

YOU CAN CONSTRUCT G -COMMUTATIVE ALGEBRAS ONE NORM AT A TIME

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ABSTRACT. We define the category of G -operads and the hierarchy of *generalized N_∞ -operads*, which are G -suboperads of Comm_G^\otimes . We exhibit an isomorphism between the category of generalized N_∞ -operads and the self-join poset

$$\text{Op}_G^{GN_\infty} \simeq \text{Ind} - \text{Sys}_G \star \text{Ind} - \text{Sys}_G,$$

where $\text{Ind} - \text{Sys}_G$ is the poset of *indexing systems* in G . This recognizes generalized N_∞ -operads as parameterizing *some commutative multiplicative transfers and possibly a commutative multiplication*. Indeed, their algebras in semiadditive Cartesian categories are incomplete Mackey functors and their algebras in Mackey functors recover incomplete Tambara functors when they are N_∞ operads, i.e. when they contain \mathbb{E}_∞ .

After this, we discuss some in-progress research. Namely, we construct a *Boardman-Vogt tensor product* of G -operads and demonstrate that tensor products of generalized N_∞ operads correspond with joins in $\text{Ind} - \text{Sys}_G \star \text{Ind} - \text{Sys}_G$ i.e. there is an $N_{(IV)_\infty}$ -monoidal equivalence

$$\text{Alg}_{N_{I_\infty}} \text{Alg}_{N_{J_\infty}} C \simeq \text{Alg}_{N_{(IV)_\infty}} C$$

for all $N_{(IV)_\infty}$ -monoidal categories C , allowing G -commutative structures to be constructed “one norm at a time.”

Foreword. The following are notes prepared for a casual talk in the [zygotop](#) seminar concerning research which is currently in-progress [cite](#). The reader should read with the understanding that they are particularly casual error-prone, as the non-cited results herein amount to the communication of a pre-draft of a paper in a casual setting.

The reader should implicitly insert the text ∞ - before the words operad and category throughout the following text.

1. INTRODUCTION

In [Dre71], the concept of a *Mackey functor* was introduced; this structure was described as consisting of functors $M_I : \mathcal{O}_G \rightarrow \mathbf{Mod}_R$ and $M_R : \mathcal{O}_G^{\text{op}} \rightarrow \mathbf{Mod}_R$ which agree on \mathcal{O}_G^\simeq and satisfying the *double coset formula*

$$R_J^H I_K^H = \prod_{x \in [J \backslash H / K]} I_{J \cap x K x^{-1}}^I \cdot \text{conj}_X R_{x^{-1} J x \cap K}$$

for all $J, K \subset H$, where $R_J^K := M_R(G/J \rightarrow G/K)$ and similar for I . The ur-example of this is the assignment $H \mapsto \mathbf{Rep}_H$ with covariant functoriality Ind and contravariant functoriality Res . This was repackaged and generalized into the modern definition of the *category of C -valued G -Mackey functors*

$$\mathcal{M}_G(C) := \text{Fun}^\times(\text{Span}(\mathbb{F}_G), C),$$

where \mathbb{F}_G denotes the category of finite G -sets.

In parallel, the concept of *transfer maps in group cohomology* was being developed in [Evens], later lifted to genuine equivariant cohomology in [Greenlees], and finally developed as a functor

$$N_H^G : \text{Sp}_H \rightarrow \text{Sp}_G$$

in [HHR16], which played a crucial role in the solution to the Kervaire invariant one problem. These were noted in [HH16] to satisfy the conditions of a *Symmetric monoidal Mackey functor*, a notion they distinguished from their notion of *G -symmetric monoidal categories* due to coherence issues.

In the broad program announced in [Bar+16], the correct notion of *G -symmetric monoidal G - ∞ -categories* (henceforth *G -symmetric monoidal categories*) was introduced:

Definition 1.1. Let C have finite products. Then, the category of G -commutative monoids in C is

$$\text{CMon}_G(C) := \mathcal{M}_G(C).$$

The category of G -symmetric monoidal categories is $\text{CMon}_G(\mathbf{Cat})$.

We similarly define the *category of small G -categories* as

$$\mathbf{Cat}_G := \mathbf{Fun}(O_G^{\text{op}}, \mathbf{Cat}) \simeq \mathbf{Cat}_{/O_G^{\text{op}}}^{\text{cocart}},$$

where the equivalence is the *straightening-unstraightening construction* of [HTT]. We may informally summarize the structure of a G -symmetric monoidal category $C^\otimes \in \mathbf{CMon}_G(\mathbf{Cat})$ as consisting of, for every conjugacy class (H) of G , a category with Weyl group action $C_H \in \mathbf{Cat}^{BW_G H}$, as well as functors

$$\begin{aligned} \otimes_H^2 : C_H^2 &\rightarrow C_H, \\ N_K^H : C_K &\rightarrow C_H, \\ \text{Res}_K^H : C_H &\rightarrow C_K \end{aligned}$$

for all subconjugacy classes (K) of (H) , which are associative, commutative, unital, and compatible with each other and the Weyl group action, together with coherence. The maps Res encode an underlying G -category C of C^\otimes , and N_K^H is pronounced “the norm from K to H .”

Given C^\otimes a G -symmetric monoidal category, we may informally define a G -commutative monoid to be a tuple of objects $(X_H)_{H \in O_G} \in \prod_{H \in O_G} C_H$ satisfying

$$X_H \simeq \text{Res}_H^G X_G$$

together with structure maps

$$\begin{aligned} \mu_H^2 : X_H^{\otimes 2} &\rightarrow X_H \\ \text{tr}_K^H : N_K^H X_K &\rightarrow X_H, \end{aligned}$$

for all $H \subset K$, together with associativity, commutativity, unitality, and coherence data. We may intuitively view these data as altogether specifying that these structure maps jointly construct a contractible space of maps

$$X^{\otimes S} \rightarrow X_H$$

for all finite H -sets $S \in \mathbb{F}_H$, where

$$X^{\otimes S} := \bigotimes_{H/K \in \text{Orb}(S)} N_K^H X_K.$$

The map tr_K^H is pronounced “the transfer from K to H 3.” When $C^\otimes = M_G(C)^\otimes$ with the *HHR norm* G -symmetric monoidal structure of [HH16], these are called *G -Tambara functors valued in C* .

This talk concerns various relaxations of the notion of G -commutative algebras. Namely, we will define a symmetric monoidal closed category \mathbf{Op}_G of (colored) G -operads, whose internal hom $\mathbf{Alg}_O(C)^\otimes$ is called the *operad of algebras under pointwise tensors*, and whose tensor product is called the *Boardman-Vogt tensor product*.

A particular example will define N_∞ operads, which interpolate between \mathbb{E}_∞ and the G -operad \mathbf{Comm}_G which encodes G -commutative algebras by adding a subset of the transfers parameterized by \mathbf{Comm}_G :

Definition 1.2. A G -transfer system is a core-preserving wide subcategory $O_G^\sim \subset T \subset O_G$ which is closed under base change, i.e. for any diagram in O_G

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow \alpha' & & \downarrow \alpha \\ U' & \longrightarrow & V' \end{array}$$

with $U \hookrightarrow V' \times_{U'} V$ a summand inclusion (pullback taken in \mathbb{F}_G) and $\alpha \in T$, we have $\alpha' \in T$.

An *indexing system* is a subcategory $I \subset \mathbb{F}_G$ induced by a transfer system under taking coproducts. A *generalized indexing system* is a core-preserving subcategory $I \subset \mathbb{F}_G$ which is either an indexing system or is attained by one by removing all non-isomorphisms of trivial G -sets. The poset of indexing systems under inclusion is denoted $\mathbf{Ind} - \mathbf{Sys}_G$, and the poset of generalized indexing systems is denoted $\mathbf{Ind} - \mathbf{Sys}_G^{\text{gen}}$.

It is not hard to see that there is an equivalence of posets

$$\widehat{\text{Ind} - \text{Sys}_G} \simeq \text{Ind} - \text{Sys}_G \star \text{Ind} - \text{Sys}_G,$$

and in particular, generalized indexing systems decompose into two different join-stable copies of indexing systems, depending on whether maps of trivial G -sets are included.

Given a generalized indexing system I , we will construct an operad called $N_{I\infty}^\otimes$ encoding precisely the maps tr_K^H such that $K \hookrightarrow H$ is in I , as well as encoding the map μ_H if and only if I is an indexing system. The main theorem of this talk characterizes the tensor products of generalized N_∞ operads.

Theorem A. *There is a fully faithful and symmetric monoidal inclusion*

$$N_{(-)\infty}^\otimes : \widehat{\text{Ind} - \text{Sys}_G} \xhookrightarrow{\Pi} \text{Op}_G^\otimes$$

whose image consists of the suboperads of Comm_G , and when restricted to the indexing systems has image consisting of operads \mathcal{O} possessing diagrams $\mathbb{E}_\infty \subset \mathcal{O} \subset \text{Comm}_G$. In particular, for \mathcal{C} an $N_{(I\vee J)\infty}$ -monoidal category, there is a canonical $N_{(I\vee J)\infty}$ -monoidal equivalence

$$\underline{\text{Alg}}_{N_{I\infty}}^\otimes \underline{\text{Alg}}_{N_{J\infty}}^\otimes \mathcal{C} \simeq \underline{\text{Alg}}_{N_{(I\vee J)\infty}}^\otimes \mathcal{C}.$$

We say an inclusion of subgroup $H \subset K$ is *atomic* if it is proper and there exist no chains of proper subgroup inclusions $H \subset J \subset K$. More generally, we say that a conjugacy class $(H) \in \text{Conj}(G)$ is an *atomic subclass* of (K) if there exists an atomic inclusion $\tilde{H} \subset \tilde{K}$ with $\tilde{H} \in (H)$ and $\tilde{K} \in (K)$, and we say that (K) is atomic if the canonical inclusion $1 \hookrightarrow K$ is atomic.

Given $(H) \subset (K)$ an atomic subclass, we refer to the N^∞ -operad corresponding to the minimal index system containing the inclusion $H \hookrightarrow K$ as $N^\infty(H, K)$. When $(H) = (1)$, we instead simply write $N^\infty(K)$.

Corollary B. *Let $1 = G_n \subset G_{n-1} \subset \dots \subset G_0 = G$ be a maximal subgroup series of a finite group, and let \mathcal{C} be a G -symmetric monoidal category. Then, there exists a canonical G -symmetric monoidal equivalence*

$$\underline{\text{Alg}}_{N^\infty(G_1, G_0)}^\otimes \cdots \underline{\text{Alg}}_{N^\infty(G_n, G_{n-1})}^\otimes \mathcal{C} \simeq \text{CAlg}_G \mathcal{C}.$$

Furthermore, if $G \simeq H \times J$, then

$$\underline{\text{CAlg}}_H^\otimes \underline{\text{CAlg}}_J^\otimes \mathcal{C} \simeq \underline{\text{CAlg}}_G^\otimes \mathcal{C}.$$

Remark. One may worry about the comparison between models for G -operads, as our notion of N_∞ -operads is ostensibly embedded deep within the world of G - ∞ -operads, which are not known to be equivalent to the ∞ -category presented by the graph model structure or by genuine G operads.

However, some work has been done to simplify the story of N_∞ operads in the model story; in particular, by [Rub21, Thm 2.16, 3.6], the full ∞ -category of the ∞ -category of *genuine* G -operads is equivalent to $\text{Ind} - \text{Sys}_G$ via a functor A which sits in a commutative diagram

$$\begin{array}{ccc} \text{Op}_G^{\text{gen}, N_\infty} & \xrightarrow{N|_{N_\infty}} & \text{Op}_G^{N_\infty} \\ & \searrow A & \downarrow A \\ & & \text{Ind} - \text{Sys}_G \end{array}$$

where we use that the functor N of [BP21] is canonically ∞ -categorical when restricted to full subcategories of Op_G^{gen} which happen to be 1-categories and map to a 1-subcategory of Op_G . Both functors named A are equivalences (c.f. [Ex 2.4.7]Nardin), and hence $N|_{N_\infty}$ is an equivalence.

2. THE IDEAS

2.1. Fibrous patterns. In order to precisely define I -operads, the most efficient way will be to go through the technology of *algebraic patterns*, a concept first defined by German mathematician Honyi Chu and the Norwegian mathematician Rune Haugseng, who generally referred to them using the letter \mathcal{O} .

Definition 2.1. An *algebraic pattern* is an ∞ -category \mathcal{F} , together with a factorization system $(\mathcal{F}^{\text{int}}, \mathcal{F}^{\text{act}})$ of \mathcal{F} and a full subcategory $\mathcal{F}^{\text{el}} \subset \mathcal{F}^{\text{int}}$. The *category of algebraic patterns* is the full subcategory

$$\text{AlgPatt} \subset \text{Fun}(D, \text{Cat})$$

spanned by algebraic patterns, where $D := \bullet \rightarrow \bullet \rightarrow \bullet \leftarrow \bullet$.

Maps in \mathcal{F}^{int} and \mathcal{F}^{act} are pronounced *inert* and *active maps*, and objects of \mathcal{F}^{el} are pronounced *elementary objects*. For instance, \mathbb{F}_* , together with its inert and active maps as defined in [HA, § 2] and elementary objects $\{ \langle 1 \rangle \}$ determines an algebraic pattern. In analogy with [HA, § 2], we will use these to develop a notion of operads, called *fibrous patterns*.

Definition 2.2. Let \mathcal{F} be an algebraic pattern. A *fibrous \mathcal{F} -pattern* is a map of algebraic patterns $\pi : \mathcal{O} \rightarrow \mathcal{F}$ such that

- (1) \mathcal{O} has π -cocartesian lifts for inert morphisms of \mathcal{F} ,
- (2) (Segal condition for colors) For every active morphism $\omega : V_0 \rightarrow V_1$ in \mathcal{F} , the functor

$$\mathcal{K}_{V_0}^{\approx} \rightarrow \lim_{\alpha \in \mathcal{F}_{V_1/}^{\text{el}}} \mathcal{O}_{\omega_{\alpha,!} V_1}^{\approx}$$

induced by cocartesian transport along ω_{α} is an equivalence, where $\omega_{(-)} : \mathcal{F}_{Y/}^{\text{el}} \rightarrow \mathcal{F}_{X/}^{\text{int}}$ is the inert morphism appearing in the inert-active factorization of $\alpha \circ \omega$, and

- (3) (Segal condition for multimorphism) for every active morphism $\omega : V_1 \rightarrow V_2$ in \mathcal{F} and all objects $X_i \in \mathcal{O}_{V_i}$, the commutative square

$$\begin{array}{ccc} \text{Map}_{\mathcal{O}}(X_0, X_1) & \longrightarrow & \lim_{\alpha \in \mathcal{F}_{V_1/}^{\text{el}}} \text{Map}_{\mathcal{O}}(X_0, \omega_{\alpha,!} X_1) \\ \downarrow & & \downarrow \\ \text{Map}_{\mathcal{F}}(V_0, V_1) & \longrightarrow & \lim_{\alpha \in \mathcal{F}_{O_1/}^{\text{el}}} \text{Map}_{\mathcal{F}}(O_0, \omega_{\alpha,!} O_1) \end{array}$$

is cartesian.

A fibrous \mathcal{F} -pattern $\pi : \mathcal{C} \rightarrow \mathcal{F}$ is a *Segal \mathcal{F} -category* if π is a cocartesian fibration. The category of fibrous \mathcal{F} -patterns is the full subcategory

$$\text{Fbrs}(\mathcal{F}) \subset \text{AlgPatt}_{\mathcal{F}}$$

spanned by fibrous patterns, and the category of Segal \mathcal{F} - ∞ -category is the full subcategory of

$$\text{Seg}_{\mathcal{F}}(\text{Cat}) \subset \text{Fbrs}(\mathcal{F}) \times_{\text{Cat}_{\mathcal{F}}} \text{Cat}_{\mathcal{F}}^{\text{cocart}}$$

spanned by Segal \mathcal{O} -categories.

We state one technical lemma:

Lemma 2.3. *All of the inclusions*

$$\text{Seg}(\mathcal{F}) \rightarrow \text{Fbrs}(\mathcal{F}) \hookrightarrow \text{AlgPatt}_{\mathcal{F}} \rightarrow \text{Cat}_{\mathcal{F}} \rightarrow \text{Cat}$$

have left adjoints; in particular, the full subcategory $\text{Fbrs}(\mathcal{F}) \subset \text{AlgPatt}_{\text{base}}$ is localizing.

We refer to the left adjoint $\text{Env} : \text{Fbrs}(\mathcal{F}) \rightarrow \text{Seg}(\mathcal{F})$ as the *Segal envelope*, and we use it analogously to the *symmetric monoidal envelope*, reducing the question of characterizing maps of fibrous patterns into Segal \mathcal{F} -categories into simply a question of characterizing maps of Segal \mathcal{F} -categories, which is much simpler.

Example 2.4:

Definition 2.5. Given the data of \mathcal{X} a category, $\mathcal{X}_b, \mathcal{X}_f$ wide subcategories, and $\mathcal{X}_0 \subset \mathcal{X}_b$ a full subcategory, we define the *span pattern* $\text{Span}_{b,f}(\mathcal{X}; \mathcal{X}_0)$ to have:

- underlying category $\text{Span}_{b,f}(\mathcal{X})$ whose objects are objects in \mathcal{X} and whose morphisms $X \rightarrow Z$ are spans

$$X \xleftarrow{B} Y \xrightarrow{F} Z$$

with $B \in \mathcal{X}_b$ and $F \in \mathcal{X}_f$.

- inert morphisms $\mathcal{X}_b^{\text{op}} \subset \text{Span}(\mathcal{X})$.
- active morphisms $\mathcal{X}_f \subset \text{Span}(\mathcal{X})$.
- Elementary objects $\mathcal{X}_0^{\text{el}} \subset \mathcal{X}_b^{\text{op}}$.

Then, for instance we have the following:

Theorem 2.6 ([BHS22]). *Pullback along the inclusion $\mathbb{F}_* \hookrightarrow \text{Span}(\mathbb{F})$ induces an equivalence on the categories of fibrous patterns and Segal categories.*

2.2. G-operads and I-operads. There is an adjunction

$$\text{Tot} : \mathbf{Cat}_G \rightleftarrows \mathbf{Cat} : \text{CoFr}^G$$

where Tot takes the total category of a cocartesian fibration and $\text{CoFr}^G(C)$ is classified by functor categories

$$\text{CoFr}^G(C)_H := \text{Fun}(O_H^{\text{op}}, C)$$

with functoriality dictated by pullback. In particular, the G -category of small G -categories $\mathbf{Cat}_G := \text{CoFr}^G(C)$ has G -fixed points given by \mathbf{Cat} .

Remark. Elmendorf's theorem may be reinterpreted in this language as the statement that the G -category of G -spaces \mathcal{S}_G is cofreely generated by \mathcal{S} .

Let $\mathbb{F}_G := \text{CoFr}^G(\mathbb{F})$ and let $\mathbb{F}_{G,*} := \text{CoFr}^G(\mathbb{F}_*)$. Then, there is an equivariant lift of [ref](#) :

Theorem 2.7 ([BHS22]). *Pullback along the composition $\mathbb{F}_{G,*} \hookrightarrow \text{Span}(\text{Tot}\mathbb{F}_G) \xrightarrow{U} \text{Span}(\mathbb{F}_G)$ induces an equivalence on the categories of fibrous patterns and Segal categories, where \mathbb{F}_G is the category of G -sets.*

Definition 2.8. The category of G -operads is the category of fibrous patterns

$$\mathbf{Op}_G := \text{Fbrs}(\text{Span}(\mathbb{F}_G)).$$

The following proposition is an exercise in category theory which was carried out in [BHS22, § 5.2].

Proposition 2.9. *An identity-on-objects functor $\pi : \mathcal{O} \rightarrow \text{Span}(\mathbb{F}_G)$ is a G -operad if and only if it satisfies the following conditions:*

- (1) \mathcal{O} has π -cocartesian lifts for inert morphisms of $\text{Span}(\mathbb{F}_G)$.
- (2) For every map of G -sets $S \rightarrow T$, the inert morphisms $\{U \leftarrow T \mid U \in \text{Orb}(T)\}$ induce equivalences

$$\text{Map}_{\mathcal{O}}(S, T) \simeq \prod_{U \in \text{Orb}(T)} \text{Map}_{\mathcal{O}}(S, U).$$

Furthermore, a cocartesian fibration $\pi : \mathcal{O} \rightarrow \text{Span}(\mathbb{F}_G)$ is a Segal $\text{Span}(\mathbb{F}_G)$ -category if and only if it unstraightens to a G -symmetric monoidal category.

We may further reorganize this through the following elementary lemma about G -sets.

Lemma 2.10. *The assignment $\varphi : T \mapsto \text{Ind}_H^G T \rightarrow G/H$ underlies an equivalence of categories*

$$\mathbb{F}_H \simeq (\mathbb{F}_G)_{/G/H}.$$

Hence we have a forgetful functor

$$O(-) : \mathbf{Op}_G^{\text{one-object}} \rightarrow \text{Fun}(\text{Tot}\mathbb{F}_G^{\simeq}, S)$$

Given $S \in \mathbb{F}_H$, we refer to $O(S)$ as the *space of S -ary operations*. We further analyze this functor in [ref](#), proving e.g. that it is conservative.

This is a straightforward, but heavy, generalization of the ∞ -operads of [HA] to the equivariant world, and we suggest the interested reader consult [BHS22] or [NS22] for more details. In particular, postcomposition along the inclusion functor $\mathbb{F} \hookrightarrow \text{Tot}\mathbb{F}_{G,*}$ induces a fully faithful functor $\mathbf{Op} \hookrightarrow \mathbf{Op}_G$.

An early observation about genuine equivariant homotopy coherent algebraic structures is that the structure of transfers *does not come canonically* from an \mathbb{E}_∞ -structure; that is, $\mathbb{E}_\infty \in \mathbf{Op}_G$ is not terminal. The failure of \mathbb{E}_∞ to be terminal is parameterized by the category of *generalized N^∞ -operads*:

Definition 2.11. Write $\text{Comm}_G^\otimes := (\text{Span}(\mathbb{F}_G) = \text{Span}(\mathbb{F}_G))$ for the terminal G -operad. A G -operad \mathcal{O}^\otimes is a *generalized N^∞ -operad* if the unique morphism $\mathcal{O}^\otimes \rightarrow \text{Comm}_G^\otimes$ is a monomorphism, i.e. $\mathcal{O}_U^\otimes \simeq *$ for all U and $\text{Map}_\mathcal{O}^\psi(x, y) \in \{*, \emptyset\}$ for all $\psi : \pi(x) \rightarrow \pi(y)$.

A generalized N^∞ operad $N_{\infty I}$ is an N^∞ operad if it admits a map

$$\mathbb{E}_\infty \rightarrow \mathcal{O}^\otimes.$$

Write $\text{Op}_G^{GN^\infty}$ for the full subcategory consisting of generalized N_∞ -operads. The following proposition is an exercise in category theory, and establishes that a map to an N_∞ operad is a *property*, not a structure.

Proposition 2.12. *Given $N_{I\infty} \in \text{Op}_G^{GN^\infty}$ a generalized N_∞ operad, the forgetful functor*

$$\text{Op}_{G, / N_{I\infty}} \rightarrow \text{Op}_G$$

is fully faithful.

Proof idea. It is equivalent to prove that $\text{Map}(\mathcal{O}, N_{I\infty}) \in \{*, \emptyset\}$ for all $\mathcal{O} \in \text{Op}_G$. In fact, there is a localizing (1-) subcategory $N : \text{Op}_{1,G} \hookrightarrow \text{Op}_G$ consisting of operads whose structure spaces are discrete, and whose localization functor $h : \text{Op}_G \rightarrow \text{Op}_{1,G}$ takes π_0 of the structure spaces. $N_{I\infty}$ evidently lies in $\text{Op}_{1,G}$, so we have

$$\text{Map}_{\text{Op}_G}(\mathcal{O}, N_{I\infty}) \simeq \text{Hom}_{\text{Op}_{1,G}}(h\mathcal{O}, N_{I\infty}).$$

Hence it suffices to check that the latter set is empty or contractible. This is easy to see in $\text{Op}_{1,G}$, since $\text{Hom}(-, *)$ and $\text{Hom}(-, \emptyset)$ are always either empty or contractible. \square

In particular, this implies that $\text{Op}_G^{GN^\infty}$ is a poset, so we'd like to identify this poset. There is a functor

$$A : \text{Op}_G \rightarrow \overline{\text{Ind} - \text{Sys}_G}$$

called the *admissible sets* with value over G/H given by

$$A(\mathcal{O})_{(G/H)} := \{S \rightarrow G/H \mid \pi_\mathcal{O}^{-1}(S \rightarrow G/H) \neq \emptyset\}$$

and extended to general G -sets by coproducts. The following proposition is an exercise in category theory (see [NS22, Ex 2.4.7], or the original references [BH15; GW18; Rub21]):

Proposition 2.13. *The restricted functor*

$$A : \text{Op}_G^{GN^\infty} \rightarrow \overline{\text{Ind} - \text{Sys}_G}$$

is an equivalence of categories.

We denote by $N_{(-)\infty}$ the composite functor

$$N_{(-)\infty} : \overline{\text{Ind} - \text{Sys}_G} \xrightarrow{A^{-1}} \text{Op}_G^{GN^\infty} \hookrightarrow \text{Op}_G$$

Using this, we finally define *I-operads*.

Definition 2.14. Let I be a generalized indexing system. Then, the *category of I-operads* is the slice category

$$\text{Op}_I := \text{Op}_{G, / N_{\infty I}^\otimes}.$$

Given $\mathcal{O}^\otimes, \mathcal{P}^\otimes \in \text{Op}_I$, the *category of \mathcal{O} -algebras in \mathcal{P}* is the full subcategory

$$\text{Alg}_\mathcal{O}(C) \subset \text{Fun}_{/ N_{\infty I}^\otimes}(\mathcal{O}^\otimes, C^\otimes)$$

spanned by maps of I -operads.

Remark. The notation $\text{Alg}_\mathcal{O}(C)$ does not include I . This presents no problem; indeed, by [proposition 2.12](#), the categories of \mathcal{O} -algebras in \mathcal{P} considered over various indexing systems (including the terminal one, i.e. in G -operads) are canonically equivalent to one another.

Example 2.15:

Let $\mathcal{F} \subset \mathcal{O}_G$ be a *family*, i.e. a collection of subgroups of G closed under sub-conjugation. Then, $\mathcal{F} \cup \mathcal{O}_G^\otimes$ is a transfer system, and we denote by $\mathcal{I}_\mathcal{F}$ the corresponding indexing system.

Let V be a real orthogonal G -representation, let \mathcal{F}_V is the family consisting of subgroups H such that $V^H \neq *$, and let $\mathcal{I}_V := \mathcal{I}_{\mathcal{F}_V}$. Then, there is an \mathcal{I}_V -operad \mathbb{E}_V of *little V -disks*, which may be informally understood to have

$$\pi_{\mathbb{E}_V}^{-1}(\text{Ind}_H^G T \rightarrow G/H) := \text{Conf}_H(T, V)$$

the space of H -equivariant embeddings of $T \hookrightarrow V$ (c.f. [Hor19]). These participate in *equivariant infinite loop space theory*, in the sense that there is an equivalence

$$\text{Alg}_{\mathbb{E}_V}(\mathcal{S}_G) \simeq \{V - \text{loop spaces}\};$$

see [Guillou-May](#) for details.

2.3. The BV tensor product. By [ref](#), the category of algebraic patterns has a cartesian monoidal structure.

Definition 2.16. The category of *symmetric monoidal algebraic patterns* is $\text{CMon}(\text{AlgPatt})$.

A symmetric monoidal structure on \mathcal{F} endows on the slice category $\text{AlgPatt}_{/\mathcal{F}}^{\otimes}$ a symmetric monoidal structure, which we may view as taking \mathcal{O}, \mathcal{P} to the tensor product

$$\mathcal{O} \times \mathcal{P} \rightarrow \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}.$$

Definition 2.17. The *Boardman-Vogt symmetric monoidal category of fibrous \mathcal{F} -patterns* is the localized symmetric monoidal structure

$$\text{Fbrs}(\mathcal{F})^{\otimes} \hookrightarrow \text{AlgPatt}_{/\mathcal{F}}^{\otimes}.$$

We may view the tensor product of fibrous \mathcal{F} -patterns as yielding the localized composite

$$\mathcal{O} \otimes \mathcal{P} := L_{\text{Fbrs}}(\mathcal{O} \times \mathcal{P} \rightarrow \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}).$$

Note that the category \mathbb{F}_G has finite products, and any indexing system \mathcal{I} is closed under products. In particular, this endows $i : \mathcal{N}_{\mathcal{I}\infty}^{\otimes} \rightarrow \text{Span}(\mathbb{F}_G)$ with the structure of a map of symmetric monoidal algebraic patterns under the so it has a cartesian monoidal structure. By [cite](#), the forgetful functor $\text{Fbrs}(\mathcal{O}) \rightarrow \text{Fbrs}(\mathcal{F})_{/\mathcal{O}}$ is an equivalence, so we may use this to define the BV tensor product of \mathcal{I} -operads.

Definition 2.18. The *Boardman-Vogt symmetric monoidal category of \mathcal{I} -operads* is

$$\text{Op}_{\mathcal{I}}^{\otimes} := \text{Fbrs}(\mathcal{N}_{\mathcal{I}\infty})$$

The following proposition is easy:

Proposition 2.19. Given an inclusion $i : \mathcal{N}_{\mathcal{I}\infty} \hookrightarrow \mathcal{N}_{\mathcal{J}\infty}$, pushforward along i yields a functor

$$i_! : \text{Op}_{\mathcal{I}}^{\otimes} \rightarrow \text{Op}_{\mathcal{J}}^{\otimes}$$

realizing $\text{Op}_{\mathcal{I}}$ as a symmetric monoidal colocalizing subcategory of $\text{Op}_{\mathcal{J}}$.

The verification of this comes down to the following fact:

Lemma 2.20. Given $f : X \rightarrow Y$ a map of commutative algebra objects in \mathcal{C} a symmetric monoidal, the associated functor $f_! : \mathcal{C}_{/X} \rightarrow \mathcal{C}_{/Y}$ lifts to a canonical symmetric monoidal functor between the slice symmetric monoidal categories.

Given $\mathcal{O}, \mathcal{P} \in \text{Op}_{\mathcal{I}}$, their BV tensor product has a mapping out property:

Proposition 2.21. The category $\text{Alg}_{\mathcal{O} \otimes \mathcal{P}}(\mathcal{Q})$ is equivalent to the category of commutative diagrams of algebraic patterns

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{P} & \longrightarrow & \mathcal{Q} \\ \downarrow \pi_{\mathcal{O}} \times \pi_{\mathcal{P}} & & \downarrow \pi_{\mathcal{Q}} \\ \mathcal{N}_{\mathcal{I}\infty}^{\otimes} \times \mathcal{N}_{\mathcal{I}\infty}^{\otimes} & \xrightarrow{\otimes} & \mathcal{N}_{\mathcal{I}\infty}^{\otimes} \end{array}$$

An \mathcal{I} -operad called the *pointwise tensor product* on $\text{Alg}_{\mathcal{P}}(\mathcal{Q})$ was constructed in [NS22]. By [ref](#), this implies the following proposition:

Proposition 2.22. There is a natural equivalence

$$\text{Alg}_{\mathcal{O} \otimes \mathcal{P}} \mathcal{Q} \simeq \text{Alg}_{\mathcal{O}} \text{Alg}_{\mathcal{P}}^{\otimes} \mathcal{Q}$$

realizing $- \otimes \mathcal{P}$ as left adjoint to $\text{Alg}_{\mathcal{P}}^{\otimes}(-)$.

2.4. Summary of the argument. We would like to construct an equivalence $N_{I\infty} \otimes N_{J\infty} \simeq N_{(I\vee J)\infty}$. Let's begin with the special case $I \subset J$; in this case, we can say something stronger.

Proposition 2.23. *If O is a one-object G -operad, then the map $N^\infty(I) \rightarrow N^\infty(I) \otimes O$ is an I -equivalence; in particular, $N^\infty(I)$ is \otimes -idempotent.*

To prove this, we use [NS22, Cor 5.3.9]; in particular, they generalize [HA] to verify that any of the following conditions are true of $\mathbf{Alg}_{N^\infty(I)}^\otimes(C)$, and we verify that the conditions are equivalent in [ref](#).

Lemma 2.24. *The following are equivalent:*

- (1) *The forgetful functor $\mathbf{CAlg}_I(C) \rightarrow C$ is an equivalence.*
- (2) *For all one-object I -operads O , the forgetful functor $\mathbf{Alg}^O(C) \rightarrow C$ is an equivalence.*
- (3) *The I -restricted operad is cocartesian*

Having proved this, we acquire a (unique) diagram

$$\begin{array}{ccc} N_{I\infty} & & \\ & \searrow & \\ & N_{I\infty} \otimes N_{J\infty} & \xrightarrow{\varphi} N_{(I\vee J)\infty} \otimes N_{(I\vee J)\infty} = N_{(I\vee J)\infty} \\ & \nearrow & \\ N_{J\infty} & & \end{array}$$

and we are tasked with proving that φ is an equivalence. An unfortunate fact is that the functor $U : \mathbf{Op}_{I\vee J} \rightarrow \mathbf{Op}_I \times \mathbf{Op}_J$ doesn't appear to be conservative in general. Our strategy will come down to trying *really hard* to make it conservative. We do so via the following two lemmas, proved as [lemmas 3.4](#) and [3.6](#).

Lemma 2.25. *Denote by $i : I \cup J \subset I \vee J$ the (non-indexing system) union of subcategories. Then, the pullback*

$$\mathbf{Fbrs}(\mathbf{Span}(I \cup J)) \rightarrow \mathbf{Op}_I \times \mathbf{Op}_J$$

is conservative. In particular, U reflects equivalences between $I \vee J$ -operads in the image of $L_{\mathbf{Fbrs}} i_!$.

Lemma 2.26. *There is an equivalence $N_{(I\vee J)\infty} \simeq L_{\mathbf{Fbrs}} i_! \mathbf{Span}(I \cup J)$.*

Proof of [theorem A](#). By the above argument, it suffices to prove that φ is an equivalence; in fact, by [lemmas 2.25](#) and [2.26](#) and symmetry it suffices to prove that the localized functor

$$\iota_J^* N_{I\cap J\infty} \otimes N_{J\infty} \rightarrow \iota_J^* N_{I\vee J}$$

is an equivalence. But $\iota_J^* N_{I\infty} \simeq N_{I\cap J\infty}$, so the above is the inclusion $N_{I\cap J\infty} \otimes N_{J\infty} \rightarrow N_{J\infty}$, which is an equivalence by [proposition 2.23](#). \square

3. TECHNICAL NONSENSE

3.1. Passing to monads is conservative. Our arguments will be reminiscent of [SY19, § 2.3-2.4]

Given $O \rightarrow \mathcal{F}$ a fibrous pattern, we define

$$\mathbf{Ar}_{\text{act/el}}^\approx(\mathcal{F}) \subset \mathbf{Ar}(\mathcal{F})$$

to be the core of the full subcategory of the arrow category consisting of active maps with elementary codomain, and we define

$$O_\Sigma := O \times_{\mathcal{F}} \mathbf{Ar}_{\text{act/el}}^\approx(\mathcal{F}),$$

which we view as the *associated symmetric sequence*.

Lemma 3.1 (C.f. [SY19, Prop 2.3.6]). *Let $\mathbf{Fbrs}_\bullet(\mathcal{F})$ denote the full subcategory of fibrous patterns whose associated maps $O^{\text{el}} \rightarrow \mathcal{F}^{\text{el}}$ are equivalences. Then, the functor*

$$(-)_\Sigma : \mathbf{Fbrs}_\bullet(\mathcal{F}) \rightarrow \mathbf{Fun}(\mathbf{Ar}_{\text{act/el}}^\approx(\mathcal{F}), S)$$

is conservative.

Proof. **Just look at the Segal condition for fibrous patterns** \square

We now specialize to the case $\mathcal{F} = \text{Span}(\mathbb{F}_G)$. Let C be a G -symmetric monoidal category, let $O \in \text{Op}_G$ be a G -operad, and let $X \in \text{Alg}_O(C)$ be an O -algebra in C . Then, the inclusion

In the case $O = \text{Span}(\mathbb{F}_G)$, note that an element of $\text{Ar}_{\text{act/el}}(\text{Span}(\mathbb{F}_G))$ is precisely a map of G -sets $S \rightarrow G/H$; but in fact, there is a unique H -set T and equivalence $\text{Ind}_H^G T \simeq S$ over G/H , highlighting an equivalence $\mathbb{F}_{G,/G/H} \simeq \mathbb{F}_H$. Hence we have

$$\text{Ar}_{\text{act/el}}(\text{Span}(\mathbb{F}_G)) \simeq \text{Tot} \mathbb{F}_G,$$

and $\text{Ar}_{\text{act/el}}^{\simeq}(\text{Span}(\mathbb{F}_G)) \simeq (\text{Tot} \mathbb{F}_G)^{\simeq}$. Setting $\bar{\Sigma}_G := (\text{Tot} \mathbb{F}_G)^{\simeq}$, the above lemma asserts that

$$(-)_{\Sigma} : \text{Op}_G \rightarrow \text{Fun}(\bar{\Sigma}_G, \mathcal{S})$$

is conservative.

Remark. Let $\underline{\Sigma}_G := \text{CoFr}^G(\mathbb{F}^{\simeq})$, so that $\bar{\Sigma}_G \simeq (\text{Tot} \underline{\Sigma}_G)^{\simeq}$. Then, the above lemma implies that the evident forgetful functor

$$U : \text{Op}_G \rightarrow \text{Fun}(\text{Tot} \underline{\Sigma}_G, \mathcal{S})$$

is conservative. The *genuine model structure* $\text{Sym}_{\bullet}^G(\text{sSet})$ of [BP22] exists and presents $\text{Fun}(\text{Tot} \underline{\Sigma}_G, \mathcal{S})$; the ∞ -category of *Genuine G -operads* are then algebras over a monad on $\text{Fun}(\text{Tot} \underline{\Sigma}_G, \mathcal{S})$ which are explicitly defined in [BP21].

In this setting, **lemma 3.1** amounts to a verification of one of the two Barr-Beck conditions expressing U as *monadic* (cf [HA, Thm 4.7.3.5]); if one can verify that U creates spit geometric realizations and characterize the associated monad along the lines of [BP21], then they may prove that one-object genuine G -operads are equivalent to one-object G -operads. The author hopes to explore this as a potential strategy for comparison results in the future.

We say that a G -operad O is *reduced* if $O_{\Sigma}(\text{Ind}_H^G T \rightarrow G/H) = *$ whenever T is empty or an orbit. In this setting, we can characterize the *monad* associated with an operad:

Proposition 3.2. *Let O be a reduced G -operad and let $C \in \text{CALG}_G(\text{Pr}_G^L)$ be a presentably G -symmetric monoidal category. Then, the forgetful map $\underline{\text{Alg}}_O(C) \rightarrow C$ is monadic, and the associated monad T_O acts on $X \in C$ as*

$$(T_O X)^H \simeq \coprod_{\substack{J \supset K \subset H \\ S \in \mathbb{F}_J}} \left(O(S) \otimes X^{\otimes (\text{Ind}_K^H \text{Res}_K^J S)} \right)_{h \text{Aut}_J S},$$

where for all $S' \in \mathbb{F}_H$, we write

$$X^{\otimes S'} := \bigotimes_{U \in \text{Orb}(S')} N_U^H X_U.$$

In fact, there is an adjunction $\text{triv} : \mathcal{S} \rightleftarrows \mathcal{S}_G : F^G$, where triv is fully faithful and bicontinuous (indeed, it has a left adjoint given by F_G) and the diagram of forgetful functors

$$\begin{array}{ccccc} \text{Alg}_O(\underline{\mathcal{S}}_G)^G & \xrightarrow{\sim} & \text{Seg}_O(\mathcal{S}_G) & \xrightarrow{F^G} & \text{Seg}_O(\mathcal{S}) \\ \downarrow U^G & & \downarrow U & & \downarrow U \\ (\underline{\mathcal{S}}_G)^G & \xrightarrow{\sim} & \mathcal{S}_G & \xrightarrow{F^G} & \mathcal{S} \end{array}$$

commutes for any G -operad O . Taking left adjoints to this yields a commutative diagram of adjunctions, and noting that fixed points of G -adjunctions are adjunctions yields the following corollary. **Justify weirdness around presentability**

Corollary 3.3. *Let O be a reduced G -operad. Then, the associated monad $T_{O,\mathcal{S}}$ acts on $X \in \mathcal{S}$ as*

$$T_{O,\mathcal{S}} X \simeq (T_{O,\mathcal{S}} X)^G \simeq \coprod_{\substack{J \supset H \\ S \in \mathbb{F}_J}} \left(O(S) \times \text{Ind}_e^{\text{Ind}_K^G \text{Res}_K^J S} X \right)_{h \text{Aut}_J S}.$$

In particular, the functor $\mathbf{Alg}_{(-)}(\mathcal{S}) : \mathbf{Op}_G^{\text{Red}} \rightarrow \mathbf{Cat}$ is conservative.

Proof. All but the final statement follow by the above analysis. Suppose $\varphi : \mathcal{O} \rightarrow \mathcal{P}$ induces an equivalence on $\mathbf{Alg}_{\mathcal{O}}(\mathcal{S}) \rightarrow \mathbf{Alg}_{\mathcal{P}}(\mathcal{S})$.

Then φ induces a natural equivalence $T_{\mathcal{O},\mathcal{S}} \Rightarrow T_{\mathcal{P},\mathcal{S}}$ respecting the summand decomposition in the above presentation. In particular, taking $K = \{e\}$, for all $S \in \mathbb{F}_J$, this induces an equivalence

$$\left(\mathcal{O}(S) \times \text{Ind}_J^S X \right)_{h \text{Aut}_J S}.$$

Choosing X a set with at least 2 points, we find that $n_S \cdot \mathcal{O}(S) \rightarrow n_S \cdot \mathcal{P}(S)$ is an equivalence for some $n_S > 0$ and all S ; this implies that $\mathcal{O}(S) \rightarrow \mathcal{P}(S)$ is an equivalence for all S , i.e. φ_Σ is an equivalence. By lemma 3.1, this implies φ is an equivalence. \square

The remainder of this subsection will be dedicated to proving proposition 3.2. We should probably integrate distributivity

Proof of proposition 3.2. Monadicity is precisely [NS22, Cor 5.1.5] when $\mathcal{T} = \mathcal{O}_G$, so it suffices to compute the associated monad in this case. \square

3.2. The conservativity lemmas. We have two conservativity lemmas to prove. The first is easier:

Lemma 3.4. Denote by $i : I \cup J \subset I \vee J$ the (non-indexing system) union of subcategories. Then, the pullback

$$\mathbf{Fbrs}(\text{Span}(I \cup J)) \rightarrow \mathbf{Op}_I \times \mathbf{Op}_J$$

is conservative. In particular, U reflects equivalences between $I \vee J$ -operads in the image of $L_{\mathbf{Fbrs}} i_!$.

Proof. Passing to the underlying symmetric sequences yields a diagram

$$\begin{array}{ccc} \mathbf{Fbrs}(\text{Span}(I \cup J)) & \xrightarrow{i^*} & \mathbf{Op}_I \times \mathbf{Op}_J \\ \downarrow & & \downarrow \\ \mathbf{Fun}(I \cup J, \mathcal{S}) & \xrightarrow{\quad} & \mathbf{Fun}(I, \mathcal{S}) \times \mathbf{Fun}(J, \mathcal{S}) \end{array}$$

The diagonal functor is a composite of two conservative arrows by ??, so it is conservative, and hence i^* is conservative. \square

The second will take a bit more work. Note that the Segal conditions for Segal $\text{Span}(I \cup J)$ -categories are a Union of those of Segal $\text{Span}(I)$ -categories and Segal $\text{Span}(J)$ -categories. That is,

Lemma 3.5. The following diagram of categories is cartesian:

$$\begin{array}{ccc} \text{Seg}_{\text{Span}(I \cup J)}(C) & \longrightarrow & \text{Seg}_{\text{Span}(I)}(C) \\ \downarrow & & \downarrow \\ \text{Seg}_{\text{Span}(J)}(C) & \longrightarrow & \text{Seg}_{\text{Span}(I \cap J)}(C) \end{array}$$

In particular, all but the top left are simply categories of product preserving functors. We use this:

Lemma 3.6. There is an equivalence $\mathcal{N}_{(I \vee J)\infty} \simeq L_{\mathbf{Fbrs}} i_! \text{Span}(I \cup J)$.

Proof. The functor $L_{\mathbf{Fbrs}} i_! \text{Span}(I \cup J)$ is left adjoint to i^* , so it suffices by lemma to verify that the following square is cartesian:

$$\begin{array}{ccc} \mathbf{Fun}^\times(\text{Span}(I \vee J), \mathcal{S}) & \longrightarrow & \mathbf{Fun}^\times(\text{Span}(I), \mathcal{S}) \\ \downarrow & & \downarrow \\ \mathbf{Fun}^{\text{times}}(\text{Span}(J), \mathcal{S}) & \longrightarrow & \mathbf{Fun}^\times(\text{Span}(I \cap J), \mathcal{S}) \end{array}$$

The property that this square is cartesian is witnessed by the equivalence

$$\text{Span}(I \vee J) \simeq \text{Span}(I) \coprod_{\text{Span}(I \cap J)} \text{Span}(J),$$

with pushout taken in the category of Cartesian categories and product preserving functors. \square

3.3. Identifying cocartesian symmetric monoidal structures. In this subsection, we want to prove the following lemma.

Lemma 3.7 (C.f. [HA, Prop 2.4.3.9]). *The following are equivalent for $C^\otimes \in \text{CMon}_I(\mathbf{Cat})$.*

- (1) *For all unital I -operads O^{otimes} , the forgetful functor $\underline{\mathbf{Alg}}_O(C) \rightarrow \underline{\mathbf{Fun}}_C(O, C)$ is an equivalence.*
- (2) *The forgetful functor $\text{CAlg}_I(C) \rightarrow C$ is an equivalence.*
- (3) *For all morphisms $f : S \rightarrow T$ in I , the action map $f_\otimes : C_S \rightarrow C_T$ is left adjoint to the pullback $f^* : C_T \rightarrow C_S$.*

We will prove this in analogy to the non-equivariant case; in particular, the implication (3) \implies (1) will closely mimic the proof of [HA, Prop 2.4.3.16].

Proof. (1) implies (2) by choosing $O = \mathcal{N}_{I\infty}$. The forgetful functor $\text{CAlg}_I(C) \rightarrow C$ is I -symmetric monoidal by construction, so by [ref](#) and [cite](#), (2) implies (3).

Let C be an I -symmetric monoidal category satisfying (3). Let $\Gamma^* \rightarrow \mathbb{F}_*$ be the functor of [HA, Const 2.4.3.1] and let $\underline{\Gamma}_G^* := \text{CoFr}^G \Gamma^*$. Then, define the category

$$\mathcal{D} := O^\otimes \times_{\mathbb{F}_G^*} \underline{\Gamma}_G^*$$

Define [Gamma](#) \square

3.4. The pointwise tensor product is an internal hom.

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