ORBITAL CATEGORIES AND WEAK INDEXING SYSTEMS

NATALIE STEWART

ABSTRACT. Fix $\mathcal{T}$ an atomic orbital ∞-category. In this exposé, we initiate the combinatorial study of the poset $\mathrm{wIndex}_{\mathcal{T}}$ of *weak $\mathcal{T}$-indexing systems*, consisting of collecitons of arities for equivariant algebraic structures which can be composed. Within this sits a natural orbital lift $\mathrm{Index}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}$ of Blumberg-Hill's *indexing systems*, consisting of arities for structures possessing binary and nullary operations. For instance, in this settings, results of Balchin-Barnes-Roitzheim quickly imply that the lattice of $C_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}_p$-indexing systems is equivalent to the infinite associahedron.

Along the way, we characterize the relationship between the posets of *unital weak indexing systems* and *indexing systems*, the latter remaining isomorphic to *transfer systems* on this level of generality. We use this to compute the poset of unital C_{p^N}-weak indexing systems for $N \in \mathbb{N} \cup \{\infty\}$.

CONTENTS

1. INTRODUCTION

Fix G a finite group. In [BH15], the notion of *$\mathcal{N}_\infty$-operads for G* was introduced, encapsulating a collection of *blueprints* for G-equivariantly commutative multiplicative structures on Mackey functors which possess underlying Green functors. They demonstrated that the ∞-category of $\mathcal{N}_\infty$-operads for G is an embedded sub-poset of the lattice of *indexing systems* Index_G.

Subsequently, the embedding $\mathcal{N}_\infty - \mathrm{Op}_G \subset \mathrm{Index}_G$ was shown to be an equivalence in several independent works [BP21; GW18; Rub21]; of particular interest is the equivalent characterization of indexing systems as a poset of wide subcategories $\mathrm{IndexCat}_G \subset \mathrm{Sub}(\mathbb{F}_G)$ (referred to as *indexing categories*) and the observation of

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Rubin that indexing categories only depend on their intersections with the orbit category $\mathcal{O}_G = \{G/H\} \subset \mathbb{F}_G$, the resulting embedded subposet

$$\begin{tikzcd} \mathrm{Index}_G \arrow[d, hook] & \mathrm{IndexCat}_G \arrow[l, "\sim"'] \arrow[rr, "\sim"] \arrow[d, hook] & & \mathrm{Transf}_G \arrow[d, hook] \\ \mathrm{FullSub}_G(\underline{\mathbb{F}}_G) & \mathrm{Sub}(\mathbb{F}_G) \arrow[l, "\mathbb{F}_{(-)}"] \arrow[r, "(-)\cap\mathcal{O}_G"'] & \mathrm{Sub}(\mathcal{O}_G) \arrow[r, "\mathrm{p.b.}"'] & \mathrm{Sub}_{\mathrm{Poset}}\mathrm{Sub}_{\mathrm{Grp}}(G) \end{tikzcd}$$

being referred to as *transfer systems*. It is in this form that the burgeoning subfield of *homotopical combinatorics* (coined in [BOOR23], where it is related to finite model category theory) has attacked enumerative problems concerning $\mathcal{N}_\infty$-algebras.

Using the synonymous language of *norm maps* and noting that $\left[\mathcal{O}_{C_{p^n}}\right] = [n+1]$, this approach was used in [BBR21] to prove that $\mathrm{Transf}_{C_{p^n}}$ is equivalent to the $(n+1)$st associahedron K_{n+1}. Furthermore, this has powered a large amount of further work on the topic; for instance, $\mathrm{Transf}_{C_{pqr}}$ is enumerated for p, q, r distinct primes in [BBPR20], with some indications on how to generalize this to arbitrary squarefree integers.

In this paper, we aim to demonstrate how one may extend these enumerative efforts in two ways:

(1) we will remove the assumption on indexing systems that they are closed under coproducts; on the side of algebra, we see in [Ste24] this corresponds with removing the assumption that algebras over the corresponding G-operad $\mathcal{N}_{I\infty}$ in Mackey functors possess underlying green functors.
(2) we will replace the orbit category $\mathcal{O}_G$ with an axiomatic version, called an *atomic orbital ∞-category*; this allows us to fluently describe equivariance under families and cofamilies, as well as extending to more general orbit categories, such as the finite-index orbit category of a compact Lie group.

For the former, we find that in Example 1.30 that the poset of *weak* indexing systems is always infinite; nevertheless, we find when we assert a unitality assumption that $\mathrm{wIndex}_G^{\mathrm{uni}}$ is finite when G is finite, and it can usually be explicitly described in terms of transfer systems and G-families (c.f. Theorem C and Corollary D). Moreover, this behaves well with joins (c.f. Proposition 2.50), and in [Ste24] we establish that joins compute tensor products of unital weak $\mathcal{N}_\infty$-operads.

We assure the skeptical reader that they may freely assume $\mathcal{T}$ is (the orbit category of) a G-family, and replace all instances of orbits $V \in \mathcal{T}$ with homogeneous G-spaces G/H for $H \in \mathcal{F}$ (or with the subgroup $H \subset G$ itself, depending on which is contextually appropriate). Nevertheless, we will now review the axiomatic setting of *(atomic) orbital ∞-categories*.

1.1. Orbital categories.

We briefly review the setting introduced in [BDGNS16].

Construction 1.1 (c.f. [Gla17])**.** Given $\mathcal{T}$ an ∞-category[1], its *finite coproduct completion* is the full subcategory $\mathbb{F}_\mathcal{T} \subset \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S})$ spanned by coproducts of representable presheaves. ◁

Example 1.2. If G is a finite group, then $\mathbb{F}_{\mathcal{O}_G}$ is equivalent to the category of finite G-sets; more generally, if $\mathcal{F} \subset \mathcal{O}_G$ is a subconjugacy-closed family of subgroups, then $\mathbb{F}_\mathcal{F} \subset \mathbb{F}_{\mathcal{O}_G}$ is equivalent to the subcategory of finite G-sets whose stabilizers lie in $\mathcal{F}$. ◁

Inspired by the above example, given $S \in \mathbb{F}_\mathcal{T}$, there is a canonical expression $S \simeq \bigoplus_I V$ for some elements $(V) \subset \mathcal{T}$. We refer to these (V) as *orbits*, and refer to the set of orbits of S as $\mathrm{Orb}(S)$. An important property of the finite coproduct completion is existence of equivalences

$$\mathbb{F}_{\mathcal{T},/S} \simeq \prod_{V \in \mathrm{Orb}(S)} \mathbb{F}_{\mathcal{T},/V}; \qquad\qquad \mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_{\mathcal{T}_{/V}}.$$

We henceforth refer to $\mathcal{T}_{/V}^{\mathrm{op}}$ simply as $\underline{V}$, and $\mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_{\mathcal{T}_{/V}}$ as $\mathbb{F}_V$. Note that, in the case $\mathcal{T} = \mathcal{O}_G$, induction furnishes an equivalence $\mathcal{O}_{G,/[G/H]} \simeq \mathcal{O}_H$, so $\mathbb{F}_{[G/H]} \simeq \mathbb{F}_H$.

Fundamental to representation theory is the *effective Burnside category*, $\mathrm{Span}(\mathbb{F}_G)$; for instance, G-Mackey functors may be presented as product-preserving functors $\mathrm{Span}(\mathbb{F}_G) \to \mathbf{Ab}$. In fact, the spectral Mackey functor theorem of [GM17] presents G-spectra as product-preserving functors of ∞-categories $\mathrm{Span}(\mathbb{F}_G) \to \mathrm{Sp}$, a perspective which has been greatly exploited e.g. in [Bar14; BGS20].

[1] 1-categories embed fully faithfully into ∞-categories, and the reader is free to safely assume all categorical terminology refer to 1-categories (and spaces as sets) if they so choose, at the expense of some examples regarding parameterization over spaces or non-discrete groups.

In $\mathrm{Span}(\mathbb{F}_G)$, composition of morphisms is accomplished via the pullback

$$\begin{tikzcd}
 & & R_{fg} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & R_g \arrow[dl] \arrow[dr] & & R_f \arrow[dl] \arrow[dr] & \\
S & & T & & Q
\end{tikzcd}$$

Indeed, given $\mathcal{T}$ an arbitrary ∞-category, the triple $(\mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}})$ is *adequate* in the sense of [Bar14] if and only if $\mathbb{F}_{\mathcal{T}}$ has pullbacks, in which case the triple is *disjunctive*. Thus, Barwick's construction [Bar14, Def 5.5] defines a $\mathcal{T}$-effective Burnside ∞-category $\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) = A^{eff}(\mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}})$ precisely if $\mathcal{T}$ is *orbital* in the sense of the following definition.

Definition 1.3 ([Nar16, Def 4.1])**.** An ∞-category is *orbital* if $\mathbb{F}_{\mathcal{T}}$ has pullbacks; an orbital ∞-category is *atomic* if all retracts in $\mathcal{T}$ are equivalences. $\triangleleft$

We will not discuss the Burnside ∞-category for the remainder of this paper.

Remark 1.4. We show in Section 2.1 that, if $\mathcal{T}$ is an atomic orbital ∞-category, then $\mathrm{ho}(\mathcal{T})$ is as well, and the main combinatorial objects of this paper are the same between $\mathcal{T}$ and $\mathrm{ho}(\mathcal{T})$; hence the reader may uniformly assume that $\mathcal{T}$ is a 1-category, at the loss of essentially none of the combinatorics. $\triangleleft$

Example 1.5. Given X a space considered as an ∞-category, X is atomic orbital; by [Gla18, Thm 2.13], the associated stable ∞-category is the Ando-Hopkins-Rezk ∞-category of parameterized spectra over X (c.f. [ABGHR14]). In particular, for $X = BG$, this recovers *spectra with G-action*. $\triangleleft$

Example 1.6. Given P a meet semilattice, P is atomic orbital, as the meets in $\mathbb{F}_P$ are easily computed in terms of meets in P. $\triangleleft$

Given G a Lie group, let $\mathcal{S}_G$ denote the ∞-category presented by orthogonal G-spaces, and let $\mathcal{O}_G \subset \mathcal{S}_G$ denote the full subcategory spanned by the homogeneous G-spaces G/H for $H \subset G$ a closed subgroup. A famous issue with equivariant homotopy theory over positive-dimensional Lie groups is that $\mathcal{O}_G$ is not *orbital*; the G-Burnside category does not exist, as $\mathbb{F}_G$ does not have pullbacks with which to define composition of spans.

Nevertheless, this has been rectified in various homotopical contexts. One particularly lucid treatment due to Cnossen-Lenz-Linskens uses the slightly more general setting of *global homotopy theory*.

Definition 1.7 ([CLL23, Def 4.2.2, 4.3.2])**.** If $\mathcal{T}$ is an ∞-category, an *atomic orbital subcategory* of $\mathcal{T}$ is a wide subcategory $\mathcal{P} \subset \mathcal{T}$ satisfying the following conditions:

(a) Denote by $\mathbb{F}_{\mathcal{T}}^{\mathcal{P}} \subset \mathbb{F}_{\mathcal{T}}$ the wide subcategory consisting of morphisms which are disjoint unions of morphisms in $\mathcal{P}$. Then, $\mathbb{F}_{\mathcal{T}}^{\mathcal{P}}$ is stable under pullbacks along arbitrary maps in $\mathbb{F}_{\mathcal{T}}$, and all such pullbacks exist.
(b) Any morphism $A \to B$ in $\mathcal{P}$ admitting a section in $\mathcal{T}$ is an equivalence. $\triangleleft$

An ∞-category is atomic orbital if and only if it's an atomic orbital subcategory of itself. We have a partial converse:

Lemma 1.8. *Suppose $\mathcal{P} \subset \mathcal{T}$ is an atomic orbital subcategory. Then, $\mathcal{P}$ is atomic orbital as an ∞-category.*

Proof. First, assume we have a square in $\mathbb{F}_{\mathcal{P}}$; taking a pullack in $\mathbb{F}_{\mathcal{T}}$, we extended it to be the outer square of the following $\mathcal{T}$-diagram

$$\begin{tikzcd}
T' \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g'"] \arrow[ddr, bend right, "f'"'] & & \\
 & T \times_S S' \arrow[r, dashed, "\pi_T"'] \arrow[d, dashed, "\pi_{S'}"] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "f"] \\
 & S' \arrow[r, "g"'] & S
\end{tikzcd}$$

To prove that $\mathcal{P}$ is orbital, it suffices to verify that the inner square is a pullback, for which it suffices to check that all of the involved maps are in $\mathcal{P}$. First note that, $\pi_{S'}$ and π_T are in $\mathcal{P}$ since $\mathcal{P} \subset \mathcal{T}$ is orbital; h is then in $\mathcal{P}$ since atomic orbital subcategories are left cancellable by [CLL23, Lem 4.3.5], so we've proved that $\mathcal{P}$ is orbital. To see that $\mathcal{P}$ is atomic, note that this immediately follows from the second condition of Definition 1.7. $\square$

Definition 1.9. Given $\mathcal{T}$ an ∞-category, a $\mathcal{T}$*-family* is a full subcategory $\mathcal{F} \subset \mathcal{T}$ satisfying the condition that, given $F : V \to W$ a morphism with $W \in \mathcal{F}$, we have $V \in \mathcal{T}$. A $\mathcal{T}$*-cofamily* is a full subcategory $\mathcal{F}^\perp \subset \mathcal{T}$ such that $\mathcal{F}^{\perp,\mathrm{op}} \subset \mathcal{T}$ is a $\mathcal{T}^{\mathrm{op}}$-family.

Given $\mathcal{T}$ an ∞-category, an *interval family* of $\mathcal{T}$ is an intersection of a family and a cofamily; equivalently, it is a full subcategory $\mathcal{F}$ with the property that whenever $U, W \in \mathcal{F}$ and there is a path $U \to V \to W$, we have $V \in \mathcal{F}$. $\triangleleft$

Observation 1.10. If $\mathcal{F} \subset \mathcal{T}$ is an interval family in an atomic orbital ∞-category satisfying the condition that, for all cospans $U \to W \leftarrow V \in \mathcal{T}$ with $U, W \in \mathcal{F}$, there is a span $U \leftarrow W' \to V$ with $W \in \mathcal{F}$, then the inclusion $\mathbb{F}_{\mathcal{F}} \subset \mathbb{F}_{\mathcal{T}}$ creates pullbacks. In particular, $\mathcal{F}$ is an atomic orbital ∞-category. $\triangleleft$

Example 1.11. Let G be a Lie group and $\mathcal{O}_G^{f.i.} \subset \mathcal{O}_G$ the wide subcategory of the orbit ∞-category spanned by projections $G/K \to G/H$ corresponding with finite-index closed subgroup inclusions $K \subset H$. Then, by [CLL23, Ex 4.2.6], $\mathcal{O}_G^{f.i.} \subset \mathcal{O}_G$ is an orbital subcategory. In fact, it follows quickly from definition that it is atomic as well; hence $\mathcal{O}_G^{f.i.}$ is an atomic orbital ∞-category. The pullbacks in $\mathbb{F}_G^{f.i.}$ are computed by a double coset formula.

In fact, by Observation 1.10, the $\mathcal{O}_G$ interval families consisting of *finite subgroups* and of *finite-index closed subgroups* are atomic orbital ∞-categories as well. The former in the case $G = \mathbb{T}$ yields the *cyclonic orbit category* of [BG16]. $\triangleleft$

Example 1.12. Given $H \subset G$ a closed subgroup, the cofamily $\mathcal{O}^{f.i.}_{G,\geq[G/H]}$ spanned by homogeneous G-spaces G/J admitting a quotient map from G/H satisfies the assumption of Observation 1.10, so it is atomic orbital; in the case $H = N \subset G$ is normal, it is equivalent to $\mathcal{O}^{f.i.}_{G/N}$. In any case, the associated stable homotopy theory is the value category of H*-geometric fixed points* with residual genuine G/H-structure (c.f. [Gla17]). $\triangleleft$

1.2. **Weak indexing systems and weak indexing categories.**

Throughout the remainder of this introduction, we fix $\mathcal{T}$ an orbital ∞-category.

1.2.1. *Weak indexing systems.* In the case $\mathcal{T} = \mathcal{O}_G$ is the orbit category of a compact Lie group G, Elmendorf's theorem [DK84; Elm83] implies that the ∞-category of G-spaces is equivalent to the functor ∞-category

$$\mathcal{S}_G \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}),$$

i.e. they are (homotopy coherent) *indexing systems of spaces*. It has become traditional to allow G to act on the *category theory* surrounding equivariant homotopy theory, culminating in the following definition.

Definition 1.13. The *2-category of $\mathcal{T}$-1-categories* is the functor 2-category[2]

$$\mathbf{Cat}_{\mathcal{T},1} \coloneqq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Cat}_1) \simeq \mathrm{Fun}(h_2\mathcal{T}^{\mathrm{op}}, \mathbf{Cat}_1),$$

where $\mathbf{Cat}_1$ is the 2-category of 1-categories and $h_2(-)$ denotes the homotopy 2-category. $\triangleleft$

We refer to the morphisms in $\mathbf{Cat}_{\mathcal{T},1}$ as $\mathcal{T}$*-functors*. Given a $\mathcal{T}$-1-category $\mathcal{C}$ and an object $V \in \mathcal{T}$, there is a V*-value* 1-category $\mathcal{C}_V \coloneqq \mathcal{C}(V)$, and given a map $V \to W$ in $\mathcal{T}$, there is an associated *restriction functor* $\mathcal{C}_W \to \mathcal{C}_V$.

Example 1.14. By [NS22, Prop 2.5.1], the ∞-category $\mathcal{T}_{/V}$ is a 1-category, so $\mathbb{F}_V \simeq \mathbb{F}_{\mathcal{T}_{/V}} \simeq \mathbb{F}_{\mathcal{T},/V}$ is a 1-category. Hence the functor $\mathcal{T}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ sending $V \mapsto \mathbb{F}_{\mathcal{T},/V}$ is a $\mathcal{T}$-1-category, which we call *the $\mathcal{T}$-1-category of finite $\mathcal{T}$-sets* and denote as $\underline{\mathbb{F}}_{\mathcal{T}}$. $\triangleleft$

Evaluation is functorial in the $\mathcal{T}$-1-category; given a $\mathcal{T}$-functor $\mathcal{C} \to \mathcal{D}$, there is a canonical functor

$$\mathrm{Res}_V^W : \mathcal{C}_V \to \mathcal{D}_V.$$

We refer to a $\mathcal{T}$-functor whose V-values are fully faithful as a *fully faithful $\mathcal{T}$-functor*; if $\iota : \mathcal{C} \to \mathcal{D}$ is a fully faithful $\mathcal{T}$-functor, we say that $\mathcal{C}$ is a *full $\mathcal{T}$-subcategory of* $\mathcal{D}$. A full $\mathcal{T}$-subcategory of $\mathcal{D}$ is uniquely determined by an equivalence-closed and restriction-stable class of objects in $\mathcal{D}$; see [Sha23] for details.

[2] Throughout this paper, *n-category* will mean $(n,1)$-category, i.e. ∞-category whose mapping spaces are $(n-1)$-truncated.

Definition 1.15 (c.f. [HHR16, § 2.2.3])**.** Fix $\mathcal{C}$ a $\mathcal{T}$-1-category. The functor $\mathrm{Ind}_U^V : \mathcal{C}_U \to \mathcal{C}_V$, if it exists, is the left adjoint to Res_U^V. Furthermore, given a V-set S and a tuple $(T_U)_{U \in \mathrm{Orb}(S)}$, the *$S$-indexed coproduct of T_U* is, if it exists, the element

$$\coprod_U^S T_U \coloneqq \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^V T_U \in \mathcal{C}_V.$$

Dually, $\mathrm{CoInd}_U^V : \mathcal{C}_U \to \mathcal{C}_V$ denote the right adjoint to Res_U^V (if it exists), and the S-indexed product is (if it exists), the element

$$\prod_U^S T_U \coloneqq \prod_{U \in \mathrm{Orb}(S)} \mathrm{CoInd}_U^V T_U \in \mathcal{C}_V. \qquad \triangleleft$$

Example 1.16. Given a subgroup inclusion $K \subset H \subset G$, the associated functor $\mathbb{F}_H \to \mathbb{F}_K$ is restriction, and hence its left adjoint $\mathbb{F}_K \to \mathbb{F}_H$ is *G-set induction*, matching the *indexed coprodudcts* of [HHR16, § 2.2.3]. $\triangleleft$

Given $S \in \mathbb{F}_V$, we write

$$\mathcal{C}_S \coloneqq \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_V;$$

we say that $\mathcal{C}$ *strongly admits finite coproducts* if $\coprod_U^S T_U$ always exists, in which case it amounts to a functor

$$\coprod_{-}^S (-) : \mathcal{C}_S \to \mathcal{C}_V.$$

It follows from construction that $\underline{\mathbb{F}}_{\mathcal{T}}$ strongly admits finite coproducts.

Definition 1.17. Given a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ and a full $\mathcal{T}$-subcategory $\mathcal{E} \subset \mathcal{D}$, we say that $\mathcal{E}$ is *closed under $\mathcal{C}$-indexed coproducts* if, for all $S \in \mathcal{C}_V$ and $(T_U) \in \mathcal{E}_S$, we have $\displaystyle\coprod_U^S T_U \in \mathcal{E}_V$. $\triangleleft$

Definition 1.18. We say that a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is *closed under self-indexed coproducts* if it is closed under $\mathcal{C}$-indexed coproducts. $\triangleleft$

Definition 1.19. Given $\mathcal{T}$ an orbital ∞-category, a *$\mathcal{T}$-weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ with V-values $\mathbb{F}_{I,V} \coloneqq (\underline{\mathbb{F}}_I)_V$ satisfying the following conditions:

- (IS-a) Whenever $\mathbb{F}_{I,V} \neq \varnothing$, we have $*_V \in \mathbb{F}_{I,V}$.
- (IS-b) $\underline{\mathbb{F}}_I$ is closed under self-indexed coproducts.

We denote by $\mathrm{wIndex}_{\mathcal{T}} \subset \mathrm{FullSub}_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$ the embedded sub-poset spanned by $\mathcal{T}$-weak indexing systems. Moreover, we say that a $\mathcal{T}$-weak indexing system *has one color* if it satisfies the following condition

- (IS-i) For all $V \in \mathcal{T}$, we have $\mathbb{F}_{I,V} \neq \varnothing$;

these span an embedded subposet $\mathrm{wIndex}_{\mathcal{T}}^{oc} \subset \mathrm{wIndex}_{\mathcal{T}}$. We say that a $\mathcal{T}$-weak indexing system is almost E-unital or (aE-unital) if it satisfies the condition

- (IS-ii) For all noncontractible V-sets $S \sqcup S' \in \mathbb{F}_{I,V}$, we have $S, S' \in \mathbb{F}_{I,V}$.

An almost E-unital $\mathcal{T}$-weak indexing system is *almost unital* if it has one color. These are denoted $\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}$. We say that a $\mathcal{T}$-weak indexing system is *E-unital* if it satisfies the condition

- (IS-iii) For all V-sets $S \sqcup S' \in \mathbb{F}_{I,V}$, we have $S, S' \in \mathbb{F}_{I,V}$.

and an E-unital $\mathcal{T}$-weak indexing system is *unital* if it has one color. We write $\mathrm{wIndex}_{\mathcal{T}}^{E\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}$. Lastly, a $\mathcal{T}$-weak indexing system is an *indexing system* if it satisfies the following condition.

- (IS-iv) The subcategory $\mathbb{F}_{I,V} \subset \mathbb{F}_V$ is closed under finite coproducts for all $V \in \mathcal{T}$.

We denote the resulting poset by $\mathrm{Index}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}}$. $\triangleleft$

Remark 1.20. The indexing systems of [BH15] are seen to be equivalent to ours when $\mathcal{T} = \mathcal{O}_G$ by unwinding definitions. The weak indexing systems of [BP21; Per18] are equivalent to our *unital* weak indexing systems when $\mathcal{T} = \mathcal{O}_G$ by [Per18, Rem 9.7] and [BP21, Rem 4.60]. $\triangleleft$

In practice, we will find that non-aE-unital weak indexing systems are not well behaved, and questions involving aE-unital weak indexing systems are usually quickly reducible to the unital case; the non-combinatorial user is encouraged to focus primarily on unital weak indexing systems for this reason.

1.2.2. *Some examples.*

Example 1.21. The terminal $\mathcal{T}$-weak indexing system is $\underline{\mathbb{F}}_{\mathcal{T}}$; the initial $\mathcal{T}$-weak indexing system is the empty subcategory; the initial one-color $\mathcal{T}$-weak indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}}$ is defined by

$$\mathbb{F}^{\mathrm{triv}}_{\mathcal{T},V} \coloneqq \{*_V\}. \qquad \triangleleft$$

Proposition 1.22. *Given $\underline{\mathbb{F}}_I$ a $\mathcal{T}$-weak indexing system, the following are $\mathcal{T}$-families:*

$$c(I) \coloneqq \{V \in \mathcal{T} \mid *_V \in \mathbb{F}_{I,V}\}$$
$$\upsilon(I) \coloneqq \{V \in \mathcal{T} \mid \varnothing_V \in \mathbb{F}_{I,V}\}$$
$$\nabla(I) \coloneqq \{V \in \mathcal{T} \mid 2*_V \in \mathbb{F}_{I,V}\}$$

Proof. This follows by noting that $\mathrm{Res}_U^V\, n \cdot *_V = n \cdot *_U$ and $\mathcal{T}$-subcategories are restriciton-stable. □

Note that $c(I) \leq \upsilon(I) \cap \nabla(I)$. The following lemma will be used ubiquitously.

Lemma 1.23. *Let $\underline{\mathbb{F}}_I$ be a $\mathcal{T}$-weak indexing system.*

(1) $\underline{\mathbb{F}}_I$ has one-color if and only if $c(I) = \mathcal{T}$.
(2) $\underline{\mathbb{F}}_I$ is E-unital if and only if $\upsilon(I) = c(I)$.
(3) $\underline{\mathbb{F}}_I$ is unital if and only if $\upsilon(I) = \mathcal{T}$.
(4) $\underline{\mathbb{F}}_I$ is an indexing system if and only if $\upsilon(I) \cap \nabla(I) = \mathcal{T}$.

Proof. (1) follows immediately by unwinding definitions. For (2), if $\underline{\mathbb{F}}_I$ is E-unital and $V \in c(I)$, then choosing $\varnothing_V \sqcup *_V \in \mathbb{F}_{I,V}$ yields $\varnothing_V \in \mathbb{F}_{I,V}$, i.e. $V \in \upsilon(I)$. Conversely, if $\upsilon(I) = c(I)$ and $S \sqcup S' \in \mathbb{F}_{i,V}$, then

$$S = \coprod_U^{S \sqcup S'} \chi_S(U), \qquad \qquad \chi_S(U) = \begin{cases} *_U & U \in S \\ \varnothing_U & U \notin S \end{cases}$$

so $\underline{\mathbb{F}}_I$ is E-unital. (3) follows by combining (1) and (2).

For (4), note that $\underline{\mathbb{F}}_I$ an indexing system implies that $\upsilon(I) \cap \nabla(I) = \mathcal{T}$ by taking nullary and binary copowers of $*_V \in \mathbb{F}_{I,V}$. Conversely, if $\upsilon(I) \cap \nabla(I) = \mathcal{T}$, then by iterating binary coproducts $(n-1)$-times, we find that $n*_V = (*_V \coprod (n-1)*_V) \in \mathbb{F}_{I,V}$ for all V and $n \in \mathbb{N}$. Applying **??**, we find that $\mathbb{F}_{I,V}$ is closed under n-ary coproudcts for all finite n, i.e. $\underline{\mathbb{F}}_I$ is an indexing system. □

In fact, the proof of (2) shows more; we may use the same argue to show the following.

Lemma 1.24. *$\underline{\mathbb{F}}_I$ is aE-unital if and only if, whenever $S \in \mathbb{F}_{I,V}$ is noncontractible, $V \in \upsilon(I)$.*

We may use c to reduce study of weak indexing systems to the one-color case via the following.

Construction 1.25. Given $\mathcal{F}$ a $\mathcal{T}$-family and $\underline{\mathbb{F}}_I$ an $\mathcal{F}$-weak indexing system, we may define the $\mathcal{T}$-weak indexing system $E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_I$ by

$$\left(E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_I\right)_V \coloneqq \begin{cases} \mathbb{F}_{I,V} & V \in \mathcal{F}; \\ \varnothing & \text{otherwise.} \end{cases}$$

this yields an embedding of posets $\mathrm{wIndex}_{\mathcal{F}} \to \mathrm{wIndex}_{\mathcal{T}}$. ◁

Proposition 1.26. *The fiber of $c\colon \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ is the image of $E_{\mathcal{F}}^{\mathcal{T}}|_{oc} \colon \mathrm{wIndex}_{\mathcal{F}}^{oc} \to \mathrm{wIndex}_{\mathcal{T}}$.*

In particular, we find that $E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}$ and $E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^{\mathrm{triv}}$ are terminal and initial among $c^{-1}(\mathcal{F})$.

Example 1.27. In [Ste24] we define the *underlying $\mathcal{T}$-symmetric sequence* $\mathcal{O}(-)$ of a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$; when $\mathcal{O}^{\otimes}$ parameterizes a type of equivariant multiplicative structures, the space $\mathcal{O}(S)$ parameterizes the *S-ary operations* endowed on an $\mathcal{O}$-algebra. We define the *arity support*

$$\mathbb{F}_{A\mathcal{O},V} \coloneqq \{S \in \mathbb{F}_V \mid \mathcal{O}(S) \neq \varnothing\};$$

in [Ste24], we show that this possesses a fully faithful right adjoint, making $\mathcal{T}$-weak indexing systems equivalent to *weak $\mathcal{N}_\infty$-$\mathcal{T}$-operads*, i.e. subterminal objects in the ∞-category of $\mathcal{T}$-operads. This inspires our naming; [Ste24] establishes that $\underline{\mathbb{F}}_{A\mathrm{triv}_{\mathcal{T}}} = \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}}$ and $\underline{\mathbb{F}}_{A\mathrm{Comm}_{\mathcal{T}}} = \underline{\mathbb{F}}_{\mathcal{T}}$.

Additionally, choosing $\mathcal{T} = \mathcal{O}_{C_p}$ with standard representation λ, we show that in [Ste24] that the *little* $\infty\lambda$*-disks* C_p*-operad* has arity support

$$\mathbb{F}_{A\mathbb{E}_{\infty\lambda},e} = \mathbb{F}_e, \qquad \mathbb{F}_{A\mathbb{E}_{\infty\lambda},C_p} = \{n \cdot [C_p/e] \mid n \in \mathbb{N}\} \sqcup \left\{*_{C_p} + n \cdot [C_p/e] \mid n \in \mathbb{N}\right\};$$

in particular, this unital weak indexing system corresponds with an interesting algebraic theory and it is *not* an indexing system. $\triangleleft$

Example 1.28. The intial unital $\mathcal{T}$-weak indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^0$ is defined by

$$\mathbb{F}_{\mathcal{T},V}^0 \coloneqq \{\varnothing_V, *_V\};$$

we will see in [Ste24] that this is equal to $\underline{\mathbb{F}}_{A\mathbb{E}_0}$. $\triangleleft$

Example 1.29. The initial $\mathcal{T}$-indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^\infty$ is defined by

$$\mathbb{F}_V^\infty \coloneqq \{n \cdot *_V \mid n \in \mathbb{N}\};$$

we will see in [Ste24] that this is equal to $\underline{\mathbb{F}}_{A\mathbb{E}_\infty}$. $\triangleleft$

Example 1.30. Let $\mathcal{T} = *$ be the terminal category. Then, a full subcategory $\underline{\mathbb{F}}_I \subset \mathbb{F}$ can be identified with a subset $n(I) \subset \mathbb{N}$, Condition (IS-a) with the condition that $n(I)$ is empty or contains 1, and condition Condition (IS-b) with the condition that $n(I)$ is closed under k-fold sums for all $k \in n(I)$. There are many such things; for instance, for each $n \in \mathbb{N}$, the set $\{1\} \cup n\mathbb{N} \subset \mathbb{N}$ gives a nonunital $*$-weak indexing system.

Nevertheless, if we assert that $\varnothing \in n(I)$ (i.e. $\underline{\mathbb{F}}_I$ is unital), then $n(I)$ is closed under summands, i.e. it is lower-closed in $\mathbb{N}$. Thus we have the following computations for $\mathcal{T} = *$:

condition	poset
indexing system	$\mathbb{F}$
unital	$\mathbb{F}^0 \longrightarrow \mathbb{F}$
almost-unital	$\mathbb{F}^{\text{triv}} \longrightarrow \mathbb{F}^0 \longrightarrow \mathbb{F}$
E-unital	$\varnothing \longrightarrow \mathbb{F}^0 \longrightarrow \mathbb{F}$
almost-E-unital	$\varnothing \longrightarrow \mathbb{F}^{\text{triv}} \longrightarrow \mathbb{F}^0 \longrightarrow \mathbb{F}$

$\triangleleft$

Example 1.31. We will see in Corollary 2.4 that when X is a space, there is a canonical equivalence $\mathrm{wIndex}_X \simeq \mathrm{wIndex}_*$ respecting our various conditions. In particular, the computations for *Borel* equivariant weak indexing systems mirror those of Example 1.30. $\triangleleft$

1.2.3. *Weak indexing categories.* With a wealth of examples under our belt, we now simplify the combinatorics.

Observation 1.32. Denote by $\mathrm{Ind}_V^{\mathcal{T}} S \to V$ the map corresponding a V-set S under the equivalence $\mathbb{F}_V \simeq \mathbb{F}_{\mathcal{T},/V}$. This equivalence implies a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is determined by the subgraph

$$I(\mathcal{C}) \coloneqq \left\{ \coprod_i \mathrm{Ind}_{V_i}^{\mathcal{T}} S_i \to V_i \;\middle|\; \forall i, \quad S \in \mathcal{C}_{V_i} \right\} \subset \mathbb{F}_{\mathcal{T}}.$$

In other words, the construction I yields an embedding of posets

$$I(-) : \mathrm{wIndex}_{\mathcal{T}} \hookrightarrow \mathrm{Sub}_{\mathrm{graph}}(\mathbb{F}_{\mathcal{T}}). \qquad \triangleleft$$

We will prove the following in Section 2.2.

Theorem A. *The image of $I(-)$ consists of the subcategories $I \subset \mathcal{C}$ satisfying the following conditions*

(IC-a) (restrictions) I is stable under arbitrary pullbacks in $\mathbb{F}_{\mathcal{T}}$;
(IC-b) (segal condition) the pair $T \to S$ and $T' \to S'$ are in I if and only if $T \sqcup T' \to S \sqcup S'$ is in I; and
(IC-c) ($\Sigma_{\mathcal{T}}$-action) if $S \in I$, then all automorphisms of S are in I.

moreover, for all numbers n, condition (IS-n) of Definition 1.19 is equivalent to condition (IC-n) below:

(IC-i) (one color) I is wide; equivalently, I contains $\mathbb{F}_{\mathcal{T}}^{\simeq}$.
(IC-ii) (aE-unital) if $S \sqcup S' \to T$ is a non-isomorphism identity in I, then $S \to T$ and $S' \to T$ are in I.
(IC-iii) (E-unital) if $S \sqcup S' \to T$ is in I, then $S \to T$ and $S' \to T$ are in I.
(IC-iv) (indexing category) the fold maps $n \cdot V \to V$ are in I for all $n \in \mathbb{N}$ and $V \in \mathcal{T}$.

We refer to the image of $I(-)$ as the *weak indexing categories* $\mathrm{wIndexCat}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_{\mathcal{T}})$. In general, we will refer to a generic weak indexing category as I and its corresponding weak indexing system as $\underline{\mathbb{F}}_I$. The following observations form the basis for the proof of Theorem A.

Observation 1.33. By a basic inductive argument, Condition (IC-b) is equivalent to the following condition:

(IC-b') $S \to T$ is in I if and only if $S_U = S \times_T U \to U$ is in I for all $U \in \mathrm{Orb}(T)$.

in particular, I is uniquely determined by the maps to orbits. ◁

Observation 1.34. By Observation 1.33, in the presence of Condition (IC-b), Condition (IC-a) is equivalent to the following condition:

(IC-a') for all Cartesian diagrams in $\mathbb{F}_{\mathcal{T}}$

$$\begin{tikzcd} T \times_V U \arrow[r] \arrow[d, "\alpha'"] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "\alpha"] \\ U \arrow[r] & V \end{tikzcd} \tag{1}$$

with $U, V \in \mathcal{T}$ and $\alpha \in I$, we have $\alpha' \in I$. ◁

One of the major reasons for this formalism is the technology of *equivariant algebra*. If $\iota : I \subset \mathbb{F}_{\mathcal{T}}$ is a pullback-stable subcategory write $\mathbb{F}_{c(I)}$ for the coproduct closure of the essential image of ι. Then $(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I)$ is an adequate triple in the sense of [Bar14], so we may form the span ∞-category

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \coloneqq A^{eff}(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I),$$

whose forward maps are I and backwards maps are arbitrary. If $\mathcal{C}$ is an ∞-category, the ∞-category of *I-commutative monoids in $\mathcal{C}$* is the product preserving functor ∞-category

$$\mathrm{CMon}_I(\mathcal{C}) \coloneqq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C});$$

the *I-symmetric monoidal 1-categories* are

$$\mathbf{Cat}_{I,1}^{\otimes} \coloneqq \mathrm{CMon}_I(\mathbf{Cat}_1),$$

where $\mathbf{Cat}_1$ denotes the 2-category of 1-categories. These are a form of *I-symmetric monoidal Mackey functors.*

$\mathcal{T}$-commutative monoids yields I-commutative monoids by neglect of structure. By [Ste24], a full $\mathcal{T}$-subcategory of a cocartesian I-symmetric monoidal category $\mathcal{C} \subset \mathcal{D}^{I-\sqcup}$ is I-symmetric monoidal if and only if it's closed under I-indexed coproducts. Hence we have the following.

Corollary B. *Fix a collection of objects $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ containing the contractible $c(I)$-sets and $I \subset \mathbb{F}_{\mathcal{T}}$ the corresponding collection of maps satisfying Condition (IC-b). Then, the following conditions are equivalent:*

(1) I is a weak indexing category;
(2) $\underline{\mathbb{F}}_I$ is a weak indexing system;
(3) $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is an I-symmetric monoidal subcategory under indexed coproducts.

We explore this further in [Ste24], wherein we frequently use that indexed coproducts of arities compute the arities of composite operations in the theory of equivariant operads.

1.3. Weak indexing categories and transfer systems.

Definition 1.35. Given $\mathcal{T}$ an orbital category, an *orbital transfer system in $\mathcal{T}$* is a core-containing wide subcategory $\mathcal{T}^{\simeq} \subset R \subset \mathcal{T}$ satisfying the "base change" condition that for all $\mathcal{T}$ digarams

$$\begin{tikzcd} V' \arrow[r] \arrow[d, "\alpha'"] & V \arrow[d, "\alpha"] \\ U' \arrow[r] & U \end{tikzcd}$$

whose associated $\mathbb{F}_{\mathcal{T}}$ map $V' \to V \times_U U$ is a summand inclusion, if $\alpha \in R$, we have $\alpha' \in R$. The associated embedded sub-poset is denoted $\mathrm{Transf}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_{\mathcal{T}})$. ◁

Observation 1.36. If I is a unital weak indexing category, the intersection $\mathfrak{R}(I) \coloneqq I \cap \mathcal{T}$ is an orbital transfer system; hence it yields a monotone map

$$\mathfrak{R}(-) : \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}}. \quad ◁$$

Transfer systems first arose due to the following phenomenon.

Proposition 1.37 ([NS22, Rmk 2.4.9]). *$\mathfrak{R}(-)$ restricts to an equivalence*

$$\mathfrak{R}(-) : \mathrm{Index}_{\mathcal{T}} \xrightarrow{\sim} \mathrm{Transf}_{\mathcal{T}}.$$

Remark 1.38. In the case $\mathcal{T} = \mathcal{O}_G$, before Nardin-Shah's result, it was shown independently in [Rub19, Thm 3.7] and [BBR21, Cor 8] that the composite inclusion $\mathrm{Sub}_{\mathbf{Grp}}(G) \hookrightarrow \mathcal{O}_G \hookrightarrow \mathbb{F}_G$ induces an embedding $\mathrm{Index}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathbf{Poset}}(\mathrm{Sub}_{\mathbf{Grp}}(G))$ whose image is identified by those subposets which are closed under restriction and conjugation, which were called *G-transfer systems*; this and Proposition 1.37, together imply that pullback along the *homogeneous G-set* functor $\mathrm{Sub}_{\mathbf{Grp}}(G) \to \mathcal{O}_G$ induces an equivalence between the poset of G-transfer systems of [BBR21; Rub19] and the orbital $\mathcal{O}_G$-transfer systems of Definition 1.35. ◁

In view of Remark 1.38, we henceforth in this paper refer to orbital transfer systems simply as *transfer systems*, never referring to the other notion. In Theorem 2.27, we will show that the composite

$$\mathrm{Transf}_{\mathcal{T}} \simeq \mathrm{Index}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$$

is a fully faithful right adjoint to $\mathfrak{R}$, i.e. the poset of unital weak indexing systems possessing a given transfer system has a terminal object, given by the unique such indexing system. However, the fibers can be quite large; for instance, in Remark 2.32, we will see that $\mathfrak{R}$ also attains a fully faithful left adjoint, which is distinct from the right adjoint over all transfer systems when $\mathcal{T}$ has a terminal object (e.g. when $\mathcal{T} = \mathcal{O}_G$).

The upshot is that unital weak indexing systems are not determined by their transitive V-sets. Nevertheless, we may say a bit more, for which the following assumption is helpful.

Definition 1.39. We say that $\mathcal{T}$ *has no self-normalizing transfers* if for all non-isomorphisms $f : V \to W$, there is a summand inclusion $2 \cdot *_V \subset \mathrm{Res}_V^W \mathrm{Ind}_V^W *_V$. ◁

Example 1.40. If G is a finite group, then the following conditions are equivalent:

(1) G is nilpotent.
(2) $\mathcal{O}_G$ has no self-normalizing transfers.

To see this, note that for all $J \subset H$, the double coset formula specializes to the fixed point formula

$$\left(\mathrm{Res}_J^H \mathrm{Ind}_J^H *_J\right)^J = [N_H(J)/J],$$

i.e. $\mathrm{Res}_J^H \mathrm{Ind}_J^H *_J$ admits an $n \cdot *_J$ summand if and only if $n \leq |N_J(H)/J|$. Thus $\mathcal{O}_G$ has no self-normalizing transfers if and only if, for all $J \subsetneq H$, J is not self-normalizing in H (i.e. $J \subsetneq N_J(H)$).

It is well known that the condition that proper subgroups of H are non-self-normalizing is equivalent to the condition that H is nilpotent; thus $\mathcal{O}_G$ has no self-normalizing transfers if and only if all subgroups of G are nilpotent. Since subgroups of nilpotent groups are nilpotent, this is equivalent to G itself being nilpotent. ◁

Construction 1.41. If $\mathcal{T}$ is an orbital ∞-category, then we define the collection of *sparse objects* $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ to have V-value spanned by the V-sets

$$\varepsilon \cdot *_V \sqcup W_1 \sqcup \cdots \sqcup W_n,$$

for $\varepsilon \in \{0, 1\}$ and $W_1, \ldots, W_n \in \underline{V}$ subject to the condition that there exist no maps $W_i \to W_j$ for $i \neq j$. ◁

Example 1.42. Let G be a finite group. Then, for $(H) \subset G$ a conjugacy class of G, the *sparse H-sets* are precisely the H-sets

$$\varepsilon \cdot *_H \sqcup [H/K_1] \sqcup \cdots \sqcup [H/K_n],$$

where none of the conjugacy classes $(K_1), \ldots, (K_n)$ include into each other. ◁

Given $\mathcal{C}^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}$, we may form the full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ generated by $\mathcal{C}^{\mathrm{sprs}}$ under $\mathcal{C}^{\mathrm{sprs}}$-indexed coproducts We say that $\mathcal{C}^{\mathrm{sprs}}$ is *closed under applicable self-indexed coproducts* if $\mathcal{C}^{\mathrm{sprs}} = \mathcal{C} \cap \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}$. Similarly, we define $\underline{\mathbb{F}}_{\mathcal{T}}^{\leq 2}$ to have V-sets consist of the objects admitting summands $n \cdot *_U$ only when $n \leq 2$, and given $\mathcal{C}^{\leq 2} \subset \underline{\mathbb{F}}_{\mathcal{T}}^{\leq 2}$ which generates $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ under $\mathcal{C}^{\leq 2}$-indexed coproducts, we say that $\mathcal{C}^{\leq 2}$ is *closed under applicable self-indexed coproducs* if $\mathcal{C}^{\leq 2} = \mathcal{C} \cap \underline{\mathbb{F}}_{\mathcal{T}}^{\leq 2}$. We prove the following in .

Theorem C. *Suppose $\mathcal{T}$ is an atomic orbital ∞-category. Then, restriciton along the inclusion $\underline{\mathbb{F}}_{\mathcal{T}}^{\leq 2} \hookrightarrow \underline{\mathbb{F}}_{\mathcal{T}}$ yields an embedding of posets*

$$\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \hookrightarrow \mathrm{Coll}(\underline{\mathbb{F}}_{\mathcal{T}}^{\leq 2}),$$

with image the aE-unital collections closed under applicable self-indexed coproducts. Furthermore, if $\mathcal{T}$ has no self-normalizing transfers, then restriction along the inclusion $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}} \hookrightarrow \underline{\mathbb{F}}_{\mathcal{T}}$ yields an embedding of posets

$$\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \subset \mathrm{Coll}(\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}})$$

whose image is spanned by the aE-unital collections which are closed under applicable self-indexed coproducts.

In Remark 3.5 , we remark that Theorem C is compatible with the conditions of Definition 1.19; namely, the conditions of almost-unitality, E-unitality, unitality, and being an indexing system correspond with the same conditions on the sparse or ≤ 2-orbital collections.

We prove in [Ste24] that the aE-unital weak indexing systems are isomorphic to the poset of $\otimes$-idempotent weak $\mathcal{N}_\infty$-operads Thus we may conclude the following.

Corollary 1.43. *If $\mathcal{T}$ is an atomic orbital ∞-category such that $\pi_0(\mathcal{T})$ is finite and $\mathcal{T}_{/V}$ is finite as a 1-category for all $V \in \pi_0(\mathcal{T})$, then there exist finitely many $\otimes$-idempotent weak $\mathcal{N}_\infty$-$\mathcal{T}$-operads.*

Remark 1.44. Let $\mathcal{T} = \mathcal{O}_G$ for G a nilpotent group. By Theorem C, one may devise an inefficient algorithm to compute $\mathrm{wIndex}_G^{\mathrm{uni}}$. Namely, given a collections sparse collection $\mathcal{C}^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_G^{\mathrm{sprs}}$, one may compute all of its self-indexed coproducts in finite time using the double coset formula in order to determine whether $\mathcal{C}^{\mathrm{sprs}}$ is closed under applicable self-indexed coproducts. One may simply iterate over the finite poset $\mathrm{Coll}(\underline{\mathbb{F}}_G^{\mathrm{sprs}})$, performing the above computation at each step to determine which collections correspond with unital weak indexing systems. $\triangleleft$

The above algorithm is quite inefficient; in practice, we instead prefer to divide and conquer, first computing Fam_G and Transf_G, then computing the fibers under $\mathfrak{R}$ and ∇. When $N \in \mathbb{N} \cup \{\infty\}$, we will state the result of this for $G = C_{p^N} = \mathrm{colim}_{n \leq N}\, \mathbb{Z}/p^n\mathbb{Z}$, but first we need notation. Given $R \in \mathrm{Transf}_G$, we define the families

$$\mathrm{Dom}(R) := \left\{ U \in \mathcal{O}_G \mid \;\; \exists U \to V \xrightarrow{f} W \;\text{ s.t. }\; f \in R \right\};$$
$$\mathrm{Cod}(R) := \left\{ U \in \mathcal{O}_G \mid \;\; \exists V \xrightarrow{f} W \leftarrow U \;\text{ s.t. }\; f \in R \right\}.$$

Given a full subcategory $\mathcal{F} \subset \mathcal{O}_G$ and a G-transfer system T, we denote by $\mathrm{Sieve}_T(\mathcal{F})$ the poset of precomposition-closed wide subcategories of $T \cap \mathcal{F}$. We let K_N be the Nth associahedron.

Corollary D. *Fix $N \in \mathbb{N} \cup \{\infty\}$. Then, there is a map of posets*

$$(\mathfrak{R}, \nabla) : \mathrm{wIndex}_{C_{p^N}}^{\mathrm{uni}} \to K_N \times [N]$$

with fibers satisfying

$$\mathfrak{R}^{-1}(R) \cap \nabla^{-1}(\mathcal{F}) = \begin{cases} \varnothing & \mathrm{Dom}(R) \not\leq \mathcal{F}; \\ * & \mathrm{Cod}(R) \leq \mathcal{F}; \\ \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) & \text{otherwise.} \end{cases}$$

Moreover, the associated surjection onto its image is a cocartesian fibration, with cocartesian transport computed along $R \leq R'$ by the inclusion

$$\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \hookrightarrow \mathrm{Sieve}_{R'}(\mathrm{Cod}(R') - \mathcal{F})$$

and computed along $\mathcal{F} \leq \mathcal{F}'$ by the restriction

$$\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \twoheadrightarrow \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}').$$

This completely determines $\mathrm{wIndex}_{C_{p^N}}^{\mathrm{uni}}$. Nevertheless, we draw this explicitly for $N \leq 2$ in Section 3.

1.4. **Why weak indexing systems?** The author finds weak indexing systems compelling for two reasons:

(1) once the algebraist is convinced that they want finite H-sets to index their G-equivariant algebraic structures, weak indexing systems are forced upon them, and our various support properties classify useful properties of algebraic theories;
(2) $\mathbb{E}_V$-ring spaces and $\mathbb{E}_V$-ring spectra frequently appear in algebraic topology, and sometimes this occurs for V a representation which has *zero-dimensional fixed points*, and hence the associated G-operad $\mathbb{E}_V$ has arities supported only on a (unital) *weak* indexing system.

Hopefully this paper and [Ste24] will demonstrate the first point handily, through the following examples:

- $\mathcal{T}$-weak indexing systems are the image of the arity support functor A, and they embed fully faithfully into the ∞-category of $\mathcal{T}$-operads $\mathrm{Op}_{\mathcal{T}}$ through the right adjoint $\mathcal{N}_{(-)\infty} : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T}}$ (c.f. [Ste24]), with image the $\mathcal{T}$-operads whose structure spaces are each either empty or contractible, which we refer to as the poset of *weak $\mathcal{N}_\infty$-operads*.
- *aE*-unital weak indexing systems classify the weak $\mathcal{N}_\infty$-operads which are $\otimes$-idempotent [Ste24]; in algebra, this states that the aE-unital locus consists of those (subcommutative) $\mathcal{T}$-equivariant algebraic theories satisfying the weak form of the *Eckmann-Hilton argument* that

$$\mathrm{CAlg}_I \underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{C}) \xrightarrow{U} \mathrm{CAlg}_I(\mathcal{C})$$

  is an equivalence.
- one-color weak indexing systems are the (subcommutative) $\mathcal{T}$-equivariant algebraic theories with underlying $\mathcal{T}$-objects.
- almost-unital weak indexing systems are the (unique) maximal locus consisting of those weak $\mathcal{N}_\infty$-operads on which $\otimes$ acts by taking joins of weak indexing systems.
- this paper and [Ste24] demonstrate that aE-unital weak indexing systems can be characterized by reducing to the case of unital weak indexing systems using the unit and color fibrations.

The author's favorite example behind the second point is the sign C_2-representation σ; as explained above, its arity-support (which is shared with $\infty\sigma$) is *not* an indexing system. Nevertheless, the evident conjectural extension of Dunn's additivity theorem [Dun88] in the equivariant setting would imply that $\mathbb{E}_\sigma^{\otimes\infty} \simeq \mathbb{E}_{\infty\sigma}$, so one should expect this structure to arise around constructions using $\mathbb{E}_\sigma$ structures (such as Real topological Hochschild homology [AGH21, § 3]).

Indeed, the author critically weak indexing systems in [Ste24] to show that whenever V is a real orthogonal C_2-representation containing infinitely main copies of σ, there is an equivalence $\mathbb{E}_V \otimes \mathbb{E}_\sigma \simeq \mathbb{E}_V$; hence the forgetful functors are equivalences of ∞-categories

$$\mathbf{Alg}_{\mathbb{E}_V} \underline{\mathbf{Alg}}^{\otimes}_{\mathbb{E}_\sigma}(\mathcal{C}) \xrightarrow{\sim} \mathbf{Alg}_{\mathbb{E}_V}(\mathcal{C}) \xleftarrow{\sim} \mathbf{Alg}^{\otimes}_{\mathbb{E}_\sigma} \underline{\mathbf{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C}).$$

This allows one to take arbitrary *iterated THR*, as it lifts THR to a C_2-symmetric monoidal endofunctor

$$\mathrm{THR} : \underline{\mathbf{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C}) \to \underline{\mathbf{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C}),$$

without the assumption that V is a complete C_2-universe.

1.5. **Notation and conventions.** There is an equivalence of categories between that of posets and that of categories whose hom sets have at most one point; we safely conflate these notions. In doing so, we use categorical terminology to describe posets.

A *sub-poset* of a poset P is an injective monotone map $P' \hookrightarrow P$, i.e. a relation on a subset of the elements of P refining the relation on P. A *embedded sub-poset* (or *full sub-poset*) is a sub-poset $P' \hookrightarrow P$ such that $x \leq_{P'} y$ if and only if $x \leq_P y$ for all $x, y \in P'$.

An *adjunction of posets* (or *monotone Galois connection*) is a pair of opposing monotone maps $L : P \rightleftarrows Q : R$ satisfying the condition that

$$Lx \leq_Q y \qquad \Longleftrightarrow \qquad x \leq_P Ry \quad \forall\, x \in P,\ y \in Q.$$

In this case, we refer to L as the *left adjoint* and R as the *right adjoint*, as L is uniquely determined by R and vice versa.

A *cocartesian fibration of posets* is a monotone map $\pi : P \to Q$ satisfying the condition that, for all pairs $q \leq q'$ and $p \in \pi^{-1}(q)$, there exists an element $t_q^{q'} p \in \pi^{-1}(q')$ characterized by the property

$$p \leq p' \qquad \Longleftrightarrow \qquad t_q^{q'} p \leq p' \quad \forall\, p' \in \pi^{-1}(q');$$

in this case, we note that $t_q^{q'} : \pi^{-1}(q) \to \pi^{-1}(q')$ is a monotone map, and we may express P as the set $\coprod_{q \in Q} \pi^{-1}(q)$ with relation determined entirely by the above formula.

Acknowledgements.

2. Weak indexing systems

2.1. **Recovering weak indexing categories from their slice categories.** Recall that the poset of weak indexing systems $\mathrm{wIndexCat} \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_{\mathcal{T}})$ is the embedded subposet spanned by those subcategories satisfying Conditions (IC-a) to (IC-c) of Theorem A.

Proposition 2.1. *If I is a $\mathcal{T}$-weak indexing category then, $I_V := I_{/V}$ is a $\mathcal{T}_{/V}$-weak indexing category.*

Proof. Condition (IC-c) for I_V follows quickly by noting that automorphisms I_V have underlying automorphisms, and Condition (IC-b) for I_V follows by unwinding definitions, noting that $\mathbb{F}_V \to \mathbb{F}_{\mathcal{T}}$ is coproduct-preserving. Lastly, Condition (IC-a) follows by unwinding definitions, noting that the pullback functor $\mathbb{F}_V \to \mathbb{F}_W$ is pullback-preserving for each $W \to V$ since $\mathbb{F}_{\mathcal{T}}$ is an ∞-topos. □

We refer to $\mathcal{T}$-1-categories whose V-values are posets as $\mathcal{T}$-*posets*.

Construction 2.2. We denote the embedded $\mathcal{T}$-subposet with V-values the $\mathcal{T}_{/V}$-weak indexing systems by

$$\underline{\mathrm{wIndexCat}}_{\mathcal{T}} \subset \underline{\mathrm{Sub}}_{\underline{\mathbf{Cat}}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}}). \qquad \triangleleft$$

Given a $\mathcal{T}$-poset $P : \mathcal{T}^{\mathrm{op}} \to \mathrm{Poset}$, we denote by $\Gamma^{\mathcal{T}} P$ the associated limit. There is a monotone map of posets

$$\gamma : \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_{\mathcal{T}}) \to \Gamma \underline{\mathrm{Sub}}_{\underline{\mathbf{Cat}}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}})$$

defined by $\gamma(\mathcal{C})_V \simeq \mathcal{C}_{/V}$ with functoriality supplied by pullback. The primary proposition of this subsection recovers $\mathrm{wIndexCat}_{\mathcal{T}}$ from $\underline{\mathrm{wIndexCat}}_{\mathcal{T}}$.

Theorem 2.3. *γ restricts to an equivalence*

$$\gamma : \mathrm{wIndexCat}_{\mathcal{T}} \xrightarrow{\sim} \Gamma \underline{\mathrm{wIndexCat}}_{\mathcal{T}}$$

Proof. Proposition 2.1 implies that γ restricts to a monotone map of posets $\gamma_W : \mathrm{wIndexCat}_{\mathcal{T}} \to \Gamma^{\mathcal{T}} \underline{\mathrm{wIndexCat}}_{\mathcal{T}}$, so it suffices to prove that this is bijective. In fact, it quickly follows from Condition (IC-b) that γ_W is injective, so it suffices to prove that it is surjective.

To do so, fix $I_{(-)} \in \Gamma^{\mathcal{T}} \underline{\mathrm{wIndexCat}}_{\mathcal{T}}$. Define the subcategory

$$I := \{T \to S \mid \forall U \in \mathrm{Orb}(S), \quad T \times_S V \to V \in I_V\} \subset \mathbb{F}_{\mathcal{T}};$$

note that I satisfies Condition (IC-b') by definition. Furthermore, since any automorphism of V is isomorphic to $*_V \in \mathbb{F}_V$, the subcategory I satisfies Condition (IC-c). Lastly, Condition (IC-a') is precisely the condition that $I_{(-)}$ is an element of $\underline{\mathrm{wIndexCat}}_{\mathcal{T}}$. Hence I is a $\mathcal{T}$-weak indexing system, proving that γ_W is an isomorphism of posets. □

Noting that spaces (as ∞-categories) have *constractible* slice categories, we find the following.

Corollary 2.4. *If X is a space, then the forgetful map $\mathrm{wIndex}_X \to \mathrm{wIndex}_*$ is an equivalence.*

Remark 2.5. The atomic orbital ∞-category $\mathcal{T}_{/V}$ has a terminal object; by [NS22, Prop 2.5.1], this implies that $\mathcal{T}_{/V}$ is a 1-category. In general for $F : J \to \mathcal{T}$ a diagram in an atomic orbital ∞-category indexed by a finite 1-category, $\mathcal{T}_{/J}$ is also a 1-category; in particular, the top arrow

$$\begin{tikzcd} \mathcal{T}_{/J} \arrow[r] \arrow[dr] & \mathrm{ho}(\mathcal{T})_{/J} \arrow[d, phantom, "\wr|"] \\ & \mathrm{ho}(\mathcal{T}_{/J}) \end{tikzcd}$$

is an equivalence. This implies that $\mathbb{F}_{\mathrm{ho}\mathcal{T}}$ has pullbacks, i.e. $\mathrm{ho}(\mathcal{T})$ is orbital; because $\mathcal{T}$ is atomic, retracts in $\mathrm{ho}(\mathcal{T})$ are isomorphisms, i.e. $\mathrm{ho}(\mathcal{T})$ is atomic orbital. $\triangleleft$

Using this and fact that the 1-category of posets is a 1-category, we an equivalence

$$\begin{tikzcd} \mathrm{Sub}(\mathbb{F}_{\mathcal{T}}) \arrow[r, "\mathrm{ho}"] & \mathrm{Sub}(\mathbb{F}_{\mathrm{ho}(\mathcal{T})}) \\ \mathrm{wIndexCat}_{\mathcal{T}} \arrow[r, "\sim"] \arrow[u, hook] & \mathrm{wIndexCat}_{\mathrm{ho}(\mathcal{T})} \arrow[u, hook] \\ \lim_{V \in \mathcal{T}^{\mathrm{op}}} \mathrm{wIndexCat}_{\mathcal{T}_{/V}} \arrow[r, "\sim"] \arrow[u, phantom, "\wr|"] & \lim_{V \in \mathrm{ho}\mathcal{T}^{\mathrm{op}}} \mathrm{wIndexCat}_{\mathrm{ho}(\mathcal{T})_{/V}} \arrow[u, phantom, "\wr|"] \end{tikzcd}$$

In other words, we have the following.

Corollary 2.6. *The homotopy category construction yields an equivalence* $\mathrm{wIndexCat}_{\mathcal{T}} \simeq \mathrm{wIndexCat}_{\mathrm{ho}(\mathcal{T})}$.

Using this, for the rest of the paper, we will assume that $\mathcal{T}$ is a 1-category.

2.2. Weak indexing categories vs weak indexing systems.

Construction 2.7. Given $I \subset \mathbb{F}_{\mathcal{T}}$ a subgraph, define the class of I*-admissible* V*-sets*

$$\mathbb{F}_{V,I} := \left\{ S \mid \mathrm{Ind}_V^{\mathcal{T}} S \to V \in I \right\} \subset \mathbb{F}_V.$$

Taken altogether, we refer to this as $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$. ◁

Recall the notation $I(-)$ used in Observation 1.32.

Observation 2.8. Given $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ a collection of objects, we have $\mathbb{F}_{V,I(\mathcal{C})} \simeq \mathcal{C}$; conversely, if $I \subset \mathbb{F}_{\mathcal{T}}$ satisfies Condition (IC-b), then $I(\underline{\mathbb{F}}_I) = I$. ◁

Observation 2.9. If $S \simeq S'$ as V-sets, then there exists an equivalence $\psi : \mathrm{Ind}_V^{\mathcal{T}} S \simeq \mathrm{Ind}_V^{\mathcal{T}} S'$ over V. Hence whenever $I \subset \mathbb{F}_{\mathcal{T}}$ is a subcategory satisfying Condition (IC-c), $\mathrm{Ind}_V^{\mathcal{T}} S' \to V$ is in I, i.e. $\mathbb{F}_{V,I} \subset \mathbb{F}_V$ is closed under equivalence; these objects determine a unique full subcategory, which we henceforth refer to by the same name.

Conversely, if $\underline{\mathbb{F}}_I$ is a $\mathcal{T}$-weak indexing system and $\mathcal{T}$ has a terminal object $*_{\mathcal{T}}$, then the fact that $\underline{\mathbb{F}}_I$ contains all automorphisms immediately implies that $I(\underline{\mathbb{F}}_I)$ contains all automorphisms. ◁

Observation 2.10. By definition, the restriction map $\mathbb{F}_V \to \mathbb{F}_W$ is implemented by the pullback

$$\begin{tikzcd} \mathrm{Ind}_W^{\mathcal{T}} \mathrm{Res}_W^V S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ind}_V^{\mathcal{T}} S \arrow[d] \\ W \arrow[r] & V \end{tikzcd}$$

thus I satisfies Condition (IC-a') if and only if $\mathrm{Res}_W^V \mathbb{F}_{V,I} \subset \mathbb{F}_{W,I}$; in particular, in this case, $\{\mathbb{F}_{V,I}\}_{V \in \mathcal{T}}$ correspond with a unique full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$. ◁

Proposition 2.11. *If* $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ *is a weak indexing system, then* $I(\mathcal{C})$ *is a weak indexing category.*

Proof. By Theorem 2.3, we may assume that $\mathcal{T}$ has a terminal object. By Observations 1.33 and 1.34, it suffices to verify Conditions (IC-a'), (IC-b') and (IC-c). Condition (IC-a') is verified by Observation 2.10; Condition (IC-b') follows immediately from construction; Condition (IC-c) is verified in Observation 2.9. □

Proposition 2.12. *If* I *is a weak indexing category, then* $\underline{\mathbb{F}}_I$ *is a weak indexing system.*

Proof. Observations 2.9 and 2.10 verify that $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a full $\mathcal{T}$-subcategory, and the fact that the identity arrow on V corresponds with the contractible V-set implies that whenever $\underline{\mathbb{F}}_{I,V} \neq \varnothing$ (i.e. $V \in I$), $*_V \in \underline{\mathbb{F}}_{I,V}$. Thus it suffices to verify that $\underline{\mathbb{F}}_I$ is closed under self-indexed coproducts.

Let $(T_U) \in \underline{\mathbb{F}}_{I,S}$ be an S-tuple in $\underline{\mathbb{F}}_I$ for some $S \in \mathbb{F}_{I,V}$. Then, the indexed coproduct of (T_U) corresponds with the composite arrow

$$\mathrm{Ind}_V^{\mathcal{T}} \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^V T_U = \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^{\mathcal{T}} T_U \to S \to V;$$

the left arrow is in I by Condition (IC-b) applied to the structure maps for each T_U and the right arrow is in I by assumption. Thus the composite is in I, i.e. $\coprod_U^S T_U \in \underline{\mathbb{F}}_I$, as desired. □

Proof of Theorem A. By Propositions 2.11 and 2.12, $I : \mathrm{wIndex}_{\mathcal{T}} \rightleftarrows \mathrm{wIndexCat}_{\mathcal{T}} : \underline{\mathbb{F}}_{(-)}$ are well defined monotone maps; by Observation 2.8, they are inverse to each other, so I is an isomorphism onto its image $\mathrm{wIndexCat}_{\mathcal{T}}$.

What remains is to verify that (IC-n) is equivalent to (IS-n) in Definition 1.19 and Theorem A. For $n = i$, this follows immediately by noting that $V \in I \iff \mathrm{id}_V \in I \iff *_V \in \mathbb{F}_{I,V} \iff \mathbb{F}_{I,V} \neq \varnothing$. For $n = ii$ and $n = iii$, this follows by unwinding definitions using by Condition (IC-b'). For $n = iv$, this follows by noting that the fold map $n \cdot V \to V$ corresponds with the element $n \cdot *_V \in \mathbb{F}_V$. □

2.3. **Joins, closures, color-support, color-borelification, and coinduction.**

2.3.1. *Prerequisites on cocartesian fibrations.* Recall that a monotone map $\pi : \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration if and only if, for all related pairs $D \leq D'$ in $\mathcal{D}$ and elements $C \in \pi^{-1}(D)$, there is an element $t_D^{D'} C \in \pi^{-1}(D')$ satisfying the property

$$C \leq C' \qquad \iff \qquad t_C^{C'} C \leq C' \quad \forall\, C' \text{ s.t. } D' \leq \pi(C')$$

Proposition 2.13. *Suppose $\pi : \mathcal{C} \to \mathcal{D}$ is a monotone map possessing a left adjoint L and $\mathcal{C}$ has binary joins. Then, π is a cocartesian fibration with*

$$t_C^{C'} C = L(D) \vee C.$$

Proof. This follows immediately from the propery

$$L(D') \vee C \leq C' \iff L(D') \leq C' \text{ and } C \leq C',$$

noting that $L(D') \leq C'$ by assumption. □

Lemma 2.14. *Let $\pi : \mathcal{C} \to \mathcal{D}$ be a monotone map. The following are equivalent.*

(a) π possesses a fully faithful left adjoint L.
(b) For all $D \in \mathcal{D}$, the preimage $\pi^{-1}(\mathcal{D}_{\geq D})$ possesses an initial object $L(D)$ with $\pi L(D) = D$.
(c) For all $D \in \mathcal{D}$, the fiber $\pi^{-1}(D)$ has an initial object $L(D)$, and $D \leq D'$ implies $L(D) \leq L(D')$.

Furthermore, the element $L(D)$ agrees between these three constructions.

Proof. By definition, π has a left adjoint L if and only if there are initial objects to $\pi^{-1}(\mathcal{D}_{\leq D})$, which are $L(D)$. By the usual category theoretic nonsense, L is fully faithful if and only if the unit relation $D \leq \pi L(D)$ is an equiality, i.e. $L(D) \in \pi^{-1}(D)$; hence (a) $\iff$ (b). To see (b) $\iff$ (c), it follows to note that $L(D) \leq C'$ if and only if $D \leq C'$ if and only if $L(D) \leq L\pi(C')$. □

2.3.2. *Closures and joints of weak indexing systems.*

Construction 2.15. Given $\mathcal{D}, \mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ full $\mathcal{T}$-subcategories, inductively define $\mathrm{Cl}_{\mathcal{D},0}(\mathcal{C}) \coloneqq \mathcal{C}$ and

$$\mathrm{Cl}_{\mathcal{D},n}(\mathcal{C})_V = \left\{ \coprod_U^S T_U \mid (T_U) \in \mathrm{Cl}_{n-1}(\mathcal{C})_S, \;\; S \in \mathcal{D} \right\},$$

with $\mathrm{Cl}_{\mathcal{D},\infty}(\mathcal{C}) \coloneqq \bigcup_n \mathrm{Cl}_{\mathcal{D},n}(\mathcal{C})$. and $\mathrm{Cl}_n(\mathcal{C}) \coloneqq \mathrm{Cl}_{\mathcal{C},n}(\mathcal{C})$. We call this the *$n$-step closure of $\mathcal{C}$ under $\mathcal{D}$-indexed coproducts* or just the *closure of $\mathcal{C}$ under $\mathcal{D}$-indexed coproducts* when $n = \infty$. ◁

Observation 2.16. If $\mathcal{D}$ is a weak indexing system, then the canonical inclusion

$$\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C}) \subset \mathrm{Cl}_{\mathcal{D}}(\mathcal{C})$$

is an equality for all $\mathcal{C}$. ◁

Let $\mathrm{FullSub}^*_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}}) \subset \mathrm{FullSub}_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$ denote the full subposet of elements satisfying Condition (IS-a).

Lemma 2.17. *The fully faithful map $\iota : \mathrm{wIndex}_{\mathcal{T}} \hookrightarrow \mathrm{FullSub}^*_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$ is right adjoint to Cl_∞.*

Proof. If $\mathrm{Cl}_\infty(\mathcal{C})$ is a weak indexing system, then it is clearly minimal among those containing $\mathcal{C}_S$, so it suffices to prove that it's a weak indexing system. Note that $\mathrm{Cl}_\infty(\mathcal{C})_V \neq \varnothing$ iff $\mathcal{C}_V \neq \varnothing$ iff $*_V \in \mathcal{C}_V$ iff $*_V = \coprod_{*_V}^{*_V} *_V \in \mathrm{Cl}_\infty(\mathcal{C})_V$, so it suffices to prove that $\mathrm{Cl}_\infty(\mathcal{C})$ is closed under self-indexed coproducts.

In fact, if by a basic inductive argument, we find that $\mathrm{Cl}_i(\mathcal{C})$-indexed coproducts of elements of $\mathrm{Cl}_j(\mathcal{C})$ lie in $\mathrm{Cl}_{i+j}(\mathcal{C}) \subset \mathrm{Cl}_\infty(\mathcal{C})$, so the result follows by taking a union. □

Observation 2.18. If $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ was a collection of objects *not* closed under restriction, then there is a corresponding minimal full $\mathcal{T}$-subcategory $\widehat{\mathcal{C}} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ containing $\mathcal{C}$, with objects consisting of the restrictions of objects in $\mathcal{C}$. In general, we will write $\mathrm{Cl}_\infty(\mathcal{C}) \coloneqq \mathrm{Cl}_\infty(\widehat{\mathcal{C}})$, and note that this is the minimal weak indexing system containing the objects in $\mathcal{C}$. ◁

Given $S \in \mathbb{F}_V$, let $\mathbb{F}_{I_S,V}$ be the closure of $\{*_V\}$ under S-indexed coproducts; more generally, let $\mathbb{F}_{I_S,W} \coloneqq \bigcup_{f:W \to V} \mathrm{Res}_W^V \mathbb{F}_{I_S,V}$, and let $\left(\underline{\mathbb{F}}_{I_S}\right)_W \coloneqq \mathbb{F}_{I_S,W}$.

Proposition 2.19. *Given $S \in \mathbb{F}_V$, we have $\mathrm{Cl}_\infty(\{S\}) = \underline{\mathbb{F}}_{I_S}$.*

Proof. First, note that $\underline{\mathbb{F}}_{I_S} \subset \mathrm{Cl}_\infty(\{S\})$. By Lemma 2.17, it suffices to prove that $\underline{\mathbb{F}}_{I_S}$ is weak indexing system containing S.

By construction, $\underline{\mathbb{F}}_{I_S}$ is a full $\mathcal{T}$-full subcategory satisfying the property that

$$*_W \in \mathbb{F}_{I_S,W} \qquad \Longleftrightarrow \qquad \exists f : W \to V \qquad \Longleftrightarrow \qquad \varnothing \neq \mathbb{F}_{I_S,W}.$$

Hence it suffices to prove that $\underline{\mathbb{F}}_{I_S}$ is closed under self-indexed coproducts.

First, note that if $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is closed under T-indexed coproducts and X_U-indexed coproducts for $(X_U) \in \mathbb{F}_T$, then $\mathcal{C}$ is closed under $\coprod_U^T X_U$-indexed coproducts; hence $\mathbb{F}_{I_S,V}$ is closed under $\mathbb{F}_{I_S,V}$-indexed coproducts.

Second, note that if $\mathcal{C}_W$ is generated under restrictions by $\mathcal{C}_U$ and $\mathcal{C}_U$ is closed under T-indexed coproducts, then $\mathcal{C}_W$ is closed under $\mathrm{Res}_W^U T$-indexed coproducts; hence $\underline{\mathbb{F}}_{I_S}$ is closed under self-indexed coproducts, as desired. □

Proposition 2.20. *$\mathrm{wIndex}_{\mathcal{T}}$ is a lattice; the meets in $\mathrm{wIndex}_{\mathcal{T}}$ are intersections, and the joins are*

$$\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = \bigcup_{n \in \mathbb{N}} \overbrace{\mathrm{Cl}_I \mathrm{Cl}_J \cdots \mathrm{Cl}_I \mathrm{Cl}_J}^{2n}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J).$$

Proof. By Lemma 2.17, $\mathrm{wIndex}_{\mathcal{T}}$ has meets computed in $\mathrm{FullSub}^*_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$, which are clearly given by intersections. Furthermore, Lemma 2.17 implies that $\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J)$. Thus is suffices to note that, for arbitrary $\mathcal{C}, \mathcal{D}, \mathcal{E}$, we have

$$\mathrm{Cl}_{\mathcal{C} \cup \mathcal{D}, \infty}(\mathcal{E}) = \bigcup_{n \in \mathbb{N}} \overbrace{\mathrm{Cl}_{\mathcal{C}} \mathrm{Cl}_{\mathcal{D}} \cdots \mathrm{Cl}_{\mathcal{C}} \mathrm{Cl}_{\mathcal{D}}}^{2n}(\mathcal{E}),$$

and set $\mathcal{C} = \underline{\mathbb{F}}_I$, $\mathcal{D} = \underline{\mathbb{F}}_J$, and $\mathcal{E} = \underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J$. □

Observation 2.21. For any full $\mathcal{T}$-subcategory $\mathcal{C}$ and family $\mathcal{F} \supset c(\mathcal{C})$, we have $\mathrm{Cl}_{\underline{\mathbb{F}}_{\mathcal{F}}^{\mathrm{triv}}}(\mathcal{C}) = \mathcal{C}$; in particular, when $\underline{\mathbb{F}}_I$ is a weak indexing system, this implies that

$$\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_{\mathcal{F}}^{\mathrm{triv}} = \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_{\mathcal{F}}^{\mathrm{triv}}. \qquad ◁$$

2.3.3. *The color-support fibration.* Recall the map c from Proposition 1.22.

Proposition 2.22. *The monotone map $c : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ has a fully faithful left adjoint $\underline{\mathbb{F}}_{(-)}^{\mathrm{triv}}$ and a fully faithful right adjoint $\underline{\mathbb{F}}_{(-)}$.*

Proof. By Lemma 2.14 it suffices to note that $\underline{\mathbb{F}}_{c(\underline{\mathbb{F}}_I)}^{\mathrm{triv}} \leq \underline{\mathbb{F}}_I \leq \underline{\mathbb{F}}_{c(\underline{\mathbb{F}}_I)}$. □

The following proposition follows by undwinding definitions, and allows us to reduce our analysis entirely to the one-object case.

Proposition 2.23. *The fiber $c^{-1}(\mathrm{Fam}_{\mathcal{T}, \leq \mathcal{F}})$ is equivalent to $\mathrm{wIndex}_{\mathcal{F}}^{\mathrm{oc}}$, and the associated fully faithful functor $E_{\mathcal{F}}^{\mathcal{T}} : \mathrm{wIndex}_{\mathcal{F}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ is left adjoint to $\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \coloneqq (-) \cap \underline{\mathbb{F}}_{\mathcal{F}}$.*

Thus, applying Proposition 2.13 and Observation 2.21, we arrive at the following.

Corollary 2.24. *Let $\mathcal{T}$ be an orbital category.*

(1) The map $c : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ is a cocartesian fibration with fiber $c^{-1}(\mathcal{F}) = \mathrm{wIndex}_{\mathcal{F}}^{\mathrm{oc}}$ and with cocartesian transport along $\mathcal{F} \leq \mathcal{F}'$ sending $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_{\mathcal{F}'}^{\mathrm{triv}}$.
(2) The map $c : \mathrm{wIndex}_{\mathcal{T}}^{E\mathrm{uni}} \to \mathrm{Fam}_{\mathcal{T}}$ is a cocartesian fibration with fiber $c^{-1}(\mathcal{F}) = \mathrm{wIndex}_{\mathcal{F}}^{\mathrm{uni}}$ and cocartesian transport along $\mathcal{F} \leq \mathcal{F}'$ sending $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_{\mathcal{F}'}^{\mathrm{triv}}$.

(3) *The map* $c : \mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \to \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration with fiber* $c^{-1}(\mathcal{F}) = \mathrm{wIndex}_{\mathcal{F}}^{a\mathrm{uni}}$ *and cocartesian transport along* $\mathcal{F} \le \mathcal{F}'$ *sending* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_{\mathcal{F}'}^{\mathrm{triv}}$.

,

Remark 2.25. Entailed in this corollary is the statement that $\underline{\mathbb{F}}_I$ is E-unital if and only if $\underline{\mathbb{F}}_I = E_{c(I)}^{\mathcal{T}} \mathrm{Bor}_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_I$ and $\mathrm{Bor}_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_I$ is unital; in particular, we find that the E-unital weak indexing systems are those which are E of unital weak indexing systems. ◁

2.3.4. *Coinduction.* If it exists, the right adjoint to $\mathrm{Res}_V^W : \mathrm{wIndex}_W \to \mathrm{wIndex}_V$ is denoted CoInd_V^W.

Proposition 2.26. *Let* $\underline{\mathbb{F}}_I$ *be a weak indexing system. Then,* $\mathrm{CoInd}_V^W \underline{\mathbb{F}}_I$ *exists and is computed by*

$$\left(\mathrm{CoInd}_V^W \underline{\mathbb{F}}_I\right)_U = \left\{ S \in \mathbb{F}_U \;\middle|\; \forall U \leftarrow U' \to V, \;\; \mathrm{Res}_{U'}^U S \in \mathbb{F}_{I,U'} \right\}$$

Proof. Denote by $\mathcal{C}$ the right hand side of the above equation. Note that $\mathcal{C} \subset \underline{\mathbb{F}}_W$ is the maximum full $\mathcal{T}$-subcategory such that $\mathrm{Res}_V^W \mathcal{C} \le \underline{\mathbb{F}}_I$. Indeed, if $S \in \mathbb{F}_U - \left(\mathrm{CoInd}_V^W \underline{\mathbb{F}}_I\right)_U$, then for some $U' \to V$, we have $\mathrm{Res}_{U'}^U S \notin \mathbb{F}_{I,U'}$; thus when $\underline{\mathbb{F}}_J \not\le \mathrm{Res}_V^W \underline{\mathbb{F}}_I$, $\underline{\mathbb{F}}_J \not\le \underline{\mathbb{F}}_I$. Hence it suffices to prove that $\mathcal{C}$ is a weak indexing system.

First, suppose that $S \in \mathcal{C}_U$; then, $\mathrm{Res}_{U'}^U S \in \mathbb{F}_{I,U}$ for all $U' \to V$, so $*_{U'} = \mathrm{Res}_{U'}^U *_U \in \mathbb{F}_{I_U}$ for all $U' \to V$. Hence $*_U \in \mathcal{C}_U$. Now, fix $(T_X) \in \mathcal{C}_S$ an S-tuple. What remains is to show for all $U' \to V$ that

$$\mathrm{Res}_{U'}^U \coprod_X^S T_X \in \mathbb{F}_{I,U'}.$$

For every orbit $X' \in \mathrm{Orb}(\mathrm{Res}_U^{U'} S)$, there is a corresponding orbit $o(X') \in \mathrm{Orb}(S)$. Note that

$$\mathrm{Res}_{U'}^U \coprod_X^S T_X = \coprod_{X'}^{\mathrm{Res}_{U'}^U S} \left(X' \times_{\mathrm{Res}_{U'}^U o(X')} \mathrm{Res}_{U'}^U T_{o(X')} \right);$$

in particular, by assumption, $\mathrm{Res}_{U'}^U S$ and $X' \times_{\mathrm{Res}_U^{U'} o(X)} \mathrm{Res}_{U'}^U T_{o(X')}$ are in $\mathbb{F}_{I,U'}$, so $\mathrm{Res}_{U'}^U \coprod_X^S T_X \in \mathbb{F}_{I,U'}$; this implies that $\coprod_X^S T_X \in \mathcal{C}_U$, i.e. $\mathcal{C}$ is closed under self-indexed coproducts, as desired. □

We will use this in [Ste24] to see that $\mathrm{CoInd}_V^W A\mathcal{O} = A\mathrm{CoInd}_V^W \mathcal{O}$. This is significant, as norms of $\mathcal{O}$-algebras bear natural structures of algebras over $\mathrm{CoInd}_V^W \mathcal{O}$; for instance, norms of I-commutative algebras bear natural structures as $\mathrm{CoInd}_V^W I$-commutative algebras.

2.4. **The transfer system fibration.** Recall that the monotone map $\mathfrak{R} : \mathrm{wIndexCat}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}}$ is defined by $\mathfrak{R}(I) = I \cap \mathcal{T}$; we denote the composite $\mathrm{wIndex}_{\mathcal{T}} \simeq \mathrm{wIndexCat}_{\mathcal{T}} \to \mathrm{Transf}_{\mathcal{T}}$ as $\mathfrak{R}$ as well. Given R a transfer system, define the weak indexing system

$$\overline{\underline{\mathbb{F}}}_R \coloneqq \mathrm{Cl}_\infty(\{\mathrm{Res}_V^W U \mid U \to W \in R, \;\; V \to W \in \mathcal{T}\})$$

This subsection is primarily dedicated to proving the following.

Theorem 2.27. *The map of posets* $\mathfrak{R} : \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}}$ *has fully faithful right adjoint given by the composite* $\mathrm{Transf}_{\mathcal{T}} \simeq \mathrm{Index}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ *and fully faithful left adjoint given by* $\overline{\underline{\mathbb{F}}}_{(-)}$.

Corollary 2.28. *If* I, J *are unital weak indexing categories, then* $\mathfrak{R}(I) \vee \mathfrak{R}(J) = \mathfrak{R}(I \vee J)$ *and* $\mathfrak{R}(I) \cap \mathfrak{R}(J) = \mathfrak{R}(I \cap J)$.

We begin with an easy technical lemma concerning closures and transfer systems.

Lemma 2.29. $\mathfrak{R}(\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C})) = \mathfrak{R}(\mathrm{Cl}_{R(\mathcal{D}),1} \mathfrak{R}\mathcal{C})$.

Proof. It suffices to note that whenever $\coprod_U^S T_U$ is an orbit, there is exactly one T_U which is nonempty, in which case $\mathrm{Ind}_U^V T_U = \coprod_U^S T_U$, implying that T_U is an orbit. □

We use this to give compatibility of $\mathfrak{R}$ with joins in a restricted setting.

Proposition 2.30. *If I, J unital satisfy $\mathfrak{R}(I) \leq \mathfrak{R}(J)$, then $\mathfrak{R}(I \vee J) = \mathfrak{R}(J)$.*

Proof. Note that $\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J$ is closed under J-indexed induction, so we have

$$\mathfrak{R}(\mathrm{Cl}_{\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J, 1}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J)) = \mathfrak{R}(\mathrm{Cl}_{\mathfrak{R}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J),1}(\mathfrak{r}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J))) = \mathfrak{R}(\mathrm{Cl}_{\mathfrak{R}(J),1}(\mathfrak{R}(J)))) = \mathfrak{R}(J).$$

Iterating this and taking a union, we find that

$$\mathfrak{R}(I \vee J) = \mathfrak{R}\mathrm{Cl}_{\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J, \infty}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) = \mathfrak{R}(J). \qquad \square$$

We additionally note the following.

Proposition 2.31. *$\overline{\underline{\mathbb{F}}}_R$ is the initial element of $\mathfrak{R}^{-1}(R)$.*

Proof. The only nontrivial part is showing that $\mathfrak{R}(\overline{\underline{\mathbb{F}}}_R) = R$; in fact, this follows by unwinding definitions and applying Lemma 2.29. □

Proof of Theorem 2.27. By Lemma 2.14, we're tasked with proving that $\mathfrak{R}^{-1}(R)$ has an initial and terminal object; the former is is Proposition 2.31, so we re left with proving the latter. By Proposition 2.30, the indexing system $I^\infty_{\mathcal{T}} \vee I$ satisfies $\mathfrak{R}(I^\infty_{\mathcal{T}} \vee I) = \mathfrak{R}(I)$, and is an upper bound for I. In fact, $I^\infty_{\mathcal{T}} \vee I$ is an indexing system, so it is the *unique* indexing system with $\mathfrak{R}(I \vee I^\infty_{\mathcal{T}}) = \mathfrak{R}(I)$, and by the above argument, it is an upper bound for all J such that $\mathfrak{r}(J) = \mathfrak{R}(J)$, as desired. □

Remark 2.32. If $\mathcal{T}$ has a terminal object V, then $2*_V$ is not in $\overline{\underline{\mathbb{F}}}_R$ for any R, since $2*_V$ is not a summand in the restriction of any transitive W-sets for any $W \in \mathcal{T}$. Hence $\overline{\underline{\mathbb{F}}}_R$ is not an indexing system, or equivalently, $\mathfrak{R}^{-1}(R)$ has multiple elements. We may interpret this as saying that unital weak indexing systems are seldom determined by their transitive V-sets. ◁

2.5. **The unit and fold map fibrations.** Recall the maps υ and ∇ of Proposition 1.22.

2.5.1. *The unit fibration.* We study the map υ using the following.

Proposition 2.33. *The map $\upsilon : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ has fully faithful left adjoint given by $E^{\mathcal{T}}_{-}\underline{\mathbb{F}}^0_{(-)}$.*

Proof. In view of Lemma 2.14, we're tasked with proving that $E^{\mathcal{T}}_{\mathcal{F}}\underline{\mathbb{F}}^0_{\mathcal{F}} \in \upsilon^{-1}(\mathcal{F})$ is initial, which follows by unwinding definitions. □

Corollary 2.34. *The restricted map $\upsilon_a : \mathrm{wIndex}^{a\mathrm{uni}}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ is a cocartesian fibration with fiber $\upsilon_a^{-1}(\mathcal{F}) = \mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{F}}$ embedded along $\underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{T}} \cup E^{\mathcal{T}}_{\mathcal{F}}(-)$. Moreover, the cocartesian transport map $t^{\mathcal{F}'}_{\mathcal{F}} : \upsilon_a^{-1}(\mathcal{F}) \to \upsilon_a^{-1}(\mathcal{F}')$ is implemented by*

$$t^{\mathcal{F}'}_{\mathcal{F}}\underline{\mathbb{F}}_I = \underline{\mathbb{F}}^0_{\mathcal{F}'} \cup E^{\mathcal{F}'}_{\mathcal{F}}\underline{\mathbb{F}}_I$$

Proof. The property $\upsilon_a^{-1}(\mathcal{F}) = \mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{F}}$ follows by unwinding definitions using Proposition 1.22. For the remaining property, we're tasked with proving that $\underline{\mathbb{F}}^0_{\mathcal{F}'} \cup E^{\mathcal{F}'}_{\mathcal{F}}\underline{\mathbb{F}}_I \in \mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{F}'}$ is the initial unital $\mathcal{F}'$-weak indexing system which embeds $\underline{\mathbb{F}}_I$ after each are embedded into $\mathrm{wIndex}^{a\mathrm{uni}}_{\mathcal{T}}$ along $\underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{T}} \cup E^{\mathcal{T}}_{-}(-)$. The fact that it is a unital $\mathcal{F}'$-weak indexing system follows by unwinding definitions, and the relation is implemented by noting combining the universal property of $\mathbb{E}^{\mathcal{F}'}_{\mathcal{F}}$ from Corollary 2.24 with the fact that $\underline{\mathbb{F}}^0_{\mathcal{F}'}$ is the initial unital $\mathcal{F}'$-weak indexing system. □

Proposition 2.35. *Given $\mathcal{F} \in \mathrm{Fam}_{\mathcal{T}}$, the fiber $\upsilon^{-1}(\mathcal{F})$ has a terminal object computed by*

$$\mathbb{F}_{\mathcal{F}^\perp - nu, V} = \begin{cases} \mathbb{F}_V & V \in \mathcal{F}; \\ \mathbb{F}_V - \{S \mid \forall U \in \mathrm{Orb}(S), U \in \mathcal{F}\} & V \notin \mathcal{F} \end{cases}$$

Proof. We begin by noting that $\underline{\mathbb{F}}_{\mathcal{F}^\perp - nu}$ contains all $\mathcal{T}$-weak indexing systems with unit family $\mathcal{F}$; indeed, if $\underline{\mathbb{F}}_J$ satisfies $\upsilon(J) \geq \mathcal{F}$ and there is some $S \in \mathbb{F}_{J,V} - \mathbb{F}_{\mathcal{F}^\perp - nu, V}$, then we must have $U \in \mathcal{F} \subset \upsilon(J)$ for all $U \in \mathrm{Orb}(S)$ and $V \notin \mathcal{F}$, so

$$\coprod_U^S \varnothing_U = \varnothing_V \in \mathbb{F}_{J,V},$$

implying that $\upsilon(J) < \mathcal{F}$. By contraposition, if $\upsilon(J) = \mathcal{F}$, we have $\underline{\mathbb{F}}_J \subset \underline{\mathbb{F}}_{\mathcal{F}^\perp - nu}$. Thus it suffices to verify that $\underline{\mathbb{F}}_{\mathcal{F}^\perp - nu}$ is a $\mathcal{T}$-weak indexing system. Since it contains all contractible V-sets, it suffices to prove that it's closed under self-indexed coproducts.

Fix some $S \in \mathbb{F}_{\mathcal{F}^\perp - nu,V}$ and $(T_U) \in \mathbb{F}_{\mathcal{F}^\perp - nu,S}$. If $V \in \mathcal{F}$, then there is nothing to prove, so suppose $V \notin \mathcal{F}$. Then, note that

$$\mathrm{Orb}\left(\coprod_U^S T_U\right) = \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Orb}(T_U).$$

S must contain some orbit U outside of $\mathcal{F}$, and by assumption, T_U contains an orbit outside of $\mathcal{F}$; thus $\coprod_U^S T_U$ contains an orbit outside of $\mathcal{F}$, as desired. □

Warning 2.36. υ does not admit a right adjoint, as it is not even compatible with binary joins; for instance, if $\mathcal{T} = \mathcal{O}_G$, then note that the weak indexing system $\underline{\mathbb{F}}_{\varnothing^\perp - nu,V}$ consists of all nonempty H-sets, and $E_{BG}^G \underline{\mathbb{F}}_{BG}^0$ contains only the e-sets $\{\varnothing_e, *_e\}$. Nevertheless, the join $\underline{\mathbb{F}}_{\varnothing^\perp - nu,V} \vee E_{BG}^G \underline{\mathbb{F}}_{BG}^0$ contains the norm $N_e^H \varnothing_e = \varnothing_H$, so it is equal to the complete indexing system $\underline{\mathbb{F}}_G$. Thus when G is nontrivial, we have a proper family inclusion

$$\upsilon(\underline{\mathbb{F}}_{\varnothing^\perp - nu}) \cup \upsilon(E_{BG}^G \underline{\mathbb{F}}_{BG}^0) = BG \subsetneq \mathcal{O}_G = \upsilon(\underline{\mathbb{F}}_{\varnothing^\perp - nu} \vee E_{BG}^G \underline{\mathbb{F}}_{BG}^0).$$

◁

Remark 2.37. Despite Warning 2.36, υ is *lax*-compatible with joins, in the sense that there is a relation

$$\upsilon(I) \cup \upsilon(J) \le \upsilon(I \vee J);$$

this follows by simply noting that $I \vee J$ contains I and J. In particular, by Proposition 1.22, we find that joins of unital weak indexing systems are unital. ◁

Observation 2.38. If $\underline{\mathbb{F}}_I$ is E-unital, then there is an equality

$$\underline{\mathbb{F}}_I = E_{\upsilon(I)}^{\mathcal{T}} \mathrm{Bor}_{\upsilon(I)}^{\mathcal{T}} \underline{\mathbb{F}}_I = E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(I)}^{\mathrm{triv}} \vee E_{\upsilon(I)}^{\mathcal{T}} \mathrm{Bor}_{\upsilon(I)}^{\mathcal{T}} \underline{\mathbb{F}}_I.$$

More generally, if $\underline{\mathbb{F}}_I$ is aE-unital, then there is an equality

$$\underline{\mathbb{F}}_I = E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(I)}^{\mathrm{triv}} \cup E_{\upsilon(I)}^{\mathcal{T}} \mathrm{Bor}_{\upsilon(I)}^{\mathcal{T}} \underline{\mathbb{F}}_I = E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(I)}^{\mathrm{triv}} \vee E_{\upsilon(I)}^{\mathcal{T}} \mathrm{Bor}_{\upsilon(I)}^{\mathcal{T}} \underline{\mathbb{F}}_I$$

and $\mathrm{Bor}_{\upsilon(I)}^{\mathcal{T}} \underline{\mathbb{F}}_I$ is unital. Furthermore, note that $E_{\mathcal{F}}^{\mathcal{T}}$ sends E-unital weak indexing systems to aE-unital weak indexing systems by unwinding definitions; what we find is that E-unital weak indexing systems are the union of $E_{\mathcal{F}}^{\mathcal{T}} \mathrm{wIndex}_{\mathcal{F}}^{\mathrm{uni}}$ as $\mathcal{F}$ ranges across $\mathrm{Fam}_{\mathcal{T}}$, and that aE-unital weak indexing systems are the subcategory of $\mathrm{wIndex}_{\mathcal{T}}$ generated under joins by E-unital weak indexing systems and $E_{\mathcal{F}}^{\mathcal{T}} \mathbb{F}_{\mathcal{F}}^{\mathrm{triv}}$ as $\mathcal{F}$ ranges across $\mathrm{Fam}_{\mathcal{T}}$. ◁

2.5.2. *The fold map fibration.* We would like to prove the following.

Proposition 2.39. *The restricted map* $\nabla_u : \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Fam}_{\mathcal{T}}$ *has fully faithful left adjoint given by* $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{-}^{\mathcal{T}} \underline{\mathbb{F}}_{(-)}^{\infty}$*; hence it is a cocartesian fibration, and the cocartesian transport map* $t_{\mathcal{F}}^{\mathcal{F}'}$ *is implemented by*

$$t_{\mathcal{F}}^{\mathcal{F}'} \underline{\mathbb{F}}_I \simeq \underline{\mathbb{F}}_{\mathcal{F}}^{\infty} \vee \underline{\mathbb{F}}_I.$$

Proof. In view of Lemma 2.14 and Proposition 2.13, we're tasked with proving that $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^{\infty}$ is the initial unital $\mathcal{T}$-weak indexing system with fold family $\ge \mathcal{F}$. It follows by unwinding definitions that $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^{\infty}$ is a weak indexing system. Furthermore, it follows from Proposition 1.22 that every unital $\mathcal{T}$-weak indexing system with fold family $\ge \mathcal{F}$ contains $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^{\infty}$,and that it is unital and has fold family $\mathcal{F}$, as desired. □

Lemma 2.40. *Suppose* $\underline{\mathbb{F}}_I$ *is unital. If* $\nabla(\underline{\mathbb{F}}_I), \nabla(\mathcal{C}) \le \mathcal{F}'$*, then* $\nabla(\mathrm{Cl}_{\underline{\mathbb{F}}_I,1}(\mathcal{C})) \le \mathcal{F}'$.

Proof. Suppose $V \in \nabla(\mathrm{Cl}_{\underline{\mathbb{F}}_I,1}(\mathcal{C}))$, i.e. there exists some $S \in \mathbb{F}_{I,V}$, $(X_U) \in \mathcal{C}_S$, and $n \ge 2$ such that $\coprod_U^S X_U = n *_V$. We would like to prove that $V \in \mathcal{F}'$. Since $\underline{\mathbb{F}}_I$ is unital, writing $S = S_{ne} \sqcup S_\varnothing$ for $S_\varnothing$ the disjoint union of S-orbits over which X_U is empty, we have $S_{ne} \in \mathbb{F}_{I,V}$ and

$$\coprod_U^S X_U = \coprod_U^{S_{ne}} X_U;$$

hence we may replace S with S_{ne} and assume that X_U is nonempty for all U.

Note that, for all $U \in \mathrm{Orb}(S)$, we have $\mathrm{Ind}_U^V X_U = m *_V$ for some $m \ge 0$; in particular, this implies $U = V$. Hence $S = k *_V$ for some k. Writing our decomposition as $S = \{1, \dots, n\}$ and $X_i = m_i *_V$, we find that $n = \sum_{i=1}^k m_i \ge 2$, so either $m_i \ge 2$ for some i or $k \ge 2$. In either case, we find $V \in \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\mathcal{C}) \subset \mathcal{F}'$, as desired. □

Observation 2.41. For any nonempty set of collections $(\mathcal{C}_i)_{i\in I}$, it follows by unwinding definitions that we have $\nabla\left(\bigcup_{i\in I}\mathcal{C}_i\right) = \bigcup_{i\in I}\nabla\left(\mathcal{C}_i\right)$. $\triangleleft$

We use this to compute the fold family of a join of weak indexing systems.

Proposition 2.42. $\nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J) = \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J)$.

Proof. By Observation 2.41, we have $\nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J) = \nabla(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) \le \nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J)$, so we are tasked with proving the opposite inclusion. By Lemma 2.40, we find inductively that $\nabla \mathrm{Cl}_{\underline{\mathbb{F}}_I,1}\mathrm{Cl}_{\underline{\mathbb{F}}_J,1}\cdots\mathrm{Cl}_{\underline{\mathbb{F}}_J,1}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) \le \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J)$; applying Observation 2.41 to take a union, we find that $\nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J) \le \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J)$, as desired. $\square$

Remark 2.43. The fibers of ∇ are all nonempty by Proposition 2.39; by Observation 2.41 and Proposition 2.42, $\nabla^{-1}(\mathcal{F})$ is closed under *arbitrary* joins, so it has a terminal object, i.e. ∇ possesses a fully faithful right adjoint.

The author is not aware of a general formula for this, but there are interesting examples; for instance, if λ is a nontrivial irreducible real orthogonal C_p-representation, then we show in [Ste24] that the arity support $A\lambda$ of the C_p-weak $\mathcal{N}_\infty$-operad $\mathbb{E}_{\lambda\infty}$ is terminal among the C_p-weak indexing systems with fold maps over the trivial subgroup. In algebra, this may be intrerpreted as saying that $\mathbb{E}_{\lambda\infty}$ presents the terminal sub-C_p-commutative algebraic theory prescribing a multiplication on the underlying Borel type of a genuine C_p-object, but not on genuine C_p-fixed points. $\triangleleft$

We would like to compute examples with many transfers and few folds.

Observation 2.44. If R is a transfer system, then unwinding definitions, we find

$$\nabla\overline{\underline{\mathbb{F}}}_R = \mathrm{Dom}(R) \coloneqq \left\{U \in \mathcal{T} \mid \;\; \exists U \to W \xleftarrow{f} V \;\text{ s.t. }\; f \in R \;\text{ and }\; 2{*_U} \subset \mathrm{Res}_U^W \mathrm{Ind}_V^W {*_V}\right\}. \quad \triangleleft$$

Remark 2.45. If $\mathcal{T} = \mathcal{F} \subset \mathcal{O}_G$ is a family of normal subgroups of a finite group (e.g. any family of subgroups of a finite Dedekind group), then for every pair of proper subgroup inclusion $H, K \subset J$, the double coset formula implies that $\mathrm{Res}_K^J \mathrm{Ind}_H^J {*_H} = [K\backslash J/H] \cdot H/H\cap K$. In particular, $2{*_H} \subset \mathrm{Res}_K^J \mathrm{Ind}_H^J {*_H}$ if and only if $H \subset K$.

Unwinding definitions, we find in this case that $\mathrm{Dom}(R)$ is the family

$$\mathrm{Dom}(R) = \left\{U \in \mathcal{F} \mid \exists U \to V \xrightarrow{f} W \mid f \in R - R^{\simeq}\right\},$$

i.e. it is the family of subgroups generated by domains of nontrivial transfers. $\triangleleft$

2.5.3. *The essence fibration.* Similarly, given $\underline{\mathbb{F}}_I$ a weak indexing system, define the family

$$\epsilon(I) \coloneqq \{V \in \mathcal{T} \mid \exists S \in \mathbb{F}_{I,V} - \{*_V\}\}$$

so that $\underline{\mathbb{F}}_I$ is aE-unital if and only if $\epsilon(I) = \upsilon(I)$.

Lemma 2.46. *If $\epsilon(\mathcal{C}) \subset \epsilon(\mathcal{D})$, then*

$$\epsilon(\mathrm{Cl}_{\mathcal{C},1}(\mathcal{D})) = \epsilon(D).$$

Proof. Fix some noncontractible V-set $T \in \mathrm{Cl}_{\mathcal{C},1}(\mathcal{D})$, and express it as an S-indexed colimit

$$T = \coprod_U^S T_U$$

For $S \in \mathcal{C}$ and $T_U \in \mathcal{D}$. Since T is noncontractible, either S is noncontractible or T_U is noncontractible; either way, this implies that $V \in \epsilon(D)$, as desired. $\square$

Proposition 2.47. $\epsilon(I \vee J) = \epsilon(I) \cup \epsilon(J)$.

Proof. We may inductively prove using Lemma 2.46 that $\epsilon\left(\overbrace{\mathrm{Cl}_I\mathrm{Cl}_J\cdots\mathrm{Cl}_I\mathrm{Cl}_J}^{2n}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J)\right) = \epsilon(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) = \epsilon(I) \cup \epsilon(J)$; taking a union as $n \to \infty$ yields the desired statement. $\square$

We're finally ready to round up properties localizations to our various conditions.

Proposition 2.48. *Let $\mathcal{T}$ be an orbital ∞-category.*

(1) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \rightarrow \mathrm{wIndex}_{\mathcal{T}}$ *is right adjoint to* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}^0_{\epsilon(I)}$.
(2) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{E\mathrm{uni}} \rightarrow \mathrm{wIndex}_{\mathcal{T}}$ *is right adjoint to* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}^0_{c(I)}$.
(3) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{\mathrm{oc}} \rightarrow \mathrm{wIndex}_{\mathcal{T}}$ *is right adjoint to* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{T}}$.
(4) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \rightarrow \mathrm{wIndex}_{\mathcal{T}}$ *is right adjoint to* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^0_{\epsilon(\underline{\mathbb{F}}_I)}$.
(5) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \rightarrow \mathrm{wIndex}_{\mathcal{T}}$ *is right adjoint to* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^0_{\mathcal{T}}$.
(6) The inclusion $\mathrm{Index}_{\mathcal{T}} \rightarrow \mathrm{wIndex}_{\mathcal{T}}$ *is right adjoint to* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^{\infty}_{\mathcal{T}}$.

Furthermore, each inclusion is additionally compatible with joins.

Proof. First, note by Propositions 2.22, 2.42 and 2.47 that the maps $c, \nabla, \epsilon : \mathrm{wIndex}_{\mathcal{T}} \rightarrow \mathrm{Fam}_{\mathcal{T}}$ are compatible with joins, and by Remark 2.37 the map υ is lax-compatible with joins. This implies that the conditions that the conditions that $c(I) = \mathcal{T}$, that $\upsilon(I) = c(I)$, that $\upsilon(I) = \mathcal{T}$, $\nabla(I) = \mathcal{T}$, and are all compatible with joins, so we are left with proving that aE-unital weak indexing systems are closed under joins. But this follows by combining Observation 2.38 with the observation that $E^{\mathcal{T}}_{\mathcal{F}}\underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}} \vee E^{\mathcal{T}}_{\mathcal{F}'}\underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}'} = E^{\mathcal{T}}_{\mathcal{F}\cup\mathcal{F}'}\underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}\cup\mathcal{F}'}$. Thus we are left with constructing left adjoints.

We begin by proving (1). By Lemma 2.14, we are tasked with verifying that $\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}^0_{\epsilon(I)}$ is initial among aE-unital weak indexing systems $\mathcal{C}$ satisfying the property that $\underline{\mathbb{F}}_I \leq \mathcal{C}$. In fact, if $\underline{\mathbb{F}}_I \leq \underline{\mathbb{F}}_J$ and $\underline{\mathbb{F}}_J$ is aE-unital, then $\epsilon(I) \leq \epsilon(J) = \upsilon(J)$, and $c(I) \leq c(J)$, so we have $E^{\mathcal{T}}_{(I)}\underline{\mathbb{F}}^{\mathrm{triv}}_{c(I)}, E^{\mathcal{T}}_{\epsilon(I)}\underline{\mathbb{F}}^0_{\epsilon(I)} \leq \underline{\mathbb{F}}_J$. Taking a join, this implies that

$$\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}^0_{\epsilon(I)} = \underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}^{\mathrm{triv}}_{c(I)} \vee E^{\mathcal{T}}_{\epsilon(I)}\underline{\mathbb{F}}^0_{\epsilon(I)} \leq \underline{\mathbb{F}}_J.$$

Thus we're left with verifying that $\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}^0_{\upsilon(I)}$ is aE-unital; in particular, we have

$$\upsilon(\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}^0_{\epsilon(I)}) \geq \upsilon(E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}_{\epsilon(I)^0}) = \epsilon(I),$$

and by Proposition 2.47 we have

$$\epsilon(\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}_{\epsilon(I)^0}) = \epsilon(I).$$

Together these imply that $\epsilon(\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}^0_{\epsilon(I)}) = \upsilon(\underline{\mathbb{F}}_I \vee E^{\mathcal{F}}_{c(I)}\underline{\mathbb{F}}^0_{\epsilon(I)})$, so it is aE-unital, proving (1).

The proof of (2) is analogous, instead concluding the relation $\upsilon(\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}^0_{c(I)}) = c(\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}^0_{c(I)})$ by the same argument, replacing Proposition 2.47 with Proposition 2.22. The proof of (3) is easier, as we only need to use Proposition 2.22 to verify that $c(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{T}}) = \mathcal{T}$ Similarly, the proof of (6) uses Proposition 2.42 and Remark 2.37 to verify that $\mathcal{T} \geq \nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^{\infty}_{\mathcal{T}}) \cap \upsilon(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^{\infty}_{\mathcal{T}}) \geq \mathcal{T}$. (4) follows by combining (1) and (3), and (5) follows by combining (1) and (2). □

2.5.4. *The combined transfer-fold fibration.*

Observation 2.49. By Proposition 2.31 and Observation 2.44, If $\mathrm{Dom}(R) \not\subset \mathcal{F}$, then $\mathfrak{R}^{-1}(R) \cap \nabla^{-1}(\mathcal{F})$ is empty. In fact, by Proposition 2.42 and Observation 2.44 we find that $\overline{\underline{\mathbb{F}}}_R \vee \underline{\mathbb{F}}^{\infty}_{\mathcal{F}} \in \mathcal{F}^{-1}(R) \cap \nabla^{-1}(\mathcal{F} \cup \mathrm{Dom}(R))$ is *initial*; in particular the condition $\mathrm{Dom}(R) \subset \mathcal{F}$ is necessary and sufficient for $\mathfrak{R}^{-1}(R) \cap \nabla^{-1}(\mathcal{F})$ to be nonempty. ◁

Define the embedded subposet $(\mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}})^{\mathrm{admsbl}} \subset \mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}}$ spanned by the pairs $(R, \mathcal{F})$ such that $\mathrm{Dom}(R) \leq \mathcal{F}$. In light of Lemma 2.14, we may rephrase Observation 2.49 as follows.

Proposition 2.50. *The map* $(\mathfrak{R}, \nabla) : \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \rightarrow \mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}}$ *has image* $(\mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}})^{\mathrm{admsbl}}$, *and factors as*

$$\begin{tikzcd}
\mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \arrow[d, "{(\mathfrak{R},\nabla)}"'] \arrow[dr, "{(\mathfrak{R},\nabla)}"] & \\
(\mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}})^{\mathrm{admsbl}} \arrow[r, hook] & \mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}}
\end{tikzcd}$$

where the lefthand vertical map admits a fully faithful left adjoint computed by $(R, \mathcal{F}) \mapsto \overline{\underline{\mathbb{F}}}_R \vee \underline{\mathbb{F}}^{\infty}_{\mathcal{F}}$. *Thus the left vertical map is a cocartesian fibration with cocartesian transport computed by*

$$t^{(R',\mathcal{F}')}_{(R,\mathcal{F})}\underline{\mathbb{F}}_I = \underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'} \vee \underline{\mathbb{F}}^{\infty}_{\mathcal{F}'}.$$

2.6. **Compatible pairs of weak indexing systems.** We finish the section with a discussion of *compatible pairs of weak indexing systems*, generalizing the setting of [BH22].

Proposition 2.51. *If I_a, I_m are weak indexing categories, the following conditions are equivalent:*

(a) $\underline{\mathbb{F}}_{I_a}$ admits I_m-indexed products.
(b) $(\mathbb{F}_{\mathcal{T}}, I_m, I_a)$ is a bispan triple in the sense of [EH23, Def 2.4.3].

Proof. Note that $\mathbb{F}_{\mathcal{T}}$ is an ∞-topos, as it is an accessible localization of a presheaf topos. Hence [EH23, Rmk 2.4.7] and [EH23, Lem 2.4.6] imply that $(\mathbb{F}_{\mathcal{T}}, I_m, I_a)$ is a bispan triple if and only if, for all maps $T \to S$ in I_m, the pullback functor

$$\begin{tikzcd} I_{a,/T} \arrow[d, phantom, "\wr"] & I_{a,/S} \arrow[l, "f^*"'] \arrow[d, phantom, "\wr"] \\ \prod\limits_{U \in S} \mathbb{F}_{I_a, T_U} & \prod\limits_{U \in S} \mathbb{F}_{I_a, U} \arrow[l, "f^*"'] \end{tikzcd}$$

has a right adjoint; unwinding definitions, this is true if and only if $\underline{\mathbb{F}}_{I_a}$ admits I_m-indexed products. $\square$

Definition 2.52. A pair of one-object weak indexing categories (I_a, I_m) is *compatible* if $\underline{\mathbb{F}}_{I_a} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is closed under I_m-indexed products, i.e. $\underline{\mathbb{F}}_{I_a} \subset \underline{\mathbb{F}}_{\mathcal{T}}^{\times}$ is an I_m-symmetric monoidal full subcategory. $\triangleleft$

Given a compatible pair (I_a, I_m), Proposition 2.51 and [EH23, Notn 2.5.11] yield an ∞-category

$$P^{\mathcal{T}}_{I_a, I_m} \coloneqq \mathrm{Bispan}_{I_m, I_a}(\mathbb{F}_{\mathcal{T}}),$$

whose homotopy category recovers the category $P^G_{I_a, I_m}$ of [BH22] when I_a, I_m are $\mathcal{O}_G$-indexing systems. Furthermore, this is compatible with restrictions, and hence it yields a $\mathcal{T}$-∞-category $\underline{P}^{\mathcal{T}}_{I_a, I_m}$ equipped with a core-preserving and I_m-product-preserving $\mathcal{T}$-functor

$$\iota : \mathrm{Span}_{I_a}(\underline{\mathbb{F}}_{\mathcal{T}}) \to \underline{P}^{\mathcal{T}}_{I_a, I_m}.$$

Together, this defines a pair of ∞-categories

$$\mathrm{Mack}_{I_a}(\mathcal{C}) \coloneqq \mathrm{Fun}^{\times}(\mathrm{Span}_{I_a}(\mathbb{F}_{\mathcal{T}}), \mathcal{C}),$$
$$\mathrm{Tamb}_{I_a, I_m}(\mathcal{C}) \coloneqq \mathrm{Fun}^{\times}(P^{\mathcal{T}}_{I_a, I_m}, \mathcal{C}),$$

together with a forgetful functor

$$U : \mathrm{Tamb}_{I_a, I_m}(\mathcal{C}) \to \mathrm{Mack}_{I_a}(\mathcal{C}),$$

the codomain being modelled by $\mathrm{CAlg}_{I_a}\left(\underline{\mathrm{CoFr}}\,\mathcal{C}^{I_a - \times}\right)$ in [Ste24]. In CHLL, $\mathrm{Tamb}_{I_a, I_m}(\mathcal{C})$ is shown to additionally be modelled by $\mathrm{CAlg}_{I_m}(\mathrm{Mack}_{I_a}(\mathcal{C}))$ with respect to an *indexed Box product* I_m-symmetric monoidal structure.

Such considerations will be greatly simplified by the following generalization of [BH22, Cor 6.19].

Proposition 2.53 (Multiplicative hull)**.** *Given $\underline{\mathbb{F}}_I$ a one-object weak indexing system, the subcategories*

$$\mathbb{F}_{m(I), V} \coloneqq \{S \in \mathbb{F}_V \mid \underline{\mathbb{F}}_I \text{ closed under } S\text{-indexed products}\}$$

form an indexing system characterized by the property that, for all $I_m \in \mathrm{wIndex}_{\mathcal{T}}$, the pair (I, I_m) is compatible if and only if $I_m \leq m(I)$.

Proof of Proposition 2.53. It follows directly from construction that $I_m \leq m(I)$ if and only if (I, I_m) is compatible. Furthermore, the $*_V$-indexed product functor is the identity, so $*_V \in \mathbb{F}_{m(I), V}$ for all V. Hence it suffices to prove that $n *_V \in \mathbb{F}_{m(I), V}$ for all $n \neq 1$ and $\underline{\mathbb{F}}_{m(I)}$ is closed under self-induction.

For the first statement, empty products are terminal objects (i.e. $*_V$), so $\varnothing_V \in \mathbb{F}_{m(I), V}$ for all V. Hence it suffices to prove that $2 *_V \in \mathbb{F}_{m(I), V}$, i.e. $\mathbb{F}_{I, V}$ is closed under binary products. By distributivity of finite products and coproducts, we have

$$S \times S' = \coprod_{U \in \mathrm{Orb}(S)} U \times S' = \coprod_U^S \mathrm{Res}_U^V S',$$

which is in $\mathbb{F}_{I, V}$ by closure under self-indexed coproducts.

For the second statement, it suffices to note that

$$\coprod_{U}^{\mathrm{Ind}_U^V S} T_U = \mathrm{Ind}_U^V \coprod_{U}^{S} T_U, = \coprod_{U}^{\mathrm{Ind}_U^V *_U} \coprod_{U}^{S} T_U$$

which is in $\mathbb{F}_{I,V}$ by closure under self-indexed coproducts. ☐

The situation with fixed I_m and varying I_a is more complicated, and has been studied for indexing systems in [BH22]; we do not study it here.

3. Computational results

3.1. Sparsely indexed coproducts. Given S a V-set, let $\mathrm{Istrp}(S) \coloneqq \left\{U \in \mathcal{T}_{/V} \mid \exists \text{ summand inclusion } U \hookrightarrow S\right\}$ be the *isotropy poset*, and given $U \in \mathrm{Istrp}(S)$, write $S_{(U)} = n_{S,U} \cdot U$ for the maximal summand of S which is a multiple of U and write $S^*_{(U)} = n_{S,U} \cdot *_U \in \mathbb{F}_U$. Furthermore, write

$$\overline{S} \coloneqq \coprod_{U \in \mathrm{Istrp}(S)} U,$$

so that

$$S = \coprod_{U \in \mathrm{Istrp}(S)} S_{(U)} = \coprod_{U}^{\overline{S}} S^*_{(U)}. \tag{2}$$

In general, we write $S^V \coloneqq S_{(V)}$ for the *V-fixed points of S*.

Lemma 3.1. *If $\mathcal{T}$ is an atomic orbital ∞-category, the U-set $\mathrm{Res}_U^V \mathrm{Ind}_U^V *_U$ has a fixed point.*

Proof. We have a diagram

$$\begin{tikzcd}
U \arrow[dr, dashed] \arrow[drr, bend left, equal] \arrow[ddr, bend right, equal] & & \\
& \mathrm{Ind}_U^{\mathcal{T}} \mathrm{Res}_U^V \mathrm{Ind}_U^V *_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d] \\
& U \arrow[r] & V
\end{tikzcd}$$

Taking slices over U, the lefthand triangle establishes $*_U$ as a retract of $\mathrm{Res}_U^V \mathrm{Ind}_U^V *_U$, i.e. it is a retract of an orbital summand $*_U \rightleftarrows S \subset \mathrm{Res}_U^V \mathrm{Ind}_U^V *_U$. By the atmoic assumption, this establishes $*_U = S$, as desired. ☐

Proposition 3.2. *If $\mathcal{T}$ is an atomic orbital ∞-category and I is aE-unital, then*

$$\underline{\mathbb{F}}_I = \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\leq 2}).$$

Proof. Fix $S \in \underline{\mathbb{F}}_I$. We will prove that $S \in \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\leq 2}) \subset \underline{\mathbb{F}}_I$. Note that $\overline{S} \in \underline{\mathbb{F}}_I^{\leq 2}$, so by Eq. (2), it suffices to prove that $S^*_{(U)} = n_{S,U} \cdot *_U \in \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\leq 2})_U$ for all U. In fact, by Lemma 3.1, there is a summand inclusion $S^*_{(U)} \subset \mathrm{Res}_U^V S_{(U)}$; if $S^*_{(U)}$ is noncontractible, then $\mathrm{Res}_U^V S_{(U)}$ is noncontractible, so the aE-unitality assumption guarantees that $S^*_{(U)} \in \underline{\mathbb{F}}_I$ for all U.

If $S^*_{(U)} \in \mathbb{F}_{I,U}^{\leq 2}$, then we are done, so suppose $n_{S,U} > 2$. By the aE-unitality assumption, we have $S^*_{(U)} \supset 2 \cdot *_U \in \mathbb{F}_{I,U}^{\leq 2}$. By the argument in Lemma 1.23, we then have that $S^*_{(U)}$ is an iterated binary coproduct of $2 \cdot *_U$, i.e. it is in $\mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\leq 2})$, as desired. ☐

Proposition 3.3. *If $\mathcal{T}$ is an atomic orbital ∞-category with no self-normalizing transfers and $\underline{\mathbb{F}}_I$ is an aE-unital $\mathcal{T}$-weak indexing system, then*

$$\underline{\mathbb{F}}_I = \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\mathrm{sprs}})$$

Proof. Since $\mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\mathrm{sprs}}) \subset \underline{\mathbb{F}}_I$, it suffices to prove the opposite inclusion. We first note that $\underline{\mathbb{F}}_I \cap \underline{\mathbb{F}}_{\mathcal{T}}^{\infty} \subset \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\mathrm{sprs}})$ by Proposition 3.2. Hence it suffices to prove that $\underline{\mathbb{F}}_I$ is generated under sparse coproducts by $\underline{\mathbb{F}}_I^{\mathrm{sprs}} \cup E_{\nabla(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{\nabla(I)}^{\infty}$.

Fix some $S \in \mathbb{F}_{I,V}$, and recall that $\overline{S} \in \mathbb{F}_{I,V}^{\mathrm{sprs}}$. Furthermore, note by Lemma 3.1 that $S^*_{(U)} \in \mathbb{F}_{I,U}$. If $S^*_{(U)}$ is not sparse (i.e. $n_{S,U} > 2$), then $3 \cdot *_U \in \mathbb{F}_{I,U}$, so $U \in \nabla(I)$; hence $S^*_{(U)} \in \mathbb{F}_{I,U}^{\mathrm{sprs}} \cup \mathbb{F}_{\nabla(I),U}^{\infty}$. Applying Eq. (2), we find that S is a sparse coproduct of elements of $\underline{\mathbb{F}}_I^{\mathrm{sprs}} \cup E_{\nabla(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{\nabla(I)}^{\infty}$, as desired. □

Proof of Theorem C. By Proposition 3.2, $(-)^{\leq 2}$ is a section of $\mathrm{Cl}_\infty(-)$ and a left adjoint; this implies that $(-)^{\leq 2}$ is an embedding by Lemma 2.14, with image spanned by those collections $\mathcal{C}$ satisfying $\mathcal{C} \simeq \mathrm{Cl}_\infty(\mathcal{C})^{\leq}$. Unwinding definitions, this is what we set out to prove. The second statement follows by an identical argument using Proposition 3.3. □

Observation 3.4. If $\underline{\mathbb{F}}_I$ is aE-unital and contains the sparse V-set $S = \varepsilon *_V + V_1 + \cdots + V_n$ and the transfer $U \to V_1$, then $\underline{\mathbb{F}}_I$ contains the sparse V-set $\varepsilon *_V + U + \cdots + V_n$, as it's an S-indexed coproduct of elements of $\underline{\mathbb{F}}_I$. Thus the description in terms of sparse V-sets is not as compact as it *could* be. We exploit this for C_{p^N} in the following sections, but avoid delving into a more general story. ◁

Remark 3.5. Note that the maps $\upsilon, c, \nabla, \mathfrak{R}$ all factor as

$$\begin{tikzcd}
\mathrm{wIndex}_{\mathcal{T}} \arrow[rr, "\upsilon{,}c{,}\nabla{,}\mathfrak{R}"] \arrow[d, "-\cap\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}"'] & & \mathcal{C} \\
\mathrm{Sub}(\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}) \arrow[r, hook] & \mathrm{Sub}(\underline{\mathbb{F}}_{\mathcal{T}}) \arrow[ur, "\upsilon{,}c{,}\nabla{,}\mathfrak{R}"'] &
\end{tikzcd}$$

where $\mathcal{C} = \mathrm{Transf}_{\mathcal{T}}$ for $\mathfrak{R}$ and $\mathrm{Fam}_{\mathcal{T}}$ otherwise. By using Proposition 1.22, we find that:

(1) $\mathfrak{R}(\underline{\mathbb{F}}_I) = \mathfrak{R}(\underline{\mathbb{F}}_I^{\mathrm{sprs}})$.
(2) $\underline{\mathbb{F}}_I$ has one-color if and only if $\underline{\mathbb{F}}_I^{\mathrm{sprs}}$ has one color.
(3) $\underline{\mathbb{F}}_I$ is E-unital if and only if $\underline{\mathbb{F}}_I^{\mathrm{sprs}}$ is E-unital.
(4) $\underline{\mathbb{F}}_I$ is unital if and only if $\underline{\mathbb{F}}_I^{\mathrm{sprs}}$ is unital.
(5) $\underline{\mathbb{F}}_I$ is an indexing system if and only if $\upsilon(\underline{\mathbb{F}}_I^{\mathrm{sprs}}) \cap \nabla(\underline{\mathbb{F}}_I^{\mathrm{sprs}}) = \mathcal{T}$. ◁

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3.2. **Warmup: the (aE-)unital C_p-weak indexing systems.** The orbit category of the prime-order cyclic group $C_p = \langle x \mid x^p \rangle$ may be presented as follows:

$$\left\langle \begin{tikzcd} {[C_p]} \arrow["\tau"', loop left] \arrow[r, "r_{e,C_p}"] & *_{C_p} \end{tikzcd} \;\middle|\; \tau^p = \mathrm{id}_{[C_p]}, \quad r_{e,C_p} = r_{e,C_p}\tau \right\rangle$$

It is easy to see that there are precisely two C_p-transfer systems: R_0 contains no transfers, and R_1 contains the transfer $e \to C_p$. Thus the poset Transf_{C_p} is $R_0 \to R_1$. Furthermore, there are exactly three C_p families, and the poset is $\varnothing \to \{e\} \to \{e, C_p\}$.

We will perform the following computation.

Theorem 3.6. *The poset* $\mathrm{wIndex}_{C_p}^{aE\mathrm{uni}}$ *is presented by the following*

$$\begin{tikzcd}
\varnothing \arrow[d] & & & & \\
E_e^{C_p}\mathbb{F}^{\mathrm{triv}} \arrow[r] \arrow[d] & E_e^{C_p}\mathbb{F}^{0} \arrow[r] \arrow[d] \arrow[dd, bend left] & E_e^{C_p}\mathbb{F}^{\infty} \arrow[d] \arrow[dd, bend left] & & \\
\underline{\mathbb{F}}_{C_p}^{\mathrm{triv}} \arrow[r] & \underline{\mathbb{F}}_e^{0} \arrow[r] \arrow[d] & \underline{\mathbb{F}}_e^{0} \vee E_e^{C_p}\mathbb{F}^{\infty} \arrow[d] & & \\
& \underline{\mathbb{F}}_{C_p}^{0} \arrow[r] & \underline{\mathbb{F}}_e^{\infty} \arrow[rr] \arrow[d] & & \underline{\mathbb{F}}_{C_p}^{\infty} \arrow[d] \\
& & \overline{\mathbb{F}}_{C_p} \arrow[r] & \mathbb{F}_{A\lambda} \arrow[r] & \mathbb{F}_{C_p}
\end{tikzcd}$$

where $\left\{\underline{\mathbb{F}}_{C_p}^{\infty}, \underline{\mathbb{F}}_{C_p}\right\}$ *are the indexing systems,* $\left\{\underline{\mathbb{F}}_{C_p}^{0}, \underline{\mathbb{F}}_e^{\infty}, \overline{\mathbb{F}}_{C_p}, A_\lambda\right\}$ *are the otherwise-unital weak indexing systems,* $\left\{\underline{\mathbb{F}}_e^{0}, \underline{\mathbb{F}}_e^{0} \vee E_e^{C_p}\underline{\mathbb{F}}^{\infty}\right\}$ *are the otherwise almost-unital weak indexing systems, and* $\left\{E_e^{C_p}\mathbb{F}^0, E_e^{C_p}\mathbb{F}^\infty\right\}$ *are the otherwise* E*-unital weak indexing systems.*

Remark 3.7. Already, we see that none of $\mathrm{wIndex}_{C_p}^{\mathrm{uni}}$, $\mathrm{wIndex}_{C_p}^{a\mathrm{uni}}$, $\mathrm{wIndex}_{C_p}^{E\mathrm{uni}}$, or $\mathrm{wIndex}_{C_p}^{aE\mathrm{uni}}$ are self-dual, since each embed the poset $\bullet \to \bullet \to \bullet \leftarrow$ as a cofamily, but none embed its dual as a family. This heavily contrasts the cases of $\mathrm{Index}_G = \mathrm{Transf}_G$ and Fam_G, which are known to be self-dual for arbitrary abelian G by [FOOQW22]. ◁

Note that $\underline{\mathbb{F}}_{C_p}^{\infty} \subset \underline{\mathbb{F}}_{C_p}$ are C_p-indexing systems; thus Proposition 1.37 shows that this is the poset of indexing systems.

We've completely characterized $\nabla^{-1}(\mathcal{T}) \cap \mathfrak{R}^{-1}(-)$. We may extend this to arbitrary fibers. First, those with no transfers:

Observation 3.8. For any orbital ∞-category $\mathcal{T}$, the map $\nabla : \mathfrak{R}^{-1}(\mathcal{T}^{\simeq}) \to \mathrm{Fam}_{\mathcal{T}}$ is an equivalence by Proposition 3.2; the fibers of this are

$$\nabla^{-1}(\mathcal{F}) \cap \mathfrak{r}^{-1}(\mathcal{T}^{\simeq}) = \{\underline{\mathbb{F}}_{\mathcal{T}}^{\infty}\}.$$

◁

The only remaining case is $\nabla^{-1}(\{e\}) \cap \mathfrak{R}^{-1}(R_1)$. Unwinding definitions, we find that there are two options for unital sparse collections closed under applicable self-indexed coproducts with the specified transfers and fold maps; they each must have e-values given by $\{\varnothing_e, *_e, 2\cdot *_e\}$, and the two options for C_p-values are

$$\overline{\mathbb{F}}_{C_p}^{\mathrm{sprs}} = \left\{\varnothing_{C_p}, *_{C_p}, [C_p/e]\right\}, \qquad \mathbb{F}_{A\lambda, C_p}^{\mathrm{sprs}} = \left\{\varnothing_{C_p}, *_{C_p}, [C_p/e], *_{C_p} + [C_p/e]\right\}.$$

Furthermore, in view of Corollary 2.4, we have $\mathrm{wIndex}_{BC_p}^{\mathrm{uni}} \simeq \mathrm{wIndex}_{*}^{\mathrm{uni}}$. Applying Example 1.30, we've arrived at the following computations:

$$\mathrm{wIndex}_{BC_p}^{\mathrm{uni}} : \qquad \underline{\mathbb{F}}^{0} \longrightarrow \underline{\mathbb{F}}^{\infty}$$

$$\mathrm{wIndex}_{C_p}^{\mathrm{uni}} : \qquad \begin{tikzcd}
\underline{\mathbb{F}}_{C_p}^{0} \arrow[r] & \underline{\mathbb{F}}_e^{\infty} \arrow[rr] \arrow[d] & & \underline{\mathbb{F}}_{C_p}^{\infty} \arrow[d] \\
& \overline{\mathbb{F}}_{C_p} \arrow[r] & \mathbb{F}_{A\lambda} \arrow[r] & \mathbb{F}_{C_p}
\end{tikzcd}$$

Theorem 3.6 then follows by applying Corollaries 2.24 and 2.34.

3.3. **The fibers of the C_{p^N}-transfer-fold fibration.** Recall that when $\mathcal{F} \subset \mathcal{O}_{C_{p^N}}$ is a collection of objects and R a C_{p^N}-transfer system, we refer to precomposition-closed wide subcategories of $R \cap \mathcal{F}$ as R*-sieves on* $\mathcal{F}$. Let $\underline{\mathbb{F}}_I^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_{C_{p^N}}$ be a sparse collection which is closed under applicable self-indexed coproducts. Let $S(\underline{\mathbb{F}}_I^{\mathrm{sprs}}) \subset \mathrm{Cod}(\mathfrak{R}(\underline{\mathbb{F}}_I^{\mathrm{sprs}})) - \nabla(\underline{\mathbb{F}}_I)$ be the wide subcategory consisting of maps $U \to V$ such that $*_V + U \in \mathbb{F}_{I,V}^{\mathrm{sprs}}$.

Proposition 3.9. *The restricted map map $S : \mathfrak{R}^{-1}(R) \cap \nabla^{-1}(\mathcal{F}) \to \mathrm{Sub}_{\mathbf{Cat}}(\mathrm{Cod}(\mathfrak{R}(\underline{\mathbb{F}}_I)) - \mathcal{F})$ is embedding with image the R-sieves.*

Proof. First note that in general, when $\mathcal{T}$ has no self-normalizing transfers, a unital $\mathcal{T}$-weak indexing system lying over $(R, \mathcal{F})$ is determined by its nontrivial sparse V-sets S such that:

(a) $S^V = *_V$; and
(b) $S - S^V = U_1 \sqcup \cdots \sqcup U_n \neq \varnothing$ and there exist no maps $U_i \to U_j$ for $i \neq j$; and
(c) $V \notin \mathrm{Cod}(\mathfrak{R}(\underline{\mathbb{F}}_I)) - \mathcal{F}$.

Thus we may restrict fully faithfully to just these sparse V-sets.

In fact, since $\left[\mathcal{O}_{C_{p^N}}\right]$ is a total order, such a sparse H-set is exactly an H-set of the form $*_H + [H/J]$ for some $J \subsetneq H$. Thus S is an embedding, so it suffices to characterize its image. This follows by noting that closure under sparse self-induction is precisely the characteristic that $S(\underline{\mathbb{F}}_I)$ is closed under precomposition along maps in R, i.e. it is an R-sieve. □

In order to prove Corollary D, we need to identify $\mathrm{Transf}_{C_{p^N}}$; this was already done in [BBR21] when N is finite, and the infinite case follows immediately from e.g. Theorem 2.3.

Proposition 3.10 ([BBR21, Thm 25])**.** *For $N \in \mathbb{N} \cup \{\infty\}$, there is an equivalence of posets*

$$K_{N+1} \simeq \mathrm{Transf}_{C_{p^N}},$$

the left side denoting the Nth associahedron.

Proof of Corollary D . In view of Proposition 3.10, the combined transfer-fold fibration maps $(\mathfrak{R}, \nabla)$: $\mathrm{wIndex}^{\mathrm{uni}}_{C_{p^N}} \to K_{N+1} \times [N+1]$ After Propositions 2.50, 3.9 and 3.10, we've identified the fibers. Thus it suffices to understand cocartesian transport, which is implemented by

$$t_{(R,\mathcal{F})}^{(R',\mathcal{F}')} \underline{\mathbb{F}}_I = \underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'} \vee \underline{\mathbb{F}}^{\infty}_{\mathcal{F}'}$$

by Proposition 2.13, in terms of R-sieves. When $R = R'$, it is clear that this is given by the restriction $\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \twoheadrightarrow \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}')$, so it suffices to characterize this in the case $\mathcal{F} = \mathcal{F}'$. Unwinding definitions, we're tasked with characterizing for which $U \hookrightarrow V$, we have

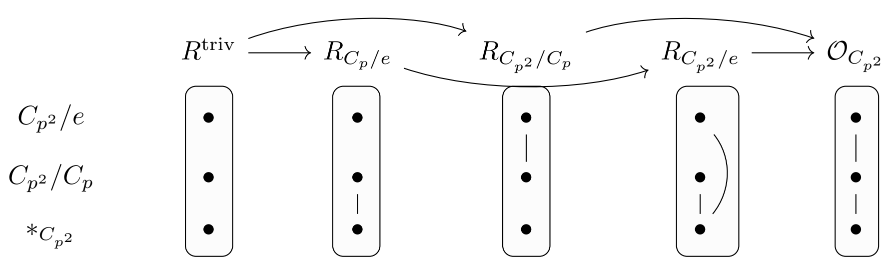
$$*_V + U \in \underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'}.$$

By Theorem C, it suffices to characterise which of these are presented as sparse indexed coproducts of elements of $\underline{\mathbb{F}}_I$ and $\overline{\underline{\mathbb{F}}}_{R'}$. Certainly the closure of the sieve for $\underline{\mathbb{F}}_I$ under precomposition along elements of R' is presented by sparse indexed coproducts of such elements; in turn, any sprase indexed coproduct ends up in such a form, proving the theorem. □

We finish by drawing this out for $N = 2$. We may illustrate $\mathcal{O}_{C_{p^2}}$ as follows

$$\begin{tikzcd} {[C_{p^2}/e]} \arrow[r] \arrow[loop below, "C_{p^2}"] & {[C_{p^2}/C_p]} \arrow[r] \arrow[loop below, "C_p"] & *_{C_p^2} \end{tikzcd}$$

with $\mathrm{Map}([C_{p^2}/e], [C_{p^2}/C_p])$ is a C_p-torsor. Yhe independent computations of [BBR21; Rub21] verify the that the following 5 transfer systems are the elements of $\mathrm{Transf}_{C_{p^2}}$



Given $R \in \mathrm{Transf}_{C_{p^2}}$, we let $\underline{\mathbb{F}}_R$ be the corresponding indexing system. Corollary D implies the following.

Corollary E. *Let $\lambda_{C_{p^N}}$ denote a nontrivial irreducible real orthogonal C_{p^N}-representation. Then, the poset of unital C_{p^2}-weak indexing systems is the following:*

$$\begin{tikzcd}
\underline{\mathbb{F}}^{\infty}_{C_{p^2}} \arrow[r] \arrow[rr, bend left] & \underline{\mathbb{F}}_{R_{C_p/e}} \arrow[rr, bend right] & \underline{\mathbb{F}}_{R_{C_{p^2}/C_p}} \arrow[rr, bend left] & \underline{\mathbb{F}}_{R_{C_{p^2}/C_p}} \arrow[r] & \underline{\mathbb{F}}_{C_{p^2}} \\
 & & \underline{\mathbb{F}}_{A\lambda_{C_p}} \arrow[u] \arrow[rr, bend left] & & \underline{\mathbb{F}}_{A\left(\lambda_{C_{p^2}}\oplus\lambda_{C_p}\right)} \arrow[u] \\
 & & & \underline{\mathbb{F}}^{\infty}_{C_p} \vee \underline{\mathbb{F}}_{A\lambda_{C_{p^2}}} \arrow[r] \arrow[uu] & \overline{\underline{\mathbb{F}}}_{C_{p^2}} \vee \underline{\mathbb{F}}_{A\lambda_{C_{p^2}}} \arrow[u] \\
\underline{\mathbb{F}}^{\infty}_{C_p} \arrow[uuu] \arrow[r] \arrow[rr, bend left] & \underline{\mathbb{F}}^{\infty}_{C_p} \vee \overline{\underline{\mathbb{F}}}_{R_{C_p/e}} \arrow[uuu] \arrow[rr, bend right] & \overline{\underline{\mathbb{F}}}_{R_{C_{p^2}/C_p}} \arrow[uu] \arrow[rr, bend left] & \underline{\mathbb{F}}^{\infty}_{C_p} \vee \overline{\underline{\mathbb{F}}}_{R_{C_{p^2}/C_p}} \arrow[u] \arrow[r] & \overline{\underline{\mathbb{F}}}_{C_{p^2}} \arrow[u] \\
 & & & \underline{\mathbb{F}}_{A\lambda_{C_{p^2}}} \arrow[uu, bend right] & \\
 & \underline{\mathbb{F}}^{0}_{\mathcal{T}} \vee E^{C_{p^2}}_{C_p} \underline{\mathbb{F}}_{A\lambda_{C_p}} \arrow[uu] \arrow[rr, bend right] & & \overline{\underline{\mathbb{F}}}_{R_{C_{p^2}/C_p}} \vee E^{C_{p^2}}_{C_p} \underline{\mathbb{F}}_{A\lambda_{C_p}} \arrow[u] & \\
\underline{\mathbb{F}}^{\infty}_{e} \arrow[uuu] \arrow[r] & \overline{\underline{\mathbb{F}}}_{R_{C_p/e}} \arrow[u] \arrow[rr, bend right] & & \overline{\underline{\mathbb{F}}}_{R_{C_{p^2}/C_p}} \arrow[u] \arrow[uuu, bend left] & \\
\underline{\mathbb{F}}^{0}_{C_{p^2}} \arrow[u] & & & &
\end{tikzcd}$$

3.4. **Questions and future directions.** To simulate further development in this area, we now pose a litany of questions concerning the structure and tabulation of weak indexing systems. The first arose to the author out of consternation concerning the apparent lack of structure arising in Corollary E.

Question 3.11. Is there a closed form expression for $\mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{O}_{C_{p^N}}}$ or $\left|\mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{O}_{C_{p^N}}}\right|$? ◁

The author believes that, akin to the strategy employed in [BBR21], this may be solved by characterizing change-of-group functors such as restriction, Borelificaiton, and inflation. In particular, given $H \subset G$ a subgroup, the cofamily $\mathcal{O}_{G/H}$ consisting of transitive G-sets on which H acts trivially is an atomic orbital ∞-category, so it possesses a well-defined theory of weak indexing systems, which should participate in an adjunction

$$\mathrm{Infl}^G_H : \mathrm{wIndex}_{G/H} \rightleftarrows \mathrm{wIndex}_G : F^G_H,$$

where F^G_H metaphorically represents "fixed points with residual genuine $W_G(H)$-action," and literally sends $\underline{\mathbb{F}}_I$ to a $\mathcal{O}_{G/H}$-weak indexing system satisfying $F^G_H \mathbb{F}_{I,V} = \mathbb{F}_{I,V}$ for all $V \in \mathcal{O}_{G/H} \subset \mathcal{O}_G$. In the setting where $N \subset G$ is normal, $\mathcal{O}_{G/N}$ is the orbit category for the group G/N, so given a choice of a *normal* subgroup, this produces an inductive procedure: characterize $\mathcal{O}_G$ weak indexing systems by picking a normal subgroup and inductively characterizing weak indexing systems for $\mathcal{O}_{G,\leq N}$ (related to $\mathcal{O}_N$ by Theorem 2.3), weak indexing systems for $\mathcal{O}_{G/N}$, and the possible transfers from outside $\mathcal{O}_{G/N}$ to inside (as well as the possible additional data of H-sets S for which N acts trivially on G/H but not on the $G/\mathrm{stab}_H(x)$ for all $x \in S$).

Outside of closed form expressions, the following question is evident as an extension of Corollary D.

Question 3.12. Is there a good combinatorial expression of $\nabla^{-1}(\mathcal{F}) \cap \mathfrak{R}^{-1}(R)$ over an arbitrary dedekind, nilpotent, or general finite group? ◁

The author expects that our techniques may be extended to a similar sieve-based presentation for $\nabla^{-1}(\mathcal{F}) \cap fR^{-1}(R)$ over more general families of groups.

Another question arises by looking closely at **??**; we were able to tabulate all 20 unital C_{p^2}-weak indexing systems using only the families $\underline{\mathbb{F}}_R$, $\overline{\underline{\mathbb{F}}}_R$, $\underline{\mathbb{F}}_{\mathcal{F}}^{\infty}$, $\underline{\mathbb{F}}_{\mathcal{F}}^{0}$, and $\underline{\mathbb{F}}_{AV}$ together with joins and the functors $E_{(-)}^{C_{p^2}}$. Thus we ask the following.

Question 3.13. Which unital weak indexing systems are realizable via tensor products of the image of $\mathbb{E}_V$ operads under various change of group functors? ◁

In particular, all instances of the right adjoint to ∇ occur as the arity support $\underline{\mathbb{F}}_{AV}$ of an $\mathbb{E}_V$-G-operad, so we ask the following.

Question 3.14. What is the right adjoint to ∇? Is it related to $\mathbb{E}_V$? ◁

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