Homotopy-coherent interchange and equivariant little disk operads

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INI Equivariant Homotopy Theory in Context

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A question you might happen upon

In trace methods for Real algebraic K-theory, THH has a Real analogue:[1]

$$\begin{tikzcd} \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) \arrow[d, "\mathrm{THH}"'] \arrow[rr, rightsquigarrow, "\text{“Reality”}"', phantom] & & \mathrm{Alg}_{\mathbb{E}_\sigma}(\mathrm{Sp}_{C_2}) \arrow[d, "\mathrm{THR}"] \\ \mathrm{Sp} \arrow[rr, rightsquigarrow, "\text{“Reality”}", phantom] & & \mathrm{Sp}_{C_2} \end{tikzcd}$$

In this, σ is the sign representation and $\mathbb{E}_\sigma$-algebras are a genuine-equivariant version of rings with anti-involution.

Question (c.f. Dotto-Moi-Patchkoria-Reeh[2] '17)

What algebraic structure does THR of highly structured C_2-ring spectra have?

[1] Emanuele Dotto. *Stable real K-theory and real topological Hochschild homology*. Thesis (Ph.D.)–Uniiversity of Copenhagen. 2012. arXiv: 1212.4310 [math.AT].

[2] Emanuele Dotto, Kristian Moi, Irakli Patchkoria, and Sune Precht Reeh. "Real topological Hochschild homology". In: *J. Eur. Math. Soc. (JEMS)* 23.1 (2021), pp. 63–152.

How to construct structure on THH

Observation

THH can be given a symmetric monoidal structure, so we may lift

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathrm{Sp}) \arrow[drr] & \simeq & \mathrm{Alg}_{\mathcal{O}}\mathrm{Alg}^{\otimes}_{\mathbb{E}_1}(\mathrm{Sp}) \arrow[r, dashed] \arrow[d, "U"'] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) \arrow[d, "U"'] \\
& & \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) \arrow[r, "\mathrm{THH}"] & \mathrm{Sp}
\end{tikzcd}$$

Theorem (Dunn[3] '88, Lurie[4] '09)

$\mathbb{E}_n \simeq \mathbb{E}_{n-1} \otimes \mathbb{E}_1$; hence THH takes $\mathbb{E}_n$-rings to $\mathbb{E}_{n-1}$-rings.

[3] Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258.

[4] Jacob Lurie. *Derived Algebraic Geometry VI: Ek Algebras*. 2009. arXiv: 0911.0018 [math.AT].

How to construct structure on THR

Observation (S.[5])

THR can be given a C_2-symmetric monoidal structure, so we may lift

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_\sigma}(\mathrm{Sp}_{C_2}) \arrow[r, phantom, "\simeq"] \arrow[drr] & \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_\sigma}(\mathrm{Sp}_{C_2}) \arrow[rr, dashed] \arrow[d, "U"'] & & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}_{C_2}) \arrow[d, "U"'] \\
& & \mathrm{Alg}_{\mathbb{E}_\sigma}(\mathrm{Sp}_{C_2}) \arrow[r, "\mathrm{THR}"] & \mathrm{Sp}_{C_2}
\end{tikzcd}$$

Conjecture

$\mathbb{E}_V \otimes \mathbb{E}_W \simeq \mathbb{E}_{V\oplus W}$

[5] Natalie Stewart. *On tensor products of equivariant commutative operads (draft)*. 2025.

Statement of the additivity theorem

For this talk, all terms are defined ∞-categorically.

Theorem (S.[5])

Given V, W orthogonal G-representations, we have

$$\mathbb{E}_V \otimes \mathbb{E}_W \simeq \mathbb{E}_{V \oplus W};$$

hence there is an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_V} \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_W}(\mathrm{Sp}_G) \simeq \mathrm{Alg}_{\mathbb{E}_{V \oplus W}}(\mathrm{Sp}_G).$$

Corollary

THR of $\mathbb{E}_{V \oplus \sigma}$-rings has a natural $\mathbb{E}_V$-ring structure.

[5] Natalie Stewart. *On homotopical additivity of equivariant little disks operads (forthcoming)*. 2025.

Myriad versions of this theorem

0 *May '72:*[6] $C_n \otimes C_m$ and C_{n+m} agree on **connected spaces**.

1 *Dunn '88:*[7] $C_1^{\otimes n} \simeq C_n$ w.r.t. a **point-set** tensor product.

2 *Brinkmeier '00:*[8] $C_n \otimes C_m \simeq C_{n+m}$ w.r.t. a **point-set** tensor product.

3 *Rourke-Sanderson '00:*[9] $D_V \otimes D_W$ and $D_{V \oplus W}$ agree on **G-connected G-spaces**.

4 *Lurie '09:*[10] $\mathbb{E}_n^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_m^{\otimes} \simeq \mathbb{E}_{n+m}^{\otimes}$ with respect to a **homotopical** tensor product.

5 *Fiedorowicz-Vogt '15:*[11] Dunn & Brinkmeier's result extends to **cofibrant $\mathbb{E}_n$-operads**.

6 *Szczesny '24:*[12] $D_V \otimes D_W \simeq D_{V \oplus W}$ w.r.t. a **point-set** tensor product.

7 *S. '25:* $\mathbb{E}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} \simeq \mathbb{E}_{V \oplus W}^{\otimes}$ with respect to a **homotopical** tensor product.

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[6] J. P. May. *The geometry of iterated loop spaces*. Vol. Vol. 271. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1972, pp. viii+175.

[7] Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258.

[8] Michael Brinkmeier. *On Operads*. Thesis (Ph.D.)–Universitat Osnabrück. 2000.

[9] Colin Rourke and Brian Sanderson. "Equivariant configuration spaces". In: *J. London Math. Soc. (2)* 62.2 (2000), pp. 544–552.

[10] Jacob Lurie. *Derived Algebraic Geometry VI: Ek Algebras*. 2009. arXiv: 0911.0018 [math.AT].

[11] Z. Fiedorowicz and R. M. Vogt. "An additivity theorem for the interchange of E_n structures". In: *Adv. Math.* 273 (2015), pp. 421–484.

[12] Ben Szczesny. *Equivariant Framed Little Disk Operads are Additive*. 2024. arXiv: 2410.20235 [math.AT].

A heavily abridged history of G-∞-categorical operads

0. *Hill-Hopkins '16:*[13] G-commutative monoids are **semi-Mackey functors**.
1. *Nardin-Shah '22:*[14] G-operads are a type of fibration over a G-categorical version $\underline{\mathbb{F}}_{G,*}$ of Segal's category Γ^{op}.
2. *Barkan-Haugseng-Steinebrunner '22:*[15] G-operads are also **fibrations over the effective Burnside 2-category $\mathrm{Span}(\mathbb{F}_G)$.**
3. *S. '25:*[16] homotopy-coherent interchange is corepresented by **BV tensor products** and G-operads are monadic over G-symmetric sequences.
4. *S. '25:*[17] Algebras in **(co)cartesian G-symmetric monoidal structures** have concrete descriptions and $\mathcal{N}_{I\infty} \otimes \mathcal{N}_{J\infty} \simeq \mathcal{N}_{I \vee J\infty}$

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[13] Michael A. Hill and Michael J. Hopkins. *Equivariant symmetric monoidal structures*. 2016. arXiv: 1610.03114 [math.AT].

[14] Denis Nardin and Jay Shah. *Parametrized and equivariant higher algebra*. 2022. arXiv: 2203.00072 [math.AT].

[15] Shaul Barkan, Rune Haugseng, and Jan Steinebrunner. *Envelopes for Algebraic Patterns*. 2022. arXiv: 2208.07183 [math.CT].

[16] Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT].

[17] Natalie Stewart. *On tensor products of equivariant commutative operads (draft)*. 2025.

How to prove equivariant Dunn additivity

$$
\begin{tikzcd}
\mathbb{P}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{P}_W^{\otimes} \arrow[d, "\alpha_V \otimes \alpha_W"'] & \mathbb{P}_V^{\otimes} \times \mathbb{P}_W^{\otimes} \arrow[l, "L_{\mathbb{P}}"'] \arrow[r, "M_{\mathbb{P}}"] \arrow[d, "\alpha_V \times \alpha_W"'] & \mathbb{P}_V^{\otimes} \wr \mathbb{P}_W^{\otimes} \arrow[r, "\widetilde{\varphi}_{\mathbb{P}}"] \arrow[d, "\alpha_V \wr \alpha_W"'] & \mathbb{P}_{V|W}^{\otimes} \arrow[r, hook] \arrow[d, "\alpha_{V|W}"'] & \mathbb{P}_{V \oplus W}^{\otimes} \arrow[d, "\alpha_{V \oplus W}"] \\
\mathbb{E}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} & \mathbb{E}_V^{\otimes} \times \mathbb{E}_W^{\otimes} \arrow[l, "L_{\mathbb{E}}"] \arrow[r, "M_{\mathbb{E}}"'] & \mathbb{E}_V^{\otimes} \wr \mathbb{E}_W^{\otimes} \arrow[r, "\varphi_{\mathbb{E}}"'] & \mathbb{E}_{V \oplus W}^{\otimes} \arrow[r, equal] & \mathbb{E}_{V \oplus W}^{\otimes}
\end{tikzcd}
$$

1. Define “G-operads” as a localizing subcategory of “G-preoperads.”
2. Define G-preoperad models $\times$ and $\wr$ for $\overset{\mathrm{BV}}{\otimes}$, reduce to showing that $\varphi_{\mathbb{E}}$ is an L_{Op_G}-equivalence.
3. Define “approximations” α, an approximated “surjection to image” $\widetilde{\varphi}_{\mathbb{P}}$ onto “decomposable little disks,” verify that $\alpha_{V|W}$ is an approximation by lifting Dunn’s argument about decomposability of little disks.
4. Use ∞-category theory to reduce to showing $\widetilde{\varphi}_{\mathbb{P}}$ is an L_{Op_G}-equivalence.
5. Verify that $\widetilde{\varphi}_{\mathbb{P}}$ by lifting Dunn’s argument about uniqueness of decompositions.

A quasi-definition of G-operads

Definition

- A **G-preoperad** is a functor $\mathcal{O}^{\otimes} \to \mathrm{Span}(\mathbb{F}_G)$ with cocartesian lifts over backwards maps.
- A **G-operad** is required to satisfy "Segal conditions."
- An **$\mathcal{O}$-algebra in $\mathcal{C}^{\otimes}$** is a functor preserving cocartesian arrows :

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "\varphi"] \arrow[dr] & & \mathcal{C}^{\otimes} \arrow[dl] \\ & \mathrm{Span}(\mathbb{F}_G) & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Op}_G \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathrm{POp}_G \arrow[l, bend right, "L_{\mathrm{Op}_G}"'] \end{tikzcd} = \mathrm{Cat}^{\mathrm{Backwards-cocart}}_{\infty,/\ \mathrm{Span}(\mathbb{F}_G)}$$

Sidebar: S-ary spaces in Mike's G-operads

When considering the *homotopy theory* of Mike's G-operads, we usually define *graph subgroups*: given $H \subset G$ and $S \in \mathbb{F}_H$ define

$$\Gamma_S = \left\{(h, \rho_S(h)) \mid h \in H, \quad \rho_S \colon H \to \Sigma_{|S|}\right\} \subset G \times \Sigma_S.$$

Definition

The *S-ary structure space of $\mathcal{O}$* is the fixed points

$$\mathcal{O}(S) = \mathcal{O}(|S|)^{\Gamma_S}.$$

A map of Mike's G-operads $\varphi \colon \mathcal{O} \to \mathcal{P}$ is a *weak equivalence* if $\mathcal{O}(S) \to \mathcal{P}(S)$ is an equivalence for all $S \in \mathbb{F}_H$.

Our theory of G-operads should factor through this...

The underlying G-symmetric sequence

Construction

Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_G$, $H \subset G$, and $S \in \mathbb{F}_G$, define the **structure space**

$$\begin{tikzcd} \mathcal{O}(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mor}\left(\mathcal{O}^{\otimes}\right) \arrow[d] \\ \left\{\mathrm{Ind}_H^G S = \mathrm{Ind}_H^G S \rightarrow [G/H]\right\} \arrow[r] & \mathrm{Mor}\left(\mathrm{Span}(\mathbb{F}_G)\right) \end{tikzcd}$$

Proposition (S.[18])

If $\mathcal{O}^{\otimes}$ has “one color” then is it conservatively identified by $\left(\mathcal{O}(S)\right)_{\substack{H \subset G \\ S \in \mathbb{F}_H}}$.

[18] Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: `2501.02129 [math.CT]`.

Ur-examples

Example

The **little V-disks G-operad** has S-ary structure space given by **H-equivariant configurations of S in V**:

$$\mathbb{E}_V(S) := \mathrm{Emb}^{H,\mathrm{Affine}}(S \cdot D(V), D(V)) \simeq \mathrm{Conf}_S^H(V).$$

Example

Given a G-symmetric monoidal category $\mathcal{C}^{\otimes}\colon \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$, its unstraightening is a G-operad. Given $X \in \mathcal{C}(G/G)$, there is an **endomorphism G-operad** with

$$\mathrm{End}_X(S) \simeq \mathrm{Map}_{\mathcal{C}(G/H)}\left(\left(\mathrm{Res}_H^G X\right)^{\otimes S}, \mathrm{Res}_H^G X\right)$$

Ur-examples

In particular, an $\mathbb{E}_V$-algebra in $\mathcal{C}$ consists of an object $X \in \mathcal{C}_G$ and homotopy-coherently compatible maps

$$\mathrm{Conf}_S^H(V) \to \mathrm{Map}_{\mathcal{C}(G/H)}\left(\left(\mathrm{Res}_H^G X\right)^{\otimes S}, \mathrm{Res}_H^G X\right).$$

Example (Horev-Klang-Zou[19] '20)

Let $\underline{\mathcal{S}}_G^{G-\times}$ be the *cartesian structure* on G-spaces. Then, for all $X \in \mathcal{S}_G$, we have $\Omega^V X \in \mathrm{Alg}_{\mathbb{E}_V}(\mathcal{S}_G)$.

Example (Horev-Klang-Zou '20, loc. cit.)

Let $\underline{\mathrm{Sp}}_G^{\otimes}$ be the HHR G-symmetric monoidal structure. If $f\colon \Omega^V X \to \underline{\mathrm{Pic}}(\underline{\mathrm{Sp}}_G)$ is a **V-loop map**, then $\mathrm{Th}(f) \in \mathrm{Alg}_{\mathbb{E}_V}(\mathrm{Sp}_G)$.

[19] Jeremy Hahn, Asaf Horev, Inbar Klang, Dylan Wilson, and Foling Zou. *Equivariant nonabelian Poincaré duality and equivariant factorization homology of Thom spectra*. 2024. arXiv: 2006.13348 [math.AT].

Modelling G-operadic constructions, two ways

Model categorist: **(co)fibrantly replace**, then apply a construction to monoids in G-symmetric sequences.

∞-categorist: apply a G-preoperadic construction, then L_{Op_G}**-localize**.

Shared goal: model corepresenting object for **pairings** (aka interchanging algebras, bifunctors, etc.) akin to May.[20]

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \arrow[r, "\text{``pairing''}"] \arrow[d, "\pi"] & \mathcal{Q}^{\otimes} \arrow[d, "\pi"] \\
\mathrm{Span}(\mathbb{F}_G) \times \mathrm{Span}(\mathbb{F}_G) \arrow[r, "\wedge"] & \mathrm{Span}(\mathbb{F}_G)
\end{tikzcd}$$

Today, we are ∞-categorists.

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[20] J. P. May. "Pairings of categories and spectra". In: *J. Pure Appl. Algebra* 19 (1980), pp. 299–346.

The Boardman-Vogt tensor product

Definition

$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} := L_{\mathrm{Op}_G}\left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \longrightarrow \mathrm{Span}(\mathbb{F}_G)^2 \xrightarrow{\wedge} \mathrm{Span}(\mathbb{F}_G)\right)$$

$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ lifts to a "pointwise" G-symmetric monoidal category $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$.

Proposition (S.[21])

$(-) \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes}$ is left adjoint to $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(-)$, so

$$\mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\mathcal{C}).$$

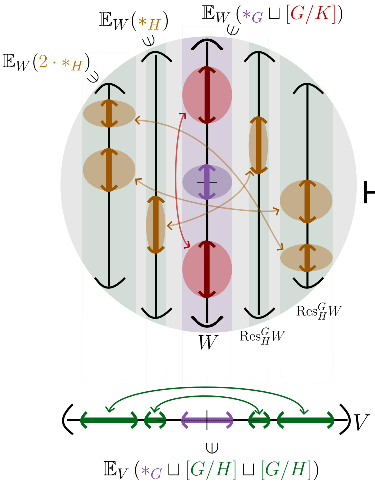
Also, have "wreath" operator $\wr$ and natural L_{Op_G}-equivalences

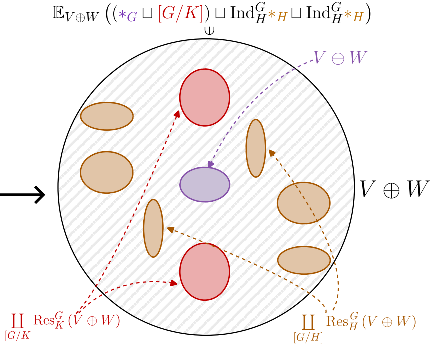
$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \leftarrow \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \rightarrow \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}.$$

[21] Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT].

The Dunn map

$$\mathbb{E}_V^{\otimes} \wr \mathbb{E}_W^{\otimes} \xrightarrow{\varphi_{\mathbb{E}}} \mathbb{E}_{V \oplus W}^{\otimes}$$





$$\mathrm{Ar}\left(\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}\right) \times_{\mathrm{ArSpan}(\mathbb{F}_G))} \left\{\mathrm{Ind}_H^G T = \mathrm{Ind}_H^G T \rightarrow [G/H]\right\}$$
$$\wr\|$$
$$\coprod_{T \to S} \mathcal{O}(S) \times \prod_{[H/K] \subset S} \mathcal{P}\left(T \times_S [H/K]\right)$$

Sidebar on G-∞-categories

Need theory of "approximations." We use *equivariant higher category theory*.

Definition

A *G-category* is a coefficient system of ∞-categories $\mathcal{C} \colon \mathcal{O}_G^{\mathrm{op}} \to \mathrm{Cat}$.

Example

The terminal G-category has $*_G(G/H) = *$.

Definition

A *G-functor* $\mathcal{C} \to \mathcal{D}$ is a natural transformation $F \colon \mathcal{C} \implies \mathcal{D}$. A *$G$-object* is a G-functor $X \colon *_G \to \mathcal{C}$.

The corresponding theory of **G-(co)limits, G-adjunctions** formalizes "H-set indexed (co)products." Moral for approximations: **model (reflective) localizations of the underlying G-category**.

Preoperad models for 1-colored G-operads

Given $\mathcal{P}^{\otimes} \in \mathrm{POp}_G$, we let $\mathcal{P}^{\mathrm{act}}_{/P}$ be the ∞-category of arrows $P' \to P$ projecting to a forward map $T = T \to S$.

Definition

A map of G-preoperads $\alpha : \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ with $\mathcal{O}^{\otimes}$ a "one color" G-operad is a **weak approximation** if

1. The G-category of colors $U\mathcal{P}$ **has a terminal G-object**, and
2. For all $P \in \mathcal{P}^{\otimes}$ and $T \to \pi(P)$, **the map of spaces**

$$B\left(\mathcal{P}^{\mathrm{act}}_{/P} \times_{\mathbb{F}_{G,/\pi(P)}} \{T \to \pi(P)\}\right) \to \prod_{[H/K] \subset \pi(P)} \mathcal{O}\left(T \times_{\pi(P)} [H/K]\right)$$

is a weak equivalence.

Preoperad models for 1-colored G-operads

Proposition (Harpaz[22] '19 + reinterpretation)

If $\alpha : \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a weak approximation, pullback is fully faithful

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{S}_G) \subset \mathrm{Alg}_{\mathcal{P}}(\mathcal{S}_G)$$

with image the $\mathcal{P}$-monoids whose "color" G-functors $U\mathcal{P} \to \underline{\mathcal{S}}_G$ are constant.

Weak approximations can be made to have many colors; a weak approximation α is a **strong approximation** if $U\mathcal{P} \to U\mathcal{O}$ is an equivalence.

Proposition (S.[23])

$\mathrm{Alg}_{(-)}(\mathcal{S}_G)$ detects L_{Op_G}-equivalences when $U\mathcal{P} \to U\mathcal{O}$ is an equivalence; in particular, **strong approximations are L_{Op_G}-equivalences**.

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[22] Yonatan Harpaz. *Little cubes algebras and factorization homology (course notes)*.

[23] Natalie Stewart. *On tensor products of equivariant commutative operads (draft)*. 2025.

The L_{Op_G}-conservativity argument, in short

Proof sketch.

Examine the free $\mathcal{O}$-G-space monad:[24]

$$(T_{\mathcal{O}}X)^H \simeq \coprod_{S \in \mathbb{F}_H} \left(\mathcal{O}(S) \times \left(X^S\right)^H\right)_{h\,\mathrm{Aut}_H S}.$$

The inclusion $\mathrm{Aut}_H(S) \subset \mathrm{End}_H(S) = \left(S^S\right)^H$ yields natural splitting

$$(T_{\mathcal{O}}S)^H \simeq \mathcal{O}(S) \sqcup \mathbf{Junk}.$$

Use monadicity of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{S}_G) \to \mathcal{S}_G$ and conservativity of $\left(\mathcal{O}(S)\right)_{\substack{H \subset G \\ S \in \mathbb{F}_H}}$. □

[24] Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT].

Approximating manifolds

Proposition (Dugger-Isaksen[25] '01)

If X is a topological space and $\mathfrak{O} \subset \mathscr{P}(X)$ a basis of contractible open subsets, then we get a weak equivalence

$$B\mathfrak{O} \xrightarrow{\sim} X$$

Corollary

Let $\mathfrak{O}_S^H(V) \subset \mathrm{Conf}_S^H(V)$ be the basis of configurations in affinely $\coprod_S D(V)$-shaped invariant subspaces of $D(V)$.We get a weak equivalence

$$B\mathfrak{O}_S^H(V) \xrightarrow{\sim} \mathrm{Conf}_S^H(V) \simeq \mathbb{E}_V(S).$$

[25] Daniel Dugger and Daniel C. Isaksen. "Topological hypercovers and A^1-realizations". In: *Math. Z.* 246.4 (2004), pp. 667–689.

$$\begin{tikzcd}
\mathbb{P}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{P}_W^{\otimes} \arrow[d, "\alpha_V \otimes \alpha_W"'] & \mathbb{P}_V^{\otimes} \times \mathbb{P}_W^{\otimes} \arrow[l, "L_{\mathbb{P}}"'] \arrow[r, "M_{\mathbb{P}}"] \arrow[d, "\alpha_V \times \alpha_W"'] & \mathbb{P}_V^{\otimes} \wr \mathbb{P}_W^{\otimes} \arrow[r, "\widetilde{\varphi}_{\mathbb{P}}"] \arrow[d, "\alpha_V \wr \alpha_W"'] & \mathbb{P}_{V|W}^{\otimes} \arrow[r, hook] \arrow[d, "\alpha_{V|W}"'] & \mathbb{P}_{V\oplus W}^{\otimes} \arrow[d, "\alpha_{V\oplus W}"] \\
\mathbb{E}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} & \mathbb{E}_V^{\otimes} \times \mathbb{E}_W^{\otimes} \arrow[l, "L_{\mathbb{E}}"] \arrow[r, "M_{\mathbb{E}}"'] & \mathbb{E}_V^{\otimes} \wr \mathbb{E}_W^{\otimes} \arrow[r, "\varphi_{\mathbb{E}}"'] & \mathbb{E}_{V\oplus W}^{\otimes} \arrow[r, equal] & \mathbb{E}_{V\oplus W}^{\otimes}
\end{tikzcd}$$

We define a G-preoperad $\mathbb{P}_V^{\otimes}$ such that $\mathbb{P}_{V,/P}^{\mathrm{act}} \simeq \mathfrak{D}_S^H(V)$, yielding a weak approximation $\alpha_V \colon \mathbb{P}_V^{\otimes} \to \mathbb{E}_V^{\otimes}$. Then, we define a $\mathbb{P}$-Dunn map fitting into the above diagram.

Here, $\mathbb{P}_{V|W}^{\otimes}$ is the "G-preoperadic image, i.e. "decomposed little disks."

We're tasked with verifying that $\varphi_{\mathbb{E}} \circ M_{\mathbb{E}}$ induces an equivalence

$$\mathbf{Alg}_{\mathbb{E}_{V\oplus W}}(\mathcal{S}_G) \xrightarrow{\sim} \mathbf{Alg}_{\mathbb{E}_V \times \mathbf{Alg}_{\mathbb{E}_W}}(\mathcal{S}_G) \xleftarrow{\sim} \mathbf{Alg}_{\mathbb{E}_V \otimes \mathbb{E}_W}(\mathcal{S}_G)$$

$$
\begin{tikzcd}
\mathbb{P}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{P}_W^{\otimes} \arrow[d, "\alpha_V \otimes \alpha_W"'] & \mathbb{P}_V^{\otimes} \times \mathbb{P}_W^{\otimes} \arrow[l, "L_{\mathbb{P}}"'] \arrow[r, "M_{\mathbb{P}}"] \arrow[d, "\alpha_V \times \alpha_W"'] & \mathbb{P}_V^{\otimes} \wr \mathbb{P}_W^{\otimes} \arrow[r, "\widetilde{\varphi}_{\mathbb{P}}"] \arrow[d, "\alpha_V \wr \alpha_W"'] & \mathbb{P}_{V|W}^{\otimes} \arrow[r, hook] \arrow[d, "\alpha_{V|W}"'] & \mathbb{P}_{V\oplus W}^{\otimes} \arrow[d, "\alpha_{V\oplus W}"] \\
\mathbb{E}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} & \mathbb{E}_V^{\otimes} \times \mathbb{E}_W^{\otimes} \arrow[l, "L_{\mathbb{E}}"] \arrow[r, "M_{\mathbb{E}}"'] & \mathbb{E}_V^{\otimes} \wr \mathbb{E}_W^{\otimes} \arrow[r, "\varphi_{\mathbb{E}}"'] & \mathbb{E}_{V\oplus W}^{\otimes} \arrow[r, equal] & \mathbb{E}_{V\oplus W}^{\otimes}
\end{tikzcd}
$$

1. $L_{\mathbb{P}}$ and $L_{\mathbb{E}}$ are $L_{\mathcal{O}\mathrm{p}_G}$-equivalences by fiat.
2. A variant of Harpaz's strategy[26] shows that $M_{\mathbb{P}}$ and $M_{\mathbb{E}}$ are $L_{\mathcal{O}\mathrm{p}_G}$-equivalences.
3. Simple ∞-category theory shows that $\alpha_V \times \alpha_W$ is a weak approximation.
4. A variant of Dunn's strategy[27] shows that $\alpha_{V|W}$ is a weak approximation.
5. Explicit 1-category theory shows that $\widetilde{\varphi}_{\mathbb{P}}$ is a strong approximation.
6. Routine bookkeeping then shows that $\mathbf{Alg}_{\mathbb{E}_{V\oplus W}}(\mathcal{S}_G) \xrightarrow{\sim} \mathbf{Alg}_{\mathbb{E}_V \times \mathbb{E}_W}(\mathcal{S}_G)$.

[26] Yonatan Harpaz. *Little cubes algebras and factorization homology (course notes)*.

[27] Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258.

Stalkwise-linearizable tangential structures

Define $\mathbb{E}^{\otimes}_{B^{\mathrm{lin}}_G \mathrm{Top}(n)}$ to have colors the linearizable G-actions on $\mathbb{R}^n$ and operations the *topological* embeddings.

Given a G-space X with stalkwise-linearizable equivariant $\mathbb{R}^n$-bundle $T_\bullet \colon X \to B^{\mathrm{lin}}_G \mathrm{Top}(n)$, we define the assembly

$$\mathbb{E}^{\otimes}_X := L_{\mathrm{Op}_G}\left(\mathbb{E}^{\otimes}_{B^{\mathrm{lin}}_G \mathrm{Top}(n)} \times_{B^{\mathrm{lin}}_G \mathrm{Top}(n)^{G-\sqcup}} X^{G-\sqcup}\right).$$

Theorem (S.[28])

There is a natural equivariant colimit expression of operads

$$\underline{\mathrm{colim}}_{x \in X} \mathbb{E}^{\otimes}_{T_x} \xrightarrow{\sim} \mathbb{E}^{\otimes}_X$$

[28] Natalie Stewart. *On homotopical additivity of equivariant little disks operads (forthcoming)*. 2025.

Stalkwise-linearizable tangential structures

Corollary (S.)

Given stalkwise-linearizable equivariant $\mathbb{R}^k$-bundles $X \to B_G^{\mathrm{lin}}\mathrm{Top}(n)$ and $Y \to B_G^{\mathrm{lin}}\mathrm{Top}(m)$, fixing the direct sum structure

$$X \times Y \longrightarrow B_G^{\mathrm{lin}}\mathrm{Top}(n) \times B_G^{\mathrm{lin}}\mathrm{Top}(m) \xrightarrow{\oplus} B_G^{\mathrm{lin}}\mathrm{Top}(n+m),$$

there is an equivalence.

$$\mathbb{E}_X^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_Y^{\otimes} \xrightarrow{\sim} \mathbb{E}_{X \times Y}^{\otimes}.$$

Other cases that are covered by the original argument:

- (Equivariant) swiss cheese
- Stratified little disks

A curious corollary about algebraic topology

Let $\underline{\mathrm{Disk}}_G^{X-\mathrm{fr},\sqcup} \subset \underline{\mathrm{Mfld}}_G^{X-\mathrm{fr},\sqcup}$ be the G-symmetric monoidal full category whose objects are disjoint unions of X-framed G-disks and whose mapping spaces are X-framed equivariant disk embeddings.

Corollary (S., c.f. Dwyer-Hess-Knudsen[29] '19)

There is a G-symmetric monoidal equivalence

$$\underline{\mathrm{Disk}}_G^{X-\mathrm{fr},\sqcup} \square \underline{\mathrm{Disk}}_G^{Y-\mathrm{fr},\sqcup} \simeq \underline{\mathrm{Disk}}_G^{X\times Y-\mathrm{fr},\sqcup}.$$

Here, $\square$ is the box product of semi-Mackey functors valued in Cat_∞.

[29] William Dwyer, Kathryn Hess, and Ben Knudsen. "Configuration spaces of products". In: *Trans. Amer. Math. Soc.* 371.4 (2019), pp. 2963–2985.

References

[1] Shaul Barkan, Rune Haugseng, and Jan Steinebrunner. *Envelopes for Algebraic Patterns*. 2022. arXiv: 2208.07183 [math.CT].

[2] Michael Brinkmeier. *On Operads*. Thesis (Ph.D.)–Universitat Osnabrück. 2000.

[3] Emanuele Dotto. *Stable real K-theory and real topological Hochschild homology*. Thesis (Ph.D.)–Uniiversity of Copenhagen. 2012. arXiv: 1212.4310 [math.AT].

[4] Emanuele Dotto, Kristian Moi, Irakli Patchkoria, and Sune Precht Reeh. "Real topological Hochschild homology". In: *J. Eur. Math. Soc. (JEMS)* 23.1 (2021), pp. 63–152.

[5] Daniel Dugger. "An Atiyah-Hirzebruch spectral sequence for *KR*-theory". In: *K-Theory* 35.3-4 (2005), 213–256 (2006).

[6] Daniel Dugger and Daniel C. Isaksen. "Topological hypercovers and A^1-realizations". In: *Math. Z.* 246.4 (2004), pp. 667–689.

[7] Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258.

[8] William Dwyer, Kathryn Hess, and Ben Knudsen. "Configuration spaces of products". In: *Trans. Amer. Math. Soc.* 371.4 (2019), pp. 2963–2985.

[9] Z. Fiedorowicz and R. M. Vogt. "An additivity theorem for the interchange of E_n structures". In: *Adv. Math.* 273 (2015), pp. 421–484.

[10] Jeremy Hahn, Asaf Horev, Inbar Klang, Dylan Wilson, and Foling Zou. *Equivariant nonabelian Poincaré duality and equivariant factorization homology of Thom spectra*. 2024. arXiv: 2006.13348 [math.AT].

[11] Yonatan Harpaz. *Little cubes algebras and factorization homology (course notes)*.

[12] Michael A. Hill and Michael J. Hopkins. *Equivariant symmetric monoidal structures*. 2016. arXiv: 1610.03114 [math.AT].

[13] Jacob Lurie. *Derived Algebraic Geometry VI: Ek Algebras*. 2009. arXiv: 0911.0018 [math.AT].

[14] J. P. May. *The geometry of iterated loop spaces*. Vol. Vol. 271. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1972, pp. viii+175.

[15] J. P. May. "Pairings of categories and spectra". In: *J. Pure Appl. Algebra* 19 (1980), pp. 299–346.

[16] Denis Nardin and Jay Shah. *Parametrized and equivariant higher algebra*. 2022. arXiv: 2203.00072 [math.AT].

[17] Colin Rourke and Brian Sanderson. "Equivariant configuration spaces". In: *J. London Math. Soc. (2)* 62.2 (2000), pp. 544–552.

[18] Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT].

[19] Natalie Stewart. *On homotopical additivity of equivariant little disks operads (forthcoming)*. 2025.

[20] Natalie Stewart. *On tensor products of equivariant commutative operads (draft)*. 2025.

[21] Ben Szczesny. *Equivariant Framed Little Disk Operads are Additive*. 2024. arXiv: 2410.20235 [math.AT].

This presentation was made in Beamer, with figures via tikz-cd and Inkscape, presented via Impressive. The tex is on my website. The title slide is $H^{\star}_{C_2}(*_{C_2}; \underline{\mathbb{Z}})$ [5].