# ECKMANN-HILTON ARGUMENTS IN EQUIVARIANT HIGHER ALGEBRA

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Abstract. Let $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ be $k$- and $\ell$-connected unital $G$-operads subject to the condition for all $S$ that $\mathcal{O}(S) = \varnothing$ if and only if $\mathcal{P}(S) = \varnothing$. We show that the Boardman-Vogt tensor product $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ is $(k+\ell+2)$-connected; equivalently, $\mathcal{O} \otimes \mathcal{P}$-monoids in any $(k+\ell+3)$-category lift uniquely to incomplete semi-Mackey functors. As a consequence, we show that the smashing localizations on unital $G$-operads correspond precisely with unital $\mathcal{N}_\infty$-operads, and hence the (finite) poset of unital weak indexing systems. Along the way we characterize $\ell$-connectivity of a unital $G$-operad $\mathcal{O}^{\otimes}$ equivalently as $\ell$-connectivity of $\mathcal{O}$-admissible Wirthmüller maps of $\mathcal{O}$-monoid spaces.

In the discrete case, under no connectivity assumptions, $\mathcal{O} \otimes \mathcal{P}$-monoids lift uniquely to incomplete semi-Mackey functors, recovering an Eckmann-Hilton argument for "$C_p$-unital magmas." In the limiting case of infinite tensor powers, we *take the loops out of equivariant infinite loop space theory*, constructing algebraic approximations to incompletely stable $G$-spectra over arbitrary transfer systems.

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## Introduction

The classical *Eckmann-Hilton argument* shows that, given a set $M$ with two unital multiplications $*, \cdot \colon M^2 \to M$ satisfying the interchange law

$$(a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d),$$

the unital magmas $(M, *)$ and $(M, \cdot)$ are isomorphic to each other and are commutative monoids. Indispensable to algebraic topologists, this fact recovers the usual proof that $\pi_n(X, x)$ is Abelian for all $n \geq 2$, as well as the same claim for $n = 1$ when $X$ admits an $H$-space structure. We will study equivariant variations of this result, beginning with a weakening of Dress' Mackey functors [Dre71].

**Definition 1.** Let $\mathcal{C}$ be a 1-category with finite products and $C_p$ the cyclic group of prime order $p$. A *$C_p$-unital magma in $\mathcal{C}$* is a unital magma $M^e$ with a $C_p$ action by unital magma homomorphisms, a unital magma $M^{C_p}$ (with trivial $C_p$-action), and $C_p$-equivariant *restriction and transfer* homomorphisms

$$r \colon M^{C_p} \to M^e, \qquad\qquad t \colon M^e \to M^{C_p}$$

subject to the condition that $r \circ t$ is multiplication by $p$. A *homomorphism* $M \to N$ is a pair of unital magma homomorphisms $F^e \colon M^e \to N^e$ and $F^{C_p} \colon M^{C_p} \to M^e$ such that $F^{C_p} \circ t = t \circ F^e$ and $F^e \circ r = r \circ F^{C_p}$. $\triangleleft$

**Example 2.** The $(\lambda + 1)$st (or $\rho$th) homotopy coefficient system of a $C_p$-space attains a natural $C_p$-unital magma structure under the evident analog of Lewis' *unstable Mackey structure* [Lew92].[1] $\triangleleft$

In this article, we prove and vastly generalize the following theorem.

**Theorem A.** *Suppose $(M, M')$ is a pair of $C_p$-unital magma structures on the same coefficient system satisfying suitable interchange relations. Then, $M \simeq M'$ and each underlie a semi-Mackey functor; in particular, if the multiplications on $M^e$ and $M^{C_p}$ are invertible, then $M$ and $M'$ are isomorphic Mackey functors.*

For instance, given $X$ a $C_2$-unital magma valued in spaces, the $C_2$-space $UX$ defined by $UX^H = X(H)$ has an induced $C_2$-unital magma structure on $\underline{\pi}_\rho(X)$ which interchanges with that of Example 2. Theorem A implies that these two structures agree and lift to a Mackey functor.

**Example 3.** The above argument confirms that the Mackey structure from Real Bott periodicity [Ati66] and the additive Mackey structure on Real vector bundles induce the same structure on $\underline{\pi}_{2\rho}\mathrm{BU}_\mathbb{R} \simeq \underline{\pi}_0\mathrm{BU}_\mathbb{R}$. $\triangleleft$

To prove Theorem A, we embed $C_p$-unital magmas in the theory of *algebras over $G$-operads* in the sense of [NS22]; in particular, we show in Section 4 that $C_p$-unital magmas are algebras over a particular $C_p$-operad $\mathbb{A}^\otimes_{2,C_p}$ in $C_p$-coefficient systems valued in $\mathcal{C}$, and spell out the correct interchange relations there. We recommend that the reader familiarizes themself with the language of equivariant higher algebra via the introductions to [Ste25a; Ste25b].

Crucially, the *Boardman-Vogt tensor product* of [Ste25a] corepresents interchanging $G$-operad algebras:

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\mathcal{D}) \simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}^\otimes_{\mathcal{P}}(\mathcal{D}).$$

In particular, pairs of interchanging $C_p$-unital magma structures correspond with $\mathbb{A}^\otimes_{2,C_p} \overset{\mathrm{BV}}{\otimes} \mathbb{A}^\otimes_{2,C_p}$-algebras.

Now, $G$-operads are manifestly homotopy-theoretic gadgets; indeed, their algebras subsume the homotopy-coherent incomplete (semi-) Mackey functors of [BH18; CLL24; Gla17] by [Mar24; Ste25b], the homotopy-coherent bi-incomplete Tambara functors of [BH22; EH23] by [Cha24; CHLL24], and the algebraic structure on equivariant iterated loop spaces and their Thom spectra by [GM11; HHKWZ24]. In particular, the first and second examples are incarnated by the *weak $\mathcal{N}_\infty$-operads* of [Ste25a], which are characterized by the fact that their nonempty structure spaces are contractible, and classified by their "arity support" weak indexing category

$$A\mathcal{O} := \{T \to S \mid \forall [G/H] \subset S,\ \mathcal{O}(T \times_S [G/H]) \neq \emptyset\} \subset \mathbb{F}_G;$$

here, $\mathcal{O}(S)$ is the "$S$-ary structure space," such as the $S$-ary structure space $\mathbb{E}_V(S) = \mathrm{Conf}^H_S(V)$ for the little $V$-disks $G$-operad incarnating the third example. See [Ste24] for an overview of weak indexing categories.

[1] Explicitly, by $V$-Mackey functor, we mean a functor $\mathscr{B}_G(V) \to \mathbf{Ab}$ sending disjoint unions to direct sums, where $\mathscr{B}_G(V)$ is Lewis' $V$-Burnside category; the transfer map $\Sigma^{\lambda+1}_+ *_{C_p} \to \Sigma^{\lambda+1}_+[C_p/e]$ is constructed by the usual $\mathbb{S}_G$-duality construction along an embedding $[C_p/e] \hookrightarrow \lambda$ (see [Wir75]). $\lambda$ refers to any nontrivial 2-dimensional $C_p$-representation.

Thankfully, $\mathcal{O}$-algebras in a $G$-symmetric monoidal $n$-category are canonically equivalent to algebras over the *homotopy $n$-operad* $h_n\mathcal{O}^{\otimes}$, whose structure spaces are $(n-1)$-truncations of the structure spaces of $\mathcal{O}^{\otimes}$ [Ste25a].[2] In particular, the structure spaces of $\mathcal{O}^{\otimes}$ are $n$-connected if and only if $h_n\mathcal{O}^{\otimes}$ possesses a (unique) equivalence with a weak $\mathcal{N}_\infty$-operad; then, $\mathcal{O}$-algebras in coefficient systems valued in an $n$-category $\mathcal{D}$ are (homotopy-coherent) incomplete semi-Mackey functors. In this situation, we say that $\mathcal{O}^{\otimes}$ is *$n$-connected*.

From this, we identify Theorem A with the statement that $\mathbb{A}_{2,C_p}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{A}_{2,C_p}^{\otimes}$ is (0-)connected, together with the easy observation that $A\mathbb{A}_{2,C_p} = \mathbb{F}_{C_p}$, so the corresponding incomplete semi-Mackey functors have all transfers. Thus it suffices to prove the following equivariant generalization of [SY19, Thm. 1.0.1].

**Theorem B** (Equivariant Eckmann-Hilton argument). *If $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ are $k$ and $\ell$-connected almost essentially unital $G$-operads with $A\mathcal{O} = A\mathcal{P}$, then $\mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes}$ is $(k+\ell+2)$-connected.*

All nonempty $G$-operads are $(-1)$-connected, so this extends Theorem A to equivariant higher algebra.

**Corollary 4** (Equivariant stabilization hypotheses). *If $\mathcal{O}^{\otimes}$ is a nonempty almost essentially unital $G$-operad, then $\mathcal{O}^{\otimes(n+1)}$ is $(n-1)$-connected; in particular, for any $G$-symmetric monoidal $n$-category $\mathcal{C}^{\otimes}$,*

$$U \colon \underline{\mathrm{CAlg}}^{\otimes}_{A\mathcal{O}}(\mathcal{C}) \xrightarrow{\ \sim\ } \overbrace{\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}} \cdots \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}}^{(n+1)\text{-fold}}(\mathcal{C}),$$

*where the $(n+1)$-fold tensor product is taken as a colimit (and composition as limit) in the case $n = \infty$.*

Here, $\mathrm{CAlg}_I$ refers to algebras over the weak $\mathcal{N}_\infty$-operad associated with $I$. For instance, Corollary 4, lax $G$-symmetric monoidality of $\underline{\pi}_0 \colon \underline{\mathrm{Sp}}_G^{\otimes} \to \underline{\mathrm{Mack}}_G^{\square}(\mathbf{Ab})$, and the results of [Cha24] or [CHLL24] together construct a natural $A\mathcal{O}$-Tambara structure on the 0th homotopy groups of $\mathcal{O}\overset{\mathrm{BV}}{\otimes}\mathcal{O}$-ring $G$-spectrum;[3] this and a forthcoming equivariant Dunn additivity result will construct a natural $AV$-Tambara structure on the 0th homotopy Mackey functors of $\mathbb{E}_{2V}$-ring $G$-spectra.

We may remove the assumption $A\mathcal{O} = A\mathcal{P}$ in Theorem B, but we will need a more refined notion of connectivity. In general, given a weak indexing category $I$, we say that $\mathcal{O}^{\otimes}$ is *$k$-connected at $I$* if, for all elements of the corresponding weak indexing system

$$T \in \mathbb{F}_{I,H} := \left\{ S \in \mathbb{F}_H \;\middle|\; \mathrm{Ind}_H^G S \to [G/H] \in I \right\},$$

the structure space $\mathcal{O}(T)$ is $k$-connected. We define the *connectivity function*

$$\mathrm{Conn}_{\mathcal{O}} \colon \mathrm{wIndexCat}_G^{a\mathrm{uni}} \longrightarrow \mathbb{Z} \cup \{\infty\}$$

by the formula $\mathrm{Conn}_{\mathcal{O}}(I) := \inf\{k \mid \mathcal{O}^{\otimes} \text{ is } k\text{-connected at } I\}$. This is a $G$-operadic version of the *connectivity dimension function* of a $G$-space (c.f. [Lew92, Def 1.1.(vi)]).

Now, $(\mathbb{Z} \cup \{\infty\})^{\mathrm{wIndexCat}_G^{a\mathrm{uni}}}$ forms a *pointwise* commutative monoidal poset, i.e.

$$f \leq g \quad \iff \quad \forall I,\ f(I) \leq g(I).$$

In this language, we will prove the following strengthening of Theorem B.

**Theorem C.** *Given $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ a pair of almost-unital $G$-operads, the following inequality holds:*

$$\mathrm{Conn}_{\mathcal{O}} + \mathrm{Conn}_{\mathcal{P}} + 2 \leq \mathrm{Conn}_{\mathcal{O} \otimes \mathcal{P}}.$$

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[2] Throughout this article, *$n$-category* will be used to refer to *$(n,1)$-categories*, i.e. $\infty$-categories whose mapping spaces are all $(n-1)$-truncated. The reader should feel free to think mostly in terms of familiar examples, such as the $n$-categoy of $(n-1)$-truncated spaces, of $(n-1)$-truncated connective spectra, of small $(n-1)$-categories, or of the hammock localization of chain complexes with homology concentrated in degrees $[d, d+n-1]$ for some uniform $d$.

[3] To construct this lax symmetric monoidality, first note that $\underline{\mathrm{Sp}}^{\otimes}_{G,\geq 0} \subset \underline{\mathrm{Sp}}^{\otimes}_G$ is closed under tensor products, so the localization $G$-functor $\underline{\mathrm{Sp}}_G \to \underline{\mathrm{Sp}}^{\otimes}_{G,\geq 0}$ is given a lax $G$-symmetric monoidal structure by Proposition 44. Moreover, to construct a lax $G$-symmetric monoidal structure on $\tau_{\leq 0} = \pi_0 \colon \underline{\mathrm{Sp}}_{G,\geq 0} \to \underline{\mathrm{Sp}}_G^{\heartsuit} = \underline{\mathrm{Mack}}_G(\mathbf{Ab})$, in light of [NS22] we need only note that indexed tensor products take $\pi_0$-equivalences to $\pi_0$-equivalences and that the resulting structure agrees with the usual one on Mackey functors; the former follows by the same fact for $G = e$, conservativity of $\prod_{(H) \subset G} \Phi^H$, and the geometric fixed point formulae of [HHR16].

To put Theorems B and C into context, note that a $G$-operad $\mathcal{O}^\otimes$ is a weak $\mathcal{N}_\infty$-operad if and only if $\mathrm{Conn}_{\mathcal{O}}$ has all values $-2$ or $\infty$; in this case, Theorem C says that weak $\mathcal{N}_\infty$-operads are closed under tensor products and $\mathcal{N}_{I\infty}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^\otimes$ is classified by a weak indexing category contained in the join $I \vee J$.

Fortunately, this incarnation of Theorem C is the difficult part of the main theorem of [Ste25b]. To explain how it is proved, we must introduce a definition: for $S \in \mathbb{F}_H$, the *$S$-indexed Wirthmüller map* in a (suitably pointed) $G$-$\infty$-category is defined to be the $S$-indexed semiadditive norm map as in [CLL24; Nar16]; that is, the $[H/K]$-indexed Wirthmüller map $W_{[H/K],X} \colon \mathrm{Ind}_K^H X \to \mathrm{CoInd}_K^H X$ is adjunct to the map

$$X \longrightarrow \mathrm{Res}_K^H \mathrm{CoInd}_K^H X \simeq \prod_{g \in [K\backslash H/K]} \mathrm{CoInd}_{H \cap gKg^{-1}}^H \mathrm{Res}_{H \cap gKg^{-1}}^H X$$

whose projection onto the factor indexed by the identity double coset is the identity and whose other projections are zero, and the $\coprod_i [H/K_i]$-indexed Wirthmüller map

$$W_{\coprod_i [H/K_i],(X_i)} \colon \coprod_{K_i}^H X_i \simeq \coprod_i \mathrm{Ind}_{K_i}^H X_i \longrightarrow \prod_i \mathrm{CoInd}_{K_i}^H X_i \simeq \prod_{K_i}^H X_i$$

is classified by the diagonal matrix whose $i$th entry is $W_{[H/K_i],X_i}$.

Key to [Ste25b] was the result that $\mathcal{O}$-monoid spaces have $I$-indexed Wirthmüller isomorphisms if and only if $\mathcal{O}^\otimes$ is $\infty$-connected at $I$. In particular, this identified $\mathcal{N}_{I\infty}^\otimes$ as the *unique* $G$-operad $\mathcal{O}^\otimes$ with $A\mathcal{O} \le I$ such that $\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{S})$ has $I$-indexed Wirthmüller isomorphisms.

Analogously, the key to this article will be to identify *$\ell$-connectivity of $\mathcal{O}^\otimes$ at $I$* within the $G$-category theory of $\mathcal{O}$-monoid spaces. To that end, we will prove the following.

**Theorem D.** *Let $\mathcal{P}^\otimes$ be a $G$-operad, $I$ an almost unital weak indexing category, and $\ell$ a natural number. Then, the following conditions are equivalent:*

*(a) $\mathcal{P}^\otimes$ is $\ell$-connected at $I$.*

*(b) For all $n$-toposes $\mathcal{C}$ (with $n \le \infty$), $I$-admissible $H$-sets $S \in \mathbb{F}_{I,H}$, and $S$-tuples of $\mathcal{P}$-monoids $(X_K) \in \prod_{[H/K] \in \mathrm{Orb}(S)} \mathrm{Mon}_{\mathrm{Res}_K^G \mathcal{P}}(\mathcal{C})$, the $S$-indexed $\mathcal{P}$-monoid Wirthmüller map*

$$W_{S,(X_K)} \colon \coprod_K^S X_K \longrightarrow \prod_K^S X_K$$

*is $\ell$-connected.*

*(c) For all $I$-admissible $H$-sets $S \in \mathbb{F}_{I,H}$ and $S$-tuples of $\mathcal{P}$-monoids $(X_K) \in \prod_{[H/K] \in \mathrm{Orb}(S)} \mathrm{Mon}_{\mathrm{Res}_K^G \mathcal{P}}(\mathcal{S})$, the $S$-indexed $\mathcal{P}$-monoid space Wirthmüller map*

$$W_{S,(X_K)} \colon \coprod_K^S X_K \longrightarrow \prod_K^S X_K$$

*is $\ell$-connected.*

For Theorem D, a morphism $g \colon X \to Y$ in an $\infty$-category $\mathcal{C}$ is *$\ell$-truncated* if, for all $Z \in \mathcal{C}$, the map of spaces $\mathrm{Map}(Z,X) \to \mathrm{Map}(Z,Y)$ is $\ell$-truncated, and $f \colon A \to B$ is $\ell$-connected if, for all diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

such that $g$ is $\ell$-truncated, the space of lifts $h$ is contractible.

**Remark 5.** In the case that $\mathcal{C}$ is an $n$-topos for some $0 \le n \le \infty$, the above definitions are equivalent to *$\ell$-truncatedness* and *$(\ell-1)$-connectiveness* in the sense of [HTT, Def. 6.5.1.10] by [SY19, Lem. 4.2.6] and [HTT, Props. 6.5.1.12, 6.5.1.19]. ◁

**Remark 6.** In the course of proving Theorem D, we will verify that Condition (b) is further equivalent to the condition that the $\mathrm{Coeff}^H\mathcal{C}$-map underlying $W_{S,(X_K)}$ is pointwise $\ell$-connected; moreover, Condition (c) is equivalent to the condition that the underlying $H$-space map is $\ell$-connected, i.e. its associated maps on $J$-fixed point spaces are surjective on path components with $\ell$-connected fiber for each $J \subset H$. ◁

**Remark 7.** When $G = e$, only the implication (a) $\implies$ (b) is argued in [SY19], but (b) $\implies$ (c) is obvious and our argument that (c) $\implies$ (a) does not involve much more than the proof of the first implication. ◁

The rest of this paper replaces the orbit category $\mathcal{O}_G$ with an arbitrary atomic orbital $\infty$-category $\mathcal{T}$; we will prove Theorems B to D in that level of generality. We encourage the reader to either globally specialize to $\mathcal{T} = \mathcal{O}_G$ or familiarize themself with the atomic orbital setting via [Ste25a] .

**Consequences in higher algebra.** Theorem B specializes to infinite tensor powers as follows.

**Corollary 8.** *Suppose $\mathcal{O}^\otimes$ is an almost-reduced $\mathcal{T}$-operad. Then, the following conditions are equivalent.*

*(a) $\mathcal{O}^\otimes$ is an almost-unital weak $\mathcal{N}_\infty$-operad.*
*(b) ($\mathcal{O}^\otimes$-EHA) the unique map $\mathrm{triv}_{\mathcal{T}}^\otimes \to \mathcal{O}^\otimes$ yields an equivalence*

$$\mathcal{O}^\otimes \simeq \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{\mathcal{T}}^\otimes \xrightarrow{\mathrm{id}\otimes !} \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{O}^\otimes.$$

*(c) (abstract $\otimes$-idempotence) there exists an equivalence $\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{O}^\otimes \simeq \mathcal{O}^\otimes$.*

*Proof.* The implication (a) $\implies$ (b) is one of the main results of [Ste25b], and is also implied by Theorem B. The implication (b) $\implies$ (c) is obvious. To see the implication (c) $\implies$ (a), note that Theorem B implies that $\mathcal{O}^\otimes$ is $\infty$-connected, i.e. all of its nonempty structure spaces are contractible. The result follows by the identification of such almost-reduced $\mathcal{T}$-operads with almost-unital weak $\mathcal{N}_\infty$-operads [Ste25a]. □

To see why we may view Condition (b) as an *Eckmann-Hilton argument*, note that it is equivalent to the condition that $\mathcal{O}^\otimes$ possesses a unital magma structure in $\mathrm{Op}_{\mathcal{T}}^\otimes$ whose multiplication map $\mu\colon \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{O}^\otimes \to \mathcal{O}^\otimes$ is an equivalence; unitality of $\mu$ is precisely the condition that the pullback natural transformation

$$\delta = \mu^*\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \longrightarrow \mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\mathcal{C})$$

is split by restriction to either $\mathcal{O}$-algebra structure, and the fact that $\mu$ is an equivalence is precisely the condition that $\delta$ is a natural equivalence, i.e. pairs of interchanging $\mathcal{O}$-algebra structures agree and $\mathcal{O}$-algebra structures interchange with themselves in an essentially unique way.

On the other hand, Condition (b) is equivalent to the assertion that $\mathcal{O}^\otimes$ admits a (unique) structure as an *idempotent algebra* in $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni},\otimes}$; taking modules yields a bijective monotone correspondence between these and the smashing localizations on $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni},\otimes}$ (see [GGN15, § 3] and [CSY20, § 5.1]). Thus Corollary 8 classifies smashing localizations on $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}$; define the full subcategory

$$\mathrm{Op}_{\mathcal{T}}^{I-\mathrm{Wirth}} := \left\{ \mathcal{O}^\otimes \;\middle|\; \forall\, S \in \underline{\mathbb{F}}_I,\, \mathcal{C}^\otimes \in \mathrm{Cat}_{\mathcal{T}}^\otimes,\ \bigotimes^S \simeq \coprod^S \text{ in } \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \right\} \subset \mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}.$$

In [Ste25b] we showed that this is the smashing localization for $\mathcal{N}_{I\infty}^\otimes$ in order to compute tensor products of $\mathcal{N}_\infty$-operads. We also showed that idempotent algebras in $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}$ are almost-reduced, yielding the following.

**Corollary E.** *The assignment $I \mapsto \mathrm{Op}_{\mathcal{T}}^{I-\mathrm{Wirth}}$ is an isomorphism of posets*

$$\mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \xrightarrow{\sim} \left\{ \text{Smashing localizations of } \mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}} \text{ under reverse inclusion} \right\}$$

Of course, we also saw in [Ste25b] that $\mathrm{Op}_{\mathcal{T}}^{I-\mathrm{Wirth}}$ consists of those $\mathcal{T}$-operads whose underlying $I$-operads are cocartesian, so we may view this in shorthand as a correspondence between smashing localizations in $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}$ and notions of either *incomplete cocartesianness of algebras* or *incomplete cocartesianness of operads.*

A striking corollary of this is that there are finitely many smashing localizations on $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}$ whenever $\mathcal{T}$ is essentially finite [Ste24]; moreover, they have rich combinatorial structure, as they are naturally cocartesian-fibered over $\mathcal{T}$-transfer systems, giving e.g. a cocartesian fibration from smashing localizations on $\mathrm{Op}_{C_{p^n}}^{a\mathrm{uni}}$ to the $(n+2)$nd associahedron, whose fibers can be explicitly described [BBR21; Ste24].

**Consequences in algebraic topology.** Let $I$ be an indexing category and $\mathrm{Sp}_I$ the $\infty$-category presented by Blumberg-Hill's stable model category of $I$-spectra [BH21].

**Definition 9.** We define the *$k$-connected, $\ell$-truncated, and $[k+1,\ell]$-concentrated* $I$-spectra by

$$\begin{aligned} \mathrm{Sp}_{I,\geq k+1} &:= \left\{E \mid \forall n \leq k, \ \ \underline{\pi}_n(E) \simeq 0\right\} \subset \mathrm{Sp}_I \\ \mathrm{Sp}_{I,\leq \ell} &:= \left\{E \mid \forall n > \ell, \ \ \underline{\pi}_n(E) \simeq 0\right\} \subset \mathrm{Sp}_I \\ \mathrm{Sp}_{I,[k+1,\ell]} &:= \mathrm{Sp}_{I,\geq k+1} \cap \mathrm{Sp}_{I,\leq \ell} \qquad\qquad \subset \mathrm{Sp}_I . \end{aligned}$$

◁

In the connected case, this category is algebraic over $G$-spaces concentrated in the same degrees.

**Lemma 10.** *If $k \geq 0$, then there exists an equivalence $\mathrm{Sp}_{I,[k+1,\ell]} \simeq \mathrm{CAlg}_I(\mathcal{S}_{G,[k+1,\ell]})$ over $\mathcal{S}_{G,[k+1,\ell]}$.*

*Proof.* Given a model $\mathcal{O}^t \in \mathrm{Op}(\mathrm{sSet}_G)$ for $\mathcal{N}^{\otimes}_{A\mathcal{O}}$, we acquire a diagram of equivalences

$$\begin{tikzcd} \mathrm{Sp}_{A\mathcal{O},[k+1,\ell]} \arrow[dr, "\Omega^\infty"'] & := & \mathrm{Alg}_{\mathcal{O}^t}\left(\mathrm{Top}_{G,[k+1,\ell]}\right)\left[\mathrm{WEQ}^{-1}\right] \arrow[r, "{[\text{Mar24}]}", "\sim"'] \arrow[dl, "U"] & \mathrm{CMon}_{A\mathcal{O}}\left(\mathcal{S}_{[k+1,\ell]}\right) \arrow[r, "{[\text{Ste25b}]}", "\sim"'] \arrow[dll, "U"] & \mathrm{CAlg}_{A\mathcal{O}}\left(\underline{\mathcal{S}}_{G,[k+1,\ell]}\right) \arrow[dlll, "U"] \\ & \mathcal{S}_{G,[k+1,\ell]} & & & \end{tikzcd}$$

Indeed, the case without connectivity and truncations is proved drectly in the cited articles, and the restriction to $[k+1,\ell]$ follows by unwinding definitions to see that the notion of concentration at degrees $[k+1,\ell]$ corresponds with the preimage of $\mathcal{S}_{G,[k+1,\ell]} \subset \mathcal{S}_G$ within each category. □

Now, *any* loop space theory with arity support $I$ reaches connected $I$-spectra after sufficient iteration.

**Corollary 11.** *Fix $\mathcal{O}^{\otimes}$ a reduced $G$-operad with $\mathcal{O}(2 \cdot *_G) \neq 0$, $0 \leq k < \ell \leq \infty$ a related pair of numbers, and $X$ a $k$-connected and $\ell$-truncated $G$-space equipped with $(\ell - k + 2)$-many interchanging $\mathcal{O}$-algebra structures. Then, $X$ is the $0$th $G$-space of an essentially unique $k$-connected (and $\ell$-truncated) $A\mathcal{O}$-spectrum compatibly with its interchanging $\mathcal{O}$-algebra structures.*

*Proof.* First note that the $G$-$\infty$-category of $k$-connected $\ell$-truncated connected $G$-spaces is a $G$-$(\ell-k+1)$-category; indeed, Elmendorf's theorem yields an equivalence

$$\left(\underline{\mathcal{S}}_{G,[k+1,\ell]}\right)_H \simeq \mathrm{Fun}\left(\mathcal{O}_H^{\mathrm{op}}, \mathcal{S}_{[k+1,\ell]}\right),$$

and $\mathcal{S}_{[k+1,\ell]}$ is an $(\ell-k+1)$-category as whenever $X$ is $k$connected and $Y$ is $\ell$-truncated, we have

$$\Omega^{\ell-k+2}\mathrm{Map}(X,Y) \simeq \mathrm{Map}(\Sigma^{\ell-k+2}X, Y) \simeq *;$$

hence [HTT, Cor. 2.3.4.8] implies that each value $\left(\underline{\mathcal{S}}_{G,[k+1,\ell]}\right)_H$ is an $(\ell-k+1)$-category. Thus Corollary 4 and Lemma 10, together construct an equivalence

$$\begin{tikzcd} \mathrm{Sp}_{A\mathcal{O},[k+1,\ell]} \arrow[dr, "\Omega^\infty"'] & \simeq \mathrm{CAlg}_{A\mathcal{O}}\left(\underline{\mathcal{S}}_{G,[k+1,\ell]}\right) \arrow[r, "U", "\sim"'] \arrow[d, dashed, "U"] & \mathrm{Alg}_{\mathcal{O}}\overbrace{\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}} \cdots \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}}^{(\ell-k+2)\text{-fold}}\left(\underline{\mathcal{S}}_{G,[k+1,\ell]}\right), \arrow[dl, "U"] \\ & \mathcal{S}_{G,[k+1,\ell]} & \end{tikzcd}$$

The composite equivalence over $\mathcal{S}_{G,[k+1,\ell]}$ is precisely what we wanted. □

**Remark 12.** Qualitatively, this is much weaker than the result we find nonequivariantly, e.g. from [SY19, Thm. 5.2.2]. This is because non-identity norms *need not exacerbate connectivity*; indeed, given $X \in \mathcal{S}_{H,*}$ such that $X^H$ is *not* $n$-connected, $\left(\mathrm{CoInd}_H^G X\right)^G \simeq X^H$ is not $n$-connected, so $\mathrm{CoInd}_H^G X$ is not $n$-connected. The author suspects instead that a version of the strong result follows for concentration in particular *regular slice degrees*, but we will not discuss that here. ◁

To construct an infinite loop space theory for $I$-spectra, one is left with the following question.

**Question 13.** Given an indexing category $I$, does there exist a reduced $G$-operad $\mathcal{O}^{\otimes}$ with $A\mathcal{O} = I$ and a space $S^I$ such that $\mathcal{O}$-monoid structures on a connected $G$-space $X$ are equivalent to $S^I$-loop space structures? ◁

**Remark 14.** Corollary 11 has a strong philosophical implication running transverse to Question 13: regardless of the topology, it constructs a flexible machine which inputs unital equivariant algebraic theories and outputs towers of $\infty$-categories converging to equivariant stable homotopy theory. For instance, iterating algebras over Rubin's free or associative $N$-opeards [Rub21] yields such a tower converging to arbitrary $\mathrm{Sp}_I$.

In essence, Corollary 11 *takes the loops out of equivariant infinite loop space theory*, extending algebraic versions of the theory to arbitrary incompletely stable categories regardless of the answer to Question 13. ◁

**Remark 15.** We chose to specialize to the connected setting for convenience; one could instead assume that there exists some $\mu \in \mathcal{O}(2 \cdot *_G)$ whose action on one of the $\mathcal{O}$-structures on $X$ induces an *invertible* magma structure on the coefficient system $\underline{\pi}_0 X$, in which case the corresponding $A\mathcal{O}$-commutative algebra has an underlying grouplike commutative monoid structure; the variation of Corollary 11 follows *mutatis mutandis.* ◁

More traditionally, we acquire $\Omega^V$-spectrum structures in a similar circumstance.

**Corollary 16.** *Fix $V$ an orthogonal $G$-representation, $0 \le k < \ell \le \infty$ related numbers, and $\mathcal{O}^\otimes$ an almost-reduced $G$-operad with $\mathcal{O}(S) \neq \varnothing$ whenever there exists an embedding $S \hookrightarrow \mathrm{Res}^G_H V$. If $X$ is a $k$-connected and $\ell$-truncated $G$-space admitting $(\ell - k + 2)$-many interchanging $\mathcal{O}$-algebra structures, then $X$ underlies a $V$-infinite loop space.*

*Proof.* The desired $V$-infinite loop space structure corresponds under the recognition principle of [GM17; RS00] with the $\mathbb{E}_{\infty V}$-structure pulled back along the unique map specified by Corollary 4:

$$\mathbb{E}^\otimes_{\infty V} \simeq \mathcal{N}^\otimes_{AV} \xrightarrow{!} \mathcal{N}^\otimes_{A\mathcal{O}} \simeq h_{\ell-k+1}\mathcal{O}^{\otimes(\ell-k+2)}. \qquad \square$$

**Philosophy.** The following significantly motivated this article and its prequels [Ste24; Ste25a; Ste25b].

**Question 17.** For what higher-algebraic, universal, and operadic reasons do $\mathcal{N}_\infty$ operads arise? ◁

Of course, there are preexisting higher-algebraic reasons: the several incomplete variants of the *spectral Mackey functor theorem* verify that $\mathcal{N}_{I\infty}$-monoids are intimately connected with $I$-admissible Wirthmüller isomorphisms, which are close to the heart of equivariant stable homotopy theory [BH18; CLL24; GM11; Mar24; Nar16]. Since the value $\underline{\mathrm{Alg}}_{\mathcal{N}_{I\infty}}(\underline{\mathcal{S}}_G)$ uniquely pins $\mathcal{N}^\otimes_{I\infty}$ as a $G$-operad [Ste25b], this characterizes $\mathcal{N}_\infty$-operads. This is not a complete answer, as it requires us to care about $I$-indexed Wirthmüller isomorphisms a-priori; that is, while our reason is higher-algebraic and universal, it is not quite operadic in its philosophy.

Now, if we admit *weak* $\mathcal{N}_\infty$-operads, a universal operadic characterization is easy to come by: in [Ste25a] we confirmed that weak $\mathcal{N}_\infty$-operads are precisely the subterminal objects of $\mathrm{Op}_G$. Unfortunately, *algebra* lives in the mapping spaces from one-object $G$-operads to $G$-symmetric monoidal $\infty$-categories, and no nontrivial $\mathcal{N}_\infty$-operads are $G$-symmetric monoidal $\infty$-categories, so the *mapping-in* property identifying weak $\mathcal{N}_\infty$-operads is not higher-algebraic in nature.

In the almost-unital locus (or, for that matter, the unital locus), Corollary 8 gives a characterization with all three properties: almost-unital weak $\mathcal{N}_\infty$-operads are characterized universally as the corepresenting $G$-operads at the limit of of (infinitary) Eckmann-Hilton arguments in equivariant higher algebra.

**Sharpness.** Theorems B and C are not sharp for all examples. One reason is the discrepancy between unions and joins of weak indexing systems.

**Example 18.** It follows by definition that

$$\mathrm{Conn}_{\mathcal{N}_{I\infty}}(J) = \begin{cases} \infty & J \subset I, \text{ and} \\ -2 & \text{otherwise;} \end{cases}$$

we also found in [Ste25b] that $\mathcal{N}^\otimes_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^\otimes_{J\infty} \simeq \mathcal{N}^\otimes_{I \vee J\infty}$. Generically, this defeats sharpness of Theorem C, as

$$\left(\mathrm{Conn}_{\mathcal{N}_{I\infty}} + \mathrm{Conn}_{\mathcal{N}_{J\infty}} + 2\right)^{-1}(\infty) = \mathrm{wIndex}^{auni}_{\mathcal{T},\le I} \cup \mathrm{wIndex}^{auni}_{\mathcal{T},\le J} \subsetneq \mathrm{wIndex}^{auni}_{\mathcal{T},\le I\vee J} = \mathrm{Conn}^{-1}_{\mathcal{N}_{I\infty}\otimes\mathcal{N}_{J\infty}}(\infty). \quad ◁$$

Another issue is topological; in forthcoming work, given $V$ an orthogonal $G$-representation, we will show that the little $V$-disks $G$-operad $\mathbb{E}^\otimes_V$ is $\ell$-connected at the minimal unital weak indexing category $I_S \vee I^0$ containing $S$ if and only if the following conditions are satisfied:

(a) For all orbits $[H/K] \subset S$ and intermediate inclusions $K \subset J \subset H$, we have $\dim V^J \ge \dim V^K + \ell + 2$, and
(b) if $\left|S^H\right| \ge 2$, then $\dim V^H \ge \ell + 2$.

Moreover, we will show that $\mathbb{E}_V$ is additive under tensor products, i.e. $\mathbb{E}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} \simeq \mathbb{E}_{V \oplus W}^{\otimes}$.

**Example 19.** Let $G := C_2$, with sign representation $\sigma$. Then, we have fixed point dimensions

$$\dim (a+b\sigma)^e = a+b; \qquad\qquad \dim (a+b\sigma)^{C_2} = a.$$

In particular, the connectivity function has

$$\mathrm{Conn}_{\mathbb{E}_{a+b\sigma}}(k *_e) = a+b-2$$

$$\mathrm{Conn}_{\mathbb{E}_{a+b\sigma}}(c *_{C_2} + d[C_2/e])) = \begin{cases} a-2 & d=0 \\ b-2 & c<2 \\ \min(a,b)-2 & \text{otherwise.} \end{cases}$$

where $\mathrm{Conn}(S) := \mathrm{Conn}(I_S \vee I^0)$. Note that $\mathrm{Conn}_{\mathbb{E}_{a+b\sigma}}(c *_{C_2} + d[C_2/e])$ is as non-additive as is possible in the last case; indeed, the examples $1+b\sigma$ and $a'+\sigma$ have the same arity-support, but when $a', b > 1$, we have

$$\begin{aligned} \mathrm{Conn}_{1+b\sigma}(2 *_{C_2} + [C_2/e]) + \mathrm{Conn}_{a'+\sigma}(2 *_{C_2} + [C_2/e]) - 2 &= 0 \\ &< \min(a',b)-1 \\ &= \mathrm{Conn}_{a'+1+(b+1)\sigma}(2 *_{C_2} + [C_2/e]). \end{aligned}$$ ◁

Nevertheless, equality is sometimes attained.

**Example 20.** For all orthogonal $G$-representations $V$, it follows from the above description that

$$\mathrm{Conn}_{\mathbb{E}_V \otimes \mathbb{E}_V} = \mathrm{Conn}_{\mathbb{E}_{2V}} = 2\mathrm{Conn}_{\mathbb{E}_V} - 2.$$ ◁

**The strategy.** In Section 3.3 we reduce Theorems B and C to the case of Theorem C where $\mathcal{O}, \mathcal{P}$ are *unital* and $\mathcal{T}$ has a terminal object. In this case, we perform a similar reduction to [SY19]; namely, by examining the free $\mathcal{O}$-algebra monad, we reduce this to $(k+1)$-*connectivity of the reduced endomorphism $A\mathcal{O}$-operad in* $\underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{C})^{I-\times}$ in the case $\mathcal{C}$ is the $\mathcal{T}$-$\infty$-category of coefficient systems in a presheaf $\infty$-topos.

We express the structure space $\mathrm{End}_X^{\mathrm{red}}\left(\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C})^{I-\times}\right)(S)$ as the spaces of lifts of $\Delta\colon X^{\sqcup S} \to X$ along the $S$-indexed Wirthmüller map $W_{X,S}\colon X^{\sqcup S} \to X^{\times S}$, which is directly related to truncatedness of $X$ and connectedness of $W_{X,S}$ [SY19]; hence it suffices to prove Theorem D in the unital case.

We finish by directly relating $\ell$-connectivity of $W_{X,S}$ in $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ and $\mathrm{Mon}_{\mathcal{O}}(\tau_{\le \ell}\mathcal{C})$, reducing Theorem D to the fact that $\mathrm{Mon}_{\mathcal{O}}(\tau_{\le \ell}\mathcal{C})$ is $I$-semiadditive when $\mathcal{O}$ is $\ell$-connected at $I$, which we verified in [Ste25b].

**Acknowledgements.** This article is greatly influenced by the work of Schlank-Yanovski [SY19], which recovers almost all of the results and ideas in this article in the case that $G$ is the trivial group, and has additionally been influential to my thinking in the previous articles [Ste25a; Ste25b]. In general, I'd like to thank my advisor Mike Hopkins for several helpful conversations on this material.

## 1. Preliminaries

Throughout this article, we fix $\mathcal{T}$ an atomic orbital $\infty$-category in the sense of [NS22]; that is, we assume that all retracts in $\mathcal{T}$ are equivalences and that the finite coproduct completion $\mathbb{F}_{\mathcal{T}} := \mathcal{T}^{\sqcup}$ has pullbacks.

We begin in Sections 1.1 and 1.2 by recalling the simultaneous generalization and weakening of Blumberg-Hill's $G$-indexing systems and $I$-Mackey functors to $\mathcal{T}$-*weak* indexing systems and $I$-commutative monoids. We go on to Section 1.3 where we recall the relevant background from [NS22; Ste25a; Ste25b] on $\mathcal{T}$-operads; we use this in Section 3.3 to reduce the main theorems in this paper to Corollary 65.

### 1.1. Preliminaries on $\mathcal{T}$-$\infty$-categories and weak indexing systems.

Recall that a $\mathcal{T}$-*coefficient system* is a functor out of $\mathcal{T}^{\mathrm{op}}$:

$$\mathrm{Coeff}^{\mathcal{T}}(\mathcal{C}) := \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C}).$$

Generalizing Elmendorf's theorem, we define $d$-truncated $\mathcal{T}$-spaces and $\mathcal{T}$-$d$-categories as coefficient systems:

$$\mathcal{S}_{\mathcal{T},\le d} := \mathrm{Coeff}^{\mathcal{T}}(\mathcal{S}_{\le d}); \qquad\qquad \mathrm{Cat}_{\mathcal{T},d} := \mathrm{Coeff}^{\mathcal{T}}(\mathrm{Cat}_d).$$

We write $\mathrm{Cat}_{\mathcal{T}} := \mathrm{Cat}_{\mathcal{T},\infty}$ and $\mathcal{S}_{\mathcal{T}} := \mathcal{S}_{\mathcal{T},\le\infty}$. Given a $\mathcal{T}$-$\infty$-category $\mathcal{C}$, we write $\mathcal{C}_V$ for the value $\mathcal{C}(V)$ and $\mathrm{Res}_V^W\colon \mathcal{C}_W \to \mathcal{C}_V$ for the functoriality under a map $V \to W$. The $\infty$-category of $\mathcal{T}$-coefficient systems lifts to a $\mathcal{T}$-$\infty$-category with $V$-value the $\mathcal{T}_{/V}$-coefficient systems

$$\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{C})_V := \mathrm{Coeff}^{\mathcal{T}_{/V}}(\mathcal{C});$$

the functoriality is given by restriction. We acquire $\mathcal{T}$-$\infty$-categories $\underline{\mathcal{S}}_{\mathcal{T},\le d}$ and $\underline{\mathrm{Cat}}_{\mathcal{T},d}$ similarly.

**Example 21.** We may define a $\mathcal{T}$-$\infty$-category by $\underline{\mathbb{F}}_{\mathcal{T}}$ by values

$$(\underline{\mathbb{F}}_{\mathcal{T}})_V := \mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_{\mathcal{T}_{/V}}$$

with functoriality given by pullback. We write $\mathbb{F}_V := \mathbb{F}_{\mathcal{T},/V}$. Note that this is a $\mathcal{T}$-1-category since $\mathcal{T}_{/V}$ is a 1-category [NS22, Prop. 2.5.1]. ◃

**Example 22.** Given $\mathcal{C}$ an arbitrary $n$-category, $\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{C})$ is a $\mathcal{T}$-$n$-category [HTT, Cor. 2.3.4.8]. In particular, if $\mathcal{C}$ is an $\infty$-topos and $\tau_{\le n-1}\mathcal{C}$ its $n$-topos of $(n-1)$-truncated objects, then $\underline{\mathrm{Coeff}}^{\mathcal{T}}(\tau_{\le n-1}\mathcal{C})$ is a $\mathcal{T}$-$n$-category. ◃

**Example 23.** The $\infty$-category of $\mathcal{T}$-$\infty$-categories is Cartesian closed with internal hom characterized by values

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})_V \simeq \mathrm{Fun}_{\mathcal{T}_{/V}}(\mathrm{Res}^{\mathcal{T}}_V \mathcal{C}, \mathrm{Res}^{\mathcal{T}}_V \mathcal{D}),$$

where $\mathrm{Res}^{\mathcal{T}}_V \colon \mathrm{Cat}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}_{/V}}$ is pullback and $\mathrm{Fun}_{\mathcal{T}}(-,-)$ denotes the evident $\infty$-category of natural transformations [BDGNS16]. By unwinding definitions and applying [HTT, Cor. 2.3.4.8], we find that whenever $\mathcal{D}$ is a $\mathcal{T}$-$n$-category, $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$ is a $\mathcal{T}$-$n$-category. ◃

**Example 24.** We refer to the adjunction between limits and constant diagrams as the *inflation and fixed point* adjunction

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend left, "\mathrm{Infl}^{\mathcal{T}}_e"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_{\mathcal{T}} \arrow[l, bend left, "\Gamma^{\mathcal{T}}"] \end{tikzcd}$$

In the case that $\mathcal{T}$ has a terminal object $V$, the image of $\mathrm{Infl}^{\mathcal{T}}_e$ consists of the $\mathcal{T}$-$\infty$-categories whose restriction functors $\mathrm{Res}^V_W$ are all equivalences. In any case, we may string together natural equivalences

$$\begin{aligned} \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}^{\mathcal{T}}_e K, \underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}\right)_V &\simeq \mathrm{Fun}_V\left(\mathrm{Infl}^{\mathcal{T}_{/V}}_e K, \underline{\mathrm{Coeff}}^{\mathcal{T}_{/V}}\mathcal{C}\right) \\ &\simeq \mathrm{Fun}\left(K, \mathrm{Fun}\left((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{C}\right)\right) \\ &\simeq \mathrm{Fun}\left((\mathcal{T}_{/V})^{\mathrm{op}}, \mathrm{Fun}(K,\mathcal{C})\right) \\ &\simeq \underline{\mathrm{Coeff}}^{\mathcal{T}}\left(\mathcal{C}^K\right)_V \end{aligned}$$

to construct a $\mathcal{T}$-equivalence $\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}^{\mathcal{T}}_e K, \underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}\right) \simeq \underline{\mathrm{Coeff}}^{\mathcal{T}}\left(\mathcal{C}^K\right)$; in particular, choosing $\mathcal{C} = \mathcal{K}$, $\mathcal{T}$-coefficient systems in presheaves of spaces on $K$ can equivalently be realized as $\mathcal{T}$-equivariant presheaves of $\mathcal{T}$-spaces on $K$ with trivial $\mathcal{T}$-equivariant structure. We henceforth write

$$\underline{\mathcal{S}}^K_{\mathcal{T},\le n} := \underline{\mathrm{Coeff}}^{\mathcal{T}}\left(\mathcal{S}^K_{\le n}\right); \qquad \underline{\mathcal{S}}^K_{\mathcal{T}} := \underline{\mathrm{Coeff}}^{\mathcal{T}}\left(\mathcal{S}^K\right).$$ ◃

Given $V \in \mathcal{T}$ an orbit and $S \in \mathbb{F}_V$ a finite $V$-set, we write $\varphi_{SV} \colon \mathrm{Ind}^{\mathcal{T}}_V S \to V$ for the corresponding map in $\mathbb{F}_{\mathcal{T}}$, and we write

$$\mathcal{C}_S := \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_U \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathrm{Ind}^{\mathcal{T}}_V S, \mathcal{C}\right).$$

Pullback along the structure map $\varphi_{SV}$ yields an *indexed diagonal* functor

$$\Delta^S \colon \mathcal{C}_V \to \mathcal{C}_S;$$

its values are $\Delta^S X = (\mathrm{Res}^V_U X)_{U \in \mathrm{Orb}(S)}$. The $S$*-indexed coproduct* (if it exists) is the left adjoint $\coprod^S \colon \mathcal{C}_S \to \mathcal{C}_V$ to $\Delta^S$, and the $S$*-indexed product* $\prod^S \colon \mathcal{C}_S \to \mathcal{C}_V$ is the right adjoint.

**Notation 25.** In the case $U \to V$ is a map of orbits, considered as an element of $\mathbb{F}_V = \mathbb{F}_{\mathcal{T},/V}$, we write

$$\mathrm{Ind}^V_U(-) := \coprod^{U \to V}(-); \qquad \mathrm{CoInd}^V_U(-) := \prod^{U \to V}(-),$$

so that $\mathrm{Ind}_U^V \dashv \mathrm{Res}_U^V \dashv \mathrm{CoInd}_U^V$ we refer to these as *induction and coinduction*. In particular, $\mathrm{Ind}_U^V \colon \mathbb{F}_{\mathcal{T},/U} \to \mathbb{F}_{\mathcal{T},/V}$ is *postcomposition*. We call $S^V = \mathrm{Hom}_{\mathbb{F}_V}(*_V, S)$ the *fixed points* and define the *distinguished fixed point*

$$\begin{tikzcd}
U = \mathrm{Ind}_U^{\mathcal{T}} *_U \arrow[dr, dashed, "\mathrm{Ind}_U^{\mathcal{T}}\delta"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, equal] & & \\
& \mathrm{Ind}_U^{\mathcal{T}} \mathrm{Res}_U^V \mathrm{Ind}_U^V *_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U = \mathrm{Ind}_V^{\mathcal{T}} \mathrm{Ind}_U^V *_U \arrow[d] \\
& U \arrow[r] & V
\end{tikzcd}$$

Note that, since $\mathcal{T}$ is atomic, $\delta \colon *_U \to \mathrm{Res}_U^V \mathrm{Ind}_U^V *_U$ is a summand inclusion. In analogy to equivariant homotopy theory, we suggest the reader view $\delta_U$ as "the identity coset fixed point." More generally, this produces a map

$$\delta \colon S^U \to \left(\mathrm{Res}_U^V \mathrm{Ind}_U^V S\right)^U,$$

which in fact occurs as a specialization of a similarly defined orbit type-preserving map

$$\delta \colon \mathrm{Orb}(S) \to \mathrm{Orb}\left(\mathrm{Res}_U^V \mathrm{Ind}_U^V S\right)^U. \qquad \triangleleft$$

These are the ur-examples of *equivariantly indexed operations*, whose combinatorics we control using *weak indexing systems*.

**Definition 26.** A *one-color weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ which is closed under $\underline{\mathbb{F}}_I$-indexed coproducts and contains $*_V$ for all $V \in \mathcal{T}$. A *one-color weak indexing category* is a pullback-stable wide subcategory $I \subset \mathbb{F}_{\mathcal{T}}$ subject to the condition that $\coprod_i (T_i \to S_i)$ lies in $I$ if and only if each map $T_i \to S_i$ lies in $I$. $\triangleleft$

Given $I$ a one-color weak indexing category, we define the *$I$-admissible $V$-sets* as

$$\underline{\mathbb{F}}_I := \left\{ S \mid \mathrm{Ind}_V^{\mathcal{T}} S \to V \in I \right\} \subset \underline{\mathbb{F}}_{\mathcal{T}};$$

we verified in [Ste24] that $\underline{\mathbb{F}}_{(-)}$ furnishes an equivalence between one-color weak indexing systems and one-color weak indexing categories, so we safely conflate these notions. For the following example, a full subcategory $\mathcal{F} \subset \mathcal{T}$ is called a *$\mathcal{T}$-family* if, whenever there exists a morphism $V \to W$ with $W \in \mathcal{F}$, we have $V \in \mathcal{F}$.

**Example 27.** The terminal one-color weak indexing system is $\underline{\mathbb{F}}_{\mathcal{T}}$. We define the following other examples, where $\mathcal{F} \subset \mathcal{T}$ is a fixed $\mathcal{T}$-family:

$$\begin{aligned}
(\underline{\mathbb{F}}_{\mathrm{triv}})_V &:= \{*_V\} \\
\left(\underline{\mathbb{F}}_{0,\mathcal{F}}\right)_V &:= \begin{cases} \{\varnothing_V, *_V\} & V \in \mathcal{F} \\ \{*_V\} & \text{otherwise.} \end{cases} \\
(\underline{\mathbb{F}}_{\infty})_V &:= \{n \cdot *_V \mid n \in \mathbb{N}\}.
\end{aligned}$$

The corresponding one-color weak indexing categories are denoted $I_{\mathrm{triv}}, I_{0,\mathcal{F}}, I_\infty$. $\triangleleft$

**Construction 28.** We write

$$\upsilon(I) := \left\{ V \in \mathcal{T} \mid \varnothing_V \in (\underline{\mathbb{F}}_I)_V \right\} \subset \mathcal{T}.$$

This is a $\mathcal{T}$-family, called the *unit family* of $I$ [Ste24]. $\triangleleft$

We say that $\underline{\mathbb{F}}_I$ is *almost-unital* if, whenever $\{*_V\} \subsetneq \mathbb{F}_{I,V}$, we have $\varnothing_V \in \mathbb{F}_{I,V}$; that is, $\underline{\mathbb{F}}_I$ is unital over all orbits for which $\underline{\mathbb{F}}_I$ has nontrivial arities. We say $\underline{\mathbb{F}}_I$ is *unital* if $\varnothing_V \in \mathbb{F}_{I,V}$ for all $V$.

## 1.2. Preliminaries on $I$-commutative monoids and $I$-symmetric monoidal $\infty$-categories.

Let $I$ be a one-color weak indexing category. The pair $(\mathbb{F}_{\mathcal{T}}, I)$ is a *span pair* in the sense of [EH23] (i.e. $(\mathbb{F}_{\mathcal{T}}, I, I)$ is an *adequate triple* in the sense of [Bar14]), so it yields a wide subcategory

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \hookrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$$

of the effective Burnside ∞-category whose morphisms are given by spans $X \leftarrow R \xrightarrow{f} Y$ with $f \in I$. Given $I$ a one-color weak indexing category and $\mathcal{C}$ an ∞-category, we define the ∞-category of *$I$-commutative monoids* in $\mathcal{C}$ as

$$\mathrm{CMon}_I(\mathcal{C}) \coloneqq \mathrm{Fun}^\times(\mathrm{Span}_I(\mathbb{F}_\mathcal{T}), \mathcal{C}).$$

We define the ∞-category of *small $I$-symmetric monoidal ∞-categories* as

$$\mathrm{Cat}_I^\otimes \coloneqq \mathrm{CMon}_I(\mathrm{Cat}).$$

We henceforth ignore size issues and omit the adjective "small." Given an $I$-symmetric monoidal ∞-category $\mathcal{C}$ and $S \in \mathbb{F}_{I,V}$ an $I$-admissible $V$-set, we denote the functoriality of $\mathcal{C}^\otimes$ under the structure map $\mathrm{Ind}_S^\mathcal{T} S = \mathrm{Ind}_S^\mathcal{T} S \to V$ by

$$\bigotimes^S : \mathcal{C}_S \to \mathcal{C}_V.$$

If $I$ is almost-unital, $S \in \mathbb{F}_{I,V}$ is $I$-admissible, and $1_U \in \mathcal{C}_U$ is initial whenever it exists, then given an $S$-indexed tuple $(X_U) \in \mathcal{C}_S$ in an $I$-symmetric monoidal ∞-category with $S$-indexed coproducts, we define an *$S$-indexed tensor Wirthmüller map*

$$W_{S,(X_U)} \colon \coprod_U^S X_U \longrightarrow \bigotimes_U^S X_U$$

by defining its composite map $\mathrm{Ind}_W^V X_W \hookrightarrow \coprod_U^S X_U \to \bigotimes_U^S X_U$ to be adjunct to the map

$$\iota_W \colon X_W \simeq X_W \otimes \bigotimes_U^{\mathrm{Res}_W^V S - \delta(W)} 1_U \xrightarrow{(\mathrm{id},\eta)} X_W \otimes \bigotimes_U^{\mathrm{Res}_W^V S-\delta(W)} X_U \simeq \mathrm{Res}_W^V \bigotimes_U^S X_U;$$

intuitively, on the $W$'th factor, $W_{S,(X_U)}$ takes $x$ to the simple tensor with $x$ in the $W$'th place and units elsewhere. Given $J \subset I$, we say that $\mathcal{C}$ is *$J$-cocartesian* if $W_{S,(X_U)}$ is an equivalence for all $S \in \underline{\mathbb{F}}_I$ and $(X_U) \in \mathcal{C}_S$, and we say that $\mathcal{C}$ is *$J$-cartesian* if its "vertical opposite"

$$\mathrm{Span}_I(\mathbb{F}_\mathcal{T}) \xrightarrow{\mathcal{C}^\otimes} \mathrm{Cat} \xrightarrow{\mathrm{op}} \mathrm{Cat}$$

is a $J$-cocartesian $I$-symmetric monoidal ∞-category..

In [Ste25b], given $\mathcal{C}$ a $\mathcal{T}$-∞-category with $I$-indexed (co)products, we constructed essentially unique *(co)cartesian* $I$-symmetric monoidal structures on $\mathcal{C}$ and verified that $\mathcal{C}$ is *$I$-semiadditive* in the sense of [CLL24] if and only if $\mathcal{C}^{I-\times}$ is cocartesian, or equivalently, there exists an equivalence $\mathcal{C}^{I-\sqcup} \simeq \mathcal{C}^{I-\times}$ lying over the identity endofunctor.

1.3. **Naive preliminaries on $I$-operads.** In [NS22], an ∞-category $\mathrm{Op}_\mathcal{T}$ of *$\mathcal{T}$-operads* was introduced, and in [Ste25a; Ste25b] it was given a symmetric monoidal closed $\mathcal{T}$-∞-category structure $\underline{\mathrm{Op}}_\mathcal{T}^\otimes$. We review the relevant formal properties here; in particular, outside of a the verification of another formal property in Proposition 44, we will only use formal properties of $\underline{\mathrm{Op}}_\mathcal{T}^\otimes$, instead probing its objects via the various functors

$$\begin{tikzcd} \mathrm{Cat}_\mathcal{T}^\otimes \arrow[r, hook] & \mathrm{Op}_\mathcal{T} \arrow[r, "\mathrm{sseq}"] \arrow[dl, "U"] \arrow[d, dotted, "\underline{\mathrm{Alg}}_{(-)}(\mathcal{C})"] \arrow[dr, "\underline{\mathrm{Alg}}_\mathcal{P}(-)"] & \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_\mathcal{T}, \mathcal{S}) \\ \mathrm{Cat}_\mathcal{T} & \mathrm{Cat}_\mathcal{T} & \mathrm{Cat}_\mathcal{T} \end{tikzcd}$$

In this way, this paper can be considered agnostic to the presentation of $\underline{\mathrm{Op}}_\mathcal{T}^\otimes$ and the above functors.

1.3.1. *$\mathcal{T}$-symmetric sequences and $I$-operads.* Writing $\underline{\Sigma}_\mathcal{T}$ for the composite $\mathcal{T}$-∞-category

$$\mathcal{T}^{\mathrm{op}} \xrightarrow{\mathbb{F}_\mathcal{T}} \mathrm{Cat} \xrightarrow{(-)^\simeq} \mathcal{S} \hookrightarrow \mathrm{Cat}$$

and writing $\mathrm{Tot} \colon \mathrm{Cat}_\mathcal{T} \simeq \mathrm{Cat}^{\mathrm{cocart}}_{/\mathcal{T}^{\mathrm{op}}} \to \mathrm{Cat}$ for the total category functor, in [Ste25a] we defined a *underlying $\mathcal{T}$-symmetric sequence* functor

$$\mathcal{O}(-) \colon \mathrm{Op}_\mathcal{T} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_\mathcal{T}, \mathcal{S}).$$

To characterize this, we need a definition.

**Definition 29.** We say that an $I$-operad $\mathcal{O}^{\otimes}$ *has at least one color* if $\mathcal{O}(*_V) \neq \varnothing$ for all $V \in \mathcal{T}$ and *has one color* if $\mathcal{O}(*_V) \simeq *$ for all $V \in \mathcal{T}$, ◁

**Proposition 30** ([Ste25a])**.** *The functor* $\mathcal{O}(-)\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ *has a left adjoint* $\mathrm{Fr}$*; in particular, letting* $\mathrm{Fr}_{\mathrm{Op}}(S)$ *be the free* $\mathcal{T}$*-operad on the left Kan extended* $\mathcal{T}$*-symmetric sequence*

$$\begin{tikzcd} \{S\} \arrow[r, "*"] \arrow[d, hook] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{S} \\ \mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \arrow[ur, dashed, "\mathrm{Fr}_{\Sigma,S}(*)"'] & \end{tikzcd}$$

*the adjunctions construct a natural equivalence*

$$\mathrm{Alg}_{\mathrm{Fr}_{\mathrm{Op}}(S)}(\mathcal{O}) \simeq \mathcal{O}(S).$$

*Moreover, the restricted functor* $\mathcal{O}(-)\colon \mathrm{Op}_{\mathcal{T}}^{\mathrm{oc}} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ *is monadic.*

In particular, identifying an object of $\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}$ with a pair $(V,S)$ where $V \in \mathcal{T}$ and $S \in \mathbb{F}_V$, $\mathcal{T}$-operads are identified conservatively by the functor

$$\mathcal{O} \mapsto \prod_{V,S} \mathcal{O}(S).$$

Intuitively, we view $\mathcal{O}(S)$ as the space of $S$*-ary operations* $\left(\mathrm{Res}^{\mathcal{T}}_V X\right)^{\otimes S} \to \mathrm{Res}^{\mathcal{T}}_V X$ borne by an $\mathcal{O}$-algebra $X$. This technology allowed us to define the *arity support* functor

$$A\mathcal{O} := \left\{ T \to S \;\middle|\; \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T \times_S U) \neq \varnothing \right\} \subset \mathbb{F}_{\mathcal{T}};$$

which we verified in [Ste25a] to be a weak indexing category. In fact, we verified that the essential surjection associated with $A$ possesses a fully faithful right adjoint

$$\begin{tikzcd} \mathrm{Op}_{\mathcal{T}} \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathrm{wIndexCat}_{\mathcal{T}}; \arrow[l, bend left, "\mathcal{N}^{\otimes}_{(-)\infty}"] \end{tikzcd} \tag{1}$$

we refer to the $\mathcal{T}$-operad $\mathcal{N}^{\otimes}_{I\infty}$ as the *weak* $\mathcal{N}_\infty$*-operad associated with* $I$. Now, we further verified in [Ste25a] that, given a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$, the unique map $\mathcal{O}^{\otimes} \to \mathrm{Comm}^{\otimes}_{\mathcal{T}}$ is a monomorphism if and only if the counit map $\mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{A\mathcal{O}}$ is an equivalence; in particular, we acquire an equality of full subcategories

$$\mathrm{Op}_{\mathcal{T},/\mathcal{N}^{\otimes}_{I\infty}} = A^{-1}(\mathrm{wIndexCat}_{\mathcal{T},\leq I}) \;\subset\; \mathrm{Op}_{\mathcal{T}},$$

and a full subcategory of $\mathrm{Op}_{\mathcal{T}}$ has a terminal object if and only if it is of this form. We refer to $\mathrm{Op}_I := \mathrm{Op}_{\mathcal{T},/\mathcal{N}^{\otimes}_{I\infty}}$ as the $\infty$-category of $I$*-operads*; see [Ste25a] for an intrinsic characterization of $\mathrm{Op}_I$.

Monomorphisms are right-cancellable, so all inclusions $I \subset J$ induce monomorphisms $\iota_I^J\colon \mathcal{N}^{\otimes}_{I\infty} \to \mathcal{N}^{\otimes}_{J\infty}$; in other words, the push-pull adjunction

$$\begin{tikzcd} \mathrm{Op}_I \arrow[r, bend left, "E_I^J = \iota^J_{I!}"] \arrow[r, phantom, "\perp"] & \mathrm{Op}_J \arrow[l, bend left, "\mathrm{Bor}_I^J = \iota_I^{J*}"] \end{tikzcd}$$

witnesses $\mathrm{Op}_I \subset \mathrm{Op}_J$ as a colocalizing subcategory. Moreover, it behaves well with $\overset{\mathrm{BV}}{\otimes}$.

**Proposition 31** ([Ste25a])**.** *Suppose* $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ *have at least one color. Then, there is an equality*

$$A(\mathcal{O} \otimes \mathcal{P}) \simeq A\mathcal{O} \vee A\mathcal{P}.$$

*In particular,* $\mathrm{Op}_I \subset \mathrm{Op}_{\mathcal{T}}$ *is a symmetric monoidal full subcategory.*

1.3.2. *Restrictions of $\mathcal{T}$-operads.* The $\mathcal{T}$-category of coefficient systems has a universal property

$$\mathrm{Fun}_{\mathcal{T}}(\mathcal{C}, \underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}) \simeq \mathrm{Fun}(\mathrm{Tot}^{\mathcal{T}}\mathcal{C}, \mathcal{D});$$

in particular, this yields a *restriction functor*

$$\begin{tikzcd} \mathrm{Fun}\left(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S}\right) \arrow[r, "\mathrm{Res}_V^{\mathcal{T}}"] & \mathrm{Fun}\left(\mathrm{Tot}\underline{\Sigma}_V, \mathcal{S}\right) \\ \mathrm{Fun}_{\mathcal{T}}\left(\underline{\Sigma}_{\mathcal{T}}, \mathcal{S}_{\mathcal{T}}\right) \arrow[u, "\wr"] \arrow[r] & \mathrm{Fun}_V\left(\underline{\Sigma}_V, \mathcal{S}_V\right) \arrow[u, "\wr"] \end{tikzcd}$$

so that, given a map $W \to V$ and an $W$-set $S$, $\mathrm{Res}_V^{\mathcal{T}} \mathcal{O}(S) \simeq \mathcal{O}(S)$. By [Ste25a, § 2.3], this lifts to a restriction functor on $\mathcal{T}$-operads

$$\begin{tikzcd} \mathrm{Op}_{\mathcal{T}} \arrow[r, "\mathrm{Res}_V^{\mathcal{T}}"] \arrow[d] & \mathrm{Op}_V \arrow[d] \\ \mathrm{Fun}\left(\mathrm{Tot}^{\mathcal{T}}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S}\right) \arrow[r] & \mathrm{Fun}\left(\mathrm{Tot}^{V}\underline{\Sigma}_V, \mathcal{S}\right) \end{tikzcd}$$

assembling to an equivalence $\mathrm{Op}_{\mathcal{T}} \simeq \Gamma^{\mathcal{T}} \underline{\mathrm{Op}}_{\mathcal{T}}$; we will refer to the induced tensor product on $\mathrm{Op}_{\mathcal{T}}$ as $\overset{\mathrm{BV}}{\otimes}$.

1.3.3. *$I$-symmetric monoidal categories and $\mathcal{O}$-algebras.* [NS22] constructed a (non-full) subcategory inclusion

$$\iota \colon \mathrm{Cat}_I^{\otimes} \to \mathrm{Op}_{\mathcal{T}};$$

$\mathcal{T}$-operad maps between $I$-symmetric monoidal categories are called *lax $I$-symmetric monoidal functors*, and morphisms in the image of $\iota$ are called *$I$-symmetric monoidal functors.*

Moreover, given $\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$, we define *$\mathcal{O}$-algebras in $\mathcal{C}^{\otimes}$* to be $\mathcal{T}$-operad maps $\mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$, which naturally fit into an $\infty$-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. These have a *pointwise $\mathcal{T}$-operad structure* $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ given by the internal hom in a presentably symmetric monoidal structure on $\mathrm{Op}_{\mathcal{T}}$, whose tensor product we write as $\overset{\mathrm{BV}}{\otimes}$ [Ste25a; Ste25b]. The unit for this symmetric monoidal structure is the $\mathcal{T}$-operad $\mathrm{triv}_{\mathcal{T}}^{\otimes} := \mathcal{N}_{I^{\mathrm{triv}}\infty}^{\otimes}$ [Ste25a], i.e. there is a canonical equivalence

$$\underline{\mathrm{Alg}}_{\mathrm{triv}_{\mathcal{T}}}^{\otimes}(\mathcal{O}) \simeq \mathcal{O}^{\otimes} \tag{2}$$

Moreover, we verified in [Ste25a] that whenever $\mathcal{C}^{\otimes}$ is an $I$-symmetric monoidal $\infty$-category, $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is as well, and given a $\mathcal{T}$-operad map $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ and an $I$-symmetric monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$, the induced lax $I$-symmetric monoidal functors

$$\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}); \qquad \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{D})$$

are $I$-symmetric monoidal. In particular, when $\mathcal{C}^{\otimes}$ is an $I$-symmetric monoidal $\infty$-category and $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are $I$-operads, there are natural $I$-symmetric monoidal equivalence

$$\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \simeq \underline{\mathrm{Alg}}_{\mathcal{O}\otimes\mathcal{P}}^{\otimes}(\mathcal{C}) \simeq \underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \tag{3}$$

1.3.4. *The underlying $\mathcal{T}$-$\infty$-category.* An $I$-operad $\mathcal{O}^{\otimes}$ has an underlying $\mathcal{T}$-$\infty$-category $U\mathcal{O}$ [NS22]; indeed, $\mathcal{T}$-operads are equivariantizations of the classical notions of *colored operads*, and $U\mathcal{O}$ the *$\infty$-category of colors.* Moreover, the composite functor $\mathrm{Cat}_I^{\otimes} \to \mathrm{Op}_I \xrightarrow{U} \mathrm{Cat}_{\mathcal{T}}$ is the usual *underlying $\mathcal{T}$-$\infty$-category* functor.

$U$ behaves well with respect to $\underline{\mathrm{Alg}}^{\otimes}$; indeed, we verified in [Ste25a] that the underlying $\mathcal{T}$-$\infty$-category has values

$$U\left(\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})\right)_V \simeq \mathrm{Alg}_{\mathrm{Res}_V^{\mathcal{T}}\mathcal{O}}\left(\mathrm{Res}_V^{\mathcal{T}}\mathcal{C}\right),$$

where $\mathrm{Res}_V^{\mathcal{T}} \colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Op}_V$ is a restriction functor, and furthermore

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \simeq \Gamma^{\mathcal{T}} U \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}).$$

It was observed in [NS22] that the composite functor $\mathrm{Op}_{I^{\mathrm{triv}}} \subset \mathrm{Op}_{\mathcal{T}} \xrightarrow{U} \mathrm{Cat}_{\mathcal{T}}$ is an equivalence, and that $U$ factors as $\mathrm{Op}_{\mathcal{T}} \xrightarrow{\mathrm{Bor}_{I^\infty}^{\mathrm{triv}}} \mathrm{Op}_{I^{\mathrm{triv}}} \simeq \mathrm{Cat}_{\mathcal{T}}$. We write $\mathrm{triv}(-)^{\otimes}$ for the composite functor

$$\mathrm{triv}(-)^{\otimes} \colon \mathrm{Cat}_{\mathcal{T}} \xrightarrow{U^{-1}} \mathrm{Op}_{I^\infty} \hookrightarrow \mathrm{Op}_{\mathcal{T}};$$

unwinding definitions, we find that there is a natural equivalence

$$\underline{\mathrm{Alg}}_{\mathrm{triv}(\mathcal{C})}(\mathcal{O}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, U\mathcal{O});$$

that is, $\mathrm{triv}(\mathcal{C})$ algebras are simply $\mathcal{C}$-indexed diagrams of objects.

1.3.5. *$\mathcal{T}$-operadic inflation and fixed points.* In [Ste25a] we constructed an equivalence

$$\varphi\colon \mathrm{Op}_{I_\infty} \xrightarrow{\sim} \mathrm{Coeff}^{\mathcal{T}}\mathrm{Op}$$

exhibiting natural equivalences $\varphi\mathcal{O}_V(n) \simeq \mathcal{O}(n\cdot *_V)$. Limits and constant diagrams yields an *inflation and fixed point* adjunction

$$\begin{tikzcd} \mathrm{Op} \arrow[r, bend left, "\mathrm{Infl}_e^{\mathcal{T}}"] \arrow[r, phantom, "\perp"] & \mathrm{Op}_{I_\infty} \arrow[l, bend left, "\Gamma^{\mathcal{T}}"] \arrow[r, bend left, "E_{I_\infty}^{\mathcal{T}}"] \arrow[r, phantom, "\perp"] & \mathrm{Op}_{\mathcal{T}}; \arrow[l, bend left, "\mathrm{Bor}_{I_\infty}^{\mathcal{T}}"] \end{tikzcd}$$

we refer to the composite adjunction $\mathrm{Op} \rightleftarrows \mathrm{Op}_{\mathcal{T}}$ also as $\mathrm{Infl}_e^{\mathcal{T}} \dashv \Gamma^{\mathcal{T}}$. For instance we have

$$\mathrm{Alg}_{\mathrm{Infl}(\mathcal{O})}(\mathcal{P}) \simeq \mathrm{Alg}_{\mathcal{O}}\left(\Gamma^{\mathcal{T}}\mathcal{P}\right); \tag{4}$$

moreover, we can identify the image of $\mathrm{Infl}_e^{\mathcal{T}}$ easily: they are the $I_\infty$-operads $\mathcal{O}^\otimes$ whose underlying $\mathcal{T}$-$\infty$-category is inflated and whose restriction maps

$$\mathcal{O}(C;D) \to \mathcal{O}(\mathrm{Res}_U^V C; \mathrm{Res}_U^V D)$$

are all equivalences.

**Example 32.** The above description yields a natural equivalence $\mathrm{Infl}_e^{\mathcal{T}}(\mathrm{triv}(\mathcal{C})^\otimes) \simeq \mathrm{triv}(\mathrm{Infl}_e^{\mathcal{T}}\mathcal{C})^\otimes$. ◁

**Example 33.** The $\mathcal{T}$-operads $\mathbb{E}_0^\otimes := \mathcal{N}_{I_{0,\mathcal{T}}}^\otimes$ and $\mathbb{E}_\infty^\otimes := \mathcal{N}_{I_\infty}^\otimes$ are inflated from operads of the same names; in particular, unwinding definitions, we may identify $\mathbb{E}_0$-algebras by the formula

$$\underline{\mathrm{Alg}}_{\mathbb{E}_0}(\mathcal{C})_V \simeq \mathcal{C}_{V,1_V/}.$$

If $1_V$ is terminal for all $V \in \mathcal{T}$, then this is the $\mathcal{T}$-category of pointed objects $\mathcal{C}_*$. ◁

1.3.6. *Unital $I$-operads.* Assume that $I$ is an almost unital weak indexing category. In [Ste25b] we introduced the following gamut of definitions, each of which will be useful.

**Definition 34.** We say that an $I$-operad $\mathcal{O}^\otimes$

- is *almost unital* if it has at least one color and whenever there exists some $S \in \mathbb{F}_V$ such that $\mathcal{O}(S) \neq \varnothing$, we have $\mathcal{O}(\varnothing_V) \simeq *$,
- is *unital* if it has at least one color and $\mathcal{O}(\varnothing_V) \simeq \mathcal{N}_{I\infty}(\varnothing_V)$ for all $V \in \mathcal{T}$, and
- is *almost reduced* if it is almost unital and has one color, and
- is *reduced* if it is unital and has one color. ◁

A $\mathcal{T}$-operad is almost unital if and only if it's a unital $I$-operad for *some* almost-unital weak indexing category $I$. For this reason, we'll usually focus on either unital $I$-operads or almost-unital $\mathcal{T}$-operads. It will be important to keep the $I$-symmetric monoidal case in mind.

**Example 35.** We verified in [Ste25b] that an $I$-symmetric monoidal $\infty$-category $\mathcal{C}^\otimes$ is a unital $I$-operad if and only if, for all $V \in v(I)$, the unit object $1_V \in \mathcal{C}_V$ is initial. ◁

Write $\mathbb{E}_{0,v(I)}^\otimes := \mathcal{N}_{I_{0,v(I)}}^\otimes$. We will largely use the following result of [Ste25b] to access unital $I$-operads.

**Proposition 36** ([Ste25b])**.** *The full subcategory $\mathrm{Op}_I^{\mathrm{uni}} \subset \mathrm{Op}_I$ of unital $I$-operads is both a localizing and colocalizing subcategory, i.e. the inclusion participates in a double adjunction*

$$\begin{tikzcd} \mathrm{Op}_I \arrow[r, bend left, "(-)\overset{\mathrm{BV}}{\otimes}\mathbb{E}_{0,v(I)}^\otimes"] \arrow[r, bend right, "\underline{\mathrm{Alg}}_{\mathbb{E}_{0,v(I)}}^\otimes(-)"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathrm{Op}_I^{\mathrm{uni}}. \arrow[l, hook'] \end{tikzcd}$$

*In particular, if $\mathcal{O}^{\otimes}$ and $\mathcal{C}^{\otimes}$ are unital, then there are natural equivalences*

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) \simeq \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}\otimes\mathbb{E}_{0,v(I)}}(\mathcal{C});$$
$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{D}) \simeq \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{0,v(I)}}(\mathcal{D}).$$

We accomplished this in part by recognizing an equality of full subcategories $\mathrm{Op}_I^{\mathrm{uni}} = \mathrm{Op}_I^{I_{0,v(I)}-\mathrm{Wirth}}$; that is, an $I$-operad is unital if and only if its $I$-symmetric monoidal $\infty$-categories of algebras have $V$-units which are initial for each $V \in v(I)$, which is true if and only if they are unital by Example 35. Moreover, since the $\overset{\mathrm{BV}}{\otimes}$-unit $\mathrm{triv}_{\mathcal{T}}^{\otimes}$ is initial among one color $I$-operads, this yields the following easy corollary.

**Corollary 37.** $\mathbb{E}^{\otimes}_{0,v(I)}$ *is initial among reduced $I$-operads.*

1.3.7. *Cartesian and cocartesian $I$-symmetric monoidal $\infty$-categories.* In [Ste25b], given $\mathcal{C}$ a $\mathcal{T}$-$\infty$-category with $I$-indexed (co)products, we defined *cocartesian* and *cartesian* $I$-symmetric monoidal $\infty$-categories $\mathcal{C}^{I-\sqcup}$ and $\mathcal{C}^{I-\times}$, which are determined by the properties that their $I$-indexed tensor products are canonically equivalent to indexed (co)products. We gave algebras in cartesian $I$-symmetric monoidal $\infty$-categories an explicit presentation generalizing the $\mathcal{O}$*-monoids* of [HA] (as $\mathcal{T}$-functors satisfying "Segal conditions") which we will not mention explicitly here; as a relic of this, we will simply use the notation

$$\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{D}) := \underline{\mathrm{Alg}}_{\mathcal{O}}\left(\mathcal{D}^{I-\times}\right); \qquad \mathrm{Mon}_{\mathcal{O}}(\mathcal{D}) := \mathrm{Alg}_{\mathcal{O}}\left(\mathcal{D}^{I-\times}\right). \tag{5}$$

The associated $I$-symmetric monoidal structure is cartesian [Ste25b]. When $\mathcal{C}$ is an $\infty$-category, we will write

$$\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}) := \underline{\mathrm{Mon}}_{\mathcal{O}}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}\right); \qquad \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) := \mathrm{Mon}_{\mathcal{O}}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}\right). \tag{6}$$

instead we will use their monadic presentation, which goes as follows.

**Proposition 38** ([Ste25a])**.** *Suppose $\mathcal{C}$ is a presentable and cartesian closed $\infty$-category. Then, the monad $T_{\mathcal{O}}$ associated with the monadic functor $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ has fixed points*

$$(T_{\mathcal{O}}X)^W \simeq \coprod_{S \in \mathbb{F}_{I,W}} \left(\mathrm{Fr}_{\mathcal{C}}\,\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S)} X^U\right)_{h\mathrm{Aut}_W(S)},$$

*where $\mathrm{Fr}_{\mathcal{C}} \colon \mathcal{S} \to \mathcal{C}$ is the unique left adjoint sending $*$ to the terminal object of $\mathcal{C}$.*

Moreover, in the case that $\mathcal{O}^{\otimes}$ is unital, we characterized cocartesian algebras simply as diagrams

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}\left(\mathcal{C}^{I-\sqcup}\right) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{O},\mathcal{C})^{I-\sqcup}; \tag{7}$$

in fact, $\mathcal{C}^{I-\sqcup}$ still exists as an $I$-operad with the above algebras in when $\mathcal{C}$ is not assumed to have $I$-indexed coproducts. In particular, in the unital case, we acquire a double adjunction

$$\begin{tikzcd} \mathrm{Cat}_{\mathcal{T}} \arrow[r, bend left, "\mathrm{triv}(-)^{\otimes}\overset{\mathrm{BV}}{\otimes}\mathbb{E}_{0,v(I)}"] \arrow[r, bend right, "(-)^{I-\sqcup}"'] & \mathrm{Op}_I^{\mathrm{uni}}. \arrow[l, "U"'] \end{tikzcd} \tag{8}$$

**Example 39.** In [Ste25b] we gave a general formula for $\mathcal{C}^{I-\sqcup}$, but the mapping-in property makes it easy enough to determine this in the case that $\mathcal{C}$: there is an equivalence

$$\mathrm{Alg}_{\mathcal{O}}\left(*_{\mathcal{T}}^{I-\sqcup}\right) \simeq * \simeq \mathrm{Alg}_{\mathcal{O}}\left(\mathcal{N}^{\otimes}_{I\infty}\right),$$

natural in the unital $I$-operad $\mathcal{O}^{\otimes}$, constructing an equivalence $\mathcal{N}^{\otimes}_{I\infty} \simeq *_{\mathcal{T}}^{I-\sqcup}$ by Yoneda's lemma. ◁

**Example 40** ([Ste25b])**.** Given $\mathcal{C}^{\otimes}$ a $\mathcal{T}$-operad, $\mathrm{Bor}_I^{\mathcal{T}}\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{C})$ is a cocartesian $I$-operad. ◁

1.3.8. *$I$-$d$-operads.* In [Ste25a], we defined the full subcategory $\mathrm{Op}_{\mathcal{T},d} \subset \mathrm{Op}_{\mathcal{T}}$ of *$\mathcal{T}$-$d$-operads* to be those such that $\mathcal{O}(S)$ is a $(d-1)$-truncated space for all $S \in \underline{\mathbb{F}}_{A\mathcal{O}}$, and verified the following.

**Proposition 41** ([Ste25a])**.** *Fix $d \geq -1$ and $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$.*

*(1) The inclusion $\mathrm{Op}_{\mathcal{T},d} \subset \mathrm{Op}_{\mathcal{T}}$ has a left adjoint $h_d \colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T},d}$ satisfying*

$$h_d\mathcal{O}(S) \simeq \tau_{\leq d-1}\mathcal{O}(S).$$

*(2) The unit of the $h_0$-localization adjunction is the map $\mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{A\mathcal{O}}$; in particular, $\mathcal{N}^{\otimes}_{(-)\infty}$ factors through an equivalence*

$$\mathrm{wIndexCat}_{\mathcal{T}} \simeq \mathrm{Op}_{\mathcal{T},0}.$$

*(3) When $\mathcal{P}^{\otimes}$ is a $\mathcal{T}$-$d$-operad, there is a natural equivalence*

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) \simeq \underline{\mathrm{Alg}}^{\otimes}_{h_d\mathcal{O}}(\mathcal{P}),$$

*and each are $\mathcal{T}$-$d$-operads.*

*(4) An $I$-symmetric monoidal $\infty$-category $\mathcal{C}^{\otimes}$ is a $\mathcal{T}$-$d$-operad if and only if $U\mathcal{C}$ is a $\mathcal{T}$-$d$-category.*

We call $h_d\mathcal{O}^{\otimes}$ the *homotopy $d$-operad of $\mathcal{O}^{\otimes}$.*

1.3.9. *$\mathcal{O}$-algebras in $I$-symmetric monoidal 1-categories.* Fix $\mathcal{C}^{\otimes}$ an $I$-symmetric monoidal 1-category; in light of Proposition 41, to characterize $\mathcal{O}$-algebras in $\mathcal{C}^{\otimes}$, we may equivalently characterise $h_1\mathcal{O}$-algebras in $\mathcal{C}$, so assume $\mathcal{O}^{\otimes}$ is an $I$-1-operad, i.e. its structure spaces are sets.

We gave a simple combinatorial model for $I$-1-operads in [Ste25a], which we will not relitigate here, instead focusing only on algebras. Given a $\mathcal{T}$-object $X \in \Gamma^{\mathcal{T}}\mathcal{C}$, we defined the *unreduced endomorphism $I$-operad* $\mathrm{End}_X(\mathcal{C})$ as a one-colored $I$-1-operad with structure sets

$$\mathrm{End}_X(\mathcal{C})(S) \simeq \mathrm{Hom}_{\mathcal{C}_V}\left(X^{\otimes S}_{\underline{V}}, X_{\underline{V}}\right),$$

where $X_{\underline{V}} \in \mathcal{C}_V$ is the $V$-object underlying $X$. 1-categorical algebras take a familiar form.

**Proposition 42** ([Ste25a])**.** *Given $\mathcal{O}^{\otimes} \in \mathrm{Op}^{\mathrm{oc}}_{I,1}$, $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is a 1-category whose objects are pairs $(X \in \Gamma^{\mathcal{T}}\mathcal{C}, \varphi \colon \mathcal{O} \to \mathrm{End}_X(\mathcal{C}))$ and whose morphisms are $\Gamma^{\mathcal{T}}\mathcal{C}$-maps $f \colon X \to Y$ such that the corresponding diagram commutes*

$$\begin{tikzcd} & \mathrm{End}_X(\mathcal{C}) \arrow[d, "\mathrm{End}_f"] \\ \mathcal{O}^{\otimes} \arrow[ur] \arrow[r] & \mathrm{End}_Y(\mathcal{C}) \end{tikzcd}$$

Moreover, we may exploit this to explicitly describe interchange.

**Corollary 43** ([Ste25a])**.** *Given $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes} \in \mathrm{Op}^{\mathrm{oc}}_{I,1}$, an $\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{P}$-algebra structure on $X$ is precisely a pair of $\mathcal{O}$-algebra and $\mathcal{P}$-algebra structures such that, for all $\mu \in \mathcal{O}(S)$, the corresponding $\mathcal{C}$-map $X^{\otimes S}_{\underline{V}} \to X_{\underline{V}}$ is a morphism of $\mathcal{P}$-algebras; a morphism of $\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{P}$-algebras is a $\Gamma^{\mathcal{T}}\mathcal{C}$-map which is separately an $\mathcal{O}$-algebra and $\mathcal{P}$-algebra morphism.*

1.3.10. *The doctrinal adjunction.* The following result of [Ste25a] will be useful crucial.

**Proposition 44** (Doctrinal adjunction)**.** *Suppose $L^{\otimes} \colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is an $I$-symmetric monoidal functor whose underlying $\mathcal{T}$-functor $L$ admits a right adjoint $R$. Then, $R$ lifts to a canonical lax $I$-symmetric monoidal right adjoint $R^{\otimes} \vdash L^{\otimes}$. Moreover, for any $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ the postcomposition lax $I$-symmetric monoidal functors partake in a lax $I$-symmetric monoidal adjunction*

$$L^{\otimes}_* \colon \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \rightleftarrows \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{D}) \colon R^{\otimes}_*$$

*such that $L^{\otimes_*}$ is $I$-symmetric monoidal. If $R^{\otimes}$ is symmetric monoidal then $R^{\otimes}_*$ is symmetric monoidal; if $R$ is also fully faithful, then $R^{\otimes}_*$ is fully faithful.*

# 2. $I$-operads

In this section, we establish the $I$-operadic results necessary to prove Theorems B to D. In particular, in Section 2.1 we prove the universal property for algebras *out of* cocartesian $I$-operads, showing compatibility between cocartesian structures and the formation of homotopy $n$-operads. Using this, in Section 2.2 we give the necessary recognition results on $h_n$-equivalences and $h_n$-cocartesian $I$-operads for the rest of the paper. Last, in Section 2.3 we characterize the reduced endomorphism $I$-operad, ultimately forming the main content of the reduction of Theorems B and C to Theorem D.

## 2.1. The mapping-out property for cocartesian structures.

**Proposition 45.** *Given $\mathcal{C} \in \mathrm{Cat}_{\mathcal{T}}$, $\mathcal{C}^{I-\sqcup}$ is identified by the mapping-out property*

$$\mathrm{Alg}_{\mathcal{C}^{I-\sqcup}}(\mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{CAlg}}_I(\mathcal{D})\right);$$

*in particular, we acquire a triple adjunction*

$$\begin{tikzcd} \mathrm{Cat}_{\mathcal{T}} \arrow[rr, bend left=50, "\mathrm{triv}(-)^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}^{\otimes}_{0,v(I)}"] \arrow[rr, shift right=1.5, "(-)^{I-\sqcup}"'] & & \mathrm{Op}_I^{\mathrm{uni}}. \arrow[ll, shift right=1.5, "U"'] \arrow[ll, bend left=50, "\underline{\mathrm{CAlg}}_I^{\otimes}(-)"] \end{tikzcd} \tag{9}$$

*Moreover, $\mathrm{triv}(-)^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{0,v(I)}$ and $(-)^{I-\sqcup}$ are fully faithful.*

*Proof.* For the first statement, simply apply the equivalences

$$\begin{aligned} \mathrm{Alg}_{\mathcal{C}^{I-\sqcup}}(\mathcal{D}) &\simeq \mathrm{Alg}_{\mathcal{C}^{I-\sqcup} \otimes \mathcal{N}_{I\infty}}(\mathcal{D}) && n = \infty \text{ case,} \\ &\simeq \mathrm{Alg}_{\mathcal{C}^{I-\sqcup}} \underline{\mathrm{CAlg}}_I^{I-\sqcup}(\mathcal{D}) && n = \infty \text{ case and Eq. (3)} \\ &\simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{CAlg}}_I(\mathcal{D})\right) && \text{Eq. (7).} \end{aligned}$$

and Yoneda's lemma under the equivalence $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})^{\simeq} \simeq \mathrm{Map}_{\mathrm{Op}_{\mathcal{T}}}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes})$. The bottom adjunction follows by taking cores, and the remaining adjunctions by Eq. (8). Fully faithfulness for the former follows from the latter, which is itself follows by combining the mapping out property with Eq. (7) and taking cores. □

**Remark 46.** The case $I = \mathcal{T}$ is proved in [Yan25, Lem 4.1.10], but it is used as *input to* rather than a corollary of computations of cocartesian algebras, so their techniques are more difficult. ◁

**Remark 47.** Proposition 45 is the restriction of the *defining* property of $\underline{\mathrm{CAlg}}_{\mathcal{T}}^I(\mathcal{C})$ in [LLP25], who left implicit a comparison with the atomic orbital setting; Proposition 45 together with the identification of the two notions of cocartesian structure gives a slick identification of the two [Ste25b]. ◁

We easily acquire compatibility of $h_n$ with cocartesian structures.

**Corollary 48.** *Given $\mathcal{C}$ a $\mathcal{T}$-category, there exists an equivalence $h_n\left(\mathcal{C}^{I-\sqcup}\right) \simeq (h_n\mathcal{C})^{I-\sqcup}$.*

*Proof.* Proposition 41 constructs a commutative diagram

$$\begin{tikzcd} \mathrm{Op}_{\mathcal{T},d} \arrow[r, "\underline{\mathrm{CAlg}}_I(-)"] \arrow[d, hook] & \mathrm{Cat}_{\mathcal{T},d} \arrow[d, hook] \\ \mathrm{Op}_{\mathcal{T}} \arrow[r, "\underline{\mathrm{CAlg}}_I(-)"] & \mathrm{Cat}_{\mathcal{T}} \end{tikzcd}$$

The result follows by taking left adjoints. □

The following proposition is not necessary for the present work, but it is nevertheless enlightening.

**Proposition 49** ([Ste25b])**.** *$(U-)^{I-\sqcup}$ is an $\overset{\mathrm{BV}}{\otimes}$-smashing localization associated with $\mathcal{N}_{I\infty}^{\otimes}$, and the associated symmetric monoidal structure on $\mathrm{Cat}_{\mathcal{T}}$ is cartesian. Moreover, the top two three functors of Eq. (9) lift to a nonunital symmetric monoidal double adjunction whose leftmost two functors lift to symmetric monoidal adjoints.*

**Remark 50.** $(-)^{I-\sqcup}$ is easily seen to seldom be compatible with units as $\eta\colon \mathbb{E}_{0,\upsilon(I)}^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$ does not admit a retract if there exists some $S \in \mathbb{F}_{I,V} - \{\varnothing_V, *_V\}$; under the same circumstances,

$$\Gamma^{\mathcal{T}} \underline{\mathrm{CAlg}}_I(\mathbb{E}_{0,\upsilon(I)}^{\otimes}) \simeq \varnothing \overset{\nexists}{\dashleftarrow} *$$

so the bottom functor is also not unital. $\underline{\mathrm{CAlg}}_I$ often does not even admit a nonunital oplax symmetric monoidal structure; indeed, choosing $J_1, J_2 \subsetneq I$ such that $I = J_1 \vee J_2$, we have

$$\begin{tikzcd} \Gamma^{\mathcal{T}}\left(\underline{\mathrm{CAlg}}_I\left(\mathcal{N}_{J_1\infty}\right) \times \underline{\mathrm{CAlg}}_I\left(\mathcal{N}_{J_2\infty}\right)\right) \arrow[d, phantom, "\wr"] & \Gamma^{\mathcal{T}}\underline{\mathrm{CAlg}}_I(\mathcal{N}_{J_1\infty} \otimes \mathcal{N}_{J_2\infty}) \arrow[l, dotted, "\nexists"'] \arrow[d, phantom, "\wr"] \\ \varnothing & * \arrow[l, dotted, "\nexists"'] \simeq \Gamma^{\mathcal{T}}\underline{\mathrm{CAlg}}_I(\mathcal{N}_{I\infty}) \end{tikzcd}$$

◁

### 2.2. **Recognizing $I$-local $h_n$-equivalences.**

2.2.1. *Detection via algebras.* Theorem D recognizes morphisms of $\mathcal{T}$-operads which become equivalences after applying $h_{n+1}$, so we now spell out some of its antecedents.

**Proposition 51.** *Let $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ be a morphism of $\mathcal{T}$-operads. The following are equivalent:*

*(a) for all $S \in \underline{\mathbb{F}}_{A\mathcal{O}} \cup \underline{\mathbb{F}}_{A\mathcal{P}}$, the map of spaces*

$$\varphi(S)\colon \mathcal{O}(S) \to \mathcal{P}(S)$$

*is an $n$-equivalence;*

*(b) $\varphi$ is an $h_{n+1}$-equivalence;*

*(c) for all $\mathcal{T}$-symmetric monoidal $(n+1)$-categories $\mathcal{C}$, the pullback $\mathcal{T}$-symmetric monoidal functor*

$$\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$$

*is an equivalence;*

*(d) the pullback functor*

$$\mathrm{Mon}_{\mathcal{P}}(\mathcal{S}_{\leq n}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{S}_{\leq n})$$

*is an equivalence; and*

*(e) for all $\infty$-categories $K$, the pullback map of spaces*

$$\mathrm{Mon}_{\mathcal{P}}\left(\mathcal{S}_{\leq n}^K\right)^{\simeq} \to \mathrm{Mon}_{\mathcal{O}}\left(\mathcal{S}_{\leq n}^K\right)^{\simeq}$$

*is an equivalence.*

To prove this, we apply the following lemma.

**Lemma 52.** *Given a $\mathcal{T}$-operad $\mathcal{P}^{\otimes}$ and a pair of $\infty$-categories $\mathcal{D}, K$ such that $\mathcal{D}$ admits finite products, there is an equivalence*

$$\underline{\mathrm{Mon}}_{\mathcal{P}}\left(\mathcal{D}^K\right) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}} K, \underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{D})\right),$$

*natural in functors of $K$, product-preserving functors of $\mathcal{D}$, and $\mathcal{T}$-operad maps of $\mathcal{P}$; in particular, taking $\mathcal{T}$-fixed points yields a natural equivalence of categories*

$$\mathrm{Mon}_{\mathcal{P}}\left(\mathcal{D}^K\right) \simeq \mathrm{Mon}_{\mathcal{P}}(\mathcal{D})^K.$$

*Proof.* We construct a chain of equivalences

$$\begin{aligned}
\underline{\mathrm{Mon}}_{\mathcal{P}}\left(\mathcal{D}^K\right) &\simeq \underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{D}^K)^{\mathcal{T}-\times}) && \text{Eqs. (5) and (6)}\\
&\simeq \underline{\mathrm{Alg}}_{\mathcal{P}}\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}} K, \underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}\right)^{\mathcal{T}-\times} && \text{Example 24}\\
&\simeq \underline{\mathrm{Alg}}_{\mathcal{P}}\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}(\mathrm{Infl}_e^{\mathcal{T}} K)}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}^{\mathcal{T}-\times}\right) && \text{Eq. (2)}\\
&\simeq \underline{\mathrm{Alg}}_{\mathcal{P}}\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{Infl}_e^{\mathcal{T}}\mathrm{triv}(K)}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}^{\mathcal{T}-\times}\right) && \text{Example 32}\\
&\simeq \underline{\mathrm{Alg}}_{\mathrm{Infl}_e^{\mathcal{T}}\mathrm{triv}(K)}\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}^{\mathcal{T}-\times}\right) && \text{Eq. (3)}\\
&\simeq \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}} K, \underline{\mathrm{Alg}}_{\mathcal{P}}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}, \mathcal{D}^{\mathcal{T}-\times}\right)\right) && \text{Eq. (4)}\\
&\simeq \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}} K, \underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{D})\right) && \text{Eqs. (5) and (6)}
\end{aligned}$$

The remaining equivalence follows by noting that $\Gamma^{\mathcal{T}}\mathrm{Infl}_e^{\mathcal{T}}\mathcal{C} \simeq \mathcal{C}$, naturally in $\mathcal{C}$. □

*Proof of Proposition 51.* A generalization of the equivalence between Conditions (a) to (d) was proved in [Ste25a], and Condition (c) clearly implies Condition (e). Moreover, fixing $\mathcal{D} = \mathcal{S}_{\leq n}$ and taking cores of Lemma 52 yields a natural equivalence

$$\mathrm{Mon}_{\mathcal{P}}\left(\mathcal{S}_{\leq n}^K\right)^{\simeq} \simeq \mathrm{Map}_{\mathrm{Cat}}(K, \mathrm{Mon}_{\mathcal{P}}(\mathcal{S}_{\leq n}))$$

so Condition (e) and Yoneda's lemma together imply Condition (d). □

2.2.2. *The smashing localization on $\mathcal{T}$-$n$-operads associated with $\mathcal{N}_{I\infty}^{\otimes}$.* Note the following.

**Proposition 53.** *If $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is an $h_n$-equivalence and $\mathcal{Q}^{\otimes}$ is a $\mathcal{T}$-operad, then the induced map*

$$\mathcal{Q}^{\otimes} \overset{\mathrm{BV}}{\otimes} \varphi\colon \mathcal{Q}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes} \longrightarrow \mathcal{Q}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$$

*is an $h_n$-equivalence.*

*Proof.* By Proposition 51, pullback along $\varphi \otimes \mathcal{Q}^{\otimes}$ yields an equivalence

$$\begin{array}{ccc}
\mathrm{Mon}_{\mathcal{Q}}\underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{S}_{\leq n}) & \longrightarrow & \mathrm{Mon}_{\mathcal{Q}}\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{S}_{\leq n})\\
\wr\! \| & & \wr\! \|\\
\mathrm{Mon}_{\mathcal{Q}\otimes\mathcal{P}}(\mathcal{S}_{\leq n}) & \longrightarrow & \mathrm{Mon}_{\mathcal{Q}\otimes\mathcal{O}}(\mathcal{S}_{\leq n})
\end{array}$$

Applying Proposition 51 once more shows that $\varphi \otimes \mathcal{Q}^{\otimes}$ is an $h_n$-equivalence. □

In particular, Proposition 53 and [HA, Prop 2.2.1.8] construct a symmetric monoidal structure on $\mathrm{Op}_{\mathcal{T},n}$ together with a symmetric monoidal structure on $h_n$. The tensor product for this structure is $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes}_n \mathcal{P}^{\otimes} \simeq h_n \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$, and in particular, Proposition 49 shows that $\mathcal{N}_{I\infty}^{\otimes} \in \mathrm{Op}_{\mathcal{T},n}$ is an idempotent algebra. It's easy to identify its smashing localization, and in fact, its $h_n$-preimages.

**Corollary 54.** *Suppose $\mathcal{O}^{\otimes}$ is an almost-unital $\mathcal{T}$-operad. Then, the following conditions are equivalent:*

- *(b') The map $\mathrm{Bor}_I^{\mathcal{T}}\mathcal{O}^{\otimes} \to (h_n U\mathcal{O})^{I-\sqcup}$ is an $h_n$-equivalence.*
- *(f') For all $\mathcal{T}$-$(n+1)$-operads $\mathcal{P}^{\otimes}$, the $\mathcal{T}$-operad $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P})$ is cocartesian at $I$.*
- *(g') For all $A\mathcal{O}$-symmetric monoidal $(n+1)$-categories $\mathcal{C}^{\otimes}$, the $A\mathcal{O}$-symmetric monoidal $(n+1)$-category $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ is cocartesian at $I$.*
- *(h') The $\mathcal{T}$-$(n+1)$-category $\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{S}_{\leq n})$ is $I$-semiadditive.*
- *(j') The unit map tensors to an $h_n$-equivalence*

$$h_n(\mathrm{id}\otimes !)\colon h_n\mathcal{O}^{\otimes} \simeq h_n\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{\mathcal{T}}^{\otimes}\right) \xrightarrow{\ \sim\ } h_n\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}\right).$$

*Proof.* We will use that the $n = \infty$ case was proved in [Ste25b]. The implications Condition (b') $\implies$ Condition (f') $\implies$ Condition (g') $\implies$ Condition (h') $\implies$ Condition (j') were covered in [Ste25b]. Moreover, Corollary 48 and the $n = \infty$-case together show that Condition (j') implies Condition (b'), as we have

$$h_n\mathrm{Bor}_I^{\mathcal{T}}\left(\mathcal{O}^{\otimes}\right) \simeq h_n\mathrm{Bor}_I^{\mathcal{T}}\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}\right) \simeq h_n(U\mathcal{O})^{I-\sqcup} \simeq (h_n U\mathcal{O})^{I-\sqcup}.$$ □

2.3. **The reduced endomorphism $I$-operad as a right adjoint.** In [Ste25b], we introduced the *reduced endomorphism $I$-operad* of a $\mathcal{T}$-operad for the purpose of lifting the disintegration and assembly process of [HA]. In this section, we gain explicit computational control over reduced endomorphism $I$-operads of unital $I$-symmetric monoidal $\infty$-categories.

**Proposition 55.** *The inclusion $\mathrm{Op}_I^{\mathrm{red}} \simeq \mathrm{Op}^{\mathrm{red}}_{I,\mathbb{E}_{0,v(I)}/} \hookrightarrow \mathrm{Op}^{\mathrm{uni}}_{I,\mathbb{E}_{0,v(I)}/}$ has a right adjoint computed by the pullback*

$$\begin{tikzcd} \mathrm{End}_X^{I,\mathrm{red}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes} \arrow[d, "\eta"] \\ \mathcal{N}^{\otimes}_{I\infty} \arrow[r, "\{X\}"'] & \mathcal{O}^{\mathcal{T}-\sqcup} \end{tikzcd} \tag{10}$$

*In the case that $\mathcal{C}^{\otimes}$ is a unital $I$-symmetric monoidal $\infty$-category and $X \in \mathcal{C}_V$ is a $V$-object, mapping in from the free unital $I$-operad $\mathrm{Fr}_{\mathrm{Op}}(S) \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{0,v(I)}$ on an operation in arity $S \in \mathbb{F}_{I,V}$ yields a pullback*

$$\begin{tikzcd} \mathrm{End}_X^{I,\mathrm{red}}(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_{\mathcal{C}_V}\left(X^{\otimes S}, X\right) \arrow[d, "W^*_{S,X}"] \\ \{\nabla\} \arrow[r] & \mathrm{Map}_{\mathcal{C}_V}\left(X^{\sqcup S}, X\right) \end{tikzcd}$$

*i.e. $\mathrm{End}_X^{I,\mathrm{red}}(S)$ is equivalent to the space of lifts along the following dashed arrow in $\mathcal{C}_V$*

$$\begin{tikzcd} X^{\sqcup S} \arrow[r, "\nabla"] \arrow[d, "W_{S,X}"'] & X \arrow[d, "!"] \\ X^{\otimes S} \arrow[r, "!"'] \arrow[ur, dashed] & * \end{tikzcd}$$

*Proof.* We will apply the general reduction procedure of [SY19, Prop 2.1.5], applied to the *sliced* adjunction

$$U_* \colon \mathrm{Op}^{\mathrm{uni}}_{I,\mathbb{E}^{\otimes}_{0,v(I)}/} \rightleftarrows \mathrm{Cat}_{\mathcal{T},*} \colon \eta^*(-)^{I-\sqcup},$$

whose right adjoint is $(-)^{I-\sqcup}$ together with the precomposed structure map

$$\mathbb{E}^{\otimes}_{0,v(I)} \xrightarrow{\eta} \mathcal{N}^{\otimes}_{I\infty} \simeq *_{\mathcal{T}}^{I-\sqcup} \to \mathcal{C}^{I-\sqcup}.$$

Indeed, $\mathrm{Cat}_{\mathcal{T},*}$ admits an initial object $*_{\mathcal{T}} \simeq U\mathbb{E}_{0,v(I)}$, and $\mathrm{Op}^{\otimes}_{I,\mathbb{E}^{\otimes}_{0,v(I)}/}$ admits all limits, which are preserved by $U$ since it is a right adjoint by Eq. (8). Moreover, $\mathbb{E}_{0,v(I)} \in \mathrm{Op}_I^{\mathrm{red}}$ is initial by Corollary 37, there is a unique equivalence $\mathcal{N}^{\otimes}_{I\infty} \simeq *_{\mathcal{T}}^{I-\sqcup}$ by Eq. (1) and Example 39, and $\mathcal{O}^{\otimes} \in \mathrm{Op}^{\mathrm{uni}}_{I,\mathbb{E}_{0,v(I)}/}$ corresponds with a reduced $I$-operad if and only if $U\mathcal{O}^{\otimes} \in \mathrm{Cat}_{\mathcal{T},*}$ is initial, so the first claim follows by [SY19, Prop 2.1.5].

To acquire the second pullback square, one need only note that the natural equivalences

$$\mathrm{Map}_{\mathrm{Op}_{\mathcal{T}}}\left(\mathrm{Fr}_{\mathrm{Op}}(S) \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{0,v(I)}, \mathcal{C}^{\otimes}\right) \simeq \mathrm{Map}_{\mathcal{C}_V}\left(X^{\otimes S}, X\right),$$
$$\mathrm{Map}_{\mathrm{Op}_{\mathcal{T}}}\left(\mathrm{Fr}_{\mathrm{Op}}(S) \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{0,v(I)}, \mathcal{N}^{\otimes}_{I\infty}\right) \simeq *$$

follow by Propositions 30 and 36. What remains is to verify that the right vertical arrow is $W^*_{S,X}$ and the bottom arrow includes the fold map $\nabla$; both facts were verified in [Ste25b]. □

In fact, [SY19, Prop 4.2.8] introduced a result on connectivity of such spaces of lifts, immediately yielding the following corollary.

**Corollary 56.** *If $X \in \mathcal{C}_V$ is a $(k+\ell+2)$-truncated object and the Wirthmüller map $W_{S,X} \colon X^{\sqcup S} \to X^{\otimes S}$ is $\ell$-connected, then the space $\mathrm{End}_X^{I,\mathrm{red}}(\mathcal{C})(S)$ is $k$-truncated.*

In general, reduction is an incarnation of the *disintegration and assembly* procedure of [HA; Ste25b]; given a reduced $I$-operad $\mathcal{P}^\otimes$ and a $V$-object $X \in \mathcal{O}_V$, applying $\mathcal{P}$-algebras to Eq. (10) yields a pullback

$$\begin{tikzcd} \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{P}} \mathrm{End}^{I,\mathrm{red}}_X(\mathcal{O}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{O})_V \arrow[d, "U"] \\ \{X\} \arrow[r, hook] & U\mathcal{O}_V \end{tikzcd} \tag{11}$$

In the case that $U\mathcal{O}$ is a $\mathcal{T}$-space, $U$ is a automatically cocartesian fibration, so $\mathcal{O}$-algebras are $U\mathcal{O}$-indexed diagrams of $\mathrm{End}^{I,\mathrm{red}}_X(\mathcal{O})$-algebras. Unfortunately, this is far from our case; the best we can do is take cores of the above pullback square, resulting in the following proposition.

**Proposition 57.** *Suppose $\mathcal{P}^\otimes \to \mathcal{Q}^\otimes$ is a morphism of $I$-operads inducing an equivalence of spaces*

$$\varphi^{*,\simeq}_X \colon \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{Q}} \mathrm{End}^{I,\mathrm{red}}_X(\mathcal{O})^\simeq \longrightarrow \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{P}} \mathrm{End}^{I,\mathrm{red}}_X(\mathcal{O})^\simeq$$

*for all $V \in \mathcal{T}$ and $X \in U\mathcal{O}_V$. Then, the induced map of $\mathcal{T}$-spaces*

$$\underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{O})^\simeq \to \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{O})^\simeq$$

*is an equivalence; in particular, passing to $\mathcal{T}$-fixed points, the induced map of spaces*

$$\mathrm{Alg}_{\mathcal{Q}}(\mathcal{O})^\simeq \to \mathrm{Alg}_{\mathcal{P}}(\mathcal{O})^\simeq$$

*is an equivalence.*

*Proof.* Taking cores of Eq. (11), we find that that $\varphi^{*,\simeq}_X$ is the induced map on the homotopy fiber over $X$ of the following map of $\mathcal{T}$-spaces over $U\mathcal{O}$:

$$\begin{tikzcd} \underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{O})^\simeq \arrow[rr, "\varphi^{*,\simeq}"] \arrow[dr] & & \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{O})^\simeq \arrow[dl] \\ & U\mathcal{O} & \end{tikzcd}$$

$\varphi^{*,\simeq}$ is an equivalence if and only if its $V$-fixed points are an equivalence for all $V \in \mathcal{T}$, and the homotopy fibers of $\varphi^{*,\simeq,V}$ are contractible by the above argument, so $\varphi^{*,\simeq,V}$ is an equivalence for all $V$. Hence $\varphi^{*,\simeq}$ is an equivalence, proving the proposition. □

## 3. Connectivity and Eckmann-Hilton arguments

We now prove Theorems B to D, beginning with a recognition result for $\ell$-connected $\mathcal{O}$-monoid maps.

### 3.1. Connectivity of algebras can be detected in the value topos.

Fix $\mathcal{C}$ an $n$-topos for some $n \le \infty$.

**Lemma 58.** *A map $f \colon C \to D$ in $\mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ is $\ell$-connected if and only if, for all $V \in \mathcal{T}^{\mathrm{op}}$, the fixed point map $C^V \to D^V$ is $\ell$-connected.*

*Proof.* Per Remark 5, it is equivalent to prove that $\ell$-connectiveness of a morphism in $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C})$ is measured elementwise. Indeed, since (co)limits in $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C})$ are computed elementwise, effective epimorphisms and diagonals are as well. The former proves the statement for $(-2)$-connectiveness, and the latter together with the diagonal presentation of [HTT, Prop. 6.5.1.18] shows that the statement for $(\ell-1)$-connectiveness implies the statement for $\ell$-connectiveness, so the lemma follows by induction. □

**Proposition 59.** *Given a map $f \colon X \to Y$ in $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$, if the underlying map $Uf$ in $\mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ is $\ell$-connected, then $f$ is $\ell$-connected.*

*Proof.* In view of [SY19, Lem 4.4.1], it suffices to verify that the monad $T_{\mathcal{O}} \colon \mathrm{Coeff}^{\mathcal{T}}\mathcal{C} \to \mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ preserves $\ell$-connected morphisms; by Lemma 58, it suffices to verify that whenever each $\mathcal{C}$-diagram $X^V \to Y^V$ is $\ell$-connected, each induced map $T_{\mathcal{O}}X^W \to T_{\mathcal{O}}X^W$ is $\ell$-connected. But by Proposition 38, it suffices to note that $\ell$-connected morphisms in an $\infty$-topos are closed under cartesian products and colimits [HTT, Cor. 6.5.1.13, Prop. 5.2.8.6]. □

For instance, $U$ preserves the terminal object and is conservative, so it also reflects the property of being terminal; applying Proposition 59 in the case $Y = *$ shows that $U$ reflects $n$-connectivity of objects.

**Remark 60.** Since $U$ is a right adjoint, it preserves $n$-truncatedness and $n$-truncated objects. ◃

**Warning 61.** Proposition 59 is delicate for a few reasons.

(1) 1f $\mathcal{O}$ is not $n$-connected, then the free $\mathcal{O}$-algebra monad $T_{\mathcal{O}}\colon \mathcal{C}_V \to \mathcal{C}_V$ may itself may fail to preserve $n$-connected objects; indeed, we have $T_{\mathcal{O}} *_V \simeq \coprod_{S \in \mathbb{F}_V} \mathrm{Fr}_{\mathcal{C}} \mathcal{O}(S)_{h \mathrm{Aut}_V S}$, which is often not much more highly connected than the individual spaces $\mathcal{O}(S)_{h \mathrm{Aut}_V S}$.

(2) $U$ does not generally *preserve* $\ell$-connectivity of objects or morphisms for instance, given an $\ell \geq (k+1)$-connected space $X$, the equivalence $\Omega^k\colon \mathcal{S}_{*,\geq k+1} \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S}_{\geq 1})$ exhibits $\Omega^k$ as an $\ell$-connected $\mathbb{E}_k$-algebra such that $U\Omega^n$ is only in general $(\ell - k)$-connected.

(3) For a similar reason, $U$ does not usually reflect $\ell$-truncatedness of morphisms or objects. ◃

3.2. **The proof of Theorem D.** We now begin to reduce Theorem D to the case $n \leq \ell + 1$ with the following.

**Lemma 62.** *The truncation functor $\tau_{\leq \ell}\colon \mathcal{C} \to \tau_{\leq \ell}\mathcal{C}$ extends to a $\mathcal{T}$-functor*

$$\tau_{\mathcal{O}}\colon \underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Mon}}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C})$$

*satisfying $\tau_{\mathcal{O}} W_{S,X} = W_{S,\tau_{\mathcal{O}} X}$. Moreover, the inclusion $\iota\colon \tau_{\leq \ell}\mathcal{C} \to \mathcal{C}$ extends to a fully faithful $\mathcal{T}$-functor*

$$\iota_{\mathcal{O}}\colon \underline{\mathrm{Mon}}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C}) \hookrightarrow \underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C})$$

*such that $\tau_{\mathcal{O}} W_{S,\iota_{\mathcal{O}} X} = W_{S,X}$.*

*Proof.* Since $\tau_{\leq \ell}$ is product-preserving [HTT, Lem. 6.5.1.2], $\tau_{\leq \ell}\colon \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{C} \to \underline{\mathrm{Coeff}}^{\mathcal{T}} \tau_{\leq \ell \mathcal{C}}$ is a $\mathcal{T}$-symmetric monoidal left adjoint for the cartesian structure [Ste25b]; everything other than the equalities involving $W_{S,X}$ then follows straightforwardly from Proposition 44.

In particular, $\tau_{\mathcal{O}}$ is a $\mathcal{T}$-functor which preserves indexed products and coproducts; this implies that $\tau_{\mathcal{O}} W_{S,X} = W_{S,\tau_{\mathcal{O}} X}$. The remaining equality follows from fully faithfulness by noting that

$$\tau_{\mathcal{O}} W_{S,\iota_{\mathcal{O}} X} = W_{S,\tau_{\mathcal{O}}\iota_{\mathcal{O}} X} = W_{S,X}. \qquad \square$$

We say that a map $f\colon X \to Y$ in an $n$-topos is an *$\ell$-equivalence* if it is a $\tau_{\leq \ell}$-equivalence; if $f$ admits a section, this is equivalent to $f$ being $\ell$-connected (see [SY19, Prop. 4.3.5] or note that this follows by splitting the long exact sequence in homotopy). We apply this by equivariantizing [SY19, Lem. 5.1.1].

**Lemma 63.** *If $\mathcal{C}^{I-\times}$ is a Cartesian $I$-symmetric monoidal $\infty$-category and $S \in \underline{\mathbb{F}}_I$, then the image of the $\mathcal{O}$-algebra Wirthmüller map $W_{X,S}\colon \coprod_U^S X_U \to \prod_U^S X_U$ under $U\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_V \to \mathcal{C}_V$ admits a section.*

*Proof of Lemma 63.* Let $i_U\colon Y_U \to \mathrm{Res}_U^V \coprod_{U'}^S Y_{U'}$ be adjunct to the inclusion $\mathrm{Ind}_U^V Y_U \hookrightarrow \coprod_{U'}^S Y_{U'}$ and fix some operation $\mu \in \mathcal{O}(S)$. We verify that the following diagram commutes, giving a section $\mu\sigma_1 f$ for $W_{X,S}$.

$$\begin{tikzcd}
\prod_U^S \left(\mathrm{Res}_U^V \coprod_U^V X_U\right) \arrow[r, "\sigma_1", "\sim"'] & \left(\coprod_U^S X_U\right)^{\times S} \arrow[r, "\mu"] \arrow[d, "h=\left(W_{\mathrm{Res}_U^V X, \mathrm{Res}_U^V S}\right)_{U \in \mathrm{Orb}(S)}"] & \coprod_U^S X_U \arrow[d, "W_{X,S}"] \\
 & \left(\prod_U^S X_U\right)^{\times S} \arrow[r, "\mu"] \arrow[d, "\sigma_2"', "\sim"] & \prod_U^S X_U \arrow[d, equal] \\
\prod_U^S X_U \arrow[uu, "f=(i_U)_{U \in \mathrm{Orb}(S)}"] \arrow[r, "g=(\iota_U)_{U \in \mathrm{Orb}(S)}"] \arrow[rr, bend right, equal] & \prod_U^S X_U^{\times \mathrm{Res}_U^V S} \arrow[r, "\mu"] & \prod_U^S X_U
\end{tikzcd}$$

Note that the top right square is commutative by the fact that $W_{S,X}$ is an $\mathcal{O}$-algebra morphism and the bottom right follows by unwinding the definition of $\mu$.

Now, note that $\mu \circ g$ is the external product of a collection of endomorphisms $X_U \xrightarrow{\iota_U} X_U^{\times \mathrm{Res}_U^V S} \xrightarrow{\mu} X_U$; unwinding definitions, $\iota_U$ is the inclusion of a unit on all but one factor:

$$\begin{tikzcd}
X_U \arrow[rr, "\iota_U"] \arrow[d, equal] & & X_U^{\times \mathrm{Res}_U^V S} \arrow[r, "\mu"] \arrow[d, equal] & X_U \\
X_U \times \prod_W^{\mathrm{Res}_U^V S - \{\alpha\}} 1_W \arrow[rr, "(\mathrm{id},\eta)"] & & X_U \times \prod_W^{\mathrm{Res}_U^V S - \{\alpha\}} X_W \arrow[ur] &
\end{tikzcd}$$

in particular, $\mu \circ \iota_U$ is homotopic to the identity, so $\mu \circ g$ is homotopic to the identity, and the bottom triangle commutes.

To characterize the composite morphism of the left rectangle, we may equivalently characterize the composite map $\pi_U \sigma_2 h \sigma_1 f \colon \prod_U^S X_U \to \mathrm{CoInd}_U^V X_U^{\times \mathrm{Res}_U^V S}$; in fact, under the expression $X_U^{\times \mathrm{Res}_U^V S} \simeq \prod_W^{\mathrm{Res}_U^V S} \mathrm{Res}_W^U X_U$, it suffices to characterize the composite morphism $\prod_U^S X_U \to \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U$ and verify that it is homotopic to the relevant projection of $g$ for each $W, U$.

In particular, relevant projection of $g$ is the composite morphism

$$\prod_U^S X_U \twoheadrightarrow \mathrm{CoInd}_U^V X_U \xrightarrow{\delta_{U,W}} \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U$$

where $\delta_{U,W}$ is a Kronecker delta

$$\delta_{U,W} = \begin{cases} \mathrm{id} & U = W; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, note that the projection $\pi_U \sigma_2 h \sigma_1 \colon \prod_U^S X_U \to X_U^{\times \mathrm{Res}_U^V S}$ itself factors as

$$\prod_U^S \left( \mathrm{Res}_U^V \coprod_U^V X_U \right) \twoheadrightarrow \mathrm{CoInd}_U^V X_U \xrightarrow{\widetilde{f}_U} \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U,$$

so we're tasked with verifying that $\widetilde{f}_U$ is homotopic to $\delta_{U,W}$. Indeed, this follows by examining the following diagram:

$$\begin{tikzcd}
\prod_U^S X_U \arrow[r, "f"] \arrow[d, two heads] & \prod_U^S \left( \mathrm{Res}_U^V \coprod_U^V X_U \right) \arrow[dr, two heads] & \simeq \left( \coprod_U^S X_U \right)^{\times S} \arrow[rr, "h"] & & \left( \prod_U^S X_U \right)^{\times S} \simeq & \prod_U^S X_U^{\times \mathrm{Res}_U^V S} \arrow[d, two heads] \\
\mathrm{CoInd}_U^V X_U \arrow[rr, "\mathrm{CoInd}_U^V i_U"] \arrow[drrr, bend right, "\delta_{U,W}"] & & \mathrm{CoInd}_U^V \mathrm{Res}_U^V \coprod_U^V X_U \arrow[r, "\mathrm{CoInd}_U^V W"] & \mathrm{CoInd}_U^V \mathrm{Res}_U^V \prod_U^V X_U \arrow[d, two heads] & \simeq & X_U^{\times \mathrm{Res}_U^V S} \\
& & & \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U & &
\end{tikzcd}$$

$\square$

*Proof of Theorem D.* Assume $\mathcal{O}^{\otimes}$ is $\ell$-connected at $I$, i.e. Condition (a). We study the behavior of $W_{S,X}$ under the following diagram:

$$\begin{tikzcd}
\underline{\mathrm{Mon}}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C}) \arrow[r, "\iota_{\mathcal{O}}"] \arrow[d, "U_{\leq \ell}"] & \underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "L_{\mathcal{O}}"] \arrow[d, "U"] & \underline{\mathrm{Mon}}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C}) \arrow[d, "U_{\leq \ell}"] \\
\underline{\mathrm{Coeff}}^{\mathcal{T}} \tau_{\leq \ell}\mathcal{C} \arrow[r, "\iota"] & \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{C} \arrow[r, "L"] & \underline{\mathrm{Coeff}}^{\mathcal{T}} \tau_{\leq \ell}\mathcal{C}
\end{tikzcd}$$

In particular, by Proposition 51 and Lemma 62, $L_{\mathcal{O}} W_{S,X} = W_{S,L_{\mathcal{O}}X}$ is an equivalence, so $U_{\leq \ell} L_{\mathcal{O}} W_{S,X} = LUW_{S,X}$ is an equivalence, i.e. $UW_{S,X}$ is an $\ell$-equivalence. In turn, by Lemma 63 this implies that $UW_{S,X}$ is $\ell$-connected, so Proposition 59 implies that $W_{S,X}$ is $\ell$-connected, i.e. Condition (b).

The implication Condition (b) $\implies$ Condition (c) is immediate, so assume Condition (c), i.e. fix the case $\mathcal{C} := \mathcal{S}$ and assume that that $W_{S,X}$ is $\ell$-connected for all $X \in \mathrm{Alg}_{\mathcal{O}}\mathcal{S}$ and $S \in \underline{\mathbb{F}}_I$. We may invert the above argument: this time, we find that $UW_{S,\iota_{\mathcal{O}}Y}$ is an $\ell$-equivalence for all $Y \in \mathrm{Alg}_{\mathcal{O}}\mathcal{S}_{\leq \ell}$, so $LUW_{S,Y} = U_{\leq \ell} L_{\mathcal{O}} W_{S,\iota_{\mathcal{O}}Y} = U_{\leq \ell} W_{S,Y}$ is an equivalence. By conservativity of $U_{\leq \ell}$, this implies that $W_{S,Y}$ is an equivalence, so $\mathcal{O}^{\otimes}$ is $\ell$-connected at $I$ by Proposition 51, proving Condition (a). □

### 3.3. The proof of Theorems B and C.

**Proposition 64.** *If $\mathcal{P}^{\otimes}$ is $\ell$-connected at $I$, then for all $(k+\ell+2)$-toposes $\mathcal{C}$, the reduced endomorphism $I$-operad $\mathrm{End}_X\left(\underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{C})^{I-\times}\right)$ is an $I$-$(k+1)$-operad.*

*Proof.* Since $\mathcal{C}$ is a $(k+\ell+2)$-category, $X$ is $(k+\ell+2)$-truncated, and Theorem D implies that $W_{X,S}$ is $\ell$-connected, so the result follows from Corollary 56. □

We quickly acquire a slightly weakened version of Theorem C.

**Corollary 65.** *Suppose $\mathcal{T}$ is an atomic orbital $\infty$-category with a terminal object, $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ are unital $\mathcal{T}$-operads and $I$ is a unital weak indexing category. If $\mathcal{O}^{\otimes}$ is $k$-connected at $I$ and $\mathcal{P}^{\otimes}$ is $\ell$-connected at $I$, then $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ is $(k+\ell+2)$-connected at $I$.*

*Proof.* By Proposition 64, we know that $\mathrm{End}_X\left(\underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{C})^{\mathcal{T}-\times}\right)$ is $(k+1)$-connected at $I$; by Corollary 54 this shows that $\mathrm{Mon}_{\mathcal{O}}\mathrm{End}_X\left(\underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{C})^{\mathcal{T}-\times}\right)$ is $I$-cocartesian, so in particular, we have

$$\mathrm{CMon}_I \mathrm{Mon}_{\mathcal{O}} \mathrm{Mon}_{\mathcal{P}}(\mathcal{C})^{\simeq} \xrightarrow{\sim} \mathrm{Mon}_{\mathcal{O}} \mathrm{Mon}_{\mathcal{P}}(\mathcal{C})^{\simeq}.$$

By Proposition 51, this implies that the map

$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{\mathcal{T}}^{\otimes} \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes !} \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$$

is an $h_{k+\ell+2}$-equivalence, so Corollary 54 shows that $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ is $(k+\ell+2)$-connected at $I$. □

From this, we finish our main theorems.

*Proof of Theorems B and C.* Restriction assembles to a (tautologically symmetric monoidal) equivalence

$$\mathrm{Op}_{\mathcal{T}}^{\otimes} \simeq \lim_{V \in \mathcal{T}} \mathrm{Op}_V^{\otimes}$$

such that, given a morphism $V \to W$ in $\mathcal{T}$ and $S$ a finite $V$-set, $\mathrm{Res}_V^{\mathcal{T}} \mathcal{O}(S) \simeq \mathcal{O}(S)$. In particular, Theorems B and C may be verified after restriction to each to $V \in \mathcal{T}$, in which case the base $\infty$-category $\mathcal{T}_{/V}$ has a terminal object.

Moreover, each $\mathcal{O}(S)$ and $\mathcal{P}(S)$ are easily determined by arity support except in the case $V \in \upsilon(\mathcal{O}) = \upsilon(\mathcal{P})$, and arity support is additive in the predicted way by [Ste25b]; thus Theorems B and C may be verified after restriction to each $V \in \upsilon(\mathcal{O})$, in which case $\mathrm{Res}_V^{\mathcal{T}} \mathcal{O}^{\otimes}$ and $\mathrm{Res}_V^{\mathcal{T}} \mathcal{P}^{\otimes}$ are unital. This and Corollary 65 together yield Theorem C, and Theorem B follows by setting $I := A\mathcal{O}$. □

## 4. The $C_p$-operad $\mathbb{A}_{2,C_p}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{A}_{2,C_p}^{\otimes}$ and Theorem A

For the rest of this article, we specialize to $\mathcal{T} = \mathcal{O}_{C_p}$, where $C_p$ is the group of prime order $p$, and $\mathcal{C}$ is a 1-category. As in Proposition 30, let $\mathrm{Fr}_{\Sigma}(S)$ denote the free $C_p$-symmetric sequence on an operation in arity $S$. Now, the pointwise formula for left Kan extensions yields equivalences

$$\mathrm{Fr}_{\Sigma, p \cdot *_{C_p}}(*)(p \cdot *_e) \simeq \Sigma_p;$$
$$\mathrm{Fr}_{\Sigma, [C_p/e]}(*)(p \cdot *_e) \simeq \Sigma_p.$$

We define the $C_p$-symmetric sequence of sets $F_{2,C_p}$ as the coequalizer

$$F_{2,C_p} := \mathrm{CoEq}\left(\Sigma_p[p \cdot *_e] \rightrightarrows \left(\mathrm{Fr}_{\Sigma,[C_p/e]}(*) \sqcup \mathrm{Fr}_{\Sigma, p \cdot *_{C_p}(*)}\right)\right),$$

where $\Sigma_p[p \cdot *_e]$ is the $C_p$-symmetric sequence defined by

$$\Sigma_p[p \cdot *_e](S) := \begin{cases} \Sigma_p & S = p \cdot *_e; \\ \varnothing & \text{otherwise.} \end{cases}$$

and the two arrows are the inclusions of $\Sigma_p[p \cdot *_e$. We define the unital $C_p$-operad $\mathbb{A}^{\otimes}_{2,C_p}$ by the Boardman-Vogt tensor product

$$\mathbb{A}^{\otimes}_{2,C_p} := \mathbb{E}_0^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Fr}_{\mathrm{Op}}\left(F_{2,C_p}\right).$$

As promised, we verify that $\mathbb{A}_{2,C_p}$-monoids are the same as $C_p$-unital magmas.

**Proposition 66.** *There is an equivalence between* $\mathrm{Mon}_{\mathbb{A}_{2,C_p}}(\mathcal{C})$ *and* $C_p$*-unital magmas in* $\mathcal{C}$*.*

*Proof.* By Example 33 and Proposition 36 we have

$$\mathrm{Mon}_{\mathbb{A}_{2,C_p}}(\mathcal{C}) \simeq \mathrm{Mon}_{\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})} \underline{\mathrm{Mon}}^{\otimes}_{\mathbb{E}_0}(\mathcal{C}) \simeq \mathrm{Mon}_{\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})} \mathcal{C}_*.$$

Moreover, by Proposition 42, the data of an $\mathbb{A}_{2,C_p}$-monoid structure on $X \in \mathsf{Coeff}^{C_p}\mathcal{C}$ is equivalently viewed as a map $\eta\colon *_{C_p} \to X$ (which we identify with an element $\widetilde{X} \in \mathsf{Coeff}^{C_p}\mathcal{C}_*$) and an element of

$$\begin{aligned} \mathrm{Mon}_{\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})}(\mathrm{End}_{\widetilde{X}}(\mathcal{C}_*))^{\simeq} &\simeq \mathrm{Hom}_{\mathrm{Fun}\left(\mathrm{Tot}\underline{\Sigma}_{C_p}, \mathcal{S}\right)}\left(F_{2,C_p}, \mathrm{End}_{\widetilde{X}}(\mathcal{C}_*)\right) \\ &\simeq \mathrm{Hom}_{\mathsf{Coeff}^{C_p}\mathcal{C}_*}\left(\widetilde{X}^p, \widetilde{X}\right) \times_{\mathrm{Hom}_{\mathcal{C}_*}\left((\widetilde{X}^e)^p, \widetilde{X}^e\right)} \mathrm{Hom}_{\mathsf{Coeff}^{C_p}\mathcal{C}_*}\left(\mathrm{CoInd}_e^{C_p} \widetilde{X}^e, \widetilde{X}\right). \end{aligned}$$

We're left with interpreting this concretely: by a standard argument, $\mathrm{Hom}_{\mathsf{Coeff}^{C_p}\mathcal{C}_*}(\widetilde{X}^p, \widetilde{X})$ corresponds bijectively with the set of unital magma structures on $X$ with unit $\eta$, and this corresponds bijectively with the pairs of unital magma structures on $X^{C_p}$ and $X^e$ with unit maps $\eta^{C_p}$ and $\eta^e$ such that the restriction map is a homomorphism. Under this bijection, the forgetful map $\mathrm{Hom}_{\mathsf{Coeff}^{C_p}\mathcal{C}_*}\left(\widetilde{X}^p, \widetilde{X}\right) \to \mathrm{Hom}_{\mathcal{C}_*}\left(\left(\widetilde{X}^e\right)^p, \widetilde{X}\right)$ simply forgets the data of $X^{C_p}$ and the restriction.

Similarly, since $C_p$-coefficient coinduction is presented by the coefficient system $X^p \overset{\Delta}{\longleftarrow} X$ with permutation action, $\mathrm{Hom}_{\mathsf{Coeff}^{C_p}\mathcal{C}^*}\left(\mathrm{CoInd}_e^{C_p} \widetilde{X}^e, \widetilde{X}\right)$ corresponds bijectively with the set of unital $C_p$-equivariant transfers $t\colon X^e \to X^{C_p}$ and unital magma structures on $X^e$ with unit $\eta^e$ satisfying the condition that the following diagram commutes.

$$\begin{tikzcd} X^e \arrow[r, "t"] \arrow[d, "\Delta"] & X^{C_p} \arrow[d, "r"] \\ (X^e)^p \arrow[r, "*"] & X^e \end{tikzcd}$$

Once again, the forgetful map restricts to the unital magama structure on $\eta^e$; thus the fiber product corresponds exactly with $G$-unital magma structures on $X$ with units $\eta^e$ and $\eta^{C_p}$.

Now, what we've described is a bijective assignment of *sets* $\mathrm{Ob}\,\mathrm{Mon}_{\mathbb{A}_{2,C_p}}(\mathcal{C}) \to \mathrm{Ob}\,\mathrm{Magma}^{\mathrm{uni}}_{C_p}(\mathcal{C})$ over $\mathrm{Ob}\,\mathcal{C}$. To conclude, it suffices to prove that a $\mathsf{Coeff}^{C_p}\mathcal{C}$ morphism between a pair of $C_p$-unital magmas is a $C_p$-unital magma homomorphism if and only if it's an $\mathbb{A}_{2,C_p}$-algebra homomorphism.

To prove this, note that an $\mathbb{A}_{2,C_p}$-monoid morphism is equivalently a $\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})$-monoid morphism of pointed objects, i.e. a pair of maps $F^e\colon M^e \to N^e$ and $F^{C_p}\colon M^{C_p} \to N^{C_p}$ which are compatible with units, satisfying $F^{C_p} \circ t = t \circ F^e$ and $F^e \circ r = r \circ F^{C_p}$ together with $p$-degree additivity

$$\begin{tikzcd} \left(M^{C_p}\right)^p \arrow[r] \arrow[d] & \left(N^{C_p}\right)^p \arrow[d] \\ M^{C_p} \arrow[r] & N^{C_p} \end{tikzcd} \qquad \begin{tikzcd} (M^e)^p \arrow[r] \arrow[d] & (M^e)^p \arrow[d] \\ M^e \arrow[r] & N^e \end{tikzcd}$$

It suffices to note that a map between the pointed sets underlying unital magmas is a homomorphism if and only if it intertwines with $n$ary addition for *some* $n \geq 2$; indeed, one can simply identify binary addition with $n$-ary addition whose first $(n-2)$-factors are the unit. ☐

We now spell out the interchange relations explicitly.

**Proposition 67.** *There is an equivalence between* $\mathrm{Mon}_{\mathbb{A}_{2,C_p} \otimes \mathbb{A}_{2,C_p}}(\mathcal{C})$ *and pairs of $G$-unital magma structures* $(M, *, \bullet, t_*, t_\bullet)$ *in* $\mathcal{C}$ *satisfying the interchange relations* $1_* = 1_\bullet$ *and*

$$\begin{tikzcd} (X^p)^p \arrow[r, "(\bullet)"] \arrow[d, "(*)"'] & X^p \arrow[d, "*"] \\ X^p \arrow[r, "\bullet"] & X \end{tikzcd} \qquad \begin{tikzcd} X^{C_p} \arrow[d, "r"'] & X^e \arrow[l, "t_\bullet"'] \arrow[r, "t_*"] \arrow[d, "\Delta"'] & X^{C_p} \arrow[d, "r"] \\ X^e & (X^e)^p \arrow[l, "*"] \arrow[r, "\bullet"'] & X^e \end{tikzcd} \qquad \begin{tikzcd} (X^e)^p \arrow[r, "(t_\bullet)"] \arrow[d, "*"'] & \left(X^{C_p}\right)^p \arrow[d, "*"'] \\ X^e \arrow[r, "t_\bullet"'] & X^{C_p} \end{tikzcd} \qquad \begin{tikzcd} (X^e)^p \arrow[r, "(t_*)"] \arrow[d, "\bullet"'] & \left(X^{C_p}\right)^p \arrow[d, "\bullet"'] \\ X^e \arrow[r, "t_*"'] & X^{C_p} \end{tikzcd}$$

*Proof.* Example 33 and Proposition 36 yields an equivalence.

$$\mathrm{Mon}_{\mathbb{A}_{2,C_p}^{\otimes 2}}(\mathcal{C}) \simeq \mathrm{Mon}_{\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})^{\otimes 2}}(\mathcal{C}_*).$$

This is characterized explicitly by Corollary 43 and Proposition 66; it suffices to note that the specified interchange relations correspond precisely with the conditions that $t_\bullet$ and $\bullet$ are $C_p$-unital magma homomorphisms. ☐

We conclude the following form of Theorem A.

**Corollary 68.** *Given* $\mathcal{C}$ *a 1-category, the forgetful functor*

$$\begin{aligned} \mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_{C_p}), \mathcal{C}) \longrightarrow & \mathrm{Mon}_{\mathbb{A}_{2,C_p} \otimes \mathbb{A}_{2,C_p}}(\mathcal{C}) \\ & \simeq \left\{ \textit{Interchanging pairs of } C_p\textit{-unital magmas in } \mathcal{C} \right\} \end{aligned}$$

*is an equivalence of categories.*

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