Kan Seminar Notes

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This will be a rough collection of live-LaTeXed notes covering the Kan seminar talks given in Fall 2021. I'll make no promises that the contents of this are readable, or without significant clerical error. Exercise skepticism, and don't use these as a replacement for the papers. Last update: September 24, 2021.

Contents

1 Gabrielle Li: Serre, Cohomologie modulo 2 des complexes d'Eilenberg Mac Lane (i)

This talk was delivered September 15, 2021 by Gabrielle Li. Throughout, $H^*(-) := H^*(-;\mathbb{F}_2)$.

1.1 Steenrod operations

The *Steenrod operations* are a family of cohomology operations $\mathrm{Sq}^n : H^*(X) \to H^{*+n}(X)$ such that:

(1) Each Sq^n is natural in X.

(2) Each Sq^n is stable: $\mathrm{Sq}^n(\Sigma X) = \Sigma\,\mathrm{Sq}^n(X)$.

(3) When $|x| = n$, $\mathrm{Sq}^n(x) = x \cup x$.

(4) $\mathrm{Sq}^0 = \mathrm{id}$.

We give a basis for these:

Definition 1.1. A sequence $I = (i_1) \subset \mathbb{Z}_{>0}$ is *admissable* if $i_k \geq 2i_{k+1}$ for each k. We define the *degree* $n(I) := \sum i_k$ and the *excess* $e(I) = \sum(i_k - i_{k+1}) = 2i_i - n(I)$ (padding with zeros).

1.2 Borel's theorem

Let $F \hookrightarrow E \to B$ be a Serre fibration. Recall that, in the cohomological Serre spectral sequence, we have transgression morphisms $\tau : E_r^{0,r-1} \to E_r^{r,0}$, whose domain is a subset of $H^{r-1}(F)$ and whose codomain is a quotient of $H^r(B)$. This is an additive relation between $H^{r-1}(F)$ and $H^r(B)$. We say that $x \in H^{r-1}(F)$ is *transgressive* if it survives to the r page.

We hold off on proving the following proposition until the next talk:

Proposition 1.2. *τ commutes with Steenrod operations.*

We need a bit more language to use this:

Definition 1.3. For a space X, an ordered family of elements $(x_i) \subset H^*(X)$ is a *simple system of generators* if:

(1) Each x_i is homogeneous.

(2) The increasing products $x_{i_1} \cdots x_{i_j}$ (for $i_k < i_{k+1}$) form a $\mathbb{F}_2$-basis of $H^*(X)$.

The following examples are important:

> **Example 1.4:**
> $\mathbb{F}_2[x_1, x_2, \ldots]$ has simple system of generators $(x_j^{2^i})$. Similar systems apply to the exterior algebra $E[x]$ and the truncated poltnomial algebra $\mathbb{F}_2[x]/(x^{2^i})$.

We're finally ready to state our theorem:

Theorem 1.5 (Borel)**.** *Given a fibration $F \hookrightarrow E \to B$ satisfying the following properties:*

(1) $E_2^{s,t} = H^s(B) \otimes H^t(F)$ (for instance, when B is 1-connected and $H^*(B), H^*(F)$ are f.g.).

(2) $H^i(E) = 0$ for $i > 0$.

(3) $H^*(F)$ have a simple system of transgressive generators (x_i).

Then, $H^*(B)$ is a polynomial algebra generated (independently) by the any choice of representatives $y_i \in H^*(B)$ which map to $\tau(x_i)$ in $E_*^{*,0}$.

Note that, whenever $H^*(F)$ is a polynomial algebra generated by z_i, we know that $H^*(F)$ has a simple system of generators $z_i^{2^r}$. In order to use this, we introduce a bit of notation:

Notation. $L(a,r) := \{2^{r-1}a, 2^{r-2}a, \cdots, 2a, a\}$.

Note that $z_i^{2^r} = \mathrm{Sq}^{L(n_i,r)}(z_i)$. Hence

$$\tau\left(z_i^{2^r}\right) = \mathrm{Sq}^{L(n_i,r)}\, t_i$$

where $t_i := \tau(z_i)$. Hence $H^*(B)$ is a polynomial algebra generated by $\mathrm{Sq}^{L(n_i,r)}(z_i)$.

1.3 Performing the calculation

We will use Borel's theorem soon, but first, a lemma:

Lemma 1.6. *An admissible sequence $J = \{j_1, \ldots, j_k\}$ with $e(J) < q-1$. Then, we may define a sequence*

$$J' := \left\{2^{r-1}s_J, 2^{r-2}s_J, \ldots, s_J, j_1, j_2, \ldots, j_k\right\},$$

where $s_J = q-1+n(J)$. Then, J' is admissible, with $e(J') < q$; furthermore, all admissible sequences of excess $< q$ arise this way.

The reversal is surprisingly easy; simply take the longest prefix satisfying $j_1 = 2j_2 = \cdots = 2^i j_i$.

We will need a few more constructions to prepare for the calculation:

(1) There is a fibration $K(\mathbb{F}_2, q-1) \hookrightarrow E \to K(\mathbb{F}_2, q)$ where E is contractible.

(2) By Hurewicz, $H^q(K(\mathbb{F}_2,q)) = \mathbb{F}_2$, with a generator that we call u_q.

Theorem 1.7. *$H^*(K(\mathbb{Z}/2,q),\mathbb{Z}/2)$ is a polynomial algebra (independently) generated by $\mathrm{Sq}^I(u_q)$ where I runs over the admissible sequences of excess $e(I) < q$.*

Proof. We prove this via induction. The $q = 1$ case is easy, as we have $K(\mathbb{F}_2, 1) = \mathbb{RP}^\infty$, and $H^*(\mathbb{RP}^\infty) = \mathbb{F}_2[u_q]$ via the usual computation.

For the inductive step, assume we've proven the theorem for $q-1$. We use the fibration from (1). For an admissible sequence J, let

$$S_J := \left|\mathrm{Sq}^J(u_{q-1})\right| = q-1+n(J).$$

We have transgression additive relation $H^{q-1}(K(\mathbb{F}_2, q-1)) \rightsquigarrow H^q(K(\mathbb{F}_2,q))$. Note that the transgression sends $\tau(u_{q-1}) = u_q$ (this will be justified later). Using our trick,

$$\tau(\mathrm{Sq}^J(u_{q-1})) = \mathrm{Sq}^J\, u_q.$$

By Borel, the $H^*(K(\mathbb{F}_2,q))$ is generated by $\mathrm{Sq}^{L(s_J,r)}\,\mathrm{Sq}^J\, u_q = \mathrm{Sq}^{L(s_j,r)J}\, u_q = \mathrm{Sq}^I\, u_q$, where I is an admissible sequence with $e(I) < q$, and all such I are generated this way. $\square$

The other computations are routine and similar.

2 Weixiao Lu: Serre, Cohomologie modulo 2 des complexes d'Eilenberg Mac Lane (ii)

This talk was delivered September 17, 2021 by Weixiao Lu. We'll first cover some preliminaries.

2.1 Preliminaries

Theorem 2.1 (Serre spectral sequence). *Let $F \hookrightarrow E \xrightarrow{p} B$ be a Serre fibration. Then, there is a spectral sequence*

$$E_2^{s,t} = H^s(B; H^t(p^{-1}(-); G)) \implies H^{s+t}(E; G).$$

If $\pi_1(B)$ acts trivially on $H^n(p^{-1}(-))$, then

$$E_2^{2,t} = H^s(B; H^t(F; G)).$$

Proof sketch. If F^*C^* is a filtered cochain complex, we have an SS,

$$E_0^{s,t} = \mathrm{gr}^s(C^{s+t}) \implies H^{s+t}(C^*).$$

Assume B is a CW complex with n-skeleton B^n. Then, $E_n := p^{-1}(B^n)$. We have $F^sS^*(E) = S^*(E, E_{s-1}) = \ker(S^*(E) \to S^*(E_{s-1}))$, which gives the right E_0 page. □

In any upper-right quadrant SS, we have a transgression morphism $d^n : E^{0,n-1} \to E_n^{n,0}$. Note that $E_n^{0,n-1} \subset E_{n-1}^{0,n-1} \subset \cdots \subset H^{n-1}(F)$. The transgressive elements of $H^{n-1}(F)$ map to some quotient of $H^n(B)$.

We can create a diagram

$$\begin{tikzcd}
H^n(B,b) \arrow[r, "p^*"] \arrow[d, "\sim"] & H^n(E,F) & \\
H^n(B) \arrow[ur] & & H^{n-1}(F) \arrow[ul, "\partial"']
\end{tikzcd}$$

Theorem 2.2 (Transgression theorem). *The transgression relation coincides with this diagram.*

This comes down to how the Serre SS was constructed.

Proposition 2.3. *The Steenrod square Sq_i "commutes" with transgression in the sense that any $x \in H^{n-1}(F; \mathbb{Z}/2)$ transgressive has $\mathrm{Sq}^i x$ transgressive, and $\tau(\mathrm{Sq}^i x) = \mathrm{Sq}^i(\tau x)$.*

Proof. Recall that a functor is stable iff it commutes with coboundary operators, so Sq_1 commutes with coboundary operators. Further, recall that it's natural. Hence the following diagram commutes, so Sq^i "commutes with the transgression relation" (is a morphism of cospans):

$$\begin{tikzcd}
& H^{n+i}(E,F) & \\
H^{n+i}(B) \arrow[ur, "p^*"] & & H^{n+i-1}(F) \arrow[ul, "\partial"'] \\
& H^n(E,F) \arrow[uu, "\mathrm{Sq}^i"] & \\
H^n(B \arrow[uu, "\mathrm{Sq}^i"] \arrow[ur, "p^*"] & & H^{n-1}(F) \arrow[ul, "\partial"'] \arrow[uu, "\mathrm{Sq}^i"']
\end{tikzcd}$$

□

Reclal that for G a f.g. Abelian group,

1. $H^*(K(G \times H; q)) = H^*(K(G;q)) \otimes H^*(K(H;q))$.
2. $H^*(K(\mathbb{F}_2;q)) = \mathbb{F}_2[\mathrm{Sq}^I u_q \mid I \text{ admissible, s.t. } e(I) < q]$.
3. $H^*(K(\mathbb{F}_2;q)) = \mathbb{F}_2[\mathrm{Sq}^I u_q \mid I \text{ admissible, s.t. } e(I) < q, 1 \text{ does not appear in } i]$.
4. $H^*(K(\mathbb{F}_2^h;q)) = \mathbb{F}_2[\mathrm{Sq}^I u_q, \mathrm{Sq}^J k_{q+1}]$ where $k_{q+1} \in H^{q+1}(K(\mathbb{F}_{2^h}, q))$ for admissibles $e(I) < q$, $e(J) \leq q$ where no Sq^1 term appears in both Sq^I and Sq^J. This comes from a fibration fill in from notes later .
5. $H^*(K(\mathbb{F}_{p^h};q)) = \mathbb{Z}/2$ for p odd with $q > 0$.

Remark. We have a different choice of generators related to universal classes, but as graded $\mathbb{F}_2$-algebras,

$$H^*(K(\mathbb{F}_{2^h};q)) \simeq H^*(K(\mathbb{F}_2;q)).$$

We will aim towards the following theorem:

Theorem 2.4. *For all $n > 1$, there are infinitely many indices i at which $\pi_i(S^n)$ has nonzero 2-torsion.*

Our tool will be Poincaré series. The accents in Poincaré's name are to be understood from here on out.

2.2 Poincaré series

For L_* a finite type graded k-vector space, define the series

$$L(t) = \sum_{n \in \mathbb{N}} \dim L^n t^n \in \mathbb{Z}[[t]].$$

This is called the *Poincare series*, called $\theta(G;q;t)$ in the case of $H^*(K(G;q))$.

Example 2.5:
For $L^* = \mathbb{Z}/2[u]$, we have

$$L(t) = \frac{1}{1 - t^m}.$$

Note that $(N^* \otimes M^*)(t) = L(t)M(t)$. Hence $L^{'*} = k[u_1, \dots]$ with finite type has

$$L(t) = \prod_{n \geq 1} \frac{1}{1 - t^{\deg u_i}}$$

which converges t-adically.

Hence

$$\theta(\mathbb{F}_2, q, t) = \prod_{e(I) < q} \frac{1}{1 - t^{\deg(\mathrm{Sq}^I u_q)}} = \prod_{e(I) < q} \frac{1}{1 + t^{q + n(I)}}.$$

We can give this another combinatorial description:

Proposition 2.6.

$$\theta(\mathbb{F}_2, q, t) = \prod_{n_1 \geq n_2 \geq \dots \geq n_{q-1} \geq 0} \frac{1}{1 - t^{2^{n_1} + \dots 2^{n_{q-1}} + 1}}.$$

The radius of convergence of this is 1 considered as a complex power series. We can continue to analyze this series along these lines:

Theorem 2.7.

$$\lim_{x \to \infty} \frac{\log_2 \theta\left(\mathbb{F}_2, q, 1 - 2^{-x}\right)}{x^q / q!} = 1.$$

In general there is an essential singularity at 1. Serre used this replacement to reign it in, but we won't work with it very explicitly.

2.3 Applications

Theorem 2.8. *Suppose X is a 1-connected space satisfying the following conditions:*

1. $H_*(X;\mathbb{Z})$ *is of finite type.*
2. $H_i(X;\mathbb{F}_2) = 0$ *for* $i \gg 0$.

Then, for infinitely many indices i, $\pi_i(X)$ has a subspace isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/2$.

This directly implies Theorem 2.4 once you know that only finitely many homotopy groups of spheres are infinite.

We see this using a whitehead tower

$$\begin{tikzcd}
\cdots \arrow[d] & \\
X_{n+1} \arrow[d] \arrow[dr] & \\
X_n \arrow[r] \arrow[d] & X \\
X_{n-1} \arrow[d] \arrow[ur] & \\
\cdots &
\end{tikzcd}$$

where X_n is n-connected, and a π_i iso to X and X_{n-1} for $i > n$. We'll use another piece of machinery, seen by the Serre SS directly.

Lemma 2.9. *For $F \hookrightarrow E \to B$ a Serre fibration with B simply connected, $B(t)F(t) \geq E(t)$.*

Proof of Theorem 2.8. Otherwise, there is some largest q with $\pi_q(X) \otimes \mathbb{Z}/2 \neq 0$. Then, there is some j smallest such that $H_j(X;\mathbb{Z}/2) \neq 0$. Then, $\pi_j(X) \otimes \mathbb{Z}/2 \neq 0$.

In the whitehead tower, $X_q \to X_{q-1}$ is trivial on $\pi_*(-) \otimes \mathbb{Z}/2$, so $H^*(X_q, \mathbb{Z}/2)$ is trivial. Using the fibration $X_q \hookrightarrow X_{q-1} \to K(\pi_q(X), q)$ from the whitehead tower, we must have $H^*(X_{q-1}) = H^*(K(\pi_q(X), q))$. Then,

$$X_{q-1}(t) = \theta(\pi_q(x), q, t).$$

Further, the fibbrations in the whitehead series imply that

$$X_{i+1}(t) \leq X_i(t)\theta(\pi_{i+1}(X), i, t)$$

for each i, Chaining these together forever, what we get is

$$\theta(\pi_q(X), q, t) \leq X_1(t)\theta(\pi_2(X), 1, t) \cdots \theta(\pi_{q-1}(x), q-2, t).$$

Note that $X_1(t)$ is a polynomial, so bounded on $[0, 1]$. Applying our asymptotic bound on θ yields a contradiction. $\square$

3 Zihong Chen: Moore, Semi-simplicial complexes and Postnikov systems

This talk was delivered September 20, 2021 by Zihong (Peter) Chen.

3.1 Review of simplicial sets

The talk began with a very brief review of simplicial sets: let Δ be the category of finite ordered sets and order preserving maps. Recall that such maps are generated by distinguished maps $\delta_i : [n] \to [n+1]$ and $s_i : [n+1] \to [n]$, called the *face and degeneracy maps*.

Definition 3.1. A *simplicial set* is a functor $X : \Delta^n \to \mathbf{Set}$.

The morphism set is completely characterised by their images on face and degeneracy maps, which must satisfy a collection of combinatorial relations, which I won't write down here.

Example 3.2:
The *standard n-simplex* is given by the representable functor $\Delta[n] := \mathrm{Hom}(-, [n])$.

By Yoneda's lemma, $X_n = \mathrm{Hom}(\Delta[n], X)$, where $X_n = X([n])$.

Example 3.3:
If $X \in \mathbf{Top}$, the singular simplicial set $\mathrm{Sing}(X)$ is familiar. It participates in an adjunction, with left adjoint $|\cdot|$ the *Geometric realization*.

Example 3.4:
Define the *ith face* $\delta_i : \Delta[n-1] \to \Delta[n]$. The *ith horm* is $\bigvee_i^n := \cup_{k \neq i} \delta_i$. The *boundary* is $\partial\Delta[n] = \bigcup_i \delta_i$.

This allows us to define the combinatorial equivalent of a topological space:

Definition 3.5. A simplicial set X is a *Kan complex* if every morphism $\bigvee_k^n \to X$ factors through $\Delta[n] \to X$; you can *fill any horn* (not necessarily uniquely).

A morphism $p : E \to B$ is a *Kan fibration* if it has the right lifting property against horn inclusions:

$$\begin{tikzcd}
\bigvee_k^n \arrow[r] \arrow[d] & E \arrow[d] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

Examples of this include $\mathrm{Sing}(X)$, and any simplicial group (which we won't prove).

Definition 3.6. For X a Kan complex, define the *path components* $\pi_0(X) = X_0/\sim$ where $x \sim y$ if there exists some p with $d_1 p = x$ and $d_0 p = y$.

This is in fact an equivalence relation: you can do this via horn filling, which was drawn on the board, but which I will not spell out. We can define higher homotopy groups after defining the internal hom:

Definition 3.7. For $A \subset X$ and $B \subset Y$, define the *mapping object*

$$\mathrm{Map}((X, A), (Y, B)) = \mathrm{Hom}(\Delta[n] \times (X, A), (Y, B))$$

i.e. the maps $\Delta[n] \times X \to Y$ restricting to a map $\Delta[n] \times A \to B$. The maps $\Delta[i] \to \mathbf{Set}$ form a covariant functor, so this is a contravariant functor, i.e. a simplicial set.

We use the following Theorem of Kan:

Theorem 3.8 (Kan)**.** *If Y, B are Kan complexes, then so is* $\mathrm{Map}((X, A), (Y, B))$.

We finally define homotopy groups.

Definition 3.9. If X is a Kan complex, define $\pi_n(X,x) := \pi_0(\mathrm{Map}((\Delta[n], \partial\Delta[n]), (X,x)))$. A Kan complex is $K(\Pi, n)$ if $\pi_q(X,x) = \Pi$ when $q = n$ and 0 otherwise.[1]

We will use these to decompose Kan complexes.

3.2 Postnikov systems

Let $\Delta[q]_n$ be the n-skeleton of $\Delta[q]$. For X a Kan complex, define the complex $X^{(n)}$ via

$$X_q^{(n)} = X_q/\sim \qquad\qquad x \sim y \iff x|_{\Delta[q]_n} = y|_{\Delta[q]_n}.$$

The maps are induced by X. We have the following properties:

1. $X^{(n)}$ is a Kan complex.
2. There is a quotient Kan fibration $X^{(n)} \xrightarrow{p} X^{(k)}$ if $n > k$.
3. $\pi_q(X^{(n)}, x) = 0$ if $q > n$.
4. $p_* : \pi_q(X^{(n)}, x) \xrightarrow{\sim} \pi_q(X^{(k)}, x)$ is an iso if $n \geq k \geq q$.

As in topology, Kan fibrations induce LES of homotopy groups; hence the fiber $F^{(n+1)} \hookrightarrow X^{(n+1)} \xrightarrow{p} X^{(n)}$ is a $K(\pi_{n+1}(X), x+1)$. We finally give this a name:

Definition 3.10. $(X^0, X^{(1)}, \dots)$ is called the *natural Postnikov system* of X.

This motivates a question: How far is X from $\prod_n K(\pi_n, n)$? It's always a colimit, but we'll measure how complex it is in the following section.

The idea is that $K(\pi_{n+1}, n+1) \hookrightarrow X^{(n+1)} \to X^{(n+1)}$ will be seen as something like a "principal $K(\pi_{n+1}, n+1)$-bundle." We will construct something like a "classifying space" $\overline{W}(K(\pi_{n+1}, n+1)) = K(\pi_{n+1}, n+2)$, and derive algebraic invariants from this. Let's actually do this now:

3.3 Principally twisted cartesian products

Definition 3.11. A *principally twisted Cartesian product* (PTCP) with simplicial group G and base G is written

$$E(T) = G \times_T B$$

where $E(T)_n = G_n \times B_n$ with degeneracy maps all the same, except that

$$\partial_0(g, b) = (T(b) \cdot d_0 g, d_0 b)$$

and T is a *twisting function* $B_q \to G_{q-1}$ for $q \geq 1$.

This is a combinatorial version of *holonomy*, as per a comment from Prof. Miller.

Definition 3.12. A PTCP is *of type (W)* if $B_0 = \{b_0\}$ and

$$\partial_0|_{\{e_q\} \times B_q} : [e_q] \times B_q \xrightarrow{\sim} E(T)_{q-1}$$

is an iso. Let $\int$ be its inverse.

Theorem 3.13. *If $G \times_T B$, $G' \times_{T'} B'$, and $\gamma : G \to G'$ is a morphism of simplicial group, then there exists a unique γ-equivariant map $\theta : G \times_T B \to G' \times_{T'} B'$ and Some condition holds of θ–fill in later .*

I couldn't follow this part; use $\int$ to construct this "upwards" from b_0, or something like that.

Corollary 3.14. *A PTCP of type (W) with group G is unique, if it exists.*

[1] This *actually* has a requirement of minimality, but we handwave this away.

Theorem 3.15. *If $E(T)$ is PTCP of type (W), it is contractible.*

They do exist! We can construct them by $B := \overline{W}(G)$, $W(G) = G \times_{T(G)} \overline{W(G)}$, where $\overline{W}_n(G) = G_{n-1} \times \cdots \times G_0$ for $n \geq 1$, and terminal for $n = 0$. put face and degen maps here . It has twisting function

$$T(G)[g_n, \ldots, g_0] = g_n.$$

It can be checked explicitly that this is type (W).[2]

Corollary 3.16. *Every PTCP with group G is by*

$$B \xrightarrow{\pi} \overline{W}(G)$$

with $\pi(b) = \left[T(b), T(\partial_0 b), \ldots, T(\partial_0^{n-1} b)\right]$.

This is a simplicial version of the bar construction??

This allows us to explicitly construct $K(\pi, n)$! Define $K(\pi, 0)$ to be π in each degree and $\partial_i s_i$ all identity. Define $K(\pi, n) = \overline{W}(K(\pi, n-1))$ inductively. We can see this is in fact a $K(\pi_1)$ via a fibration

$$K(\pi, n) \to W(K(\pi, n)) \to \overline{W}(K(\pi, n)),$$

where we know $W(*)$ to be contractible.

The main technical result follows:

Lemma 3.17. *Suppose there is no nontrivial morphism $\pi_1 \to \mathrm{Aut}(\pi_n)$. Then, $X^{(n)}$ is a PTCP with group $K(\pi_{n+1}, n+1)$.*[3]

To handwave, the idea for this is that minimal Kan fibrations are fiber bundles. Given the π_1 assumption, the structure group is $K(\pi_{n+1}, n+1)$. Then, a "principal G-bundle" is the same thing as a PTCP, in some intuitive way.

We can define the k-invariants via the fibrations $K(\pi_{n+1}, n+1) \hookrightarrow X^{(n+1)} \to X^{(n)}$: there is a universal class

$$u \in H^{n+2}(K(\pi_{n+1}, n+2))$$

and via the map $X^{(n+1)} \xrightarrow{f^{n+2}} \overline{W}(K(\pi_{n+1}, n+1)) = K(\pi_{n+1,n+2})$, we can define k-invariants as $\left(f^{n+2}\right)^* u = k^{n+2}$.

[2] This was written down in class.

[3] Per a comment of Prof. Miller, we only need simplicity, not total nontriviality of morphisms $\pi_1 \to \mathrm{Aut}(\pi_n)$.

4 Dylan Pentland: Borel, La cohomologie modulo 2 de certains espaces homogenes

This talk was delivered September 22, 2021 by Dylan Pentland.

4.1 Motivation and preprequisites

Characteristic classes We have a functor

$$\mathrm{Bun}_{O(n)} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$$

sending X to the isomorphism classes of principal $O(n)$ bundles mod isomorphisms. We know that this is representable, i.e. expressable as $\mathrm{Bun}_{O(n)}(-) = \mathrm{Hom}(-, \mathrm{BO}(n))$ (in the homotopy category).

Definition 4.1. A *characteristic class* is a natural transformation $\mathrm{Bun}_{O(n)} \implies H^i(-)$, where coefficients are understood mod 2. By the Yoneda lemma, this is the same thing as an element of $H^i(\mathrm{BO}(n))$.

We're going to characterize these via a cohomology computation. The main theorem is as follows: let $Q(n) \subset O(n)$ be the diagonal matrices. From this inclusion, we get a projection $\mathrm{BQ}(n) \xrightarrow{\rho} \mathrm{BO}(n)$, wich yields an induced map

$$\rho^* : H^*(\mathrm{BO}(n)) \to H^*(\mathrm{BQ}(n)) \simeq \mathbb{F}_2[x_1, \ldots, x_n].$$

Theorem 4.2. *The map ρ^* satisfies the following properties:*

- *ρ^* is injective.*
- *the image of ρ^* is $\mathbb{F}_2[x_1, ;x_n]^{\Sigma_{in}}$.*
- *$p^*(w_i) = e_i$.*

The splitting principle We give a modern POV on this:

Theorem 4.3. *Let X be paracompact and $E \to X$ a bundle. There is an undiced bundle $f : \mathrm{Fl}(E) \to X$ so that*

$$f^* : H^*(X) \to H^*(\mathrm{Fl}(E))$$

is injective, and f^*E splits into a direct sum of line bundles.

This winds up telling you the injectivity of Theorem 4.2, but not the image statement (only a containment). Either way, the proof is not much easier.

Spectral sequences breaking down We'll keep some assumptions about finite type cohomology. Dylan stated the requirement of simply connected spaces or principal G-bundle.[4]

Theorem 4.4. *Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration. The associated Serre spectral sequence (SSS) is trivial if and only if $H^*(E) \to H^*(F)$ is surjective. In this case, we say that F is* totally non-homologous to zero*, and we have the following properties:*

- *p^* is injective.*
- *$P(E) = P(B) \cdot P(F)$.*

The condition is called totally non-homologous to zero because the dual condition $H_*(F) \hookrightarrow H_*(E)$ makes sense for this name.

[4]Haynes had some comments about this; principality is not enough in general. There's secretly some connectedness condition.

4.2 Cohomology of $\mathrm{BO}(n)$

Outline of the proof of Theorem 4.2 Let $F_n = O(N)/Q(N)$. We will use the fibration

$$F_n \hookrightarrow \mathrm{BQ}(n) \xrightarrow{\rho} \mathrm{BO}(n).$$

We call this fibration $(\star)$. We follow the following steps:

(1) $H^*(F_n) = \langle H^1(F_n)\rangle$ so that $P(F_n) = (1-t)\cdots(1-t)^n \cdot (1-t)^{-n}$.

(2) The SSS for $(\star)$ is trivial, so $P(\mathrm{BQ}(n)) = P(\mathrm{BO}(n))P(F_n)$, giving injectivity.

(3) $\operatorname{im}\rho^* \subset \mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_{in}}$, and dimensions yield that this is an equality.

General rule of theorem: every spectral sequence written today will be trivial.

Step 1: cohomology of F_n. We use induction via the fibration

$$F_{n-1} \hookrightarrow F_n \to \mathbb{P}^{n-1}$$

Lemma 4.5. $\dim H^1(F_n) \geq n-1$.

Write ${}^nE_r = \bigoplus_{s+t=n} E_r^{s,t}$.

Proof. Recall the fibration $F_n \hookrightarrow \mathrm{BSQ}(n) \to \mathrm{BSO}(n)$. The base space is simply connected, so

$$E_2^{1,0} = H^1(\mathrm{BSO}(n), H^0(F_n)) = 0.$$

Hence

$$\dim{}^1 E_2 \geq \dim{}^1 E_\infty = \dim H^1(\mathrm{BSQ}(n)) = n-1.$$

This implies that $E_2^{0,1} = H^1(F_n)$. □

Proposition 4.6. $P(F_n) = (1-t)\cdots(1-t^n)(1-t)^{-n}$ *and* $H^*(F_n) = \langle H^1(F_n)\rangle$.

Proof. Return to the fibration from the beginning of the fibration. We know the Poincaré polynomial for projective space, and we just have to prove that the SSS is trivial.[5] Write $H^*(F_n) \xrightarrow{i^*} H^*(F_{n-1})$. Assume both claims for $n-1$, so $\dim H^1(F_{n-1}) = n-2$. Note the following:

- $\dim E_2^{1,0} = \dim H^1(\mathbb{P}^{n-1}) = 1$.
- $\dim E_2^{0,1} = \dim H^0(\mathbb{P}^{n-1}, H^1(F_n)) \leq n-2$.

Look at $\dim H^1(F_n) = {}^1E_\infty \leq n-1$; combined with our previous bound, we have $\dim H^1(F_n) = n-1$. This implies that ${}^1E_n = {}^1E_\infty$ since they have equal dimensions. This implies that $E_2^{0,1}$ are cocycles for differentials.

Further, note that $\operatorname{im} i^*|_{\deg 1} = E_\infty^{0,1} = H^1(F_{n-1})$. Since cohomology of the codomain is generated in degree 1, this implies that i^* is surjective, so the SSS is trivial. This implies the Poincaré polynomial is as we said it is, by a familiar technique. □

Step 2: triviality of the SSS of $(\star)$.

Proposition 4.7. *The SSS for $(\star)$ is trivial.*

Proof. Note that $\dim{}^1 E_2 = \dim H^1(\mathrm{BO}(n), H^0(F_n)) + \dim H^0(\mathrm{BO}(n), H^1(F_n))$. The first is equal to 1, and the second is $\leq n+1$, and the second is $\leq n-1$, so the total is $\leq n$.

Now look at $\dim{}^1E_\infty \leq \dim{}^1 E_2$, which is an equality for dimension reasons. We have $\dim{}^1 E_2 \geq \dim{}^1 E_\infty$, and hence $H^0(\mathrm{BO}(n), H^1(F_n)) = H^1(F_n)$. For reasons relating to generation t degree 1, we also have $H^0(\mathrm{BO}(n), H^k(F_n)) = H^k(F_n)$. Hence $H^*(\mathrm{BQ}(n)) \twoheadrightarrow H^*(F_n)$. Hence the SSS is trivial. □

This allows us to immediately compute the Poincaré polynomial

$$P(\mathrm{BO}(n)) = \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)}.$$

[5] In particular, $P(\mathbb{P}^{n-1}) = \frac{1-t^n}{1-t}$, so since passing to the associated graded preserves graded dimension, triviality implies that the Poincaré series are multiplicative, and we can prove the Poincaré series computation inductively. I'll skip this.

Step 3: containment of the image of ρ^* in the symmetric polynomials. Combinatorics exists:

Lemma 4.8. $P(\mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_{in}}) = P(\mathrm{BO}(n))$.

Use Schur polynomials. Back to the topology.

Proposition 4.9. in $\rho^* \subset \mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_{in}}$.

Proof. $\Sigma_{in} = N_n/Q(n)$. Write down the classifying space fibration:

$$Q(n) \hookrightarrow EQ(n) \to BQ(n)$$

N_n acts on this, and acts on polynomials by permuting the generators in $\mathbb{F}_2[x_1, \dots, x_n]$. The normalizer N_n also acts on

$$O(n) \to EO(n) \to \mathrm{BO}(n).$$

the action on $\mathrm{BO}(n)$ is homotopically trivial[6], which we could use... Instead, we know the groups, so we can check concretely that this acts trivially on the cohomology, which gives the image containment. □

Hidden step 4: talking about $p^*(w_i) = e_i$. Once we know the $p^*(w_i) = e_i$ statement, universal relations on Steifel-Whitney classes come down to relations on $H^*(\mathrm{BO}(n))$.

[6] This is a general fact stated by Haynes, which comes down to some categories thing I didn't catch...

5 Mikayel Mkrtchyan: Milnor, The Steenrod algebra and its dual

This talk was delivered September 24, 2021 by Mikayel Mkrtchyan.

5.1 Refresher on the Steenrod algebra and Hopf algebras

For the next 48 minutes or so, we set $p = 2$ and work with the mod-2 Steenrod algebra. Recall that the *Steenrod algebra* $\mathcal{A}^*$ is a graded commutative algebra of mod-2 cohomology operations generated as an algebra by $\mathrm{Sq}^n \in \mathcal{A}^n$ for $n \geq 1$. For a finite sequence $I = (i_1, \ldots, i_r)$, define

$$\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_r}.$$

Recall that a sequence I is admissible if $a_i \geq 2a_{i-1}$. We have a basis made of these:

Theorem 5.1 (Serre-Cartan). *The set $\{\mathrm{Sq}^I \mid I \text{ admissible}\}$ is an $\mathbb{F}_2$-basis for $\mathcal{A}^*$.*

This is proved via the following relation:

Theorem 5.2 (Adem relation). *For all $0 < n < 2m$,*

$$\mathrm{Sq}^n \mathrm{Sq}^m = \sum_{k=1}^{n/2} \binom{m-k-1}{n-2k} \mathrm{Sq}^{n+m-k} \mathrm{Sq}^k,$$

and these generate all relations in a presentation of $\mathcal{A}^*$.[7]

As a preview of what's to come, define $\mathcal{A}_*$ to be the dual coalgebra.

Theorem 5.3. *$\mathcal{A}^*$ is a graded connected Hopf algebra, and its dual satisfies $\mathcal{A}_* \simeq \mathbb{F}_2[\zeta_1, \zeta_2, \ldots]$.*

As an application, we'll show that all elements in $\mathcal{A}^{>0}$ are nilpotent.
The base field of $\mathbb{F}_2$ is to be understood.

Definition 5.4. A connected graded Hopf algebra is a graded associative algebra B^* s.t. $B_0 = \mathbb{F}_2$, endowed with a coassociative *comultiplication* map

$$B^* \xrightarrow{\psi} B^* \otimes B^*$$

s.t. $\psi(b) = b \otimes 1 + 1 \otimes b + \sum b_i' \otimes b_i''$ for all $b \in B^{>0}$.

Projecting to the 0th graded part is the "augmentation" (counit), and you can define the antipode uniquely given this data.

5.2 Coalgebra structure on the Steenrod algebra

We want to define a map $\psi : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$ extending the map

$$\psi(\mathrm{Sq}^n) = \sum_{i+j=n} \mathrm{Sq}^i \otimes \mathrm{Sq}^j.$$

We will prove that this is well defined using the Cartan formula

$$\mathrm{Sq}^n(a \times b) = \sum_{i+j=n} \mathrm{Sq}^i(a) \times \mathrm{Sq}^j(b).$$

We'll also use the following

[7] The presentation statement was said verbally but not written.

Lemma 5.5. *Fix some n. There exists a space U with finite-type cohomology and a class $u \in H^*(U)$ such that*

$$\sigma : \mathcal{A}^* \to H^*(U)$$

given by $a \mapsto a \cdot u$ is injective on $\mathcal{A}^{\le n}$.

Proof sketch. This is given by $U = K(\mathbb{Z}/2, n+1)$ with $u \in H^{n+1}(K(\mathbb{Z}/2, n+1))$. Use the result of Gabi's talk. $\square$

Lemma 5.6. *There exists a lift*

$$\begin{tikzcd} T(\{\mathrm{Sq}^n\}) \arrow[r, two heads] \arrow[d, "\psi"] & \mathcal{A}^* \arrow[dl, dashed] \\ \mathcal{A}^* \otimes \mathcal{A}^* & \end{tikzcd}$$

Proof. Choose (U, u, σ) as in the lemma. Form the diagram

$$\begin{tikzcd} T(\{\mathrm{Sq}^n\}) \arrow[rr, two heads] \arrow[d, "\psi"] & & \mathcal{A}^* \arrow[dll, dashed] \arrow[d, "\text{act on } u\times u"] \\ \mathcal{A}^* \otimes \mathcal{A}^* \arrow[r, "\sigma\otimes\sigma"] & H^*(U) \otimes H^*(U) \simeq & H^*(U \times U). \end{tikzcd}$$

Then, note that $\sigma \otimes \sigma$ is injective at degrees $\le n$, and this commutes, hence we can choose a lift in lower degrees; this allows you to define it degreewise by picking high enough n. $\square$

Corollary 5.7. *$\mathcal{A}^*$ is a cocommutative Hopf algebra.*[8]

5.3 Housekeeping

Suppose X is a finite CW complex. This is routine, formal, and not talked about explicitly.

1. We have an action
$$\mathcal{A}^* \otimes H^* \to H^*.$$
2. This yields a dual operation
$$H_* \otimes \mathcal{A}* \to H_*.$$
3. We can dualize this:
$$\lambda : H^* \to H^* \otimes \mathcal{A}_*,$$
since the homology and $\mathcal{A}$ are both finite type.
4. Note that λ makes H^* into a $\mathcal{A}_*$-comodule.
5. The following proof was omitted:

   Lemma 5.8. *λ is an $\mathbb{F}_2$-algebra homomorphism.*

Let's work an example.

> **Example 5.9:**
> Let $X := \mathbb{RP}^\infty = K(\mathbb{Z}/2, 1)$, with $u \in H^1(X)$.[a]
> **Lemma 5.10.**
> $$\mathrm{Sq}^n(x^{2^m}) = \begin{cases} x^{2^{m+1}} & n = 2^m \\ 0 & \textit{otherwise} \end{cases}$$
>
> *Proof sketch.* Define $\mathrm{Sq} := \sum_i \mathrm{Sq}_i$. Note that $\mathrm{Sq}(u) = u + u^2$, so $\mathrm{Sq}(u^{2^m}) = u^{2^m} + u^{2^{m+1}}$. $\square$

[8] He called it coassociative, but I omit this as this is the convention for Hopf algebras in general.

Corollary 5.11. *$\lambda : H^*(X) \to H^*(X) \otimes \mathcal{A}_*$ is given by*

$$\lambda(u) = \sum_k u^{2^j} \otimes \zeta_k$$

where $\langle \zeta_i, \mathrm{Sq}^I \rangle = 0$ unless $I = I_k := (2^{i-1}, 2^{k-2}, \dots, 1, 0)$.

I'm lagging a bit behind, so expect this next bit to be choppy.

[a] We'll see why we don't have to care that X is finite.

5.4 Algebra structure on the dual Steenrod algebra

Let I be an admissible sequence, and define

$$\gamma(I) = (i_1 - 2i_2, i_2 - 2i_3, \dots, i_r, 0).$$

Let R be a sequence.

Proposition 5.12. *Let I, J be admissible sequences of the same degree. Then,*

$$\langle \zeta^{\gamma(J)}, \mathrm{Sq}^I \rangle = \begin{cases} 1 & I = J \\ 0 & I < J \end{cases}$$

where $<$ denotes the lexicographic order.[9]

Proof. We prove this by induction. Let $J = (aj_1, \dots, a_k, 0)$ and similar for I and b. define

$$J' = (a_1 - 2^{k-1}, a_2 - 2^{k-2}, \dots, 0).$$

Then,

$$\gamma(J) = \gamma(J') + (\text{a } 1 \text{ in the } k\text{th spot.}).$$

Hence

$$\zeta^{\gamma(J)} = \zeta^{\gamma(J')} \cdot \zeta_k,$$

so that

$$\langle (\zeta^{\gamma(J)}, \mathrm{Sq}^I \rangle = \langle \zeta^{\gamma(J)} \otimes \zeta_k, \psi(\mathrm{Sq}^I) \rangle = \langle \zeta^{\gamma(J')} \otimes \zeta_k, \sum \mathrm{Sq}^{I_1} \otimes \mathrm{Sq}^{I_2} \rangle.$$

If you work out the nitty gritty, this concludes the proof by induction. □

Corollary 5.13. $\mathcal{A}_* \simeq \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$.

Proof. The last proposition proved that $\zeta^{\gamma(J)}$ form an $\mathbb{F}_2$-basis, which is exactly equivalent to $\mathcal{A}_*$ being a polynomial algebra in ζ_i. □

We now characterize the comultiplication.

Theorem 5.14. *The comultiplication map $\varphi_* : \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ is given by*

$$\zeta_n \mapsto \sum_{k \geq 0} \zeta_{n-k}^{2^k} \otimes \zeta_k.$$

This is some measure of the basis we gave being nice.

Proof. We have coassociativity:

$$\begin{tikzcd} H^* \arrow[r] \arrow[d] & H^* \otimes \mathcal{A}_* \arrow[d, "\lambda \otimes \mathrm{id}"] \\ H^* \otimes \mathcal{A}_* \arrow[r, "\mathrm{id} \otimes \varphi_*"] & H^* \otimes \mathcal{A}_* \otimes \mathcal{A}_* \end{tikzcd}$$

We perform a diagram chase for $X := \mathbb{RP}^\infty$. □

[9] It hasn't been mentioned what happens when $I > J$.

5.5 Positive-degree homogeneous elements of the Steenrod algebra are nilpotent

Define $J_n \subset \mathcal{A}_*$ by $(\zeta_1^{2k}, \zeta_2^{k-1}, \ldots, \zeta_{k-1}^2, \zeta_{k+1}, \ldots)$. Observe that $\varphi_*(J_n) \subset J_n \otimes \mathcal{A}_*$ by our characterization of the Milnor diagonal, and hence $\mathcal{A}_*/J_n$ is a Hopf algebra quotient of $\mathcal{A}_*$ of finite dimension. By duality, this corresponds with a f.d. Hopf subalgebra, and expanding n threatens to swallow $\mathcal{A}_*$:

Corollary 5.15. *$\mathcal{A}^*$ is the union of its finite dimensional Hopf subalgebras.*

By degree arguments, a positive dimension homogeneous element either is nilpotent or spans an infinite dimensional Hopf subalgebra, so this gives the nilpotency statement.

5.6 A sketch of the $p > 2$ case

In the odd p case, we have Lens spaces instead of $\mathbb{RP}^\infty$, and there are more cohomology elements:

Theorem 5.16.

$$\mathcal{A}_*^p = \mathbb{F}_p[\zeta_1, \ldots, \zeta_i] \otimes \bigwedge{}^* [\tau_0, \tau_1, \ldots] .$$