε; this is canonically a V-framed open G-submanifold of S^V. We define

$$\mathrm{THH}_{S,V}(A) := \int_{\tau S^{\succ(S,V)}} A : \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C}) \to \mathcal{C}^{\otimes}.$$

We're left with verifying the tensor product formula, which itself will be an application of Horev's G-$\otimes$-excision result [Hor19, Prop 5.2.3]. We need a collar decomposition; define the function $f : S^V \to [-1,1]$ by

$$f(v) = \begin{cases} -1 & |v| < 2\varepsilon; \\ \frac{|V| - 3\varepsilon}{\varepsilon} & 2\varepsilon \le |v| \le 4\varepsilon; \\ 1 & |v| > 4\varepsilon. \end{cases}$$

The restriction of this to $\tau S^{\succ(S,V)}$ is a collar decomposition with positive and negative part framed-diffeomorphic to $D(V)$ and with interior V-framed diffeomorphic to the indexed disjoint union $S \cdot D(V)$. $\otimes$-excision yields

$$\begin{aligned}\mathrm{THH}_{S,V}(A) &\simeq \int_{\tau S_+^{\succ(S,V)}} A \otimes_{\int_{\tau S_+^{\succ(S,V)} \cap \tau S^{\succ(S,V)}} A} \int_{\tau S_-^{\succ(S,V)}} A; \\ &\simeq \int_{D(V)} A \otimes_{\int_{S\cdot D(V)} A} \int_{D(V)^{\mathrm{op}}} A \\ &\simeq A \otimes_{A^{\otimes S}} A^{\mathrm{op}}. \end{aligned}$$

$\square$

Appendix A. Technicalities on (co)cartesian I-symmetric monoidal ∞-categories

For the duration of this appendix, we assume the notation of [Ste25, § A]. Fix $\mathcal{P} \subset \mathcal{T}$ an atomic orbital subcategory and $I \subset \mathbb{F}_{\mathcal{T}}^{\mathcal{P}}$ an almost-unital weak indexing category. We define the ∞*-category of* Γ*-*I*-preoperads*

$$\mathrm{PreOp}_I^{\Gamma} \coloneqq \mathrm{Cat}^{\mathrm{int-cocart}}_{\mathcal{T},/\underline{\mathbb{F}}_{I,*}},$$

so that the results of [BHS22] recognize $\mathrm{Op}_I \subset \mathrm{PreOp}_I^{\Gamma}$ as a localizing subcategory.[7]

This appendix can be understood as a lift of [HA, § 2.4.1-2.4.3] to the setting of (co)cartesian I-symmetric monoidal ∞-categories, working in the specific model of Γ-I-preoperads; we proceed by an essentially similar strategy, complicated only by less convenient combinatorics. We suggest only readers in need of the minutiae inquire within, and the shunt remaining readers to the summaries contained in Section 1.4.

First, define the $\mathcal{T}$-1-category $\underline{\Gamma}_I^*$ to have V-values

$$\Gamma_{I,V}^* \coloneqq \left\{ U_+ \xrightarrow{s.i.} S_+ \;\middle|\; U \in \mathcal{T}_{/V} \right\} \subset \mathrm{Ar}(\underline{\mathbb{F}}_{I,*})_V;$$

that is, the objects of $\Gamma_{I,V}^*$ are pointed I-admissible V-sets with a distinguished orbit, and the morphisms of $\Gamma_{I,V}$ preserve distinguished orbits. This possesses a *target* forgetful $\mathcal{T}$-functor $t\colon \underline{\Gamma}_I^* \to \underline{\mathbb{F}}_{I,*}$. We use this to construct an ∞-category $\mathrm{TotTot}_{\mathcal{T}}\mathcal{C}$ over $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ in Section A.1 satisfying the following universal property.

Proposition A.1. *Given* $\mathcal{C}$ *a* $\mathcal{T}$*-*∞*-category, there exists an* ∞*-category* $\mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup}$ *over* $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ *satisfying the universal property that there is a natural equivalence*

$$\mathrm{Fun}_{/\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}(\mathcal{D}, \mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup}) \simeq \mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}(\mathcal{D} \times_{\mathrm{Tot}\underline{\mathbb{F}}_{I,*}} \mathrm{Tot}\underline{\Gamma}_I^*, \mathrm{Tot}\mathcal{C});$$

Second, define the (non-full) $\mathcal{T}$-subcategory $\underline{\Gamma}_I^{\times} \subset \mathrm{Ar}(\underline{\mathbb{F}}_{I,*})$ to have V-objects given by summand inclusions of pointed V-sets $\overline{S}_+ \hookrightarrow S_+$ and morphisms of V-objects given by maps $\alpha\colon S_+ \to T_+$ with the property that $\alpha^{-1}\left(\overline{T}_+\right) \subset \overline{S}_+$. In Section A.1 we prove the following.

Proposition A.2. *Given* $\mathcal{C}$ *a* $\mathcal{T}$*-*∞*-category, there exists an* ∞*-category* $\mathrm{TotTot}_{\mathcal{T}}\widetilde{\mathcal{C}}^{I-\times}$ *over* $\underline{\mathbb{F}}_{I,*}$ *satisfying the universal property that there is a natural equivalence*

$$\mathrm{Fun}_{/\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}(K, \mathrm{TotTot}_{\mathcal{T}}\widetilde{\mathcal{C}}^{I-\times}) \simeq \mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}(K \times_{\mathrm{Tot}\underline{\mathbb{F}}_{I,*}} \mathrm{Tot}\underline{\Gamma}_I^{\times}, \mathrm{Tot}\mathcal{C}).$$

Note that there is an equivalence

$$\{S_+\} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_I^{\times} \simeq \mathscr{P}_{\underline{V}}(S),$$

where $\mathscr{P}_{\underline{V}}(S)$ is the V-poset with U-value given by subsets of $\mathrm{Res}_U^V S$ ordered under inclusion. In particular, for $S_+ \in \underline{\mathbb{F}}_{I,*}$, we view objects in $\widetilde{\mathcal{C}}^{I-\times}_{S_+}$ as V-functors $\mathscr{P}_{\underline{V}}(S)^{\mathrm{op}} \to \mathcal{C}_V$. Let $\mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\times} \subset \mathrm{TotTot}_{\mathcal{T}}\widetilde{\mathcal{C}}^{I-\times}$ be the full subcategory whose objects over V are spanned by those functors $F\colon \mathscr{P}_{\underline{V}}(S)^{\mathrm{op}} \to \mathcal{C}_{\underline{V}}$ satisfying the property that, for all $U \to V$ and $T \subset \mathrm{Res}_U^V S$, the maps $\mathrm{Res}_W^V F(T) \to F(W)$ exhibit $F(T)$ as the T-indexed product $F(T) \simeq \prod_W^T F(U)$ in $\mathcal{C}$.

Following Section A.1, we construct cocartesian lifts and characterize algebras and I-symmetric monoidal functors into $\mathcal{C}^{I-\sqcup}$ and $\mathcal{C}^{I-\times}$ in Sections A.2 and A.3. We spell out a corollary in Section A.6 relating L_{Op_I}-equivalences to the Morita theory of algebraic patterns.

A.1. Quasicategories modeling $\mathcal{C}^{I-\sqcup}$ and $\mathcal{C}^{I-\times}$.

[7] Here, Γ is a reference to Segal's category Γ, whereas the undecorated version centers the *effective Burnside category*.

A.1.1. *The cocartesian case.* Let $\mathcal{T}^{\mathrm{op}}$ be a quasicategory and $\mathrm{Tot}\mathcal{C} \in \mathrm{sSet}^{\mathrm{cocart}}_{/\mathcal{T}}$ a cocartesian fibration to $\mathcal{T}$. There exists a simplicial set $\mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup}$ satisfying the universal property

$$\mathrm{Hom}_{/\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}(K, \mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup}) \simeq \mathrm{Hom}_{/\mathcal{T}^{\mathrm{op}}}(K \times_{\mathrm{Tot}\underline{\mathbb{F}}_{I,*}} \mathrm{Tot}\underline{\Gamma}^*_I, \mathrm{Tot}\mathcal{C}), \qquad \forall K \in \mathrm{sSet}^{\mathrm{cocart}}_{/\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}. \tag{15}$$

Lemma A.3. *The map* $\mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup} \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ *is an inner fibration; hence* $\mathrm{Tot}\mathcal{C}^{I-\sqcup}$ *is a quasicategory.*

Proof. The proof is exactly analogous to [HA, Prop 2.4.3.3]: apply the universal property

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "f_0"] \arrow[d] & \mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup} \arrow[d] & & \Lambda^n_i \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I \arrow[d] & \simeq & \coprod\limits_{\substack{U \in \mathrm{Orb}(S) \\ f(U) \in S^{\circ}_{n,+}}} \Lambda^n_i \arrow[r] \arrow[d] & \mathrm{Tot}\mathcal{C} \arrow[d] \\
\Delta^n \arrow[r, "(S_{0,+} \to \cdots \to S_{n,+})"'] \arrow[ur, dashed] & \mathrm{Tot}\underline{\mathbb{F}}_{I,*} \arrow[r, phantom, "\leftrightsquigarrow"] & & \Delta^n \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I & \simeq & \coprod\limits_{\substack{U \in \mathrm{Orb}(S) \\ f(U) \in S^{\circ}_{n,+}}} \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{T}^{\mathrm{op}}
\end{tikzcd}$$

to note that inner horn lifts of $\mathrm{Tot}\mathcal{C}^{I-\sqcup} \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ correspond with tuples of inner horn lifts along $\mathrm{Tot}\mathcal{C} \to \mathcal{T}^{\mathrm{op}}$, which exist by assumption that it is a cocartesian fibration (hence an inner fibration). The remaining claim follows by noting that $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ is a quasicategory, so the composite map $\mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup} \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*} \to *$ is an inner fibration. □

Proof of Proposition A.1. We've verified that $\mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup}$ is a quasicategory over $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$. Fixing some quasicategory $\mathcal{D}$ over $\underline{\mathbb{F}}_{I,*}$ and applying Eq. (15) for $K := \mathcal{D} \times \Delta^n$, we find that $\mathrm{Fun}(\mathcal{D}, \mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup}) \simeq \mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}(\mathcal{D} \times_{\mathrm{Tot}\underline{\mathbb{F}}_{I,*}} \mathrm{Tot}\underline{\Gamma}^*_I, \mathrm{Tot}\mathcal{C})$. The result then follows by replacing "quasicategory" with "∞-category." □

A.1.2. *The cartesian case.* Now, we define $\mathrm{TotTot}_{\mathcal{T}}\widetilde{\mathcal{C}}^{I-\times} \in \mathrm{sSet}_{/\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}$ by the universal property

$$\mathrm{Hom}_{/\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}(K, \mathrm{TotTot}_{\mathcal{T}}\widetilde{\mathcal{C}}^{I-\times}) \simeq \mathrm{Hom}_{/\mathcal{T}^{\mathrm{op}}}(K \times_{\mathrm{Tot}\underline{\mathbb{F}}_{I,*}} \mathrm{Tot}\underline{\Gamma}^{\times}_I, \mathrm{Tot}\mathcal{C}), \qquad \forall K \in \mathrm{sSet}^{\mathrm{cocart}}_{/\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}. \tag{16}$$

Recollection A.4 ([NS22, Def 2.1.2])**.** A morphism f in $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ from $S_+ \in \mathbb{F}_{I,*,U}$ to $T_+ \in \mathbb{F}_{I,*,V}$ may be modelled as a morphism of spans

$$\begin{tikzcd}
S \arrow[d] & & f^{-1}(T) \arrow[ll] \arrow[r, "f^{\circ}"] \arrow[dl, hook', "\iota_f"] \arrow[dd] & T \arrow[dd] \\
& \mathrm{Res}^V_U S \arrow[ul] \arrow[dr] & & \\
U & & V \arrow[ll] \arrow[r, equal] & V
\end{tikzcd}$$

such that $f^{\circ} \in I$ (c.f. Construction 1.17). Such a morphism is $\pi_{\underline{\mathbb{F}}_{I,*}}$-cocartesian if f° and ι_f are both equivalences, i.e. it witnesses an equivalence $\mathrm{Res}^V_U S_+ \xrightarrow{\sim} T_+$. ◁

Let $f\colon T_+ \to S_+$ be a map in $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ lying over an orbit map $U \to V$ and let $\overline{S} \subset S$ be an element of $\underline{\Gamma}^{\times}_I$ lying over S_+. We would like to construct a Cartesian edge landing on $\overline{S} \subset S$; we do so by setting $\overline{T} := f^{-1}(\mathrm{Res}^V_U \overline{S}) \subset f^{-1}(\mathrm{Res}^V_U S) \subset T$, and letting the associated map $t\colon \left(f^{-1}(\mathrm{Res}^V_U \overline{S}) \subset T\right) \to \left(\overline{S} \subset S\right)$ be the canonical one. The following lemma then follows by unwinding definitions, where $U\colon \underline{\Gamma}^{\times}_I \to \underline{\mathbb{F}}_{I,*}$ denotes the forgetful functor.

Lemma A.5. *t is a U-cartesian arrow; in particular, U is a cartesian fibration.*

The following lemma then follows from [HTT, Cor 3.2.2.12].

Lemma A.6. *Let* $\widetilde{p}\colon \widetilde{\mathcal{C}}^{I-\times} \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ *be the projection and let* $\tilde{\alpha}\colon F \to G$ *be a* $\widetilde{\mathcal{C}}^{I-\times}$*-morphism lying over a* $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$*-morphism* $\alpha : T_+ \to S_+$ *lying over an orbit map* $U \to V$*. Then,* $\tilde{\alpha}$ *is* $\widetilde{p}$*-cocartesian if and only if, for all* $T' \subset T$*, the induced map*

$$F(\alpha^{-1}(\mathrm{Res}^V_U T')) \to \mathrm{Res}^V_U G(T')$$

is an equivalence; in particular, $\widetilde{p}$ *is a cocartesian fibration, so* $\widetilde{\mathcal{C}}^{I-\times}$ *is a quasicategory.*

Proof of Proposition A.2. We concluded in Lemma A.6 that $\mathrm{TotTot}_{\mathcal{T}}\widetilde{\mathcal{C}}^{I-\times}$ is a quasicategory satisfying Eq. (16), so it models an ∞-category satisfying our universal property by the same argument as Proposition A.1. □

A.2. **$\mathcal{O}$-comonoids and (co)cartesian rigidity.** An object of $\mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup}$ may be viewed as S_+ a pointed V-set and $\mathbf{C} = (C_W) \in \mathcal{C}_S$ an S-tuple of elements of $\mathcal{C}$; a morphism $f\colon \mathbf{C} \to \mathbf{D}$ may be viewed as a $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$-map $(S_+ \to V_{S,+}) \xrightarrow{f} (T_+ \to V_{T,+})$ together with a collection of maps

$$\left\{ f_W\colon \mathrm{Ind}_W^U C_W \to D_U \mid W \in f^{-1}(U) \right\}$$

for all $U \in \mathrm{Orb}(T)$. Unwinding the universal property for cocartesian arrows, we find the following.[8]

Proposition A.7. *A map $f\colon \mathbf{C} \to \mathbf{D}$ is π-cocartesian if and only if $\{f_W\}$ witness D_U as the indexed coproduct*

$$\coprod_W^{f^{-1}(U)} C_W \xrightarrow{\sim} D_U$$

for all $U \in \mathrm{Orb}(T)$. In particular, f is inert if and only if the following conditions are satisfied:

(a) *The projected morphism $\pi(f)\colon S \to T$ is inert.*
(b) *The associated map $C_{f^{-1}(U)} \to D_U$ is an equivalence for all $U \in \mathrm{Orb}(T)$.*

Hence $\pi\colon \mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup} \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ presents a Γ-I-preoperad $\mathcal{C}^{I-\sqcup}$.

Corollary A.8. *$\mathcal{C}^{I-\sqcup}$ is an I-operad which is an I-symmetric monoidal ∞-category if and only if $\mathcal{C}$ admits I-indexed coproducts.*

Proof. It follows from Proposition A.7 that $\mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup} \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ is a cocartesian fibration if and only if $\mathcal{C}$ admits I-indexed coproducts, so it suffices to verify the following conditions:

(b) cocartesian transport yields an equivalence

$$\mathcal{C}_S \simeq \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_U;$$

(c) cocartesian transport yields an equivalence

$$\mathrm{Map}^{T \to S}_{\mathrm{Tot}\mathcal{O}^{\otimes}}(\mathbf{C}, \mathbf{D}) \simeq \prod_{U \in \mathrm{Orb}(S)} \mathrm{Map}^{T_U \to U}_{\mathrm{Tot}\mathcal{O}^{\otimes}}\left(\mathbf{C}_{\underline{U}}, D\right).$$

In fact, each condition follows from Proposition A.7. □

Observation A.9. It follows from Proposition A.7 that the indexed tensor product functor $\otimes^S\colon \mathcal{C}_S^{I-\sqcup} \to \mathcal{C}_V^{I-\sqcup}$ is left adjoint to Δ^S, i.e. indexed tensor products in $\mathcal{C}^{I-\sqcup}$ are indexed coproducts. ◁

Given $\mathcal{O}^{\otimes}$ a unital I-operad, define a diagram of Cartesian squares in $\mathrm{Cat}_{\mathcal{T}}$.

$$\begin{tikzcd}
\mathcal{O} \arrow[r, hook, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes}_{\Gamma} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes} \arrow[d] \\
*_{\mathcal{T}} \arrow[r, hook] & \underline{\Gamma}^*_I \arrow[r] & \underline{\mathbb{F}}_{I,*}
\end{tikzcd}$$

Note that the objects of $\mathcal{O}^{\otimes}_{\Gamma,V}$ consist of triples (S_+, U, X) where $U \in \mathrm{Orb}(S)$ and $X \in \mathcal{O}_S$, and the image of ι is equivalent to the triples where $S \in \mathcal{T}_{/V}$, hence $U = S$.

Further note that cocartesian transport along the inert morphism $U_+ \hookrightarrow S_+$ induces an equivalence

$$\mathrm{Map}_{\mathcal{O}^{\otimes}_{\Gamma,V}}(\iota Y, (S_+, U, X)) \simeq \mathrm{Map}_{\mathcal{O}^{\otimes}_{\Gamma,V}}(\iota Y, (U_+, U, X_U))$$

for all $Y \in \mathcal{O}$.[9] In particular, ι witnesses $\mathcal{O}$ as a *colocalizing* $\mathcal{T}$-subcategory, with colocalization $\mathcal{T}$-functor

$$R(S_+, U, X) \simeq (U_+, U, X_U).$$

This interacts with Kan extensions via the following lemmas.

[8] It is here that we use almost-unitality for the cocartesian setting; if I was not almost essentially unital, then there would exist some S whose I-admissible orbits do not together cover S, so $\mathcal{C}^{I-\sqcup} \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ would not be an inert-cocartesian fibration.

[9] This utilizes unitality of $\mathcal{O}^{\otimes}$, as we implicitly use that, for each orbit $U' \in \mathrm{Orb}(S)$ other than U, the space $\mathcal{O}(\varnothing_{U'}; X_{U'})$ is contractible.

Lemma A.10. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{T}$-functor and $L\colon \mathcal{C} \to \mathcal{E}$ is $\mathcal{T}$-left adjoint to $R\colon \mathcal{E} \to \mathcal{C}$. Then, FR is the $\mathcal{T}$-left Kan extension of F along L.*

Proof. Using [Sha23, Thm 10.5] we simply repeat the nonequivariant proof: Yoneda's lemma yields

$$\begin{aligned}\mathrm{Nat}_{\mathcal{T}}(L_! F, G) &\simeq \mathrm{Nat}_{\mathcal{T}}(F, GL)\\ &\simeq \mathrm{Nat}_{\mathcal{T}}(FR, G),\end{aligned}$$

so another application of Yoneda's lemma constructs a natural equivalence $L_! F \sim FR$. □

Lemma A.11. *Fix a $\mathcal{T}$-functor $A\colon \mathcal{O}^{\otimes}_{\Gamma} \to \mathcal{C}$. Then, the following are equivalent*

(a) The corresponding map $\mathcal{O}^{\otimes} \to \mathcal{C}^{I-\sqcup}$ is a functor of I-operads.
(b) For all morphisms α in $\mathcal{O}^{\otimes}_{\Gamma}$ whose image in $\mathcal{O}^{\otimes}$ is inert, $A(\alpha)$ is an equivalence in $\mathcal{C}$.
(c) If $f\colon (S_+, U, X) \to (U_+, U, X_U)$ is a cocartesian lift of the corresponding inert morphism, then $A(f)$ is an equivalence.
(d) A is $\mathcal{T}$-left Kan extended from $\mathcal{O}$.

Furthermore, every functor $F : \mathcal{O} \to \mathcal{C}$ admits a left Kan extension along $\mathcal{O} \hookrightarrow \mathcal{O}^{\otimes}_{\Gamma}$; in particular, the forgetful functor $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_G(\mathcal{O}, \mathcal{C})$ is an equivalence.

Proof. (a) $\iff$ (b) follows immediately from Proposition A.7. (b) $\iff$ (c) is immediate by definition. (c) $\iff$ (d) and the remaining statement both follow by Lemma A.10. □

A.3. **$\mathcal{O}$-monoids.** We start with the following.

Proposition A.12. $\mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\times} \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ *is a cocartesian fibration, so in particular, it presents a Γ-I-preoperad $\mathcal{C}^{I-\times}$; moreover, its straightening is an I-symmetric monoidal ∞-category if and only if $\mathcal{C}$ admits I-indexed products, in which case the indexed tensor functors in $\mathcal{C}$ are indexed products.*

Proof. For the first statement, it suffices to observe that the cocartesian arrows described in Lemma A.6 lie in $\mathrm{TotTot}_{\mathcal{T}}\mathcal{C}^{I-\times}$. For the second, note by unwinding definitions that cocartesian transport induces a fully faithful functor

$$\mathcal{C}_S \to \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_U$$

Moreover, this is essentially surjective if and only if $\mathcal{C}$ admits I-indexed products, as desired. □

We organize ourselves around the following observations.

Observation A.13. The projection $\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes} \times_{\mathrm{Tot}\underline{\mathbb{F}}_{I,*}} \mathrm{Tot}\underline{\Gamma}^{\times}_{I} \to \mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes}$ admits a left adjoint L sending $X \in \mathcal{O}^{\otimes}_{S_+}$ to $(X, S \subset S)$; the unit map of this adjunction is evidently an equivalence, so $L\colon \mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^{\times}_{I}$ is fully faithful. ◁

Observation A.14. The section $L\colon \mathrm{Tot}\underline{\mathbb{F}}_{I,*} \to \mathrm{Tot}\underline{\Gamma}^{\times}_{I}$ induces a natural transformation $K \simeq K \times_{\mathrm{Tot}\underline{\mathbb{F}}_{I,*}} \mathrm{Tot}\underline{\mathbb{F}}_{I,*} \to K \times_{\mathrm{Tot}\underline{\mathbb{F}}_{I,*}} \mathrm{Tot}\underline{\Gamma}^{\times}_{I}$, which induces a natural transformation $\mathrm{TotTot}_{\mathcal{T}}\widetilde{\mathcal{C}}^{I-\times} \to \mathrm{Tot}\mathcal{C}$ under Yoneda's lemma. Unwinding definitions using Lemma A.6, this presents a $\mathcal{T}$-functor $\mathrm{Tot}_{\mathcal{T}}\widetilde{\mathcal{C}}^{I-\times} \to \mathcal{C}$. ◁

Given a $\mathcal{T}$-functor $\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \xrightarrow{\varphi} \mathrm{Tot}_{\mathcal{T}}\widetilde{\mathcal{C}}^{I-\times}$, we acquire a corresponding functor

$$\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes} \xrightarrow{L} \mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \times_{\mathrm{Tot}\underline{\mathbb{F}}_{I,*}} \mathrm{Tot}\underline{\Gamma}^{\times}_{I} \xrightarrow{\varphi'} \mathrm{Tot}\mathcal{C}$$

over $\mathcal{T}^{\mathrm{op}}$. Now, the following observation is important.

Observation A.15. The description of cocartesian arrows of Lemma A.6 and Proposition A.12 implies that φ' and L are unstraightened from $\mathcal{T}$-functors. ◁

Now, given a Γ-I-preoperad $\mathcal{O}^{\otimes}$, we say that an *$\mathcal{O}$-monoid in $\mathcal{C}$* is a $\mathcal{T}$-functor $\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathcal{C}$ satisfying the condition that, for all $X \in \mathcal{C}_S$, the maps $\mathrm{Res}^V_U F(X) \to F(X_U)$ induced by cocartesian transport witness $F(X)$ as the indexed product

$$F(X) \simeq \prod_{U}^{S} F(X_U).$$

Proposition A.16. *Fix $\mathcal{C}$ a $\mathcal{T}$-∞-category and $\mathcal{O}^{\otimes}$ a Γ-I-preoperad. Then, the postcomposition functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\times}) \to \mathrm{Fun}_{\mathcal{T}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}, \mathcal{C})$ is fully faithful with image spanned by the $\mathcal{O}$-monoids.*

Lemma A.17. *The following conditions are equivalent:*

(a) φ is a map of Γ-I-preoperads.

(b) For all morphisms α in $\mathrm{Tot}\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \times_{\mathrm{Tot}\underline{\mathbb{F}}_{I,*}} \mathrm{Tot}\underline{\Gamma}_I^{\times}$ whose image in $\mathcal{O}^{\otimes}$ is inert $\varphi'(\alpha)$ is an equivalence..

(c) If $f: (\overline{S}_+ \to V_+, \overline{S}, F, X) \to (S_+ \to V_+, \overline{S}, F, X)$ is a cocartesian lift of the corresponding inert morphism, then $\varphi(f)$ is an equivalence.

(d) The composite $\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \xrightarrow{\varphi} \mathrm{Tot}_{\mathcal{T}}\widetilde{\mathcal{C}}^{I-\times} \to \mathcal{C}$ is homotopic to φ'.

(e) The composite $\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \xrightarrow{\varphi} \mathrm{Tot}_{\mathcal{T}}\widetilde{\mathcal{C}}^{I-\times} \to \mathcal{C}$ is $\mathcal{T}$-right Kan extended from φ' along L.

Proof. Lemma A.6 immediately implies that (a) $\iff$ (b) $\iff$ (c) $\iff$ (d). (d) $\iff$ (e) follows from Lemma A.10. $\square$

The following lemma follows by unwinding definitions.

Lemma A.18. *Suppose φ is a functor of Γ-I-preoperads. Then, the following conditions are equivalent:*

(a) φ factors through a Γ-I-preoperad map $\overline{\varphi}: \mathcal{O}^{\otimes} \to \mathcal{C}^{I-\times}$.

(b) φ' is an $\mathcal{O}$-monoid.

Proof of Proposition A.16. Since $\mathrm{Tot}_{\mathcal{T}}\mathcal{C}^{I-\times} \hookrightarrow \mathrm{Tot}_{\mathcal{T}}\tilde{\mathcal{C}}^{I-\times}$ is fully faithful, the first of the following functors is fully faithful

$$\mathrm{Alg}_{\mathcal{O}}\left(\mathcal{C}^{I-\times}\right) \hookrightarrow \mathrm{Alg}_{\mathcal{O}}\left(\widetilde{\mathcal{C}}^{I-\times}\right) \hookrightarrow \mathrm{Fun}_{\mathcal{T}}\left(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}, \mathcal{C}\right).$$

By Lemma A.17, the second is fully faithful, and by Lemma A.17 the image of the composite functor consists of the $\mathcal{O}$-mononids. $\square$

We may additionally characterize I-symmetric monoidal functors via a lift of [HA, Prop 2.4.1.7], which also follows immediately from Lemma A.6.

Lemma A.19. *Suppose $\mathcal{C}$ has I-indexed products, $\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \underline{\mathbb{F}}_{I,*}$ is a cocartesian fibration, and Lemma A.18 is satisfied. Then, the following conditions are equivalent.*

(a) $\overline{\varphi}$ is a $\mathcal{T}$-functor of cocartesian fibrations over $\underline{\mathbb{F}}_{I,*}$.

(b) If $f: \mathbf{D} \to \mathbf{D}'$ is an active arrow in $\mathrm{Tot}\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}$, then map $\varphi'(f)$ is an equivalence.

Now, we will also lift [HA, Prop 2.4.1.6]. We have a fully faithful $\mathcal{T}$-functor $\iota: \mathcal{O} \hookrightarrow \mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}$.

Lemma A.20. *Suppose φ satisfies the conditions of Lemma A.19 and the action maps $\otimes^S: \mathcal{O}_S \to \mathcal{O}_V$ are right adjoint to the restriction maps $\Delta^S: \mathcal{O}_V \to \mathcal{O}_S$. Then, the functor $\widetilde{\varphi}: \mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathcal{C}$ is right Kan-extended from the I-product-preserving functor $\mathcal{O} \to \mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathcal{C}$ along ι.*

Proof. The assumptions imply that there is a right $\mathcal{T}$-adjoint $R: \mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathcal{O}$ to ι, sending $(X_U) \mapsto \bigotimes_U^S X_U \simeq \prod_U^S X_U$. The $\mathcal{O}$-monoid assumption shows that $\widetilde{\varphi} \sim \varphi \circ \iota \circ R$, which shows that $\widetilde{\varphi}$ is right Kan extended from $\varphi \circ \iota$ along ι; moreover, the $\mathcal{O}$-monoid assumption shows that $\varphi \circ \iota$ is I-product-preserving. $\square$

A.4. **(Co)cartesian rigidity.**

Proposition A.21. *Suppose $\mathcal{O}^{\otimes}$ is an I-symmetric monoidal ∞-category such that $\otimes^S: \mathcal{O}_S \to \mathcal{O}_V$ is right adjoint to $\Delta^S: \mathcal{O}_V \to \mathcal{O}_S$ for all $S \in \mathbb{F}_{I,V}$. Then, the forgetful functor*

$$U: \mathrm{Fun}_I^{\otimes}\left(\mathcal{O}^{\otimes}, \mathcal{C}^{I-\times}\right) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \mathcal{C})$$

is fully faithful with essential image spanned by the I-product preserving $\mathcal{T}$-functors.

Proof. Let $\widetilde{\underline{\mathrm{Fun}}}_{\mathcal{T}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}, \mathcal{C}) \subset \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}, \mathcal{C})$ be the equivalent image of $\mathrm{Fun}_I^{\otimes}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$. Lemma A.20 constructs the solid portion of a diagram

$$\begin{tikzcd}
\mathrm{Fun}_{\mathcal{T}}^{I-\times}(\mathcal{O},\mathcal{C}) \arrow[r, hook, dashed] \arrow[rr, bend left, "\iota_*"] \arrow[dr, equal] & \widetilde{\underline{\mathrm{Fun}}}_{\mathcal{T}}\left(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes},\mathcal{C}\right) \arrow[r, hook] \arrow[d, "\iota^*"] \arrow[dr, "U"] & \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes},\mathcal{C}\right) \arrow[d, "\iota^*"] \\
& \mathrm{Fun}_{\mathcal{T}}^{I-\times}(\mathcal{O},\mathcal{C}) \arrow[r, hook] & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})
\end{tikzcd}$$

It suffices to verify that the dashed arrow exists, i.e. right Kan extensions of I-product-preserving functors along ι satisfy the conditions of Lemma A.19; but this follows by unwinding definitions. $\square$

Corollary A.22. *If $\mathcal{C}$ has I-indexed products, then there exists a unique I-symmetric monoidal equivalence $\mathcal{C}^{I-\times} \simeq \left((\mathcal{C}^{\mathrm{vop}})^{I-\sqcup}\right)^{\mathrm{vop}}$ lying over the equivalence $\mathcal{C} \simeq (\mathcal{C}^{\mathrm{vop}})^{\mathrm{vop}}$; if $\mathcal{C}$ has I-indexed coproducts, then there exists a unique I-symmetric monoidal equivalence $\mathcal{C}^{I-\sqcup} \simeq \left((\mathcal{C}^{\mathrm{vop}})^{I-\times}\right)^{\mathrm{vop}}$ lying over the equivalence $\mathcal{C} \simeq (\mathcal{C}^{\mathrm{vop}})^{\mathrm{vop}}$.*

Proof. By conservativity of the underlying category (see Lemma 1.11), it suffices to construct a unique I-symmetric monoidal *functor* lying over the identity in each case. For the first case, by Proposition A.21 it suffices to note that $\left((\mathcal{C}^{\mathrm{vop}})^{I-\sqcup}\right)^{\mathrm{vop}}$ has $\Delta^S \dashv \otimes^S$. The second case follows from the first under the following equivalence of arrows, where $\mathcal{D} := \mathcal{C}^{\mathrm{vop}}$.

$$\begin{tikzcd}
\mathrm{Fun}_I^{\otimes}\left((\mathcal{D}^{\mathrm{vop}})^{I-\sqcup}, \left(\mathcal{D}^{I-\times}\right)^{\mathrm{vop}}\right) \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Fun}_I^{\otimes}\left(\left((\mathcal{D}^{\mathrm{vop}})^{I-\sqcup}\right)^{\mathrm{vop}}, \left(\left(\mathcal{D}^{I-\times}\right)^{\mathrm{vop}}\right)^{\mathrm{vop}}\right) \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Fun}_I^{\otimes}\left(\left((\mathcal{D}^{\mathrm{vop}})^{I-\sqcup}\right)^{\mathrm{vop}}, \mathcal{D}^{I-\times}\right) \arrow[d] \\
\mathrm{Fun}_{\mathcal{T}}\left(\mathcal{D}^{\mathrm{vop}}, \mathcal{D}^{\mathrm{vop}}\right) \arrow[r, phantom, "\simeq"] & \mathrm{Fun}_{\mathcal{T}}\left((\mathcal{D}^{\mathrm{vop}})^{\mathrm{vop}}, (\mathcal{D}^{\mathrm{vop}})^{\mathrm{vop}}\right) \arrow[r, phantom, "\simeq"] & \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{D}, (\mathcal{D}^{\mathrm{vop}})^{\mathrm{vop}}\right)
\end{tikzcd}$$

$\square$

Corollary A.23. *Suppose $\mathcal{O}^{\otimes}$ is an I-symmetric monoidal ∞-category such that $\otimes^S : \mathcal{O}_S \to \mathcal{O}_V$ is left to $\Delta^S : \mathcal{O}_V \to \mathcal{O}_S$ for all $S \in \mathbb{F}_{I,V}$. Then, the forgetful functor*

$$U : \mathrm{Fun}_I^{\otimes}\left(\mathcal{O}^{\otimes}, \mathcal{C}^{I-\sqcup}\right) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \mathcal{C})$$

is fully faithful with essential image spanned by the I-coproduct preserving $\mathcal{T}$-functors.

Proof. This follows by taking vertical opposites of Proposition A.21 in light of Corollary A.22. $\square$

We are now ready to prove our main generalization of Theorem A' (see p. 21).

Proof of Theorem A'. We begin with the cartesian cases. To see that $(-)^{I-\times}$ is fully faithful, it suffices to combine Propositions A.12 and A.21. The compatibility with U is obvious, and the description of the image follows immediately from Proposition A.21. The cocartesian case follows by the same argument using Corollary A.23. $\square$

A.5. **Wirtmüller maps.** Suppose $\mathcal{O}^{\otimes}$ is an I-symmetric monoidal ∞-category and $\mathcal{C}$ has I-indexed coproducts. The equivalence $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\sqcup}) \simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \mathcal{C})$ embeds I-symmetric monoidal functors $\mathcal{O}^{\otimes} \to \mathcal{C}^{I-\sqcup}$ as a full subcategory of $\mathcal{T}$-functors $\mathcal{O} \to \mathcal{C}$. We now record the property identifying this full subcategory.

Observation A.24. Let $F : \mathcal{D}^{\otimes} \to \mathcal{E}^{\otimes}$ be a lax I-symmetric monoidal functor. Then, the universal property for cocartesian arrows constructs, for each active arrow $\mathrm{Ind}_V^{\mathcal{T}} S \to V$, an arrow

$$\mu_S : \bigotimes_U^S F(-) \implies F\left(\bigotimes_U^S -\right)$$

such that F is I-symmetric monoidal if and only if μ_S is an equivalence for all $S \in \underline{\mathbb{F}}_I$. In particular, in the case of the lax I-symmetric monoidal functor $F : \mathcal{O}^{\otimes} \to \mathcal{C}^{I-\sqcup}$ classified by a functor $G : \mathcal{O} \to \mathcal{C}$, the arrow μ has the type

$$\coprod_U^S G(-) \implies G\left(\bigotimes_U^S -\right);$$

moreover, unwinding definitions, in the case that $\mathcal{O}^{\otimes} = \mathcal{C}^{\otimes}$ is an I-symmetric monoidal structure on $\mathcal{C}$ and, G is the identity, and $1_{\bullet} \in \Gamma^{v(I)}\mathcal{O}$ is initial, this map is precisely the $\otimes$-Wirthmüller map constructed in Construction 1.63. In particular, we've observed that the identity classifies an I-symmetric monoidal equivalence $\mathcal{C}^{\otimes} \xrightarrow{\sim} \mathcal{C}^{I-\coprod}$ if and only if $\mathcal{C}^{\otimes}$ has I-admissible $\otimes$-Wirthmüller isomorphisms. $\triangleleft$

A.6. **A technical corollary on n-Morita equivalences.** A Segal morphism of algebraic patterns $\varphi\colon \mathfrak{O} \to \mathfrak{P}$ is called an *n-Morita equivalence* if, for all complete $(n+1)$-categories $\mathcal{C}$, the induced functor

$$f^*\colon \mathrm{Seg}_{\mathfrak{P}}(\mathcal{C}) \to \mathrm{Seg}_{\mathfrak{O}}(\mathcal{C})$$

is an equivalence; in fact, it suffices to check this in the case $\mathcal{C} = \mathcal{S}_{\leq n}$ [Bar23, Prop 2.1.9]. We have the following corollary.

Corollary A.25. *Suppose $\varphi\colon \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a morphism of Γ-I-preoperads such that the induced $\mathcal{T}$-functor $U\mathcal{P} \to U\mathcal{O}$ is essentially surjective. Then, φ is an n-Morita equivalence if and only if the associated map of $\mathcal{T}$-operads $h_{n+1}L_{\mathrm{Op}_{\mathcal{T}}}\mathcal{P}^{\otimes} \to h_{n+1}L_{\mathrm{Op}_{\mathcal{T}}}\mathcal{O}^{\otimes}$ is an equivalence.*

Proof. There is an equivalence $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\times}) \simeq \mathrm{Alg}_{L_{\mathrm{Op}_{\mathcal{T}}}\mathcal{O}}(\mathcal{C}^{I-\times})$ natural in Segal morphisms over $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$, so the result follows from the recognition result for n-equivalences of I-operads [Ste25]. ☐

Now, we define the *I-preoperads* $\mathrm{PreOp}_I := \mathrm{Cat}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}$. In [Ste25, § A.1] we proved that, under the assumption that $\mathcal{O}^{\otimes}$ is an I-operad, the canonical strong Segal morphism $s^*\mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes}$ is a Morita equivalence and $s^*\mathcal{O}^{\otimes}$ is soundly extendable; in fact, we only used that $\mathcal{O}^{\otimes}$ is an I-operad to conclude that $s^*\mathcal{O}^{\otimes}$ is soundly extendable, and the same proof shows that $s^*\mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes}$ underlies a natural Morita equivalence when $\mathcal{O}^{\otimes}$ is an I-preoperad, and in particular, a natural n-Morita equivalence.

Moreover, this underlies the counit map of I-preoperads $\varepsilon\colon s_!s^*\mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes}$, which is an L_{Op_G}-equivalence. In particular, ε becomes an equivalence after $\mathrm{Alg}_{(-)}(\underline{\mathcal{S}}_{G,\leq n+1})$. This yields a chain of natural equivalences.

$$\mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{G,\leq n+1}) \simeq \mathrm{Alg}_{s_!s^*\mathcal{O}}(\underline{\mathcal{S}}_{G,\leq n+1}) \simeq \mathrm{Alg}_{s^*\mathcal{O}}(\underline{\mathcal{S}}_{G,\leq n+1}) \simeq \mathrm{Seg}_{s^*\mathcal{O}}(\mathcal{S}_{\leq n+1}) \simeq \mathrm{Seg}_{\mathcal{O}}(\mathcal{S}_{\leq n+1})$$

In particular, the same proof as Corollary A.25 yields the following.

Corollary A.26. *Suppose $\varphi\colon \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a morphism of I-preoperads such that the induced $\mathcal{T}$-functor $U\mathcal{P} \to U\mathcal{O}$ is essentially surjective. Then, φ is an n-Morita equivalence if and only if the associated map of $\mathcal{T}$-operads $h_{n+1}L_{\mathrm{Op}_{\mathcal{T}}}\mathcal{P}^{\otimes} \to h_{n+1}L_{\mathrm{Op}_{\mathcal{T}}}\mathcal{O}^{\otimes}$ is an equivalence.*

Appendix B. I-operadic disintegration and assembly

In this appendix, we assume familiarity with the minutiae of [BHS22; Ste25] and prove Corollary 3.8.

B.1. **The algebraic pattern for families of I-operads.** Fix $\mathcal{C}$ a $\mathcal{T}$-∞-category.

Construction B.1. The *totally inert pattern structure on* $\mathcal{C}$ has

$$\left(\mathrm{Tot}^{\mathrm{int}}\mathcal{C}\right)^{\mathrm{el}} := \left(\mathrm{Tot}^{\mathrm{int}}\mathcal{C}\right)^{\mathrm{int}} := \mathrm{Tot}\mathcal{C}; \qquad \left(\mathrm{Tot}^{\mathrm{int}}\mathcal{C}\right)^{\mathrm{act}} := (\mathrm{Tot}\mathcal{C})^{\simeq}.$$

We define the *$\mathcal{C}$-family pattern*

$$\mathrm{Tot}^{\mathrm{int}}\left(\mathcal{C} \times \underline{\mathbb{F}}_{I,*}\right) := \mathrm{Tot}\mathcal{C}^{\mathrm{int}} \times_{\mathcal{T}^{\mathrm{op,int}}} \mathrm{Tot}\underline{\mathbb{F}}_{I,*}. \qquad \triangleleft$$

The following observation is as crucial as it is immediate.

Observation B.2. If $\mathcal{C}$ is a $\mathcal{T}$-space, then $\mathrm{Tot}^{\mathrm{int}}(\mathcal{C} \times \underline{\mathbb{F}}_{I,*}) \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ is an inert-cocartesian fibration, and the domain has the induced pattern structure. In particular, in this case, $\mathrm{Tot}^{\mathrm{int}}(\mathcal{C} \times \underline{\mathbb{F}}_{I,*})$ is the pattern underlying the Γ-I-preoperad $\mathcal{C} \times \underline{\mathbb{F}}_{I,*}$. $\triangleleft$

We begin by identifying $\mathcal{C}$-indexed diagrams of I-operads.

Proposition B.3. *There exists a natural equivalence*

$$\mathrm{Fbrs}\left(\mathrm{Tot}^{\mathrm{int}}\left(\mathcal{C} \times \underline{\mathbb{F}}_{I,*}\right)\right) \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{Op}}_I\right),$$

so that when $\mathcal{C} \simeq *$, this is the usual equivalence $\mathrm{Fbrs}(\underline{\mathbb{F}}_{I,*}) \simeq \underline{\mathrm{Op}}_I$.

To prove this, we use the equifibered theory, focusing on the following lemmas.

Lemma B.4. *There exists a natural equivalence* $\mathrm{Seg}_{\mathrm{Tot}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*})}(\mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{CMon}}_I(\mathcal{D})\right)$.

Proof. First off, we get an embedding

$$\begin{aligned}\mathrm{Seg}_{\mathrm{Tot}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*})}(\mathcal{D}) &\subset \mathrm{Fun}(\mathrm{Tot}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*}),\mathcal{D});\\ &\simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*},\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{D}));\\ &\simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C},\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\underline{\mathbb{F}}_{I,*},\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}\right)\right)\end{aligned}$$

characterized by the Segal condition that the restricted functor $\mathrm{Tot}\left(\mathcal{C}\times\underline{\mathbb{F}}_{I,*}^{\mathrm{int}}\right)\to\mathcal{D}$ is right Kan extended from $\mathrm{Tot}\left(\mathcal{C}\times\underline{\mathbb{F}}_{I,*}^{\mathrm{el}}\right)$. Now, unwinding conditions, this corresponds with the condition that the value V-functors $\mathbb{F}_{I_V,*}^{\mathrm{int}}\to\underline{\mathrm{Coeff}}^{V}\mathcal{D}$ are each right-Kan extended from $\mathbb{F}_{I_V,*}^{\mathrm{int}}$, i.e. the corresponding functor factors through $\underline{\mathrm{CMon}}_I(\mathcal{D})\subset\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\underline{\mathbb{F}}_{I,*},\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}\right)$ (c.f. [Nar17, Ex 1.17]). $\square$

Lemma B.5. $\mathrm{EnvTot}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*})$ *corresponds naturally with the constant* $\mathcal{T}$*-functor over* $\underline{\mathbb{F}}_{I,*}^{I-\sqcup}$*; a natural transformation* $F\to\mathrm{Env}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*})$ *is equifibered if and only if it is pointwise-equifibered.*

Proof. This follows by explicitly identifying the active arrows in $\mathrm{Tot}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*})$ as products of equivalences and active arrows of $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$. $\square$

Proof of Proposition B.3. The above work constructs a string of natural equivalences

$$\begin{aligned}\mathrm{Fbrs}(\mathrm{Tot}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*})) &\simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C},\underline{\mathrm{CMon}}_I(\mathcal{D})\right)^{\mathrm{pointwise-equifibered}}_{/\Delta\underline{\mathbb{F}}_{I,*}}\\ &\simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C},\underline{\mathrm{CMon}}_I(\mathcal{D})^{\mathrm{equifibered}}_{/\underline{\mathbb{F}}_{I,*}}\right)\\ &\simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C},\underline{\mathrm{Op}}_I\right).\end{aligned}$$ $\square$

Now, this is closely related to $(-)^{I-\sqcup}$, as described by the following construction.

Construction B.6. Fix $\mathcal{C}$ a $\mathcal{T}$-∞-category. Then, pullback along the projection $(-)\times_{\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}\mathrm{Tot}\underline{\Gamma}_I^*\to(-)$ determines a natural transformation

$$\mathrm{Fun}_{/\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}\left(-,\mathrm{Tot}\mathrm{Tot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup}\right)\simeq\mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}\left(-\times_{\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}\mathrm{Tot}\underline{\Gamma}_I^*,\mathcal{C}\right)\leftarrow\mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}(-,\mathcal{C})\simeq\mathrm{Fun}_{/\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}\left(-,\mathcal{C}\times\mathrm{Tot}\underline{\mathbb{F}}_{I,*}\right),$$

which corresponds with a functor $\gamma\colon\mathrm{Tot}\mathcal{C}\times\mathrm{Tot}\underline{\mathbb{F}}_{I,*}\to\mathrm{Tot}\mathrm{Tot}_{\mathcal{T}}\mathcal{C}^{I-\sqcup}$ under Yoneda's lemma. Note that $\gamma(C,S)\simeq\Delta^S C$. Moreover, this is compatible with Γ-I-preoperadic structure in the case $\mathcal{C}=\mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes}$:

$$\begin{tikzcd}
\underline{\mathbb{F}}_{I,*}\times\mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes} \arrow[rrr, "\mathrm{id}\times\pi_{\mathcal{P}}"] \arrow[dr, "\gamma"] \arrow[ddr, swap, "\mathrm{pr}_1"] & & & \underline{\mathbb{F}}_{I,*}\times\underline{\mathbb{F}}_{I,*} \arrow[d, "\wedge"] \\
& (\mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes})^{I-\sqcup} \arrow[r, "\pi_{\mathcal{P}}^{I-\sqcup}"] \arrow[d, "\pi"] & \left(\underline{\mathbb{F}}_{I,*}\right)^{I-\sqcup} \arrow[r, "\vee"] & \underline{\mathbb{F}}_{I,*} \\
& \underline{\mathbb{F}}_{I,*} & &
\end{tikzcd}$$

Pulling back to $\mathrm{Span}_I(\mathbb{F}_V)$, we acquire a simpler diagram

$$\begin{tikzcd}
\mathrm{Span}_I(\mathbb{F}_V)\times\mathrm{Tot}\mathcal{P}^{\otimes} \arrow[r, "\mathrm{id}\times\pi_{\mathcal{P}}"] \arrow[dr, "\gamma"] \arrow[ddr, swap, "\mathrm{pr}_1"] & \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})\times\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \arrow[dr, "\wedge"] & \\
& (\mathrm{Tot}_V\mathcal{P}^{\otimes})^{I-\sqcup} \arrow[r, "\rho"] \arrow[d, "\pi"] & \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \\
& \mathrm{Span}_I(\mathbb{F}_V) &
\end{tikzcd}$$

so in particular, γ becomes a *bifunctor* under the alternative structure map ρ. ◁

In this paper, we mostly care about the case that $\mathcal{C}$ is a generic $\mathcal{T}$-space. We will use the following specialization of Barkan's morita equivalence recognition result (c.f. [HA, Thm 2.3.3.23, Thm 2.3.3.26]).[10]

[10] To see this as a specialization of Barkan's result, note that by [Ste25], $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ is soundly extendable, so $\mathrm{Tot}_{\mathcal{T}}$ of an I-operad is soundly extendable by [BHS22, Lem 4.1.15]. The remaining modifications necessary are the observation that $\mathcal{O}^{\mathrm{el}}\simeq\mathrm{Tot}U\mathcal{O}$ (so condition

Proposition B.7 ([BHS22, Prop 3.1.16, Thm 5.1.1])**.** *Suppose $f\colon \mathcal{O}^\otimes \to \mathcal{P}^\otimes$ is a strong Segal morphism of Γ-I-preoperads such that $\mathcal{P}^\otimes$ presents an I-operad and the following conditions hold:*

(a) the $\mathcal{T}$-functor $U\mathcal{O} \to U\mathcal{P}$ is an equivalence, and
(b) for every $O \in U\mathcal{O}$, the map of spaces $\left(\mathcal{O}^{\mathrm{act}}_{/O}\right)^{\simeq} \to \left(\mathcal{P}^{\mathrm{act}}_{/f(O)}\right)^{\simeq}$ is an equivalence.

Then, the functors $f^*\colon \mathrm{Mon}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ and $f^*\colon \mathrm{Op}_{I,/\mathcal{P}^\otimes} \to \mathrm{Fbrs}(\mathcal{O})$ are equivalences.

Proposition B.8. *When X is a $\mathcal{T}$-space, the functor $\gamma\colon \mathrm{Tot}^{\mathrm{int}}\left(X \times \mathrm{Tot}\underline{\mathbb{F}}_{I,*}\right) \to \mathrm{Tot}\mathrm{Tot}_{\mathcal{T}} X^{I-\sqcup}$ satisfies the conditions of Proposition B.7; in particular, γ is an L_{Op_I} localization map.*

Proof. By unwinding definitions we find that $\mathrm{Tot}\gamma$ is an iso-segal morphism, and in particular it is a strong Segal morphism. Moreover, condition (a) follows simply by unwinding definitions.

For condition (b), note that the elements of $\left(\mathrm{Tot}_I X^{I-\sqcup}\right)^{\mathrm{act}}_{/\gamma(x,S)}$ correspond with maps $f\colon T \to S$ in $\mathbb{F}_{\mathcal{T}}$ together with elements $(y_U)_T \in X^T$ with distinguished paths $y_U \sim x$ within X^U; in particular, by contracting paths, we may construct a deformation retract of $\left(\left(\mathrm{Tot}_I X^{I-\sqcup}\right)^{\mathrm{act}}_{/\gamma(x,S)}\right)^{\simeq}$ onto the subspace $\mathbb{F}^{\simeq}_{I,/S} \subset \left(\left(\mathrm{Tot}_I X^{I-\sqcup}\right)^{\mathrm{act}}_{/\gamma(x,S)}\right)^{\simeq}$ of identity paths.

Similarly, we may perform a deformation retract of $\left(\left(X \times \underline{\mathbb{F}}_{i,*}\right)^{\mathrm{act}}_{/(x,S)}\right)^{\simeq}$ onto the summand $\mathbb{F}^{\simeq}_{I,/S} \subset \left(\left(X \times \underline{\mathbb{F}}_{i,*}\right)^{\mathrm{act}}_{/(x,S)}\right)^{\simeq}$ of identity paths. It follows by unwinding definitions that these are taken isomorphically onto each other; alternatively, one may note that the induced endomorphism of $\mathbb{F}^{\simeq}_{I,/S} \simeq \prod_{U \in \mathrm{Orb}(S)} \coprod_{T \in \mathbb{F}_{I,U}} B\mathrm{Aut}_U(T)$ is a product of coproducts of maps classified by torsor maps $\mathrm{Aut}_U(T) \to \mathrm{Aut}_U(T)$, which are automatically isomorphisms, implying that our map of 1-truncated spaces is an isomorphism on π_0 and on π_1 at all basepoints. $\square$

Warning B.9. A closely related analog of Proposition B.8 is claimed in [HA, Rmk 2.4.3.6] in the case $\mathcal{T} = *$ *without the assumption that $\mathcal{C}$ is a space*; as pointed out in [KK24, Rmk 2.3] Lurie's claim (and hence proof) is incorrect in general, but the claim was verified in *loc. cit.* when $\mathcal{C}$ is a space. $\triangleleft$

We finish with the following proposition.

Proposition B.10. *If $\mathrm{Tot}_{\mathcal{T}}\mathcal{P}^\otimes \to \mathrm{Tot}^{\mathrm{int}}\left(X \times \underline{\mathbb{F}}_{I,*}\right)$ is a fibrous pattern, then $L_{\mathrm{Op}_I}\mathcal{P}^\otimes$ is the $\mathcal{T}$-colimit of the $\mathcal{T}$-functor $X \to \underline{\mathrm{Op}}_I$ associated with $\mathcal{P}^\otimes$.*

Proof. Note that $L_{\mathrm{Op}_I}\gamma_!\pi_{X^{I-\sqcup}!} \dashv \gamma^*\pi^*_{X^{I-\sqcup}}$, and the latter is equivalent to $\Delta\colon \mathrm{Op}_I \to \mathrm{Fun}_{\mathcal{T}}(X, \underline{\mathrm{Op}}_I)$; the above presentation for the left adjoint is $L_{\mathrm{Op}_I}\mathcal{P}^\otimes$, and indexed colimits are also left adjoint to Δ, so the claim follows from uniqueness of left adjoints. $\square$

We now apply this in the language of *disintegration and assembly.*

B.2. **Disintegration and assembly.** Given $X \in \mathcal{S}_{\mathcal{T}}$ and $\mathcal{O}^\otimes \in \mathrm{Op}_{I,/X^{I-\sqcup}}$, define the pullback Γ-I-preoperad

$$\begin{tikzcd}
\mathrm{dis}^I(\mathcal{O}^\otimes) \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tot}_{\mathcal{T}}\mathcal{O}^\otimes \arrow[d] \\
X \times \underline{\mathbb{F}}_{I,*} \arrow[r, "\gamma"] & \mathrm{Tot}_I X^{I-\sqcup}
\end{tikzcd}$$

We refer to $\mathrm{dis}^I(-)$ as the *disintegration functor* and α as the *assembly map.*

Proposition B.11. *α is an L_{Op_I}-localization map.*

Proof. We verified in [Ste25, § A] that the conditions of Proposition B.7 are pullback-stable, so α is a Morita equivalence; by Corollary A.25 it is then an L_{Op_I}-equivalence. By assumption, $\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^\otimes$ is L_{Op_I}-local, proving the proposition. $\square$

(1) implies condition (1) of [BHS22, Thm 5.1.1]) and the identifications $\mathrm{Fbrs}(\mathcal{P}) \simeq \mathrm{Op}_{I,/\mathcal{P}^\otimes}$ of [Ste25] and [BHS22, Cor 4.1.17] as well as $\mathrm{Mon}_{\mathcal{P}}(\mathcal{C}) \simeq \mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$ of Corollary 1.54.

Proposition B.12. *$\mathcal{O}^\otimes$ is the $\mathcal{T}$-colimit of the $\mathcal{T}$-functor $X \to \underline{\mathrm{Op}}_I$ associated with $\mathrm{dis}^I(\mathcal{O}^\otimes)$.*

Proof. This is a straightforward application of Propositions B.10 and B.11. □

We spell out the following corollary, which summarizes the full power of what we've proved.

Corollary B.13. *Let X be a $\mathcal{T}$-space. The assignment $x \mapsto \mathcal{O}_x := \mathrm{Res}^{\mathcal{T}}_V \mathcal{O}^\otimes \times_{\mathrm{Res}^{\mathcal{T}}_V X^{I-\sqcup}} \mathcal{N}^{I-\sqcup}_{I\infty}$, yields an equivalence*

$$\underline{\mathrm{Op}}_{I,/X^{I-\sqcup}} \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(X, \underline{\mathrm{Op}}_I).$$

The unit of this equivalence specifies a natural equivalence.

$$\mathcal{O}^\otimes \simeq \underline{\mathrm{colim}}_{x\in X} \mathcal{O}^\otimes_x.$$

Proof. The first claim follows by combining Observation B.2 and Propositions B.3, B.7 and B.8. The remaining claim is proved identically to Proposition B.12. □

Appendix C. Algebraic patterns and the I-symmetric monoidal structure on overcategories

In this appendix, we repeat the arguments of [HA, § 2.2.2] in the setting of algebraic patterns, assuming familiarity with the minutiae of [HA, § 2.2.2] and of [BHS22].

C.1. **The fibrous pattern case.**

We fix $\mathfrak{O}$ an algebraic pattern and make the following temporary definitions.

Definition C.1. An $\mathfrak{O}$-*monoidal ∞-category* is a fibrous pattern $\mathfrak{C} \to \mathfrak{O}$ which is also a cocartesian fibration. If $\mathfrak{C} \to \mathfrak{O}$ is a fibrous pattern, then the ∞-category of $\mathfrak{O}$-*algebras in* $\mathfrak{C}$ is

$$\mathrm{Alg}_{/\mathfrak{O}}(\mathfrak{C}) := \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathfrak{O}}(\mathfrak{O}, \mathfrak{C}). \qquad \triangleleft$$

Remark C.2. When $\mathfrak{O}$ is sound, [BHS22, Prop 4.1.7] shows that $\mathfrak{O}$-monoidal ∞-categories are synonymous with Segal fibrations to $\mathfrak{O}$ in the sense of [CH21]. ◁

Warning C.3. If $\mathcal{O}^\otimes$ underlies a $\mathcal{T}$-operad, $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ and $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ should not be confused; the latter consists of algebras *over* $\mathcal{O}^\otimes$. Nevertheless, in the case that $\mathcal{O}^\otimes \to \mathrm{Comm}^\otimes_{\mathcal{T}}$ is a monomorphism (i.e. $\mathcal{O}^\otimes$ is a weak $\mathcal{N}_\infty$-operad), these agree. ◁

For the duration of this appendix, we fix $q\colon \mathfrak{C} \to \mathfrak{O}$ a fibrous pattern and $p\colon K \to \mathrm{Alg}_{/\mathfrak{O}}(\mathfrak{C})$ a K-indexed diagram of $\mathfrak{O}$-algebras in $\mathfrak{C}$. Let $q''\colon \mathfrak{C}_{/p_{\mathfrak{O}}} \longrightarrow \mathfrak{O}$ be the construction made in [HA, Def 2.2.2.1], interpreted as a functor of ∞-categories via [HA, Lem 2.2.2.6]. The proof of the following theorem will involve essentially no new ideas over that of [HA, Thm 2.2.2.4].

Theorem C.4. *q'' exhibits $\mathfrak{C}_{/p_{\mathfrak{O}}}$ as a fibrous $\mathfrak{O}$ pattern, which is an $\mathfrak{O}$-monoidal ∞-category if $\mathfrak{C}$ is.*

Proof. We apply [HA, Lem 2.2.2.7-9] on opposite categories. In particular, given an arrow $g\colon X \to q''(Y)$ in $\mathfrak{O}$ over which $\mathfrak{C}$ has a cocartesian lift, [HA, Lem 2.2.2.8] supplies a lift

$$\begin{tikzcd} * \arrow[r, "\{Y\}"] \arrow[d, hook] & \mathfrak{C}_{/p_{\mathfrak{O}}} \arrow[d, "q''"] \\ *^{\triangleright} \arrow[r, "g"'] \arrow[ur, dashed, "\overline{g}"] & \mathfrak{O} \end{tikzcd}$$

such that $\overline{g}$ is a q-colimit diagram; moreover, [HA, Lem 2.2.2.7] guarantees that $\overline{g}$ is a q''-cocartesian lift of g. Since $\mathfrak{C}$ has inert-cocartesian lifts, so does $\mathfrak{C}_{/p_{\mathfrak{O}}}$, and when q is a cocartesian fibration, so is q''.

We're left with verifying the Segal condition(s) for fibrous patterns; we use that of [BHS22, Prop 4.1.6]. As in the proof of [HA, Thm 2.2.2.4], it follows from a simple application of [HA, Lem 2.2.2.9] that each of the relevant diagrams are limit diagrams, as they project to limit diagrams in $\mathfrak{C}$. □

Remark C.5. As in [HA, Thm 2.2.2.4.(2)], it follows from the above diagram that, given an arrow f in $\mathfrak{C}_{/p_{\mathfrak{O}}}$ such that $\mathfrak{C}$ admits a $q''(f)$-cocartesian arrow, f is q''-cocartesian if and only if its image in $\mathfrak{C}$ is q-cocartesian. In particular the inert arrows in $\mathfrak{C}_{/p_{\mathfrak{O}}}$ are the preimages of the inert morphisms of $\mathfrak{C}$, and if $\mathfrak{C}$ is $\mathfrak{O}$-monoidal, then the cocartesian arrows in $\mathfrak{C}_{/p_{\mathfrak{O}}}$ are the preimages of cocartesian arrows in $\mathfrak{C}$. ◁

It is worthwhile to explicitly record following immediate corollary of Remark C.5, in part because it establishes the "pointwise" nature of the coherences for slice $\mathfrak{O}$-monoidal structure.

Corollary C.6. *If $\mathfrak{C}$ is an $\mathfrak{O}$-monoidal ∞-category, then the unslicing functor $\mathfrak{C}_{/p_{\mathfrak{O}}} \to \mathfrak{C}$ is an $\mathfrak{O}$-monoidal functor, i.e. it is a functor of cocartesian fibrations over $\mathfrak{O}$.*

As claimed in [HA], we may pass through *opposite categories* to establish the following result about undercategories without additional argument, noting that the additional assumption comes from the asymmetric assumptions of [HA, Lem 2.2.2.7, 8]. Let $q'\colon \mathfrak{C}_{p\mathfrak{O}/} \to \mathfrak{O}$ be the construction of [HA, Def 2.2.2.1].

Theorem C.7. *q' exhibits $\mathfrak{C}_{p\mathfrak{O}/}$ as a fibrous $\mathfrak{O}$-pattern; moreover, if $\mathfrak{C}$ is an $\mathfrak{O}$-monoidal ∞-category and the value functors $\mathfrak{O} \to \mathfrak{C}$ are all $\mathfrak{O}$-monoidal functors, then q' exhibits $\mathfrak{C}_{p\mathfrak{O}/}$ as an $\mathfrak{O}$-monoidal ∞-category.*

C.2. **The I-symmetric monoidal case.** We explicitly specialize Theorem C.4 to $\mathfrak{O} := \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$.

Corollary C.8. *Let $\mathcal{C}^{\otimes}$ be an I-symmetric monoidal ∞-category and $A \in \mathrm{CAlg}_I(\mathcal{C})$ an I-commutative algebra in $\mathcal{C}$. Then, Theorem C.4 supplies an I-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}_{/A}$ such that*

(1) The underlying $\mathcal{T}$-∞-category of $\mathcal{C}^{\otimes}_{/A}$ agrees with Shah's slice $\mathcal{T}$-∞-category $\mathcal{C}_{/(A,\mathcal{T}^{\mathrm{op}})}$ [Sha22]; moreover, if $\mathbb{N}*_V \subset \mathbb{F}_{I,V}$, then the induced symmetric monoidal structure on $\mathcal{C}^{\otimes}_{/A,\mathrm{Res}^{\mathcal{T}}_V A}$ agrees with Lurie's with respect to the restricted $\mathbb{E}_\infty$-algebra $\mathrm{Res}^{\mathcal{T}}_V \in \mathrm{CAlg}(\mathcal{C}_V)$.

(2) The S-indexed tensor functor in $\mathcal{C}^{\otimes}_{/A}$ takes a tuple of maps $\left(f_U\colon X_U \to \mathrm{Res}^{\mathcal{T}}_U A\right)_S$ to the map

$$\bigotimes_U^S X_U \xrightarrow{\bigotimes_U^S f_U} \bigotimes_U^S \mathrm{Res}^{\mathcal{T}}_U A \xrightarrow{\mu} \mathrm{Res}^{\mathcal{T}}_V A.$$

Proof. (1) is functoriality of the relative slice construction with respect to pullback of the base ∞-category; this follows straightforwardly from the defining universal property.

For (2), we may apply the universal property of [HA, Def 2.2.2.1] along the functor $\Delta^1 \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ classifying an I-admissible active arrow $\psi\colon \mathrm{Ind}^{\mathcal{T}}_V S = \mathrm{Ind}^{\mathcal{T}}_V S \to V$: active arrows lying over ψ are in correspondence with dashed arrows (and homotopies) making the following diagram commute.

$$\begin{tikzcd}
\Delta^1 \arrow[r, hook] \arrow[d, "\psi"] & \Delta^1 \times *^{\triangleleft} \arrow[r, two heads] \arrow[d, dashed] & \Delta^1 \arrow[d, "\psi"] \\
\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \arrow[r, "\{A\}"'] & \mathcal{C}^{\otimes} \arrow[r, "\pi"'] & \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})
\end{tikzcd}$$

That is, active arrows over ψ correspond with commuting diagrams

$$\begin{tikzcd}
(X_U) \arrow[r] \arrow[d] & Y \arrow[d] \\
(\mathrm{Res}^{\mathcal{T}}_U A) \arrow[r] & \mathrm{Res}^{\mathcal{T}}_V A
\end{tikzcd}$$

(with homotopies witnessing commutativity, between the bottom arrow and the active arrow $(\mathrm{Res}^{\mathcal{T}}_U A) \to \mathrm{Res}^{\mathcal{T}}_V A$, and between the underlying arrows in $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ and ψ). To compute the S-indexed tensor functor, we are tasked with exhibiting *cocartesian* active arrows, and by Remark C.5 it suffices to construct an active arrow whose top arrow is cocartesian. Indeed, the outer diagram of the following suffices.

$$\begin{tikzcd}
(X_U) \arrow[r] \arrow[d] & \bigotimes_U^S X_U \arrow[r, equal] \arrow[d] & \bigotimes_U^S X_U \arrow[d] \\
(\mathrm{Res}^{\mathcal{T}}_U A) \arrow[r] & \bigotimes_U^S \mathrm{Res}^{\mathcal{T}}_U A \arrow[r] & \mathrm{Res}^{\mathcal{T}}_V A
\end{tikzcd}$$

□

Appendix D. The I-symmetric monoidal structure on I-operadic left Kan extension

In the first arXiv copy of this paper, we incorrectly claimed that operadic left Kan extension along an arbitrary G-operad map $\mathrm{triv}(\mathcal{A})^{\otimes} \to \mathrm{triv}(\mathcal{B})^{\otimes}$ yielded a G-symmetric monoidal structure on G-left Kan with respect to the pointwise structure. This neglected a compatibility assumption between the target G-symmetric monodial category and the colimits appearing in the pointwise formula for left Kan extension, which is nevertheless satisfied in all examples of interest. We rectify this now, asserting the following compatibility.

Definition D.1. Let $\mathcal{K}$ be a collection of $\mathcal{T}$-categories. A I-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is *compatible with $\mathcal{K}$-colimits* if, for all I-admissible S, the S-indexed tensor V-functor

$$\bigotimes^{S} \colon \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_{\underline{U}} \to \mathcal{C}_{\underline{V}}$$

is compatible with $\mathcal{K}$-colimits. ◁

Proposition D.2. *Let $J \subset I \subset \underline{\mathbb{F}}_{I,*}$ be weakly extensive subcategories, suppose $\mathcal{C} \in \mathrm{Cat}_I^{\otimes}$ is such that $\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C})$ is compatible with $\mathcal{K}$-colimits, and let $\varphi \colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ be a J-operad map such that*

(a) the slice V-categories $\mathrm{Env}_J(\mathcal{O})_{\underline{V}/P}$ lie in $\mathcal{K}$ for all $P \in \mathcal{P}_V$, and

(b) the pullback $\mathcal{T}$-functor $\varphi^* \colon \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})$ admits a $\mathcal{T}$-left adjoint satisfying the Beck-Chevalley condition that the following diagram of $\mathcal{T}$-functors commutes

$$\begin{tikzcd}
\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "\varphi_!"] \arrow[d, "U_{\mathrm{Env}}"] & \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C}) \arrow[d, "U_{\mathrm{Env}}"] \\
\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathrm{Env}_J(\mathcal{O}),\mathcal{C}) \arrow[r, "\mathrm{Env}_I(\varphi)_!"] & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathrm{Env}_J(\mathcal{P}),\mathcal{C})
\end{tikzcd}$$

Then $\varphi^* \colon \underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ admits an I-symmetric left adjoint lifting $\varphi_!$, naturally in $\mathcal{K}$-colimit preserving I-symmetric monoidal functors in $\mathcal{C}$.

Before proving this, we note how it implies a desirable corollary.

Corollary D.3. *Suppose $\mathcal{T}$ is atomic orbital, I a $\mathcal{T}$-weak indexing category, and $\mathcal{C}^{\otimes} \in \mathrm{Cat}_I^{\otimes}$ distributive.*

(1) If $\varphi \colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a J-operad map satisfying Condition (a) for $\mathcal{T}$-sifted diagrams for some $J \subset I$, then $\mathcal{T}$-operadic left Kan extension lifts to an I-symmetric monoidal adjunction

$$\begin{tikzcd}
\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \arrow[r, shift left, "\varphi_!"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}), \arrow[l, shift left, "\varphi^*"]
\end{tikzcd}$$

natural in $\mathcal{T}$-sifted $\mathcal{T}$-colimit preserving I-symmetric monoidal functors in $\mathcal{C}$.

(2) If $F \colon \mathcal{A} \to \mathcal{B}$ is a $\mathcal{T}$-functor such that, for all $V \in \mathcal{T}$ and $A \in \mathcal{A}_V$, the slice V-category $\mathcal{B}_{\underline{V}/A}$ is $\mathcal{T}$-sifted, then the left Kan extension functor $F_! \colon \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{A},\mathcal{C}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{B},\mathcal{C})$ underlies an I-symmetric monoidal left adjoint to F^* under the pointwise I-symmetric monoidal structure, naturally on $\mathcal{T}$-sifted colimit preserving $\mathcal{T}$-symmetric monoidal functors in $\mathcal{C}$.

Proof. We begin by noting that statement (2) is a special case of statement (1) for the initial one-color weak indexing category $J = I^{\mathrm{triv}}$, as the I^{triv}-symmetric monoidal envelope is simply the identity.

Now, for statement (1), note that $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{D},\mathcal{C})^{\otimes-\mathrm{ptws}}$ is distributive by [NS22, Ex. 3.3.4]. The proof of [NS22, Thm. 5.1.4] then shows that $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{D},\mathcal{C})^{\otimes-\mathrm{ptws}}$ is compatible with $\mathcal{T}$-sifted $\mathcal{T}$-colimits. Moreover, $U \colon \underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{O},\mathcal{C})$ is a conservative $\mathcal{T}$-sifted $\mathcal{T}$-colimit preserving I-symmetric monoidal functor, which implies that it *reflects* compatibility with $\mathcal{T}$-sifted $\mathcal{T}$-colimits; hence $\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C})$ is compatible with $\mathcal{T}$-sifted $\mathcal{T}$-colimits.

Additionally, [NS22, Thm. 4.3.4] constructs a left adjoint to φ on each fixed point category; then, [HA, Prop. 7.3.2.6] constructs a $\mathcal{T}$-left adjoint to φ. We're left with proving that the Beck-Chevalley transformation

$$\mathrm{Env}_I(\varphi)_! U \implies U \varphi_!$$

is an equivalence; this follows by unwinding [NS22, Rmk. 4.3.6], or by mimicking the proof of [LLP25, Lem 3.40]. □

Now, we'll prove Proposition D.2 in two steps: first we construct an *oplax* I-symmetric monoidal structure, then we prove that it is I-symmetric monoidal. The oplax symmetric monoidal structure comes from general considerations around the doctrinal adjunction, so we briefly increase generality.

D.1. **A doctrinal adjunction for $\mathfrak{O}$-monoidal ∞-categories.** Fix $\mathfrak{O}$ an algebraic pattern. If $\mathfrak{C}, \mathfrak{D} \to \mathfrak{O}$ are $\mathfrak{O}$-monoidal ∞-categories, we refer to morphisms of fibrous $\mathfrak{O}$-patterns $\mathfrak{C} \to \mathfrak{D}$ as *lax $\mathfrak{O}$-monoidal functors*. Given $F\colon \mathfrak{C} \to \mathfrak{D}$ a lax $\mathfrak{O}$-monoidal functor, $f\colon P \to O$ an active arrow in $\mathfrak{O}$, and $X \in \mathfrak{C}_P$ a P-object in $\mathfrak{C}$, we refer to the dashed arrow

$$\begin{tikzcd} FX \arrow[d, "c_{f,FX}"'] \arrow[dr, bend left, "Fc_{f,X}"] & \\ f_\otimes F_X \arrow[r, dashed, "\ell_{F,f,X}"] & Ff_\otimes X \end{tikzcd}$$

supplied by the universal property for cocartesian arrows as the *f-indexed lax structure map of F at X*; here, $f_\otimes$ is cocartesian transport along f and $c_{f,Y}$ is the cocartesian arrow lifting f with domain Y. A lax $\mathfrak{O}$-monoidal functor F is $\mathfrak{O}$-monoidal if and only if $\ell_{F,f,X}$ is an equivalence for all f, X.

Now, postcomposition with the involution $(-)^{\mathrm{op}}\colon \mathsf{Cat} \to \mathsf{Cat}$ yields an involution $(-)^{\mathrm{vop}}\colon \mathsf{Cat}^{\otimes}_{\mathfrak{O}} \to \mathsf{Cat}^{\otimes}_{\mathfrak{O}}$ called the *vertical opposite*. We will refer to lax $\mathfrak{O}$-monoidal functors $\mathfrak{C}^{\mathrm{vop}} \to \mathfrak{D}^{\mathrm{vop}}$ as *oplax $\mathfrak{O}$-monoidal functors from $\mathfrak{C}$ to $\mathfrak{D}$*; an oplax $\mathfrak{O}$-monoidal functor $F\colon \mathfrak{C} \to \mathfrak{D}$ is $\mathfrak{O}$-monoidal if and only if the f-indexed oplax structure map of F at X, $o_{F,f,X}\colon Ff_\otimes X \longrightarrow f_\otimes FX$, is an equivalence for all f, X.

We refer to relative adjunctions between$\mathfrak{O}$-monodial ∞-categories $F\colon \mathfrak{C} \rightleftarrows \mathfrak{D}\colon R$ as *lax $\mathfrak{O}$-monoiadal* (resp. oplax $\mathfrak{O}$-monoidal, $\mathfrak{O}$-monoidal) if F and R are lax $\mathfrak{O}$-monoidal (oplax $\mathfrak{O}$-monoidal, $\mathfrak{O}$-monodial).

Proposition D.4 (Doctrinal adjunction for algebraic patterns)**.** *Suppose $F^{\otimes}\colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ is an $\mathfrak{O}$-monoidal functor whose underlying $\mathfrak{O}^{\mathrm{el,op}}$-functor F map admits an $\mathfrak{O}^{\mathrm{el,op}}$-right adjoint R. Then, there is a lax $\mathfrak{O}$-monoidal right adjoint $R^{\otimes} \vdash F^{\otimes}$ lifting R, such that*

(a) For all $O \in \mathfrak{O}$, the following commutes

$$\begin{tikzcd} \mathcal{C}_O \arrow[r, "R^{\otimes}_O"] \arrow[d, phantom, "\wr"] & \mathcal{D}_O \arrow[d, phantom, "\wr"] \\ \lim_{E \in \mathfrak{O}^{\mathrm{el}}_{/O}} \mathcal{C}_E \arrow[r, "\lim_E R_E"'] & \lim_{E \in \mathfrak{O}^{\mathrm{el}}_{/O}} \mathcal{D}_E \end{tikzcd}$$

(b) Given $f\colon P \to O$ in $\mathfrak{O}^{\mathrm{act}}$, the f-indexed lax structure map of R at $X \in \mathcal{C}_P$ is an adjunct arrow

$$\begin{tikzcd} & RX \arrow[dl, "c"] \arrow[dr, "gc"] & \\ f_\otimes RX \arrow[rr, dashed, "\ell_{R,f,X}"] & & Rf_\otimes X \end{tikzcd}$$

$$\Updownarrow$$

$$Ff_\otimes RX \simeq f_\otimes FRX \xrightarrow{f_\otimes \epsilon_{F,X}} f_\otimes X$$

where $\epsilon_{F,X}$ is the counit of the adjunction $R_P \vdash F_P$ applied at X.

If alternatively, F admits an $\mathfrak{O}^{\mathrm{el,op}}$-left adjoint L, then there is a oplax $\mathfrak{O}$-monoidal left adjoint $L^{\otimes} \dashv f^{\otimes}$ lifting L which is vertical-opposite to the lax $\mathfrak{O}$-monoidal left adjoint of $f^{\otimes,\mathrm{vop}}$; in particular,

(co-a) for all $O \in \mathfrak{O}$, the following commutes

$$\begin{tikzcd} \mathcal{C}_O \arrow[r, "L^{\otimes}_O"] \arrow[d, phantom, "\wr"] & \mathcal{D}_O \arrow[d, phantom, "\wr"] \\ \lim_{E \in \mathfrak{O}^{\mathrm{el}}_{/O}} \mathcal{C}_E \arrow[r, "\lim_E L_E"'] & \lim_{E \in \mathfrak{O}^{\mathrm{el}}_{/O}} \mathcal{D}_E \end{tikzcd}$$

(co-b) The oplax structure map $o_{L,f,X}\colon Lf_\otimes X \to f_\otimes LX$ is adjunct to the unit

$$f_\otimes X \xrightarrow{f_\otimes \eta_{F,X}} f_\otimes FLX \simeq Ff_\otimes LX.$$

Proof. First note that the second half of this proposition follows from the first half by taking vertical opposites. For the first half, we perform a similar argument to [HA, Cor. 7.3.2.7]; the Segal condition implies that the arrow R_O constructed above is a right adjoint to F_O (e.g. with unit given by $\lim_E \eta_E$) and these are compatible with inert-cocartesian transport, so [HA, Prop. 7.3.2.6] constructs a relative right adjoint $R^\otimes \dashv L^\otimes$ over $\mathfrak{O}$ which preserves inert-cocartesian arrows and satisfies condition (a); in particular, we've constructed a lax $\mathfrak{O}$-monoidal adjunction.

Now, to prove part (b), it suffices to note that the specified adjunct makes the triangle diagram commute; the universal property for cocartesian arrows then constructs a homotopy between $\ell_{R,f,X}$ and the adjunct to $f_\otimes, \epsilon_{F,X}$. $\square$

Before moving on, we point out a quick corollary for adjunctions between $\mathcal{T}$-categories of algebras.

Corollary D.5. *Suppose $F^\otimes\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is an I-symmetric monoidal functor whose underlying $\mathcal{T}$-functor admits a right adjoint $R\colon \mathcal{D} \to \mathcal{C}$. Then, there is a canonical lax I-symmetric monoidal lift $R^\otimes$ for R, inducing a natural lax I-symmetric monoidal adjunction*

$$F_*^\otimes\colon \underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\mathcal{C}) \rightleftarrows \underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\mathcal{D})\colon R_*^\otimes$$

by postcomposition for any $\mathcal{O}^\otimes$, so that $F_*^\otimes$ is I-symmetric monoidal. If R is I-symmetric monoidal, then so is $R_*^\otimes$; if R is also fully faithful, then so is $R_*^\otimes$.

Proof. The lax I-symmetric monoidal right adjoint is supplied by Proposition D.4, and the (lax) I-symmetric functors on $\mathcal{O}$-algebras are supplied by functoriality of $\underline{\mathrm{Alg}}_{\mathcal{O}}(-)$ developed in [Ste25, § 3.2].

Now, the counit for $F^\otimes \dashv R^\otimes$ is a lax I-symmetric monoidal natural transformation $\epsilon\colon F^\otimes R^\otimes \implies \mathrm{id}_{\mathcal{D}^\otimes}$, i.e. a functor $\Delta^1 \to \mathrm{Alg}_{\mathcal{D}}(\mathcal{D})$. A similar construction holds for η.

The construction of lifts to algebras yields a lift $\epsilon^\otimes\colon \Delta^1 \to \mathrm{Alg}_{\underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\mathcal{C})}(\underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\mathcal{C}))$ for ϵ and similarly a lift $\eta^\otimes$ for η; functoriality of this construction shows the triangle identities for $\eta^\otimes$ and $\epsilon^\otimes$, yielding the desired lax I-symmetric monoidal adjunction. Symmetric monoidality of $R_*^\otimes$ is proved in [Ste25, § 3.2]; fully faithfulness follows by the counit expression for fully faithful right $\mathcal{T}$-adjoints. $\square$

For instance, this applies to easily construct a lax I-symmetric monoidal right adjoint to

$$\mathbb{S}[-] = \Sigma_*^\infty\colon \underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes\left(\underline{\mathcal{S}}_G\right) \to \underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes\left(\underline{\mathrm{Sp}}_G\right).$$

D.2. **Assembly for I-operadic left Kan extension.** We're ready to conclude.

Proof of Proposition D.2. Let $\varphi_!^\otimes$ be the oplax I-symmetric monoidal left adjoint supplied by Proposition D.4. Now, the underlying natural transformation of the oplax structure map, $U_{\mathrm{Env}}o_{\varphi_!,\mathrm{Ind}_V^{\mathcal{T}} S \to V,(X_U)_{U\in\mathrm{Orb}(S)}}(P)$, is given by the map

$$\underline{\mathrm{colim}}_{O\in\mathrm{Env}_J(\mathcal{O})_{\underline{V},/P}} \bigotimes_U^S X_U(O) \longrightarrow \bigotimes_U^S \underline{\mathrm{colim}}_{O\in\mathrm{Env}_J(\mathcal{O})_{\underline{V},/P}} X_U(O)$$

adjunct the unit of the adjunction $\mathrm{Env}_J(\varphi)_! \dashv \mathrm{Env}_J(\varphi)^*$; unwinding definitions, it is the assembly map for the indicated $\mathcal{T}$-colimits. This is an equivalence by assumption, so conservativity of U implies that $o_{\varphi_!,\mathrm{Ind}_V^{\mathcal{T}} S\to V,(X_U)}$ is a natural equivalence for all $S,(X_U)$, implying that $\varphi_!^\otimes$ is I-symmetric monoidal. The result then follows from Proposition D.4. $\square$

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