# YOU CAN CONSTRUCT $G$-COMMUTATIVE ALGEBRAS ONE NORM AT A TIME

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Abstract. We define the category of $G$-operads and the hierarchy of *generalized $\mathcal{N}_\infty$-operads*, which are $G$-suboperads of $\mathrm{Comm}_G^{\otimes}$. We exhibit an isomorphism between the category of generalized $\mathcal{N}_\infty$-operads and the self-join poset

$$\mathrm{Op}_G^{GN\infty} \simeq \mathrm{Ind}-\mathrm{Sys}_G \star \mathrm{Ind}-\mathrm{Sys}_G,$$

where $\mathrm{Ind}-\mathrm{Sys}_G$ is the poset of *indexing systems* in $G$. This recognizes generalized $\mathcal{N}_\infty$-operads as parameterizing *some commutative multiplicative transfers and possibly a commutative multiplication*. Indeed, their algerbas in semiadditive Cartesian categories are incomplete Mackey functors and their algebras in Mackey functors recover incomplete Tambara functors when they are $N_\infty$ *operads*, i.e. when they contain $\mathbb{E}_\infty$.

After this, we discuss some in-progress research. Namely, we construct a *Boardman-Vogt tensor product* of $G$-operads and demonstrate that tensor products of genereralized $\mathcal{N}_\infty$ operads correspond with joins in $\mathrm{Ind}-\mathrm{Sys}_G \star \mathrm{Ind}-\mathrm{Sys}_G$ i.e. there is an $\mathcal{N}_{(I\vee J)\infty}$-monoidal equivalence

$$\mathbf{Alg}_{\mathcal{N}_{I\infty}} \mathbf{Alg}_{\mathcal{N}_{J\infty}} \mathcal{C} \simeq \mathbf{Alg}_{\mathcal{N}_{(I\vee J)\infty}} \mathcal{C}$$

for all $\mathcal{N}_{(I\vee J)\infty}$-monoidal categories $\mathcal{C}$, allowing $G$-commutative structures to be constructed "one norm at a time."

**Foreword.** The following are notes prepared for a casual talk in the zygotop seminar concerning research which is currently in-progress cite . The reader should read with the understanding that they are particularly casual error-prone, as the non-cited results herein amount to the communication of a pre-draft of a paper in a casual setting.

The reader should implicitly insert the text $\infty-$ before the words operad and category throughout the following text.

## 1. Introduction

In [Dre71], the concept of a *Mackey functor* was introduced; this structure was described as consisting of functors $M_I : \mathcal{O}_G \to \mathbf{Mod}_R$ and $M_R : \mathcal{O}_G^{\mathrm{op}} \to \mathbf{Mod}_R$ which agree on $\mathcal{O}_G^{\simeq}$ and satisfying the *double coset formula*

$$R_J^H I_K^H = \prod_{x\in[J\backslash H/K]} I^J_{J\cap xKx^{-1}} \cdot \mathrm{conj}_X R_{x^{-1}Jx\cap K}$$

for all $J, K \subset H$, where $R_J^K := M_R(G/J \to G/K)$ and similar for $I$. The ur-example of this is the assignment $H \mapsto \mathbf{Rep}_H$ with covariant functoriality Ind and contravariant functoriality Res. This was repackaged and generalized into the modern definition of the *category of $\mathcal{C}$-valued $G$-Mackey functors*

$$\mathcal{M}_G(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_G), \mathcal{C}),$$

where $\mathbb{F}_G$ denotes the category of finite $G$-sets.

In parallel, the concept of *transfer maps in group cohomology* was being developed in [**Evens**], later lifted to genuine equivariant cohomology in [**Greenlees**], and finally developed as a functor

$$N_H^G : \mathrm{Sp}_H \to \mathrm{Sp}_G$$

in [HHR16], which played a crucial role in the solution to the Kervaire invariant one problem. These were noted in [HH16] to satisfy the conditions of a *Symmetric monoidal Mackey functor*, a notion they distinguished from their notion of *$G$-symmetric monoidal categories* due to coherence issues.

In the broad program announced in [Bar+16], the correct notion of *$G$-symmetric monoidal $G$-$\infty$-categories* (henceforth $G$-symmetric monoidal categories) was introduced:

**Definition 1.1.** Let $\mathcal{C}$ have finite products. Then, the category of $G$-commutative monoids in $\mathcal{C}$ is

$$\mathrm{CMon}_G(\mathcal{C}) := \mathcal{M}_G(\mathcal{C}).$$

The category of $G$-symmetric monoidal categories is $\mathrm{CMon}_G(\mathbf{Cat})$.

We similarly define the *category of small G-categories* as

$$\mathbf{Cat}_G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Cat}) \simeq \mathbf{Cat}^{\mathrm{cocart}}_{/\mathcal{O}_G^{\mathrm{op}}},$$

where the equivalence is the *straightening-unstraightening construction* of [HTT]. We may informally summarize the structure of a $G$-symmetric monoidal category $C^{\otimes} \in \mathrm{CMon}_G(\mathbf{Cat})$ as consisting of, for every conjugacy class $(H)$ of $G$, a category with Weyl group action $C_H \in \mathbf{Cat}^{BW_G H}$, as well as functors

$$\begin{aligned} \otimes^2_H &: C_H^2 \to C_H, \\ N_K^H &: C_K \to C_H, \\ \mathrm{Res}_K^H &: C_H \to C_K \end{aligned}$$

for all subconjugacy classes $(K)$ of $(H)$, which are associative, commutative, unital, and compatible with each other and the Weyl group action, together with coherence. The maps Res encode an underlying $G$-category $C$ of $C^{\otimes}$, and $N_K^H$ is pronounced "the norm from $K$ to $H$."

Given $C^{\otimes}$ a $G$-symmetric monoidal category, we may informally define a *G-commutative monoid* to be a tuple of objects $(X_H)_{H \in \mathcal{O}_G} \in \prod_{H \in \mathcal{O}_G} C_H$ satisfying

$$X_H \simeq \mathrm{Res}_H^G X_G$$

together with structure maps

$$\begin{aligned} \mu^2_H &: X_H^{\otimes 2} \to X_H \\ \mathrm{tr}_K^H &: N_H^K X_K \to X_H, \end{aligned}$$

for all $H \subset K$, together with associativity, commutativity, unitality, and coherence data. We may intuitively view these data as altogether specifying that these structure maps jointly construct a contractible space of maps

$$X^{\otimes S} \to X_H$$

for all finite $H$-sets $S \in \mathbb{F}_H$, where

$$X^{\otimes S} := \bigotimes_{H/K \in \mathrm{Orb}(S)} N_K^H X_K.$$

The map $\mathrm{tr}_K^H$ is pronounced "the transfer from $K$ to $H3$." When $C^{\otimes} = \mathcal{M}_G(C)^{\otimes}$ with the *HHR norm G-symmetric monoidal structure* of [HH16], these are called *G-Tambara functors valued in $C$*.

This talk concerns various relaxations of the notion of $G$-commutative algebras. Namely, we will define a symmetric monoidal closed category $\mathrm{Op}_G$ of *(colored) G-operads*, whose internal hom $\underline{\mathbf{Alg}}_{\mathcal{O}}(C)^{\otimes}$ is called the *operad of algebras under pointwise tensors*, and whose tensor product is called the *Boardman-Vogt tensor product*.

A particular example will define $\mathcal{N}_\infty$ *operads*, which interpolate between $\mathbb{E}_\infty$ and the $G$-operad $\mathrm{Comm}_G$ which encodes $G$-commutative algebras by adding a subset of the transfers parameterized by $\mathrm{Comm}_G$:

**Definition 1.2.** A *G-transfer system* is a core-preserving wide subcategory $\mathcal{O}_G^{\simeq} \subset T \subset \mathcal{O}_G$ which is closed under base change, i.e. for any diagram in $\mathcal{O}_G$

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\alpha'"] & V \arrow[d, "\alpha"] \\ U' \arrow[r] & V' \end{tikzcd}$$

with $U \hookrightarrow V' \times_{U'} V$ a summand inclusion (pullback taken in $\mathbb{F}_G$) and $\alpha \in T$, we have $\alpha' \in T$.

An *indexing system* is a subcategory $I \subset \underline{\mathbb{F}}_G$ induced by a transfer system under taking coproducts. A *generalized indexing system* is a core-preserving subcategory $I \subset \underline{\mathbb{F}}_G$ which is either an indexing system or is attained by one by removing all non-isomorphisms of trivial $G$-sets. The poset of indexing systems under inclusion is denoted $\mathrm{Ind}-\mathrm{Sys}_G$, and the poset of generalized indexing systems is denoted .

It is not hard to see that there is an equivalence of posets

$$\widehat{\mathrm{Ind}-\mathrm{Sys}}_G \simeq \mathrm{Ind}-\mathrm{Sys}_G \star \mathrm{Ind}-\mathrm{Sys}_G,$$

and in particular, generalized indexing systems decompose into two different join-stable copies of indexing systems, depending on whether maps of trivial $G$-sets are included.

Given a generalized indexing system $I$, we will construct an operad called $\mathcal{N}^{\otimes}_{I\infty}$ encoding precisely the maps $\mathrm{tr}_K^H$ such that $K \hookrightarrow H$ is in $I$, as well as encoding the map $\mu_H$ if and only if $I$ is an indexing system. The main theorem of this talk characterizes the tensor products of generalized $\mathcal{N}_\infty$ operads.

**Theorem A.** *There is a fully faithful and symmetric monoidal inclusion*

$$\mathcal{N}^{\otimes}_{(-)\infty} : \widehat{\mathrm{Ind}-\mathrm{Sys}}_G^{\amalg} \hookrightarrow \mathrm{Op}_G^{\otimes}$$

*whose image consists of the suboperads of* $\mathrm{Comm}_G$*, and when restricted to the indexing systems has image consisting of operads $\mathcal{O}$ possessing diagrams $\mathbb{E}_\infty \subset \mathcal{O} \subset \mathrm{Comm}_G$. In particular, for $\mathcal{C}$ an $\mathcal{N}_{(I\vee J)\infty}$-monoidal category, there is a canonical $\mathcal{N}_{(I\vee J)\infty}$-monoidal equivalence*

$$\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{N}_{I\infty}} \underline{\mathbf{Alg}}^{\otimes}_{\mathcal{N}_{J\infty}} \mathcal{C} \simeq \underline{\mathbf{Alg}}^{\otimes}_{\mathcal{N}_{(I\vee J)\infty}} \mathcal{C}.$$

We say an inclusion of subgroup $H \subset K$ is *atomic* if it is proper and there exist no chains of proper subgroup inclusions $H \subset J \subset K$. More generally, we say that a conjugacy class $(H) \in \mathrm{Conj}(G)$ is an *atomic subclass* of $(K)$ if there exists an atomic inclusion $\tilde{H} \subset \tilde{K}$ with $\tilde{H} \in (H)$ and $\tilde{K} \in (K)$, and we say that $(K)$ is atomic if the canonical inclusion $1 \hookrightarrow K$ is atomic.

Given $(H) \subset (K)$ an atomic subclass, we refer to the $\mathcal{N}^\infty$-operad corresponding to the minimal index system containing the inclusion $H \hookrightarrow K$ as $\mathcal{N}^\infty(H,K)$. When $(H) = (1)$, we instead simply write $\mathcal{N}^\infty(K)$.

**Corollary B.** *Let $1 = G_n \subset G_{n-1} \subset \cdots \subset G_0 = G$ be a maximal subgroup series of a finite group, and let $\mathcal{C}$ be a $G$-symmetric monoidal category. Then, there exists a canonical $G$-symmetric monoidal equivalence*

$$\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{N}^\infty(G_1,G_0)} \cdots \underline{\mathbf{Alg}}^{\otimes}_{\mathcal{N}^\infty(G_n,G_{n-1})} \mathcal{C} \simeq \mathrm{CAlg}_G \mathcal{C}.$$

*Furthermore, if $G \simeq H \times J$, then*

$$\underline{\mathrm{CAlg}}^{\otimes}_H \underline{\mathrm{CAlg}}^{\otimes}_J \mathcal{C} \simeq \underline{\mathrm{CAlg}}^{\otimes}_G \mathcal{C}.$$

*Remark.* One may worry about the comparison between models for $G$-operads, as our notion of $N_\infty$-operads is ostensibly embedded deep within the world of $G$-$\infty$-operads, which are not known to be equivalent to the $\infty$-category presented by the graph model structure or by genuine $G$ operads.

However, some work has been done to simplify the story of $N_\infty$ operads in the model story; in particular, by [Rub21, Thm 2.16, 3.6], the full $\infty$-category of the $\infty$-category of *genuine* $G$-operads is equivalent to $\mathrm{Ind}-\mathrm{Sys}_G$ via a functor $A$ which sits in a commutative diagram

$$\begin{tikzcd}
\mathrm{Op}_G^{\mathrm{gen},N\infty} \arrow[r, "N|_{N\infty}"] \arrow[dr, "A"'] & \mathrm{Op}_G^{N\infty} \arrow[d, "A"] \\
& \mathrm{Ind}-\mathrm{Sys}_G
\end{tikzcd}$$

where we use that the functor $N$ of [BP21] is canonically $\infty$-categorical when restricted to full subcategores of $\mathrm{Op}_G^{\mathrm{gen}}$ which happen to be 1-categories and map to a 1-subcategory of $\mathrm{Op}_G$. Both functors named $A$ are equivalences (c.f. **??**Ex 2.4.7]Nardin), and hence $N|_{N\infty}$ is an equivalence.

## 2. The ideas

2.1. **Fibrous patterns.** In order to precisely define $I$-operads, the most efficient way will be to go through the technology of *algebraic patterns*, a concept first defined by German mathematician Honyi Chu and the Norwegian mathematician Rune Haugseng, who generally referred to them using the letter $\mathcal{O}$.

**Definition 2.1.** An *algebraic pattern* is an $\infty$-category $\mathfrak{O}$, together with a factorization system $(\mathfrak{O}^{\mathrm{int}}, \mathfrak{O}^{\mathrm{act}})$ of $\mathfrak{O}$ and a full subcategory $\mathfrak{O}^{\mathrm{el}} \subset \mathfrak{O}^{\mathrm{int}}$. The *category of algebraic patterns* is the full subcategory

$$\mathrm{AlgPatt} \subset \mathrm{Fun}(D, \mathbf{Cat})$$

spanned by algebraic patterns, where $D := \bullet \to \bullet \to \bullet \leftarrow \bullet$.

Maps in $\mathfrak{O}^{\mathrm{int}}$ and $\mathfrak{O}^{\mathrm{act}}$ are pronounced *inert and active maps*, and objects of $\mathfrak{O}^{\mathrm{el}}$ are pronounced *elementary objects*. For instance, $\mathbb{F}_*$, together with its inert and active maps as defined in [HA, § 2] and elementary objects $\{\langle 1 \rangle\}$ determines an algebraic pattern. In analogy with [HA, § 2], we will use these to develop a notion of operads, called *fibrous patterns*.

**Definition 2.2.** Let $\mathfrak{O}$ be an algebraic pattern. A *fibrous $\mathfrak{O}$-pattern* is a map of algebraic patterns $\pi : \mathcal{O} \to \mathfrak{O}$ such that

(1) $\mathcal{O}$ has $\pi$-cocartesian lifts for inert morphisms of $\mathfrak{O}$,
(2) (Segal condition for colors) For every active morphism $\omega : V_0 \to V_1$ in $\mathfrak{O}$, the functor

$$\mathcal{K}_{V_0}^{\simeq} \to \lim_{\alpha \in \mathfrak{O}^{\mathrm{el}}_{V_1/}} \mathcal{O}^{\simeq}_{\omega_{\alpha,!} V_1}$$

induced by cocartesian transport along $\omega_\alpha$ is an equivalence, where $\omega_{(-)} : \mathfrak{O}^{\mathrm{el}}_{Y/} \to \mathfrak{O}^{\mathrm{int}}_{X/}$ is the inert morphism appearing in the inert-active factorization of $\alpha \circ \omega$, and
(3) (Segal condition for multimorphism) for every active morphism $\omega : V_1 \to V_2$ in $\mathfrak{O}$ and all objects $X_i \in \mathcal{O}_{\mathfrak{O}_{V_i}}$, the commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{O}}(X_0, X_1) \arrow[r] \arrow[d] & \lim_{\alpha \in \mathfrak{O}^{\mathrm{el}}_{V_1/}} \mathrm{Map}_{\mathcal{O}}(X_0, \omega_{\alpha,!} X_1) \arrow[d] \\
\mathrm{Map}_{\mathfrak{O}}(V_0, V_1) \arrow[r] & \lim_{\alpha \in \mathfrak{O}^{\mathrm{el}}_{O_1/}} \mathrm{Map}_{\mathfrak{O}}(O_0, \omega_{\alpha,!} O_1)
\end{tikzcd}$$

is cartesian.

A fibrous $\mathfrak{O}$-pattern $\pi : \mathcal{C} \to \mathfrak{O}$ is a *Segal $\mathfrak{O}$-category* if $\pi$ is a cocartesian fibration. The category of fibrous $\mathfrak{O}$-patterns is the full subcategory

$$\mathrm{Fbrs}(\mathfrak{O}) \subset \mathrm{AlgPatt}_{/\mathfrak{O}}$$

spanned by fibrous patterns, and the category of Segal $\mathfrak{O}$-$\infty$-category is the full subcategory of

$$\mathrm{Seg}_{\mathfrak{O}}(\mathbf{Cat}) \subset \mathrm{Fbrs}(\mathfrak{O}) \times_{\mathbf{Cat}_{/\mathfrak{O}}} \mathbf{Cat}^{\mathrm{cocart}}_{/\mathfrak{O}}$$

spanned by Segal $\mathcal{O}$-categories.

We state one technical lemma:

**Lemma 2.3.** *All of the inclusions*

$$\mathrm{Seg}(\mathfrak{O}) \to \mathrm{Fbrs}(\mathfrak{O}) \hookrightarrow \mathrm{AlgPatt}_{/\mathfrak{O}} \to \mathbf{Cat}_{/\mathfrak{O}} \to \mathbf{Cat}$$

*have left adjoints; in particular, the full subcategory* $\mathrm{Fbrs}(\mathfrak{O}) \subset \mathrm{AlgPatt}_{/base}$ *is localizing.*

We refer to the left adjoint $\mathrm{Env} : \mathrm{Fbrs}(\mathfrak{O}) \to \mathrm{Seg}(\mathfrak{O})$ as the *Segal envelope*, and we use it analogously to the *symmetric monoidal envelope*, reducing the question of characterizing maps of fibrous patterns into Segal $\mathfrak{O}$-categories into simply a question of characterizing maps of Segal $\mathfrak{O}$-categories, which is much simpler.

**Example 2.4:**

**Definition 2.5.** Given the data of $\mathcal{X}$ a category, $\mathcal{X}_b, \mathcal{X}_f$ wide subcategories, and $\mathcal{X}_0 \subset \mathcal{X}_b$ a full subcategory, we define the *span pattern* $\mathrm{Span}_{b,f}(\mathcal{X}; \mathcal{X}_0)$ to have:

- underlying category $\mathrm{Span}_{b,f}(\mathcal{X})$ whose objects are objects in $\mathcal{X}$ and whose morphisms $X \to Z$ are spans

$$X \xleftarrow{B} Y \xrightarrow{F} Z$$

with $B \in \mathcal{X}_b$ and $F \in \mathcal{X}_f$.
- inert morphisms $\mathcal{X}_b^{\mathrm{op}} \subset \mathrm{Span}(\mathcal{X})$.
- active morphisms $\mathcal{X}_f \subset \mathrm{Span}(\mathcal{X})$.
- Elementary objects $\mathcal{X}_0^{\mathrm{el}} \subset \mathcal{X}_b^{\mathrm{op}}$.

Then, for instance we have the following:

**Theorem 2.6** ([BHS22])**.** *Pullback along the inclusion $\mathbb{F}_* \hookrightarrow \mathrm{Span}(\mathbb{F})$ induces an equivalence on the categories of fibrous patterns and Segal categories.*

2.2. **$G$-operads and $\mathcal{I}$-operads.** There is an adjunction

$$\mathrm{Tot} : \mathbf{Cat}_G \rightleftarrows \mathbf{Cat} : \mathrm{CoFr}^G$$

where Tot takes the total category of a cocartesian fibration and $\mathrm{CoFr}^G(\mathcal{C})$ is classified by functor categories

$$\mathrm{CoFr}^G(\mathcal{C})_H := \mathrm{Fun}(\mathcal{O}_H^{\mathrm{op}}, \mathcal{C})$$

with functoriality dictated by pullback. In particular, the *$G$-category of small $G$-categories* $\underline{\mathbf{Cat}}_G := \mathrm{CoFr}^G(\mathcal{C})$ has $G$-fixed points given by **Cat**.

*Remark. Elmendorf's theorem* may be reinterpreted in this language as the statement that the *$G$-category of $G$-spaces* $\mathcal{S}_G$ is cofreely generated by $\mathcal{S}$.

Let $\underline{\mathbb{F}}_G := \mathrm{CoFr}^G(\mathbb{F})$ and let $\underline{\mathbb{F}}_{G,*} := \mathrm{CoFr}^G(\mathbb{F}_*)$. Then, there is an equivariant lift of ref :

**Theorem 2.7** ([BHS22])**.** *Pullback along the composition $\underline{\mathbb{F}}_{G,*} \hookrightarrow \mathrm{Span}(\mathrm{Tot}\underline{\mathbb{F}}_G) \xrightarrow{U} \mathrm{Span}(\mathbb{F}_G)$ induces an equivalence on the categories of fibrous patterns and Segal categories, where $\mathbb{F}_G$ is the category of $G$-sets.*

**Definition 2.8.** The *category of $G$-operads* is the category of fibrous patterns

$$\mathrm{Op}_G := \mathrm{Fbrs}(\mathrm{Span}(\mathbb{F}_G)).$$

The following proposition is an exercise in category theory which was carried out in [BHS22, § 5.2].

**Proposition 2.9.** *An identity-on-objects functor $\pi : \mathcal{O} \to \mathrm{Span}(\mathbb{F}_G)$ is a $G$-operad if and only if it satisfies the following conditions:*

*(1) $\mathcal{O}$ has $\pi$-cocartesian lifts for inert morphisms of $\mathrm{Span}(\mathbb{F}_G)$.*
*(2) For every map of $G$-sets $S \to T$, the inert morphisms $\{U \leftarrow T \mid U \in \mathrm{Orb}(T)\}$ induce equivalences*

$$\mathrm{Map}_{\mathcal{O}}(S, T) \simeq \prod_{U \in \mathrm{Orb}(T)} \mathrm{Map}_{\mathcal{O}}(S, U).$$

*Furthermore, a cocartesian fibration $\pi : \mathcal{O} \to \mathrm{Span}(\mathbb{F}_G)$ is a Segal $\mathrm{Span}(\mathbb{F}_G)$-category if and only if it unstraightens to a $G$-symmetric monoidal category.*

We may further reorganize this through the following elementary lemma about $G$-sets.

**Lemma 2.10.** *The assignment $\varphi : T \mapsto \mathrm{Ind}_H^G T \to G/H$ underlies an equivalence of categories*

$$\mathbb{F}_H \simeq (\mathbb{F}_G)_{/G/H}.$$

Hence we have a forgetful functor

$$\mathcal{O}(-) : \mathrm{Op}_G^{\mathrm{one-object}} \to \mathrm{Fun}(\mathrm{Tot}\underline{\mathbb{F}}_G^{\simeq}, \mathcal{S})$$

Given $S \in \mathbb{F}_H$, we refer to $\mathcal{O}(S)$ as the *space of $S$-ary operations*. We further analyze this functor in ref , proving e.g. that it is conservative.

An early observation about genuine equivariant homotopy coherent algebraic structures is that the structure of transfers *does not come canonically* from an $\mathbb{E}_\infty$-structure; that is, $\mathbb{E}_\infty \in \mathrm{Op}_G$ is not terminal. The failure of $\mathbb{E}_\infty$ to be terminal is parameterized by the category of *generalized $N^\infty$-operads*:

**Definition 2.11.** Write $\mathrm{Comm}_G^{\otimes} := (\mathrm{Span}(\mathbb{F}_G) = \mathrm{Span}(\mathbb{F}_G))$ for the terminal $G$-operad. A $G$-operad $\mathcal{O}^{\otimes}$ is a *generalized $\mathcal{N}^{\infty}$-operad* if the unique morphism $\mathcal{O}^{\otimes} \to \mathrm{Comm}_G^{\otimes}$ is a monomorphism, i.e. $\mathcal{O}_U^{\otimes} \simeq *$ for all $U$ and $\mathrm{Map}_{\mathcal{O}}^{\psi}(x,y) \in \{*, \emptyset\}$ for all $\psi : \pi(x) \to \pi(y)$.

A generalized $\mathcal{N}^{\infty}$ operad $\mathcal{N}_{\infty I}$ is an *$N^{\infty}$ operad* if it admits a map

$$\mathbb{E}_{\infty} \to \mathcal{O}^{\otimes}.$$

Write $\mathrm{Op}_G^{GN\infty}$ for the full subcategory consisting of generalized $\mathcal{N}_{\infty}$-operads. The following proposition is an exercise in category theory, and establishes that a map to an $\mathcal{N}_{\infty}$ operad is a *property*, not a structure.

**Proposition 2.12.** *Given $\mathcal{N}_{I\infty} \in \mathrm{Op}_G^{GN\infty}$ a generalized $\mathcal{N}_{\infty}$ operad, the forgetful functor*

$$\mathrm{Op}_{G,/\mathcal{N}_{I\infty}} \to \mathrm{Op}_G$$

*is fully faithful.*

*Proof idea.* It is equivalent to prove that $\mathrm{Map}(\mathcal{O}, \mathcal{N}_{I\infty}) \in \{*, \emptyset\}$ for all $\mathcal{O} \in \mathrm{Op}_G$ In fact, there is a localizing (1-) subcategory $N : \mathrm{Op}_{1,G} \hookrightarrow \mathrm{Op}_G$ consisting of operads whose structure spaces are discrete, and whose localization functor $h : \mathrm{Op}_G \to \mathrm{Op}_{1,G}$ takes $\pi_0$ of the structure spaces. $\mathcal{N}_{I\infty}$ evidently lies in $\mathrm{Op}_{1,G}$, so we have

$$\mathrm{Map}_{\mathrm{Op}_G}(\mathcal{O}, \mathcal{N}_{I\infty}) \simeq \mathrm{Hom}_{\mathrm{Op}_{1,G}}(h\mathcal{O}, \mathcal{N}_{I\infty}).$$

Hence it suffices to check that the latter set is empty or contractible. This is easy to see in $\mathrm{Op}_{1,G}$, since $\mathrm{Hom}(-,*)$ and $\mathrm{Hom}(-,\emptyset)$ are always either empty or contractible. □

In particular, this implies that $\mathrm{Op}_G^{GN\infty}$ is a poset, so we'd like to identify this poset. There is a functor

$$A : \mathrm{Op}_G \to \widehat{\mathrm{Ind} - \mathrm{Sys}}_G$$

called the *admissible sets* with value over $G/H$ given by

$$A(\mathcal{O})_{/(G/H)} := \left\{ S \to G/H \mid \pi_{\mathcal{O}}^{-1}(S \to G/H) \neq \emptyset \right\}$$

and extended to general $G$-sets by coproducts. The following proposition is an exercise in category theory (see [NS22, Ex 2.4.7], or the original references [BH15; GW18; Rub21]):

**Proposition 2.13.** *The restricted functor*

$$A : \mathrm{Op}_G^{GN\infty} \to \widehat{\mathrm{Ind} - \mathrm{Sys}}_G$$

*is an equivalence of categories.*

We denote by $\mathcal{N}_{(-)\infty}$ the composite functor

$$\mathcal{N}_{(-)\infty} : \widehat{\mathrm{Ind} - \mathrm{Sys}}_G \xrightarrow{A^{-1}} \mathrm{Op}_G^{GN\infty} \hookrightarrow \mathrm{Op}_G$$

Using this, we finally define *I-operads*.

**Definition 2.14.** Let $I$ be a generalized indexing system. Then, the *category of $I$-operads* is the slice category

$$\mathrm{Op}_I := \mathrm{Op}_{G,/\mathcal{N}_{\infty I}^{\otimes}}.$$

Given $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes} \in \mathrm{Op}_I$, the *category of $\mathcal{O}$-algebras in $\mathcal{P}$* is the full subcategory

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}) \subset \mathrm{Fun}_{/\mathcal{N}_{\infty I}^{\otimes}}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$$

spanned by maps of $\mathcal{I}$-operads.

*Remark.* The notation $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$ does not include $\mathcal{I}$. This presents no problem; indeed, by proposition 2.12, the categories of $\mathcal{O}$-algebras in $\mathcal{P}$ considered over various indexing systems (including the terminal one, i.e. in $G$-operads) are canonically equivalent to one another.

A useful property of these are that $G$ operads *fibered* over $\mathcal{O}^{\otimes}$ have an intrinsic description in terms of $\mathcal{O}$. We may state these in the language of fibrous patterns.

**Proposition 2.15** (cite ). *Let $\mathcal{O}$ be a fibrous $\mathcal{E}$-pattern. Then, the pushforward functor $\pi_! : \mathrm{AlgPatt}_{/\mathcal{O}} \to \mathrm{AlgPatt}_{/\mathcal{E}}$ preserves fibrous patterns, and the associated functor*

$$\pi_! : \mathrm{Fbrs}(\mathcal{O}) \to \mathrm{Fbrs}(\mathcal{E})_{/\mathcal{O}}$$

*is an equivalence of categories.*

In particular, the category of $I$-operads is covariantly functorial in $I$, and it possesses an intrinsic expression along the lines of ref .

> **Example 2.16:**
> Let $\mathcal{F} \subset \mathcal{O}_G$ be a *family*, i.e. a collection of subgroups of $G$ closed under sub-conjugation. Then, $\mathcal{F} \cup \mathcal{O}_G^{\simeq}$ is a transfer system, and we denote by $\mathcal{I}_{\mathcal{F}}$ the corresponding indexing system.
>
> Let $V$ be a real orthogonal $G$-representation, let $\mathcal{F}_V$ is the family consisting of subgroups $H$ such that $V^H \neq *$, and let $\mathcal{I}_V := \mathcal{I}_{\mathcal{F}_V}$. Then, there is an $I_V$-operad $\mathbb{E}_V$ of *little $V$-disks*, which may be informally understood to have
> $$\pi_{\mathbb{E}_V}^{-1}(\mathrm{Ind}_H^G T \to G/H) := \mathrm{Conf}_H(T, V)$$
> the space of $H$-equivariant embeddings of $T \hookrightarrow V$ (c.f. [Hor19]). These participate in *equivariant infinite loop space theory*, in the sense that there is an equivalence
> $$\mathbf{Alg}_{\mathbb{E}_V}(\mathcal{S}_G) \simeq \{V - \textit{loop spaces}\}\,;$$
> see Guillou-May for details.

2.3. **The BV tensor product.** By ref , the category of algebraic patterns has a cartesian monoidal structure.

**Definition 2.17.** The category of *symmetric monoidal algebraic patterns* is CMon(AlgPatt).

A symmetric monoidal structure on $\mathcal{E}$ endows on the slice category $\mathrm{AlgPatt}^{\otimes}_{/\mathcal{E}}$ a symmetric monoidal structure, which we may view as taking $\mathcal{O}, \mathcal{P}$ to the tensor product

$$\mathcal{O} \times \mathcal{P} \to \mathcal{E} \times \mathcal{E} \to \mathcal{E}.$$

**Definition 2.18.** The *Boardman-Vogt symmetric monoidal category of fibrous $\mathcal{E}$-patterns* is the localized symmetric monoidal structure

$$\mathrm{Fbrs}(\mathcal{E})^{\otimes} \leftrightarrows \mathrm{AlgPatt}^{\otimes}_{/\mathcal{E}}.$$

We may view the tensor product of fibrous $\mathcal{E}$-patterns as yielding the localized composite

$$\mathcal{O} \otimes \mathcal{P} := L_{\mathrm{Fbrs}}(\mathcal{O} \times \mathcal{P} \to \mathcal{E} \times \mathcal{E} \to \mathcal{E}).$$

Note that the category $\mathbb{F}_G$ has finite products, and any indexing system $\mathcal{I}$ is closed under products. In particular, this endows $i : \mathcal{N}^{\otimes}_{\mathcal{I}\infty} \to \mathrm{Span}(\mathbb{F}_G)$ with the structure of a map of symmetric monoidal algebraic patterns under the so it has a cartesian monoidal structure. By cite , the forgetful functor $\mathrm{Fbrs}(\mathcal{O}) \to \mathrm{Fbrs}(\mathcal{E})_{/\mathcal{O}}$ is an equivalence, so we may use this to define the BV tensor product of $I$-operads.

**Definition 2.19.** The *Boardman-Vogt symmetric monoidal category of $I$-operads* is

$$\mathrm{Op}_{\mathcal{I}}^{\otimes} := \mathrm{Fbrs}(\mathcal{N}_{\mathcal{I}\infty})$$

The following proposition is easy:

**Proposition 2.20.** *Given an inclusion $i : \mathcal{N}_{\mathcal{I}\infty} \hookrightarrow \mathcal{N}_{\mathcal{J}\infty}$, pushforward along $i$ yields a functor*

$$i_! : \mathrm{Op}_{\mathcal{I}}^{\otimes} \to \mathrm{Op}_{\mathcal{J}}^{\otimes}$$

*realizing $\mathrm{Op}_{\mathcal{I}}$ as a symmetric monoidal colocalizing subcategory of $\mathrm{Op}_{\mathcal{J}}$.*

The verification of this comes down to the following fact:

**Lemma 2.21.** *Given $f : X \to Y$ a map of commutative algebra objects in $C$ a symmetric monoidal, the associated functor $f_! : C_{/X} \to C_{/Y}$ lifts to a canonical symmetric monoidal functor between the slice symmetric monoidal categories.*

The BV tensor product satisfies a mapping out property; namely, we review in ref  the construction due to [NS22, § 5.3] of the operad $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{Q})$, and we prove the following theorem.

**Theorem 2.22.** *There is a natural equivalence of operads*

$$\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}\otimes\mathcal{P}}\mathcal{Q} \simeq \underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{P}}\mathcal{Q}$$

*realizing* $\mathbf{Alg}^{\otimes}_{\mathcal{P}}(-)$ *as an internal hom for the BV tensor product.*

2.4. **Summary of the argument.** We would like to construct an equivalence $\mathcal{N}_{I\infty} \otimes \mathcal{N}_{J\infty} \simeq \mathcal{N}_{(I\vee J)\infty}$. Let's begin with the special case $I \subset J$; in this case, we can say something stronger.

**Proposition 2.23.** *If $\mathcal{O}$ is a one-object G-operad, then the map $\mathcal{N}^{\infty}(I) \to \mathcal{N}^{\infty}(I) \otimes \mathcal{O}$ is an I-equivalence; in particular, $\mathcal{N}^{\infty}(I)$ is $\otimes$-idempotent.*

To prove this, we use [NS22, Cor 5.3.9]; in particular, they generalize [HA] to verify that any of the following conditions are true of $\mathbf{Alg}^{\otimes}_{\mathcal{N}^{\infty}(I)}(\mathcal{C})$, and we verify that the conditions are equivalent in ref .

**Lemma 2.24.** *The following are equivalent:*

(1) *The forgetful functor* $\mathrm{CAlg}_I(\mathcal{C}) \to \mathcal{C}$ *is an equivalence.*
(2) *For all one-object I-operads $\mathcal{O}$, the forgetful functor* $\mathbf{Alg}^{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ *is an equivalence.*
(3) *The I-restricted operad is cocartesian*

Having proved this, we acquire a (unique) diagram

$$\begin{tikzcd}
\mathcal{N}_{I\infty} \arrow[dr] \arrow[drr] & & \\
& \mathcal{N}_{I\infty} \otimes \mathcal{N}_{J\infty} \arrow[r, "\varphi"] & \mathcal{N}_{(I\vee J)\infty} \otimes \mathcal{N}_{(I\vee J)\infty} = \mathcal{N}_{(I\vee J)\infty} \\
\mathcal{N}_{J\infty} \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

and we are tasked with proving that $\varphi$ is an equivalence. An unfortunate fact is that the functor $U : \mathrm{Op}_{I\vee J} \to \mathrm{Op}_I \times \mathrm{Op}_J$ doesn't appear to be conservative in general. Our strategy will come down to trying *really hard* to make it conservative. We do so via the following two lemmas, proved as lemmas 3.4 and 3.6.

**Lemma 2.25.** *Denote by $i : I \cup J \subset I \vee J$ the (non-indexing system) union of subcategories. Then, the pullback*

$$\mathrm{Fbrs}(\mathrm{Span}(\mathcal{I} \cup \mathcal{J})) \to \mathrm{Op}_I \times \mathrm{Op}_J$$

*is conservative. In particular, $U$ reflects equivalences between $\mathcal{I} \vee \mathcal{J}$-operads in the image of $L_{\mathrm{Fbrs}} i_!$.*

**Lemma 2.26.** *There is an equivalence $\mathcal{N}_{(I\vee J)\infty} \simeq L_{\mathrm{Fbrs}} i_! \,\mathrm{Span}(I \cup J)$.*

*Proof of theorem A.* By the above argument, it suffices to prove that $\varphi$ is an equivalence; in fact, by lemmas 2.25 and 2.26 and symmetry it suffices to prove that the localized functor

$$\iota_J^* \mathcal{N}_{I\cap J\infty} \otimes \mathcal{N}_{J\infty} \to \iota_J^* \mathcal{N}_{I\vee J}$$

is an equivalence. But $\iota_J^* \mathcal{N}_{I\infty} \simeq \mathcal{N}_{I\cap J\infty}$, so the above is the inclusion $\mathcal{N}_{I\cap J\infty} \otimes \mathcal{N}_{J\infty} \to \mathcal{N}_{J\infty}$, which is an equivalence by proposition 2.23. □

## 3. Technical nonsense

3.1. **Passing to monads is conservative.** Our arguments will be reminiscent of [SY19, § 2.3-2.4]

Given $\mathcal{O} \to \mathscr{C}$ a fibrous pattern, we define

$$\mathrm{Ar}^{\simeq}_{\mathrm{act/el}}(\mathscr{C}) \subset \mathrm{Ar}(\mathscr{C})$$

to be the core of the full subcategory of the arrow category consisting of active maps with elementary codomain, and we define

$$\mathcal{O}_{\Sigma} := \mathcal{O} \times_{\mathscr{C}} \mathrm{Ar}^{\simeq}_{\mathrm{act/el}}(\mathscr{C}),$$

which we view as the *associated symmetric sequence.*

**Lemma 3.1** (C.f. [SY19, Prop 2.3.6])**.** *Let $\mathrm{Fbrs}_\bullet(\mathscr{O})$ denote the full subcategory of fibrous patterns whose associated maps $O^{\mathrm{el}} \to \mathscr{O}^{\mathrm{el}}$ are equivalences. Then, the functor*

$$(-)_\Sigma : \mathrm{Fbrs}_\bullet(\mathscr{O}) \to \mathrm{Fun}\left(\mathrm{Ar}^{\simeq}_{\mathrm{act/el}}(\mathscr{O}), \mathcal{S}\right)$$

*is conservative.*

*Proof.* Just look at the Segal condition for fibrous patterns $\square$

We now specialize to the case $\mathscr{O} = \mathrm{Span}(\mathbb{F}_G)$. Let $C$ be a $G$-symmetric monoidal category, let $O \in \mathrm{Op}_G$ be a $G$-operad, and let $X \in \mathbf{Alg}_O(C)$ be an $O$-algebra in $C$. Then, the inclusion

In the case $O = \mathrm{Span}(\mathbb{F}_G)$, note that an element of $\mathrm{Ar}_{\mathrm{act/el}}(\mathrm{Span}(\mathbb{F}_G))$ is precisely a map of $G$-sets $S \to G/H$; but in fact, there is a unique $H$-set $T$ and equivalence $\mathrm{Ind}_H^G T \simeq S$ over $G/H$, highlighting an equivalence $\mathbb{F}_{G,/G/H} \simeq \mathbb{F}_H$. Hence we have

$$\mathrm{Ar}_{\mathrm{act/el}}(\mathrm{Span}(\mathbb{F}_G)) \simeq \mathrm{Tot}\underline{\mathbb{F}}_G,$$

and $\mathrm{Ar}^{\simeq}_{\mathrm{act/el}}(\mathrm{Span}(\mathbb{F}_G)) \simeq (\mathrm{Tot}\underline{\mathbb{F}}_G)^{\simeq}$ Setting $\overline{\Sigma}_G := (\mathrm{Tot}\underline{\mathbb{F}}_G)^{\simeq}$, the above lemma asserts that

$$(-)_\Sigma : \mathrm{Op}_G \to \mathrm{Fun}(\overline{\underline{\Sigma}}_G, \mathcal{S})$$

is conservative.

*Remark.* Let $\underline{\Sigma}_G := \mathrm{CoFr}^G(\mathbb{F}^{\simeq})$, so that $\overline{\Sigma}_G \simeq (\mathrm{Tot}\underline{\Sigma}_G)^{\simeq}$. Then, the above lemma implies that the evident forgetful functor

$$U : \mathrm{Op}_G \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_G, \mathcal{S})$$

is conservative. The *genuine model structure* $\mathrm{Sym}^G_\bullet(\mathrm{sSet})$ of [BP22] exists and presents $\mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_G, \mathcal{S})$; the $\infty$-category of *Genuine $G$-operads* are then algebras over a monad on $\mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_G, \mathcal{S})$ which are explicitly defined in [BP21].

In this setting, lemma 3.1 amounts to a verification of one of the two Barr-Beck conditions expressing $U$ as *monadic* (cf [HA, Thm 4.7.3.5]); if one can verify that $U$ creates spit geometric realizations and characterize the associated monad along the lines of [BP21], then they may prove that one-object genuine $G$-operads are equivalent to one-object $G$-operads. The author hopes to explore this as a potential strategy for comparison results in the future.

We say that a $G$-operad $O$ is *reduced* if $O_\Sigma(\mathrm{Ind}_H^G T \to G/H) = *$ whenever $T$ is empty or an orbit. In this setting, we can characterize the *monad* associated with an operad:

**Proposition 3.2.** *Let $O$ be a reduced $G$-operad and let $C \in \mathrm{CAlg}_G(\mathrm{Pr}^L_G)$ be a presentably $G$-symmetric monoidal category. Then, the forgetful map $\underline{\mathbf{Alg}}_O(C) \to C$ is monadic, and the associated monad $T_O$ acts on $X \in C$ as*

$$(T_O X)^H \simeq \coprod_{\substack{J \supset K \subset H \\ S \in \mathbb{F}_J}} \left(O(S) \otimes X^{\otimes\left(\mathrm{Ind}_K^H \mathrm{Res}_K^J S\right)}\right)_{h\,\mathrm{Aut}_J S},$$

*where for all $S' \in \mathbb{F}_H$, we write*

$$X^{\otimes S'} := \bigotimes_{U \in \mathrm{Orb}(S')} N_U^H X_U.$$

In fact, there is an adjunction $\mathrm{triv} : \mathcal{S} \rightleftarrows \mathcal{S}_G : F^G$, where triv is fully faithful and bicontinuous (indeed, it has a left adjoint given by $F_G$) and the diagram of forgetful functors

$$\begin{tikzcd}
\mathbf{Alg}_O(\underline{\mathcal{S}}_G)^G \arrow[r, "\sim"] \arrow[d, "U^G"] & \mathrm{Seg}_O(\mathcal{S}_G) \arrow[r, "F^G"] \arrow[d, "U"] & \mathrm{Seg}_O(\mathcal{S}) \arrow[d, "U"] \\
(\underline{\mathcal{S}}_G)^G \arrow[r, "\sim"] & \mathcal{S}_G \arrow[r, "F^G"] & \mathcal{S}
\end{tikzcd}$$

commutes for any $G$-operad $\mathcal{O}$. Taking left adjoints to this yields a commutative diagram of adjunctions, and noting that fixed points of $G$-adjunctions are adjunctions yields the following corollary. Justify weirdness around presentability

**Corollary 3.3.** *Let $\mathcal{O}$ be a reduced G-operad. Then, the associated monad $T_{\mathcal{O},\mathcal{S}}$ acts on $X \in \mathcal{S}$ as*

$$T_{\mathcal{O},\mathcal{S}} X \simeq \left(T_{\mathcal{O},\mathcal{S}} X\right)^G \simeq \coprod_{\substack{J \supset H \\ S \in \mathbb{F}_J}} \left(\mathcal{O}(S) \times \mathrm{Ind}_e^{\mathrm{Ind}_K^G \mathrm{Res}_K^J S} X\right)_{h \mathrm{Aut}_J S}.$$

*In particular, the functor $\mathbf{Alg}_{(-)}(\mathcal{S}) : \mathrm{Op}_G^{\mathrm{Red}} \to \mathbf{Cat}$ is conservative.*

*Proof.* All but the final statement follow by the above analysis. Suppose $\varphi : \mathcal{O} \to \mathcal{P}$ induces an equivalence on $\mathbf{Alg}_{\mathcal{O}}(\mathcal{S}) \to \mathbf{Alg}_{\mathcal{P}}(\mathcal{S})$..

Then $\varphi$ induces a natural equivalence $T_{\mathcal{O},\mathcal{S}} \implies T_{\mathcal{P},\mathcal{S}}$ respecting the summand decomposition in the above presentation. In particular, taking $K = \{e\}$, for all $S \in \mathbb{F}_J$, this induces an equivalence

$$\left(\mathcal{O}(S) \times \mathrm{Ind}_J^S X\right)_{h \mathrm{Aut}_J S}.$$

Choosing $X$ a set with at least 2 points, we find that $n_S \cdot \mathcal{O}(S) \to n_S \cdot \mathcal{P}(S)$ is an equivalence for some $n_S > 0$ and all $S$; this implies that $\mathcal{O}(S) \to \mathcal{P}(S)$ is an equivalence for all $S$, i.e. $\varphi_\Sigma$ is an equivalence. By lemma 3.1, this implies $\varphi$ is an equivalence. □

The remainder of this subsection will be dedicated to proving proposition 3.2. We should probably integrate distributivity

*Proof of proposition 3.2.* Monadicity is precisely [NS22, Cor 5.1.5] when $\mathcal{T} = \mathcal{O}_G$, so it suffices to compute the associated monad in this case.

□

3.2. **The conservativity lemmas.** We have two conservativity lemmas to prove. The first is easier:

**Lemma 3.4.** *Denote by $i : I \cup J \subset I \vee J$ the (non-indexing system) union of subcategories. Then, the pullback*

$$\mathrm{Fbrs}(\mathrm{Span}(\mathcal{I} \cup \mathcal{J})) \to \mathrm{Op}_I \times \mathrm{Op}_J$$

*is conservative. In particular, $U$ reflects equivalences between $\mathcal{I} \vee \mathcal{J}$-operads in the image of $L_{\mathrm{Fbrs}} i_!$.*

*Proof.* Passing to the underlying symmetric sequences yields a diagram

$$\begin{tikzcd} \mathrm{Fbrs}(\mathrm{Span}(I \cup J)) \arrow[r, "i^*"] \arrow[d, tail] & \mathrm{Op}_I \times \mathrm{Op}_J \arrow[d] \\ \mathrm{Fun}(I \cup J, \mathcal{S}) \arrow[r, tail] & \mathrm{Fun}(I, \mathcal{S}) \times \mathrm{Fun}(J, \mathcal{S}) \end{tikzcd}$$

The diagonal functor is a composite of two conservative arrows by **??**, so it is conservative, and hence $i^*$ is conservative. □

The second will take a bit more work. Note that the Segal conditions for Segal Span($I \cup J$)-categories are a *Union* of those of Segal Span($I$)-categories and Segal Span($J$)-categories. That is,

**Lemma 3.5.** *The following diagram of categories is cartesian:*

$$\begin{tikzcd} \mathrm{Seg}_{\mathrm{Span}(I \cup J)}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Seg}_{\mathrm{Span}(I)}(\mathcal{C}) \arrow[d] \\ \mathrm{Seg}_{\mathrm{Span}(J)}(\mathcal{C}) \arrow[r] & \mathrm{Seg}_{\mathrm{Span}(I \cap J)}(\mathcal{C}) \end{tikzcd}$$

In particular, all but the top left are simply categories of product preserving functors. We use this:

**Lemma 3.6.** *There is an equivalence $\mathcal{N}_{(I \vee J)\infty} \simeq L_{\mathrm{Fbrs}} i_! \,\mathrm{Span}(I \cup J)$.*

*Proof.* The functor $L_{\mathrm{Fbrs}} i_! \operatorname{Span}(I \cup J)$ is left adjoint to $i^*$, so it suffices by lemma to verify that the following square is cartesian:

$$\begin{tikzcd}
\mathrm{Fun}^{\times}(\operatorname{Span}(I \vee J), \mathcal{S}) \arrow[r] \arrow[d] & \mathrm{Fun}^{\times}(\operatorname{Span}(I), \mathcal{S}) \arrow[d] \\
\mathrm{Fun}^{times}(\operatorname{Span}(J), \mathcal{S}) \arrow[r] & \mathrm{Fun}^{\times}(\operatorname{Span}(I \cap J, \mathcal{S})
\end{tikzcd}$$

The property that this square is cartesian is witnessed by the equivalence

$$\operatorname{Span}(I \vee J) \simeq \operatorname{Span}(I) \coprod_{\operatorname{Span}(I \cap J)} \operatorname{Span}(J),$$

with pushout taken in the category of Cartesian categories and product preserving functors. ◻

3.3. **The internal hom.** Let $F : \mathcal{O}^{\otimes} \times_G \mathcal{P}^{\otimes} \to \mathcal{E}^{\otimes}$ be a bifunctor of $G$-operads and let $\mathcal{C}^{\otimes} \to \mathcal{E}^{\otimes}$ be a functor of $G$-operads. The following construction was coined in [NS22, § 5.3]. $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{E},G}(\mathcal{O}; \mathcal{C})$ was constructed as follows:

**Construction 3.7.** Define $P : \mathcal{O}^{\otimes} \times_G \operatorname{Ar}(\mathcal{O}_G^{\mathrm{op}}) \to \mathcal{O}^{\otimes}$ by cocartesian pushforward. We have a diagram

$$\mathcal{O}^{\otimes} \xleftarrow{\pi} \mathcal{O}^{\otimes} \times_G \operatorname{Ar}(\mathcal{T}) \times_G \mathcal{P}^{\otimes} \xrightarrow{P \times \mathrm{id}} \mathcal{O}^{\otimes} \times_G \mathcal{P}^{\otimes} \xrightarrow{F} \mathcal{E}^{\otimes}.$$

and an associated push-pull adjunction

$$L_{\mathrm{Fbrs}} F_! (P \times \mathrm{id})_! \pi^* : \mathrm{Op}_{G,/\mathcal{O}} \rightleftarrows \mathrm{Op}_{G,/\mathcal{E}} : \pi_* (P \times \mathrm{id})^* F^*.$$

We verify that this adjunction exists in lemma 3.8. and we define $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{E}}(\mathcal{P}; \mathcal{C}) \to \mathcal{O}^{\otimes}$ to be $\pi_*(P \times \mathrm{id})^* F^* (\mathcal{C}^{\otimes} \to \mathcal{E}^{\otimes})$.

**Lemma 3.8.** *Let $P, F, \pi$ be defined above. Then,*

(1) *$\pi$ is a strong Segal morphism, and the pullback functor*

$$\pi_* : \mathbf{Cat}_{/\mathcal{O}^{\otimes}} \to \mathbf{Cat}_{/\mathcal{O}^{\otimes} \times_G \operatorname{Ar}(\mathcal{O}_G) \times_G \mathcal{P}^{\otimes}}$$

*preserves fibrous patterns; hence $\pi_* : \mathrm{Fbrs}(\mathcal{O}^{\otimes}) \to \mathrm{Fbrs}(\mathcal{O}^{\otimes} \times_G \operatorname{Ar}(\mathcal{O}_G) \times_G \mathcal{P}^{\otimes})$ is right adjoint to $\pi^*$.*

(2) *$P$ is a Segal morphism.*

(3) *$F$ is a Segal morphism.*

*Proof.* For (1), the functor $\pi^*$ simply sends $\mathcal{Q}^{\otimes} \mapsto \mathcal{Q}^{\otimes} \times_G \operatorname{Ar}(\mathcal{O}_G) \times_G \mathcal{P}^{\otimes}$ with structure map given by the product $\pi \times \mathrm{id}$; hence this reduces to checking that (external) products of fibrous patterns are fibrous, which ref??? .

For the rest,ughhhh ◻

The resulting operad is pronounced "the operad of $G$-equivariant $\mathcal{O}$-algebras in $\mathcal{C}$ over $\mathcal{E}$" In [NS22, § 5.3], the following properties were verified.

**Proposition 3.9.** *Let $F : \mathcal{O}^{\otimes} \times_G \mathcal{P}^{\otimes} \to \mathcal{E}^{\otimes}$ be a bifunctor of $G$-operads and let $\mathcal{C}^{\otimes} \to \mathcal{E}^{\otimes}$ be a functor of $G$-operads.*

(1) *If $\mathcal{O}$ has one object, then the underlying $G$-category of $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{E}}(\mathcal{P}; \mathcal{C})$ is the usual $G$-category $\underline{\mathbf{Alg}}_{\mathcal{E}}(\mathcal{P}; \mathcal{C})$.*

(2) *If $\mathcal{C}^{\otimes}$ is $\mathcal{E}$-monoidal, then $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{E}}(\mathcal{P}; \mathcal{C})$ $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ is $\mathcal{O}$-monoidal, and there is a $\mathcal{O}$-monoidal lift $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{E}}(\mathcal{P}; \mathcal{C}) \to \mathcal{C}^{\otimes}$ to the forgetful functor.*

We specialize to the case that $\mathcal{E}^{\otimes} = \mathcal{O}^{\otimes} = \mathrm{Comm}_G^{\otimes}$, in which case we write

$$\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) := \underline{\mathbf{Alg}}^{\otimes}_{\mathrm{Comm}_G}(\mathcal{P}; \mathcal{C}).$$

Then, the above diagram instead reads as

$$\mathrm{Comm}_G^{\otimes} \xleftarrow{\pi} \mathrm{Comm}_G^{\otimes} \times_G \operatorname{Ar}(\mathcal{O}_G^{\mathrm{op}}) \times_G \mathcal{P}^{\otimes} \xrightarrow{P \times \mathrm{id}} \mathrm{Comm}_G^{\otimes} \times_G \mathcal{P}^{\otimes} \xrightarrow{F} \mathrm{Comm}_G^{\otimes}.$$

So that the left adjoint is computed by the fibrous localization of the map $\mathcal{Q} \times_G \mathcal{P} \to \mathrm{Comm}_G^\otimes$ in the following:

$$\begin{tikzcd}
\pi^*(P \times \mathrm{id})_! \mathcal{Q} \arrow[r, phantom, "\simeq"] \arrow[d] & \mathcal{Q} \times_G \mathcal{P} \arrow[dl, "\pi_{\mathcal{Q}} \times \mathrm{id}"] \\
\mathrm{Comm}_G^\otimes \times_G \mathcal{P} \arrow[d, "\mathrm{id} \times \pi_{\mathcal{P}}"] \arrow[dr, "F"] & \\
\mathrm{Comm}_G^\otimes \times \mathrm{Comm}_G^\otimes \arrow[r] & \mathrm{Comm}_G^\otimes
\end{tikzcd}$$

in fact, by definition, this is precisely $\mathcal{Q} \otimes \mathcal{P}$. This concludes the proof of theorem 2.22.

3.4. **Identifying cocartesian symmetric monoidal structures.** In this subsection, we want to prove the following lemma.

**Lemma 3.10** (C.f. [HA, Prop 2.4.3.9])**.** *The following are equivalent for $\mathcal{C}^\otimes \in \mathrm{CMon}_{\mathcal{I}}(\mathbf{Cat})$.*

*(1) For all unital $I$-operads $\mathcal{O}^{otimes}$, the forgetful functor $\underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_G(\mathcal{O}, \mathcal{C})$ is an equivalence.*

*(2) The forgetful functor $\mathrm{CAlg}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*

*(3) For all morphisms $f : S \to T$ in $\mathcal{I}$, the action map $f_\otimes : \mathcal{C}_S \to \mathcal{C}_T$ is left adjoint to the pullback $f^* : \mathcal{C}_T \to \mathcal{C}_S$.*

We will prove this in analogy to the non-equivariant case; in particular, the implication (3) $\implies$ (1) will closely mimic the proof of [HA, Prop 2.4.3.16].

*Proof.* (1) implies (2) by choosing $\mathcal{O} = \mathcal{N}_{I\infty}$. The forgetful functor $\mathrm{CAlg}_I(\mathcal{C}) \to \mathcal{C}$ is $I$-symmetric monoidal by construction, so by ref and cite , (2) implies (3).

Let $\mathcal{C}$ be an $I$-symmetric monoidal category satisfying (3). Let $\Gamma^* \to \mathbb{F}_*$ be the functor of [HA, Const 2.4.3.1] and let $\underline{\Gamma}^*_G := \mathrm{CoFr}^G \Gamma^*$. Then, define the category

$$\mathcal{D} := \mathcal{O}^\otimes \times_{\underline{\mathbb{F}}_{G,*}} \underline{\Gamma}^*_G$$

Define Gamma ◻

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