YOU CAN CONSTRUCT G-COMMUTATIVE ALGEBRAS ONE NORM AT A TIME

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Abstract. We define the category of G-operads and the hierarchy of *generalized $\mathcal{N}_\infty$-operads*, which are G-suboperads of $\mathrm{Comm}_G^\otimes$. We exhibit an isomorphism between the category of generalized $\mathcal{N}_\infty$-operads and the self-join poset

$$\mathrm{Op}_G^{GN\infty} \simeq \mathrm{Ind}-\mathrm{Sys}_G \star \mathrm{Ind}-\mathrm{Sys}_G,$$

where $\mathrm{Ind}-\mathrm{Sys}_G$ is the poset of *indexing systems* in G. This recognizes generalized $\mathcal{N}_\infty$-operads as parameterizing *some commutative multiplicative transfers and possibly a commutative multiplication*. Indeed, their algerbas in semiadditive Cartesian categories are incomplete Mackey functors and their algebras in Mackey functors recover incomplete Tambara functors when they are N_∞ *operads*, i.e. when they contain $\mathbb{E}_\infty$.

After this, we discuss some in-progress research. Namely, we construct a *Boardman-Vogt tensor product* of G-operads and demonstrate that tensor products of genereralized $\mathcal{N}_\infty$ operads correspond with joins in $\mathrm{Ind}-\mathrm{Sys}_G \star \mathrm{Ind}-\mathrm{Sys}_G$ i.e. there is an $\mathcal{N}_{(I\vee J)\infty}$-monoidal equivalence

$$\mathbf{Alg}_{\mathcal{N}_{I\infty}} \mathbf{Alg}_{\mathcal{N}_{J\infty}} \mathcal{C} \simeq \mathbf{Alg}_{\mathcal{N}_{(I\vee J)\infty}} \mathcal{C}$$

for all $\mathcal{N}_{(I\vee J)\infty}$-monoidal categories $\mathcal{C}$, allowing G-commutative structures to be constructed "one norm at a time."

Foreword. The following are notes prepared for a casual talk in the zygotop seminar concerning research which is currently in-progress. The reader should read with the understanding that they are particularly error-prone, as the non-cited results herein amount to the communication of a pre-draft of a paper in a casual setting. The reader should also henceforth implicitly insert the text $\infty-$ before the words operad and category.

1. Introduction

In [Dre71], the concept of a *Mackey functor* was introduced; this structure was described as consisting of functors $M_I : \mathcal{O}_G \to \mathbf{Mod}_R$ and $M_R : \mathcal{O}_G^{\mathrm{op}} \to \mathbf{Mod}_R$ which agree on $\mathcal{O}_G^{\simeq}$ and satisfying the *double coset formula*

$$R_J^H I_K^H = \prod_{x\in[J\backslash H/K]} I^J_{J\cap xKx^{-1}} \cdot \mathrm{conj}_X R_{x^{-1}Jx\cap K}$$

for all $J, K \subset H$, where $R_J^K := M_R(G/J \to G/K)$ and similar for I. The ur-example of this is the assignment $H \mapsto A(H)$, where $A(H)$ is the representation ring of H, with covariant functoriality Ind and contravariant functoriality Res. This was repackaged and generalized into the modern definition of the *category of $\mathcal{C}$-valued G-Mackey functors*

$$\mathcal{M}_G(\mathcal{C}) := \mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_G), \mathcal{C}),$$

where $\mathbb{F}_G$ denotes the category of finite G-sets.

In parallel, the concept of *transfer maps in group cohomology* was being developed in [Eve63], later lifted to genuine equivariant cohomology in [GM97], and finally developed as a functor

$$N_H^G : \mathrm{Sp}_H \to \mathrm{Sp}_G$$

in [HHR16], which played a crucial role in the solution to the Kervaire invariant one problem. This functor is meant to represent the *indexed tensor power*, e.g. $\mathrm{Res}_e^G N_e^G X \simeq X^{|G|}$, and the associated action is meant to represent the permutation action of G on the factors. These were noted in [HH16] to satisfy the conditions of a *Symmetric monoidal Mackey functor*, a notion they distinguished from their notion of *G-symmetric monoidal categories* due to coherence issues.

In the broad program announced in [Bar+16], the correct notion of *G-symmetric monoidal G-∞-categories* (henceforth G-symmetric monoidal categories) was introduced:

Definition 1.1. Let $\mathcal{C}$ have finite products. Then, the category of G-commutative monoids in $\mathcal{C}$ is

$$\mathrm{CMon}_G(\mathcal{C}) := \mathcal{M}_G(\mathcal{C}).$$

The category of G-symmetric monoidal categories is $\mathrm{CMon}_G(\mathbf{Cat})$.

We similarly define the *category of small G-categories* as

$$\mathbf{Cat}_G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Cat}) \simeq \mathbf{Cat}^{\mathrm{cocart}}_{/\mathcal{O}_G^{\mathrm{op}}},$$

where the equivalence is the *straightening-unstraightening construction* of [HTT], and $\mathcal{O}_G^{\mathrm{op}} \subset \mathbb{F}_G$ denotes the full subcategory of transitive G-sets, henceforth referred to as the *orbit category*. We may informally summarize the structure of a G-symmetric monoidal category $\mathcal{C}^\otimes \in \mathrm{CMon}_G(\mathbf{Cat})$ as consisting of, for every conjugacy class (H) of G, a category with Weyl group action $\mathcal{C}_H \in \mathbf{Cat}^{BW_GH}$, as well as functors

$$\begin{aligned} \otimes_H^2 &: \mathcal{C}_H^2 \rightarrow \mathcal{C}_H, \\ N_K^H &: \mathcal{C}_K \rightarrow \mathcal{C}_H, \\ \mathrm{Res}_K^H &: \mathcal{C}_H \rightarrow \mathcal{C}_K \end{aligned}$$

for all subconjugacy classes (K) of (H). These are supplied with coherent data recognizing them as associative, commutative, unital, and compatible with each other and the Weyl group action. The maps Res encode an underlying G-category $\mathcal{C}$ of $\mathcal{C}^\otimes$, and N_K^H is pronounced "the norm from K to H."

Given $\mathcal{C}^\otimes$ a G-symmetric monoidal category, we may informally define a *G-commutative algebra in $\mathcal{C}$* to be a tuple of objects $(X_H) \in \prod_{G/H \in \mathcal{O}_G} \mathcal{C}_H$ satisfying

$$X_H \simeq \mathrm{Res}_H^G X_G$$

together with structure maps

$$\begin{aligned} \mu_H^2 &: X_H^{\otimes 2} \rightarrow X_H \\ \mathrm{tr}_K^H &: N_H^K X_K \rightarrow X_H \end{aligned}$$

for all $H \subset K$, together with coherent associativity, commutativity, and unitality data. We may intuitively view these data as altogether specifying that these structure maps jointly construct a contractible space of maps

$$X^{\otimes S} \rightarrow X_H$$

for all finite H-sets $S \in \mathbb{F}_H$, where

$$X^{\otimes S} := \bigotimes_{H/K \in \mathrm{Orb}(S)} N_K^H X_K.$$

The map tr_K^H is pronounced "the transfer from K to H." When $\mathcal{C}^\otimes = \mathcal{M}_G(\mathcal{C})^\otimes$ with the *HHR norm* G-symmetric monoidal structure of [HH16], these are called *G-Tambara functors valued in $\mathcal{C}$*.

This talk concerns various relaxations of the notion of G-commutative algebras. Namely, we will define a symmetric monoidal closed category Op_G of *(colored) G-operads*, whose internal hom $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is called the *operad of algebras under pointwise tensors*, and whose tensor product is called the *Boardman-Vogt tensor product*.

We are particularly interested in $\mathcal{N}_\infty$ *operads*, which interpolate between $\mathbb{E}_\infty$ and the G-operad Comm_G which encodes G-commutative algebras by adding a subset of the transfers parameterized by Comm_G. These transfers are required to be structured according to the notion of a *transfer system*.

Definition 1.2. A *G-transfer system* is a core-preserving wide subcategory $\mathcal{O}_G^{\simeq} \subset T \subset \mathcal{O}_G$ which is closed under subconjugacy. An *indexing system* is a wide subcategory $I \subset \underline{\mathbb{F}}_G$ induced by a transfer system under taking coproducts.

A *generalized indexing system* is a core-preserving subcategory $I \subset \underline{\mathbb{F}}_G$ which is either an indexing system or is attained by one by removing all non-isomorphisms of trivial H-sets. The poset of indexing systems under inclusion is denoted $\mathrm{Ind}-\mathrm{Sys}_G$, and the poset of generalized indexing systems is denoted $\widetilde{\mathrm{Ind}-\mathrm{Sys}}_G$.

It is not hard to see that there is an equivalence of posets

$$\widehat{\mathrm{Ind}-\mathrm{Sys}}_G \simeq \mathrm{Ind}-\mathrm{Sys}_G \star \mathrm{Ind}-\mathrm{Sys}_G,$$

and in particular, generalized indexing systems decompose into two different join-stable copies of indexing systems, depending on whether maps of trivial G-sets are included.

Given a generalized indexing system I, we will construct an operad called $\mathcal{N}_{I\infty}^{\otimes}$ encoding precisely the maps tr_K^H such that $K \hookrightarrow H$ is in I, as well as encoding the map μ_H if and only if I is an indexing system. The main theorem of this talk characterizes the tensor products of generalized $\mathcal{N}_\infty$ operads.

Theorem A. *There is a fully faithful and symmetric monoidal inclusion*

$$\mathcal{N}_{(-)\infty}^{\otimes} : \widehat{\mathrm{Ind}-\mathrm{Sys}}_G^{\amalg} \hookrightarrow \mathrm{Op}_G^{\otimes}$$

whose image consists of the G-suboperads of $\mathrm{Comm}_G^{\otimes}$*, and when restricted to the indexing systems has image consisting of G-operads $\mathcal{O}^{\otimes}$ possessing diagrams $\mathbb{E}_\infty^{\otimes} \subset \mathcal{O}^{\otimes} \subset \mathrm{Comm}_G^{\otimes}$. In particular, for $\mathcal{C}^{\otimes}$ an $\mathcal{N}_{(I\vee J)\infty}$-monoidal category, there is a canonical $\mathcal{N}_{(I\vee J)\infty}$-monoidal equivalence*

$$\underline{\mathbf{Alg}}_{\mathcal{N}_{I\infty}}^{\otimes} \underline{\mathbf{Alg}}_{\mathcal{N}_{J\infty}}^{\otimes} \mathcal{C} \simeq \underline{\mathbf{Alg}}_{\mathcal{N}_{(I\vee J)\infty}}^{\otimes} \mathcal{C}.$$

We say an inclusion of subgroup $H \subset K$ is *atomic* if it is proper and there exist no chains of proper subgroup inclusions $H \subset J \subset K$. More generally, we say that a conjugacy class $(H) \in \mathrm{Conj}(G)$ is an *atomic subclass* of (K) if there exists an atomic inclusion $\tilde{H} \subset \tilde{K}$ with $\tilde{H} \in (H)$ and $\tilde{K} \in (K)$, and we say that (K) is atomic if the canonical inclusion $1 \hookrightarrow K$ is atomic.

Given $(H) \subset (K)$ an atomic subclass, we refer to the $\mathcal{N}^\infty$-operad corresponding to the minimal index system containing the inclusion $H \hookrightarrow K$ as $\mathcal{N}^\infty(H,K)$. When $(H) = (1)$, we instead simply write $\mathcal{N}^\infty(K)$.

Corollary B. *Let $1 = G_n \subset G_{n-1} \subset \cdots \subset G_0 = G$ be a maximal subgroup series of a finite group, and let $\mathcal{C}$ be a G-symmetric monoidal category. Then, there exists a canonical G-symmetric monoidal equivalence*

$$\underline{\mathbf{Alg}}_{\mathcal{N}^\infty(G_1,G_0)}^{\otimes} \cdots \underline{\mathbf{Alg}}_{\mathcal{N}^\infty(G_n,G_{n-1})}^{\otimes} \mathcal{C} \simeq \underline{\mathrm{CAlg}}_G^{\otimes} \mathcal{C}$$

Furthermore, if $G \simeq H \times J$, then

$$\underline{\mathrm{CAlg}}_H^{\otimes} \underline{\mathrm{CAlg}}_J^{\otimes} \mathcal{C} \simeq \underline{\mathrm{CAlg}}_G^{\otimes} \mathcal{C}.$$

Remark. One may worry about the comparison between models for G-operads, as our notion of N_∞-operads is ostensibly embedded deep within the world of G-∞-operads, which are not known to be equivalent to the ∞-category presented by the graph model structure or by genuine G operads.

However, some work has been done to simplify the story of N_∞ operads in the model story; in particular, by [Rub21, Thm 2.16, 3.6], the full ∞-category of the ∞-category of *genuine* G-operads is equivalent to $\mathrm{Ind}-\mathrm{Sys}_G$ via a functor A which sits in a commutative diagram

$$\begin{tikzcd} \mathrm{Op}_G^{\mathrm{gen},N\infty} \arrow[r, "N|_{N\infty}"] \arrow[dr, "A"'] & \mathrm{Op}_G^{N\infty} \arrow[d, "A"] \\ & \mathrm{Ind}-\mathrm{Sys}_G \end{tikzcd}$$

where we use that the functor N of [BP21] is canonically ∞-categorical when restricted to full subcategores of $\mathrm{Op}_G^{\mathrm{gen}}$ which happen to be 1-categories and map to a 1-subcategory of Op_G. Both functors named A are equivalences (c.f. **??**Ex 2.4.7]Nardin), and hence $N|_{N\infty}$ is an equivalence.

2. The ideas

2.1. **Fibrous patterns.** In order to precisely define G-operads, the most efficient way will be to go through the technology of *algebraic patterns*, a concept first defined by German mathematician Honyi Chu and the Norwegian mathematician Rune Haugseng, who generally referred to them using the letter $\mathcal{O}$.

Definition 2.1. An *algebraic pattern* is an ∞-category $\mathcal{O}$, together with a factorization system $(\mathcal{O}^{\mathrm{int}}, \mathcal{O}^{\mathrm{act}})$ of $\mathcal{O}$ and a full subcategory $\mathcal{O}^{\mathrm{el}} \subset \mathcal{O}^{\mathrm{int}}$. The *category of algebraic patterns* is the full subcategory

$$\mathrm{AlgPatt} \subset \mathrm{Fun}(D, \mathbf{Cat})$$

spanned by algebraic patterns, where $D := \bullet \to \bullet \to \bullet \leftarrow \bullet$.

Maps in $\mathcal{O}^{\mathrm{int}}$ and $\mathcal{O}^{\mathrm{act}}$ are pronounced *inert and active maps*, and objects of $\mathcal{O}^{\mathrm{el}}$ are pronounced *elementary objects*. For instance, $\mathbb{F}_*$, together with its inert and active maps as defined in [HA, § 2] and elementary objects $\{\langle 1 \rangle\}$ determines an algebraic pattern. In analogy with [HA, § 2], we will use these to develop a notion of operads, called *fibrous patterns*.

Definition 2.2. Let $\mathcal{O}$ be an algebraic pattern. A *fibrous $\mathcal{O}$-pattern* is a map of algebraic patterns $\pi : O \to \mathcal{O}$ such that

(1) O has π-cocartesian lifts for inert morphisms of $\mathcal{O}$,
(2) (Segal condition for colors) For every active morphism $\omega : V_0 \to V_1$ in $\mathcal{O}$, the functor

$$O^{\simeq}_{V_0} \to \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{V_1/}} O^{\simeq}_{\omega_{\alpha,!} V_1}$$

induced by cocartesian transport along ω_α is an equivalence, where $\omega_{(-)} : \mathcal{O}^{\mathrm{el}}_{Y/} \to \mathcal{O}^{\mathrm{int}}_{X/}$ is the inert morphism appearing in the inert-active factorization of $\alpha \circ \omega$, and
(3) (Segal condition for multimorphism) for every active morphism $\omega : V_1 \to V_2$ in $\mathcal{O}$ and all objects $X_i \in O_{\mathcal{O}_{V_i}}$, the commutative square

$$\begin{tikzcd}
\mathrm{Map}_O(X_0, X_1) \arrow[r] \arrow[d] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{V_1/}} \mathrm{Map}_O(X_0, \omega_{\alpha,!} X_1) \arrow[d] \\
\mathrm{Map}_{\mathcal{O}}(V_0, V_1) \arrow[r] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O_1/}} \mathrm{Map}_{\mathcal{O}}(O_0, \omega_{\alpha,!} O_1)
\end{tikzcd}$$

is cartesian.

A fibrous $\mathcal{O}$-pattern $\pi : C \to \mathcal{O}$ is a *Segal $\mathcal{O}$-category* if π is a cocartesian fibration. The category of fibrous $\mathcal{O}$-patterns is the full subcategory

$$\mathrm{Fbrs}(\mathcal{O}) \subset \mathrm{AlgPatt}_{/\mathcal{O}}$$

spanned by fibrous patterns, and the category of Segal $\mathcal{O}$-categories is the full subcategory of

$$\mathrm{Seg}_{\mathcal{O}}(\mathbf{Cat}) \subset \mathbf{Cat}^{\mathrm{cocart}}_{/\mathcal{O}}$$

spanned by Segal $\mathcal{O}$-categories.

We state one technical lemma:

Lemma 2.3. *All of the inclusions*

$$\mathrm{Seg}(\mathcal{O}) \to \mathrm{Fbrs}(\mathcal{O}) \hookrightarrow \mathrm{AlgPatt}_{/\mathcal{O}} \to \mathbf{Cat}_{/\mathcal{O}} \to \mathbf{Cat}$$

have left adjoints; in particular, the full subcategory $\mathrm{Fbrs}(\mathcal{O}) \subset \mathrm{AlgPatt}_{\mathcal{O}}$ *is localizing.*

We refer to the left adjoint $\mathrm{Env} : \mathrm{Fbrs}(\mathcal{O}) \to \mathrm{Seg}(\mathcal{O})$ as the *Segal envelope*, and we use it analogously to the *symmetric monoidal envelope*, reducing the question of characterizing maps of fibrous patterns into Segal $\mathcal{O}$-categories into simply a question of characterizing maps of Segal $\mathcal{O}$-categories, which is much simpler.

Example 2.4:

Definition 2.5. Given the data of $\mathcal{X}$ a category, $\mathcal{X}_b, \mathcal{X}_f$ wide subcategories, and $\mathcal{X}_0 \subset \mathcal{X}_b$ a full subcategory, we define the *span pattern* $\mathrm{Span}_{b,f}(\mathcal{X}; \mathcal{X}_0)$ to have:

- underlying category $\mathrm{Span}_{b,f}(\mathcal{X})$ whose objects are objects in $\mathcal{X}$ and whose morphisms $X \to Z$ are spans

$$X \xleftarrow{B} Y \xrightarrow{F} Z$$

with $B \in \mathcal{X}_b$ and $F \in \mathcal{X}_f$.

- inert morphisms $\mathcal{X}_b^{\mathrm{op}} \subset \mathrm{Span}(\mathcal{X})$.
- active morphisms $\mathcal{X}_f \subset \mathrm{Span}(\mathcal{X})$.
- Elementary objects $\mathcal{X}_0^{\mathrm{el}} \subset \mathcal{X}_b^{\mathrm{op}}$.

Then, for instance we have the following:

Theorem 2.6 ([BHS22])**.** *Pullback along the inclusion $\mathbb{F}_* \hookrightarrow \mathrm{Span}(\mathbb{F})$ induces an equivalence on the categories of fibrous patterns and Segal categories.*

2.2. **G-operads and $\mathcal{I}$-operads.** There is an adjunction

$$\mathrm{Tot} : \mathbf{Cat}_G \rightleftarrows \mathbf{Cat} : \mathrm{CoFr}^G$$

where Tot takes the total category of a cocartesian fibration and $\mathrm{CoFr}^G(\mathcal{C})$ is classified by functor categories

$$\mathrm{CoFr}^G(\mathcal{C})_H := \mathrm{Fun}(\mathcal{O}_H^{\mathrm{op}}, \mathcal{C})$$

with functoriality dictated by pullback. In particular, the *G-category of small G-categories* $\underline{\mathbf{Cat}}_G := \mathrm{CoFr}^G(\mathcal{C})$ has G-fixed points given by $\mathbf{Cat}$.

Remark. Elmendorf's theorem may be reinterpreted in this language as the statement that the *G-category of G-spaces* $\underline{\mathcal{S}}_G$ is G-cofreely generated by $\mathcal{S}$.

Let $\underline{\mathbb{F}}_G := \mathrm{CoFr}^G(\mathbb{F})$ and let $\underline{\mathbb{F}}_{G,*} := \mathrm{CoFr}^G(\mathbb{F}_*)$. Then, there is an equivariant lift of theorem 2.6.

Theorem 2.7 ([BHS22])**.** *Pullback along the composition $\underline{\mathbb{F}}_{G,*} \hookrightarrow \mathrm{Span}(\mathrm{Tot}\underline{\mathbb{F}}_G) \xrightarrow{U} \mathrm{Span}(\mathbb{F}_G)$ induces an equivalence on the categories of fibrous patterns and Segal categories, where $\mathbb{F}_G$ is the category of G-sets.*

Definition 2.8. The *category of G-operads* is the category of fibrous patterns

$$\mathrm{Op}_G := \mathrm{Fbrs}(\mathrm{Span}(\mathbb{F}_G)).$$

If $\mathcal{O}, \mathcal{P}$ are G-operads, the *category of $\mathcal{O}$-algebras in $\mathcal{P}$* is the functor category of algebraic patterns

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{P}) := \mathrm{Fun}_{\mathrm{AlgPatt}}(\mathcal{O}, \mathcal{P}).$$

We may equivalently characterize $\mathcal{O}$-algebras in $\mathcal{P}$ as functors which preserve cocartesian lifts of inert morphisms. In order to identify G-operads, we use the following exercise in category theory which was carried out in [BHS22, § 5.2].

Proposition 2.9. *An identity-on-objects functor $\pi : \mathcal{O} \to \mathrm{Span}(\mathbb{F}_G)$ is a G-operad if and only if it satisfies the following conditions:*

(1) $\mathcal{O}$ has π-cocartesian lifts for inert morphisms of $\mathrm{Span}(\mathbb{F}_G)$.
(2) For every map of G-sets $S \to T$, the inert morphisms $\{U \leftarrow T \mid U \in \mathrm{Orb}(T)\}$ induce equivalences

$$\mathrm{Map}_{\mathcal{O}}(S, T) \simeq \prod_{U \in \mathrm{Orb}(T)} \mathrm{Map}_{\mathcal{O}}(S, U).$$

Furthermore, a cocartesian fibration $\pi : \mathcal{O} \to \mathrm{Span}(\mathbb{F}_G)$ is a Segal $\mathrm{Span}(\mathbb{F}_G)$-category if and only if it unstraightens to a G-symmetric monoidal category.

We may further reorganize this through the following elementary lemma about G-sets.

Lemma 2.10. *The assignment $\varphi : T \mapsto \mathrm{Ind}_H^G T \to G/H$ underlies an equivalence of categories*

$$\mathbb{F}_H \simeq (\mathbb{F}_G)_{/G/H}.$$

Write $\underline{\Sigma}_G \simeq \mathrm{CoFr}^G(\mathbb{F}^{\simeq})$. By applying lemma 2.10 and taking cores of slice categories, we construct a forgetful functor

$$\mathcal{O}_{\mathrm{sseq}} : \mathrm{Op}_G^{\mathrm{one-object}} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_G, \mathcal{S})$$

with value on $S \in \mathbb{F}_H$ given by $\pi_{\mathcal{O}}^{-1}(\mathrm{Ind}_H^G S \to G/H)$. We refer to $\mathcal{O}(S) := \mathcal{O}_{\mathrm{sseq}}(S)$ as the *space of S-ary operations*. This functor is further analyzed in section 3.1, where e.g. it is shown to be conservative.

An early observation about genuine equivariant homotopy coherent algebraic structures is that the structure of transfers *does not come canonically* from an $\mathbb{E}_\infty$-structure; that is, $\mathbb{E}_\infty \in \mathrm{Op}_G$ is not terminal. The failure of $\mathbb{E}_\infty$ to be terminal is parameterized by the category of *generalized N^∞-operads*:

Definition 2.11. Write $\mathrm{Comm}_G^\otimes := \left(\mathrm{Span}(\mathbb{F}_G) = \mathrm{Span}(\mathbb{F}_G)\right)$ for the terminal G-operad. A G-operad $\mathcal{O}^\otimes$ is a *generalized* $\mathcal{N}^\infty$*-operad* if the unique morphism $\mathcal{O}^\otimes \to \mathrm{Comm}_G^\otimes$ is a monomorphism, i.e. it has one object and

$$\mathcal{O}(S) \in \{*, \varnothing\}$$

for all $S \in \mathbb{F}_H$.

A generalized $\mathcal{N}^\infty$ operad $\mathcal{N}_{\infty I}$ is an N^∞ *operad* if it admits a map

$$\mathbb{E}_\infty \to \mathcal{O}^\otimes,$$

i.e. $\mathcal{O}(S) \simeq *$ whenever $S \in \mathbb{F}_H$ has trivial H-action.

Write $\mathrm{Op}_G^{GN\infty}$ for the full subcategory consisting of generalized $\mathcal{N}_\infty$-operads. The following proposition is an exercise in category theory, and establishes that a map to an $\mathcal{N}_\infty$ operad is a *property*, not a structure.

Proposition 2.12. *Given $\mathcal{N}_{I\infty} \in \mathrm{Op}_G^{GN\infty}$ a generalized $\mathcal{N}_\infty$ operad, the forgetful functor*

$$\mathrm{Op}_{G,/\mathcal{N}_{I\infty}} \to \mathrm{Op}_G$$

is fully faithful.

Proof idea. It is equivalent to prove that $\mathrm{Map}(\mathcal{O}, \mathcal{N}_{I\infty}) \in \{*, \varnothing\}$ for all $\mathcal{O} \in \mathrm{Op}_G$ In fact, there is a localizing (1-) subcategory $N : \mathrm{Op}_{1,G} \hookrightarrow \mathrm{Op}_G$ consisting of operads whose structure spaces are discrete, and whose localization functor $h : \mathrm{Op}_G \to \mathrm{Op}_{1,G}$ takes π_0 of the structure spaces. $\mathcal{N}_{I\infty}$ evidently lies in $\mathrm{Op}_{1,G}$, so we have

$$\mathrm{Map}_{\mathrm{Op}_G}(\mathcal{O}, \mathcal{N}_{I\infty}) \simeq \mathrm{Hom}_{\mathrm{Op}_{1,G}}(h\mathcal{O}, \mathcal{N}_{I\infty}).$$

Hence it suffices to check that the latter set is empty or contractible. This is easy to see in $\mathrm{Op}_{1,G}$, since $\mathrm{Hom}(-,*)$ and $\mathrm{Hom}(-,\varnothing)$ are always either empty or contractible. □

In particular, this implies that $\mathrm{Op}_G^{GN\infty}$ is a poset, so we'd like to identify this poset. There is a functor

$$A : \mathrm{Op}_G \to \widehat{\mathrm{Ind}-\mathrm{Sys}}_G$$

called the *admissible sets* with value over G/H given by

$$A(\mathcal{O})_{/(G/H)} := \left\{S \to G/H \mid \pi_{\mathcal{O}}^{-1}(S \to G/H) \neq \varnothing\right\}$$

and extended to general G-sets by coproducts. The following proposition is an exercise in category theory (see [NS22, Ex 2.4.7], or the original references [BH15; GW18; Rub21]):

Proposition 2.13. *The restricted functor*

$$A : \mathrm{Op}_G^{GN\infty} \to \widehat{\mathrm{Ind}-\mathrm{Sys}}_G$$

is an equivalence of categories.

We denote by $\mathcal{N}_{(-)\infty}$ the composite functor

$$\mathcal{N}_{(-)\infty} : \widehat{\mathrm{Ind}-\mathrm{Sys}}_G \xrightarrow{A^{-1}} \mathrm{Op}_G^{GN\infty} \hookrightarrow \mathrm{Op}_G$$

Using this, we finally define *I-operads*.

Definition 2.14. Let I be a generalized indexing system. Then, the *category of I-operads* is the slice category

$$\mathrm{Op}_I := \mathrm{Op}_{G,/\mathcal{N}_{\infty I}^\otimes}.$$

Given $\mathcal{O}^\otimes, \mathcal{P}^\otimes \in \mathrm{Op}_I$, the *category of $\mathcal{O}$-algebras in $\mathcal{P}$* is the full subcategory

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}) \subset \mathrm{Fun}_{/\mathcal{N}_{\infty I}^\otimes}(\mathcal{O}^\otimes, \mathcal{C}^\otimes)$$

spanned by maps of $\mathcal{I}$-operads.

Remark. The notation $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$ does not include $\mathcal{I}$. This presents no problem; indeed, by proposition 2.12, the categories of $\mathcal{O}$-algebras in $\mathcal{P}$ considered over various indexing systems (including the terminal one, i.e. in G-operads) are canonically equivalent to one another.

A useful property of these are that G operads *fibered* over $\mathcal{O}^\otimes$ have an intrinsic description in terms of $\mathcal{O}$. We may state these in the language of fibrous patterns.

Proposition 2.15 ([BHS22, Cor 4.1.17])**.** *Let $\mathcal{O}$ be a fibrous $\mathscr{C}$-pattern. Then, the pushforward functor $\pi_! : \mathrm{AlgPatt}_{/\mathcal{O}} \to \mathrm{AlgPatt}_{/\mathscr{C}}$ preserves fibrous patterns, and the associated functor*

$$\pi_! : \mathrm{Fbrs}(\mathcal{O}) \to \mathrm{Fbrs}(\mathscr{C})_{/\mathcal{O}}$$

is an equivalence of categories.

In particular, the category of I-operads is covariantly functorial in I, and it possesses an intrinsic expression along the lines of **??**.

Example 2.16:
Let $\mathcal{F} \subset \mathcal{O}_G$ be a *family*, i.e. a collection of subgroups of G closed under sub-conjugation. Then, $\mathcal{F} \cup \mathcal{O}_G^{\simeq}$ is a transfer system, and we denote by $\mathcal{I}_{\mathcal{F}}$ the corresponding indexing system.

Let V be a real orthogonal G-representation, let $\mathcal{F}_V$ is the family consisting of subgroups H such that $V^H \neq *$, and let $\mathcal{I}_V := \mathcal{I}_{\mathcal{F}_V}$. Then, there is an I_V-operad $\mathbb{E}_V$ of *little V-disks*, which may be informally understood to have S-ary operations the H-equivariant embeddings $S \hookrightarrow V$:

$$\mathbb{E}_V(S) \simeq \mathrm{Conf}_H(S, V).$$

This along with a computation of the G-symmetric monoidal envelope was carried out in **??**. These participate in *equivariant infinite loop space theory*, in the sense that there is a fully faithful embedding

$$\{V - \textit{loop spaces}\} \hookrightarrow \mathbf{Alg}_{\mathbb{E}_V}(\mathcal{S}_G)$$

with image given by the $\mathbb{E}_V$ spaces satisfying a grouplike condition, up to model categorical weirdness. See [GM11] for details.

2.3. **The BV tensor product.** By lemma 2.3, the category of algebraic patterns has a cartesian monoidal structure such that the *underlying category* functor $U : \mathrm{AlgPatt}^{\times} \to \mathbf{Cat}^{\times}$ is symmetric monoidal.

Definition 2.17. The category of *symmetric monoidal algebraic patterns* is CMon(AlgPatt).

By [HA, § 2.2], a symmetric monoidal structure on $\mathscr{C}$ endows on the slice category $\mathrm{AlgPatt}^{\otimes}_{/\mathscr{C}}$ a symmetric monoidal structure, which we may view as taking $\mathcal{O}, \mathcal{P}$ to the tensor product

$$\mathcal{O} \times \mathcal{P} \to \mathscr{C} \times \mathscr{C} \to \mathscr{C}.$$

Definition 2.18. The *Boardman-Vogt symmetric monoidal category of fibrous $\mathscr{C}$-patterns* is the localized symmetric monoidal structure

$$\mathrm{Fbrs}(\mathscr{C})^{\otimes} \leftrightarrows \mathrm{AlgPatt}^{\times}_{/\mathscr{C}}.$$

We may view the tensor product of fibrous $\mathscr{C}$-patterns as yielding the localized composite

$$\mathcal{O} \otimes \mathcal{P} := L_{\mathrm{Fbrs}}(\mathcal{O} \times \mathcal{P} \to \mathscr{C} \times \mathscr{C} \to \mathscr{C}).$$

Note that the category $\mathbb{F}_G$ has finite products, and any indexing system $\mathcal{I}$ is closed under products. In particular, this endows $i : \mathcal{N}^{\otimes}_{\mathcal{I}\infty} \to \mathrm{Span}(\mathbb{F}_G)$ with the structure of a map of symmetric monoidal algebraic patterns under $\mathrm{Span}(\times)$.

Definition 2.19. The *Boardman-Vogt symmetric monoidal category of I-operads* is

$$\mathrm{Op}_{\mathcal{I}}^{\otimes} := \mathrm{Fbrs}(\mathcal{N}_{\mathcal{I}\infty})$$

Proposition 2.20. *Given an inclusion $i : \mathcal{N}_{\mathcal{I}\infty} \hookrightarrow \mathcal{N}_{\mathcal{J}\infty}$, pushforward along i yields a functor*

$$i_! : \mathrm{Op}_{\mathcal{I}}^{\otimes} \to \mathrm{Op}_{\mathcal{J}}^{\otimes}$$

realizing $\mathrm{Op}_{\mathcal{I}}$ as a symmetric monoidal colocalizing subcategory of $\mathrm{Op}_{\mathcal{J}}$.

The verification of this comes down to the following fact, which follows from the results of [HA, § 2.2.2], and is almost generalized by [Bar23, p. 2.37].

Lemma 2.21. *Given $f : X \to Y$ a map of commutative algebra objects in C a symmetric monoidal category, the associated functor $f_! : C_{/X} \to C_{/Y}$ lifts to a canonical symmetric monoidal functor between the slice symmetric monoidal categories.*

We may "see" this fact by staring at the following commutative diagram:

$$\begin{tikzcd} & X \otimes X \arrow[r] \arrow[d] & X \arrow[d] \\ A \otimes B \arrow[ur] \arrow[dr] & & \\ & Y \otimes Y \arrow[r] & Y \end{tikzcd}$$

The BV tensor product satisfies a mapping-out property; namely, we review in section 3.3 the construction due to [NS22, § 5.3] of the operad $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{P}}(Q)$, and we prove the following theorem.

Theorem 2.22. *There is a natural equivalence of operads*

$$\underline{\mathbf{Alg}}^{\otimes}_{O \otimes \mathcal{P}} Q \simeq \underline{\mathbf{Alg}}^{\otimes}_{O} \underline{\mathbf{Alg}}^{\otimes}_{\mathcal{P}} Q$$

realizing $\mathbf{Alg}^{\otimes}_{\mathcal{P}}(-)$ *as an internal hom for the BV tensor product.*

2.4. **Summary of the argument.** We would like to construct an equivalence $\mathcal{N}_{I\infty} \otimes \mathcal{N}_{J\infty} \simeq \mathcal{N}_{(I \vee J)\infty}$. Let's begin with the special case $I \subset J$; in this case, we can say something stronger.

Proposition 2.23. *If O is a one-object G-operad, then the map $\mathcal{N}^{\infty}(I) \to \mathcal{N}^{\infty}(I) \otimes O$ is an I-equivalence; in particular, $\mathcal{N}^{\infty}(I)$ is $\otimes$-idempotent.*

To prove this, we use [NS22, Cor 5.3.9]; in particular, they generalize [HA] to verify that any of the following conditions are true of $\mathbf{Alg}^{\otimes}_{\mathcal{N}^{\infty}(I)}(C)$, which recognize it as *I-cocartesian*:

Lemma 2.24. *The following are equivalent:*

(1) *For all one-object I-operads O, the forgetful functor $\mathbf{Alg}^{O}(C) \to C$ is an equivalence.*
(2) *For all maps $f : S \to T$ in I, the action map $f_{\otimes} : C_S \to C_T$ is left adjoint to the pullback map $f^* : C_S \to C_T$.*

We prove this in **??**. Having proved this, we acquire a (unique) diagram

$$\begin{tikzcd} \mathcal{N}_{I\infty} \arrow[dr] \arrow[drr] & & & \\ & \mathcal{N}_{I\infty} \otimes \mathcal{N}_{J\infty} \arrow[r, "\varphi"] & \mathcal{N}_{(I \vee J)\infty} \otimes \mathcal{N}_{(I \vee J)\infty} = \mathcal{N}_{(I \vee J)\infty} \\ \mathcal{N}_{J\infty} \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

and we are tasked with proving that φ is an equivalence. An unfortunate fact is that the functor $U : \mathrm{Op}_{I \vee J} \to \mathrm{Op}_I \times \mathrm{Op}_J$ doesn't appear to be conservative in general. Our strategy will come down to trying *really hard* to make it conservative. We do so via the following two lemmas, proved as lemmas 3.5 and 3.6.

Lemma 2.25. *Denote by $i : I \cup J \subset I \vee J$ the (non-indexing system) union of subcategories. Then, the pullback*

$$\mathrm{Fbrs}(\mathrm{Span}(\mathcal{I} \cup \mathcal{J})) \to \mathrm{Op}_I \times \mathrm{Op}_J$$

is conservative and symmetric monoidal. In particular, U reflects equivalences between $\mathcal{I} \vee \mathcal{J}$-operads in the image of $L_{\mathrm{Fbrs}} i_!$.

Lemma 2.26. *There is an equivalence $\mathcal{N}_{(I \vee J)\infty} \simeq L_{\mathrm{Fbrs}} i_! \,\mathrm{Span}(I \cup J)$.*

Proof of theorem A. By the above argument, it suffices to prove that φ is an equivalence; in fact, by lemmas 2.25 and 2.26 and symmetry it suffices to prove that the localized functor

$$\iota_J^* \mathcal{N}_{I \cap J\infty} \otimes \mathcal{N}_{J\infty} \to \iota_J^* \mathcal{N}_{I \vee J}$$

is an equivalence. But $\iota_J^* \mathcal{N}_{I\infty} \simeq \mathcal{N}_{I \cap J\infty}$, so the above is the inclusion $\mathcal{N}_{I \cap J\infty} \otimes \mathcal{N}_{J\infty} \to \mathcal{N}_{J\infty}$, which is an equivalence by proposition 2.23. $\square$

3. Technical nonsense

3.1. **Passing to monads is conservative.** Our arguments will be reminiscent of [SY19, § 2.3-2.4] Let $\mathrm{Fbrs}_\bullet(\mathcal{O})$ denote the full subcategory of fibrous patterns whose associated maps $\mathcal{O}^{\mathrm{el}} \to \mathcal{O}^{\mathrm{el}}$ are equivalences. Define the functor $(-)_{\mathrm{sseq}}$ to be the composite

$$\mathrm{Fbrs}_\bullet(\mathcal{O}) \xrightarrow{\varphi} \mathrm{Fun}(\mathrm{Ar}^{\mathrm{act}}(\mathcal{O}), \mathcal{S}) \to \mathrm{Fun}(\underline{\Sigma}_{\mathcal{O}}, \mathcal{S})$$

where $\underline{\Sigma}_{\mathcal{O}} \subset \mathrm{Ar}^{\mathrm{act}}(\mathcal{O})$ is the full subcategory of active arrows whose targets are elementary objects.

Lemma 3.1 (C.f. [SY19, Prop 2.3.6])**.** *The functor $(-)_{\mathrm{sseq}}$ is conservative.*

Proof. Suppose $f : \mathcal{O} \to \mathcal{P}$ induces an equivalence $f_{\mathrm{sseq}} : \mathcal{O}_{\mathrm{sseq}} \simeq \mathcal{P}_{\mathrm{sseq}}$. By the definition of fibrous patterns, this implies that $\varphi(f)$ is an equivalence.

Note that $\mathrm{Env}_{\mathcal{O}}^{/\mathcal{A}\mathcal{O}} f = (-) \times_{\mathcal{O}} \mathrm{Ar}^{\mathrm{act}}(\mathcal{O})$ is identity on objects, so it is essentially surjective; the natural transformation $\varphi(f)$ precisely specifies the action of $\mathrm{Env}_{\mathcal{O}}^{/\mathcal{A}\mathcal{O}} f$ on morphisms, so $\mathrm{Env}_{\mathcal{O}}^{/\mathcal{A}\mathcal{O}} f$ is an equivalence. Since $\mathrm{Env}_{\mathcal{O}}^{/\mathcal{A}\mathcal{O}}$ is fully faithful, this implies that f is an equivalence. □

We now specialize to the case $\mathcal{O} = \mathrm{Span}(I)$. Note that $\underline{\Sigma}_{\mathrm{Span}(\mathbb{F}_G)} \simeq \underline{\Sigma}_G$, where $\underline{\Sigma}_G \simeq \mathrm{CoFr}^G\Sigma$. Furthermore, $\underline{\Sigma}_{\mathrm{Span}(I)} \to \underline{\Sigma}_G$ is fully faithful with image spanned by I-admissible H-sets; we refer to this as $\underline{\Sigma}_I$. Hence we may translate lemma 3.1 to the following:

Proposition 3.2. *The forgetful functor*

$$(-)_{\mathrm{sseq}} : \mathrm{Op}_I \to \mathrm{Fun}(\underline{\Sigma}_I, \mathcal{S})$$

sending $\mathcal{O}(S) := \pi_{\mathcal{O}}^{-1}(\mathrm{Ind}_H^G S \to G/H)$ for all $S \in \mathbb{F}_H \cap I$ is conservative.

Remark. The *genuine model structure* $\mathrm{Sym}_\bullet^G(\mathrm{sSet})$ of [BP22] exists and presents $\mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_G, \mathcal{S})$; the ∞-category of *Genuine G-operads* are then algebras over a monad on $\mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_G, \mathcal{S})$ which are explicitly defined in [BP21]. In this setting, lemma 3.1 amounts to a verification of one of the two Barr-Beck conditions expressing U as *monadic* (cf [HA, Thm 4.7.3.5]), and hence we view it as a step in the direction of proving that these two models are equivalent.

We say that a G-operad $\mathcal{O}^\otimes$ is *reduced* if $\mathcal{O}(T) = *$ whenever T is empty or a transitive H set. Let $\mathcal{O}^\otimes$ be a reduced G-operad, $\mathcal{C}$ a G-symmetric monoidal category, and $X : \mathrm{triv}^\otimes \to \mathcal{C}^\otimes$ a G-object. Denote by $X_{\mathrm{sseq}} \in \mathrm{Fun}_G(\underline{\Sigma}_G, \mathcal{C})$ the functor of G-categories underlying the adjunct map of G-symmetric monoidal categories to X. We can use this to characterize the *monad* associated with an operad.

We say that a symmetric monoidal category is *distributive* if the action maps $f_\otimes : \mathcal{C}_S \to \mathcal{C}_T$ preserve coproducts separately in each variable (see [NS22]).

Proposition 3.3. *Let $\mathcal{O}$ be a reduced G-operad and let $\mathcal{C}^\otimes$ be a distributive G-symmetric monoidal category. Then, the forgetful map $\underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ is monadic, and the associated monad $T_{\mathcal{O}}$ acts on $X \in \mathcal{C}$ as*

$$T_{\mathcal{O}} X := \mathrm{colim}\, X_{\mathrm{sseq}}.$$

In particular, we have

$$(T_{\mathcal{O}} X)^H \simeq \coprod_{\substack{K \subset H \\ S \in \mathbb{F}_K}} \left(\mathcal{O}(S) \otimes X^{\otimes \mathrm{Ind}_K^H S} \right)_{h\, \mathrm{Aut}_K S},$$

where for all $S' \in \mathbb{F}_H$, we write

$$X^{\otimes S'} := \bigotimes_{U \in \mathrm{Orb}(S')} N_U^H X_U.$$

Proof. Monadicity is precisely [NS22, Cor 5.1.5] when $\mathcal{T} = \mathcal{O}_G$, so it suffices to compute the associated monad in this case. Note that $X_{\mathrm{sseq}}(S) \simeq \mathcal{O}(S) \otimes X^{\otimes S}$, so the computation of $(T_{\mathcal{O}} X)^H$ follows immediately from the statement $T_{\mathcal{O}} X \simeq \mathrm{colim}\, X_{\mathrm{sseq}}$, so it suffices to prove this statement.

By [NS22, Rem 4.3.6], the left adjoint $\mathrm{Fr} : C \to \mathbf{Alg}_{O}(C)$ is computed on X by G-operadic left Kan extension of the corresponding map $\mathrm{triv}^{\otimes} \xrightarrow{X} C^{\otimes}$ along the canonical inclusion $\mathrm{triv}^{\otimes} \to O^{\otimes}$; the underlying G-functor of this is computed by the G-left Kan extension

$$\begin{tikzcd} \underline{\Sigma}_G \arrow[r, equal] \arrow[d] & \mathrm{Env}_O \mathrm{triv} \arrow[r, "X"] \arrow[d] \arrow[r, phantom, shift right=3, "\Rightarrow"] & C \\ *_G \arrow[r, equal] & O \arrow[ur, dashed, "\mathrm{Fr}\, X"'] & \end{tikzcd}$$

I.e. by the indexed colimit

$$T_O X \simeq \mathrm{colim}\, X_{\mathrm{sseq}}.$$

□

Suppose C is a finitely cocomplete Cartesian closed category, and let $\mathrm{CoFr}^G(C)$ be the G-category of G-coefficient systems valued in C, and write $C_G := \mathrm{CoFr}^G(C)^G \simeq \mathrm{Fun}(O_G^{\mathrm{op}}, C)$. By [NS22, Prop 3.2.5] (noting that all colimits involved are finite), the G-Cartesian G-symmetric monoidal structure on $\mathrm{CoFr}^G(C)$ is distributive. Using Elmendorf's theorem, we apply this to $\mathcal{S}_G$:

Corollary 3.4. *Let O be a reduced G-operad. Then, the functor $\mathbf{Alg}_{(-)}(\underline{\mathcal{S}}_G) : \mathrm{Op}_G^{\mathrm{Red}} \to \mathbf{Cat}$ is conservative.*

Proof. All but the final statement follow by the above analysis. Suppose $\varphi : O \to \mathcal{P}$ induces an equivalence on $\mathbf{Alg}_O(\mathcal{S}_G) \to \mathbf{Alg}_{\mathcal{P}_G}(\mathcal{S})$..

Then φ induces a natural equivalence $T_{O,\mathcal{S}_G} \implies T_{\mathcal{P},\mathcal{S}_G}$ respecting the summand decomposition in proposition 3.3. Choosing X a set with at least 2 points, we find that $n_S \cdot O(S) \to n_S \cdot \mathcal{P}(S)$ is an equivalence for some $n_S > 0$ and all S; this implies that $O(S) \to \mathcal{P}(S)$ is an equivalence for all S, i.e. φ_Σ is an equivalence. By lemma 3.1, this implies φ is an equivalence. □

3.2. **The conservativity lemmas.** We have two conservativity lemmas to prove.

Lemma 3.5. *Denote by $i : I \cup J \subset I \vee J$ the (non-indexing system) union of subcategories. Then, the pullback*

$$\mathrm{Fbrs}(\mathrm{Span}(\mathcal{I} \cup \mathcal{J})) \to \mathrm{Op}_I \times \mathrm{Op}_J$$

is conservative. In particular, U reflects equivalences between $\mathcal{I} \vee \mathcal{J}$-operads in the image of $L_{\mathrm{Fbrs}} i_!$.

Proof. Passing to the underlying symmetric sequences yields a diagram

$$\begin{tikzcd} \mathrm{Fbrs}(\mathrm{Span}(I \cup J)) \arrow[r, "i^*"] \arrow[d, tail] & \mathrm{Op}_I \times \mathrm{Op}_J \arrow[d] \\ \mathrm{Fun}(\Sigma_I \cup \Sigma_J, \mathcal{S}) \arrow[r, tail] & \mathrm{Fun}(\Sigma_I, \mathcal{S}) \times \mathrm{Fun}(\Sigma_J, \mathcal{S}) \end{tikzcd}$$

The left vertical arrow is conservative by proposition 3.2. Note that $\Sigma_I \cup \Sigma_J \simeq \Sigma_I \coprod_{\Sigma_{I \cap J}} \Sigma_J$, so the bottom vertical arrow is simply the inclusion

$$\mathrm{Fun}(\Sigma_I, \mathcal{S}) \times_{\mathrm{Fun}(\Sigma_{I\cap J}, \mathcal{S})} \mathrm{Fun}(\Sigma_J, \mathcal{S}) \hookrightarrow \mathrm{Fun}(\Sigma_I, \mathcal{S}) \times \mathrm{Fun}(\Sigma_J, \mathcal{S}),$$

which is conservative. Hence the diagonal composite is conservative, implying that i^* is conservative as well. □

The second is essentially similar. Note that $\mathrm{Env}_I \mathrm{Span}(J) \simeq \underline{\mathbb{F}}_J^{\amalg}$ for all $J \subset I$, and that

$$\underline{\mathbb{F}}_J^{\amalg} \coprod_{\underline{\mathbb{F}}_{I\cap J}^{\amalg}} \underline{\mathbb{F}}_I^{\amalg} \simeq \underline{\mathbb{F}}_{I \vee J}^{\amalg}, \tag{1}$$

where the coproduct is taken in the category of G-symmetric monoidal categories. We use this:

Lemma 3.6. *The canonical map $L_{\mathrm{Fbrs}} i_! \,\mathrm{Span}(I \cup J) \to \mathcal{N}_{(I \vee J)\infty}$ is an equivalence.*

Proof. By corollary 3.4, it suffices to prove that the induced map

$$\mathbf{Alg}_{\mathcal{N}_{(I\vee J)}}(\mathcal{S}_G) \to \mathbf{Alg}_{L_{\mathrm{Fbrs}} i_! \operatorname{Span}(I\cup J)}(\mathcal{S}_G) \simeq \mathbf{Alg}_{\operatorname{Span}(I\cup J)}(i^* \mathcal{S}_G)$$

is an equivalence. Unwinding definitions, this is equivalent to proving that the following diagram is cartesian:

$$\begin{tikzcd} \mathrm{Fun}^{\otimes}_G(\underline{\mathbb{F}}_{I\vee J}, \underline{\mathcal{S}}_G) \arrow[r] \arrow[d] & \mathrm{Fun}^{\otimes}_G(\underline{\mathbb{F}}_I, \underline{\mathcal{S}}_G) \arrow[d] \\ \mathrm{Fun}^{\otimes}_G(\underline{\mathbb{F}}_J, \underline{\mathcal{S}}_G) \arrow[r] & \mathrm{Fun}^{\times}(\underline{\mathbb{F}}_{I\cap J}, \underline{\mathcal{S}}_G) \end{tikzcd}$$

In fact, this is precisely (1). ◻

3.3. **The BV tensor product on fibrous patterns is closed.**

Definition 3.7. Let $\mathcal{E}$ be a symmetric monoidal algebraic pattern. Then, a *bifunctor of fibrous $\mathcal{E}$ patterns* is a diagram in $\mathrm{Fbrs}(\mathcal{E})$

$$\begin{tikzcd} \mathcal{O}\times\mathcal{P} \arrow[r] \arrow[d] & \mathcal{Q} \arrow[d] \\ \mathcal{E}\times\mathcal{E} \arrow[r] & \mathcal{E} \end{tikzcd}$$

Let $F : \mathcal{O}^{\otimes}\times\mathcal{P}^{\otimes} \to \mathcal{E}^{\otimes}$ be a bifunctor of fibrous $\mathcal{E}$-patterns and let $\mathcal{C}^{\otimes} \in \mathrm{Fbrs}(\mathcal{E})$ be a fibrous $\mathcal{E}$-pattern. The following construction generalizes [NS22, § 5.3].

Construction 3.8. Define $P : \mathcal{O}^{\otimes}\times_{\mathcal{E}^{\mathrm{el}}} \mathrm{Ar}(\mathcal{E}^{\mathrm{el}}) \to \mathcal{O}^{\otimes}$ by cocartesian pushforward. We have a diagram

$$\mathcal{O}^{\otimes} \xleftarrow{\pi} \mathcal{O}^{\otimes}\times \mathrm{Ar}(\mathcal{E}^{\mathrm{el}})\times\mathcal{P}^{\otimes} \xrightarrow{P\times\mathrm{id}} \mathcal{O}^{\otimes}\mathcal{P}^{\otimes} \xrightarrow{F} \mathcal{E}^{\otimes}.$$

and an associated push-pull adjunction

$$L_{\mathrm{Fbrs}} F_! (P\times\mathrm{id})_! \pi^* : \mathrm{Fbrs}(\mathcal{O}) \rightleftarrows \mathrm{Fbrs}(\mathcal{E}) : \pi_*(P\times\mathrm{id})^* F^*.$$

We verify that this adjunction exists in lemma 3.11. and we define $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{E}}(\mathcal{P};\mathcal{C}) \to \mathcal{O}^{\otimes}$ to be $\pi_*(P\times\mathrm{id})^* F^*(\mathcal{C}^{\otimes})$.

Products of equivalences are equivalences; this proves the following lemma.

Lemma 3.9. *External products of strong Segal morphisms are strong Segal morphisms.*

The proof of the following lemma is precisely that of [CH21, Lem 9.4].

Lemma 3.10. *Fibrous patterns are strong Segal morphisms.*

The following is an exercise in category theory:

Lemma 3.11. *Fix $(\mathcal{O},\mathcal{O}') \in \mathbf{Cat}_{/\mathcal{E}}\times\mathbf{Cat}_{/\mathcal{E}'}$. Then,*

(1) *$f\times f' : \mathcal{O}\times\mathcal{O}' \to \mathcal{E}\times\mathcal{E}'$ is a (strong-, iso-) Segal morphism if and only if f and f' are (strong-, iso-) Segal morphisms.*
(2) *$\pi_{\mathcal{O}\times\mathcal{O},*}$ preserves fibrous patterns (resp. Segal categories) if and only if $\pi_{\mathcal{O},*}$ and $\pi_{\mathcal{O}',*}$ is preserves fibrous patterns (Segal categories).*
(3) *$\mathcal{O}\times\mathcal{O}'$ is a fibrous $\mathcal{E}\times\mathcal{E}'$-pattern (resp. Segal $\mathcal{E}\times\mathcal{E}'$-category) if and only if $\mathcal{O}$ and $\mathcal{O}'$ are fibrous $\mathcal{E},\mathcal{E}'$ patterns (Segal $\mathcal{E}\times\mathcal{E}'$-categories).*

In particular, the morphisms $F, P\times\mathrm{id}, \pi$ above are strong Segal morphisms and π_* preserves firbous patterns and Segal categories.

Proof. For (1), note that the associated functor

$$\mathcal{O}^{\mathrm{el}}_{X/}\times\mathcal{O}'^{\mathrm{el}}_{X'} \to \mathcal{E}^{\mathrm{el}}_{fX/}\times\mathcal{E}'^{\mathrm{el}}_{f'X'}$$

is the product $f^{\mathrm{el}}_{X/}\times f'^{\mathrm{el}}_{X'}$, so this follows by noting that products commute with limits and that a product of functors is an equivalence if and only if the factors are equivalences.

(2) should just be a formal construction of a right adjoint…Is (2) actually true? The adjunction certainly exists by [NS22], but it's a bit unclear what it would even mean in this context.

For (3), this amounts to checking that a morphism is $\pi\times\pi'$-cocartesian if and only if it's a product of π and π'-cocartesian arrows and commuting limits past products in [BHS22, Def 4.1.2]. ◻

The following lemma follows immediately from lemma 3.11.

Lemma 3.12. *Suppose $\mathcal{C}$ is a Segal $\mathcal{E}$-category. Then, $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{E}}(\mathcal{P};\mathcal{C})$ is a Segal $\mathcal{O}$-category.*

The resulting fibrous is pronounced "the fibrous $\mathcal{E}$-pattern of G-equivariant $\mathcal{O}$-algebras in $\mathcal{C}$." We specialize to the case that $\mathcal{E}^{\otimes} = \mathcal{O}^{\otimes}$, in which case we write

$$\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) := \underline{\mathbf{Alg}}^{\otimes}_{\mathcal{E}}(\mathcal{P};\mathcal{C}).$$

Then, the above diagram instead reads as

$$\mathcal{E} \xleftarrow{\pi} \mathcal{E} \times \mathrm{Ar}(\mathcal{E}^{\mathrm{el}}) \times \mathcal{P}^{\otimes} \xrightarrow{P\times \mathrm{id}} \mathcal{E} \times \mathcal{P}^{\otimes} \xrightarrow{F} \mathcal{E}.$$

So that the left adjoint is computed by the fibrous localization of the map $Q \times \mathcal{P} \to \mathcal{E}$ in the following:

$$\begin{tikzcd}
\pi^*(P\times \mathrm{id})_! Q \arrow[d] & \simeq & Q \times \mathcal{P} \arrow[dll, "\pi_Q \times \mathrm{id}"] \\
\mathcal{E} \times \mathcal{P} \arrow[d, "\mathrm{id}\times \pi_{\mathcal{P}}"] \arrow[drr, "F"] & & \\
\mathcal{E} \times \mathcal{E} \arrow[rr, "\otimes"] & & \mathcal{E}
\end{tikzcd}$$

in fact, by definition, this is precisely $Q \otimes_{\mathcal{E}} \mathcal{P}$. This concludes the proof of theorem 2.22.

As a sanity check, we verify that our construction matches that of [NS22, § 5.3]. Draw the diagram

$$\begin{tikzcd}
& \mathcal{E}^{\otimes} \times_G \mathrm{Ar}(\mathcal{O}_G^{\mathrm{op}}) \times_G \mathcal{P}^{\otimes} \arrow[r, "P\times \mathrm{id}"] \arrow[dl, "\pi'"'] \arrow[dd, "\iota"] & \mathcal{E} \times_G \mathcal{P}^{\otimes} \arrow[dr, "f"] \arrow[dd] & \\
\mathcal{E}^{\otimes} & & & \mathcal{E}^{\otimes} \\
& \mathcal{E}^{\otimes} \times \mathrm{Ar}(\mathcal{O}_G^{\mathrm{op}}) \times \mathcal{P}^{\otimes} \arrow[ul, "\pi"] \arrow[r, "P\times \mathrm{id}"] & \mathcal{E} \times \mathcal{P}^{\otimes} \arrow[ur, "f"'] &
\end{tikzcd}$$

It suffices to verify that $\pi_* = \pi'_* \iota^*$, or equivalently, that $\pi^* \simeq \iota_! \pi'^{\star}$. But this follows from direct inspection. As a corollary, we gain [NS22, Thm 5.3.9], whuch we use heavily in the following subsection.

3.4. An I-symmetric monoidal category is cocartesian if and only if unital algebra structures are canonical.

Define the category $\underline{\Gamma}^*_G := \mathrm{CoFr}^G(\Gamma^*)$. Given $\mathcal{C}$ an I-coproduct complete G-category, define the functor $\mathcal{C}^{\amalg} \to \Gamma^*_G$ to satisfy the following equivalence:

$$\mathrm{Map}_{\mathrm{Span}(\mathbb{F}_G)}(K, \mathcal{C}^{\amalg}) \simeq \mathrm{Map}(K \times_{\underline{\mathbb{F}}_{G,*}} \Gamma^*_G, \mathcal{C}).$$

An object of $\mathcal{C}^{\amalg}$ may be viewed as $S_+ \to G/H_+$ a pointed H-set and $(C_s)_{U \in \mathrm{Orb}(S)}$ an S-tuple of elements of $\mathcal{C}$; a morphism in $\mathcal{C}^{\amalg}$ $(C_s) \to (D_t)$ may be viewed as a map $(S_+ \to G/H_+) \xrightarrow{f} (T_+ \to G/J_+)$ in $\underline{\mathbb{F}}_G$ together with a map

$$f_U : \coprod_{V \in f^{-1}(U)} N_V^U C_V \to D_U$$

for all $U \in \mathrm{Orb}(T)$. Unwinding definitions, we find the following lemma.

Lemma 3.13. *A morphism $f : (C_s)_{s\in S} \to (D_t)_{t \in T}$ is π-cocartesian if and only if f_U is an equivalence for all $U \in \mathrm{Orb}(T)$. In particular, f is inert if and only if the following conditions are satisfied:*

(1) The projected morphism $\pi(f) : S \to T$ is inert.
(2) The associated map $C_{f^{-1}(U)} \to D_U$ is an equivalence for all $U \in \mathrm{Orb}(T)$.

Having characterized this, we may draw a diagram of Cartesian squares

$$\begin{tikzcd} \mathcal{O} \arrow[r, hook, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}_{\Gamma}^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes} \arrow[d] \\ \mathcal{O}_G^{\mathrm{op}} \arrow[r, hook] & \underline{\Gamma}_G^{*} \arrow[r] & \underline{\mathbb{F}}_{G,*} \end{tikzcd}$$

Note that the objects of $\mathcal{O}_{\Gamma}^{\otimes}$ consist of triples $(S_+ \to G/H, U, X)$ where $U \in \mathrm{Orb}(S)$ and $X \in \mathcal{O}_S$, and the image of ι is equivalent to the triples where $S_+ \simeq G/K$ for some $K \subset H$ (hence $U = S$).

Note that cocartesian transport along inert morphism $U_+ \hookrightarrow S_+$ induces an equivalence

$$\mathrm{Map}_{\mathcal{O}_{\Gamma}^{\otimes}}(Y, (S_+ \to G/H, U, X))) \simeq \mathrm{Map}_{\mathcal{O}_{\Gamma}^{\otimes}}(Y, (U_+ \to G/H, U, X_U))).$$

In particular, ι witnesses $\mathcal{O}$ as a *colocalizing subcategory*, with localization functor

$$R(S_+ \to G/H, U, X) \simeq (U_+ \to G/H, U, X).$$

We use this in the following lemma characterizing $\mathcal{O}$-algebras in $\mathcal{C}^{\amalg}$.

Lemma 3.14. *TFAE for a functor $A : \mathcal{O}_{\Gamma}^{\otimes} \to \mathcal{C}$.*

(1) The corresponding map $\mathcal{O}^{\otimes} \to \mathcal{C}^{\amalg}$ is a map of I-operads.
(2) For all morphisms α in $\mathcal{O}_{\Gamma}^{\otimes}$ whose image in $\mathcal{O}^{\otimes}$ is inert, $A(\alpha)$ is an equivalence in $\mathcal{C}$.
(3) If $f : (S_+ \to G/H_+, U, X) \to (U_+ \to G/H_+, U, X_U)$ is a cocartesian lift of the inert morphism, then $A(f)$ is an equivalence.
(4) A is left Kan extended from $\mathcal{O}$.

Furthermore, every functor $F : \mathcal{O} \to \mathcal{C}$ admits a left Kan extension along $\mathcal{O} \hookrightarrow \mathcal{O}_{\Gamma}^{\otimes}$; in particular, the forgetful functor $\underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_G(\mathcal{O}, \mathcal{C})$ is an equivalence.

Proof. (1) $\iff$ (2) follows immediately from **??**. (2) $\iff$ (3) is immediate by definition. (3) $\iff$ (4) is the computation of left Kan extension along the inclusion of a colocalizing subcategory. The pointwise formula for left Kan extension is precisely the composition $RF : \mathcal{O}_{\Gamma}^{\otimes} \to \mathcal{C}$. □

We would additionally like to characterize I-symmetric monoidal functors into $\mathcal{C}^{\amalg}$. This follows quickly from lemma 3.14.

Lemma 3.15. *TFAE for a map of I-operads $\varphi : \mathcal{O}^{\otimes} \to \mathcal{C}^{\amalg}$:*

(1) φ is a map of I-symmetric monoidal categories.
(2) The underlying G-functor $F : \mathcal{O} \to \mathcal{C}$ preserves I-indexed coproducts.

In particular, restriction yields an equivalence

$$\mathrm{Fun}_I^{\otimes}(\mathcal{O}^{\otimes}, \mathcal{C}^{\amalg}) \xrightarrow{\sim} \mathrm{Fun}_I^{\amalg}(\mathcal{O}, \mathcal{C}).$$

Theorem 3.16 (C.f. [HA, Prop 2.4.3.9])**.** *The following are equivalent for $\mathcal{C}^{\otimes} \in \mathrm{CMon}_{\mathcal{I}}(\mathbf{Cat})$.*

(1) For all unital I-operads $\mathcal{O}^{\otimes}$, the forgetful functor $\underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_G(\mathcal{O}, \mathcal{C})$ is an equivalence.
(2) The forgetful functor $\mathrm{CAlg}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence.
(3) For all morphisms $f : S \to T$ in $\mathcal{I}$, the action map $f_{\otimes} : \mathcal{C}_S \to \mathcal{C}_T$ is left adjoint to the pullback $f^* : \mathcal{C}_T \to \mathcal{C}_S$.
(4) There is an I-symmetric monoidal equivalence $\mathcal{C}^{\otimes} \simeq \mathcal{C}^{\amalg}$ extending the identity on $\mathcal{C}$.

Proof of theorem 3.16. (1) $\implies$ (2) by choosing $\mathcal{O} = \mathcal{N}_{I\infty}$. (2) $\implies$ (3) is precisely [NS22, Thm 5.3.9], noting that The forgetful functor $\mathrm{CAlg}_I(\mathcal{C}) \to \mathcal{C}$ is I-symmetric monoidal by construction. (3) $\implies$ (4) follows by applying lemma 3.15 to the identity functor in the case $\mathcal{O} = \mathcal{C}$. (4) $\implies$ (1) is precisely lemma 3.14. □

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