Kan Seminar Notes

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This will be a rough collection of live-LaTeXed notes covering the Kan seminar talks given in Fall 2021. I'll make no promises that the contents of this are readable, or without significant clerical error. Exercise skepticism, and don't use these as a replacement for the papers. Last update: November 17, 2021.

Contents

1 Gabrielle Li: Serre, Cohomologie modulo 2 des complexes d'Eilenberg Mac Lane (i)

This talk was delivered September 15, 2021 by Gabrielle Li. Throughout, $H^*(-) := H^*(-;\mathbb{F}_2)$.

1.1 Steenrod operations

The *Steenrod operations* are a family of cohomology operations $\mathrm{Sq}^n : H^*(X) \to H^{*+n}(X)$ such that:

(1) Each Sq^n is natural in X.

(2) Each Sq^n is stable: $\mathrm{Sq}^n(\Sigma X) = \Sigma\,\mathrm{Sq}^n(X)$.

(3) When $|x| = n$, $\mathrm{Sq}^n(x) = x \cup x$.

(4) $\mathrm{Sq}^0 = \mathrm{id}$.

We give a basis for these:

Definition 1.1. A sequence $I = (i_1) \subset \mathbb{Z}_{>0}$ is *admissable* if $i_k \geq 2i_{k+1}$ for each k. We define the *degree* $n(I) := \sum i_k$ and the *excess* $e(I) = \sum(i_k - i_{k+1}) = 2i_i - n(I)$ (padding with zeros).

1.2 Borel's theorem

Let $F \hookrightarrow E \to B$ be a Serre fibration. Recall that, in the cohomological Serre spectral sequence, we have transgression morphisms $\tau : E_r^{0,r-1} \to E_r^{r,0}$, whose domain is a subset of $H^{r-1}(F)$ and whose codomain is a quotient of $H^r(B)$. This is an additive relation between $H^{r-1}(F)$ and $H^r(B)$. We say that $x \in H^{r-1}(F)$ is *transgressive* if it survives to the r page.

We hold off on proving the following proposition until the next talk:

Proposition 1.2. *τ commutes with Steenrod operations.*

We need a bit more language to use this:

Definition 1.3. For a space X, an ordered family of elements $(x_i) \subset H^*(X)$ is a *simple system of generators* if:

(1) Each x_i is homogeneous.

(2) The increasing products $x_{i_1} \cdots x_{i_j}$ (for $i_k < i_{k+1}$) form a $\mathbb{F}_2$-basis of $H^*(X)$.

The following examples are important:

Example 1.4:
$\mathbb{F}_2[x_1, x_2, \dots]$ has simple system of generators $(x_j^{2^i})$. Similar systems apply to the exterior algebra $E[x]$ and the truncated poltnomial algebra $\mathbb{F}_2[x]/(x^{2^i})$.

We're finally ready to state our theorem:

Theorem 1.5 (Borel)**.** *Given a fibration $F \hookrightarrow E \to B$ satisfying the following properties:*

(1) $E_2^{s,t} = H^s(B) \otimes H^t(F)$ (for instance, when B is 1-connected and $H^*(B), H^*(F)$ are f.g.).

(2) $H^i(E) = 0$ for $i > 0$.

(3) $H^*(F)$ have a simple system of transgressive generators (x_i).

Then, $H^*(B)$ is a polynomial algebra generated (independently) by the any choice of representatives $y_i \in H^*(B)$ which map to $\tau(x_i)$ in $E_*^{*,0}$.

Note that, whenever $H^*(F)$ is a polynomial algebra generated by z_i, we know that $H^*(F)$ has a simple system of generators $z_i^{2^r}$. In order to use this, we introduce a bit of notation:

Notation. $L(a,r) := \left\{2^{r-1}a, 2^{r-2}a, \cdots, 2a, a\right\}$.

Note that $z_i^{2^r} = \mathrm{Sq}^{L(n_i,r)}(z_i)$. Hence

$$\tau\left(z_i^{2^r}\right) = \mathrm{Sq}^{L(n_i,r)}\, t_i$$

where $t_i := \tau(z_i)$. Hence $H^*(B)$ is a polynomial algebra generated by $\mathrm{Sq}^{L(n_i,r)}(z_i)$.

1.3 Performing the calculation

We will use Borel's theorem soon, but first, a lemma:

Lemma 1.6. *An admissible sequence $J = \{j_1, \ldots, j_k\}$ with $e(J) < q-1$. Then, we may define a sequence*

$$J' := \left\{2^{r-1}s_J, 2^{r-2}s_J, \ldots, s_J, j_1, j_2, \ldots, j_k\right\},$$

where $s_J = q - 1 + n(J)$. Then, J' is admissible, with $e(J') < q$; furthermore, all admissible sequences of excess $< q$ arise this way.

The reversal is surprisingly easy; simply take the longest prefix satisfying $j_1 = 2j_2 = \cdots = 2^i j_i$.

We will need a few more constructions to prepare for the calculation:

(1) There is a fibration $K(\mathbb{F}_2, q-1) \hookrightarrow E \to K(\mathbb{F}_2, q)$ where E is contractible.

(2) By Hurewicz, $H^q(K(\mathbb{F}_2, q)) = \mathbb{F}_2$, with a generator that we call u_q.

Theorem 1.7. *$H^*(K(\mathbb{Z}/2, q), \mathbb{Z}/2)$ is a polynomial algebra (independently) generated by $\mathrm{Sq}^I(u_q)$ where I runs over the admissible sequences of excess $e(I) < q$.*

Proof. We prove this via induction. The $q = 1$ case is easy, as we have $K(\mathbb{F}_2, 1) = \mathbb{RP}^\infty$, and $H^*(\mathbb{RP}^\infty) = \mathbb{F}_2[u_q]$ via the usual computation.

For the inductive step, assume we've proven the theorem for $q-1$. We use the fibration from (1). For an admissible sequence J, let

$$S_J := \left|\mathrm{Sq}^J(u_{q-1})\right| = q - 1 + n(J).$$

We have transgression additive relation $H^{q-1}(K(\mathbb{F}_2, q-1)) \rightsquigarrow H^q(K(\mathbb{F}_2, q))$. Note that the transgression sends $\tau(u_{q-1}) = u_q$ (this will be justified later). Using our trick,

$$\tau(\mathrm{Sq}^J(u_{q-1})) = \mathrm{Sq}^J u_q.$$

By Borel, the $H^*(K(\mathbb{F}_2, q))$ is generated by $\mathrm{Sq}^{L(s_J,r)}\, \mathrm{Sq}^J u_q = \mathrm{Sq}^{L(s_j,r)J} u_q = \mathrm{Sq}^I u_q$, where I is an admissible sequence with $e(I) < q$, and all such I are generated this way. ☺

The other computations are routine and similar.

2 Weixiao Lu: Serre, Cohomologie modulo 2 des complexes d'Eilenberg Mac Lane (ii)

This talk was delivered September 17, 2021 by Weixiao Lu. We'll first cover some preliminaries.

2.1 Preliminaries

Theorem 2.1 (Serre spectral sequence)**.** *Let $F \hookrightarrow E \xrightarrow{p} B$ be a Serre fibration. Then, there is a spectral sequence*

$$E_2^{s,t} = H^s(B; H^t(p^{-1}(-); G)) \implies H^{s+t}(E; G).$$

If $\pi_1(B)$ acts trivially on $H^n(p^{-1}(-))$, then

$$E_2^{2,t} = H^s(B; H^t(F; G)).$$

Proof sketch. If F^*C^* is a filtered cochain complex, we have an SS,

$$E_0^{s,t} = \mathrm{gr}^s(C^{s+t}) \implies H^{s+t}(C^*).$$

Assume B is a CW complex with n-skeleton B^n. Then, $E_n := p^{-1}(B^n)$. We have $F^s S^*(E) = S^*(E, E_{s-1}) = \ker(S^*(E) \to S^*(E_{s-1}))$, which gives the right E_0 page.

In any upper-right quadrant SS, we have a transgression morphism $d^n : E^{0,n-1} \to E_n^{n,0}$. Note that $E_n^{0,n-1} \subset E_{n-1}^{0,n-1} \subset \cdots \subset H^{n-1}(F)$. The transgressive elements of $H^{n-1}(F)$ map to some quotient of $H^n(B)$.

We can create a diagram

$$\begin{tikzcd}
H^n(B,b) \arrow[r, "p^*"] \arrow[d, "\sim"] & H^n(E,F) & \\
H^n(B) \arrow[ur] & & H^{n-1}(F) \arrow[ul, "\partial"']
\end{tikzcd}$$

Theorem 2.2 (Transgression theorem)**.** *The transgression relation coincides with this diagram.*

This comes down to how the Serre SS was constructed.

Proposition 2.3. *The Steenrod square* Sq_i *"commutes" with transgression in the sense that any $x \in H^{n-1}(F; \mathbb{Z}/2)$ transgressive has $\mathrm{Sq}^i x$ transgressive, and $\tau(\mathrm{Sq}^i x) = \mathrm{Sq}^i(\tau x)$.*

Proof. Recall that a functor is stable iff it commutes with coboundary operators, so Sq_1 commutes with coboundary operators. Further, recall that it's natural. Hence the following diagram commutes, so Sq^i "commutes with the transgression relation" (is a morphism of cospans):

$$\begin{tikzcd}
 & H^{n+i}(E,F) & \\
H^{n+i}(B) \arrow[ur, "p^*"] & & H^{n+i-1}(F) \arrow[ul, "\partial"'] \\
 & H^n(E,F) \arrow[uu, "\mathrm{Sq}^i"] & \\
H^n(B \arrow[uu, "\mathrm{Sq}^i"] \arrow[ur, "p^*"] & & H^{n-1}(F) \arrow[ul, "\partial"'] \arrow[uu, "\mathrm{Sq}^i"]
\end{tikzcd}$$

Reclal that for G a f.g. Abelian group,

1. $H^*(K(G \times H; q)) = H^*(K(G; q)) \otimes H^*(K(H; q))$.
2. $H^*(K(\mathbb{F}_2; q)) = \mathbb{F}_2[\mathrm{Sq}^I u_q \mid I \text{ admissible, s.t. } e(I) < q]$.
3. $H^*(K(\mathbb{F}_2; q)) = \mathbb{F}_2[\mathrm{Sq}^I u_q \mid I \text{ admissible, s.t. } e(I) < q, 1 \text{ does not appear in } i]$.
4. $H^*(K(\mathbb{F}_2^h; q)) = \mathbb{F}_2[\mathrm{Sq}^I u_q, \mathrm{Sq}^J k_{q+1}]$ where $k_{q+1} \in H^{q+1}(K(\mathbb{F}_{2^h}, q))$ for admissibles $e(I) < q$, $e(J) \leq q$ where no Sq^1 term appears in both Sq^I and Sq^J. This comes from a fibration fill in from notes later .
5. $H^*(K(\mathbb{F}_{p^h}; q)) = \mathbb{Z}/2$ for p odd with $q > 0$.

Remark. We have a different choice of generators related to universal classes, but as graded $\mathbb{F}_2$-algebras,

$$H^*(K(\mathbb{F}_{2^h}; q)) \simeq H^*(K(\mathbb{F}_2; q)).$$

We will aim towards the following theorem:

Theorem 2.4. *For all $n > 1$, there are infinitely many indices i at which $\pi_i(S^n)$ has nonzero 2-torsion.*

Our tool will be Poincaré series. The accents in Poincaré's name are to be understood from here on out.

2.2 Poincaré series

For L_* a finite type graded k-vector space, define the series

$$L(t) = \sum_{n \in \mathbb{N}} \dim L^n t^n \in \mathbb{Z}[[t]].$$

This is called the *Poincare series*, called $\theta(G; q; t)$ in the case of $H^*(K(G; q))$.

Example 2.5:
For $L^* = \mathbb{Z}/2[u]$, we have

$$L(t) = \frac{1}{1 - t^m}.$$

Note that $(N^* \otimes M^*)(t) = L(t)M(t)$. Hence $L^{'*} = k[u_1, \dots]$ with finite type has

$$L(t) = \prod_{n \geq 1} \frac{1}{1 - t^{\deg u_i}}$$

which converges t-adically.

Hence

$$\theta(\mathbb{F}_2, q, t) = \prod_{e(I) < q} \frac{1}{1 - t^{\deg(\mathrm{Sq}^I u_q)}} = \prod_{e(I) < q} \frac{1}{1 + t^{q + n(I)}}.$$

We can give this another combinatorial description:

Proposition 2.6.

$$\theta(\mathbb{F}_2, q, t) = \prod_{n_1 \geq n_2 \geq \cdots \geq n_{q-1} \geq 0} \frac{1}{1 - t^{2^{n_1} + \cdots 2^{n_{q-1}} + 1}}.$$

The radius of convergence of this is 1 considered as a complex power series. We can continue to analyze this series along these lines:

Theorem 2.7.

$$\lim_{x \to \infty} \frac{\log_2 \theta(\mathbb{F}_2, q, 1 - 2^{-x})}{x^q / q!} = 1.$$

In general there is an essential singularity at 1. Serre used this replacement to reign it in, but we won't work with it very explicitly.

2.3 Applications

Theorem 2.8. *Suppose X is a 1-connected space satisfying the following conditions:*

1. *$H_*(X;\mathbb{Z})$ is of finite type.*
2. *$H_i(X;\mathbb{F}_2) = 0$ for $i \gg 0$.*

Then, for infinitely many indices i, $\pi_i(X)$ has a subspace isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/2$.

This directly implies Theorem 2.4 once you know that only finitely many homotopy groups of spheres are infinite.

We see this using a whitehead tower

$$\begin{tikzcd}
\cdots \arrow[d] & \\
X_{n+1} \arrow[d] \arrow[dr] & \\
X_n \arrow[r] \arrow[d] & X \\
X_{n-1} \arrow[d] \arrow[ur] & \\
\cdots &
\end{tikzcd}$$

where X_n is n-connected, and a π_i iso to X and X_{n-1} for $i > n$. We'll use another piece of machinery, seen by the Serre SS directly.

Lemma 2.9. *For $F \hookrightarrow E \to B$ a Serre fibration with B simply connected, $B(t)F(t) \geq E(t)$.*

Proof of Theorem 2.8. Otherwise, there is some largest q with $\pi_q(X) \otimes \mathbb{Z}/2 \neq 0$. Then, there is some j smallest such that $H_j(X;\mathbb{Z}/2) \neq 0$. Then, $\pi_j(X) \otimes \mathbb{Z}/2 \neq 0$.

In the whitehead tower, $X_q \to X_{q-1}$ is trivial on $\pi_*(-) \otimes \mathbb{Z}/2$, so $H^*(X_q, \mathbb{Z}/2)$ is trivial. Using the fibration $X_q \hookrightarrow X_{q-1} \to K(\pi_q(X), q)$ from the whitehead tower, we must have $H^*(X_{q-1}) = H^*(K(\pi_q(X), q))$. Then,

$$X_{q-1}(t) = \theta(\pi_q(x), q, t).$$

Further, the fibbrations in the whitehead series imply that

$$X_{i+1}(t) \leq X_i(t)\theta(\pi_{i+1}(X), i, t)$$

for each i, Chaining these together forever, what we get is

$$\theta(\pi_q(X), q, t) \leq X_1(t)\theta(\pi_2(X), 1, t) \cdots \theta(\pi_{q-1}(x), q-2, t).$$

Note that $X_1(t)$ is a polynomial, so bounded on $[0, 1]$. Applying our asymptotic bound on θ yields a contradiction.

3 Zihong Chen: Moore, Semi-simplicial complexes and Postnikov systems

This talk was delivered September 20, 2021 by Zihong (Peter) Chen.

3.1 Review of simplicial sets

The talk began with a very brief review of simplicial sets: let Δ be the category of finite ordered sets and order preserving maps. Recall that such maps are generated by distinguished maps $\delta_i : [n] \to [n+1]$ and $s_i : [n+1] \to [n]$, called the *face and degeneracy maps*.

Definition 3.1. A *simplicial set* is a functor $X : \Delta^n \to \mathbf{Set}$.

The morphism set is completely characterised by their images on face and degeneracy maps, which must satisfy a collection of combinatorial relations, which I won't write down here.

Example 3.2:
The *standard n-simplex* is given by the representable functor $\Delta[n] := \mathrm{Hom}(-, [n])$.

By Yoneda's lemma, $X_n = \mathrm{Hom}(\Delta[n], X)$, where $X_n = X([n])$.

Example 3.3:
If $X \in \mathbf{Top}$, the singular simplicial set $\mathrm{Sing}(X)$ is familiar. It participates in an adjunction, with left adjoint $|\cdot|$ the *Geometric realization.*

Example 3.4:
Define the *ith face* $\delta_i : \Delta[n-1] \to \Delta[n]$. The *ith horm* is $\bigvee_i^n := \cup_{k \neq i} \delta_i$. The *boundary* is $\partial\Delta[n] = \bigcup_i \delta_i$.

This allows us to define the combinatorial equivalent of a topological space:

Definition 3.5. A simplicial set X is a *Kan complex* if every morphism $\bigvee_k^n \to X$ factors through $\Delta[n] \to X$; you can *fill any horn* (not necessarily uniquely).

A morphism $p : E \to B$ is a *Kan fibration* if it has the right lifting property against horn inclusions:

$$\begin{tikzcd} \bigvee_k^n \arrow[r] \arrow[d] & E \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Examples of this include $\mathrm{Sing}(X)$, and any simplicial group (which we won't prove).

Definition 3.6. For X a Kan complex, define the *path components* $\pi_0(X) = X_0/\sim$ where $x \sim y$ if there exists some p with $d_1 p = x$ and $d_0 p = y$.

This is in fact an equivalence relation: you can do this via horn filling, which was drawn on the board, but which I will not spell out. We can define higher homotopy groups after defining the internal hom:

Definition 3.7. For $A \subset X$ and $B \subset Y$, define the *mapping object*

$$\mathrm{Map}((X, A), (Y, B)) = \mathrm{Hom}(\Delta[n] \times (X, A), (Y, B))$$

i.e. the maps $\Delta[n] \times X \to Y$ restricting to a map $\Delta[n] \times A \to B$. The maps $\Delta[i] \to \mathbf{Set}$ form a covariant functor, so this is a contravariant functor, i.e. a simplicial set.

We use the following Theorem of Kan:

Theorem 3.8 (Kan)**.** *If Y, B are Kan complexes, then so is* $\mathrm{Map}((X, A), (Y, B))$.

We finally define homotopy groups.

Definition 3.9. If X is a Kan complex, define $\pi_n(X, x) := \pi_0(\mathrm{Map}((\Delta[n], \partial\Delta[n]), (X, x)))$. A Kan complex is $K(\Pi, n)$ if $\pi_q(X, x) = \Pi$ when $q = n$ and 0 otherwise.[1]

We will use these to decompose Kan complexes.

3.2 Postnikov systems

Let $\Delta[q]_n$ be the n-skeleton of $\Delta[q]$. For X a Kan complex, define the complex $X^{(n)}$ via

$$X_q^{(n)} = X_q/\sim \qquad\qquad x \sim y \iff x|_{\Delta[q]_n} = y|_{\Delta[q]_n}.$$

The maps are induced by X. We have the following properties:

1. $X^{(n)}$ is a Kan complex.
2. There is a quotient Kan fibration $X^{(n)} \xrightarrow{p} X^{(k)}$ if $n > k$.
3. $\pi_q(X^{(n)}, x) = 0$ if $q > n$.
4. $p_* : \pi_q(X^{(n)}, x) \xrightarrow{\sim} \pi_q(X^{(k)}, x)$ is an iso if $n \geq k \geq q$.

As in topology, Kan fibrations induce LES of homotopy groups; hence the fiber $F^{(n+1)} \hookrightarrow X^{(n+1)} \xrightarrow{p} X^{(n)}$ is a $K(\pi_{n+1}(X), x + 1)$. We finally give this a name:

Definition 3.10. $(X^0, X^{(1)}, \dots)$ is called the *natural Postnikov system* of X.

This motivates a question: How far is X from $\prod_n K(\pi_n, n)$? It's always a colimit, but we'll measure how complex it is in the following section.

The idea is that $K(\pi_{n+1}, n + 1) \hookrightarrow X^{(n+1)} \to X^{(n+1)}$ will be seen as something like a "principal $K(\pi_{n+1}, n + 1)$-bundle." We will construct something like a "classifying space" $\overline{W}(K(\pi_{n+1}, n + 1)) = K(\pi_{n+1}, n + 2)$, and derive algebraic invariants from this. Let's actually do this now:

3.3 Principally twisted cartesian products

Definition 3.11. A *principally twisted Cartesian product* (PTCP) with simplicial group G and base G is written

$$E(T) = G \times_T B$$

where $E(T)_n = G_n \times B_n$ with degeneracy maps all the same, except that

$$\partial_0(g, b) = (T(b) \cdot d_0 g, d_0 b)$$

and T is a *twisting function* $B_q \to G_{q-1}$ for $q \geq 1$.

This is a combinatorial version of *holonomy*, as per a comment from Prof. Miller.

Definition 3.12. A PTCP is *of type (W)* if $B_0 = \{b_0\}$ and

$$\partial_0|_{\{e_q\}\times B_q} : [e_q] \times B_q \xrightarrow{\sim} E(T)_{q-1}$$

is an iso. Let $\int$ be its inverse.

Theorem 3.13. *If $G \times_T B$, $G' \times_{T'} B'$, and $\gamma : G \to G'$ is a morphism of simplicial group, then there exists a unique γ-equivariant map $\theta : G \times_T B \to G' \times_{T'} B'$ and Some condition holds of θ–fill in later .*

I couldn't follow this part; use $\int$ to construct this "upwards" from b_0, or something like that.

Corollary 3.14. *A PTCP of type (W) with group G is unique, if it exists.*

[1]This *actually* has a requirement of minimality, but we handwave this away.

Theorem 3.15. *If $E(T)$ is PTCP of type (W), it is contractible.*

They do exist! We can construct them by $B := \overline{W}(G)$, $W(G) = G \times_{T(G)} \overline{W(G)}$, where $\overline{W}_n(G) = G_{n-1} \times \cdots \times G_0$ for $n \geq 1$, and terminal for $n = 0$. put face and degen maps here . It has twisting function

$$T(G)[g_n, \dots, g_0] = g_n.$$

It can be checked explicitly that this is type (W).[2]

Corollary 3.16. *Every PTCP with group G is by*

$$B \xrightarrow{\pi} \overline{W}(G)$$

with $\pi(b) = \left[T(b), T(\partial_0 b), \dots, T(\partial_0^{n-1} b)\right]$.

This is a simplicial version of the bar construction??

This allows us to explicitly construct $K(\pi, n)$! Define $K(\pi, 0)$ to be π in each degree and $\partial_i s_i$ all identity. Define $K(\pi, n) = \overline{W}(K(\pi, n-1))$ inductively. We can see this is in fact a $K(\pi_1)$ via a fibration

$$K(\pi, n) \to W(K(\pi, n)) \to \overline{W}(K(\pi, n)),$$

where we know $W(*)$ to be contractible.

The main technical result follows:

Lemma 3.17. *Suppose there is no nontrivial morphism $\pi_1 \to \mathrm{Aut}(\pi_n)$. Then, $X^{(n)}$ is a PTCP with group $K(\pi_{n+1}, n+1)$.*[3]

To handwave, the idea for this is that minimal Kan fibrations are fiber bundles. Given the π_1 assumption, the structure group is $K(\pi_{n+1}, n+1)$. Then, a "principal G-bundle" is the same thing as a PTCP, in some intuitive way.

We can define the k-invariants via the fibrations $K(\pi_{n+1}, n+1) \hookrightarrow X^{(n+1)} \to X^{(n)}$: there is a universal class

$$u \in H^{n+2}(K(\pi_{n+1}, n+2))$$

and via the map $X^{(n+1)} \xrightarrow{f^{n+2}} \overline{W}(K(\pi_{n+1}, n+1)) = K(\pi_{n+1,n+2})$, we can define k-invariants as $\left(f^{n+2}\right)^* u = k^{n+2}$.

[2] This was written down in class.

[3] Per a comment of Prof. Miller, we only need simplicity, not total nontriviality of morphisms $\pi_1 \to \mathrm{Aut}(\pi_n)$.

4 Dylan Pentland: Borel, La cohomologie modulo 2 de certains espaces homogenes

This talk was delivered September 22, 2021 by Dylan Pentland.

4.1 Motivation and preprequisites

Characteristic classes We have a functor

$$\mathrm{Bun}_{O(n)} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$$

sending X to the isomorphism classes of principal $O(n)$ bundles mod isomorphisms. We know that this is representable, i.e. expressable as $\mathrm{Bun}_{O(n)}(-) = \mathrm{Hom}(-, \mathrm{BO}(n))$ (in the homotopy category).

Definition 4.1. A *characteristic class* is a natural transformation $\mathrm{Bun}_{O(n)} \implies H^i(-)$, where coefficients are understood mod 2. By the Yoneda lemma, this is the same thing as an element of $H^i(\mathrm{BO}(n))$.

We're going to characterize these via a cohomology computation. The main theorem is as follows: let $Q(n) \subset O(n)$ be the diagonal matrices. From this inclusion, we get a projection $\mathrm{BQ}(n) \xrightarrow{\rho} \mathrm{BO}(n)$, wich yields an induced map

$$\rho^* : H^*(\mathrm{BO}(n)) \to H^*(\mathrm{BQ}(n)) \simeq \mathbb{F}_2[x_1, \ldots, x_n].$$

Theorem 4.2. *The map ρ^* satisfies the following properties:*

- *ρ^* is injective.*
- *the image of ρ^* is $\mathbb{F}_2[x_1, ;x_n]^{\Sigma_{in}}$.*
- *$p^*(w_i) = e_i$.*

The splitting principle We give a modern POV on this:

Theorem 4.3. *Let X be paracompact and $E \to X$ a bundle. There is an undiced bundle $f : \mathrm{Fl}(E) \to X$ so that*

$$f^* : H^*(X) \to H^*(\mathrm{Fl}(E))$$

is injective, and f^*E splits into a direct sum of line bundles.

This winds up telling you the injectivity of Theorem 4.2, but not the image statement (only a containment). Either way, the proof is not much easier.

Spectral sequences breaking down We'll keep some assumptions about finite type cohomology. Dylan stated the requirement of simply connected spaces or principal G-bundle.[4]

Theorem 4.4. *Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration. The associated Serre spectral sequence (SSS) is trivial if and only if $H^*(E) \to H^*(F)$ is surjective. In this case, we say that F is* totally non-homologous to zero*, and we have the following properties:*

- *p^* is injective.*
- *$P(E) = P(B) \cdot P(F)$.*

The condition is called totally non-homologous to zero because the dual condition $H_*(F) \hookrightarrow H_*(E)$ makes sense for this name.

[4]Haynes had some comments about this; principality is not enough in general. There's secretly some connectedness condition.

4.2 Cohomology of $\mathrm{BO}(n)$

Outline of the proof of Theorem 4.2 Let $F_n = O(N)/Q(N)$. We will use the fibration

$$F_n \hookrightarrow \mathrm{BQ}(n) \xrightarrow{\rho} \mathrm{BO}(n).$$

We call this fibration $(\star)$. We follow the following steps:

(1) $H^*(F_n) = \langle H^1(F_n)\rangle$ so that $P(F_n) = (1-t)\cdots(1-t)^n \cdot (1-t)^{-n}$.

(2) The SSS for $(\star)$ is trivial, so $P(\mathrm{BQ}(n)) = P(\mathrm{BO}(n))P(F_n)$, giving injectivity.

(3) $\operatorname{im}\rho^* \subset \mathbb{F}_2[x_1,\dots,x_n]^{\Sigma_{in}}$, and dimensions yield that this is an equality.

General rule of theorem: every spectral sequence written today will be trivial.

Step 1: cohomology of F_n. We use induction via the fibration

$$F_{n-1} \hookrightarrow F_n \to \mathbb{P}^{n-1}$$

Lemma 4.5. $\dim H^1(F_n) \geq n-1$.

Write ${}^nE_r = \bigoplus_{s+t=n} E_r^{s,t}$.

Proof. Recall the fibration $F_n \hookrightarrow \mathrm{BSQ}(n) \to \mathrm{BSO}(n)$. The base space is simply connected, so

$$E_2^{1,0} = H^1(\mathrm{BSO}(n), H^0(F_n)) = 0.$$

Hence

$$\dim{}^1 E_2 \geq \dim{}^1 E_\infty = \dim H^1(\mathrm{BSQ}(n)) = n-1.$$

This implies that $E_2^{0,1} = H^1(F_n)$. ☺

Proposition 4.6. *$P(F_n) = (1-t)\cdots(1-t^n)(1-t)^{-n}$ and $H^*(F_n) = \langle H^1(F_n)\rangle$.*

Proof. Return to the fibration from the beginning of the fibration. We know the Poincaré polynomial for projective space, and we just have to prove that the SSS is trivial.[5] Write $H^*(F_n) \xrightarrow{i^*} H^*(F_{n-1})$. Assume both claims for $n-1$, so $\dim H^1(F_{n-1}) = n-2$. Note the following:

- $\dim E_2^{1,0} = \dim H^1(\mathbb{P}^{n-1}) = 1$.
- $\dim E_2^{0,1} = \dim H^0(\mathbb{P}^{n-1}, H^1(F_n)) \leq n-2$.

Look at $\dim H^1(F_n) = {}^1E_\infty \leq n-1$; combined with our previous bound, we have $\dim H^1(F_n) = n-1$. This implies that ${}^1E_n ={}^1 E_\infty$ since they have equal dimensions. This implies that $E_2^{0,1}$ are cocycles for differentials.

Further, note that $\operatorname{im} i^*|_{\deg 1} = E_\infty^{0,1} = H^1(F_{n-1})$. Since cohomology of the codomain is generated in degree 1, this implies that i^* is surjective, so the SSS is trivial. This implies the Poincaré polynomial is as we said it is, by a familiar technique. ☺

Step 2: triviality of the SSS of $(\star)$.

Proposition 4.7. *The SSS for $(\star)$ is trivial.*

Proof. Note that $\dim{}^1 E_2 = \dim H^1(\mathrm{BO}(n), H^0(F_n)) + \dim H^0(\mathrm{BO}(n), H^1(F_n))$. The first is equal to 1, and the second is $\leq n+1$, and the second is $\leq n-1$, so the total is $\leq n$.

Now look at $\dim{}^1E_\infty \leq \dim{}^1 E_2$, which is an equality for dimension reasons. We have $\dim{}^1 E_2 \geq \dim{}^1 E_\infty$, and hence $H^0(\mathrm{BO}(n), H^1(F_n)) = H^1(F_n)$. For reasons relating to generation t degree 1, we also have $H^0(\mathrm{BO}(n), H^k(F_n)) = H^k(F_n)$. Hence $H^*(\mathrm{BQ}(n)) \twoheadrightarrow H^*(F_n)$. Hence the SSS is trivial. ☺

This allows us to immediately compute the Poincaré polynomial

$$P(\mathrm{BO}(n)) = \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)}.$$

[5] In particular, $P(\mathbb{P}^{n-1}) = \frac{1-t^n}{1-t}$, so since passing to the associated graded preserves graded dimension, triviality implies that the Poincaré series are multiplicative, and we can prove the Poincaré series computation inductively. I'll skip this.

Step 3: containment of the image of ρ^* in the symmetric polynomials. Combinatorics exists:

Lemma 4.8. $P(\mathbb{F}_2[x_1, \ldots, x_n]^{\Sigma_{in}}) = P(\mathrm{BO}(n))$.

Use Schur polynomials. Back to the topology.

Proposition 4.9. in $\rho^* \subset \mathbb{F}_2[x_1, \ldots, x_n]^{\Sigma_{in}}$.

Proof. $\Sigma_{in} = N_n/Q(n)$. Write down the classifying space fibration:

$$Q(n) \hookrightarrow EQ(n) \to BQ(n)$$

N_n acts on this, and acts on polynomials by permuting the generators in $\mathbb{F}_2[x_1, \ldots, x_n]$. The normalizer N_n also acts on

$$O(n) \to EO(n) \to \mathrm{BO}(n).$$

the action on $\mathrm{BO}(n)$ is homotopically trivial[6], which we could use... Instead, we know the groups, so we can check concretely that this acts trivially on the cohomology, which gives the image containment. ☺

Hidden step 4: talking about $p^*(w_i) = e_i$. Once we know the $p^*(w_i) = e_i$ statement, universal relations on Steifel-Whitney classes come down to relations on $H^*(\mathrm{BO}(n))$.

[6] This is a general fact stated by Haynes, which comes down to some categories thing I didn't catch...

5 Mikayel Mkrtchyan: Milnor, The Steenrod algebra and its dual

This talk was delivered September 24, 2021 by Mikayel Mkrtchyan.

5.1 Refresher on the Steenrod algebra and Hopf algebras

For the next 48 minutes or so, we set $p = 2$ and work with the mod-2 Steenrod algebra. Recall that the *Steenrod algebra* $\mathcal{A}^*$ is a graded commutative algebra of mod-2 cohomology operations generated as an algebra by $\mathrm{Sq}^n \in \mathcal{A}^n$ for $n \geq 1$. For a finite sequence $I = (i_1, \dots, i_r)$, define

$$\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_r}.$$

Recall that a sequence I is admissible if $a_i \geq 2a_{i-1}$. We have a basis made of these:

Theorem 5.1 (Serre-Cartan)**.** *The set* $\left\{\mathrm{Sq}^I \mid I \text{ admissible}\right\}$ *is an* $\mathbb{F}_2$*-basis for* $\mathcal{A}^*$.

This is proved via the following relation:

Theorem 5.2 (Adem relation)**.** *For all* $0 < n < 2m$,

$$\mathrm{Sq}^n \mathrm{Sq}^m = \sum_{k=1}^{n/2} \binom{m-k-1}{n-2k} \mathrm{Sq}^{n+m-k} \mathrm{Sq}^k,$$

and these generate all relations in a presentation of $\mathcal{A}^*$.[7]

As a preview of what's to come, define $\mathcal{A}_*$ to be the dual coalgebra.

Theorem 5.3. $\mathcal{A}^*$ *is a graded connected Hopf algebra, and its dual satisfies* $\mathcal{A}_* \simeq \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$.

As an application, we'll show that all elements in $\mathcal{A}^{>0}$ are nilpotent.
The base field of $\mathbb{F}_2$ is to be understood.

Definition 5.4. A connected graded Hopf algebra is a graded associative algebra B^* s.t. $B_0 = \mathbb{F}_2$, endowed with a coassociative *comultiplication* map

$$B^* \xrightarrow{\psi} B^* \otimes B^*$$

s.t. $\psi(b) = b \otimes 1 + 1 \otimes b + \sum b_i' \otimes b_i''$ for all $b \in B^{>0}$.

Projecting to the 0th graded part is the "augmentation" (counit), and you can define the antipode uniquely given this data.

5.2 Coalgebra structure on the Steenrod algebra

We want to define a map $\psi : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$ extending the map

$$\psi(\mathrm{Sq}^n) = \sum_{i+j=n} \mathrm{Sq}^i \otimes \mathrm{Sq}^j.$$

We will prove that this is well defined using the Cartan formula

$$\mathrm{Sq}^n(a \times b) = \sum_{i+j=n} \mathrm{Sq}^i(a) \times \mathrm{Sq}^j(b).$$

We'll also use the following

[7] The presentation statement was said verbally but not written.

Lemma 5.5. *Fix some n. There exists a space U with finite-type cohomology and a class $u \in H^*(U)$ such that*
$$\sigma : \mathcal{A}^* \to H^*(U)$$
given by $a \mapsto a \cdot u$ is injective on $\mathcal{A}^{\leq n}$.

Proof sketch. This is given by $U = K(\mathbb{Z}/2, n+1)$ with $u \in H^{n+1}(K(\mathbb{Z}/2, n+1))$. Use the result of Gabi's talk.

Lemma 5.6. *There exists a lift*

$$\begin{tikzcd}
T(\{\mathrm{Sq}^n\}) \arrow[r, two heads] \arrow[d, "\psi"] & \mathcal{A}^* \arrow[dl, dashed] \\
\mathcal{A}^* \otimes \mathcal{A}^* &
\end{tikzcd}$$

Proof. Choose (U, u, σ) as in the lemma. Form the diagram

$$\begin{tikzcd}
T(\{\mathrm{Sq}^n\}) \arrow[rr, two heads] \arrow[d, "\psi"] & & \mathcal{A}^* \arrow[d, "\text{act on } u \times u"] \arrow[dll, dashed] \\
\mathcal{A}^* \otimes \mathcal{A}^* \arrow[r, "\sigma\otimes\sigma"] & H^*(U) \otimes H^*(U) \simeq & H^*(U \times U).
\end{tikzcd}$$

Then, note that $\sigma \otimes \sigma$ is injective at degrees $\leq n$, and this commutes, hence we can choose a lift in lower degrees; this allows you to define it degreewise by picking high enough n.

Corollary 5.7. *$\mathcal{A}^*$ is a cocommutative Hopf algebra.*[8]

5.3 Housekeeping

Suppose X is a finite CW complex. This is routine, formal, and not talked about explicitly.

1. We have an action
$$\mathcal{A}^* \otimes H^* \to H^*.$$
2. This yields a dual operation
$$H_* \otimes \mathcal{A}* \to H_*.$$
3. We can dualize this:
$$\lambda : H^* \to H^* \otimes \mathcal{A}_*,$$
since the homology and $\mathcal{A}$ are both finite type.
4. Note that λ makes H^* into a $\mathcal{A}_*$-comodule.
5. The following proof was omitted:

   Lemma 5.8. *λ is an $\mathbb{F}_2$-algebra homomorphism.*

Let's work an example.

Example 5.9:
Let $X := \mathbb{RP}^\infty = K(\mathbb{Z}/2, 1)$, with $u \in H^1(X)$.[a]
Lemma 5.10.
$$\mathrm{Sq}^n(x^{2^m}) = \begin{cases} x^{2^{m+1}} & n = 2^m \\ 0 & \textit{otherwise} \end{cases}$$

Proof sketch. Define $\mathrm{Sq} := \sum_i \mathrm{Sq}_i$. Note that $\mathrm{Sq}(u) = u + u^2$, so $\mathrm{Sq}(u^{2^m}) = u^{2^m} + u^{2^{m+1}}$.

[8] He called it coassociative, but I omit this as this is the convention for Hopf algebras in general.

Corollary 5.11. *$\lambda : H^*(X) \to H^*(X) \otimes \mathcal{A}_*$ is given by*

$$\lambda(u) = \sum_k u^{2^j} \otimes \zeta_k$$

where $\langle \zeta_i, \mathrm{Sq}^I \rangle = 0$ unless $I = I_k := (2^{i-1}, 2^{k-2}, \dots, 1, 0)$.

I'm lagging a bit behind, so expect this next bit to be choppy.

[a] We'll see why we don't have to care that X is finite.

5.4 Algebra structure on the dual Steenrod algebra

Let I be an admissible sequence, and define

$$\gamma(I) = (i_1 - 2i_2, i_2 - 2i_3, \dots, i_r, 0).$$

Let R be a sequence.

Proposition 5.12. *Let I, J be admissible sequences of the same degree. Then,*

$$\langle \zeta^{\gamma(J)}, \mathrm{Sq}^I \rangle = \begin{cases} 1 & I = J \\ 0 & I < J \end{cases}$$

where $<$ denotes the lexicographic order.[9]

Proof. We prove this by induction. Let $J = (aj_1, \dots, a_k, 0)$ and similar for I and b. define

$$J' = (a_1 - 2^{k-1}, a_2 - 2^{k-2}, \dots, 0).$$

Then,

$$\gamma(J) = \gamma(J') + (\text{a } 1 \text{ in the } k\text{th spot.}).$$

Hence

$$\zeta^{\gamma(J)} = \zeta^{\gamma(J')} \cdot \zeta_k,$$

so that

$$\langle (\zeta^{\gamma(J)}, \mathrm{Sq}^I \rangle = \langle \zeta^{\gamma(J)} \otimes \zeta_k, \psi(\mathrm{Sq}^I) \rangle = \langle \zeta^{\gamma(J')} \otimes \zeta_k, \sum \mathrm{Sq}^{I_1} \otimes \mathrm{Sq}^{I_2} \rangle.$$

If you work out the nitty gritty, this concludes the proof by induction.

Corollary 5.13. $\mathcal{A}_* \simeq \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$.

Proof. The last proposition proved that $\zeta^{\gamma(J)}$ form an $\mathbb{F}_2$-basis, which is exactly equivalent to $\mathcal{A}_*$ being a polynomial algebra in ζ_i.

We now characterize the comultiplication.

Theorem 5.14. *The comultiplication map $\varphi_* : \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ is given by*

$$\zeta_n \mapsto \sum_{k \geq 0} \zeta_{n-k}^{2^k} \otimes \zeta_k.$$

This is some measure of the basis we gave being nice.

Proof. We have coassociativity:

$$\begin{tikzcd} H^* \arrow[r] \arrow[d] & H^* \otimes \mathcal{A}_* \arrow[d, "\lambda \otimes \mathrm{id}"] \\ H^* \otimes \mathcal{A}_* \arrow[r, "\mathrm{id} \otimes \varphi_*"] & H^* \otimes \mathcal{A}_* \otimes \mathcal{A}_* \end{tikzcd}$$

We perform a diagram chase for $X := \mathbb{RP}^\infty$.

[9] It hasn't been mentioned what happens when $I > J$.

5.5 Positive-degree homogeneous elements of the Steenrod algebra are nilpotent

Define $J_n \subset \mathcal{A}_*$ by $(\zeta_1^{2k}, \zeta_2^{k-1}, \ldots, \zeta_{k-1}^2, \zeta_{k+1}, \ldots)$. Observe that $\varphi_*(J_n) \subset J_n \otimes \mathcal{A}_*$ by our characterization of the Milnor diagonal, and hence $\mathcal{A}_*/J_n$ is a Hopf algebra quotient of $\mathcal{A}_*$ of finite dimension. By duality, this corresponds with a f.d. Hopf subalgebra, and expanding n threatens to swallow $\mathcal{A}_*$:

Corollary 5.15. *$\mathcal{A}^*$ is the union of its finite dimensional Hopf subalgebras.*

By degree arguments, a positive dimension homogeneous element either is nilpotent or spans an infinite dimensional Hopf subalgebra, so this gives the nilpotency statement.

5.6 A sketch of the $p > 2$ case

In the odd p case, we have Lens spaces instead of $\mathbb{RP}^\infty$, and there are more cohomology elements:

Theorem 5.16.

$$\mathcal{A}_*^p = \mathbb{F}_p[\zeta_1, \ldots, \zeta_i] \otimes \bigwedge{}^* [\tau_0, \tau_1, \ldots] .$$

6 Preston Cranford: Thom, Quelques proprietes globales des varieties differentiables (i)

This talk was delivered September 29, 2021 by Preston Cranford.[10]

6.1 Motivation and overview

We're motivated by the following question

Problem 6.1 (Steenrod's problem). *Let K be a finite polyhedron. Given $z \in H_r(K)$ (over $\mathbb{F}_2$ or $\mathbb{Z}$), does there exist compact M and map $f : M \to K$ s.t. $f_*[M] = Z$?*

If so, say that z is *realized* by M. We will use the notation V^n for the n-manifold, and W^p the manifold realizing a class. We henceforth restrict to $\mathbb{F}_2$ coefficients. Our result will be that whenever $2p \leq n$, alL z are realizable.

We'll follow the following outline:

- Thom spaces, classes,...
- We show z is realizable iff another class of a Thom space is realizable.
- We will studdy $\mathrm{MO}(k)$ and show this has the homotopy type of a product of $\mathbb{F}_2$-Eilenberg Mac Lane spaces.

6.2 Thom spaces and realizability

We'll use the following:

Definition 6.2. Let $G \subset O(n)$ be a distinguished subgroup with a faithful representation. A G-strcture on M is a principal G-subbundle of the frame bundle on M. The $O(k)$ structure is associated with a metric on M.

Recall that we have a universal bundle $\mathrm{EO}(k) \xrightarrow{p} \mathrm{BO}(k)$ where $\mathrm{EO}(k)$ is weakly contractible. Let $\mathrm{AO}(k)$ be the mapping cylinder, and let $\mathrm{MO}(k)$ be $\mathrm{AO}(k)/(\mathrm{EO}(k) \sim *)$.[11] See footnote.

Recall the following: construct the map

$$\varphi^* : H^{r-k}(\mathrm{BO}(k)) \to H^r(\mathrm{AO}(k), \mathrm{EO}(k)) \simeq H^r(\mathrm{MO}(k)).$$

Theorem 6.3 (Thom isomorphism theorem). *The map φ^* is an isomorphism.*

For unit class $w \in H^0(\mathrm{BO}(k))$, call $\varphi^*(w) = u$ the *fundamental class of* $\mathrm{MO}(k)$.

We'll prove the following:

Theorem 6.4. *Let V^n be a closed n-manifold. Then, $z \in H_{n-k}(V^n)$ is realizable by W^{n-k} if and only if its Poincaré dual $u \in H^k(V^n)$ is induced by some $f : U^n \to \mathrm{MO}(k)$.*

Proof. ($\implies$) Suppose we have $z \in H_{n-k}(V^n)$ realized by $i : W^{n-k} \to V^n$. Let $N \to W^{n-k}$ be a normal tubular neighborhood of W^{n-k}, which has an $O(k)$-structure.[12] Define a filler via pullback[13]

$$\begin{tikzcd} N \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{VO}(k) \arrow[d] \\ W^{n-k} \arrow[r] & \mathrm{BO}(k) \end{tikzcd}$$

[10] I came in a bit late, so expect the beginning of these to be choppy and potentially subtly wrong.

[11] This was corrected by Haynes to more modern notation, noting that this is wrong. $\mathrm{EO}(k)$ here was supposed to be the universal sphere bundle. Haynes suggested the Thom space as the quotient of the universal disk bundle by the universal sphere bundle, and that the homotopical definition here was more general than necessary.

[12] He went over how to construct this locally, but I won't repeat it, since it's routine and I'm catching up a bit.

[13] There's notational confusion; $\mathrm{AO}(k)$ henceforth means the mapping cone of the universal sphere bundle, and $\mathrm{VO}(k)$ is the total space of the universal vector bundle.

Explicitly, this takes geodesic ball fibers to ball fibers. Taking cohomology, we have

$$\begin{tikzcd}
H^k(N, \partial N) & H^n(\mathrm{VO}(n), \mathrm{EO}(n)) \arrow[l] \\
H^0(W^{n-k}) \arrow[u] & H^0(\mathrm{BO}(n)) \arrow[l] \arrow[u]
\end{tikzcd}$$

We have a collapse map, where a is the point at infinity:

$$\begin{tikzcd}
 & N \arrow[r] & W^{n-k} \arrow[r] & \mathrm{BO}(n) \arrow[r] & \mathrm{MO}(k) & \\
 & \partial n \arrow[u, hook] \arrow[rrr] & & & \{a\} \arrow[u, hook] & \\
V^n/N \arrow[uur, hook] \arrow[rrrrr] & & & & & \{a\} \arrow[uul, hook]
\end{tikzcd}$$

I am very confused. The following diagram was on the board, but I can't decipher what it means:

$$\begin{tikzcd}
\text{p.d. to } z \arrow[r, equal] & \varphi^*(w_0) & U \arrow[l] \\
 & w_0 \arrow[u] & w_{O(k)} \arrow[l] \arrow[u]
\end{tikzcd}$$

He commented on the other direction after this.

We can identify the cohomology of Thom space as an ideal: $H^*(\mathrm{MO}(k)) = w_k H^*(\mathrm{BO}(k))$, recalling that $H^*(\mathrm{BO}(k)) = \mathbb{F}_2[w_1, \dots, w_k]$.

Recall that we have the bundle $\mathrm{VO}(n) \to \mathrm{BO}(n)$, and this expresses the fundamental group

$$\pi_1(\mathrm{MO}(k)) = \pi_1(\mathrm{BO}(k))/\operatorname{im}(\pi_1(\mathrm{VO}(k)) \to \pi_1(\mathrm{BO}(k))) = \mathbb{F}_2/\mathbb{F}_2 = 0.$$

Hence $\mathrm{MO}(k)$ is simply connected. Recall the following:

Theorem 6.5. *If X, Y are simply connected and $f : X \to Y$ is a map s.t. f^* is an iso for $r < k$ and monic at $r = k$, then there is a map $g : X_k \to Y_k$ that is a homotopy equivalence.*

This allows you to prove the following theorme:

Theorem 6.6. *There is a $2k$-equivalence $\mathrm{MO}(k) \to K(\mathbb{F}_2, k)^{e_1} \times \cdots \times K(\mathbb{F}_2, 2k)^{e_{2R}}$.*

7 Swapnil Garg: Thom, Quelques proprietes globales des varieties differentiables (ii)

This talk was delivered October 1, 2021 by Swapnil Garg.

7.1 Transversality and tubular neighborhoods

For the duration of this talk, V^n is a smooth manifold, and N^{p-q} is a smooth, compact manifold.

Definition 7.1. A map $f : V^n \to M^p$ is transversal to $N^{p-q} \subset M^p$ at a point $y \in N^{p-q}$ if for all $x \in f^{-1}(y)$, the map $DF : T_x V^n \to T_y M^p \to T_y M^p / T_y N^{p-q}$.

If d is tranverse at all Y, then we say f is *transversal to* N^{p-q}.

In this case, $f^{-1}(p-q)$ is a smooth manifold with normal bundle f^{-1}(normal bundle of N^{p-q}).

Let $p : T \to N^{p-q}$ be a tubular neighborhood.

Definition 7.2. Let H be the group of diffeomorphisms $A \in \operatorname{Aut} T$ s.t.

(a) A is the identity on ∂T.

(b) A preserves $p^{-1}(y)$ for all $y \in N^{p-q}$.

Defining the distance between diffeomorphisms by taking some supremum over the maps, derivatives, inverse, etc., H is a metrizable space.

Let $y \in N^{p-q}$ be a point, and $y \in X, X'$ balls of radius r, r' around y in N^{p-q}. These can be chosen small enough to trivialize the normal bundle. Let $D = p^{-1}(X) = X \times B^q$ and similarly for D'.

There is a diagram

$$\begin{tikzcd}
V^n \arrow[r, "f"] & M & \\
 & N \arrow[u, hook] & \\
V|_{f^{-1}(D)} \arrow[r, "f"] & D' \arrow[u, hook] \arrow[r, "k"] & B^q
\end{tikzcd}$$

If $A \circ f$ has a critical value at y, i.e. the derivative is non-surjective, then $k \circ A \circ f$ has a critical value at $k(y) = 0$ (the center of B^q).

Let $\sigma_i \subset H$ be the set of automorphisms A s.t. $k \circ A \circ f|_{f^{-1}(D) \cap K_i}$ has a critical value at 0. We'll characterize this:

Lemma 7.3. *The set σ_i is closed and has no interior points in H.*

Hence the generic diffeomorphism is regular on K_i.

Proof. Closedness. For $A \notin \sigma_i$, we want to construct a ball around A. If the absolute value of the determinant of the $q \times q$ Jacobian of A is bounded below by 2ε, then under perturbing A s.t. the Jacobian changes by at most ε, the map remains regular. Hence σ_i is closed.

Empty interior. We invoke Sard's thoemre. Take $A \in \sigma_i$. Assuming f is C^n, there exists a regular value c of $K \circ A \circ f$ which is arbitrarily close to 0. Take G_1 to be the identity on B^q, G_0 a diffeomorphism of B^q with $G_0(c) = 0$, and G_t to be a homotopy between them with varying t. Let $E(y, z) = (y, G_{d(y)}(z))$ with $d(x) = 0$, $d(\partial x') = 1$ and $d \in C^\infty$. Take $k \circ E \circ A \circ f$; this has a regular value at 0, which allows you to conclude.[14]

Varying K_i, the countable intersection of $(\sigma_i)^c$ is dense. We get a meaningful result out of this:

Theorem 7.4. *A C^n map $f : V^n \to M^p$ can be perturbed to be transversal to $N^{p-q} \subset M^p$.*

We'll use this to talk about cobordism.

[14] Not sure how this finishes.

7.2 Cobordism and L-equivalence

Definition 7.5. Let V, V' be oriented compact manifolds of dimension k. The manifolds V and V' are *cobordant* if there exists a $k+1$-manifold X with $\partial(X) = V' \coprod -V$.[15][16]

Suppose $W^{n-k} \subset V^n \subset \mathbb{R}^{n+m}$. For $x \in W^{n-k}$, let $H(kx)$ be the k-dimensional vetor space tangent to V^n and tangent to W^{n-k} inside $\mathbb{R}^{n+m}$. This yields a map $W^{n-k} \to \mathrm{Gr}_k(\mathbb{R}^{n+m})$. Taking $m \to \infty$, we have a map $\mathrm{Gr}_k(\mathbb{R}^{n+m}) \to \mathrm{BO}(k)$.

For N a tubular neighborhood of W^{n-k} in V^n, we get a map

$$\begin{array}{ccc} N & \longrightarrow & D(\mathrm{EO}(k)) \\ \downarrow & & \downarrow \\ W^{r-k} & \xrightarrow{g} & \mathrm{BO}(k) \end{array}$$

Collapsing $V - N$ to a point gives a map

$$V \xrightarrow{f} D(\mathrm{EO}(k))/S(\mathrm{EO}(k)) = \mathrm{MO}(k).$$

whose homotopy class turns out to be independent of the Riemannian metric chosen to define it.

We define a stronger equivalence:

Definition 7.6. Let $W_i^{n-k} \subset V^n$ be submanifolds. We say that they are *L-equivalent* with respect to V^n if there exists a submanifold $X^{n-k+1} \subset v^n \times [0,1]$ such that $\partial X^{n-k+1} = W_0^{n-k} \sqcup W_1^{n-k}$.[17]

If W_i^{n-k} are L-equivalent, they generate homotopic maps $V^n \to \mathrm{MO}(k)$. The main theorem of Thom's in this are shows that this is a bijection:

Theorem 7.7. *The above map $L_{n-k}(V^n) \to [V^n, \mathrm{MO}(k)]$ is a bijection.*[18]

There is some confusion about hos this depends on V^n. Taking V^n the sphere spectrum, taking a limit yields that the Cobrodism ring corresponds with the stable homotopy groups of the Thom spectrum:

Lemma 7.8. *$L_k(S^n) \simeq \mathfrak{N}^k$ is a bijection if $n \geq 2k+2$. A similar statement holds in the oriented case.*

Theorem 7.9. *$\mathfrak{N}^k \simeq L_k(S^{n+k}) \simeq \pi_{n+k}(\mathrm{MO}(n)$ for $n \geq k+2$. Similarly, $\Omega^k \simeq \pi_{n+k}(\mathrm{MSO}(n))$.*

7.3 Other remarks

Note that we know the stable homotopy groups of $\mathrm{MO}(n)$; they are $\mathbb{F}_2^{d(k)}$, as per Preston's talk.

We can further argue that cobordant manifold have the same Steifel-Whitney numbers. We argue via a big diagram:

$$\begin{array}{ccccc} S^{n+k} & \xrightarrow{f_{W'}} & \mathrm{MO}(n) & \xrightarrow{F} & K(n+k, \mathbb{F}_2) \\ & & \downarrow{\scriptstyle \mathrm{proj}} & & \\ V_\omega & \longrightarrow & \mathrm{Gr}_k & & \end{array}$$

we can trace elements of the cohomology and make an argument there. I couldn't quite follow.

[15] The sign indicates reversal of orientation.
[16] He drew the pair of pants cobordism as an example here.
[17] He drew a helpful picture; $V^n \times [0,1]$ is pictured as a solid cylinder, whose ends are disk copies of V^n, in which W^i are embedded circles. An L-equivalence is pictured as a cylinder bounded by each W_i, on opposite disk faces.
[18] I use my own notation here' Swapnil followed Thom's notation.

8 Haoshuo Fu: Hirzebruch, Topological methods in algebraic geometry

This talk was delivered on October 4, 2021 by Haoshuo Fu. The topic of this talk will be the index theorem.

8.1 The signature of a manifold

Definition 8.1. Let M be an oriented closed manifold of dimension $4k$. There is a symmetric bilinear form

$$\begin{aligned} \phi_M : H^{2k}(M, \mathbb{R}) \otimes H^{2k}(M, \mathbb{R}) &\to \mathbb{R} \\ x \otimes y &\mapsto \langle x \cup y, [M] \rangle. \end{aligned}$$

The *signature of* M is the signature

$$\tau(M) = (\#\text{ pos. eigenvals. of } \phi_M) - (\#\text{ neg. eigenvals. of } \phi_M).$$

> **Example 8.2:**
> In the case $M = \mathbb{CP}^{2k}$, then $H^{2k}(\mathbb{CP}^{2k}) = \mathbb{R}$, we have and $\varphi = I$, so $\tau(\mathbb{CP}^{2k}) = 1$.

We can characterize these:

Proposition 8.3. *The following hold:*

- $\tau(-M) = -\tau(M)$.
- $\tau(M \sqcup N) = \tau(M) + \tau(N)$.
- $\tau(M^{4k} \times N^{4\ell}) = \tau(M^{4k}) \cdot \tau(N^{4\ell})$.
- *If* $M^{4k} = \partial W^{4k+1}$, *then* $\tau(M^{4k}) = 0$.

Proof sketch. For the third bullet, note that[19]

$$h^{2k+2\ell}(M^{4k} \times N^{4\ell}) = \bigoplus_{i \in \mathbb{Z}} H^{2k+i}(M^{4k}) \oplus H^{2k-i}(N^{4\ell}) =: \bigoplus_{i \in \mathbb{Z}} V(i).$$

When $i = 0$, we have $\phi_{M \times N}|_{V(0)} = \phi_M \otimes \phi_N$. When $i > 0$, letting A be the pairing of $V(i)$ and $V(-i)$, we have

$$\phi_{M \times N}|_{V(i) \oplus V(-i)} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}.$$

Noting that $\det(\lambda I - \phi_{M \times N}|_{V(i) \oplus V(-i)} = \det(\lambda^2 I - A^2)$ allows one to conclude.

The last bullet it proved via Poincaré duality; given the morphism of exact sequences

$$\begin{tikzcd} H^{2k}(w) \arrow[r, "i^*"] \arrow[d] & H^{2k}(M) \arrow[r, "\delta"] \arrow[d] & H^{2k+1}(W, M) \arrow[d] \\ H_{2k+1}(W, M) \arrow[r] & H_{2k}(M \arrow[r] & H_{2k}(W) \end{tikzcd}$$

the snake lemma yields a short exact sequence

$$0 \to \operatorname{im} i_* \to H^{2k}(M) \to \operatorname{im} \delta \to 0.$$

Note that $\dim \operatorname{im} \delta = \dim \operatorname{im} i^*$, so $\operatorname{im} i^*$ is a subspace of $H^{2k}(M)$ of half dimension. We have $\phi_M|_{\operatorname{im} i^* \otimes \operatorname{im} i^*} = 0$, and subsequently, $\phi_M = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, gviving the fourth bullet point.

Corollary 8.4. *τ yields a homomorphism* $\tau : \Omega_{4*} \to \mathbb{Z}$.

[19] I omit the proof of the first two, as they're obvious.

8.2 Review of the structure of the cobordism ring

Take notation from the previous two talks, on Thom. Note that there is a (pulled back) map

$$\begin{tikzcd} \xi_n \oplus e \arrow[r] \arrow[d] & \xi_{n+1} \arrow[d] \\ \mathrm{BSO}(n) \arrow[r] & \mathrm{BSO}(n+1). \end{tikzcd}$$

Taking Thom spaces yields a morphism $\Sigma\,\mathrm{MSO}(n) = \mathrm{Th}(\xi_n \oplus e) \to \mathrm{MSO}(n+1)$. This yields a prespectrum MSO.[20]

Previous results yield an iso of graded Abelian grooups

$$\pi_*(\mathrm{MSO}) = \operatorname{colim} \pi_{*+n)}(\mathrm{MSO}(n)) = \Omega_*.$$

We can give MSO a ring spectrum structure via maps

$$\begin{tikzcd} \xi_m \oplus \xi_n \arrow[r] \arrow[d] & \xi_{m+n} \arrow[d] \\ \mathrm{BSO}(m) \times \mathrm{BSO}(n) \arrow[r] & \mathrm{BSO}(m+n) \end{tikzcd}$$

and a similar trick to before.

Recall the rational Hurewicz theorem:

Theorem 8.5 (Rational Hurewicz theorem)**.** *Let X be a simply connected space with $\pi_i(X) \otimes \mathbb{Q} = 0$ for all $0 \le i \le$. Then, the Hurewicz map*

$$h : \pi_i(X) \otimes \mathbb{Q} \to \widetilde{H}_i(X; \mathbb{Q})$$

is an isomorphism for $0 \le i \le 2i$.

Applying this to the Thom spectrum, noting that we have we have

$$\widetilde{H}^i(\mathrm{MSO}(n); \mathbb{Q}) = 0 \qquad\qquad i \le n$$

we can compute rational stable homotopy groups of MSO by computing homology in degree $\le 2n$:[21]

$$\pi_*(\mathrm{MSO}) \otimes \mathbb{Q} \xrightarrow{\sim} H_*(\mathrm{MSO}; \mathbb{Q}).$$

We may combine this with the Thom isomorphism

$$H^i(\mathrm{BSO}(n); \mathbb{Q}) \xrightarrow{\sim} H^{i+n}(\mathrm{MSO}(n); \mathbb{Q})$$

by taking a limit in order to yield an iso

$$H^*(\mathrm{BSO}(n); \mathbb{Q}) \xrightarrow{\sim} H^*(\mathrm{MSO}; \mathbb{Q}).$$

Taking udals yields an iso

$$H_*(\mathrm{MSO}; \mathbb{Q}) \xrightarrow{\sim} H_*(\mathrm{BSO}(n); \mathbb{Q}).$$

Hence the signature actually yields a homomorphism

$$\tau : H_*(\mathrm{BSO}; \mathbb{Q}) \to \mathbb{Q},$$

i.e. an element $L \in H^*(\mathrm{BSO}; \mathbb{Q})$. We can write this in homogeneous parts as $L_n \in H^{4n}(\mathrm{BSO}; \mathbb{Q})$.

A diagram chase yields that

$$\langle L_N(\tau_M), [M] \rangle = \tau(M).$$

[20] I would just call this a sequential spectrum.

[21] This is true of any ring spectrum, according to Haynes.

This is powerful, if we can determine the structure of L_N. We do so in a little while, but first, a bit more structure.

There is a commutative diagram

$$\begin{tikzcd} H_*(\mathrm{BSO};\mathbb{Q}) \otimes H(\mathrm{BSO};\mathbb{Q}) \arrow[r] \arrow[d] & H_*(\mathrm{BSO};\mathbb{Q}) \arrow[d] \\ \mathbb{Q} \otimes \mathbb{Q} \arrow[r, equal] & \mathbb{Q} \end{tikzcd}$$

yielding a comultiplication

$$H^*(\mathrm{BSO};\mathbb{Q}) \to H^*(\mathrm{BSO};\mathbb{Q}) \otimes H^*(\mathrm{BSO};\mathbb{Q})$$

sending $L \mapsto L \otimes L$. The Whitney sum formula yields that $p \mapsto p \otimes p$ as well.

8.3 Multiplicative sequences

We henceforth fix B a unital ring, P_i indeterminants with $P_0 = 1$ and $\deg P_i = i$.

Definition 8.6. A sequence $(K_n(P_1, \dots, P_n) \in B[P_1, \dots]_n)$ of polynomials of degree n is called a *multiplicative sequence*, or an *m-sequence*, if in the notation

$$\sum_{i=0}^{\infty} P_i \zeta^i = \left(\sum P'_j \zeta^j\right)\left(\sum P''_k \zeta^k\right)$$

we have

$$\sum_{i=0}^{\infty} K_i(P_1, \dots, P_n)\zeta^i = \left(\sum_j K_j(P'_1, \dots, P'_j)\zeta^j\right)\left(\sum_k K_k(P''_1, \dots, P''_k)\zeta^k\right).$$

Note that (L_n) is an m-sequence.[22] We will use computational tools for multiplicative sequences to work with this:

Theorem 8.7. *The sequence (K_n) is uniquely determined by*

$$Q(\xi) = K(1+\xi) = \sum_{i=0}^{\infty} b_i \zeta^i$$

where $b_i = K_i(1, 0, \dots, 0)$.

Proof. Note that

$$\sum_{i=0} P_i \zeta^i = \prod_{i=1}^{m} (1 + \beta_i \zeta)$$

for some β_i via the splitting principle. Hence

$$\sum_{i=0}^{m} K_i(P_1, \dots, P_i)\zeta^i \prod_{i=1}^{m} Q(\beta_i \zeta).$$

This determines each K_i. ☺

Let's work an example.

Example 8.8:
Recall that $H^*(\mathbb{CP}^{2k};\mathbb{Q}) = \mathbb{Q}[h]/h^{2k+1}$. We can use some algebraic geometry[a] to show that

$$\sum L_i \zeta^i = Q(h^i \zeta)^{2k+1}.$$

[22] He justified this, but I couldn't follow it.

This shows that the coefficient of ζ^k in $Q(\zeta^{2k+1})$ is 1. We will conclude with this:

Claim. *$Q(\zeta)$ is unique and $Q(\zeta) = \frac{\sqrt{\zeta}}{\sinh\sqrt{\zeta}}$.*

Proof. Note that, in the notation $Q(\zeta) = \sum_{i=0}^{\infty} b_i \zeta_i$, we have $(2k+1)b_k + \text{ lower terms } = 1$. By induction, b_k is unique, and

$$\int \frac{1}{\zeta^{k+1}} \left(\frac{\sqrt{\zeta}}{\sinh\sqrt{\zeta}} \right)^{2k+1} d\zeta = 1.$$

☺

[a] I'm not going to copy the alg geo out of laziness.

This allows us to conclude the index theorem:

Theorem 8.9 (Hirzubruch's signature theorem)**.** *If (L_n) is the m-sequence related to $\frac{\sqrt{\zeta}}{\sinh\sqrt{\zeta}}$, then*

$$\langle L_n(\tau_M), \sqrt{M} \rangle = \tau(M) \in \mathbb{Z}.$$

In particular, we have

$$L_1 = \frac{1}{3}p_i \qquad\qquad L_2 = \frac{1}{45}(7p_2 - p_1^2).$$

9 Natalie Stewart: Milnor, On manifolds homeomorphic to the 7-sphere

This talk was be delivered on October 6, 2021 by Natalie Stewart. These notes were prepared in advance.

9.1 Motivation: the generalized Poincaré conjecture

The following conjecture was proposed by Poincaré in 1904:

Conjecture 9.1 (Poincaré conjecture)**.** *Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.*

This problem alluded solution for nearly a century. Before then, analogous conjectures were proved; that closed 2-manifolds homotopy equivalent to S^2 (henceforth *topological homotopy 2-spheres*) must be homeomorphic to the 2-sphere followed from classification of surfaces. That homotopy n-spheres are homeomorphic to S^n was proved for $n \geq 5$ in 1966 by M. H. A. Newman using a technique pioneered by Stephen Smale called *PL engulfing* [6].

This motivates a class of conjectures;

Conjecture 9.2 (Generalized Poincaré conjecture)**.** *Given a category of manifolds $\mathcal{C}$, all homotopy n-spheres in $\mathcal{C}$ are isomorphic.*

In the case where $\mathcal{C} = \mathbf{Diff}$ is the category of smooth manifolds, the first known counterexample was due to Milor, who constructed a family of at least 7 pairwise nondiffeomorphic smooth structures on the topological 7-sphere [4]. Those which are not diffeomorphic to the usual smooth structure on S^7 are called *exotic 7-spheres*. We will focus on his original construction in this note and the corresponding lecture, culminating in a proof of the following theorem:

Theorem 9.3. *There are at least 3 diffeomorphism classes of exotic spheres.*

Later work of Milnor and Kervaire computed that there are exactly 27 diffeomorphism classes of exotic 7-spheres, and that exotic spheres exist in many dimensions ≥ 7, by using h-cobordism [3]. This breaks down in dimension ≤ 4, as the h-cobordism theorem fails, so this fails to classify smooth 4 or 3 spheres. The following chart summarizes the status of the generalized Poincaré conjecture:

category	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n \geq 7$
Top	True	True	True	True	True	True
PL	True	Open	True	True	True	True
Diff	True	Open ($\mathbf{PL}_{\leq 4} \simeq \mathbf{Diff}_{\leq 4}$)	True	True	False	Often false

With that context in mind, we now move on to define the invariants Milnor used in his construction.

9.2 Milnor's λ invariant

Throughout this note, set

$$H^*(X) := H^*(X; \mathbb{Z}).$$

For the duration of this section, fix a closed compact orientable 7-manifold M satisfying

$$H^3(M) = H^4(M) = 0. \tag{1}$$

By Thom [7], any closed compact oriented 7-manifold bounds a compact oriented 8-manifold (with boundary);[23] suppose that M is bounded by B, with orientation class $[B] \in H_8(B)$. Condition (1) implies that the inclusion

$$\iota : H^4(B, M) \to H^4(B)$$

[23] This comes down to the identification $\Omega^* = \pi_*(\mathrm{MSO})$ and explicit computations of the low-degree homotopy groups of this spectrum via the $2k$-equivalence of its k-th space with products of Eilenberg Mac Lane spaces.

is an isomorphism, so we may define the first Pontryagin number of B in this setting to be

$$q(B) := \langle [B], (\iota^{-1}p_1)^2 \rangle$$

Recall that the *index* $\tau(B)$ is defined to be the index of the quadratic form $\alpha \mapsto \langle [B], \alpha^2 \rangle$ on $H^4(B, M)/(\text{torsion})$, i.e. the number of positive terms minus the number of negative terms in a real diagonalization of the form.

We will build an invariant on M out of $q(B)$ and $\tau(B)$, then prove that it's independent of choice of B. First, we take a quick digression into properties of closed 8-manifolds, which may fail in the non-closed case.

Recall the following corollary of the Hirzrebruch signature theorem [2]:

Theorem 9.4 (Hirzrebruch signiature theorem, 8-dimensional case)**.** *For C a closed oriented 8-manifold and $[C] \in H_8(C)$ a fundamental class, we have*

$$\tau(C) = \langle [C], \frac{1}{45}\left(7p_2(C) - p_1^2(C)\right) \rangle.$$

In particular, the signature theorem implies that

$$2q(C) - \tau(C) = 2(q(C) + 45\tau(C)) = 2 \cdot 7p_2(C) \equiv 0 \pmod 7.$$

Since B bounds M, a nontrivial space, there is no guarantee that the analogue to this equality holds in our setting. In fact, failure of this equality is our main tool:

Definition 9.5. For M a closed 7-manifold bounded by an 8-manifold B, define the invariant

$$\lambda_B(M) := 2q(B) - \tau(B) \in \mathbb{F}_7.$$

This should worry the reader; it appears to depend on B. We assuage this fear via the following theorem:

Theorem 9.6. *The residue class $\lambda_B(M) \in \mathbb{F}_7$ doesn't depend on the choice of the manifold B.*

We henceforth simplify notation to

$$\lambda(M) := \lambda_B(M).$$

We will this to prove Theorem 9.3; for each $z \in \mathbb{F}_7$, we will construct a homeomorphism 7-sphere M with $\lambda(M) = z$.

Proof of Theorem 9.6. Suppose B, B' both bound M. Let B'' be the oriented manifold given by B' with opposite orientation. Defined the closed 3-manifold

$$C := B \cup_M B''$$

It is enough to prove the following equations:

$$\tau(C) = \tau(B) - \tau(B') \tag{2}$$
$$q(C) = q(B) - q(B'), \tag{3}$$

since then the different choices of $\lambda(M)$ differ by $q(C) - \tau(C) \equiv 0 \pmod 7$.

We first prove (2). The co-inclusion morphisms yield a commutative square

$$\begin{tikzcd}
H^n(C, M) \arrow[r, "h"] \arrow[d, "\iota_C"] & H^n(B, M) \oplus H^n(B'', M) \arrow[d, "\iota_B \oplus \iota_{B''}"] \\
H^n(C) \arrow[r, "k"] & H^n(B) \oplus H^n(B'')
\end{tikzcd}$$

compatible with the Kronecker pairing. The top horizontal arrow is always an isomorphism. The vertical arrows are clearly isos when $n = 4$ by (1), so k is an isomorphism in that case as well. Let $\alpha := \iota_C h^{-1}(\alpha, \alpha'') \in H^4(C)$ be an element. Then, we may diagram chase:

$$\begin{aligned}
\langle [C], \alpha^2 \rangle_{H^8(C)} &= \langle [C], \iota_C h^{-1}(\alpha^2, \alpha''^2) \rangle_{H^8(C)} \\
&= \langle h\iota_C^{-1}[C], (\alpha^2, \alpha''^2) \rangle_{H^8(B,M) \oplus H^8(B'',M)} \\
&= \langle ([B], [B'']), (\alpha^2, \alpha''^2) \rangle_{H^8(B,M) \oplus H^8(B'',M)} \\
&= \langle [B], \alpha^2 \rangle_{H^8(B,M)} + \langle [B''], \alpha''^2 \rangle_{H^8(B'',M)}.
\end{aligned}$$

Hence the quadratic form of C is the direct sum of that of B and B'', and hence it's the direct sum of that of B and the negative of that of B'. This yields equation (2).

For equation (3), note[24] that

$$kp_1(C) = p_1(B) \oplus p_1(B'')$$

Hence $q(C) = q(B) - q(B'')$ by an analogous argument to (2).

Now that we know that λ is well defined, note that Pontryagin classes and indices both switch sign when reversing orientation; this yields the following technical lemma:

Lemma 9.7. *Reversing the orientation of M multiplies $\lambda(M)$ by -1. Hence any M possessing $\lambda(M) \neq 0$ has no orientation reversing diffeomorphism onto itself.*

We want to use λ; our strategy will begin with the construction of a convenient family of spaces with easily computable λ invariants.

9.3 The construction of Milnor's exotic spheres

One candidate for exotic spheres is the restriction of 4-plane bundles over S^4 to their associated 3-sphere bundles; these are always 7-dimensional manifolds, and we can classify them explicitly via the *clutching construction*:

Construction 9.8 (the clutching construction)**.** Consider S^4 as the union of the upper and lower hemispheres D_+ and D_- along the equator $S^3 \subset S^4$. For a map $f : S^3 \to \mathrm{SO}(4)$, construct the 4-plane bundle B_f by gluing $D_+ \times \mathbb{R}^4$ to $D_- \times \mathbb{R}^4$ along $(x, v)_+ \sim (x, f(x)(v))_-$ for $x \in S^3$. The following theorem is well known [1]:

Theorem 9.9. *This identification descends to an isomorphism $\pi_3(\mathrm{SO}(4)) \xrightarrow{\sim} \mathrm{Vec}_{\mathbb{R},4}(S^3)$.*

The group $\mathrm{SO}(4)$ has universal cover $\pi : S^3 \to \mathrm{SO}(4)$ given by $\pi(u, w)(v) = uvw$, written using quaternionic multiplication.[25] Hence there is an isomorphism $\mathrm{Vec}_{\mathbb{R},4}(S^3) \simeq \pi_3(\mathrm{SO}(4)) \simeq \mathbb{Z}^2$.

Remark. This construction is easy to picture topologically, but not very good for determining the differentiable structure as written. Taking a hemmed gluing is more suitable; we can instead replace D_+, D_- with $\mathbb{R}^4$, glued along $\mathbb{R}^4 - \{0\}$ via a modified stereographic projection

$$(u, v) \mapsto (u', v') = \left(\frac{u}{|u|^2}, \frac{u^h v u^j}{|u|^{i+j}} \right)$$

The associated 4-plane bundle of this is isomorphic to the previously described bundle. Further, this describes the differentiable structure explicitly, and restricting to the associated sphere bundle, the transition function of the differentiable structure has the same formula.

Let $f_{h,j} : S^3 \to \mathrm{SO}(4)$ correspond with the pair $(h, -j)$. Let

$$\begin{tikzcd} S^3 \arrow[r, hook] & E_{h,j} \arrow[d, "\xi_{h,j}"] \\ & S^4 \end{tikzcd}$$

be the corresponding 3-sphere bundle. For each odd integer k, let M_k be the total space of $E_{h,j}$ where h and j are determined by the equations $h + j = 1$ and $h - j = k$. These will be our candidates; we will show that they are homeomorphic to S^7, usually with nontrivial λ invariant. First we tackle the homeomorphism, via techniques from Morse theory.

[24] One can see this via representability; pushing forward the iso

$$[C, \mathrm{BU}(n)] \simeq [B \cup_M B'', \mathrm{BU}(n)] \simeq [B, \mathrm{BU}(n)] \coprod_{[M, \mathrm{BU}(n)]} [B'', \mathrm{BU}(n)]$$

along the second Chern class morphism $\mathrm{BU}(n) \to K(\mathbb{Z}, n)$ and applying this to the complexification of tangent bundles yields the desired statement after noting that the second Chern class of a bundle on M is trivial by (1).

[25] This convention is nonstandard, but agrees with Milnor.

9.4 M_k is homeomorphic to S^7: a Morse theoretic sketch

Consider the following hypothesis on a closed manifold M:

- (H) *There exists a differentiable function $f : M \to \mathbb{R}$ having only two critical points, where each are nondegenerate.*

We prove the following:

Theorem 9.10. *A manifold M satisfying hypothesis (H) is homeomorphic to S^7.*

Proposition 9.11. *M_k satisfies hypothesis (H), and hence it is homeomorphic to S^7.*

The proofs of these are largely irrelevant to each other, so I'll sketch the concrete statement first.

Proof sketch for Proposition 9.11. We can define the function f in local coordinates, compatibly with transition functions:

$$f(u,v) = \frac{\Re(v)}{\sqrt{1+|u|^2}}$$
$$f(u',v') = \frac{\Re(u'/v')}{\sqrt{1+|u'/v'|^2}}$$

where $\Re(\cdot)$ is the real part of a quaternion. The reader can verify that this has exactly two critical points, each nondegenerate, at $(u,v) = (0, \pm 1)$.

Proof sketch for Theorem 9.10. Suppose $f : M \to \mathbb{R}$ is a function witnessing hypothesis (H). Normalize f so that $f(x_0) = 0$ and $f(x_1) = 1$. According to Morse [5], one can take local coordinates $v_1, \ldots, v_n$ in a neighborhood V of x_0 so that

$$f(x) = v_1^2 + \cdots + v_n^2 \qquad \text{on } V.$$

One may define a Riemannian metric on V via $ds^2 = dv_1^2 + \cdots + dv_n^2$, and extend this to one on all of M.

The gradient of f defines a vector field on M; consider the differential equation

$$\frac{dx}{dt} = \frac{\nabla f}{|\nabla f|^2}.$$

This equation has solutions. Suppose $x_a(t)$ is a solution equation; note that

$$f(x_a(t)).$$

so that $x_a(t) = (a_1^{1/2}(t), \ldots, a_n^{1/2}(t))$ on V. Map the unit sphere of $\mathbb{R}^n$ into M via the map

$$a \mapsto x_a(t).$$

This defines a diffeomorphism of the open n-cell onto $M - \{x_1\}$. Adding a single point yields the theorem.

9.5 Completing the proof: calculating λ invariants

We refer to elements of $H^4(S^4)$ as elements of $\mathbb{Z}$, but we will find that signs don't matter, as we always square these elements when using them.. We first show that these generate all of the possible λ invariants:

Lemma 9.12. *The λ invariant of M_k is as follows:*

(i) $p_1(\xi_{h,j}) = \pm 2(h-j)\iota$.

(ii) $\lambda(M_k) = k^2 - 1$.

Proof. Part (i). First, note that $\xi_{h,j} \oplus \xi_{h',j'}$ is stably isomorphic to $\xi_{h+h',j+j'}$: writing local coordinates[26] of $\xi_{h+h',j+j'} \oplus e$ as (u,v,w) on the preimage of D^+ with transition function

$$(u,v,w) \longmapsto (u',v',w') = \left(u, u^{h'}u^h v u^h u^{j'}, w\right)$$

and the coordinates on $\xi_{h,j} \oplus \xi_{h',j'}$ similarly as (x,y,z) with transition function

$$(x,y,z) \longmapsto (x',y',z') = \left(x, x^h y x^j, x^{h'} y x^{j'}\right),$$

there is an iso $\xi_{h+h',j+j'} \oplus e \to \xi_{h,j} \oplus \xi_{h',j'}$ given by the identity above D^+ and the map

$$(u',v',w') \mapsto \left(x', x'^{-h'} v' x'^{-j'}, x'^{h'} w' x'^{j'}\right)$$

over D_-. We can check that this is compatible with transition functions:

$$\begin{tikzcd}
(\xi_{h+j} \oplus e)|_{D_+ \cap D_-} \arrow[rrr] \arrow[ddd] & & & (\xi_{h+j} \oplus e)|_{D_+ \cap D_-} \arrow[ddd] \\
& (u,v,w) \arrow[r, mapsto] \arrow[d, mapsto] & (u, u^{h+h'} v u^{j+j'}, w) \arrow[d, mapsto] & \\
& (x,y,z) \arrow[r, mapsto] & (x, x^h y x^j, x^{h'} y x^{j'}) & \\
\xi_{h+h',j+j'}|_{D_+ \cap D_-} \arrow[rrr] & & & \xi_{h+h',j+j'}|_{D_+ \cap D_-}
\end{tikzcd}$$

Hence the Whitney sum formula yields $p_1(\xi_{h+h',j+j'}) = p_1(\xi_{h,j} + \xi_{h',j'})$, i.e. $p_1(\xi_{i,j})$ is linear in h and j. Further, there is an isomorphism $\xi_{h,j} \to \xi_{-j,-h}$; hence $p_1(\xi_{h,j}) = ah + bj = -bh - aj$ for some constants a, b, i.e. $p_1(\xi_{h,j}) = c(h-j)$ for some constant c.

To compute the constant c, it suffices to compute an example. The case $(h,j) = (1,0)$ corresponds with the ordinary quaternionic Hopf fibration, i.e. the sphere bundle associated with the tautological quaternionic line bundle

$$\begin{tikzcd}
S^3 \arrow[r, hook] \arrow[d, hook] & S^7 \arrow[d, hook] \\
\mathbb{H} \arrow[r, hook] & \mathbb{H}^2 \arrow[d, "T_{\mathbb{H}}"] \\
& \mathbb{HP} \arrow[d, equal] \\
& S^4
\end{tikzcd}$$

For any complex vector bundle ξ, one has $\xi \otimes \mathbb{C} = \xi \oplus \overline{\xi}$, and $c_i(\overline{\xi}) = (-1)^i c_i(\xi)$. In the case that ξ is 2-dimensional with vanishing first Chern class (e.g. if $H^2(B(\xi)) = 0$), the Whitney sum formula then yields

$$1 + p_1(\xi) = (1 + c_2(\xi))^2 = 1 + 2c_2(\xi)$$

and hence $p_1(\xi) = 2c_2(\xi)^2$. Hence it is enough to compute $c_2(T_{\mathbb{H}})$. We compute this via the comparison map

$$g : \mathbb{CP}^2 \to \mathbb{HP}^1$$

sending $[a+bi : c+di] \mapsto [a+bi+cj+dk]$. This yields a pullback square

$$\begin{tikzcd}
\mathbb{H}^2 \times_{\mathbb{HP}^1} \mathbb{CP}^2 \arrow[r] \arrow[d, "g^* T_{\mathbb{H}}"] & \mathbb{H}^2 \arrow[d, "T_{\mathbb{H}}"] \\
\mathbb{CP}^2 \arrow[r, "g"] & \mathbb{HP}^1
\end{tikzcd}$$

[26]Here and elsewhere, e refers to a trivial 1-dimensional vector bundle.

In fact, there is a diagram of real vector bundles

$$\begin{tikzcd}
\mathbb{C}^4 \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "T_{\mathbb{C}}^2"'] & & \\
& g^*(\mathbb{H}^2) \arrow[r] \arrow[d, "g^* T_{\mathbb{H}}"] & \mathbb{H}^2 \arrow[d, "T_{\mathbb{H}}"] \\
& \mathbb{CP}^2 \arrow[r, "g"] & \mathbb{HP}^1
\end{tikzcd}$$

where

$$\phi([a+bi : c+di], w+xi, y+zi) = ([a+bi+cj+dk], w+xi+yj+zk)$$

Note that ϕ is fiberwise-injective, and hence the canonical map $\mathbb{C}^4 \to g^*(\mathbb{H}^2)$ is fiberwise-injective. Both have fiber of real dimension 4, so this must be an isomorphism of real vector bundles, i.e. $g^* T_{\mathbb{H}} = T_{\mathbb{C}}^2$. Then, naturality of Chern classes implies that

$$g^* c_2(T_H) = c_2(T_{\mathbb{C}}^2) = c_1(T_{\mathbb{C}})^2$$

is a generator of $H^4(\mathbb{CP}^2)$. In turn, since $g : \mathbb{CP}^2 \to \mathbb{HP}^1$ is a $\mathbb{CP}^1 = S^2$ bundle, the Gysin sequence yields

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(\mathbb{HP}^1) \arrow[r] \arrow[d, equal] & H^4(\mathbb{HP}^1) \arrow[r, "g^*"] & H^4(\mathbb{CP}^2) \arrow[r] & H^2(\mathbb{HP}^1) \arrow[r] \arrow[d, equal] & \cdots \\
& 0 & & & 0 &
\end{tikzcd}$$

and g^* is an isomorphism. Hence $c_2(T_{\mathbb{H}})$ is a generator of $H^4(S^4)$, i.e. $p_1(\xi_{1,0}) = p_1(T_{\mathbb{H}}) = \pm 2$, determining the constant c.

Part (ii). Note that, for any smooth fiber bundle $F \hookrightarrow E \to B$, one has $TE = (TB)^* \oplus (TF)_*$, by decomposition of the tangent space at a point into vectors tangent and normal to the copy of F containing it.

In particular, applying this to $TE_{h,j}$ and along with the Whitney sum formula yields

$$p_1(E_{h,j}) = \xi_{h,j}^* p_1(\xi_{h,j}) + \iota_* p_1(S^3) = \xi_{h,j}^* p_1(\xi_{h,j}).$$

Again applying the Gysin sequence yields an isomorphism

$$\begin{tikzcd}
\cdots \arrow[r] & H^7(S^4) \arrow[r] \arrow[d, equal] & H^7(E_{h,j}) \arrow[r, "\xi_{h,j}^*"] & H^4(S^4) \arrow[r] & H^8(E_{h,j}) \arrow[r] \arrow[d, equal] & \cdots \\
& 0 & & & 0 &
\end{tikzcd}$$

i.e. $p_1(E_{h,j}) = \pm 2(h-j)$. Pick an orientation of $E_{h,j}$ (and hence B_k) so that this is positive. Then, $\tau(B_k) = 1$, and

$$q(B_k) = \langle [B_k], 4k^2 \rangle = 4k^2.$$

Hence we have

$$\lambda(M_k) = 8k^2 - 1 = k^2 - 1 \pmod 7,$$

as desired.

This allows us to finally conclude Theorem 9.3:

Proof of Theorem 9.3. Whenever $k \neq k' \pmod 7$, we have $\lambda(M_k) \neq \lambda(M_{k'})$, and hence M_k is not diffeomorphic to $M_{k'}$. Hence $\{M_k\}$ yields at least 7 diffeomorphism classes of spaces; by Proposition 9.11, these are all homeomorphism 7-spheres, so there are at least 7 diffeomorphism classes of homeomorphism 7-spheres.

10 Serina Hu: Brown, Cohomology theories

This talk was delivered on October 9, 2021 by Serina Hu.

10.1 Statement of Brown representability, and easy corollaries

Let $\mathcal{C}$ be a homotopy category of pointed connected CW complexes.[27] A contravariant functor $H : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}_*$ is *representable* if there is some $Y \in \mathcal{C}$ and a natural isomorpism

$$[-, Y] \simeq H(-).$$

Representing objects are unique up to iso, in this case, pointed homotopy.

Theorem 10.1. *A functor $H : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}_*$ is representable if and only if the following holds:*

- *(e) (Mayer-Vietoris axiom). The following hold:*[28]

– $H(*) = \{*\}$.

– *If $X = X_1 \cup X_2, X_1, X_2, X_1 \cap X_2 \in \mathcal{C}$ and $(X_1, X_1 \cap X_2)$ and $(X_2, X_1 \cap X_2)$ have the HEP, and there exists $u_i \in H(X_i)$ s.t. $H(j_1)u_1 = H(j_2)u_n$, then there exists some $v \in H(X)$ s.t. $H(i_1)v = u_1$ and $H(i_2)v = u_2$, in the following diagram:*

$$\begin{tikzcd} & H(X) \arrow[dl, "H(i_1)"] \arrow[dr, "H(i_2)"] & \\ H(X_1) \arrow[dr, "H(j_1)"] & & H(X_2) \arrow[dl, "H(j_2)"] \\ & H(X_1 \cap X_2) & \end{tikzcd}$$

- *(w) (wedge) For spheres S^n_α with $\iota_\beta : S^n_\beta \hookrightarrow \bigvee S^n_\alpha$, the induced map*

$$H(\bigvee S^n_\alpha) \to \prod H(S^n_\alpha)$$

is an iso.

- *(l) (limit) For $X = \bigcup X_n$ a filtration with $X_j \in \mathcal{C}$ and $X^n_n = X^n$, we have*

$$\operatorname{colim} H(\iota_n) : H(X) \to \operatorname{colim} H(X_n)$$

is a surjection.

We get some free results:

Corollary 10.2. *Singular cohomology in degree n is representable, i.e. there eixsts some $K(A, n)$ s.t.*

$$[-, K(A, n)] \simeq H^n(-, A).$$

We can chain these together into a spectral statement:

Corollary 10.3. *Let $\{H^q, \omega^q\}$ be a cohomology theory with coefficients in A.*[29] *Then, there exists an Ω-spectrum $\{Y_q, h_q\}$ representing H^q.*[30]

Let's move on to actually talking about and motivating the axioms.

[27] I will add an asterisk to $\mathbf{Set}_*$, against Serina's notation.

[28] I think this can be summarized by saying H takes weak pushouts to weak pullbacks (which is often the axiom used), but I'm not 100% sure of that.

[29] There was confusion here, but I think this is ok notation for a cohomology theory whose codomain category is $(A - \mod)^{\mathbb{Z}}$.

[30] This means that the structure maps have adjoint maps given by equivalences $Y_q \simeq \omega Y_{q+1}$ expressing this as an infinite loop space, and these structure maps intertwine the cohomology theory:

$$\omega^q T_q = T_q \omega^{q+1}.$$

10.2 Motivation for the axioms of a Brown functor: homotopy colimits

Recall the (e) axiom; this is satisfied by representable functors, since given a diagram

$$\begin{tikzcd} & {[X,Y]} \arrow[dl, "H(i_1)"'] \arrow[dr, "H(i_2)"] & \\ {[X_1,Y]} \arrow[dr, "H(j_1)"'] & & {[X_2,Y]} \arrow[dl, "H(j_2)"] \\ & {[X_1 \cap X_2, Y]} & \end{tikzcd}$$

we can form the homotopy pushout $X = X_1 \sqcup_{X_1 \cap X_2} X_2$[31] The point of forcing the HEP is that, in this case, homotopy pushouts are simply traditional pushouts.[32], and use this universal property to construct the desired element.

Similarly, the (l) axiom can be phrased via "mapping telescopes," i.e. sequential homotopy colimits; one can verify this in the representable case by simply noting that the mapping telescope of such an inclusion is X, and hence the universal property of homotopy colimits allows one to construct the desired element.[33]

Let's move on to a sketch of the proof.

10.3 Sketch of the proof of Brown representability

We'll develop this in steps:

Step (1). Fix $Y \in \mathcal{C}$, $u \in H(Y)$, and define $T(u) : [-, Y] \to H(-)$ mapping f to the pushforward $H(f)u$. Our goal will be to pick appropriate Y, u making this an iso. We start with a lemma, reducing the work drastically:

Lemma 10.4. *If $T(u) : [S^n, Y] \simeq H(S^n)$, then this is actually a natural equivalence on all of $X \in \mathcal{C}$.*

Step (2). We will inductively construct a sequence $Y_i, u_i \in H(Y_i)$ such that:

(i) $Y_{n-1} \hookrightarrow Y_n$.

(ii) $H(i_n)u_n = u_{n-1}$.

(iii) $T(u_n) : [S^m, Y_n] \to H(S^m)$ is surjective for all m, and bijective for $m \leq n$.

For the base case $n = 0$, we just need surjectivity. Let $g^m_\alpha \in H(S^m)$ be elements, and let $Y = \bigvee S^m_\alpha$. Then, fix $H(Y_0) = \prod H(S^m_\alpha)$. If $h^m_\alpha : S^m \xrightarrow{\sim} S^m_\alpha \hookrightarrow Y_0$. Pick u_0 s.t. $H(h^m_\alpha)(u_0) = g^m_\alpha = T(u_0)(h^m_\alpha)$.[34]

For the inductive step, suppose we've gotten such a pair $u_{n-1} \in H(Y_{n-1})$. We know that $T(u_{n-1}) : [S^m, Y_{n-1}] \twoheadrightarrow H(S^m)$ is bijective for $m \leq n-1$. We want to modify this only in degrees $\geq n$ s.t. the map at degree n is also a bijection.

Consider the kernel of $[S^n, Y_{n-1}] \twoheadrightarrow H(S^n)$.[35] Let $A = \bigvee S^n_\beta$, $f = \bigvee f_\beta : A \to Y_{n-1}$, and let Y_n be the mapping cone of this.[36] We have the following:

Claim. *The sequence of pointed sets*

$$H(Y_n) \xrightarrow{H(i)} H(Y_{n-1}) \xrightarrow{H(f)} H(A)$$

is exact.

---

[31] I don't know notation to use the homotopy pushout.

[32] This is true based on model category nonsense; homotopy colimits along cofibrations are just traditional colimits, assuming the index category is simple and/or reedy (really, that it admits a model structure whose homotopy equivalences are simply pointwise homotopy equivalences).

[33] I think one can spend months on homotopy colimits and not fully understand them. Lord knows I have. Sorry if this section is confusing.

[34] Haynes commented that this is a huge use of the axiom of choice. Just another reason to be pro-choice I suppose...

[35] By "kernel," we mean preimage of the basepoint.

[36] Serina drew a picture of a wizard hat, aka a mapping cone, here.

Proof. Mapping cones are homotpopy pushouts:

$$\begin{tikzcd}
& H(Y_n) \arrow[dl, "H(i_1)"'] \arrow[dr, "H(i_2)"] & \\
H(\overline{X}) \arrow[dr, "H(j_1)"] & & H(\overline{Y}) = H(Y_{n-1}) \arrow[dl, "H(j_2)"] \\
& H(\overline{A}) = H(A) &
\end{tikzcd}$$

Use axiom (e).[37]

If $u_{n-1} \in \ker(H(f))$, let $\iota_B : S^n_\beta \hookrightarrow A$ represent it. There must be some $u_n \in H(Y_n)$ s.t. $H(\iota_n)u_n = u_{n-1}$, by exactness.

For surjectivity of $T(u_n)$, just push along ι_n:

$$\begin{tikzcd}
{[S^m, Y_{n-1}]} \arrow[dr, two heads, "T(u_{n-1})"] \arrow[dd, "\iota_n"] & \\
& H(S^m) \\
{[S^m, Y_n]} \arrow[ur, "T(u_n)"] &
\end{tikzcd}$$

since the upper right arrow is surjective, the lower right arrow is as well.

For injectivity of $m < n$, a similar argument applies; for $m = n$, an element in the kernel yields a nullhomotopic map

$$S^n \xrightarrow{f_\beta} Y_{n-1} \hookrightarrow Y_n$$

for each β; hence $\ker(T_{u_{p-1}}) \subset \ker \iota_{p*}$, implying injectivity.

Step (3). Take $Y = \operatorname{colim} Y_n$, and use axiom (l) to finish.

11 Tristan Yang: Milnor, On the cobordism ring and a complex analogue

This talk was delivered on October 13, 2021 by Tristan Yang.

The talk is a trojan horse. We're talking about the stable homotopy category and the Adams spectral sequence, and we may get to the complex cobordism stuff if we have time.

11.1 Stable homotopy theory

To pass to the stable context, we will define spectra. The goal will be "inverting" the suspension functor. We'll construct a functor

$$\Sigma^\infty : \operatorname{ho} \mathbf{Top}_* \to \operatorname{ho} \mathrm{Sp}$$

into the stable homotopy category, along with a suspension functor

$$\Sigma : \operatorname{ho} \mathrm{Sp} \to \operatorname{ho} \mathrm{Sp}$$

extending the suspension on $\operatorname{ho} \mathbf{Top}_*$, in the sense that $\Sigma\Sigma^\infty = \Sigma^\infty\Sigma$.

Let's actually define our category.

[37] I missed this argument

Definition 11.1. The category ho Sp has as objects the sequences of homotopy types $\{Y_k \in \operatorname{ho}\mathbf{Top}_*\}$ together with homotopy classes of maps $\Sigma Y_k \to Y_{k+1}$. Define the suspension spectrum $\Sigma^\infty Y = \{\Sigma^k Y\}$ with the obvious identities. For Y a finite CW complex, define the class of maps

$$[\Sigma^\infty Y, X]_n := \operatorname{colim}_k \left[\Sigma^k Y, X_k\right].$$

Example 11.2:
An example includes the sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$, and note that

$$\pi_n(X) := \operatorname{colim}_k \pi_{n+k}(X_k)$$

is naturally identified with the maps $[\mathbb{S}, X]$.

Example 11.3:
Brown representability yields one direction of a bijection between Ω-spectra (spectra with adjoint structure maps that are equivalences).[a] This for instance yields the *Eilenberg-Mac Lanee* Ω-spectra, which we denote $H\mathbb{F}_p$, and represents cohomology with $\mathbb{F}_p$ coefficients.

[a] Tristan mentioned that every spectra is equivalent to an Ω-spectra. I assume that this is the normal fibrant replacement in the stable model structure. I'm a bit confused as we've only defined homotopy types of spectra, so this would mean every such homotopy type *is* an Ω-spectra (the distinction is point-set, and we haven't defined spectra in a point-set way).

The stable homotopy category satisfies the following axioms:

(i) there is a functor $\Sigma^\infty : \operatorname{ho}\mathbf{Top}_* \to \operatorname{ho}\mathrm{Sp}$.

(ii) there is a suspension auto-equivalence $\Sigma : \operatorname{ho}\mathrm{Sp} \to \operatorname{ho}\mathrm{Sp}$ s.t. $\Sigma\Sigma^\infty = \Sigma^\infty\Sigma$.

(iii) The category ho Sp is additive (it is enriched over **Ab** and has all finite products and coproducts, which agree). To give the enrichment, apply suspensions twice and use the normal (homotopy) Abelian group structure.

(iv) The category ho Sp is triangulated.[38]

(v) There is a symmetric monoidal structure $\otimes$ on ho Sp extending $\wedge$, having.[39]

With the last axiom in mind, we can define generalized homology corresponding with a spectrum to be

$$\widetilde{E}_n(X) := \pi_n(E \otimes X).$$

With this in mind, we can define the Adams spectral sequence:

11.2 The Adams spectral sequence

Our goal is to compute $[Y, X]_*$. The idea is that we might be able to compute this via the homology. We can't do this super naively, but maybe we can get a bit of a foothold this way.

Let's take a special case. Suppose $X = H\mathbb{F}_p$. Then,

$$\begin{aligned} [Y, H\mathbb{F}_p] &\simeq H^*Y \\ &= \operatorname{Hom}_{\mathbb{F}_p}(H_*Y, \mathbb{F}_p) \\ &= \operatorname{Hom}_{A_*}(H_*Y, \mathbb{F}_p \otimes A_*) \\ &= \operatorname{Hom}_*(H_*Y, H_*H\mathbb{F}_p). \end{aligned}$$

[38] There are distinguished triangles, generalizing LESs, which are nice. Details not given, and I don't have time to write out the axioms, which are not easy to get meaning out of at first anyways.

[39] This is actually monoidal closed.

The general case is harder. We can get a bit more foothold by just noting that

$$\left[Y, \bigoplus \Sigma^{|b_i|H\mathbb{F}_p}\right] \simeq \operatorname{Hom}(H_*Y, A_* \otimes H_*X) \qquad \text{whenever } H_*X \text{ has basis } \{b_i\}$$

noting that we can equivalently write

$$\bigoplus \Sigma^{|b_i|} H\mathbb{F}_p \simeq H\mathbb{F}_p \otimes X.$$

Warning. This doesn't generalize to arbitrary homology theories: the iso $E_*(E \otimes X) \simeq E_*E \otimes_{E_*} E_*X$, which requires a flatness condition. We only really care about $\mathbb{F}_p$-(co)homology in this talk, so we won't worry about this.

We can form a tower

$$\begin{tikzcd}
X \arrow[d] & \overline{H\mathbb{F}_p} \otimes X \arrow[l] \arrow[d] & \cdots \arrow[l] \\
H\mathbb{F}_p \otimes X \arrow[ur, dashed] & H\mathbb{F}_p \otimes \overline{H\mathbb{F}_p} \otimes X \arrow[ur, dashed] &
\end{tikzcd}$$

where the dashed lines have degree +1.Taking $[Y, -]$ of this tower yields a diagram

$$\begin{tikzcd}
{[Y, X]} \arrow[d] & {\left[Y, \overline{H\mathbb{F}_p} \otimes X\right]} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
{[Y, H\mathbb{F}_p \otimes X]} \arrow[ur, dashed] & {\left[Y, H\mathbb{F}_p \otimes \overline{H\mathbb{F}_p} \otimes X\right]} \arrow[ur, dashed] &
\end{tikzcd}$$

which rolls (via direct sums) into an exact couple. The associated spectral sequence is called the *Adams spectral sequence*, or the *ASS*.[40]

Proposition 11.4. *The ASS has E_2 page given by*[41]

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{A_*}(H_*Y, H_Y X).$$

We have a convergence theorem:

Theorem 11.5. *The Adams Spectral sequence converges to the p-completion of homotopy:*

$$E_r^{s,t} \implies \left([Y, X]_{t-s}\right)^\wedge_p$$

Intuition about the completion. The given filtration $X \leftarrow X_1 \leftarrow \cdots$ is not actually exhaustive: $X_\infty := \lim X_n$ does not in general vanish.

We can take cofibers to get a new filtration:

$$\begin{tikzcd}
 & & & & X \arrow[d] \arrow[llll, bend right] \arrow[lll, bend right] \arrow[ll, bend right] \\
X \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_\infty \arrow[d] \\
Z \arrow[r] & Z_1 \arrow[r] & Z_2 \arrow[r] & \cdots \arrow[r] & Z_\infty = 0
\end{tikzcd}$$

and note that the two yield the same SS.

[40] This is not my choice of abbreviation.
[41] Tristan used the notation $\operatorname{Ext}^s(-)_t$ here.

11.3 Applications to complex cobordism

Define the spectrum MU to be $(*, \mathrm{MU}(1), \Sigma\,\mathrm{MU}(1), \mathrm{MU}(2), \dots)$. We have the following, which comes from the Thom iso:

Proposition 11.6. *$H_*\,\mathrm{MU} = \mathbb{F}_p[b_1, \dots]$ as an $\mathbb{F}_2$-module, where $|b_i| = 2i$.*

Recall that $A_* \simeq \mathbb{F}_p[\xi_i] \otimes \bigwedge(\tau_i)$, where $|\xi_i| = 2p^i - 2$ and $|\tau_i| = 2p^i - 1$. Let the first summand be P.

Proposition 11.7. *$H_*\,\mathrm{MU}$ is a P-comodule.*

Proof. This follows by noting that all elements of $H_*\,\mathrm{MU}$ have even degree, so odd-degree elements of A_* coact trivially. ☺

We can do better:

Proposition 11.8. *As an A_*-comodule,*

$$H_*\,\mathrm{MU} \simeq P \otimes \mathbb{F}_p\left[b_i \mid i \neq p^h - 1\right].$$

Proof. $H_*\,\mathrm{MU} \to H_*\,\mathrm{MU} \otimes P \to P$ surjects onto P (check this). The Thom iso map $\widetilde{H}_{*+2}\mathbb{CP}^\infty \to H_*\,\mathrm{MU}$ has codomain $\mathbb{F}_p[\beta_0, \beta_1, \dots]$. ☺

We ran out of time here. The upshot is that we can compute that the E_2 page has only even degrees, which is enough to show that there's no odd torsion.

12 Carina Hong: Dyer and Lashof, A topological proof of the Bott periodicity theorem

This talk was delivered on October 15, 2021 by Carina Hong.

We will prove Bott periodicity, which we now state. Define $\mathrm{O} := \operatorname{colim} \mathrm{O}(n)$ and similar for the other classic groups.

Theorem 12.1 (Bott periodicity). *$\pi_n(O) \simeq \pi_{n+4}(\mathrm{Sp}) \simeq \pi_{n+8}(O)$; further, the groups on low degree are given by*

	$n=0$	1	2	3	4	5	6	7
π_n	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Similarly, $\pi_n(U) = \pi_{n+2}(U)$.

Our strategy will be a collection of homotopy equivalences exhibiting various classical groups as loop spaces of each other.[42]

Our goal will be proving the following theorem, which is the hard step in complex Bott periodicity:

Theorem 12.2. *There exists a homology-isomorphism $\mathrm{BU} \to \Omega\,\mathrm{SU}$ of H-spaces.*

We need to give some definitions to work this out fully.

12.1 Background on our classical groups and on H-spaces

Definition 12.3. An H-space is a space X together with a multiplication map $\mu : X \times X \to X$ and a unit $e : * \to X$ exhibiting the usual unit laws.[43]

Now that we have that out of the way, let's define some groups. We define the space[44]

$$\mathrm{BU} := \mathrm{U}\,/\,\mathrm{U} \times \mathrm{U} = \operatorname{colim} \mathrm{U}(2n)/(\mathrm{U}(n) \times \mathrm{U}(n)).$$

We want to give an H-space structure on these. Define the H-space structure on $\Omega\,\mathrm{SU}$ via pointwise multiplication.[45] Define the H-space structure on BU by an action of the *linear isometries operad*:

$$\mu(T_1, T_2) : \mathbb{C}^{\infty \oplus 2} \xrightarrow{\xi^{-1} \oplus \xi^{-1}} \mathbb{C}^{\infty \oplus 4} \xrightarrow{T_1 \oplus T_2} \mathbb{C}^{\infty \oplus 4} \xrightarrow{\mathrm{id} \oplus \gamma \oplus \mathrm{id}} \mathbb{C}^{\infty \oplus 4} \xrightarrow{\xi \oplus \xi} \mathbb{C}^{\infty \oplus 2}$$

The board work of this talk is too fast to be reasonably live-texed. Proceed with caution, as there will be many holes.

12.2 Characterizing the map $\mathrm{BU} \to \Omega\,\mathrm{SU}$

We're going to prove that our map is a homology iso. We need the following to do:

Claim. *$H_*(\mathrm{BU}) = \mathbb{Z}[z_2, z_4, \dots]$, where z_{2i} are the images of an additive basis of $H_*(\mathbb{CP}^\infty)$.*

In addition to this, we need to compute $H_*(\Omega\,\mathrm{SU})$, which reduces to computing $H_*(\mathrm{SU})$ via the following dual to a special case of Borel's theorem.

Theorem 12.4 (Dual to Borel's theorem). *If X is an H-space s.t. $H_*(X)$ is a transgressively generated exterior algebra generated in odd degrees by (γ_{2n+i}), then $H_*(\Omega X)$ is a polynomial algebra generated by transgressions of γ_{2n+1}.*

[42] I am behind in copying this, so I missed the constructions of the maps here.

[43] I think this only needs to hold up to homotopy sometimes, and for the way we'll use these.

[44] There was contention on how to define the maps of this colimit.

[45] According to Haynes, this is equivalent to doing loop composition.

We can compute this via the following diagram:

$$\begin{tikzcd}
\mathbb{CP}^{n-1} \arrow[r] \arrow[d, "i"] & \Omega\,\mathrm{SU}(n) \arrow[d, "\Omega i"] \\
\mathbb{CP}^{n} \arrow[r] \arrow[d, "\rho"] & \Omega\,\mathrm{SU}(n+1) \arrow[d, "\Omega\pi"] \\
S^{2n} \arrow[r, "h"] & \Omega S^{2n+1}
\end{tikzcd}$$

You can show that $\Omega\pi\circ\alpha = i$ is trivial, so that $\Omega\pi\circ\alpha$ factors as $h\circ\rho$ The colimit $\hat{h}$ of h is a homeomorphism.[46][47]

The upshot of this is the following computation:

Claim. $H_*(\mathrm{SU}(n+1)) \simeq \bigwedge[\gamma_3, \gamma_5, \dots, \gamma_{2n+1}]$.

This is true for $\mathrm{SU}(2) = S^3$, so you can prove it inductively from there. The key is to use the cohomological SSS, which collapses at the second page, for degree reasons, and use some Hopf algebra technology to turn this into a homology statement.

Taking a colimit yields that $H_*(\mathrm{SU})$ is an exterior algebra which, via Theorem 12.4, yields $H_*(\Omega\,\mathrm{SU})$ is abstractly isomorphic to $H_*(\mathrm{BU})$.[48]

[46] I think this is what $\hat{h}$ means, but not sure.

[47] Haynes suggests that you can use this to inductively compute $H_*(\omega\,\mathrm{SU}(n))$ and take a colimit, noting that the cofiber sequence on the left is meaningful for the interpretation of the transgressions that you use in such a computation.

[48] I'm confused about how to pull that our map is a H_*-isomorphism; once you do this, use simplicity coming from the H-space structure to conclude that it's a weak equivalence.

13 Eunice Sukarto: Atiyah, K-theory (no notes today)

This talk was delivered on October 20, 2021 by Eunice Sukarto.

I woke up far too tired to take good notes on this today. See Atiyah's book, or any standard notes on K theory.

14 Gabrielle Li: Adams and Atiyah, K-theory and the Hopf invariant

This talk was delivered on October 22, 2021 by Gabrielle Li.

14.1 Motivation and review from K-theory

Our motivation is the usual one: a structure of a real division algebra on $\mathbb{R}^n$ yields a structure of S^{n-1} as an H-space, which yields a Hopf invariant 1 element in $\pi_{2n-1}(S^n)$. It looked approximately like the nlab reference.

We review some facts from Eunice's talk:

Proposition 14.1. *$\widetilde{K}^0(S^n)$ is $\mathbb{Z}$ for even n and 0 for odd n, and it's generated by the* bott element $\wedge^{n/2}b$ *where $b = H^* - 1$, with $b^2 = 0$. In either case, $\widetilde{K}^0(S^n)$ has trivial cup product.*

We also have the splitting principle.

14.2 The statement of the Hopf invariant 1 problem

Let $f : S^{2n-1} \to S^n$ be a map (really, an element of $\pi_{2n-1}(S^n)$). Form the homotopy cofiber

$$S^{2n-1} \to S^n \to Cf.$$

The LES in reduced K theory yields

$$\widetilde{K}^0(S^{n+1}) \to \widetilde{K}^0(S^{2n}) \to \widetilde{K}^0(Cf) \to \widetilde{K}^0(S^n) \to 0 = K^0(s^{2n-1})$$

Let v be a generator of $\widetilde{K}^0(S^n)$ and β a preimage of this in the cofiber. Let α be the image of a generator of $\widetilde{K}^0(S^n)$ in the cofiber.

Note that $i^*(\beta^2) = i^*(\beta)^2 = 0$. Hence β^2 is in the image of the boundary map, i.e. there is some h s.t. $\beta^2 = h\alpha$.

Definition 14.2. The *Hopf invariant* of an element $f \in \pi_{2n-1}(S^n)$ is $h \in \widetilde{K}^0(Cf)$, as defined above.

Note that whenever n is odd, $\beta^2 = 0$ by graded commutativity.[49]

If n is even and $\beta' = \beta + m\alpha$, then ${\beta'}^2 = \beta^2 + 2m\alpha\beta$. Note that

$$i^*(\alpha\beta) = i^*(\alpha)i^*(\beta) = 0$$

and hence $\alpha\beta = i\alpha$ for some i, so that $i\alpha\beta = i^2\alpha = \alpha\beta^2 = h\alpha^2$, which is then 0, by graded commutativity and torsion-freeness. Hence this invariant is well-defined.

We can state the theorem:

Theorem 14.3 (Hopf invariant one problem)**.** *If $n \neq 1, 2, 4$, then there is no Hopf invariant 1 element in $\pi_{2n-1}(S^n)$.*

We'll now develop some techniques to prove this.

14.3 Adams operations

We'll define some cohomology operations, which extend exterior powers: recall that there is an exterior power of a vector bundle $\wedge^k V$ which is fiberwise given by the exterior power of vector spaces. This descends to a natural endomorphism of K-theory λ^k.[50]

Recall from the theory of symmetric polynomials that there is an expression of $x_1^k + \cdots + x_n^k = s_k(\sigma_1, \ldots, \sigma_k)$ where σ_i are elementary symmetric polynomials. We can define the Adams operations:

[49] This is true mod 2 by graded-commutativity. To get it to a total statement needs some torsion free statement, which is not immediately clear to me.

[50] This is not multiplicative, right?

Definition 14.4. The ring endomorphism $\psi^k : \widetilde{K}^0(K) \to \widetilde{K}^0(X)$ by $E \mapsto s_k(\lambda^1(E), \ldots, \lambda^k(E))$.

These have nice properties:

Proposition 14.5 (Properties of the Adams operations)**.** *The following hold for ψ^k the kth Adams operations:*

(1) ψ^k is natural.

(2) For a line bundle L, $\psi^k(L) = L^{\otimes k}$.

(3) Each ψ^k is a ring homomorphism.

(4) $\psi^k\psi^l = \psi^{kl}$, so in particular, Adams operations commute.

(5) For prime p, $\psi^p(E) = E^p$ mod p.

Proof. (1) is clear.

For (2), we prove the slightly stronger statement that $\psi^k(L_1 \oplus L_2) = L_1^{\otimes k} \oplus L_2^{\otimes k}$. We claim that $\lambda^k(L_1 \oplus L_2) = \sigma_k(L_1 L_2)$ since $\lambda^k(L_i) = 0$ whenever $k \geq 2$.[51] Slotting this in yields $\psi^k = s_k(\sigma_1(L_1, L_2), \ldots, \sigma_k(L_1, L_2)) = L_1^k \oplus L_2^k$.

For (3)'s additivity, simply apply the splitting principle and naturality; pulling back the Adams operations along the splitting of the thing you're applying to yields the right splitting of the result of the Adams operations.

We skipped (4). For (5), apply the splitting principle again.

These are particularly nice on the K-theory of the sphere:

Lemma 14.6. *On $\widetilde{K}^0(S^{2m})$, we have*

$$\psi^k(a) = k^m a.$$

Proof. Fix $m = 1$ and $b = H^* - 1$. Then,

$$\psi^k(b) = (H^*)^k - 1 = (b+1)^k - 1 = kb.$$

Now, work inductively, and assume it's true for $m - 1$. Write the product map

$$\widetilde{K}^0(S^2) \otimes \widetilde{K}^0(S^{2m-2}) \to \widetilde{K}^0(S^{2m}).$$

Note that this maps b to kb and $\lambda^{n-1} b \mapsto k^{m-1} b$; then, the ring homomorphism structure yields that $\lambda^m b \mapsto k^m b$.

We've assembled enough tools to prove our main theorem.

14.4 Proof of the hopf invariant 1 theorem

Proof of Theorem 14.3. As mentioned earlier, we can assume n is even, i.e. $n = 2m$. Recall that $\psi^k(\alpha) = k^{2m}\alpha$ and $\psi^k(\beta) = k^m\beta = \mu_k\alpha$.

Use commutativity of ψ^3 and ψ^2; this expresses

$$6^m\beta + 3^m\mu_2\alpha + 2^m\mu_3\alpha = \psi^2\psi^3(\beta) = \psi^3\psi^2(\beta) = 5^m\beta + 2^m\mu_3\alpha + 3^{2m}\mu_{2\alpha}.$$

We can cancel additively, then recall that torsion-freeness allows us to cancel α; this implies that

$$3^m(1 - 3^m)\mu_2 = 2^m(1 - 2^m)\mu_3,$$

and in particular, 2^m divides $3^m(1-3^m)\mu_2$; however, noting that $\mu^2(\beta) = 2^m\beta + \mu_2 \equiv \beta^2 \pmod 2$, so $\mu_2 \equiv h \pmod 2$. Hence a Hopf invariant one implies that 2^m divides $3^m(1 - 3^m)$, which s only true when $m = 1, 2, 4$, i.e. when $n = 2, 4, 8$.

[51] This is trivial when $k \geq 2$.

15 Weixiao Lu: Quillen, The spectrum of an equivariant cohomology ring

This talk was delivered on October 27, 2021 by Weixiao Lu.

15.1 Motivation: the rational case

Let G be a compact connected Lie group. A question arises: what is $H^*(BG;\mathbb{Q})$?[52] We have the following theorem, where W is the Weyl group, i.e. the automorphisms of a maximal torus $T \subset G$ induced by $t \mapsto gtg^{-1}$ for some g:

Theorem 15.1. *$H^*(BG;\mathbb{Q}) = H^*(BT;\mathbb{Q})^W$. Hene $H^*(BG;\mathbb{Q})$ is a polynomial ring with $n = \operatorname{rank}(T)$ variables, and $\dim H^*(BG;\mathbb{Q}) = n$.*

Here, dim denotes Krull dimension. We can reinterpret this:

Construction 15.2. We construct the category $\mathcal{T}$ with objects the tori in G, and we have

$$\operatorname{Mor}(T,T') = \left\{\varphi : T \to T' \mid \exists g \in G \text{ s.t. } \varphi(t) = gtg^{-1}\right\}$$

We have the following reinterpretation:

Theorem 15.3. *$H^*(BG;\mathbb{Q}) = \lim_{\mathcal{T}} H^*(BT;\mathbb{Q})$.*

Todat, we will replace $\mathbb{Q}$ by $\mathbb{F}_p$, and answer the following conjecture, due to Atiyah & Swan:

Conjecture 15.4. *If G is a compact Lie group, then $\dim H^*(BG;\mathbb{F}_q)/\{nil\}$ is the rank of the maximal p-torus in G.*

We can verify this in some cases.
We will state Quillen's theorem:

Definition 15.5. Let $\varphi : R \to S$ be a morphism of $\mathbb{F}_p$ graded-commutative algebras. φ is called an *F-isomorphism* if:

(i) for all $x \in \ker\varphi$, $x^{p^n} = 0$ for some n.

(ii) for all $y \in S$, $y^{p^n} \in \operatorname{im}\varphi$ for some n.

Theorem 15.6 (Quillen)**.** *Let G be a compact Lie group. Define the category $\mathcal{A}$ to have objects the elementary p-subgroups of G, with morphisms the morphisms induced by conjugation. Then, there is an F-*isomorphism

$$H^*(BG;\mathbb{F}_p) \to \lim_{A\in\mathcal{A}} H^*(BA;\mathbb{F}_p).$$

Let's get to proving this. First, some preliminaries.

15.2 Preliminaries: sheaf cohomology and equivariant cohomology

Sheaf cohomology A *sheaf* is a local coefficient system.[53] Given X a space and $\mathcal{F}$ a sheaf on X, define the *sheaf cohomology*

$$H^i(X;\mathcal{F}) := R^i\Gamma(X,\mathcal{F}).$$

We have some examples:

> **Example 15.7:**
> $H^0(X;\mathcal{F}) = \Gamma(X;\mathbb{F})$ is the group of compatible systems of coefficients.

[52] A glut of examples were written on the side board. I won't try to copy them down.
[53] This wasn't defined rigorously.

Example 15.8:
If $\mathcal{F}$ is a constant sheaf on a ring A, then sheaf cohomology is just integral cohomology:

$$H^i(X; \mathcal{F}) = H^i(X; A)$$

There are operations on sheaf cohomology, including pushforwards and pullbacks: given a map $f : X \to Y$, the pushforward $f_*\mathcal{F}$ is a sheaf we may construct on Y s.t., in good cases,

$$(R^p f_* \mathcal{F})_y \simeq H^p(f^{-1}(y); \mathcal{F}|_{f^{-1}(y)}).$$

Equivariant cohomology Let X be a G-space. We'd like to define $H^*_G(X)$ to be $H^*(X/G)$, but this is not well-defined homotopically. We have to do some replacement with a homotopy quotient; it's enough to demand a free action of the homotopy type of G on the homotopy type of X, which we do by forming the *homotopy quotient* $EG \times_G X \to BG$.

Fixing a coefficient ring, we may define equivariant cohomology:

Definition 15.9.
$$H^*(G) := H^*(X \times_G EG).$$

This is well-behaved:

Proposition 15.10. *The following properties hold of equivariant cohomology:*

(1) $H^*_G(X)$ is independent of a choice of EG.

(2) When G acts on X freely, in sheaf cohomology $H^*_G(X) = H^*(X/G)$. In good cases, this holds for singular cohomology.

(3) $H^*_G(*) = H^*(* \times_G EG) = H^*(BG)$.

(4) $H^*_G(X)$ is functorial in both X and G.

We will use the following sheaf cohomology SS:

Proposition 15.11 (Leray spectral sequence)**.** *Let $f : X \to Y$ be a map and $\mathcal{F}$ a sheaf on X. Then, there is a spectral sequence*

$$E_2^{p,q} = H^p(Y; R^q f_* \mathcal{F}) \implies H^{p+q}(X; \mathcal{F}).$$

We can apply this in our case: the projection $EG \times_G X \xrightarrow{\pi} X/G$ yields a LSS

$$E_2^{s,t} = H^s(X/G; \mathcal{H}^t_G) \implies H^{s+t}_G(X)$$

where $\mathcal{H}^t_G = R^t \pi_* \Lambda$.

15.3 The main theorem

Let $\mathrm{cd}(X)$ be the cohomological dimension of X mod p.[54] Set $\Lambda := \mathbb{F}_p$.

Theorem 15.12. *Let X be a paracompact s.t. $\mathrm{cd}(X) < \infty$. Then, there is an F-isomorphism*

$$H^*_G(X) \xrightarrow{F-iso} \mathcal{A}^*_G$$

where $\mathcal{A}^*_G$ is a "limit."

Our proof strategy is as follows:

(1) Give an F-iso $H^*_G(X) \to H^0(X/G, \mathcal{H}^t_G)$ for good X.

(2) Give an F-iso $H^0(X/G; \mathcal{H}^t_G) \to \mathcal{A}^*_G(X)$ for good X.

[54] I found this confusing; check Quillen for a full definition.

(3) Use a descent argument to get the rest.

We start with part (1):

Proposition 15.13. *Suppose X is paracompact with $\mathrm{cd}(X) < \infty$. Then, there is an F-isomorphism*

$$\begin{aligned} H^*_G(X) &\to H^0_G(X/G; \mathcal{H}^t_G) \\ u &\mapsto (x \mapsto \textit{pullback of } u \textit{ to fiber}). \end{aligned}$$

Proof. It's a fact that this is an edge morpihsm in the above LSS. Let $u \in H^*_G(X)$ have $\tilde{u} = 0$. We want to prove that $u^n = 0$; in fact, we have $u \in F^1 H^*_G(X)$, so $u^{d+1} \in F^{d+1} H^*_G(X) = 0$ for $d \geq \mathrm{cd}(X)$. Hence $u^n = 0$ for $n \gg 0$.

If $s \in E_2^{0,t}$, then $d^2(s^p) = pd(s)^{p-1} \in E_2^{0,*}$, and we can keep doing this; $s^{p^{d+1}} \in E_\infty^{0,*}$, which is enough. ☺

In order to do part (2), we should finally define $\mathcal{A}$:

Definition 15.14. Let X be a G-space, and $\mathcal{F}$ a family of subgroups of G. Define the categofy $\mathcal{C}$ to have objectsthe elements of $\mathcal{F}$ and morphisms induced by conugation. Define the ring

$$\mathcal{F}^*(X) = \lim_k \left\{\text{locally constant function} X^k \to H^*(BK)\right\}$$

Note that

$$\mathcal{F}^t(X) = \left\{(f^k \mid X^K \to H^t(BK))_k \text{ s.t. compatibility condition}\right\}.$$

These are called *compatible families of functions* in Quillen. Let $\mathcal{A}$ be the group of elementary p-subgroups. There is a map

$$H^0(X/G; \mathcal{H}^t) \to \mathcal{A}^t(X).$$

Theorem 15.15. *If X is Hausdorff with locally finite orbit structure, and $\mathcal{A}$ contains each of the isotropy groups of X, then the above map is an isomorphism.*

This is a verification, which we skip.

For part (3), let X be paracompact and of finite cohomological dimension. G acts on $U(n)$, and for $F := U/\text{maximal torus}$, G acts on F. Then, $X \times F$ satisfies the condition. We can form a sequence

$$X \times F \times F \rightrightarrows X \times F \to X.$$

This yields a diagram

$$\begin{array}{ccccc} H^*_G(X) & \longrightarrow & H^*_G(X \times F) & \rightrightarrows & H^*_G(X \times X \times F) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A}^*_G(X) & \longrightarrow & \mathcal{A}^*_G(X \times F) & \rightrightarrows & \mathcal{A}^*_G(X \times F \times F) \end{array}$$

Two out of three vertical arrows are F-isos. You can directly check that each row is an equalizer, so the third arrow is also an F-iso, which is what we wanted to show.

16 Dylan Pentland: Quillen, The cohomology and K-theory of the general linear groups over a finite field

This talk was delivered on October 29, 2021 by Dylan Pentland. We'll mostly focus on algebraic K-theory.

16.1 Algebraic K-theory: the + construction

We'd like to define groups $K_n(R)$ for R a commutative ring, generalizing the following examples:

- $K_0(R)$ shold be the grothendieck group of the category of finitely generated projective R-modules.
- Let $\mathrm{GL}(R) := \operatorname{colim} \mathrm{GL}_n(R)$. Define[55]

$$K_1(R) := \mathrm{GL}(R)^{\mathrm{ab}}$$

We will generalize these using the *plus construction*:

Definition 16.1. Let X be a pointed CW complex and $N \subset \pi_1(X)$ a perfect normal subgroup. The plus construction, if it exists, satisfies the universal property that it represents homotopy classes of maps out of X killing N:[56]

$$\begin{tikzcd} X \arrow[r, "i_N"] \arrow[dr, "f"'] & X^+ \arrow[d, dashed, "f^+"] \\ & Y \end{tikzcd}$$

such that, if f_* is an iso on H_*, then so is f_*^+.[57]

We can use this construction to define algebraic K-theory:

Definition 16.2. $K_n(R) := \pi_n(\mathrm{BGL}(R)^+)$, taken relative to $[\mathrm{GL}(R), \mathrm{GL}(R)]$.

Our main theorem will be the following computation:

Theorem 16.3.

$$\begin{aligned} K_{2n-1}(\mathbb{F}_q) &= \mathbb{Z}/(q^n - 1)\mathbb{Z}, \\ K_{2n}(\mathbb{F}_q) &= 0. \end{aligned}$$

While this construction is computable explicitly from definition, it's not easy. Our idea will be to construct a space $F\Psi^q$ and exhibit a homotopy equivalence $\mathrm{BGL}(\mathbb{F}_q)^+ \to F\Psi^q$.

16.2 The space $F\Psi^q$

Produce maps $BG \to BU$ from representations over $\mathbb{F}_q$, using a map $R_{\mathbb{F}_q}(G) \to R_{\mathbb{C}}(G)$, called *Brouwer lifting*. Given an $\mathbb{F}_q$-representation V, fix $\rho : \overline{\mathbb{F}_q}^\times \hookrightarrow \mathbb{C}^\times$. Define

$$\chi_V(g) = \sum_{\alpha \text{ eigenval w mult.}} \rho(\alpha)$$

and use this for the lifting.

Produce a map $R_{\mathbb{C}}(G) \to [BG, BU]$ by sending V to $EG \times_G V \to BG$, which corresponds with $R_{\mathbb{C}}(G) \to K^0(BG)$, which corresponds to $[BG, BU \times \mathbb{Z}] = [BG, BU]$.[58] We use the following claim:

[55] This can be motivated by a clutching-type thing.
[56] This looks like a quotient by N to me.
[57] This exists in good enough situations; probably this entire situation.
[58] The last part follows by connectivity of BG.

Claim. *There is a map $\Psi^q : BU \to BU$ such that the map $R_{\mathbb{F}_q}(G) \to [BG, BU]$ factors through the fixed points $[BG, BU]^{\Psi^q}$.*

For X where $[X, \Omega\,\mathrm{BU}] = 0$, we then have

$$[X, \mathrm{BU}]^{\Psi^q} = [X, F\Psi^q]\,.$$

We can apply this to get a map $R_{\mathbb{F}_q}(G) \to [BG, F\Psi^q]$.

Note that $\mathrm{GL}_n(\mathbb{F}_q)$ acts on $\mathbb{F}_q^n$ via the standard representation; hence there are maps $\theta_n : \mathrm{BGL}_n(\mathbb{F}_q) \to F\Psi^q$, which in turn paste together and plus to maps $\theta^+ : \mathrm{BGL}(\mathbb{F}_q)^+ \to F\psi^q$.

The space $F\Psi^q$ is formally defined as the pullback for some mysterious reason, the diagram I'm trying to draw is not working. It's a pullback expressing it as homotopy fixed points.

We have the following lemma powering our analysis:

Lemma 16.4.

(i) $F\Psi^q \simeq \mathrm{ho\,fib}(1 - \Psi^q)$*, for the map* $1 - \Psi^q$ *suitably defined.*

(ii) if $[X, \Omega BU] = 0$*, then there is an iso* $[X, F\Psi^q] \xrightarrow{\sim} [X, BU]^{\Psi^q}$.

(iii) $\pi_{2n}(F\Psi^q) = 0$ *and* $\pi_{2n-1}(F\Psi^q) = \mathbb{Z}/(q^n - 1)\mathbb{Z}$.

By the last part of this lemma, it's enough to give a homotopy equivalence $F\Psi^q \simeq \mathrm{BGL}(\mathbb{F}_q)^+$. It's enough to check this in homology:

16.3 The homotopy equivalence $F\Psi^q \simeq \mathrm{BGL}(\mathbb{F}_q)^+$

Note that both of our spaces are simple. We can use the following theorem:

Theorem 16.5. *Let X, Y be simple spaces. A map $f : X \to Y$ is a weak homotopy equivalence if it's an iso in integral homology.*

It's enough to show it's an iso in $\mathbb{Q}$ and $\mathbb{F}_p$ coefficients.
We do this in three cases:

- For $\mathbb{Q}$, $H^*(BG; \mathbb{Q}) = 0$ when $|G|$ is finite. Hence $H_*(BG; \mathbb{Q}) = 0$. Also, $H_*(F\Psi^q)$ by rational Hurewicz. This gives the $\mathbb{Q}$-homology iso.
- A similar argument applies for mod ℓ Hurewicz and the fact that there are no Sylow ℓ-subgroups, when ℓ is a prime not dividing p.
- The rest is $\mathbb{F}_q$, where the (co)homology is interesting.

Set $q := p^d$. We start the $\mathbb{F}_q$ case:

Theorem 16.6. $H^i(\mathrm{BGL}_n(\mathbb{F}_q); \mathbb{F}_p) = 0$ *whenever* $0 < i < d(p-1)$.

Proof. Fix a p-Sylow subgroup $U_n \subset B\,\mathrm{GL}_n(\mathbb{F}_q)$, and note that this induces an inclusion $H^i(BGL_n(\mathbb{F}_q)) \hookrightarrow H^i(U_n)^T$, but the latter is 0 in these cases, which is argued in Quillen. ☺

In order to define $H^*(F\Psi^q$, we will use the Eilenberg-Moore spectral sequence. This spectral sequence has signiature

$$E_2^{s,t} = \mathrm{Tor}_{s,t}^{H^*(\mathrm{BU}^2)}(H^*(BU^I), H^*(BU)) \implies H^*(F\Psi^q).$$

This tells you the associated graded of $H^*(F\Psi^q)$ is $P(c_{j,r}) \otimes \Lambda\,[e_{j,r}]$ for some things coming from Chern classes (in even degrees).[59]

I got lost and missed the end here. It's more cohomology computations.

[59] This is a free graded-commutative algebra. Supposedly it looks like it comes from the Künneth SS.

17 Zihong Chen: Quillen, Higher Algebaic K-theory I

This talk was delivered on Nov 3, 2021 by Zihong (Peter) Chen. Our main goal will be to define the Q-construction and sketch Quillen's Dévissage and Localization theorems. We start with a review of the classifying space of a category.

17.1 The classifying space of a category

The definition We began with a review of the geometric realization $|S_\bullet|$ of a simplicial set $S_\bullet$, but I'll skip it. In order to define the classifying space of a category, we first have to define its nerve:

Definition 17.1. Let $\mathcal{C}$ be a category. The *nerve* of $\mathcal{C}$ is the simplicial set $N_\bullet \mathcal{C}$ with simplices

$$N_n\mathcal{C} := \{\text{length } n+1 \text{ paths } x_0 \to \cdots \to x_n\}.$$

The ith face map composes the map $x_{i-1} \to x_i \to x_{i+1}$ and the ith degeneracy map introduces a copy of the identity after x_i.

The *classifying space* of $\mathcal{C}$ is

$$B\mathcal{C} := |N\mathcal{C}|.$$

The fundamental group Recall that $\pi_1(BG) = \mathrm{Free}(G)/([g]\,[h] = [gh]) = G$.

We give a similar computation: fix $\mathcal{C}$ a small category, $o \in \mathcal{C}$ an object. π_n only depends on the connected components, so we might as well assume that $\mathcal{C}$ is weakly connected. Fix a maximal tree T in the underlying undirected graph of $\mathcal{C}$. We have the following:

Claim.

$$\pi_1(B\mathcal{C}) = \frac{\textit{free}([F] \in \mathrm{Mor}(\mathcal{C}))}{1 = [t] \in T, \quad [g]\,[f] = [gf]}.$$

Functoriality Note that a functor $F : \mathcal{C} \to \mathcal{D}$ induces a map $BF : B\mathcal{C} \to B\mathcal{D}$ by functoriality of the nerve and geometric realization. First note that the classifying space commutes with products. We use this:

Claim. *A natural transformation $F \implies G$ induces a homotopy from BF to BG; that is, B is 2-functorial.*

Proof sketch. The main perspective is that such an n.t. is a functor $\mathcal{C} \times [1] \to \mathcal{D}$ with the right restriction, which geometrically realizes to a map $B\mathcal{C} \times I \to B\mathcal{D}$ with the right restrictions. ☺

17.2 Quillen's Theorems A and B

We will consider the conditions of the induced map from a functor being a homotopy equivalence or a quasifibration. Note that, for $Y \in \mathcal{D}$, the undercategory $Y\backslash F$ forms the strict fiber above Y:

$$\begin{tikzcd} Y\backslash F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d] \\ \{Y\} \arrow[r, hook] & \mathcal{D} \end{tikzcd} \tag{4}$$

We have some relatively strong results about this: first, trivial fibers (not trivial homotopy fibers) are enough to conclude that a map induced by a functor is an equivalence:

Theorem 17.2 (Theorem A)**.** *If for all $Y \in \mathcal{D}$, the undercategory $Y\backslash F$ is contractible, then BF is an equivalence.*

Second, the strict fiber $Y\backslash F$ is a homotopy fiber if all of the transfer maps are homotopy equivalences:

Theorem 17.3 (Theorem B)**.** *If for all maps $Y \to Y'$ in $\mathcal{D}$, the induced maps $Y'\backslash F \to Y\backslash F$ are homotopy equivalences, then (4) is a homotopy cartesian square.*

These were only stated in the talk. We now move on to the Q construction.

17.3 Quillen's Q construction

The axioms for an exact category were stated and very briefly discussed. We call the initial map in a SES an *admissible mono*, and the other one an *admissible epi*. The following theorem was stated:

Theorem 17.4 (Quillen-Gabriel embedding theorem)**.** *Every exact category is an extension-closed additive, exact subcategoy of an Abelian category.*

Every such subcategory is exact; hence our perspective ought to be that exact categories are extension-closed additive fully faithful subcategories of an Abelian category.

With this in mind, we can go on to our construction:

Construction 17.5. Fix $\mathcal{A}$ an exact category. We construct the category $Q\mathcal{A}$ having objects $\operatorname{Ob}\mathcal{A}$, and morphisms given by equivalence classes of spans $A \twoheadleftarrow C \rightarrowtail B$, where the special arrows denote admissible epis and monos. Equivalence corresponds with isos of diagrams

$$\begin{tikzcd}
& C \arrow[dl, two heads] \arrow[dr, tail] \arrow[dd, "\sim"] & \\
A & & B \\
& C' \arrow[ul, two heads] \arrow[ur, tail] &
\end{tikzcd}$$

Composition is given by pullbacks: the normal diagram for pullback composition of spans goes here. For admissible epi $A \twoheadleftarrow B$, define the corresponding map in $Q\mathcal{A}$ via the same notation to be $A \twoheadleftarrow B = B$, and similar for admissible monos.

Now we can state a theorem which verifies that this is a meaningful construction:

Theorem 17.6. $\pi_1(BQ\mathcal{A}) = K_0(\mathcal{A})$.

Proof sketch. The arrows $[0 \rightarrowtail A]$ form a maximal tree. The composition $[0 \rightarrowtail A]\,[A \rightarrowtail B] = [0 \rightarrowtail B]$ verifies that $[A \rightarrowtail B] = 1$. Hence $[A \twoheadleftarrow B \rightarrowtail C] = [A \twoheadleftarrow B]$. Note further that $[0 \twoheadleftarrow A]\,[A \twoheadleftarrow B] = [0 \twoheadleftarrow B]$, so that π_1 is actually generated by $\{[0 \twoheadleftarrow A]\}$.

This shows that the morphism $\operatorname{Free}(\mathcal{A}) \to \pi_1(BQ\mathcal{A})$ is surjective. One can show directly that the relations induced by passing to $K_0(\mathcal{A})$ is respected (i.e. additivity is respected), which can be seen by drawing a diagram.This diagram is the pullback composition of $0 \rightarrowtail C$ with $C \twoheadleftarrow B$. ☺

We can now naturally define higher algebraic K theory for an Abelian category!

Definition 17.7. The higher algebraic K theory is defined by

$$K_i(\mathcal{A}) := \pi_{i+1}(BQ\mathcal{A}) = \pi_i(\Omega BQA).$$

We can verify that this is coherent with our previous definition:

Theorem 17.8 ("$Q = +$")**.** *For R a ring and $P(R)$ the category of f.g. projectiv emodules,*[60]

$$\Omega BQP(R) \simeq B\,\mathrm{GL}^+(R).$$

We can now work on some characterization:

[60] There was a tiny bit of doubt that there was no assumption necessary on the ring.

17.4 Dévissage and localization

Let's state some nice theorems.

Theorem 17.9 (Dévissage)**.** *Fix $\mathcal{B} \subset \mathcal{A}$ an Abelian subcategory closed under subobjects and quotients, such that for all $M \in \mathcal{A}$, there is a finite filtration*

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$$

with $X_i/X_{i-1} \in \mathcal{B}$. Then, the map $Q\mathcal{B} \to Q\mathcal{A}$ corresponding with the embedding is an iso, and hence $K_n(\mathcal{B}) \simeq K_n(\mathcal{A})$.

Proof sketch. We use theorem A. It suffices to prove the functor $Q\mathcal{B} \xrightarrow{f} Q\mathcal{A}$ has contractible (strict) fibers. Note that $\{(N, u : N \to M) \mid N \in Q\mathcal{B}, u \in \operatorname{Mor} Q\mathcal{A}\} = M\backslash f$. Observve that ,similarly, $M\backslash f \simeq J(M)$, where $J(M)$ is the poset of admissible subobjects of M; we can show that $J(M)$ is contractible inductively by proving that, for all $M' \subset M$ with $M/M' \in \mathcal{B}$, $J(M') \to J(M)$. This is complicated, and we didn't write it out very explicitly.

We can apply this: Let $\mathcal{A} := \mathbf{Ab}_p$ be the category of f.g. Abelian p-groups and let $B := \operatorname{Vec}^{\text{fin}}(\mathbb{F}_p)$. Then, we can apply Dévissage using the filtration $0 = p^N M \subset p^{n-1} M \subset \cdots \subset M$ to conclude that $Q\mathbf{Ab}_p \simeq Q\operatorname{Vec}^{\text{fin}}(\mathbb{F}_p)$.

Recall the construction of $\mathcal{A}/\mathcal{B}$, where $\mathcal{B}$ is a Serre subcategory[61] of an Abelian category $\mathcal{A}$. These play nicely with Q:

Theorem 17.10. *There is a homotopy fiber sequence*

$$Q\mathcal{B} \to Q\mathcal{A} \to Q(\mathcal{A}/\mathcal{B}).$$

Hence there is an associated LES in higher algebraic K-theory.

For instance, let $\mathcal{A} := \mathbf{Ab}$, and let $B := \mathbf{Ab}_p$. It's not hard to chekc that $\mathcal{A}/\mathcal{B} = \mathbb{Z}[1/p] - \mathbf{Mod}$. Hence there is a homotopy fiber sequence

$$Q(\mathbb{Z}/p) = Q(\mathbf{Ab}_p) \to Q\mathbb{Z} \to \mathbb{Q}(\mathbb{Z}[1/p]).$$

[61] I.e. exact subcategory closed under quotients, subobjects, and extensions

18 Mikayel Mkrtchyan: Griffiths and Morgan, Rational homotopy theory and differential forms

This talk was delivered on Nov 5, 2021 by Mikayel Mkrtchyan. The title of the talk is simply "Rational homotopy theory."

18.1 Definitions: rational equivalence and CDGAs

Homology will always have $\mathbb{Q}$ coefficients today. We begin with the following theorem:

Theorem 18.1 (Serre-Whitehead)**.** *Suppose $f : X \to Y$ is a map of simply connected spaces. Then, $f_* : \pi_*(X) \otimes \mathbb{Q} \xrightarrow{\sim} \pi_*(Y) \otimes \mathbb{Q}$ is an iso iff $f_* : H_*(X) \xrightarrow{\sim} H_*(Y)$ is an iso.*

This motivates the following definition:

Definition 18.2. A map $f : X \to Y$ is a rational equivalence if it induces an iso on $\pi_*(-) \otimes \mathbb{Q}$.

We again always work over $\mathbb{Q}$.

Definition 18.3. A CDGA is a pair (A^*, d) where A^* is a graded-commutative $\mathbb{Q}$-algebra and $d : A^* \to A^{*+1}$; that is, we have the relations

$$\begin{aligned} ab &= (-1)^{|a||b|} \\ d^2 &= 0 \\ d(ab) &= da \cdot b + (-1)^{|a|} a \cdot db. \end{aligned}$$

The homology of a CDGA is a CGA. Let $\mathbf{Spc}^{\geq 2}$ be the category of simply connected spaces with rational homology of finite type. Then, there is a diagram of functors

$$\begin{tikzcd} \mathbf{Spc}^{\geq 2} \arrow[rr, "A_{\mathrm{PL}}"] \arrow[dr, "H^*"'] & & \mathbf{CDGA} \arrow[dl, "H^*"] \\ & \mathbf{CGA} & \end{tikzcd}$$

where A_{PL} is to be defined. We have the following theorem:

Theorem 18.4. *For notation defined above,*

1. *The diagram commutes*
2. *Given $X \in \mathbf{Spc}^{\geq 2}$, its rational homotopy type can be recovered froM $A_{PL}(X)$.*
3. *Any CDGA with $H^0 = \mathbb{Q}$ and $H^1 = 0$ arises from such an element, up to quasi-isomorphism.*

18.2 Minimal CDGAs and some computations

Definition 18.5. A CDGA (A^+, d) is *minimal* if

(i) $A^* \simeq \wedge(V^*)$ for some GVS V^*.

(ii) $d : A^{>0} \to A^{>0} \cdot A^{>0}$.

A nonexample of a minimal model was given. An example now follows, which is a minimal quotient of this:

> **Example 18.6:**
> $(A^*, d) = (\wedge(e_{2n})/(e_{2n}^2 = 0), d = 0)$.

A minimal model of a CDGA is a minimal CDGA quasi-isomorphic to this. An example of a minimal model was given. We have the following theorem: We assume that CDGAs have finite-type cohomology throughout.

Theorem 18.7.

1. *Any CDGA has a unique up to iso minimal model $m = (\wedge(V^*), d)$.*
2. *Given a space $X \in \mathbf{Spc}^{\geq 2}$, let $m = (\wedge(V^*), d)$, be a minimal model of $A_{PL}(X)$. Then, $\pi_n(X) \otimes \mathbb{Q} \simeq (V^n)^\vee$.*

We can work an example:

Example 18.8:
Let $X = S^{2n+1}$ be an odd sphere. There is a map from a minimal CDGA $m = (\wedge(e_{2n+1}), d = 0) \to A_{\mathrm{PL}}(S^{2n+1})$ sending e_{2n+1} to a cocycle generator x of H^{2n+1}. This is a map of CDGAs which exhibits m as a minimal model for $A_{\mathrm{PL}}(S^{2n+1})$. We have the following corollary:
Corollary 18.9. $\dim(\pi_i(S^{2n+1}) \otimes \mathbb{Q}) = \delta_{i,2n+1}$.

Example 18.10:
Let $X = S^{2n}$ be an even sphere. Then, $H^*(X) = \wedge(e_{2n})/(e_{2n}^2)$. We can construct a map

$$m = (\wedge(e_{2n}, e_{4n-1}), de_{2n=0}, de_{4n-1} = e_{2n}^2) \to A_{\mathrm{PL}}(S^{2n}),$$

which is in fact a map of CDGAs exhibiting m as a minimal model. Hence we have the following computation:
Corollary 18.11. *The only nonzero terms of $\pi_i(S^{2n}) \otimes \mathbb{Q}$ are in degrees $2n$ and $4n-1$, where there are $\mathbb{Q}$.*

We can do the same thing with $\mathbb{CP}^n$.

We've been passing to cohomology very quickly. We have a definition that can justify this:

Definition 18.12. A CDGA (A^*, d) is *formal* if it's quasi-isomorphic to $(H^*(A), d = 0)$.

Note the following observation:

Proposition 18.13. *If $H^*(A^*) = \wedge(V^*)$, then (A^*, d) is formal.*

Proof. By freeness, we can just define a quasi-iso $\wedge(V^*) \xrightarrow{\sim} (A^*, d)$. ☺

This yields the following corollary, by Hopf's theorem:

Corollary 18.14. *If $X \in \mathbf{Spc}^{\geq 2}$ is an H-space, then it is formal.*

Proposition 18.15. $H^*(K(\mathbb{Z}, n)) \simeq \wedge(e_n)$.

Proof. We do this by induction, using the Serre SS on $K(\mathbb{Z}, n-1) \to * \to K(\mathbb{Z}, n)$. ☺

Corollary 18.16. *H-spaces are rationally equivalent to some product of Eilenberg Mac Lane spaces.*

Corollary 18.17. *If G is a simply connected Lie group, then $H^*(BG)$ is a polynomial algebra.*

Proof. Let $m = (\wedge(V^*), d)$ be a minimal model for G, and $n = (\wedge(W^*), d)$ for BG. Note that V^* is odd, so W^* is even, which hence must have trivial differential.[62] ☺

We can work another example.

Example 18.18:
Let X be a $K3$ surface. Then, $b_1 = b_3 = 0$ and $b_2 = 22$. These are formal, and $H^*(X) = \mathbb{Q}$ in degree 0, $\mathbb{Q}^{\oplus 22}$ in 2, $\mathbb{Q}$ in degree 4, and 0 otherwise. There is some intersection form $B : H^2 \times H^2 \to H^4$.

We can construct a minimal model to compute the rational homotopy groups of a $K3$ surface this way.

We then gave an example of a non-minimal model. Then we went through $S^3 \vee S^3$.

[62] Not sure why V^* is odd.

18.3 The CDGA associated with a space

Definition 18.19. let

$$A_n := \Big(\mathbb{Q}[t_0, \ldots, t_n]/(\sum t_i) = 1\Big) \otimes \Big(\langle dx_0, \ldots, dx_n\rangle/(\sum dx_i = 0)\Big).$$

Observe that $\{A_n\}$ is a simplicial CDGA, and that

$$\mathrm{Hom}_{s\mathbf{Set}}(Y, A_*)$$

is a CDGA for any $Y \in s\mathbf{Set}$. We may define the functor A_{PL} to be the composition

$$\mathbf{Spc}^{\geq 2} \xrightarrow{\mathrm{Sing}} s\mathbf{Set} \xrightarrow{\mathrm{Hom}(-,A_*)} \mathbf{CDGA}.$$

If X is a simplicial complex, we may alternatively define $\widetilde{A}_{\mathrm{PL}}(X)$ to be the PL forms on a space. This is "not very functorial."

Recall the notion of Postnikov towers. There is a CDGA analog:

Definition 18.20. Let (A^*, d) be a CDGA, and V a graded vector space concentrated in degree n. Let $k : V^n \to Z^{n+1}(A^*)$ be a map. The associated *Hirsch extension* of (A^*, d) by V is the CDGA

$$(A^* \otimes \wedge(V).d_k)$$

where $dv|_{A^*} = d$ and $d(v) = k(v)$ for $v \in V$. Note that Hirch extensions are iso iff the associated maps from V^n to cohomology agree. Use the k invariants from a Postnikov tower to construct a tower of the minimal model by Hirsch extensions.

19 Preston Cranford: Quillen, Homotopical Algebra

This talk was delivered on November 10, 2021 by Preston Cranford. We'll do model categories, with the examples in mind of $s\mathbf{Set}$ and $\mathbf{Top}$.

19.1 Model categories

We aim at defining 3 families: fibrations (fib), cofibrations (cofib), and weak equivalences (weq). We say that an *acyclic (co)fibration* is a morphism that is both a cofibration and a weak equivalence. First, lifting properties:

Definition 19.1. We say that f has the *left lifting property* (LLP) with respect to g if, whenever you have a comutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"] & B \arrow[d, "f"] \\ C \arrow[r] \arrow[ur, dashed, "h"] & D \end{tikzcd}$$

there is a filler h.

Also, we have to define retracts

Definition 19.2. A morphism f is a *retract of* g if there's a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, equal] & B \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ A' \arrow[r] \arrow[rr, bend right, equal] & B' \arrow[r] & A' \end{tikzcd}$$

With this in mind, we can define a model category:

Definition 19.3. A *model category* is a category $\mathcal{C}$ with distinguished classes $\mathrm{fib}, \mathrm{cofib}, \mathrm{weq}$ s.t.

1. (bicompleteness) $\mathcal{C}$ is bicomplete, i.e. it has small limits and colimits.[63]
2. (left lifting) Acyclic cofibrations have the LLP wrt fibrations and cofibrations have the LLP wrt acyclic fibrations.
3. (retracts) If g is in a distinguished a family and f is a retract of g, then f is in that family.
4. (factorization) All morphisms factor as $p \circ i$, where i is an acyclic cofibration and p is a fibration; similarly, morphisms factor where i is a cofibration and p is an acyclic fibration.[64]
5. (2 out of 3) If two of $f, g, g \circ f$ are weak equivalences, then so is the third.

Serre would call this a closed model category, but this fell out of favor, and we now usually assume all model categories are closed. We can define functorial factorization now:

Definition 19.4. Define $\mathrm{Map}\,\mathcal{C} := \mathbf{Cat}(\cdot \to \cdot, \mathcal{C})$. These have objects given by morphisms on $\mathcal{C}$ and arrows given by commuting squares. A model category has *functorial factorization* if the (co)fibrant factorization is given by a functor $\mathrm{map}\,\mathcal{C} \to \mathrm{Map}\,\mathcal{C}$.

We can give examples:

Example 19.5:
One example is the *Serre model structure.* This has

- $\mathcal{C} = \mathbf{Top}$,

[63] The word "bicomplete" here is mine.
[64] In practice, we're going to assume that we have a functorial factorization, which we'll define later.

- fib = Serre fibrations,
- weq = weak homotopy equivalences,
- cofib = morphisms with LLP against fib $\cap$ weq.

This dependent definition of cofib seems weird at first; however, it turns out that 2 out of the families determine the third, so all model categories can be defined with one of the families determined in such a way. Let's work another example, this one somewhat explicit:

Example 19.6:
We can do homotopy theory of simplicial sets:

- $\mathcal{C} = s\mathbf{Set}$,
- fib = Kan fibrations,
- weq = morphisms which geometrically realize to weak homotopy equivalences,
- cofib = injections.

One last example:

Example 19.7:
Fix R a commutative ring.

- $\mathrm{Ch}_{>0} R$,
- fib = maps surjective in each degree > 1,
- weq = quasi-isos,
- cofib = injections with projective cokernel.

Let's define (co)fibrant replacement now.

Definition 19.8. The model category $\mathcal{C}$ is *pointed* if the map $\varnothing \to *$ from the initial to final object is an iso.

Say that an object X is *cofibrant* if the map $\varnothing \to X$ is a cofibration, and dually, *fibrant* if $X \to *$ is a fibration.

Definition 19.9. Given X an object in $\mathcal{C}$ the map $* \to X$ factorizes as $* \to QX \to X$ as a cofibration followed by an acyclic fibration. We call QX the *cofibrant replacement.* Dually, given a factorization $X \to RX \to *$ with first an acyclic cofibration, we call RX the *fibrant replacement* of X.

Now, given this, we can finally define the homotopy category.

19.2 The homotopy category of a model category

Definition 19.10 (The idea of localization)**.** The *free category* $F(\mathcal{C}, W^{-1})$ has objects $\mathcal{C}$ and morphisms $(f_1, \ldots, f_n)$ where $f_i \in \operatorname{Mor} \mathcal{C} \cup W^{\mathrm{op}}$. The homotopy category is the quotient category of $F(\mathcal{C}, W^{-1})$ by sensible identifications concerning composition and elimination of formal inverses.

This definition is easier to state, but harder to understand. We'll go through a painstaking definition instead.

Definition 19.11. A *cylinder object* of an object $B \in \mathcal{C}$ is a factorization $B \sqcup B \to B \times I \to B$ where the last morphism is required to be some acyclic fibration (and the first a cofibration). Dually, a *path object* is a factorization $X \to PX \to X \times X$ where the first map is an acyclic cofibration.

We can go on to define homotopies.

Definition 19.12. We say that $f, g : B \to A$ are *left homotopic* if the associated map $B \sqcup B \to A$ factors through $B \times I$. There is a dual notion of *right homotopy.*

Let's finally state a result:

Proposition 19.13. *Let $\mathcal{C}_c$ be the cofibrant objects. There is a diagram of inclusions*

$$\begin{tikzcd} & \mathcal{C}_c \arrow[dr] & \\ \mathcal{C}_{cf} \arrow[ur] \arrow[dr] & & \mathcal{C} \\ & \mathcal{C}_f \arrow[ur] & \end{tikzcd}$$

which descends to a diagram of equivalences

$$\begin{tikzcd} & \mathrm{ho}\,\mathcal{C}_c \arrow[dr, "\sim"] & \\ \mathrm{ho}\,\mathcal{C}_{cf} \arrow[ur, "\sim"] \arrow[dr, "\sim"] & & \mathrm{ho}\,\mathcal{C} \\ & \mathrm{ho}\,\mathcal{C}_f \arrow[ur, "\sim"] & \end{tikzcd}$$

Using this, homotopies define a nice equivalence relation $\sim$ on $\mathcal{C}_{cf}$ s.t. $\mathrm{ho}\,\mathcal{C} = \mathrm{ho}\,\mathcal{C}_{cf}/\sim$. Let's define relationships between model categories now.

19.3 Quillen equivalences

Definition 19.14. A functor $F : \mathcal{C} \to \mathcal{D}$ is a *left quillen functor* if it has a right adjoint, and it preserves cofibrations and acyclic cofibrations. The dual is a right quillen functor.

These induce adjoint pairs:

Definition 19.15. A *total left derived functor* is a composition $\mathrm{ho}\,\mathcal{C} \to \mathrm{ho}\,\mathcal{C}_c \to \mathrm{ho}\,\mathcal{D}$.

Definition 19.16. An adjoint pair $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$ between model categories is a *Quillen equivalence* if its a Quillen functor pair whose derived functors are adjoint equivalences.

The principle example of this is the Quillen equivalence

$$\mathrm{sing} : s\mathbf{Set} \leftrightarrows \mathbf{Top} : \Delta.$$

20 Swapnil Garg: Hill, Hopkins, Ravenel, On the nonexistence of elements of Kervaire invariant one

This talk was delivered on November 12, 2021 by Swapnil Garg. We'll be talking about orthogonal spectra.

20.1 Motivation and the equivariant Spanier-Whitehead category

Fix $\mathcal{T}$ the category of CGWH pointed topological spaces. We will want to construct a model category $\mathcal{C}$ with homotopy category $\mathcal{SHC}$ together with:

1. a Quillen adjunction $\Sigma^\infty : \mathcal{T} \leftrightarrows \mathcal{C} : \Omega^\infty$, such that
2. an identification $[L\Sigma^\infty A, L\Sigma^\infty B] = \operatorname{colim}\,[\Sigma^n A, \Sigma^n B]$,
3. A smash product satisfying $L\Sigma^\infty A \wedge L\Sigma^\infty B = L\Sigma^\infty(AB)$,

and some others.

We naively could take sequential spectra. This doesn't work, since e.g. $\tau_{S^1 S^1}$ the twist map is not homotopic to the identity; one can see this by passing to homology. Hence the natural notion of smash product on sequential spectra can't yield a symmetric monoidal product on $\mathcal{SHC}$. We need a different model.

Fix G a Lie group(?) .

Definition 20.1. The G-equivariant Spanier-Whitehead category $\mathcal{SW}^G$ has objects the finite pointed G-CW complexes and morphisms

$$\{X, Y\}^G = \operatorname{colim}_V \left[S^V \wedge X, S^V \wedge Y\right]^G$$

where V is indexed over the poset of orthogonal representations of G.

We will expand this, but we need some category theory first.

20.2 Some category theory: symmetric monoidal categories

Definition 20.2. A *symmetric monoidal category* is a category V together with a bifunctor $\otimes : V \times V \to V$ and a unit object $1 \in V$ satisfying associativity, commutativity, and coherence axioms.

$(V, \otimes, I)$ is *monoidal closed* if the functor $(-) \otimes B$ has a right adjoint $(-)^B$:

$$[A \otimes B, C] \simeq \left[A, C^B\right].$$

This is the setting we need in order to define enriched categories:

Definition 20.3. A $\mathcal{V}$-category $\mathcal{C}$ has, for each $x, y \in \operatorname{Ob}\mathcal{C}$, a *morphism object* $\mathcal{C}(x, y) \in \mathcal{V}$, together with an identity morphism $I \to \mathcal{C}(X, Y)$ and a composition law[65]

$$\mathcal{C}(y, z) \otimes \mathcal{C}(x, y) \to \mathcal{C}(x, z).$$

Note that closed symmetric monoidal categories are enriched over themselves. Note that $(\mathcal{T}, T, S^0)$ is a closed symmetric monoidal category.

20.3 Orthogonal spectra

Definition 20.4. Let $O(V, W)$ be the space of linear isometric embeddings $V \hookrightarrow W$.

Note that $O(V, V) = O(V)$. We can think of $O(V, W)$ as $O(W)/O(W - V)$. This allows us to define a category:

[65]Beware the many omitted coherence axioms.—

Definition 20.5. The $\mathcal{T}$-category J has objects $V = \mathbb{R}^n$ finite-dimensional vector spaces,[66] and morphisms

$$J(V, W) = \mathrm{Thom}(O(V, W); V - W).$$

As a remark, given elts $V \xrightarrow{(f,v_1)} W \xrightarrow{(g,v_2)} U$, the composition is given by $(g \circ f, g(v_1) + v_2)$.

Definition 20.6. An *orthogonal spectrum* X is a functor

$$X : J \to \mathcal{T}.$$

Explicitly, X consists of spaces $\{X_V\}$, and for each $V, W \in J$, a map $\mathrm{Thom}(O(V, W); W - V) \to \mathcal{T}(X_V, X_W)$; instead, we can equivalently supply the adjoint map $\mathrm{Thom}(O(V, W); W - W) \wedge X_V \to X_W$.

The category $\mathcal{OS}$ of orthogonal spectra is the $\mathcal{T}$-presheaf category $\mathcal{T} - \mathbf{Cat}(J, \mathcal{T})$.

For each linear isometric embedding $V \hookrightarrow W$, we get an $O(W - V) \times O(V)$-equivariant map

$$S^{W-V} \wedge X_V \to X_W,$$

and this various continuously in the linear isometric embeddings $V \hookrightarrow W$ compatibly with composition.

For $X, Y : J \to \mathcal{T}$, the map $X \wedge Y : J \to \mathcal{T}$ is given by the left Kan extension of $(V, W) \mapsto X_V \wedge Y_W : J \times J \to T$ along the functor $\oplus : J \times J \to J$ sending; explicitly, this is the universal filling

$$\begin{tikzcd} J \times J \arrow[rr, "x \wedge y"] \arrow[dr, "\oplus"'] & & T \\ & J \arrow[ur, dashed] & \end{tikzcd}$$

with a universal 2-cell going "down." This is an instance of day convolution.

We can given an aternative definition:

Definition 20.7. For $V \in J$, let $S^{-V} = J(V, -)$ be the corepresentable by V.

By the enriched Yoneda lemma, we get $\mathcal{OS}(S^{-V}, X) = X_V$. We get a reflexive coequalizer

$$\bigvee_{V,W} S^{-1} \wedge J(V, W) \wedge X_V \rightrightarrows \bigvee_V S^{-V} \wedge X_V \to X.$$

Write this as $X = \mathrm{colim}_V\, S^{-V} \wedge X_V$.

Note that $S^{-V} \wedge S^{-W} = S^{-V \oplus W}$. We also have $X \wedge Y = \mathrm{colim}_{V,W}\, S^{-V \oplus W} \wedge X_V \wedge Y_W$.

Definition 20.8. The kth stable homotopy groups are

$$\pi_k X = \mathrm{colim}_{V > -k}\, \pi_{V+k} X_V.$$

A *weak equivalence of orthogonal spectra* is a map yielding a π_*-isomorphism.

These satisfy 2-out-of-6:

Definition 20.9. A homotopical category is a category with weak equivalences satisfying the 2 our of 6 property; for maps $\cdot \xrightarrow{u} \cdot \xrightarrow{v} \cdot \xrightarrow{w} \cdot$, if vu, wv are weak equivalences, so are all compositions in this diagram.

We get the desired properties of the adjunction $\Sigma^\infty : \mathcal{T} \leftrightarrows \mathcal{OS} : \Omega^\infty$ where $(\Sigma^\infty X)_V = S^V \wedge X$, and $\Omega^\infty X = X_{\{0\}}$.

The total left derived functor $L\Sigma^\infty$ extends to a fully faithful embedding of $\mathcal{SW}$ into $\mathrm{ho}\,\mathcal{OS}$.

[66] Alternatively, objects are $\mathbb{N}$.

21 Haoshuo Fu: Bousfield, Localization of spectra with respect to homology

This talk was delivered on November 17, 2021 by Haoshuo Fu. We'll talk about Bousfield localization today. First, we need some technology.

21.1 Reflective localization

Definition 21.1. A localization functor is *reflective* if it admits a fully faithful right adjoint.

We began with a confusing example; for categories with all objects cofibrant, fibrant replacement may or may not be such an example. Let's move on to another example:

> **Example 21.2:**
> Let $\mathcal{C}$ be a model category and $\mathcal{C}'$ be another model structure with the same category with the same cofirbations and more weak equivalences. We get a quillen equivalence $\mathcal{C} \rightleftarrows \mathcal{C}'$ given by the identity in each direction. The left derived functor $L : \mathrm{ho}\,\mathcal{C} \to \mathrm{ho}\,\mathcal{C}'$ is then a reflective localization, with adjoint the right-derived functor.
>
> When $\mathcal{C} = \mathbf{Top}$ and W' are H-isos, this is the Bousfield localization of spaces.

We can apply this to get some ∞ nonsense.

> **Example 21.3:**
> For $\mathcal{C} = s\mathbf{Set}$ having the Joyal model structure,[a] For $\mathcal{C}' = s\mathbf{Set}$ under the model structure "transferred" from the Quillen model structure on **Top**.
>
> By the above procedure, we get a left adjoint functor $L : \infty - \mathbf{Cat} \to \mathrm{Kan}$, generalizing the group completion.
>
> [a]Cofibrations are inclusions, weak equivalences are categorical equivalence, fibrant objects are ∞-categories, whatever this means.

We can level up from HTT things to HA things:

> **Example 21.4:**
> Let $\mathcal{C}$ be a stable $(\infty, 1)$-cat together with a t-structure $(\mathcal{C}_{\leq 0}, \mathcal{C}_{\geq 0})$. The inclusion $\mathcal{C}_{\geq n} \hookrightarrow \mathcal{C}$ admits a left adjoint, *trunation*, $\tau_{\leq n} : \mathcal{C} \to \mathcal{C}_{\leq n}$. We get a cofiber sequence
>
> $$\tau_{\geq n+1}X \to X \to \tau_{\leq n}X.$$
>
> When $\mathcal{C} = \mathrm{Sp}$ is the $(\infty, 1)$-category of spectra and $\tau_{\leq n}$ a Postnikov tower, we have $\mathrm{Sp}^{\heartsuit} = \mathbf{Ab}$, given by Eilenberg-Mac Lane spectra.

That's enough examples.

21.2 E_* local spectra

One can define a categorical monoidal product on the category of presentable $(\infty, 1)$-categories and colimit preserving functors; Sp is the unit of this, so it's a monoid, which descends to a symmetric monoidal structure on Sp, called $\wedge$.

For $E \in \mathrm{Sp}$, we define the *E-homology* by

$$E_*X := \pi_*(E \wedge X).$$

This generalizes singular homology, which is given by HG-homology.

Definition 21.5. A spectrum X is *E_*-acyclic* if $E_*X = 0$, which is true iff $E \wedge X \simeq 0$. A spectrum X is *E_*-local* if $[C, X] = 0$ for all E_*-acyclic spectra C.

We now give a long list of properties:

Proposition 21.6. $\mathrm{Map}(E, X)$ *is E_*-local.*

This uses the tensor-hom adjunciton.

Proposition 21.7. *For E a ring spectrum and X an E-module, X is E_*-local.*

Any map $C \xrightarrow{f} X$ factor through $C \wedge E \xrightarrow{f} X \wedge E$, but $C \wedge E$ is contractible. This gives a few quick corollaries.

Corollary 21.8. *HG is H_*-local.*

Corollary 21.9. *If X is connected, then $\tau_{\leq n}X$ is H_*-local.*

These play nicely with limits:

Proposition 21.10. *The limit of E_*-local objects is E_*-local.*[67]

See this by pulling the limit out of $[,]$ explicitly. We can use this:

Proposition 21.11. *$i : E_* - local \hookrightarrow \mathrm{Sp}$ preserves limits.*

Follow your nose to prove this formally, since the domains limits in Sp end up contained in the domain directly. We need one more thing in order to use the Adjoint functor theorem, saying this is an *accessible* ∞-functor:

Proposition 21.12. *i preserves filtered colimits.*

Using the Adjoint ufnctor theorem allows us to conclude:

Corollary 21.13. *There is a left adjoint $L_E : \mathrm{Sp} \to E_* - local$ to i.*

We can characterize this:

Proposition 21.14. $[X, Y] = [L_E X, Y]$ *if Y is E_*-local. The fiber $G_E = \mathrm{fib}(X \to L_E X)$*

This is unique:

Proposition 21.15. *The triangle $G_E X \to X \to L_E X \to \Sigma G_E X$ is unique.*

To see this, if Y is E_*-acyclic and E_*-local, then $[Y, Y] = 0$, and hence $Y = 0$.
Now let's see some examples.

21.3 Examples of Bousfield localization of spectra

Definition 21.16. if G is an Abelian group, let SG be the *Moore spectrum*, defined to be the connective spectrum with $\pi_0 SG = H_0 SG = G$ and $H_{>0} SG = 0$.

Example 21.17:
Fix $G = \mathbb{Z}_{(J)}$ where J is a set or primes (a localization).[a] $H_*(SG \wedge SG) = G \otimes_{\mathbb{Z}} G = G$. Hence $SG \wedge SG = SG$, and SG is a right spectrum. This implies that $SG \wedge X$ is SG_*-local.

Applying the fact that SG is idempotant, we also have $SG_*(SG \wedge X) = SG_* X$, so that $L_{SG} X = SG \wedge X$ is the SG-localization of X. Note that $\pi_*(SG \wedge X) = G \otimes_{\mathbb{Z}} \pi_* X$, so that X is SG_*-local iff $\pi_* X = G \otimes_{\mathbb{Z}} \pi_* X$ iff $\pi_* X$ is p-divisible for $p \notin J$.

[a]Haoshuo stated that SG is a wedge of spheres, which Haynes contested.

We can work a similar example.

[67]This is an ∞-categorical limit! If thinking model-categorically, one should think about homotopy limits instead.

Example 21.18:

Instead, fix $G = \mathbb{Z}/p$, so that $SG = \mathrm{cofib}(\mathbb{S} \xrightarrow{p} \mathbb{S})$. Note that X is SG_*-acyclic iff cofib)$(X \xrightarrow{p} X) = 0$ iff $\pi_* X \xrightarrow{p} \pi_* X$ is an isomorphism iff $\pi_* X$ is p-divisible. Hence we can compute the localization by the previous example.

There was some more analysis after this, but I couldn't follow.

22 Natalie Stewart: May, The Geometry of Iterated Loop Spaces

22.1 Operads and their algebras

Motvation: little cubes One of the most important structures in algebraic topology is the *nth homotopy group*, i.e. a group structure on the pointed set

$$[S^n, X] = [*, \Omega^n X] = \pi_0 \Omega^n X.$$

When we construct such a group structure, we tend to make a non-canonical choice concerning how to compose n-spheres, constructing a magma structure on $\Omega^n X$, which fails to be unital, associative, or commutative on the nose, but which descends to a group structure on homotopy classes of maps, i.e. $\pi_0 \Omega^n X$. The construction usually goes as follows:

Construction 22.1 (Homotopy groups)**.** We construct a group structure on $\Omega^n X = \mathrm{Maps}((I^n, \partial I^n), (X, *))$. Such a map α is drawn in such a diagram: little cubes diagram . The composition is $\alpha \circ \beta$ by "shrinking" each map by half rectilinarly in one direction, then sticking them together, as drawn in the following diagram: little cubes diagram . Associativity is witness by a homotopy between diagrams: little cubes diagram . Unitality is witnessed by a similar homotopy, and in the case $n \geq 2$, we have the *Eckmann-Hilton argument*, which is another such homotopy of diagrams: little cubes diagram .

A good mantra to take is to *take homotopy as late as possible.* Even this construction was defined point-set, although the group we reference is gotten via a formal process (taking homotopy) and lives in a category which is not very concrete (ho $\mathbf{Top}_*$).

This paragraph is wonky We can obey this mantra even better: we could have defined the composition of two maps in any way "homotopic" to the way we did, so there is actually a coherent family of ways of compose two maps, as well as ways to compose m many maps: if $C_n(m)$ is the space of rectilinear embeddings of m disjoint n-cubes into an n-cube, then all of our choices are jointly summarized by an $\mathbb{N}$-indexed collection of maps

$$C_n(m) \times (\Omega^n X)^m \to \Omega^n X$$

compatibly with the "slotting in" composition maps

$$C_n(m) \times \prod_{i=1}^m C_n(k_i) \to C_n(k_1 + \cdots + k_m).$$

We will formalize this, along with equivariance, through the notion of an *operad*.

Operads and algebras For the duration of this section, fix $(\mathcal{V}, I, \otimes)$ a cocomplete symmetric monoidal closed category. We define the notion formalizing a *compatible collection of symmetric n-ary operations.*

Definition 22.2. An *operad* (sometimes called a *symmetric operad*) O in $\mathcal{V}$ is the data:

- for each $n \in \mathbb{N}$, an object $O(n)$,
- for each index n, a right-action of the symmetric group $S_n \to \mathcal{V}(O(n), O(n))$, and
- for each tuple $(n_1, \ldots, n_k)$ with $n = \sum_i n_i$, a *composition* morphism

$$\gamma : O(k) \otimes O(n_1) \otimes \cdots \otimes O(n_k) \to O(n),$$

- an distinguished *identity* element $I \to O(1)$,

subject to the following conditions:

(i) (Composition of operations is S_*-equivariant) the composition γ coequalizes the following pair:

$$O(k) \otimes \bigotimes O(n_i) \rightrightarrows O(k) \otimes \bigotimes O(n_i) \xrightarrow{\gamma} O(n)$$

where one of the arrows acts on $O(k)$ via the prescribed action, and the other acts by permuting tensor powers.[68]

[68] In this setting, "coequalizes" means that the precomposition of either action before γ yields the same result, i.e. you can either permute tensor powers or act on the space $O(k)$.

(ii) (Composition of operations is associative) for tuples $(d_1, \ldots, d_k)$ and $(e_1, \ldots, e_j)$, defining $f_i := \gamma(d_i, e_{j_1+\cdots+j_{i-1}+1}, \ldots, e_{j_1+\cdots+j_s}$ we have

$$\gamma(\gamma(c; d_1, \ldots, d_k); e_1, \ldots, e_j) = \gamma(c; f_1, \ldots, f_k).$$

(iii) (Composition of operations is unital)

$$\gamma(1; d) = d$$

for all $d \in O(j)$ and

$$\gamma(c; 1^k)$$

for $c \in O(k)$ and $1^k = (1, \ldots, 1) \in O(1)^{\otimes k}$.

A *morphism of operads* is an S_*-equivariant morphism of graded objects $f_* : O(*) \to O'(*)$ which is unital and compatible with composition:

$$f(1) = 1$$
$$f(\gamma(\theta; \theta_1, \ldots, \theta_n)) = \gamma(f(\theta); f(\theta_1), \ldots, f(\theta_n)).$$

This has a graphical calculus. Draw some trees sometime.

Operads are defined as a blueprint for a type of algebraic object. To understand the significance of these, we need to understand how to make objects that comply with this blueprint. To do so, we need an example and a definition.

Example 22.3:
Let $X \in \mathcal{V}$ be an object. Then, the *endomorphism operad on* X, denoted $\mathcal{E}_X$, has:

- Objects $\mathcal{E}_X(n) := \mathcal{V}(X^{\otimes n}, X)$,
- symmetric action induced by the symmetric action on $X^{\otimes n}$,
- composition given by composition of endomorphism

$$\gamma(\theta; \theta_1, \ldots, \theta_n) : X^{\otimes n} = X^{\otimes n_1} \otimes \cdots \otimes X^{\otimes n_k} \xrightarrow{\bigotimes \theta_i} X^{\otimes k} \xrightarrow{\theta} X.$$

It's easy to verify that this is an operad.

Definition 22.4. Let O be an operad. An *algebra over* O is an object X and a morphism of operads

$$O \to \mathcal{E}_X.$$

That is, the set $O(n)$ represents the *possible* n*-ary operations of an algebraic theory*, and an algebra assigns these to particular n-ary operations on an object, with behavior dictated by the symmetry, composition, and unitality. We can flesh this out via an example:

Example 22.5:
The *commutative monoid operad* Comm has objects $\mathrm{Comm}(n) = I$, composition morphisms induced by the identity $I = I$, and trivial symmetric action. It's algebras are canonically identified with *commutative monoid objects in* $\mathcal{V}$, with the unique map $M^{\otimes n} \to M$ given by the unique n-ary operation given by repeated application of the monoid law.

Hence we've subsumed the construction of commutative rings/algebras, commutative monoid, topological commutative monoid, etc. We can get around the commutativity condition by adding extra elements:

Example 22.6:
Suppose $\mathcal{V}$ has finite coproducts. The *monoid operad* Assoc has objects $\mathrm{Assoc}(n) = \coprod_{S_n} I$, with shuffling comultiplicands, with composition chosen uniquely by equivariance, and with identity given by the identity map $I = I$.

Hence we've subsumed monoids. One such example of this will be a monoid in a category of endofunctors, which in turn will describe all operad algebras:

22.2 Operad algebras are monad algebras

Bad Definition 22.7. Fix $\mathcal{C}$ a category. A *monad in* $\mathcal{C}$ is a monoid in the monoidal category of endofunctors $(\mathcal{C}^{\mathcal{C}}, \circ, \mathrm{id})$.

This is a bad definition for two reasons: it takes nontrivial time to unroll, and it doesn't tell us how to define monad algebras, which are what we really care about. We could talk about these as *modules over an algebra over an operad* where the operad is Assoc in an enofunctor category, but this is inefficient. We might as well unroll our definition into something readable:

Good Definition 22.8. Fix $\mathcal{C}$ a category. A *monad in* $\mathcal{C}$ is the data:

1. An endofunctor $O : \mathcal{C} \implies \mathcal{C}$,
2. A *multiplication* natural transformation $\mu : O \circ O \implies O$, and
3. A *unit* natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \implies O$.

subject to associativity and unitality relations:

$$\begin{tikzcd} O^3X \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"] & O^2X \arrow[d, "\mu"] \\ O^2X \arrow[r, "\mu"] & OX \end{tikzcd} \qquad \begin{tikzcd} OX \arrow[r, "\mathrm{id}\otimes\eta"] \arrow[d, "\eta\otimes\mathrm{id}"'] \arrow[dr, equal] & O^2X \arrow[d, "\mu"] \\ O^2X \arrow[r, "\mu"] & OX \end{tikzcd}$$

A *algebra over a monad* O is an object $X \in \mathcal{C}$ together with an action $OX \to X$ which is associative and unital:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, equal] & OX \arrow[d, "a"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} O^2X \arrow[r, "\mathrm{id}\otimes a"] \arrow[d, "\mu"] & OX \arrow[d, "a"] \\ OX \arrow[r, "a"] & X \end{tikzcd}$$

The following establishes that monads are strongly related to adjoint functors:

Example 22.9:
Suppose that $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ is a pair of adjoint functors. Then, there is a monad structure on the endofunctor RL on $\mathcal{D}$, whose multiplication is induced by the counit $\varepsilon : LR \implies \mathrm{id}$:

$$RLRL \xrightarrow{R\circ\varepsilon\circ l} RL$$

with unit given by the adjunction unit.

However, we can go the other direction: for O a monad and X an object, there is an O-algebra structure on OX given by exactly the structure of O, and this is functorial; one can verify that this yields a pair of adjoint functors:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{Free}_O"] \arrow[r, phantom, "\perp"] & O(\mathcal{C}) \arrow[l, bend left, "U"] \end{tikzcd}$$

where U is the forgetful functor taking an operad algebra and $O(\mathcal{C})$ is the category of O-algebras.[69]

Definition of the monad associated with an operad.

[69] For $O = RL$ induced by a pair of adjoint functors, it is not always the case that the free-forgetful adjunction is identified with (L, R). There is a recognition theorem of when this is true, called *Beck's monadicity theorem*; the condition is that the right adjoint is conservatve and preserves certain coequalizers.

22.3 Little cubes and approximation, and a handwave of the recognition principle

In this section, we relate the monad C_n associated with the operad $\mathcal{C}_n$ to the monad $\Omega^n\Sigma^n$:

Theorem 22.10 (The approximation theorem)**.** *There is a morphism of monads $\alpha_n : C_n \to \Omega^n\Sigma^n$ such that, for $X \in \mathbf{Top}_*$ a connected space, the associated morphism $C_nX \to \Omega^n\Sigma^nX$ is a weak homotopy equivalence.*

As an upshot, this gives a combinatorial model for $\Omega^n\Sigma^nX$.

In order to construct this, recall from the beginning of this note that $\mathcal{C}_n$ acts on $\Omega^n\Sigma^nX = \mathrm{Map}((I^n, \partial I^n), (\Sigma X, *))$; this action "slots in" an ordered tuple of j-many maps maps $(I^n, \partial I^n) \to (\Sigma X, *)$ into an element of $\mathcal{C}_n(j)$ to define a single map $(I^n, \partial I^n) \to (\Sigma X, *)$. For C_n the monad associated with $\mathcal{C}_n$, this corresponds with an action $\mu_n : C_n\Omega^n\Sigma^nX \to \Omega^n\Sigma^nX$. We can define the composition map

$$\alpha_n = \left(C_nX \xrightarrow{C_n\eta_n} C_n\Omega^n\Sigma^nX \xrightarrow{\mu_n} \Omega^n\Sigma^nX\right),$$

where α_0 is defined to be the identity. It suffices to prove the following proposition:

Proposition 22.11. *For each $1 \leq n < \infty$ and $X \in \mathbf{Top}_*$, there is a commuting diagram*

$$\begin{tikzcd} C_nX \arrow[r, hook] \arrow[d, "\alpha_n"] & E_nX \arrow[r, two heads] \arrow[d, "\delta_n"] & C_{n-1}\Sigma X \arrow[d, "\alpha_{n-1}\Sigma"] \\ \Omega^n\Sigma^nX \arrow[r, hook] & P\Omega^{n-1}\Sigma^nX \arrow[r, two heads, "p"] & \Omega^{n-1}\Sigma^nX \end{tikzcd} \tag{5}$$

where the bottom row is the loop space fibration and E_nX is contractible. If X is connected, this diagram may be constructed so that the top row is a quasifibration, and hence each α_n is a weak equivalence.

Clearly, this proposition implies Theorem 22.10 in the case $n < \infty$. Taking a colimit yields Theorem 22.10 in the case $n = \infty$.

Unfortunately, the proof of Proposition 22.11 is lengthy and technical. We will construct the diagram and sketch the proof that the top row is a quasifibration; the reader may refer to Cite for full details.

Proof sketch of Proposition 22.11. We start with a general construction of a space associated with a pair, which will specialize to E_nX.

Construction 22.12. Fix (X, A) a pair in $\mathbf{Top}_*$. The space $E_n(j; X, A) \subset C_n(j) \times X^j$ is defined to be the subspace of little n cubes $f' \times f'' \to J \times J^{n-1} \to J \times J^{n-1}$ consisting of points $(\langle c_1, \ldots, c_j\rangle, x_1, \ldots, x_j)$ such that, whenever $x_r \notin A$, there are no cubes $c_s(J^n)$ intersecting the "shadow" $(c_r'(0), 1) \times c_r''(J^{n-1})$ for $r \neq s$. let $E_n(X, A)$ be the union of the image of $E_n(j; X, A)$ in C_nX as j ranges over $\mathbb{N}$.

This construction is functorial, as maps of pairs $(X, A) \to (X', A')$ induce evident maps $E_n(X, A) \to E_n(X', A')$, compatibly with composition.

Because A is closed in X, $E_n(X, A)$ is closed in C_nX. Observe that $E_n(X, X) = C_nX$. On the other end, there's a more delicate relationship between $E_n(X, *)$ and $C_{n-1}X$: define the (surjective, based) map $v_n : E_n(X, *) \to C_{n-1}X$ by the formula

$$v_1 [c, x] = x$$
$$v_n [c' \times c'', x] = [c'', x].$$

For a map of pairs $(Y, A) \to (X, *)$, there is an associated map $E_n(Y, A) \to C_{n-1}X$; fixing TX the cone over X, define the map $\pi_n : E_n(TX, X) \to C_{n-1}\Sigma X$ to be the map associated with the collapse map $(TX, X) \to (\Sigma X, X)$.

Assemble the diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\eta_n"] & TX \arrow[r, "\pi"] \arrow[d, "\bar{\eta}_n"] & \Sigma X \arrow[d, "\eta_{n-1}"] \\ \Omega^n\Sigma^nX \arrow[r, hook] & P\Omega^{n-1}\Sigma^nX \arrow[r, "p"] & \Omega^{n-1}\Sigma^nX \end{tikzcd}$$

This induces the diagram

$$\begin{tikzcd}
C_nX \arrow[r, hook] \arrow[d, "\alpha_n"] & E_n(TX,X) \arrow[r, two heads] \arrow[d, "\delta_n"] & C_{n-1}\Sigma X \arrow[d, "\alpha_{n-1}\Sigma"] \\
C_n\Omega^n\Sigma^n X \arrow[r, hook] & E_n(P\Omega^{n-1}\Sigma^n X, \Omega^n\Sigma^n X) \arrow[r, two heads, "p"] & C_{n-1}\Omega^{n-1}\Sigma^n X
\end{tikzcd}$$

Then, to define diagram (5), we need to paste this together with another diagram from the following lemma applied to ΣX:

Lemma 22.13. *There is a map θ making the following diagram commute:*

$$\begin{tikzcd}
C_n\Omega^n X \arrow[r, hook] \arrow[d, "\theta_n"] & E_n(P\Omega^{n-1}X, \Omega^n X) \arrow[r, two heads] \arrow[d, "\bar{\theta}_n"] & C_{n-1}X \arrow[d, "\theta_{n-1}"] \\
\Omega^n X \arrow[r, hook] & P\Omega^{n-1}X \arrow[r, two heads, "p"] & \Omega^{n-1}X
\end{tikzcd}$$

Then, to prove Proposition 22.11, and hence conclude Theorem 22.10, it suffices to prove that the top right map of (5) is a quasifibration and $E_n(TX,X)$ is contractible.

The general strategy for the quasifibration statement is to use the following lemma of Dold and Thom:

Lemma 22.14. *Let $p : E \to B$ be a map onto a filtered space B. Say that a subset $U \subset B$ is distinguished if $p^{-1}(U) \to U$ is a quasifibration. Then, each F_jB is distinguished and p is a quasifibration if:*

1. *F_0B and every open subset of $F_jB - F_{j-1}B$ is distinguished.*
2. *For each $j > 0$, there is an open neighborhood $F_{j-1}B \subset U \subset F_jB$ together with a homotopies $h_t : U \times I \to U$ and $H_t : p^{-1}(U) \times I \to p^{-1}(U)$ satisfying:*

(a) *$h_0 = \mathrm{id}$, $h_t(F_{j-1}B) \subset F_{j-1}B$, and $h_1(U) \subset F_{j-1}B$; that is, h deformation retracts U into $F_{j-1}B$;*

(b) *$H_0 = \mathrm{id}$, and H covers h in the sense that $pH_t = h_tp$; and*

(c) *$H_1 : p^{-1}(x) \to p^{-1}(h_1(x))$ is a weak homotopy equivalence for all $x \in U$.*

A few pages of painstaking proof shows that the filtration

$$F_jE_n(TX,X) = \mathrm{im}\left(C_n(j) \times X^j \to C_nTX\right) \cap E_n(TX,X)$$

satisfies this property, proving the proposition, and hence Theorem 22.10.

The contents of Theorem 22.10 are that, up to weak equivalence, free algebras over C_n are the same as free algebras over $\Omega^n\Sigma^n$. Note that Ω^nY has the structure of an $\Omega^n\Sigma^n$-algebra via the counit:

$$\Omega^n\Sigma^n\Omega^nY \xrightarrow{\Omega^n\varepsilon_n} \Omega^nY$$

We can ask an analogous question for general algebras, and it turns out there's an affirmative answer:

Theorem 22.15 (The recognition principle)**.** *A space has the weak homotopy type of a loop space if and only if it can be endowed with the structure of a $\mathcal{C}_n$-algebra.*

In order to prove this, we'll need to construct an explicit delooping of a $\mathcal{C}_n$-algebra. Our trick will be to "simplicially resolve" the associated $\Omega^n\Sigma^n$-algebra by free $\Omega^n\Sigma^n$-algebras, and attempt to deloop that data. This is fleshed out in the *bar construction*, which we go into now.

22.4 The two-sided bar construction

We have a nice handle on free C_n-algebras and $\Omega^n\Sigma^n$ algebras. One fruitful idea is that we may *simplicially resolve* a monad algebra by free algebras; we now give some technology necessary in order to do so.

For the duration of this section, fix $\mathcal{V}$ a category. We've previously defined *algebras for a monad* as objects that a monad acts on from the left. We generalize this heavily:

Definition 22.16. Let (T, μ, η) be a monad on $\mathcal{V}$. Then, a *left T-functor* is a functor $E : \mathcal{U} \to \mathcal{V}$ together with a natural transformation $\xi : TE \implies E$ making the following commute:

$$\begin{tikzcd} E \arrow[r, "\eta"] \arrow[dr, equal] & TE \arrow[d, "\xi"] \\ & E \end{tikzcd} \qquad \begin{tikzcd} T^2E \arrow[r, "T\xi"] \arrow[d, "\mu E"'] & TE \arrow[d, "xi"] \\ TE \arrow[r, "\xi"] & E \end{tikzcd}$$

Dually, a *right T-functor* is a functor $F : \mathcal{V} \to \mathcal{W}$ with a right action $\lambda : FT \implies F$ satisfying the analogous unitality and associativity conditions.

Define the category $\mathcal{B}(\mathcal{U}, \mathcal{V}, \mathcal{W})$ to have objects given by triples $((F, \lambda), (T, \mu, \eta), (E, \xi))$, abbreviated as (F, T, E), where T is a monad on $\mathcal{V}$, $F : \mathcal{V} \to \mathcal{W}$ is a right T-functor, and $E : \mathcal{U} \to \mathcal{V}$ is a left T-functor. Morphisms in $\mathcal{B}(\mathcal{U}, \mathcal{V}, \mathcal{W})$ are compatible triples of morphisms.

This connects to our previous notion of algebras for a monad; a T-algebra is precisely a T-functor whose domain is $*$, the terminal category.[70] We use this to power the main construction of the recognition principle, called the *(two-sided monadic) simplicial bar construction.*

Definition 22.17. Let $(F, T, E) \in \mathcal{B}(\mathcal{U}, \mathcal{V}, \mathcal{W})$ be a monad together with right and left T-functors. Define the *simplicial bar construction* to be the simplicial functor $B_\bullet(F, T, E) \in s\,[\mathcal{U}, \mathcal{W}]$ given by

$$\begin{aligned} B_n(F,T,E) &= FT^nE \\ d_i &= \begin{cases} \lambda T^{n-1}E & i = 0 \\ FC^{i-1}\mu C^{n-i-1}E & 0 < i < n \\ FC^{n-1}\lambda & i = n \end{cases} \\ s_i &= FC^i\eta C^{n-i}E \end{aligned}$$

Note that when E is a T-algebra in particular, $B_\bullet(F, T, E) \in s\,[*, \mathcal{W}] = s\mathcal{W}$, so we get a simplicial object rather than a simplicial functor. We henceforth restrict to this case.

We don't always like working with simplicial objects; In the particular case that $\mathcal{U} = *$ and $\mathcal{W} = \mathbf{Top}_*$, we end up with a simplicial space, which we know how to turn into a space:

Definition 22.18. Let $(F, T, X) \in \mathcal{B}(*, \mathcal{V}, \mathbf{Top}_*)$. Then, define the *bar construction* to be the space

$$B(F, T, X) := |B_\bullet(F, T, X)|\,.$$

Examples of this include the bar complex in homological algebra (after applying Dold-Kan), the simplicial nerve and geometric realization of a category or topological monoid, and many other nice constructions.Elaborate

The composite of action maps yields an *augmentation* $\varepsilon : B_\bullet(F, T, X) \to X_\bullet$, where $X_\bullet$ is the simplicial object with $X_n = X$ and identity structure maps. When $F = T$, this behaves nicely:

Proposition 22.19. *There is a homotopy in $s\mathcal{W}$ exhibiting the augmentation map $\varepsilon : B_\bullet(T, T, X) \to X_\bullet$ as a strong deformation retract.*

This establishes that we've made some sort of free resolution, the n-simplices of $B_\bullet(T, T, X)$ are free T-algebras for each n. The following is clear, and useful for *delooping*.

[70]This category has a single object, with only the identity morphism. One has $[*, \mathcal{C}] \simeq \mathcal{C}$.

Proposition 22.20. *Suppose $L : \mathcal{V} \leftrightarrows \mathcal{W} : R$ is an adjoint pair, T a monad, and X a T-algebra. Then, there is a* de-*R*ing *isomorphism*

$$R_\bullet B_\bullet(L, T, X) = B_\bullet(RL, T, X).$$

In view of Propositions 22.19 and 22.20, given X a space with the weak homotopy type of a C_n algebra, we may replace X with $B_\bullet(C_n, C_n, X)$, then $B_\bullet(\Omega^n\Sigma^n, \Omega^n\Sigma^n X)$, then $\Omega^n_\bullet B_\bullet(\Sigma^n, \Omega^n\Sigma^n, X)$, each time up to pointwise weak equivalence of simplicial spaces. Translating this simplicial delooping into one of spaces takes a good amount of technical work, but it is the main idea of the recognition principle.

22.5 The recognition principle

Theorem 22.21 (The connected C_n, $n < \infty$ recognition principle)**.** *Let X be a C_n-algebra. The diagram*

$$\begin{tikzcd} B(C_n, C_n, X) \arrow[r, "\alpha"] \arrow[d, "\varepsilon"] & B(\Omega^n\Sigma^n, C_n, X) \arrow[d, "\gamma"] \\ X \arrow[r, dashed] & \Omega^n B(\Sigma^n, C_n, X) \end{tikzcd}$$

satisfying the following properties:

1. *ε is a strong deformation retract with right inverse induced by the unit of C_n.*
2. *$B(\alpha_n, 1, 1)$ is a weak homotopy equivalence if X is connected.*
3. *γ is a weak homotopy equivalence for all X.*
4. *The composite $X \to \Omega^n B(\Sigma^n, C_n, X)$ is adjoint to the map $\Sigma^n X \to B(\Sigma^n, C_n, X)$ induced by the unit of C_n.*[71]
5. *The space $B(\Sigma^n, D, X)$ is $(m+n)$-connected if X is m-connected.*

In particular, when X is connected, the map $X \to \Omega^n B(\Sigma^n, C_n, X)$ is a weak homotopy equivalence, and hence X has the weak homotopy type of an n-fold loop space.

Furthermore, when Y is n-connected, the map $B(\Sigma^n, C_n, \Omega^n Y) \to Y$ induced by counits is a weak homotopy equivalence, so the choice of $B(\Sigma^n, C_n, X)$ as an $(m+n)$-connected delooping of X is unique up to weak homotopy equivalence.

The major piece missing for the proof of this theorem is some technical fudging with simplicial spaces. There exists a definition of a *simply proper* simplicial space which I shouldn't define .

Lemma 22.22. *Let $f : X \to Y$ be a morphism of simply proper simplicial spaces such that each f_q is a weak homotopy equivalence and either $|X|$ and $|Y|$ are simply connected or $|f|$ is an H-map between connected H-spaces. Then, $|f|$ is a weak homotopy equivalence.*

2. *Let $\mathcal{C}$ be an operad on $\mathbf{Top}_*$ with associated monad C. Then, there is a natural homeomorphism $\nu : |C_* X| \xrightarrow{\sim} C\,|X|$ for all $X \in s\mathbf{Top}_*$, compatible with the simplicial and topological unit and multiplication.*
3. *Realization commutes with suspension, i.e. $|\Sigma_\bullet X| = \Sigma\,|X|$.*
4. *For all $X \in s\mathbf{Top}_*$, the space $|P_\bullet X|$ is contractible, and there are maps making the following diagram commute:*

$$\begin{tikzcd} {|\Omega_\bullet X|} \arrow[r, hook] \arrow[d] & {|P_\bullet X|} \arrow[r] \arrow[d] & {|X|} \arrow[d, equal] \\ \Omega\,|X| \arrow[r, hook] & P\,|X| \arrow[r] & X \end{tikzcd}$$

where the bottom row is the path space fibration. Moreover, if X is proper and each X_q is connected, then the top row is a quasifibration sequence, and hence γ is a weak homotopy equivalence.

We can strengthen this in the $n < \infty$ case, by defining a class of operads which are suitably equivalent to $\mathcal{C}_n$.

[71] This composition begins in diagrammatic order by the right inverse of ε, which is induced by the unit of C_n.

Definition 22.23. An operad is Σ-*free* if S_j acts freely on $\mathcal{C}(j)$ for all j. A morphism of operads $\mathcal{O} \to \mathcal{O}'$ of operads is a *local equivalence* if it induces weak equivalenes $\mathcal{O}(j) \xrightarrow{\sim} \mathcal{O}'(j)$ for each j, and a *local* Σ-*equivalence* if it is a local equivalence which is S_j-equivariant on the jth degree.

Fix the operads $\mathcal{M} = \text{Assoc}$ and $\mathcal{N} = \text{Comm}$ on $\mathbf{Top}_*$. The overcategory $\text{Operads}(\mathbf{Top}_*)_{/\mathcal{M}}$ is called the *operads over* $\mathcal{M}$, and similar for $\mathcal{N}$. We have the following proposition, which uses the H-space structure and the (nontrivial!) fact that the maps induce isos in integral homology:

Proposition 22.24. *Let $\psi : \mathcal{C} \to \mathcal{C}'$ be a morphism of operads over $\mathcal{M}$ or $\mathcal{N}$, and assume that either ψ is a local Σ-equivalence of ψ is a local equivalence and $\mathcal{C}$ or $\mathcal{C}'$ are Σ-free. Then, the associated maps $\psi : CX \to C'X$ are weak homotopy equivalences for al connected spaces X.*

With this, the results of Theorem 22.21 can be stated for D a Σ-free operad locally equivalent to C_n. Say a few words about A_∞ operads

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