ON TENSOR PRODUCTS OF EQUIVARIANT COMMUTATIVE OPERADS

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Abstract. We lift the Boardman-Vogt tensor product to a symmetric monoidal closed G-∞-category $\mathrm{Op}_G^{\otimes}$ of G-operads, whose underlying ∞-category recovers Nardin-Shah's $\mathcal{O}_G$-∞-*operads*. This possesses a fully faithful G-functor

$$\mathcal{N}_{(-)\infty} : \underline{\mathrm{wIndex}}_G \hookrightarrow \underline{\mathrm{Op}}_G$$

with image the *weak* $\mathcal{N}_\infty$-G-*operads* and left adjoint constructing the *arity support* weak indexing system. We precisely characterize the $\otimes$-idempotent weak $\mathcal{N}_\infty$-G-operads as those satisfying a weak unitality assumption, called *aE-unitality*. We show that the G-subcategory

$$\underline{\mathrm{wIndex}}_G^{aE-\mathrm{uni}} \hookrightarrow \underline{\mathrm{Op}}_G$$

is symmetric monoidal and combinatorially characterize its tensor products; in particular, the symmetric monoidal G-subcategory of *unital* weak $\mathcal{N}_\infty$-G-operads is cocartesian, i.e. its tensor products are joins of (unital) weak indexing systems.

Blumberg-Hill's $\mathcal{N}_\infty$-operads correspond with a join-closed sub-poset $\mathrm{Index}_G \subset \mathrm{wIndex}_G^{\mathrm{uni}}$, so we confirm a conjecture of Blumberg-Hill. In particular, for I, J unital weak indexing systems and $\mathcal{C}$ an $I \vee J$-symmetric monoidal ∞-category, we construct a canonical $I \vee J$-symmetric monoidal equivalence

$$\underline{\mathrm{CAlg}}_I^{\otimes} \underline{\mathrm{CAlg}}_J^{\otimes} \mathcal{C} \simeq \underline{\mathrm{CAlg}}_{I \vee J}^{\otimes} \mathcal{C}.$$

From this we recover derived additivity of the equivariant little disks operads in a variety of infinitary cases.

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Introduction

Background and motivation. Let $\mathcal{C}$ be a 1-category with finite products.[1] Recall that a *commutative monoid in* $\mathcal{C}$ is the data

$$A \in \mathrm{Ob}(\mathcal{C}); \qquad \text{multiplication } \mu : A \times A \to A; \qquad \text{unit } \eta : * \to A,$$

subject to the usual unitality, associativity, and commutativity assumptions; more generally, if $\mathcal{C}$ is a symmetric monoidal 1-category, a *commutative algebra in* $\mathcal{C}$ is the data of

$$R \in \mathrm{Ob}(\mathcal{C}); \qquad \text{multiplication } \mu : R \otimes R \to R; \qquad \text{unit } \eta : 1 \to R.$$

When $\mathcal{C} = \mathbf{Set}$, this recovers the traditional theory of commutative monoids, and when $\mathcal{C} = \mathbf{Ab}$ with the tensor product of Abelian groups, this recovers the traditional theory of commutative rings.

These have been the subject of a great deal of homotopy theory, in three guises:

(1) We may define the $(2,1)$-category $\mathsf{Span}(\mathbb{F})$ to have objects the finite sets, morphisms from X to Y the spans of finite sets $X \leftarrow R \rightarrow Y$, 2-cells the isomorphisms of spans

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] \arrow[dd, dashed, "\sim"] & \\ X & & Y, \\ & R' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

and composition the pullback of spans

$$\begin{tikzcd} & & R_{XZ} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", rotate=-45] & & \\ & R_{XY} \arrow[dl] \arrow[dr] & & R_{YZ} \arrow[dl] \arrow[dr] & \\ X & & Y & & Z. \end{tikzcd}$$

If $\mathcal{C}$ is an ∞-category, then we define the *∞-category of commutative monoids in* $\mathcal{C}$ as the models of the associated Lawvere theory; that is, we define the product-preserving functor category

$$\mathsf{CMon}(\mathcal{C}) := \mathsf{Fun}^{\times}(\mathsf{Span}(\mathbb{F}), \mathcal{C}),$$

noting that products in $\mathsf{Span}(\mathbb{F})$ correspond with disjoint unions of finite sets. Indeed, if $\mathcal{C}$ is a 1-category and A a commutative monoid in $\mathcal{C}$, we flesh this out with the dictionary

$$\begin{aligned} ([2] = [2] \to [1]) \quad &\longmapsto \quad \mu : X^{\times 2} \to X; \\ (\varnothing = \varnothing \to [1]) \quad &\longmapsto \quad \eta : * \simeq X^{\times 0} \to X; \\ ([1] \leftarrow [2] = [2]) \quad &\longmapsto \quad \Delta : X \to X^{\times 2} \\ ([1] \leftarrow \varnothing = \varnothing) \quad &\longmapsto \quad ! : X \to * \simeq X^{\times 0}. \end{aligned}$$

Unitality, associativity, and commutativity are conveniently packaged by functoriality. This turns out to be equivalent to Graeme Segal's *special* Γ *spaces* [Seg74] when $\mathcal{C} = \mathcal{S}$, and for general $\mathcal{C}$, it recovers the anologously defined theory in $\mathcal{C}$ (c.f. [BHS22, Ex 3.1.6, Prop 3.1.16, Pf. of prop 5.2.14]).

[1] In this paper we will call ∞-categories ∞-*categories* and ∞-categories with discrete mapping spaces *1-categories*, as their theory is equivalent to the traditional theory of categories. More generally, we will call ∞-categories whose mapping spaces are $(d-1)$-truncated *d-categories*.

(2) We say that an ∞-category is *semiadditive* if it has finite products and coproducts and for all finite sets S, the canonical natural transformation $\coprod_{s \in S}(-) \implies \prod_{s \in S}(-)$ is an equivalence. Then, the full subcategory $\mathbf{Cat}^{\oplus} \subset \mathbf{Cat}^{\times}$ of *semiadditive ∞-categories and product-preserving functors* possesses a localization functor $L_{\oplus} : \mathbf{Cat}^{\times} \to \mathbf{Cat}^{\oplus}$, which we study.

(3) Let Op denote the ∞-category of operads.[2] Then, there is a terminal operad $\mathrm{Comm}^{\otimes} \simeq \mathbb{E}_{\infty}^{\otimes}$; given $\mathcal{C}$ a symmetric monoidal ∞-category, we may form the ∞-category of *commutative algebra objects*

$$\mathrm{CAlg}(\mathcal{C}) := \mathbf{Alg}_{\mathrm{Comm}}(\mathcal{C}) \simeq \mathbf{Alg}_{\mathbb{E}_{\infty}}(\mathcal{C}).$$

We may study this and its specialization to the cartesian symmetric monoidal structure.

These perspectives each present the same ∞-category, i.e. [Cra11; GGN15] show that

$$\mathrm{CMon}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C}^{\times}) \simeq L_{\oplus}\mathcal{C}.$$

As a result, translating between these perspectives has proved invaluable; for instance, [GGN15] uses Perspectives 2 and 3 to construct an essentially unique symmetric monoidal structure on $\mathrm{CMon}(\mathcal{C})$ and [CHLL24a] uses Perspectives 1 and 3 to model commutative algebras in $\mathrm{CMon}(\mathcal{C})^{\otimes}$ as models for the Lawvere theory of *commutative semirings*.

Crucially, Perspective 3 may be used to construct homotopical lifts of the *Eckmann-Hilton argument*; for instance, in [HA], it is shown that for *any* reduced operad $\mathcal{O}^{\otimes}$, the forgetful functors

$$\mathrm{CAlg}\mathbf{Alg}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C}) \leftarrow \mathbf{Alg}_{\mathcal{O}}\mathrm{CAlg}^{\otimes}(\mathcal{C}),$$

are equivalences for the "pointwise" symmetric monoidal structure on algebras. Such a task may be accomplished by recognizing the far left and far right side each as algebras over the *Boardman-Vogt tensor product* $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes}$ and each arrow as pullback along the unique map $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes} \to \mathrm{Comm}^{\otimes}$; the above statement is equivalent to the statement that the object $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes} \in \mathrm{Op}$ is terminal, which may be checked using the theorem that *tensor products of commutative algebras are coproducts*.

This result is used ubiquitously to replace (lax) symmetric monoidal functors $\mathbf{Alg}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathcal{C}^{\otimes}$ with symmetric monodial endofunctors

$$\mathrm{CAlg}^{\otimes}(\mathcal{C}) \simeq \mathrm{CAlg}^{\otimes}\mathbf{Alg}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathrm{CAlg}^{\otimes}(\mathcal{C});$$

for instance, this underlies the symmetric monoidal structure on left-modules [HA] and the multiplicative structure on various invariants such as factorization homology [HA, Thm 5.5.3.2], THH and TC [NS18, § IV.2], and higher algebraic K-theory [BGT15].

This paper concerns the analog of Perspective 3 in the equivariant theory of algebra stemming from Hill-Hopkins-Ravanel's use of *norms of G-spectra* on the Kervarire invariant one problem, as well as the resulting theory of *indexed tensor products and (co)products* (c.f. [HH16]).

For the rest of this introduction, fix G a finite group; in G-equivariant homotopy theory, the point is replaced with elements of the *orbit category* $\mathcal{O}_G \subset \mathbf{Set}_G$, whose objects are homogeneous G-sets $[G/H]$ and whose morphisms are G-equivariant maps; indeed, Elmendorf's theorem realizes G-spaces as coefficient systems $\mathcal{S}_G \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S})$.[3] In G-equivariant higher category theory, ∞-categories are thus replaced with G-∞-categories

$$\mathbf{Cat}_G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Cat}).$$

In G-equivariant higher algebra, following Perspective 1, we may form the effective Burnside 2-category $\mathrm{Span}(\mathbb{F}_G)$ whose objects are finite G-sets, whose morphisms are spans, whose 2-cells are isomorphisms of spans, and whose composition is pullback; the following central definition is the heart of this subject.

Definition. The *∞-category of G-commutative monoids in $\mathcal{C}$* is the product-preserving functor ∞-category

$$\mathrm{CMon}_G(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_G), \mathcal{C});$$

the *∞-category of small G-symmetric monoidal ∞-categories* is

$$\mathbf{Cat}_G^{\otimes} := \mathrm{CMon}_G(\mathbf{Cat}). \qquad \triangleleft$$

[2] This is unambiguous; c.f. [HM23].

[3] Maps $[G/K] \to [G/H]$ may equivalently be presented as elements of g such that $gKg^{-1} \subset H$, modulo K; see e.g. [**Dieck**] for details.

Rather than commutative monoids, these are a homotopical lift of Dress' *Mackey functors* [Dre71].[4] Indeed, given $\mathcal{C}^\otimes \in \mathbf{Cat}_G^\otimes$ a G-symmetric monoidal ∞-category, the product-preserving functor

$$\iota_H : \mathrm{Span}(\mathbb{F}) \xrightarrow{* \mapsto G/H} \mathrm{Span}(\mathbb{F}_G)$$

constructs a symmetric monoidal ∞-category $\mathcal{C}_H^\otimes := \iota_H^* \mathcal{C}^\otimes$ whose underlying ∞-category $\mathcal{C}_H$ is the *value* of $\mathcal{C}^\otimes$ on the orbit G/H.[5] For all subgroups $K \subset H \subset G$, the covariant and contravariant functoriality of $\mathcal{C}^\otimes$ then yield symmetric monoidal *restriction* and *norm* functors

$$\begin{aligned} \mathrm{Res}_K^H : \mathcal{C}_H^\otimes \to \mathcal{C}_K^\otimes, \\ N_K^H : \mathcal{C}_K^\otimes \to \mathcal{C}_H^\otimes, \end{aligned}$$

which satisfy a form of Mackey's *double coset formula*.

Example. In Section 1.5, we recall a theorem of [BH21a; CHLL24b]: there exists a unique presentably G-symmetric monoidal ∞-category $\underline{\mathrm{Sp}}_G^\otimes$ such that:

- the H-value of $\underline{\mathrm{Sp}}_G^\otimes$ is the symmetric monoidal ∞-category $\left(\underline{\mathrm{Sp}}_G^\otimes\right)_H \simeq \mathrm{Sp}_H^\otimes$ of *genuine H-spectra* under the usual tensor product;
- the restriction functors $\mathrm{Res}_K^H : \mathrm{Sp}_H^\otimes \to \mathrm{Sp}_K^\otimes$ are the usual restriction functors; and
- the norm functors $N_K^H : \mathrm{Sp}_K^\otimes \to \mathrm{Sp}_H^\otimes$ are the *norm* of [HHR16]. $\triangleleft$

In fact, this symmetric monoidal structure is completely determined by its unit objects $S^0 \in \mathrm{Sp}_H^\otimes$.

If $H \subset G$ is a subgroup and $S \in \mathbb{F}_H$ a finite H-set, we may form the induced G-set $\mathrm{Ind}_H^G S \to [G/H]$, and the covariant functoriality then yields an *S-indexed tensor map*

$$\bigotimes_K^S : \mathcal{C}_S \to \mathcal{C}_H,$$

where $\mathcal{C}_S := \prod_{[H/K] \in \mathrm{Orb}(S)} \mathcal{C}_K$. By factoring the structure map of S as $\mathrm{Ind}_H^G S \to \coprod_{[H/K] \in \mathrm{Orb}(S)} [G/H] \to [G/H]$, we find that this possesses a canonical natural equivalence

$$\bigotimes_K^S X_K \simeq \bigotimes_{[H/K] \in \mathrm{Orb}(S)} N_K^H X_K$$

(c.f. [HHR16, Prop A.29]). In particular, pre-composing with the canonical map $[G/H] \xrightarrow{\Delta} \coprod_{[H/K] \in \mathrm{Orb}(S)} [G/H] \to S$, the role of *$n$-fold tensor power* is played by the *S-fold tensor power* of an object $X_H \in \mathcal{C}_H$, defined by

$$X_H^{\otimes S} := \bigotimes_K^S \mathrm{Res}_H^K X_H \simeq \bigotimes_{[H/K] \in \mathrm{Orb}(S)} N_K^H \mathrm{Res}_K^H X_K.$$

For instance, given a subgroup $C_2 \subset G$, the spectrum $\mathrm{MU}^{((G))}$ of [HHR16, § 5.1] is a $[G/C_2]$-indexed tensor power of $\mathrm{MU}_{\mathbb{R}}$.

In this setting, we may just as well define *indexed (co)products*; indeed, if $\mathcal{C}$ is a G-∞-category, the *contravariant* functoriality along $\mathrm{Ind}_H^G S \to [G/H]$ yields an *S-indexed diagonal*

$$\Delta^S : \mathcal{C}_H \to \mathcal{C}_S.$$

If this possesses a pointwise left adjoint at the tuple $(X_K)_{[H/K] \in \mathrm{Orb}(S)}$, we refer to the value as the *S-indexed coproduct* $\coprod_K^S X_K$, and dually, its pointwise right adjoint as the *S-indexed product* $\prod_K^S X_K$. These possess a

[4] Indeed, when the underlying ∞-category is a 1-category, these differ from Hill-Hopkins' *symmetric monoidal Mackey functors* only by asserting some reasonable coherence diagrams for the double coset formula isomorphisms; see Section 5.2 for details.

[5] In this paper, "orbits" refer to transitive G-sets, i.e. objects of the orbit category $\mathcal{O}_G \subset \mathbf{Set}_G$ spanned by transitive G-sets; We write $[G/H] \in \mathcal{O}_G$ for the homogeneous G-set given by cosets of G modulo H.

canonical natural transformation

$$\coprod_K^S (-) \implies \prod_K^S (-) : \mathcal{C}_S \to \mathcal{C};$$

we say that $\mathcal{C}$ is *G-semiadditive* if this transformation is an equivalence for all subgroups $H \subset G$ and finite H-sets $S \in \mathbb{F}_H$. More generally, if $I \subset \mathbb{F}_G$ is a subcategory, we say that $\mathcal{C}$ is *I-semiadditive* if its S-indexed coproducts and products are equivalent by the above natural transformation whenever $\mathrm{Ind}_H^G S \to [G/H]$; we verify in Section 1.2 that the subcategory of indices over which $\mathcal{C}$ is semiadditive is pullback-stable, contains all isomorphisms, and is extended from maps to orbits by coproducts, i.e. it is a *weak indexing categories* in the sense of [Ste24], so we might as well restrict our attention to semiadditivity over weak indexing categories.

In this level of generality, Perspectives 1 and 2 have been studied; indeed, the *semiadditive closure* theorem of [CLL24, Thm B] demonstrates that $\mathbf{Cat}_G^{I-\oplus} \subset \mathbf{Cat}_G^{I-\times}$ is a smashing localization implemented by

$$L_{I-\oplus}(\mathcal{C}) \simeq \underline{\mathrm{CMon}}_I(\mathcal{C}) \coloneqq \underline{\mathrm{Fun}}_G^{\times}(\mathrm{Span}_I(\underline{\mathbb{F}}_G), \mathcal{C}),$$

in particular, when $\mathcal{C}$ is a G-∞-category of coefficient systems

$$\underline{\mathrm{Coeff}}^G(\mathcal{D})_H \coloneqq \mathrm{Fun}(\mathcal{O}_H^{\mathrm{op}}, \mathcal{D}),$$

by [CLL24, Thm C], we have the formula

$$\underline{\mathrm{CMon}}_I(\underline{\mathrm{Coeff}}^G(\mathcal{D}))_H \simeq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_H), \mathcal{D}),$$

where $\mathrm{Span}_I(\mathbb{F}_H) \subset \mathrm{Span}(\mathbb{F}_H)$ is the wide subcategory of spans whose forward maps lie in the restriction of I to $\mathbb{F}_H$. Thus, we set the notation $\mathrm{CMon}_I(\mathcal{D}) \coloneqq \underline{\mathrm{CMon}}_I(\underline{\mathrm{Coeff}}^G(\mathcal{D}))_G \simeq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_G), \mathcal{D})$, and make the followign definition.

Definition. For I is a weak indexing category, the *∞-category of small I-symmetric monoidal ∞-categories* is

$$\mathbf{Cat}_I^{\otimes} \coloneqq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_G), \mathbf{Cat}). \qquad \triangleleft$$

Following through on Perspective 3, algebraic objects in an G-symmetric monoidal ∞-category should possess *S-ary operations* subject to various conditions, controlled by a theory of *G-operads*. To that end, in Section 2.3 we recall the definition of an ∞-category Op_G of *$\mathcal{O}_G$-∞-operads* (henceforth G-operads) equivalent to that of [NS22]. Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_G$ a G-operad, $S \in \mathbb{F}_H$ an H-set for some $H \subset G$, and T_i a finite K_i-set for all orbits $[H/K_i] \subset S$, we construct a *space of S-ary operations* $\mathcal{O}(S)$, together with *operadic composition maps*

$$\gamma : \mathcal{O}(S) \otimes \bigotimes_{[H/K_i] \in \mathrm{Orb}(S)} \mathcal{O}(T_i) \to \mathcal{O}\left(\coprod_{[H/K_i] \in \mathrm{Orb}(S)} \mathrm{Ind}_{K_i}^H T_i \right), \tag{1}$$

operadic restriction maps

$$\mathrm{Res} : \mathcal{O}(S) \to \mathcal{O}\left(\mathrm{Res}_K^H S\right), \tag{2}$$

and *equivariant symmetric group action*

$$\rho : \mathrm{Aut}_H(S) \times \mathcal{O}(S) \to \mathcal{O}(S). \tag{3}$$

Eqs. (2) and (3) together lift to a structure of a *G-symmetric sequence*; we go on to show in Corollary 2.77 that Op_G is *monadic* over G-symmetric sequences under a reducedness assumption.

Definition. We say that $\mathcal{O}^{\otimes}$ *has at least one color* if $\mathcal{O}(*_H)$ is nonempty for all subgroups $H \subset G$, and we say $\mathcal{O}^{\otimes}$ *has at most one color* if $\mathcal{O}(*_H) \in \{*, \emptyset\}$ for all $H \subset G$.[6] We say that $\mathcal{O}^{\otimes}$ *has one color* if it has at least one color and at most one color. $\triangleleft$

When $\mathcal{O}^{\otimes}$ has one color, an $\mathcal{O}$-algebra in the G-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ can intuitively be viewed as a tuple $(X_H \in \mathcal{C}_H^{BW_G(H)})_{G/H \in \mathcal{O}_G}$ satisfying $X_K \simeq \mathrm{Res}_K^H X_H$, together with $\mathcal{O}(S)$-actions

$$\mu_S : \mathcal{O}(S) \otimes X_H^{\otimes S} \to X_H \tag{4}$$

for all $H \subset G$ and $S \in \mathbb{F}_H$, homotopy-coherently compatible with the maps Eqs. (1) to (3).[7]

[6] Throughout this paper, $*_H$ refers to the terminal H-set, i.e. $*_H = [H/H]$ is the H-orbit with one point.
[7] Here, $W_G(H) = N_G(H)/H$ is the *Weyl group* of $H \subset G$, i.e. the automorphism group of the homogeneous G-set $[G/H]$.

Example. There exists a terminal G-operad $\mathrm{Comm}_G^{\otimes}$, which is characterized up to (unique) equivalence by the property that $\mathrm{Comm}_G(S)$ is contractible for all $S \in \mathbb{F}_H$; its algebras are endowed with contractible spaces of maps $X_H^{\otimes S} \to X_H$ for all $S \in \mathbb{F}_H$, as well as coherent homotopies witnessing their compatibility. We call these *G-commutative algebras*.

On one hand, we see in Section 5.2 that Comm_G-algebras present a homotopical lift of Hill-Hopkins' *G-commutative monoids* [HH16, § 4], though we prefer to reserve this name for the Cartesian case, following the convention of [HA]. On the other hand, our model agrees with that of [CHLL24b], so the recent *homotopical Tambara functor theorem* of Cnossen, Lenz, and Linskens [CHLL24b, Thm B] presents G-commutative algebra objects in $\underline{\mathrm{Sp}}_G^{\otimes}$ as *spectral G-Tambara functors.* ◁

Example. Let V be a real orthogonal G-representation; then, there is a *little disks V-operad* $\mathbb{E}_V^{\otimes}$ whose structure spaces are *spaces of equivariant configurations*:

$$\mathbb{E}_V(S) \simeq \mathrm{Conf}_S^H(V)$$

(see [Hor19]). This is modelled by the *Steiner graph G-operad*, so e.g. pointed G-spaces of the form $X = \Omega^V Y := \mathrm{Map}_*\left(S^V, Y\right)$ lift to $\mathbb{E}_V$-spaces by composition of loops [GM11]; hence many $\mathbb{E}_V$-algebras may be constructed in Sp_G as equivariant Thom spectra of V-fold loop spaces. ◁

In this paper, we are primarily concerned with indexed tensor products of $\mathcal{O}$-algebras, as well as $\mathcal{P}$-algebras in the resulting G-symmetric monoidal ∞-category. Mirroring the nonequivariant case, we accomplish this by constructing a *Boardman Vogt tensor product* and studying tensor products of G-operads of interest. In particular, we focus on the following example.

Example. Let $I \subset \mathbb{F}_G$ be an *indexing category*, i.e. a weak indexing category containing the fold maps $n \cdot [G/H] \to [G/H]$; this recovers the notion from [HH16], and so it is canonically induced from a *transfer system*, i.e. a subconjugacy-closed and restriction-stable sub-poset of the subgroup lattice $\mathrm{Sub}_{\mathbf{Grp}}(G)$ [BBR21; Rub19], consisting of the inclusions $K \subset H$ whose corresponding quotient maps $[G/K] \to [G/H]$. Blumberg and Hill conjectured that these live fully faithfully within G-operads [BH15] as *$\mathcal{N}_\infty$-operads*, and this conjecture was independently confirmed by Bonventre-Periera [BP21], Gutierrez-White [GW18], and Rubin [Rub21a].

The structure of an $\mathcal{N}_{I\infty}$-ring spectrum is intutively viewed as commutative ring structures on each spectrum X_H, connected by multiplicative I-indexed norms, suitably compatible with the restriction and (additive) transfer structures inherent to G-spectra.

Many examples of $\mathcal{N}_{I\infty}$-ring structures (which are not necessarily G-commutative) arise from the observation that Bousfield localizations of Sp_G need not respect norms, i.e. they need not be G-symmetric monoidal; for instance, of the $\mathbb{N} \cup \{\infty\} \times \mathbb{N} \cup \{\infty\}$-indexed collection of chromatic C_p-equivariant localizations, only those corresponding with (n,n) and $(n+1,n)$ are C_p-symmetric monodial [Hil17, Cor 4.1.3]. ◁

It will quickly follow from the definition of the Boardman-Vogt tensor product of G-operads that there is a pairing $\mathcal{N}_{I\infty} \otimes \mathcal{N}_{J\infty} \to \mathcal{N}_{I \vee J\infty}$, where $I \vee J$ is the join in the poset of transfer systems; intuitively, this says that given an algebra with $I \vee J$-indexed norms, we may separate these into I-indexed norms and J-indexed norms, satisfying an interchange law when applicable. Moreover, the transfer system for $I \vee J$ consists of thsoe inclusions $K \subset H$ which can be factored as

$$K \subset K_{I1} \subset K_{J1} \subset K_{I2} \subset \cdots \subset K_{Jn} \subset H$$

where $K_{I\ell} \subset K_{J\ell}$ is in I and $K_{J\ell} \subset K_{I(\ell+1)}$ is in J [Rub21b, Prop 3.1]; intuition would then suggest that we may combine interchanging $\mathcal{N}_{I\infty}$ and $\mathcal{N}_{J\infty}$-structures to construct an $\mathcal{N}_{I \vee J\infty}$ structure. Thus Blumberg and Hill conjectured that there is an equivalence $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{I\vee J\infty}^{\otimes}$ [BH15, Conj 6.26]; the main theorem of this paper confirms their conjecture.

Summary of main results. Following [BS24a], we will characterize the Boardman Vogt symmetric monoidal structure on G-operads using the tensor product of G-symmetric monoidal ∞-categories; beginning by equivariantly lifting the uniqueness of tensor products of commutative monoids shown in [GGN15, Thm 5.1]. In order to do so, we define a *symmetric monoidal G-∞-category* to be a commutative monoid object in $\mathbf{Cat}_G$.

Theorem A. *If $\mathcal{C}$ is a presentably symmetric monoidal ∞-category, then there exists a unique presentably symmetric monoidal structure $\underline{\mathrm{CMon}}_G^{\otimes-\mathrm{mode}}(\mathcal{C})$ on $\underline{\mathrm{CMon}}_G(\mathcal{C})$ such that the free G-commutative monoid*

G-functor

$$\underline{\mathrm{Coeff}}_G\mathcal{C} \to \underline{\mathrm{CMon}}_G(\mathcal{C})$$

possesses a (necessarily unique) symmetric monoidal structure.

In Section 1.3, we generalize Theorem A to *presentable G-∞-categories*, e.g. as developed in [CLL23b; Hil24]. We use this to define the coherences on a *Boardman-Vogt symmetric monoidal structure on G-operads.*

Theorem B. *There exists a unique symmetric monoidal structure $\underline{\mathrm{Op}}_G^{\otimes}$ on $\underline{\mathrm{Op}}_G$ attaining a (necessarily unique) symmetric monoidal structure on the fully faithful G-functor*

$$\mathrm{Env}^{/\underline{\mathbb{F}}_G^{G-\sqcup}} : \underline{\mathrm{Op}}_G^{\otimes} \to \underline{\mathbf{Cat}}^{\otimes-\mathrm{mode}}_{G,/\underline{\mathbb{F}}_G^{G-\sqcup}}$$

of [BHS22; NS22]. Furthermore, $\underline{\mathrm{Op}}_G^{\otimes}$ satisfies the following properties.

(1) In the case $G = e$ is the trivial group, there is a canonical symmetric monoidal equivalence $\mathrm{Op}_e^{\otimes} \simeq \mathrm{Op}^{\otimes}$, under the symmetric monoidal structure of [BS24a]; in particular, the underlying tensor product is equivalent to the Boardman-Vogt tensor product of [BV73; HM23; HA].

(2) The underlying tensor functor $- \overset{BV}{\otimes} \mathcal{O} : \mathrm{Op}_G \to \mathrm{Op}_G$ possesses a right adjoint $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(-)$, whose underlying G-∞-category is the G-∞-category of algebras $\underline{\mathbf{Alg}}_{\mathcal{O}}(-)$; the associated ∞-category is the ∞-category of algebras $\mathbf{Alg}_{\mathcal{O}}(-)$.

(3) The $\overset{BV}{\otimes}$-unit of $\mathrm{Op}_G^{\otimes}$ is the G-operad $\mathrm{triv}_G^{\otimes}$ of [NS22]; hence $\underline{\mathbf{Alg}}^{\otimes}_{\mathrm{triv}_G}(\mathcal{O}) \simeq \mathcal{O}^{\otimes}$.

(4) When $\mathcal{C}^{\otimes}$ is a G-symmetric monoidal ∞-category, $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is a G-symmteric monoidal ∞-category; furthermore, when $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a map of G-operads, the pullback lax G-symmetric monoidal functor

$$\underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \to \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$$

is G-symmetric monoidal; in particular, if $\mathcal{O}^{\otimes}$ has one object, then pullback along the unique map $\mathrm{triv}_G^{\otimes} \to \mathcal{P}^{\otimes}$ presents the unique natural transformation of operads

$$\underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \to \mathcal{C}^{\otimes},$$

and this is G-symmetric monoidal when $\mathcal{C}$ is G-symmetric monoidal.

(5) When $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a G-symmetric monoidal functor, the induced lax G-symmetric monoidal functor

$$\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{D})$$

is G-symmetric monoidal.

Remark. In analogy to [BV73], in Observation 2.33 we interpret algebras over the BV-tensor product $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ in a G-symmetric monoidal category $\mathcal{C}^{\otimes}$ as *bifunctors of G-operads $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathcal{C}^{\otimes}$*; unwinding definitions in the case $\mathcal{C}^{\otimes}$ is G-symmetric monoidal, we interpret these as *interchanging pairs of $\mathcal{O}$- and $\mathcal{P}$-algebras structures on an object of $\mathcal{C}$* in Observation 5.21; we show that this fully determines $\overset{\mathrm{BV}}{\otimes}$ in Corollary 4.4.

Furthermore, by Yoneda's lemma, the G-operad $\underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C})$ itself is determined by the property that its $\mathcal{O}$-algebras are interchanging pairs of $\mathcal{O}$- and $\mathcal{P}$-algebra structures on an object in $\mathcal{C}$; we show in Philosophical remark 4.1 that G-symmetric monoidal ∞-categories are determined by their underlying G-operads, so this fully determines $\underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C})$ as a G-symmetric monoidal ∞-category.

Lastly, in Proposition 4.25 we show that, under the *G-symmetric monoidal envelope* equivalence $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Fun}_G^{\otimes}(\mathrm{Env}\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$, the G-symmetric monoidal structure on algebras corresponds with the *pointwise G-symmetric monoidal structure* of [NS22, § 3.3]; intuitively, indexed tensor products of $\mathcal{O}$-algebras are simply indexed tensor products of their underlying H-objects with the "diagonal" $\mathcal{O}$-algebra structure. ◁

Remark. After this introduction, we replace $\mathcal{O}_G$ with an atomic orbital ∞-category $\mathcal{T}$; we prove Theorem B as well as other theorems in this introduction in this setting, greatly generalizing the stated results at the cost of ease of exposition. ◁

Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_G^{\mathrm{oc}}$ a G-operad with one color and $\psi : T \to S$ a map of finite H-sets, we also define the *space of multimorphisms*[8]

$$\mathrm{Mul}_{\mathcal{O}}^{\psi}(T;S) := \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T \times_S U).$$

We then define the *arity support* subcategory[9] $A\mathcal{O} \subset \mathbb{F}_G$ by

$$A\mathcal{O} := \left\{ \psi : T \to S \;\middle|\; \mathrm{Mul}_{\mathcal{O}}^{\psi}(T;S) \neq \varnothing \right\} \subset \mathbb{F}_G.$$

In essence, taking tensor products of Eq. (4) yields an action

$$\mathrm{Mul}_{\mathcal{O}}^{\psi}(T;S) \otimes X_H^{\otimes T} \to X_H^{\otimes S},$$

and $A\mathcal{O}$ consists of the *pairs of equivariant (multi-)arities* over which $\mathcal{O}^{\otimes}$ produces structure on X.

The fact that $\varnothing$ accepts no maps from nonempty sets obstructs construction of maps matching Eqs. (1) and (2), so $A\mathcal{O}$ can't be an arbitrary subcategory. In order to state restrictions on $A\mathcal{O}$, we introduce some terminology: we say that a G-operad $\mathcal{O}^{\otimes}$ is *E-unital* if for all subgroups $H \subset G$, we have

$$\mathcal{O}(\varnothing_H) = \begin{cases} * & \mathcal{O}(*_H) \neq \varnothing; \\ \varnothing & \mathcal{O}(*_H) = \varnothing. \end{cases}$$

We say that $\mathcal{O}^{\otimes}$ is *unital* if it is E-unital and has at least one color, and we say that $\mathcal{O}^{\otimes}$ is *reduced* if it is E-unital and has one color. More generally, we say that $\mathcal{O}^{\otimes}$ is *almost E-unital* (henceforth aE-unital) if, for all subgroups $H \subset G$ and non-contractible finite H-sets $S \in \mathbb{F}_H$ with $\mathcal{O}(S) \neq \varnothing$, we have $\mathcal{O}(\varnothing_H) = *$; we say that $\mathcal{O}^{\otimes}$ is *almost-unital* if it is almost E-unital and has at least one color. We denote the associated full subcategories by

$$\mathrm{Op}_G^{\mathrm{uni}} \subset \mathrm{Op}_G^{a\mathrm{uni}}, \mathrm{Op}_G^{E\mathrm{uni}} \subset \mathrm{Op}_G^{aE\mathrm{uni}} \subset \mathrm{Op}_G$$

Furthermore, we say that a G-operad is a *G-d-operad* if, for all subgroups $H \subset G$ and all finite H-sets $S \in \mathbb{F}_H$, the space $\mathcal{O}(S)$ is $(d-1)$-truncated.[10]

Theorem C. *The following posets are each equivalent:*

1. *The poset* $\mathrm{Sub}_{\mathrm{Op}_G}(\mathrm{Comm}_G)$ *of* sub-commutative G-operads.
2. *The poset* $\mathrm{Op}_{G,0}$ *of* G-0-operads.
3. *The essential image* $A(\mathrm{Op}_G) \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_G)$
4. *The embedded sub-poset* $\mathrm{wIndexCat}_G \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_G)$ *spanned by subcategories $I \subset \mathbb{F}_G$ which are closed under base change and automorphisms and satisfy the Segal condition that*

$$T \to S \in I \qquad \Longleftrightarrow \qquad \forall U \in \mathrm{Orb}(S), \quad T \times_S U \to U \in I$$

5. *The sub-poset* $\mathrm{wIndex}_G \subset \mathrm{FullSub}_G(\underline{\mathbb{F}}_G)$ *spanned by full G-subcategories $\mathcal{C} \subset \underline{\mathbb{F}}_G$ which are closed under self-indexed coproducts and have $*_H \in \mathcal{C}_H$ whenever $\mathcal{C}_H \neq \varnothing$.*

[8] We only make the assumption that $\mathcal{O}^{\otimes}$ has one color for ease of exposition; throughout the remainder of text following the introduction, we will not make this assumption.

[9] Throughout this paper, we say *subobject* to mean monomorphism in the sense of [HTT, § 5.5.6] and we write $\mathrm{Sub}_{\mathcal{C}}(X)$ for the poset of subobjects of X in $\mathcal{C}$; in the case the ambient ∞-category is a 1-category, this agrees with the traditional notion.

In the case our objects are in the ∞-category $\mathbf{Cat}$ of small ∞-categories, we call this a *subcategory*; in the case that the containing ∞-category is a 1-category, this is canonically expressed as a *core-preserving wide subcategory of a full subcategory*, i.e. it is a *replete subcategory*. Hence it is uniquely determined by its morphisms, so we will implicitly identify subcategories of $\mathcal{C}$ a 1-category with their corresponding subsets of $\mathrm{Mor}(\mathcal{C})$.

[10] A space is -1-*truncated* if it is either empty or contractible; for all $k \geq 0$, a space X is *truncated* if it is a disjoint union of connected spaces $(X_\alpha)_{\alpha \in A}$ such that, for each $\ell > k$ and $\alpha \in A$, the ℓth homotopy group $\pi_\ell(X_\alpha)$ is trivial.

Furthermore, there are equalities of sub-posets

$$\begin{aligned} \mathrm{IndCat}_G &= A\mathrm{Op}^{\mathrm{uni}}_{G,\geq \mathbb{E}_\infty} \subset \mathrm{wIndexCat}_G, \\ \mathrm{wIndexCat}^{\mathrm{uni}}_G &= A\mathrm{Op}^{\mathrm{uni}}_G \subset \mathrm{wIndexCat}_G \\ \mathrm{wIndexCat}^{a\mathrm{uni}}_G &= A\mathrm{Op}^{a\mathrm{uni}}_G \subset \mathrm{wIndexCat}_G \\ \mathrm{wIndexCat}^{E\mathrm{uni}}_G &= A\mathrm{Op}^{E\mathrm{uni}}_G \subset \mathrm{wIndexCat}_G \\ \mathrm{wIndexCat}^{aE\mathrm{uni}}_G &= A\mathrm{Op}^{aE\mathrm{uni}}_G \subset \mathrm{wIndexCat}_G. \end{aligned}$$

where $\mathrm{IndCat}_G \simeq \mathrm{Index}_G$ *denotes the* indexing categories *of [BH15; BP21; GW18; Rub21a] and the remaining notation is that of [Ste24].*

References. The equivalence between Poset (4) and Poset (5) is handled in [Ste24]; nevertheless, the composite map from Poset (1) to Poset (5) is shown to be furnished by the *self-indexed symmetric monoidal envelope* in Example 2.68. We then characterize the image of A, constructing an equivalence between Poset (3) and Poset (4) in Proposition 2.43 and Corollary 3.10.

Poset (2) and Poset (3) are shown to be equivalent in Corollary 3.10 by constructing a fully faithful right adjoint $\mathcal{N}^\otimes_{(-)\infty}$ to A:

$$\begin{tikzcd} \mathrm{Op}_G \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathrm{wIndexCat}_G \arrow[l, bend left, "\mathcal{N}^\otimes_{I\infty}"] \end{tikzcd} \tag{5}$$

with image the G-0-operads. Along the way, in Remark 3.9 we show that Poset (1) and Poset (2) are equivalent as subcategories. Finally, the remaining identites follow by Observation 3.11 ☐

Remark. The equivalence between Poset (4) and Poset (5) is implemented in [Ste24] by the construction

$$\mathbb{F}_{I,H} \coloneqq (\underline{\mathbb{F}}_I)_H \coloneqq \left\{ S \in \mathbb{F}_H \;\middle|\; \mathrm{Ind}^G_H S \to G/H \in I \right\}.$$

We refer to elements of $(\underline{\mathbb{F}}_I)_H$ as *I-admissible H-sets*, and note that we may view the arity support as the collection of S-sets over which $\mathcal{O}$-algebras have structure. ◁

We call the operads $\mathcal{N}^\otimes_{I\infty}$ constructed by Eq. (5) *weak $\mathcal{N}_\infty$-operads*. By Theorem C, we find that a slice category $\mathrm{Op}_{G,/\mathcal{O}^\otimes} \to \mathrm{Op}_G$ is a full subcategory if and only if $\mathcal{O}^\otimes$ is a weak $\mathcal{N}_\infty$-operad, in which case we write

$$\mathrm{Op}_I \coloneqq \mathrm{Op}_{G,/\mathcal{N}^\otimes_{I\infty}} \simeq A^{-1}(\mathrm{wIndexCat}_{G,\leq I});$$

explicitly, a map $\mathcal{P}^\otimes \to \mathcal{N}^\otimes_{I\infty}$ is a *property* of $\mathcal{P}^\otimes$, and this property is the arity support condition $A\mathcal{P} \leq I$.

We may understand $\mathcal{N}^\otimes_{I\infty}$ in a hands-on manner in a number of ways; for instance, it is constructed explicitly in Proposition 2.43. On the other hand, the equivalence between Poset (2) and Poset (5) of Theorem C shows that $\mathcal{N}^\otimes_{I\infty}$ is uniquely identified by the property

$$\mathcal{N}_{I\infty}(S) = \begin{cases} * & S \in \mathbb{F}_{I,H}; \\ \varnothing & \text{otherwise.} \end{cases} \tag{6}$$

There are many weak $\mathcal{N}_\infty$-G-operads of interest which are not $\mathcal{N}_\infty$-G-operads:

Example. Given $\mathcal{F} \subset \mathcal{O}^{\mathrm{op}}_G$ a G-family[11], let $\underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}}$ be the E-unital weak indexing system

$$\mathbb{F}^{\mathrm{triv}}_{\mathcal{F},H} = \begin{cases} \{*_H\} & H \in \mathcal{F}; \\ \varnothing & H \notin \mathcal{F}. \end{cases}$$

If $I^{\mathrm{triv}}_{\mathcal{F}}$ is the corresponding weak indexing category, then the G-operad $\mathbf{triv}^\otimes_{\mathcal{F}} \coloneqq \mathcal{N}^\otimes_{I^{\mathrm{triv}}_{cF}\infty}$ is characterized by a natural equivalence

$$\underline{\mathbf{Alg}}^\otimes_{\mathrm{triv}_{\mathcal{F}}}(\mathcal{C}) = \mathrm{Bor}^G_{\mathcal{F}}(\mathcal{C}^\otimes)$$

[11] By a *G-family*, we mean a subconjugacy closed family of subgroups. These correspond canonically with full subcategories $\mathcal{F} \subset \mathcal{O}_G$ satisfying the property that for all $V \in \mathcal{F}$ and maps $U \to V$ in $\mathcal{O}_G$, $U \in \mathcal{F}$; we will safely conflate these notions.

in Corollary 3.14, where $\mathrm{Bor}_{\mathcal{F}}^G$ is the *color Borelification* discussed in Section 3.2. ◃

Example. Given $\mathcal{F} \subset \mathcal{O}_G^{\mathrm{op}}$ a G-family, define the almost-unital weak indexing system

$$\mathbb{F}^0_{\mathcal{F},H} = \begin{cases} \{\varnothing_H, *_H\} & H \in \mathcal{F}; \\ \{*_H\} & H \notin \mathcal{F}. \end{cases}$$

with corresponding weak indexing category $I^0_{\mathcal{F}}$ and weak $\mathcal{N}_\infty$ operad $\mathbb{E}^{\otimes}_{\mathcal{F}0} := \mathcal{N}^{\otimes}_{I^0_{\mathcal{F}}\infty}$. In Section 3.3, $\mathbb{E}^{\otimes}_{\mathcal{F}0}$ is characterized by a natural equivalence

$$\mathbf{Alg}_{\mathbb{E}_{\mathcal{F}0}}(\mathcal{C}) \simeq \left(\Gamma^{\mathcal{F}}\mathcal{C}\right)^{1/} \times_{\Gamma^{\mathcal{F}}\mathcal{C}} \Gamma^G \mathcal{C},$$

where $\Gamma^{\mathcal{F}}\mathcal{C}^{\otimes}$ is the symmetric monoidal ∞-category of $\mathcal{F}$*-objects*

$$\Gamma^{\mathcal{F}}\mathcal{C}^{\otimes} \simeq \lim_{V \in \mathcal{F}^{\mathrm{op}}} \mathcal{C}_V^{\otimes}.$$ ◃

Example. Given $\mathcal{F} \subset \mathcal{O}_G^{\mathrm{op}}$ a G-family, define the unital weak indexing system

$$\mathbb{F}^{\infty}_{\mathcal{F},H} = \begin{cases} \{n \cdot *_H \mid n \in \mathbb{N}\} & H \in \mathcal{F}; \\ \{\varnothing_H, *_H\} & H \notin \mathcal{F}. \end{cases}$$

with corresponding weak indexing category $I^{\infty}_{\mathcal{F}}$ and weak $\mathcal{N}_\infty$ operad $\mathbb{E}^{\otimes}_{\mathcal{F}\infty} := \mathcal{N}^{\otimes}_{I^{\infty}_{\mathcal{F}}\infty}$. In Section 3.3, $\mathbb{E}^{\otimes}_{\mathcal{F}\infty}$ is characterized by a natural equivalence

$$\mathbf{Alg}_{\mathbb{E}_{\mathcal{F}\infty}}(\mathcal{C}) \simeq \mathrm{CAlg}(\Gamma^{\mathcal{F}}\mathcal{C}) \times_{(\Gamma^{\mathcal{F}}\mathcal{C})^{1/}} \mathcal{C}_G^{1/}.$$ ◃

We say a real orthogonal G-representation V is a *weak universe* if it admits an equivalence $V \simeq V \oplus V$.

Example. Given V a weak G-universe, we verify in Section 3.3 that $\mathbb{E}_V^{\otimes}$ is a weak $\mathcal{N}_\infty$-operad whose arity support $\underline{\mathbb{F}}^V := \underline{\mathbb{F}}_{A\mathbb{E}_V}$ is computed by

$$S \in \mathbb{F}_H^V \qquad \Longleftrightarrow \qquad \exists\ H-\text{equivariant embedding } S \hookrightarrow V.$$

In particular, if λ is a nontrivial irreducible C_p-representation, we use this to compute $A\mathbb{E}^{\otimes}_{\infty\lambda}$ in Section 3.3, verifying that $\mathbb{E}^{\otimes}_{\infty\lambda}$ is *not* an $\mathcal{N}_\infty$-operad in the sense of [BH15]. Thus $\infty\lambda$-fold loop spaces and their Thom spectra provide a rich topological source of examples of weak $\mathcal{N}_\infty$-algebras which are not $\mathcal{N}_\infty$-algebras. ◃

We show in Proposition 2.47 that I-symmetric monoidal ∞-categories have underlying I-operads; for $\mathcal{C} \in \mathbf{Cat}_I^{\otimes}$, we define the ∞-category of I*-commutative algebras* in $\mathcal{C}$ as

$$\mathrm{CAlg}_I(\mathcal{C}) := \mathbf{Alg}_{\mathcal{N}_{I\infty}}(\mathcal{C}).$$

We'd like to relate CAlg_I and CMon_I, for which we use the following construction.

Theorem D. *When I is almost-unital, there are fully faithful embeddings $(-)^{I-\sqcup}$and$(-)^{I-\times}$ making the following commute:*

$$\begin{tikzcd} \mathbf{Cat}_I^{\sqcup} \arrow[r, hook, "(-)^{I-\sqcup}"] \arrow[dr, "U"'] & \mathbf{Cat}_I^{\otimes} \arrow[d, "U"] & \mathbf{Cat}_I^{\times} \arrow[l, hook', "(-)^{I-\times}"'] \arrow[dl, "U"] \\ & \mathbf{Cat}_G & \end{tikzcd}$$

The image of $(-)^{I-\sqcup}$ is spanned by the I-symmetric monoidal ∞-categories whose indexed tensor products are indexed coproducts and the image of $(-)^{I-\times}$ is spanned by those whose indexed tensor products are indexed products.

We refer to I-symmetric monoidal ∞-categories of the form $\mathcal{C}^{I-\times}$ as *cartesian*, and $\mathcal{C}^{I-\sqcup}$ *cocartesian*. In Theorem 1.51 and Corollary 1.83, we characterize the ∞*-category of I-commutative monoids* in $\mathcal{C}$ a complete ∞-category as an ∞-category of I-commutative algebras, integrating Perspectives 1 to 3:

$$\mathrm{CMon}_I(\mathcal{C}) \simeq \mathrm{CAlg}_I(\mathcal{C}^{I-\times}).$$

Suppose $\mathcal{O}^{\otimes}$ has at most one object; if $\mathcal{O}^{\otimes}$ is additionally aE-unital, we say that it is *aE-reduced.* Under an equivariant distributivity assumption, $\underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C})$ is (fiberwise)-monadic over $\mathcal{C}$ [NS22], for a monad we explicitly describe in Section 4.1; in particular, this implies that the forgetful G-functor $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}^{I-\times}) \to \mathcal{C}^{I-\times}$ preserves indexed tensor products, preserves indexed products, and reflects equivalences. Hence $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}^{I-\times})$ is cartesian.

We will show that I-indexed tensor products in $\underline{\mathrm{CAlg}}_I^{\otimes}\mathcal{C}$ are indexed coproducts (i.e. its underlying I-symmetric monoidal ∞-category is *cocartesian*) and that this completely characterizes $\mathcal{N}_{I\infty}^{\otimes}$. The heart of our stragegy will use the explicit monadic description of Proposition 4.2 to reduce this to the case $\mathcal{C}^{\otimes} \simeq \underline{\mathcal{S}}_G^{G-\times}$ is the *cartesian G-symmetric monoidal ∞-category of G-spaces*; in this case, we may easily see that the I-symmetric monoidal ∞-category $\underline{\mathrm{CAlg}}_I^{\otimes}(\underline{\mathcal{S}}_G^{G-\times}) \simeq \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_G)^{I-\times}$ is cocartesian, as its underlying G-∞-category is I-semiadditive by [CLL24, Thm B-C]. Thus we will conclude the following.

Theorem E. *Let $\mathcal{O}^{\otimes}$ be an aE-reduced G-operad. Then, the following conditions are equivalenent.*

(a) The G-∞-category $\underline{\mathbf{Alg}}_{\mathcal{O}}\underline{\mathcal{S}}_G$ is $A\mathcal{O}$-semiadditive.

(b) The unique map $\mathcal{O}^{\otimes} \to \mathcal{N}_{A\mathcal{O}\infty}^{\otimes}$ is an equivalence.

Furthermore, for all aE-unital weak indexing categories I and I-symmetric monoidal ∞-categories $\mathcal{C}^{\otimes}$, the I-symmetric monoidal ∞-category $\underline{\mathrm{CAlg}}_I^{\otimes}\mathcal{C}$ is cocartesian.

For the following theorem, we say that an I-operad $\mathcal{O}^{\otimes}$ is *reduced* if, for all $S \in \mathbb{F}_H$ which is empty or contractible, the unique map $\mathcal{O}^{\otimes} \to \mathcal{N}_{I\infty}$ induces an equivalence

$$\mathcal{O}(S) \simeq \mathcal{N}_{I\infty}(S)$$

(c.f. Eq. (6)). We completely characterize algebras in cocartesian I-symmetric monoidal categories in Theorem 4.11, and from this Theorem E entirely characterizes the tensor products of reduced I-operads with $\mathcal{N}_{I\infty}^{\otimes}$ in the almost-E-unital setting.

Corollary F. *$\mathcal{N}_{I\infty}^{\otimes} \otimes \mathcal{N}_{I\infty}^{\otimes}$ is a weak $\mathcal{N}_\infty$-operad if and only if I is aE-unital. In this case, if $\mathcal{O}^{\otimes}$ is a reduced I-operad, then the unique map*

$$\mathcal{O}^{\otimes} \otimes \mathcal{N}_{I\infty}^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$$

is an equivalence.

Idempotent algebras correspond with smashing localizations, i.e. they classifiy $\otimes$-absorptive properties [HA, § 4.8.2]; in view of Corollary F, when $I \le J$ are almost-unital, we would like to characterize the smashing localization that $\mathcal{N}_{I\infty}^{\otimes}$ induces on $\mathrm{Op}_J^{\mathrm{red}}$ using the adjunction $- \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes} \dashv \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(-)$. Namely, in Section 3.2, we construct a right adjoint to the natural inclusion $E_I^J : \mathrm{Op}_I \to \mathrm{Op}_J$, called the *$I$-borelification* Bor_I^J and note that the I-indexed tensor products in $\mathcal{C}^{\otimes}$ and $\mathrm{Bor}_I^J\mathcal{C}^{\otimes}$ agree for all $\mathcal{C}^{\otimes} \in \mathbf{Cat}_J^{\otimes}$; thus, in Theorem 4.14, we conclude that the smashing localization corresponding with $\mathcal{N}_{I\infty}^{\otimes} \in \mathrm{Op}_J^{\mathrm{red}}$ classifies the property of *having commutative Borel I-type*:

$$\begin{aligned}
\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{O}^{\otimes} \quad &\Longleftrightarrow \quad \mathrm{Bor}_I^J\mathcal{O}^{\otimes} \simeq \mathcal{N}_{I\infty}^{\otimes}, \\
&\Longleftrightarrow \quad \forall \mathcal{C}^{\otimes} \in \mathbf{Cat}_J^{\otimes}, \quad \forall S \in \mathbb{F}_{I,V}, \quad \coprod_U^S \simeq \bigotimes_U^S : \underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C})_S \to \underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C})_V, \\
&\Longleftrightarrow \quad \underline{\mathbf{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_G) \text{ is } I\text{-semiadditive.}
\end{aligned}$$

Tensor products of idempotents algebras are themselves idempotent algebras, and they classify the conjunction of the properties classified by their factors [CSY20, Prop 5.1.8]. We leverage this to completely characterize indexed tensor products of almost-E-unital weak $\mathcal{N}_\infty$-operads, affirming Conjecture 6.27 of [BH15].

Theorem G. *The functor $\mathcal{N}_{(-)\infty}^{\otimes} : \mathrm{wIndex}_G \to \mathrm{Op}_G$ lifts to a fully faithful symmetric monoidal G-right adjoint*

$$\begin{tikzcd}
\underline{\mathrm{wIndex}}_G^{aE\mathrm{uni}} \arrow[rr, bend right, "\mathcal{N}_{(-)\infty}^{\otimes}"'] \arrow[rr, phantom, "\perp"] & & \underline{\mathrm{Op}}_G^{aE\mathrm{uni}} \arrow[ll, bend right, "A"']
\end{tikzcd}$$

Furthermore, the resulting tensor product of weak $\mathcal{N}_\infty$-operads are computed by the Borelified join

$$\mathcal{N}_I^\otimes \overset{BV}{\otimes} \mathcal{N}_J^\otimes \simeq \mathcal{N}^\otimes_{\mathrm{Bor}^G_{c(I\cap J)}(I\vee J)}.$$

In particular, this implies that $\mathcal{N}^\otimes_{(-)\infty}$ is compatible with products, restriction, and coinduction; hence norms of I-commutative algebras are $\mathrm{CoInd}_H^G I$-commutative algebras, and when I,J are almost-unital weak indexing categories and $\mathcal{C}^\otimes$ is an $I\vee J$-symmetric monoidal ∞-category, there is a canonical equivalence of $I\vee J$-symmetric monoidal ∞-categories

$$\underline{\mathrm{CAlg}}_I^\otimes \underline{\mathrm{CAlg}}_J^\otimes(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}^\otimes_{I\vee J}(\mathcal{C}).$$

Remark. The reader interesting in computing tensor products of G-operads may benefit from reading the combinatorial characterization of joins of weak indexing systems in terms of *closures* in [Ste24]; there, we prove that the join of weak indexing systems $\mathbb{F}_I \vee \mathbb{F}_J$ is computed by closing the union $\mathbb{F}_I \cup \mathbb{F}_J$ under iterated I and J-indexed coproducts. ◁

We conclude an infinitary case of an equivariant homotopical lift of Dunn's additivity theorem [Dun88].

Corollary H (Equivariant infinitary Dunn additivity)**.** *Let V and W be real orthogonal G-representations satisfying at least one of the following conditions:*

(a) V,W are weak G-universes, or
(b) the canonical map $\mathbb{E}_V^\otimes \simeq \mathbb{E}^\otimes_{V\oplus W}$ is an equivalence.

Then the canonical map

$$\mathbb{E}_V^\otimes \overset{BV}{\otimes} \mathbb{E}_W^\otimes \to \mathbb{E}^\otimes_{V\oplus W}$$

is an equivalence; equivalently, for any G-symmetric monoidal ∞-category $\mathcal{C}^\otimes$, the pullback functors

$$\mathbf{Alg}_{\mathbb{E}_V}\underline{\mathbf{Alg}}^\otimes_{\mathbb{E}_W}(\mathcal{C}) \leftarrow \mathbf{Alg}_{\mathbb{E}_{V\oplus W}}(\mathcal{C}) \to \mathbf{Alg}_{\mathbb{E}_W}\underline{\mathbf{Alg}}^\otimes_{\mathbb{E}_V}(\mathcal{C})$$

are equivalences.

For instance, we may set $\mathcal{C}^\otimes \coloneqq \underline{\mathcal{S}}_G^{G-\times}$ to recover a result about $\mathbb{E}_V$-spaces (which are not necessarily grouplike), or we may set $\mathcal{C} \coloneqq \underline{\mathrm{Sp}}_G^\otimes$ to conclude additivity of $\mathbb{E}_V$-ring spectra

$$\mathbf{Alg}_{\mathbb{E}_V}\underline{\mathbf{Alg}}^\otimes_{\mathbb{E}_W}(\underline{\mathrm{Sp}}_G) \simeq \mathbf{Alg}_{\mathbb{E}_{V\oplus W}}(\underline{\mathrm{Sp}}_G) \simeq \mathbf{Alg}_{\mathbb{E}_W}\underline{\mathbf{Alg}}^\otimes_{\mathbb{E}_V}(\underline{\mathrm{Sp}}_G)$$

under either of the assumptions in Corollary H.

Remark. In the thesis [Szc23], an ostensibly-similar result to Corollary H is proved: given D_V the *little Disks graph G-operad*, Szczesny constructs a non-homotopical Boardman-Vogt tensor product $\otimes$ and a canonical map $D_V \otimes D_W \to D_{V\oplus W}$, which he shows to be a weak equivalence of graph G-operads in [Szc23, Thm 4.5.5]. Neither this result nor Corollary H imply each other.

On one hand, Szczesny's result concerns a tensor product with no known homotopical properties, so it is incomparable with results concerning ∞-categories of algebras satisfying *homotopical* universal properties. On the other hand, while Corollary H is homotopical, it only concerns cases where at least one of the representations induces I-symmetric monoidal ∞-categories of algebras whose indexed tensor products are indexed coproducts; this property will not be satisfied for any nontrivial indexed tensor products in the finite-dimensional case, so the range of representations in Szczesny's result is significantly larger. ◁

Along the way, we quickly acquire various corollaries in equivariant higher algebra. For instance, in Section 4.4 we use Corollary H to define iterated Real topological Hochschild homology for $\mathbb{E}_V$-algebras whenever V admits an $\infty\sigma$ summand, and we express it as a S^σ-indexed colimit when $V = \infty\rho$. We go on in Corollary 5.3 to lift Bonventre's genuine operadic nerve to a conservative functor of ∞-categories, and we verify in Proposition 5.6 that it restricts to an equivalence between the two categories of discrete G-operads, giving traditional presentations for all of our objects in the discrete setting.

Notation and conventions. We assume that the reader is familiar with the technology of higher category theory and higher algebra as developed in [HTT] and [HA, § 2-3], though we encourage the reader to engage with such technologies via a "big picture" perspective akin to that of [Gep19, § 1-2] and [Hau23, § 1-3]. In particular, we only pierce the veil to use non-homotopical aspects of quasicategory theory in Appendix B.2.

Throughout this paper, we frequently describe conditions which may be satisfied by objects parameterized over some ∞-category $\mathcal{T}$. If P is a property, in the instance where there exists Borelification adjunctions

$$E_{\mathcal{F}}^{\mathcal{T}} : \mathcal{C}_{\mathcal{F}} \rightleftarrows \mathcal{C}_{\mathcal{T}} : \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}}$$

along family inclusions $\mathcal{F} \subset \mathcal{T}$, we say that $X \in \mathcal{C}_{\mathcal{T}}$ is E-P when there exists some $\overline{X} \in \mathcal{C}_{\mathcal{F}}$ which is P such that $X \simeq E_{\mathcal{F}}^{\mathcal{T}} \overline{X}$. We say that X is *almost* E-P (or aE-P) if $\mathcal{C}_{\mathcal{F}}$ has a terminal object $*_{\mathcal{F}}$ for all $\mathcal{F}$, and there is a pushout expression

$$X \simeq *_{\mathcal{F}'} \sqcup_{*_{\mathcal{F}}} *_{\mathcal{F}'}$$

for some $\mathcal{F}' \subset \mathcal{F}$; we say that X is *almost* P (or a-P) if it's almost E-P and $\mathcal{F}' = \mathcal{T}$ in the above.

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1. Equivariant symmetric monoidal categories

In this section, we review and advance the equivariant ∞-category theory of of *homotopical incomplete (semi)-Mackey functors* for a weak indexing system I, which we call I*-commutative monoids*. To that end, we begin in Section 1.1 by reviewing our equivariant higher categorical setup. We go on to cite and prove some basic facts about I-commutative monoids in Section 1.2. In Section 1.3 we then endow the $\mathcal{T}$-∞-category of I-commutative monoids with its *mode* symmetric monoidal structure, and prove that this is uniquely determined as a presentable symmetric monoidal structure by the free functor from coefficient systems, confirming an atomic orbital lift of Theorem A and identifying the resulting symmetric monoidal structure with the *localized Day convolution structure*. Following this, in Section 1.4 we quickly develop a framework for $\mathcal{T}$-symmetric monoidal d-categories. We finish the section in Section 1.5 with a tour through the gamut of existing examples of I-symmetric monidal ∞-categories.

1.1. Recollections on $\mathcal{T}$-∞-categories.

We center on the following definition.

Definition 1.1. An ∞-category $\mathcal{T}$ is

(1) *orbital* if the finite coproduct completion $\mathbb{F}_{\mathcal{T}} := \mathcal{T}^{\sqcup}$ has all pullbacks, and
(2) *atomic orbital* if it is orbital and every map in $\mathcal{T}$ possessing a section is an equivalence. ◁

We view the setting of atmoic orbital ∞-categories as a natural axiomatic home for higher algebra centered around the Burnside category (see [Nar16, § 4]), generalizing the orbit categories of a finite groups. The reader who is exclusively interested in equivariant homotopy theory is encouraged to assume every atomic orbital ∞-category is the orbit category of a family of subgroups of a finite group.

Definition 1.2. Let $\mathcal{T}$ be an ∞-category. Then, a full subcategory $\mathcal{F} \subset \mathcal{T}$ is a $\mathcal{T}$*-family* if whenever $V \in \mathcal{F}$ and $W \to V$ is a map, we have $W \in \mathcal{F}$.[12] The poset of $\mathcal{T}$-families under inclusion is denoted $\mathsf{Fam}_{\mathcal{T}}$.

Similarly, a full subcategory $\mathcal{F} \subset \mathcal{T}$ is a $\mathcal{T}$*-cofamily* if its opposite $\mathcal{F}^{\mathrm{op}} \subset \mathcal{T}^{\mathrm{op}}$ is a $\mathcal{T}^{\mathrm{op}}$-family. ◁

Example 1.3. Let G be a topological group, let $\mathcal{S}_G$ be the ∞-category of G-spaces, and let $\mathcal{O}_G \subset \mathcal{S}_G$ be the full subcategory spanned by homogeneous G-spaces $[G/H]$, where $H \subset G$ is a closed subgroup. The following are all atomic orbital ∞-categories (c.f. [Ste24]).

(1) For G is a topological group, the full subcategory $\mathcal{O}_G^{fin} \subset \mathcal{O}_G$ spanned by G/H for H finite.

[12] These are named *families* after subconjugacy closed families of subgroups, which frequently occur in equivariant homotopy; these are referred to as *sieves* in [BH15; NS22] and *upwards-closed subcategories* in [Gla17].

(2) If G is a topological group, the wide subcategory $\mathcal{O}_G^{f.i.} \subset \mathcal{O}_G$ whose morphisms are projections $G/K \to G/H$ for $K \subset H$ finite index inclusion of closed subgroups.
(3) If G is a topological group, the full subcategory $\mathcal{O}_G^{f.i.sb} \subset \mathcal{O}_G$ spanned by G/H for $H \subset G$ a finite-index closed subgroup.
(4) X a space, considered as an ∞-category.
(5) P a meet semilattice.
(6) If $\mathcal{T}$ is atomic orbital, $\mathrm{ho}(\mathcal{T})$.
(7) If $\mathcal{T}$ is atomic orbital, $\mathcal{F} \subset \mathcal{T}$ a full subcategory satisfying the following conditions:
   (a) For all $U, W \in \mathcal{F}$ and paths $U \to V \to W$ in $\mathcal{T}$, $V \in \mathcal{F}$.
   (b) For all $U, W \in \mathcal{F}$ and cospans $U \to V \leftarrow W$ in $\mathcal{T}$, there is a span $U \leftarrow V' \to V$ in $\mathcal{F}$.

   For instance, $\mathcal{F}$ may be the intersection of a family and a cofamily whose connected components have weakly initial objects (e.g. $\mathcal{T}_{\geq V}$).
(8) If $\mathcal{T}$ is atomic orbital and $V \in \mathcal{T}$, the ∞-category $\mathcal{T}_{/V}$. ◃

In this section, we briefly summarize some relevant elements of parameterized and equivariant higher category theory in the setting of atomic orbital ∞-categories. Of course, this theory has advanced far past that which is summarized here; for instance, further details can be found in the work of Barwick-Dotto-Glasman-Nardin-Shah [BDGNS16a; BDGNS16b; Nar16; Sha22; Sha23], Cnossen-Lenz-Linskens [CLL23a; CLL23b; CLL24; Lin24; LNP22], and Hilman [Hil24].

1.1.1. *The $\mathcal{T}$-∞-category of small $\mathcal{T}$-∞-categories.*

Example 1.4. Let $\mathcal{F} \subset \mathcal{O}_G$ be a subconjugacy-closed family of subgroups and let $\mathcal{S}_{\mathcal{F}}$ be the ∞-category of $\mathcal{F}$-spaces, constructed e.g. by inverting $\mathcal{F}$-weak equivalences between topological G-spaces. Then, a version of Elmendorf's theorem [DK84; Elm83] states that the *total $\mathcal{F}$-fixed points* functor yields an equivalence

$$\mathcal{S}_{\mathcal{F}} \simeq \mathrm{Fun}(\mathcal{F}^{\mathrm{op}}, \mathcal{S}).$$ ◃

Definition 1.5. The *∞-category of small $\mathcal{T}$-∞-categories* as

$$\mathbf{Cat}_{\mathcal{T}} := \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Cat}_\infty).$$ ◃

Remark 1.6. We show in Example 2.14 that $\mathbf{Cat}_{\mathcal{T}}$ is equivalently presented as *complete Segal obects* in the ∞-topos $\mathcal{S}_{\mathcal{T}} := \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Cat}_\infty)$. ◃

Remark 1.7. The Grothendieck Construction, imported to ∞-category theory as the Straightening-Unstraightening equivalence in [HTT, Thm 3.2.0.1], provides an equivalence

$$\mathbf{Cat}_{\mathcal{T}} \simeq \mathbf{Cat}^{\mathrm{cocart}}_{/\mathcal{T}^{\mathrm{op}}},$$

the latter denoting the (non-full) subcategory of functors to $\mathcal{T}^{\mathrm{op}}$ whose objects are cocartesian fibrations and whose morphisms are functors over $\mathcal{T}^{\mathrm{op}}$ which preserve cocartesian arrows. ◃

Given $\mathcal{T}$ an ∞-category, we may define the *$\mathcal{T}$-functor category* to be the full subcategory

$$\mathrm{Fun}_{\mathcal{T}}(\mathcal{C}, \mathcal{D}) := \mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{T}^{\mathrm{op}}}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}(\mathcal{C}, \mathcal{D})$$

consisting of functors over $\mathcal{T}$ which preserve cocartesian lifts of the structure maps.

Example 1.8. For any object $V \in \mathcal{T}$, the forgetful functor $(\mathcal{T}_{/V})^{\mathrm{op}} \to \mathcal{T}^{\mathrm{op}}$ is a cocartesian fibration classified by the representable presheaf $\mathrm{Map}_{\mathcal{T}}(-; V)$. We refer to the associated $\mathcal{T}$-category as $\underline{V}$. This is covariantly functorial in $\mathcal{T}$, since the shriek pushforward (i.e. Ind) yields functors $\mathcal{T}_{/V} \to \mathcal{T}_{/W}$ for all maps $f : V \to W$. ◃

When G is a finite group, it is well known that the category $\mathcal{O}_G$ has objects homogeneous G-spaces G/H and morphisms G-equivariant maps $G/K \to G/H$; it is not hard to see that such maps are equivalent to maps $xKx^{-1} \subset G/H$, and in particular, the endomorphism monoid is the Weyl group $W_G H = N_G(H)/H$.

It is well known that when G is a finite group, the map $\mathrm{Ind}_H^G : \mathcal{O}_H \to \mathcal{O}_{G,/(G/H)}$ is an equivalence of categories, so replacing $\mathcal{O}_G$ with its overcategories does not escape genuine equivariant mathematics.

Notation 1.9. In the setting that $\mathcal{T} = \mathcal{O}_G$, the elements of $\mathcal{T}$ are canonically expressed as homogenous G-spaces G/H; in this case, we refer to $\underline{G/H}$ simply as $\underline{H}$, we refer to $\mathcal{O}_G$-∞-categories as G-∞-categories, and we refer to the value of a G-∞-category $\mathcal{C}$ on G/H as $\mathcal{C}_H$. ◃

The representable $\mathcal{T}$-categories have total categories of a particularly nice form.

Proposition 1.10 ([NS22, Prop 2.5.1])**.** *If an atomic orbital ∞-category $\mathcal{T}$ has a terminal object, then it is a 1-category; in particular, $\mathcal{T}_{/V}$ is a 1-category.*

They play an important role in equivariant higher category theory.

Proposition 1.11 ([BDGNS16b, Thm 9.7])**.** *The cartesian product $\mathcal{C} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{D}$ on $\mathbf{Cat}_{\mathcal{T}}$ has exponential objects $\underline{\mathrm{Fun}}_{\mathcal{T}^{\mathrm{op}}(\mathcal{C},\mathcal{D})}$ which are classified by the functor*

$$V \mapsto \mathrm{Fun}_{\underline{V}}(\mathcal{C} \times_{\mathcal{T}^{\mathrm{op}}} \underline{V}, \mathcal{D} \times_{\mathcal{T}^{\mathrm{op}}} \underline{V}).$$

We refer to monomorphisms in $\mathbf{Cat}_{\mathcal{T}}$ as *$\mathcal{T}$-subcategories*, and $\mathcal{T}$-functors which are fiberwise-fully faithful as *full $\mathcal{T}$-subcategories*, or *$\mathcal{T}$-fully faithful functors*.

Observation 1.12. By the fiberwise expression for limits in functor categories (c.f. [HTT, Cor 5.1.2.3]), a $\mathcal{T}$-functor is a $\mathcal{T}$-subcategory inclusion if and only if it is fiberwise a $\mathcal{T}$-subcategory inclusion. $\triangleleft$

Observation 1.13. The terminal $\mathcal{T}$-category $\underline{*}$ is classified by the constant functor $V \mapsto *$. The poset of *sub-terminal objects in* $\mathbf{Cat}_{\mathcal{T}}$ (i.e. monomorphisms) is in monotone isomorphism with $\mathrm{Fam}_{\mathcal{T}}$, and the category $\underline{*}_{\mathcal{F}}$ associated with $\mathcal{F}$ is determined by the values

$$\underline{*}_{\mathcal{F},V} \simeq \begin{cases} * & V \in \mathcal{F}; \\ \varnothing & \text{otherwise}. \end{cases}$$

To see this, it suffices to note that when $X \in \{\varnothing, *\}$, the mapping space $\mathrm{Map}(Y, X)$ is empty if $X = \varnothing$ and $Y \neq \varnothing$, and contractible otherwise. $\triangleleft$

The ∞-category $\mathbf{Cat}_{\mathcal{T}}$ participates in an adjunction

$$\mathrm{Tot} : \mathbf{Cat}_{\mathcal{T}} \rightleftarrows \mathbf{Cat} : \underline{\mathrm{Coeff}}^{\mathcal{T}}$$

whose left adjoint Tot is the total category of cocartesian fibrations, and whose right adjoint has fibers

$$(\underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{C})_V \simeq \mathrm{Fun}\left((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{C}\right)$$

where the functoriality on f is given by $(f_!)^*$ (see [BDGNS16b, Thm 7.8]).

Example 1.14. There is an equivalence $\underline{*} = \mathrm{Coeff}^{\mathcal{T}} * \in \mathbf{Cat}_{\mathcal{T}}$, since right adjoints preserve terminal objects. $\triangleleft$

Example 1.15. Set $\underline{\mathcal{S}}_{\mathcal{T}} := \mathrm{Coeff}^{\mathcal{T}} \mathcal{S}$. Then, in the case $\mathcal{T} = \mathcal{O}_G$, Elmendorf's theorem [Elm83] may be reinterpreted as an equivalence

$$\underline{\mathcal{S}}_G \simeq \mathrm{Coeff}^G \mathcal{S}. \qquad \triangleleft$$

Definition 1.16. The *$\mathcal{T}$-∞-category of small $\mathcal{T}$-∞-categories* is $\underline{\mathbf{Cat}}_{\mathcal{T}} := \mathrm{Coeff}^{\mathcal{T}}(\mathbf{Cat})$. $\triangleleft$

Notation 1.17. Fix $\mathcal{C} \in \mathbf{Cat}_{\mathcal{T}} = \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Cat}_\infty)$. We refer to the value of $\mathcal{C}$ at $V \in \mathcal{T}^{\mathrm{op}}$ as the *V-value category of* $\mathcal{C}$, written as $\mathcal{C}_V$; given $f : V \to W$, we refer to the associated functor as *restriction*

$$\mathrm{Res}_V^W : \mathcal{C}_W \to \mathcal{C}_V.$$

In the case $\mathcal{T} = \mathcal{O}_G$, we write Res_K^H in place of $\mathrm{Res}_{G/K}^{G/H}$. $\triangleleft$

Example 1.18. The *V-value of* $\underline{\mathbf{Cat}}_{\mathcal{T}}$ is $(\underline{\mathbf{Cat}}_{\mathcal{T}})_V = \mathbf{Cat}_{\underline{V}}$; we henceforth refer to this as $\mathbf{Cat}_V$. The restriction functor $\mathrm{Res}_V^W : \mathbf{Cat}_W \to \mathbf{Cat}_V$ is presented from the perspective of cocartesian fibrations by the pullback

$$\begin{tikzcd}
\mathrm{Res}_W^V \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d] \\
\underline{V}^{\mathrm{op}} \arrow[r] & \underline{W}^{\mathrm{op}}
\end{tikzcd}$$

$\triangleleft$

We additionally construct the *associated ∞-category*

$$\Gamma^{\mathcal{T}} \mathcal{C} := \mathrm{Fun}_{\mathcal{T}}(\underline{*}, \mathcal{C}),$$

whose objects consist of cocartesian sections of the structure functor $\mathcal{C} \to \mathcal{T}$. We refer to this as the *category of $\mathcal{T}$-objects in* $\mathcal{C}$. For instance, if $\mathcal{T}$ has a terminal object V, [BDGNS16b, Lemma 2.12] shows that we have an equivalence

$$\Gamma^{\mathcal{T}} \mathcal{C} \simeq \mathcal{C}_V;$$

more generally, this implies that $\Gamma^{\mathcal{T}}\mathcal{C} \simeq \lim_{V\in\mathcal{T}^{\mathrm{op}}}\mathcal{C}_V$, i.e. it is the $\mathcal{T}$*-fixed points*, (the limit of $\mathcal{C}$ viewed as a $\mathcal{T}^{\mathrm{op}}$ functor). Defining the $\mathcal{T}$*-inflation* by

$$\left(\mathrm{Infl}_e^{\mathcal{T}}\mathcal{D}\right)_V := \mathcal{D}$$

for any $\mathcal{D} \in \mathbf{Cat}$ and $V \in \mathcal{T}$, the adjunction between limits and diagonals immediately yields the following.

Proposition 1.19. *The functor* $\mathrm{Infl}_e^{\mathcal{T}} : \mathbf{Cat} \to \mathbf{Cat}_{\mathcal{T}}$ *is left adjoint to* $\Gamma^{\mathcal{T}} : \mathbf{Cat}_{\mathcal{T}} \to \mathbf{Cat}$.

1.1.2. *Join, slice, and (co)limits.* We summarize some elements of [Sha22; Sha23].

Definition 1.20. Let $\iota : \mathcal{T}^{\mathrm{op}} \times \partial\Delta^1 \hookrightarrow \mathcal{T}^{\mathrm{op}} \times \Delta^1$ be the evident inclusion. Then, the $\mathcal{T}$*-join* is the top horizontal functor

$$\begin{tikzcd}
\mathbf{Cat}_{\mathcal{T}}^2 \arrow[rr, "- \star_{\mathcal{T}} -"] \arrow[d, hook] & & \mathbf{Cat}_{\mathcal{T}} \arrow[d, hook] \\
\mathbf{Cat}_{/\mathcal{T}^{\mathrm{op}}\times\partial\Delta^1} \arrow[r, "\iota^*"] & \mathbf{Cat}_{/\mathcal{T}\times I} \arrow[r, "\pi_!"] & \mathbf{Cat}_{/\mathcal{T}^{\mathrm{op}}}
\end{tikzcd}$$

which exists by [Sha22, Prop 4.3]. We write

$$K^{\underline{\triangleright}} := K \star_{\mathcal{T}} \underline{*} \qquad\qquad \text{and} \qquad\qquad K^{\underline{\triangleleft}} := \underline{*} \star_{\mathcal{T}} K$$

If $p : K \to \mathcal{C}$ is a $\mathcal{T}$-functor, then we define the $\mathcal{T}$*-slice categories* as slice categories $\mathcal{T}$-categories

$$\mathcal{C}^{(p,\mathcal{T})/} := \underline{\mathrm{Fun}}_{\mathcal{T},K/}(K^{\underline{\triangleright}}, \mathcal{C});$$
$$\mathcal{C}^{/(p,\mathcal{T})} := \underline{\mathrm{Fun}}_{\mathcal{T},K/}(K^{\underline{\triangleleft}}, \mathcal{C}) \qquad \triangleleft$$

In the case $p : \underline{*}_{\mathcal{T}} \to \mathcal{C}$ corresponds with the $\mathcal{T}$-object $X \in \Gamma^{\mathcal{T}}\mathcal{C}$, we simply write $\mathcal{C}^{X/} := \mathcal{C}^{(p,\mathcal{T})/}$ and similar for overcategories. In general, the categories $\mathcal{C}^{(p,\mathcal{T})/}$ take part in a functor out of $\mathbf{Cat}_{\mathcal{T},K/}$. Of fundamental importance is the adjoint relationship between these functors:

Theorem 1.21 ([Sha23, Cor 4.27])**.** *The $\mathcal{T}$-join forms the left adjoint in a pair of adjunctions*

$$K \star_{\mathcal{T}} - : \mathbf{Cat}_{\mathcal{T}} \rightleftarrows \mathbf{Cat}_{\mathcal{T},K/} : (-)^{(-,\mathcal{T})/},$$
$$- \star_{\mathcal{T}} K : \mathbf{Cat}_{\mathcal{T}} \rightleftarrows \mathbf{Cat}_{\mathcal{T},K/} : (-)^{/(-,\mathcal{T})}.$$

We say a $\mathcal{T}$-functor $\underline{p} : K^{\underline{\triangleleft}} \to \mathcal{C}$ extends $p : K \to \mathcal{C}$ if the composite $K \to K^{\underline{\triangleleft}} \to \mathcal{C}$ is homotopic to p.

Definition 1.22. Let $\mathcal{C}$ be a $\mathcal{T}$-category. A $\mathcal{T}$-object $X \in \Gamma^{\mathcal{T}}\mathcal{C}$ is *final* if for all $V \in \mathcal{T}$, the object $X_V \in \mathcal{C}_V$ is final; if $\underline{p} : K^{\underline{\triangleleft}} \to \mathcal{C}$ is a $\mathcal{T}$-functor extending $p : K \to \mathcal{C}$ and the corresponding cocartesian section $\sigma_{\underline{p}} : *_{\mathcal{T}} \to \mathcal{C}^{/(p,\mathcal{T})}$ is a final $\mathcal{T}$-object, then we say $\underline{p}$ is a *limit diagram for* p. $\triangleleft$

The *fiberwise opposite* (or vertical opposite) functor $\mathbf{Cat}_{\mathcal{T}} \to \mathbf{Cat}_{\mathcal{T}}$ is the $\mathcal{T}$ functor induced under $\mathrm{Coeff}^{\mathcal{T}}$ by the opposite functor $\mathbf{Cat} \to \mathbf{Cat}$; the notions of initial $\mathcal{T}$-objects and $\mathcal{T}$-colimits are defined dually as final $\mathcal{T}$-objects and $\mathcal{T}$-limits in the fiberwise opposite.

In many cases, these are familiar; for instance, *trivially indexed* colimits are non-equivariant in nature.

Proposition 1.23 ([Sha22, Thm 8.6])**.** *Suppose $\underline{K}$ is a $\mathcal{T}$-category such that, for all morphisms $V \to W$ in $\mathcal{T}$, the associated restriction (i.e. cocartesian transport) functor $\underline{K}_W \to \underline{K}_V$ is an equivalence. Then, a diagram $\underline{p} : K^{\underline{\triangleleft}} \to \mathcal{C}$ is a limit diagram for $p : K \to \mathcal{C}$ if and only if for all V, the associated diagram $\underline{p}_V : K_V^{\triangleleft} \to \mathcal{C}_V$ is a limit diagram for p_V.*

Definition 1.24. Let $\mathcal{C}$ be a $\mathcal{T}$-category and let $\underline{\mathcal{K}}_{\mathcal{T}} = (\mathcal{K}_V)_{V\in\mathcal{T}} \subset \underline{\mathbf{Cat}}_{\mathcal{T}}$ be a restriction-stable collection of $\underline{V}$-categories. We say that $\mathcal{C}$ *strongly admits $\mathcal{K}$-shaped limits* if for each $V \in \mathcal{T}$, each $\underline{V}$-category $K \in \mathcal{K}_V$ and each $\underline{V}$-functors $p : K \to \mathcal{C}_{\underline{V}}$, there exists a limit diagram for p. We say $\mathcal{C}$ is $\mathcal{T}$*-complete* if it strongly admits $\underline{\mathbf{Cat}}_{\mathcal{T}}$-shaped limits.

If $\mathcal{C}, \mathcal{D}$ are $\mathcal{T}$-categories which strongly admit all $\mathcal{K}$-shaped limits and $F : \mathcal{C} \to \mathcal{D}$ is a $\mathcal{T}$, functor, we say F *strongly preserves all $\mathcal{T}$-limits* if for all $V \in \mathcal{T}$ and all $K \in \mathcal{K}_V$, postcomposition with the $\underline{V}$-functor $F_{\underline{V}} : \mathcal{C}_{\underline{V}} \to \mathcal{D}_{\underline{V}}$ sends K-shaped limits diagrams to limits diagrams. $\triangleleft$

An important class of examples is *indexed (co)products.*

Notation 1.25. Consider $S \in \mathbb{F}_V$, considered as a V-category under the unique coproduct-preserving inclusion $\mathbf{Set}_V \hookrightarrow \mathbf{Cat}_V$. Then, we refer to S-shaped V-limits as *S-indexed products* and S-shaped V-colimits as *S-indexed coproducts*.

If $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a full $\mathcal{T}$-subcategory, we refer to $\mathcal{T}$-colimits of the corresponding class as *$\mathcal{C}$-indexed coproducts*; similarly, following [Ste24], if $I \subset \mathbf{Set}_{\mathcal{T}}$ is a pullback-stable full subcategory, we define the full $\mathcal{T}$-subcategory $\underline{\mathbf{Set}}_I \subset \underline{\mathbf{Set}}_{\mathcal{T}}$ of *I-admissible sets* by

$$\left(\underline{\mathbf{Set}}_I\right)_V := \mathbf{Set}_{I,V} := \left\{S \mid \mathrm{Ind}_V^{\mathcal{T}} S \to V \in I\right\} \subset \underline{\mathbf{Set}}_{\mathcal{T}}.$$

We refer to the class of $\underline{\mathbf{Set}}_I$-indexed coproudcts as *$I$-indexed coproducts*, and use the dual language for I-indexed products.

Given $\mathcal{C}$ a $\mathcal{T}$-category and $S \in \mathbf{Set}_{\mathcal{T}}$, we write

$$\mathcal{C}_S := \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_U,$$

where $\mathrm{Orb}(S)$ is the multiset of *orbits* expressing S as a disjoint union of elements of $\mathcal{T}$. Given $S \in \mathbf{Set}_{I,V} := \left(\underline{\mathbf{Set}}_I\right)_V$, and $(X_U) \in \mathcal{C}_S$, we denote the S-indexed products and coproducts as

$$\prod_U^S X_U \in \mathcal{C}_V, \qquad \coprod_U^S X_U \in \mathcal{C}_V.$$

If $\mathcal{D}$ strongly admits $\underline{\mathbf{Set}}_I$-shaped limits, we simply say $\mathcal{D}$ *admits I-indexed coproducts*; if $I = \mathbb{F}_{\mathcal{T}}$, we say that $\mathcal{D}$ *admits finite indexed coproducts*, and if $I = \mathbf{Set}_{\mathcal{T}}$, we say that $\mathcal{D}$ *admits all indexed coproducts.* ◁

Observation 1.26. If $\mathcal{C} \in \mathbf{Cat}_{\mathcal{T}}$ admits all indexed coproducts, $S \in \mathbf{Set}_V$, and $(X_U) \in \mathcal{C}_S$, then $\coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^V X_U$ satisfies the universal property for S-indexed coproducts; hence there is a natural equivalence

$$\coprod_U^S X_U \simeq \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^V X_U,$$

and the dual argument characterizes indexed products similarly. ◁

In nonequivariant homotopy theory, all colimits are coequalizers of geometric realizations. The equivariant version of this states that $\mathcal{T}$-colimits are coequalizers of indexed coproducts. An example is the following result of Shah.

Proposition 1.27 ([Sha23, Cor 12.15])**.** *Let $\mathcal{T}$ be an orbital ∞-category. Then, $\mathcal{C}$ is $\mathcal{T}$-cocomplete if and only if it admits all geometric realizations and indexed coproducts.*

We will need notation for strongly (co)limit-presereving functors.

Notation 1.28. Let $I \subset \mathbb{F}_{\mathcal{T}}$ be a pullback-stable full subcategory. Following and slightly extending [Sha22, Notn 1.15], we use the following notation for the described distinguished full $\mathcal{T}$-subcategories of $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$:

(1) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{\mathcal{K}-L}(\mathcal{C}, \mathcal{D})$: the $\underline{V}$-functors which strongly preserve $\underline{\mathcal{K}_V}$-indexed colimits;
(2) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{\mathcal{K}-R}(\mathcal{C}, \mathcal{D})$: the $\underline{V}$-functors which strongly preserve $\underline{\mathcal{K}_V}$-indexed limits;
(3) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{L}(\mathcal{C}, \mathcal{D})$: the $\underline{V}$-functors which strongly preserve small $\underline{V}$-colimits;
(4) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{R}(\mathcal{C}, \mathcal{D})$: the $\underline{V}$-functors which strongly preserve small $\underline{V}$-limits;
(5) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\sqcup}(\mathcal{C}, \mathcal{D})$: the $\underline{V}$-functors which (strongly) preserve I-indexed coproducts;
(6) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\times}(\mathcal{C}, \mathcal{D})$: the $\underline{V}$-functors which (strongly) preserve I-indexed products.
(7) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{\sqcup}(\mathcal{C}, \mathcal{D})$: the $\underline{V}$-functors which (strongly) preserve ordinary coproducts;
(8) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{\times}(\mathcal{C}, \mathcal{D})$: the $\underline{V}$-functors which (strongly) preserve ordinary products. ◁

In many cases of interest, it is easy to verify these properties. Given $\mathcal{K} \subset \mathbf{Cat}$, define $\underline{\mathcal{K}}_V \subset \mathbf{Cat}_{\mathcal{T}^V}$ to consist of $\underline{V}$-categories whose fibers lie in $\mathcal{K}$, and define $\underline{\mathcal{K}} := (\underline{\mathcal{K}}_V) \subset \underline{\mathbf{Cat}}_{\mathcal{T}}$.

Proposition 1.29 ([Sha22, Thm 8.6])**.** *Let $\mathcal{C}, \mathcal{D}$ be ∞-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor.*

(1) $\underline{\mathrm{Coeff}}^G \mathcal{C}$ strongly admits $\underline{\mathcal{K}}$-shaped limits if and only if $\mathcal{C}$ admits $\mathcal{K}$-shaped limits, and

(2) $\underline{\mathrm{Coeff}}^G F : \underline{\mathrm{Coeff}}^G \mathcal{C} \to \underline{\mathrm{Coeff}}^G \mathcal{D}$ *strongly preserves* $\underline{\mathcal{K}}$*-shaped limits if and only if* F *preserves* $\mathcal{K}$*-shaped limits.*

Definition 1.30. A $\mathcal{T}$-functor $L : \mathcal{C} \to \mathcal{D}$ is *left adjoint* to $R : \mathcal{D} \to \mathcal{C}$ if the associated functors $L_V : \mathcal{C}_V \to \mathcal{D}_V$ are left adjoint to $R_V : \mathcal{D}_V \to \mathcal{C}_V$ for all $V \in \mathcal{T}$. ◃

These are the same as *relative adjoints* over $\mathcal{T}^{\mathrm{op}}$ by [HA, Prop 7.3.2.1], and they satisfy a parameterized version of the adjoint functor theorem [Hil24, Thm 6.2.1]. In the case that $K = \underline{*}_{\mathcal{T}}$, the results [HTT, Lem 6.1.1.1], Proposition 1.23, and Proposition 1.29 together with [Sha23, Lem 4.8] immediately imply the following.

Lemma 1.31. *The* $\mathcal{T}$*-functor* $\mathrm{Ar}(\mathcal{C}) \xrightarrow{\mathrm{ev}_1} \mathcal{C}$ *is a Cartesian fibration if and only if* $\mathcal{C}$ *admits* $\mathcal{T}$*-pullbacks; in this case, for* $\alpha : X \to Y$ *a morphism of* $\mathcal{T}$*-objects in* $\mathcal{C}$*, there exists an adjunction*

$$\alpha_! : \mathcal{C}^{/X} \rightleftarrows \mathcal{C}^{/Y} : \alpha^*$$

where $\alpha^*(Z) \simeq Z \times_Y X$*.*

Additionally, we can make adjunctions *non-genuine* directly using [HA, Prop 7.3.2.1]

Proposition 1.32. *If* $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$ *are adjoint* $\mathcal{T}$*-functors, then* $\Gamma L : \Gamma\mathcal{C} \rightleftarrows \Gamma\mathcal{D} : \Gamma R$ *are adjoint.*

Fortunately, $\underline{\mathrm{Coeff}}^{\mathcal{T}}(-)$ is compatible with adjunctions.

Lemma 1.33. *Suppose* $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ *is an adjunction of* ∞*-categories. Then,*

$$\underline{\mathrm{Coeff}}^{\mathcal{T}} L : \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{C} \rightleftarrows \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{D} : \underline{\mathrm{Coeff}}^{\mathcal{T}} R$$

is an adjunction of $\mathcal{T}$*-*∞*-categories.*

Proof. This follows from the fiberwise description of $\underline{\mathrm{Coeff}}^{\mathcal{T}} R$; indeed, the V-values

$$L_* : \mathrm{Fun}((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{C}) \rightleftarrows \mathrm{Fun}((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{D}) : R_*$$

are adjoint. □

Example 1.34. We may use Lemma 1.33 to e.g. realize the full subcategory of $\mathcal{T}$-spaces whose fixed points are d-truncated and d-connected as (co)localizing subcategories

$$\mathcal{S}_{\mathcal{T},\geq d} \rightleftarrows \mathcal{S}_{\mathcal{T}} \rightleftarrows \mathcal{S}_{\mathcal{T},\leq d}.$$

Under the assumption that $\mathcal{T}$ is orbital, the author believes that most of the results of [LM06] may be carried out on this level of generality; later on, we will use this line of thought to understand *truncatedness and connectedness of* $\mathcal{T}$*-operads and* $\mathcal{T}$*-symmetric monoidal categories.* ◃

Example 1.35. By Lemma 1.33, the *classifying space and core* double adjunction $(-)_{\simeq} \dashv \iota \dashv (-)^{\simeq}$ yields

$$\begin{tikzcd} \underline{\mathbf{Cat}}_{\mathcal{T}} \arrow[r, bend left, "(-)_{\simeq}"] \arrow[r, bend right, "(-)^{\simeq}"'] & \underline{\mathcal{S}}_{\mathcal{T}} \arrow[l, hook'] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Cat}_{\mathcal{T}} \arrow[r, bend left, "(-)_{\simeq}"] \arrow[r, bend right, "(-)^{\simeq}"'] & \mathcal{S}_{\mathcal{T}} \arrow[l, hook'] \end{tikzcd}$$

a double $\mathcal{T}$-adjunction and double adjunction. ◃

1.2. I-commutative monoids.

Following [Bar14], we say that an *adequite triple* is the data of two core-preserving wide subcategories $\mathcal{X}_b \subset \mathcal{X} \supset \mathcal{X}_f$ of an ∞-category such that cospans $X \xrightarrow{\varphi_f} Y \xleftarrow{\varphi_b} Z$ satisfying $\varphi_f \in \mathcal{X}_f$ and $\varphi_b \in \mathcal{X}_b$ lift to pullback diagrams

$$\begin{tikzcd} & X \times_Y Z \arrow[dl, "\psi_b"'] \arrow[dr, "\psi_f"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ X \arrow[dr, "\varphi_f"'] & & Z \arrow[dl, "\varphi_b"] \\ & Y & \end{tikzcd}$$

satisfying $\psi_b \in \mathcal{X}_b$ and $\psi_f \in \mathcal{X}_f$. Given an adequate triple $\mathcal{X}_b \subset \mathcal{X} \supset \mathcal{X}_f$, we define the *span category* to be

$$\mathrm{Span}_{b,f}(\mathcal{X}) := A^{eff}(\mathcal{X}, \mathcal{X}_b, \mathcal{X}_f).$$

In particular, the objects of $\mathrm{Span}_{b,f}(\mathcal{X})$ are precisely those of $\mathcal{X}$, and the morphisms from X to Z are the spans $X \xleftarrow{\varphi_b} Y \xrightarrow{\varphi_f} Z$ with $\varphi_b \in \mathcal{X}_b$ and $\varphi_f \in \mathcal{X}_f$, with composition defined by taking pullbacks. [13]

Example 1.36. For $\mathcal{T}$ an orbital ∞-category and $I \subset \mathbb{F}_{\mathcal{T}}$ a pullback-stable wide subcategory, $\mathbb{F}_{\mathcal{T}} = \mathbb{F}_{\mathcal{T}} \hookleftarrow I$ is an adequate triple; write

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) := \mathrm{Span}_{all,I}(\mathbb{F}_{\mathcal{T}}).$$ ◁

Warning 1.37. Even when $\mathbb{F}_{\mathcal{T}}$ is a 1-category (i.e. $\mathcal{T}$ is a 1-category), $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ will seldom be a 1-category; indeed, in this case, $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ is a 2-category whose 2-cells given by the isomorphisms of spans

$$\begin{tikzcd} & Y' \arrow[dl] \arrow[dr] \arrow[d, dashed, "\sim"] & \\ X & Y \arrow[l] \arrow[r] & Z \end{tikzcd}$$

◁

In this subsection, we review the cartesian algebraic theory these corepresent, called *I-commutative monoids*. We will find that, in the same way that CMon is easily characterized via *semiadditivity* (c.f. [GGN15]), CMon_I is easily characterized via *I-semiadditivity*. Little of this subsection is original, simultaneously forming a slight generalization of [Nar16] and a massive specialization of [CLL24].

1.2.1. *Weak indexing systems.* We briefly review the setting of *weak indexing systems* introduced in [Ste24], which we view as assumptions on I which cut out the intersection of category theoretic and algebraic notions of I-commutative monoids.

Definition 1.38. A *$\mathcal{T}$-weak indexing category* is a subcategory $I \subset \mathbb{F}_{\mathcal{T}}$ satisfying the following conditions:

- (IC-a) (restrictions) I is stable under arbitrary pullbacks in $\underline{\mathbb{F}}_{\mathcal{T}}$;
- (IC-b) (segal condition) $T \to S$ and $T' \to S$ are both in I if and only if $T \sqcup T' \to S \sqcup S'$ is in I; and
- (IC-c) ($\Sigma_{\mathcal{T}}$-action) if $S \in I$, then all automorphisms of S are in I.

A *$\mathcal{T}$-weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ satisfying the following conditions:

- (IS-a) whenever $\mathbb{F}_{I,V} \neq \varnothing$, we have $*_V \in \mathbb{F}_{I,V}$.
- (IS-b) $\underline{\mathbb{F}}_I$ is closed under $\underline{\mathbb{F}}_I$-indexed coproducts. ◁

Observation 1.39. By a basic inductive argument, condition (IC-b) is equivalent to the condition that $S \to T$ is in I if and only if $T_U = T \times_S U \to U$ is in I for all $U \in \mathrm{Orb}(S)$; in particular, I is determined by its slice categories over *orbits*. ◁

We denote the *I-admissible sets* by $\underline{\mathbb{F}}_I := \underline{\mathbf{Set}}_I$ as in Notation 1.25. Inspired by Observation 1.39, in [Ste24] we prove the following.

Proposition 1.40. *The assignment $I \mapsto \underline{\mathbb{F}}_I$ implements an equivalence between the posets of $\mathcal{T}$-weak indexing categories and $\mathcal{T}$-weak indexing systems.*

We say that $\underline{\mathbb{F}}_I$ is *unital* if it contains the V-set $\varnothing_V$ is I-admissible for all $V \in \mathcal{T}$; we say that $\underline{\mathbb{F}}_I$ is an *indexing system* if $n \cdot *_V$ is I-admissible for all $V \in \mathcal{T}$ and all $n \in \mathbb{N}$. When $\mathcal{T} = \mathcal{O}_G$, this recovers the notion given the same name in [BH15]; see [Ste24] for details.

These come up for two main reasons: Theorem C will establish that these enumerate the *weak $\mathcal{N}$-∞-operads* which form the basis of the main results of this paper, and [Ste24] established that these are precisely the data consisting of full $\mathcal{T}$-subcategories $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ which are I-symmetric monoidal subcategories; we will see throughout the remainder of this paper that the I-indexed coproducts in $\underline{\mathbb{F}}_I$ appear frequently as the arities of *compositions* of I-indexed algebraic structures.

[13] Those readers more familiar with [EH23] may note that this specializes to the notion of a *span pair*, when backwards maps are $\mathcal{X}_b = \mathcal{X}$, in which case $\mathrm{Span}_f(\mathcal{X})$ recovers that of [EH23], and hence lifts to an $(\infty, 2)$-category with a universal property that we will not use.

Remark 1.41. By Observation 1.39, in the presence of Condition (IC-b), Condition (IC-a) is equivalent to the condition that for all Cartesian diagrams in $\mathbb{F}_{\mathcal{T}}$

$$\begin{tikzcd} T \times_V U \arrow[r] \arrow[d, "\alpha'"] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "\alpha"] \\ U \arrow[r] & V \end{tikzcd} \tag{7}$$

with $U, V \in \mathcal{T}$ and $\alpha \in I$, we have $\alpha' \in I$. ◃

One important source of examples of weak indexing categories is *semiadditivity.*

Definition 1.42. We say that a $\mathcal{T}$-∞-category $\mathcal{D}$ is *I-semiadditive* if it possesses I-indexed coproducts and its I-indexed coproducts and products agree. ◃

Given $\mathcal{D}$ a $\mathcal{T}$-∞-category admitting finite products and coproducts, we define the *semiadditive locus*

$$s(\mathcal{D}) = \left\{ T \to S \mid \coprod \simeq \prod : \mathcal{D}_T \to \mathcal{D}_S \right\}.$$

Proposition 1.43. *$s(\mathcal{D})$ is a $\mathcal{T}$-weak indexing category.*

Proof. The functor $\mathcal{D}_T \to \mathcal{D}_S$ is an (external) product of the functors $\mathcal{D}_{T_U} \to \mathcal{D}_U$ as U ranges across $\mathrm{Orb}(S)$, so $s(D)$ satisfies **??**. Furthermore, unary products and coproducts are modelled by the identity functor, so $s(D)$ satisfies **??**. In view of Remark 1.41, to conclude the proposition, we need that whenever the coproduct and product functors $\mathcal{D}_T \to \mathcal{D}_V$ agree, the same is true for the restriction $\mathcal{D}_{\mathrm{Res}_U^V T} \to \mathcal{D}_U$; this follows quickly from restriction-stability of (co)limits (e.g. [Sha23, Prop 5.5]). □

By [Ste24], the poset $\mathrm{wIndexCat}_{\mathcal{T}}$ has joins, which we write as $- \vee -$.

Corollary 1.44. *$\mathcal{D}$ is $I \vee J$-semiadditive if and only if it is I-semiadditive and J-semiadditive.*

Proof. This follows by noting that $\mathcal{D}$ is I'-semiadditive if and only if $I' \le s(\mathcal{D})$ and applying the universal property for joins. □

1.2.2. *I-commutative monoids as the I-semiadditivization.* Let $\mathrm{Trip}^{\mathrm{adeq}} \subset \mathrm{Fun}(\bullet \to \bullet \leftarrow \bullet, \mathbf{Cat})$ be the full subcategory spanned by adequate triples. By definition [Bar14, Def 3.6], $\mathrm{Span}_{-,-}(-)$ forms a functor $\mathrm{Trip}^{\mathrm{adeq}} \to \mathbf{Cat}$.

Write $\underline{\mathbb{F}}_V := \mathbb{F}_{\underline{V}}$ and write $I_V := I_{/V}$. If $I \subset \mathbb{F}_{\mathcal{T}}$ is pullback-stable, then the slice categories form a $\mathcal{T}$-subcategory $\underline{I} \subset \underline{\mathbb{F}}_{\mathcal{T}}$, yielding a functor $\mathcal{T}^{\mathrm{op}} \to \mathrm{Trip}^{\mathrm{adeq}}$. We use this to define the composite $\mathcal{T}$-functor

$$\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}}) : \mathcal{T}^{\mathrm{op}} \xrightarrow{(\underline{I}, \underline{\mathbb{F}}_{\mathcal{T}}, \underline{\mathbb{F}}_{\mathcal{T}})} \mathrm{Trip}^{\mathrm{adeq}} \xrightarrow{\mathrm{Span}} \mathbf{Cat}.$$

Definition 1.45. If $\mathcal{C}$ is a $\mathcal{T}$-∞-category admitting finite $\mathcal{T}$-products, then the *$\mathcal{T}$-∞-category of I-commutative monoids in $\mathcal{C}$* is

$$\underline{\mathrm{CMon}}_I(\mathcal{C}) := \underline{\mathrm{Fun}}^{\times}_{\mathcal{T}}(\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}}), \mathcal{C}).$$ ◃

Definition 1.46. We say that a functor $F : \mathcal{D} \to \mathcal{C}$ is the *I-semiadditive completion* if $\mathcal{D}$ is I-semiadditive and for all I-semiadditive $\mathcal{T}$-categories $\mathcal{E}$, postcomposition along F yields an equivalence

$$\underline{\mathrm{Fun}}^{I-\times}(\mathcal{E}, \mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}^{I-\times}(\mathcal{E}, \mathcal{C}).$$ ◃

Remark 1.47. The article *the* above is justified by the fact that Yoneda's lemma yields an equivalence over $\mathcal{C}$ between any two I-semiadditive completions. ◃

Theorem 1.48 ([CLL24, Thm B])**.** *$U : \underline{\mathrm{CMon}}_I(\mathcal{C}) \to \mathcal{C}$ is the I-semiadditive completion.*

1.2.3. *Commutative monoids in $\mathcal{T}$-objects.* Let $I^\infty \subset \mathbb{F}_{\mathcal{T}}$ denote the smallest core-preserving wide subcategory containing the fold maps $n \cdot V \to V$ for all $V \in \mathcal{T}$ and $n \in \mathbb{N}$; this is precisely the indexing category corresponding with the minimal indexing system. We set the notation

$$\underline{\mathrm{CMon}}_{\nabla}(\mathcal{C}) \coloneqq \underline{\mathrm{CMon}}_{I^\infty}(\mathcal{C}).$$

Observation 1.49. The I^∞-indexed products are precisely *trivially* indexed products; by Proposition 1.23 the I^∞-indexed product preserving functors are precisely the fiberwise product-preserving $\mathcal{T}$-functors. Furthermore, a $\mathcal{T}$-category is ∇-semiadditive if and only if, for each $V \in \mathcal{T}$, the ∞-category $\mathcal{C}_V$ is semiadditive. ◁

Thus, we have equivalences $\mathbf{Cat}_{\mathcal{T}}^{\times} \simeq \mathrm{Coeff}^{\mathcal{T}}(\mathbf{Cat}^{\times})$ and $\mathbf{Cat}_{\mathcal{T}}^{\oplus} \simeq \mathrm{Coeff}^{\mathcal{T}}(\mathbf{Cat}^{\oplus})$ compatible with the inclusions; this together with Lemma 1.33 directly implies that the ∇-semiadditive closure satisfies

$$\underline{\mathrm{CMon}}_{\nabla}(\mathcal{C}) \simeq \left(\mathcal{T}^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathbf{Cat}^{\times} \xrightarrow{\mathrm{CMon}} \mathbf{Cat}^{\oplus}\right);$$

Cnossen-Lenz-Linsken's semiadditive closure theorem (i.e. Theorem 1.48) then yields the following.

Corollary 1.50. *There is a canonical equivalence* $\mathrm{CMon}_{\nabla}(\mathcal{C}) \simeq \mathrm{CMon}(\Gamma\mathcal{C})$.

1.2.4. *I-commutative monoids in ∞-categories.* We recall a special case of Cnossen-Lenz-Linsken's Mackey functor theorem.

Theorem 1.51 ([CLL24, Thm C])**.** *For every presentable category $\mathcal{C}$, there exists a natural equivalence*

$$\mathrm{CMon}_I(\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{C})) \simeq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C});$$

furthermore, given a map $f : V \to W$, the associated restriction functor

$$\mathrm{Res}_V^W : \mathrm{Fun}(\mathrm{Span}_{I_W}(\mathbb{F}_W), \mathcal{C}) \to \mathrm{Fun}(\mathrm{Span}_{I_V}(\mathbb{F}_V), \mathcal{C})$$

is given by $(f_!)^*$.

Definition 1.52. If $\mathcal{C}$ is an ∞-category with finite products, then the *$\mathcal{T}$-∞-category of I-commutative monoids in $\mathcal{C}$* is

$$\underline{\mathrm{CMon}}_I(\mathcal{C}) \coloneqq \underline{\mathrm{CMon}}_I(\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{C})). \quad ◁$$

Similar to the case of $\underline{\mathrm{Coeff}}^{\mathcal{T}}$, this construction is compatible with adjunctions.

Lemma 1.53. *Let $I \subset \mathcal{T}$ be a pullback-stable wide subcategory of an orbital ∞-category.*

(1) If $f : \mathcal{C} \to \mathcal{D}$ is a product-preserving functor, then postcomposition yields a $\mathcal{T}$-functor

$$f_* : \underline{\mathrm{CMon}}_I\mathcal{C} \to \underline{\mathrm{CMon}}_I\mathcal{D}.$$

(2) If $L : \mathcal{C} \rightleftarrows : R$ is an adjunction whose right adjoint R is product preserving, then

$$L_* : \underline{\mathrm{CMon}}_I\mathcal{C} \rightleftarrows \underline{\mathrm{CMon}}_I\mathcal{D} : R_*$$

is a $\mathcal{T}$-adjunction.

Proof. (1) follows by noting that f_* exists since f is product preserving, and it is compatible with restriction because postcomposition and precomposition commute. (2) follows by noting that the associated functors

$$L_* : (\mathrm{CMon}_I\mathcal{C})_V \simeq \mathrm{Fun}^{\times}\left(\mathrm{Span}_{I_V}(\mathbb{F}_V), \mathcal{C}\right) \rightleftarrows \mathrm{Fun}^{\times}\left(\mathrm{Span}_{I_V}(\mathbb{F}_V), \mathcal{D}\right) = (\mathrm{CMon}_I\mathcal{D})_V) : R_*$$

are adjoint. □

Construction 1.54. Let $X \in \mathrm{CMon}_I\mathcal{C}$ be a a I-commutative monoid, and let $V \in \mathcal{T}$ be an orbit. Let $\iota_V : \mathbb{F} \to \mathbb{F}_{\mathcal{T}}$ send $* \mapsto V$, and let $I_V := I \times_{\mathbb{F}_{\mathcal{T}}} \mathbb{F}$, where the map $\mathbb{F} \to \mathbb{F}_{\mathcal{T}}$ is ι_V. Then, the V*-value* is the pullback

$$\begin{tikzcd} \mathrm{CMon}_I\mathcal{C} \arrow[r, "(-)_V"] \arrow[d, phantom, "\wr"] & \mathrm{CMon}_{I_V}\mathcal{C} \arrow[d, phantom, "\wr"] \\ \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C}) \arrow[r, "\iota_V^*"] & \mathrm{Fun}^{\times}(\mathrm{Span}_{I_V^\infty}(\mathbb{F}), \mathcal{C}) \end{tikzcd}$$

In particular, when I contains all fold maps (e.g. I is an indexing system in the sense of [BH15; Ste24]) and X is an I-commutative monoid, X_V is a commutative monoid in $\mathcal{C}$. ◁

Construction 1.55. Fix $X \in \mathrm{CMon}_{\mathcal{T}} I$ and $f : V \to W$ a map in I. There exists a natural transformation $\alpha_f : \iota_V \to \iota_W$ whose value on n is the copower map $n \cdot V \to n \cdot W$; this induces a natural transformation $N_V^W : (-)_V \implies (-)_W$, which we refer to as the *norm map*. ◁

Construction 1.56. Let $f : V \to W$ be a morphism in $\mathcal{T}$. Recall that $\mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_V$; hence there is a double adjunction

$$\begin{tikzcd} \mathrm{Fun}(\mathrm{Span}(\mathbb{F}_V),\mathcal{C}) \arrow[r, "f^*"] & \mathrm{Fun}(\mathrm{Span}(\mathbb{F}_W),\mathcal{C}) \arrow[l, bend right=40, "f_!"'] \arrow[l, bend left=40, "f_*"] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"] \end{tikzcd}$$

We prove in **??** that all three of these functors restrict to $\mathcal{T}$-commutative monoids. Write $\mathrm{Ind}_V^W = f_!$, $\mathrm{CoInd}_V^W = f_*$, and $\mathrm{Res}_V^W := f^*$. ◁

1.2.5. *I-symmetric monoidal ∞-categories.* We refer to

$$\mathbf{Cat}_I^{\otimes} := \underline{\mathrm{CMon}}_I \mathbf{Cat}$$

as the *$\mathcal{T}$-∞-category of I-symmetric monoidal ∞-categories.* In the case $I = \mathbb{F}_{\mathcal{T}}$, we refer to these simply as *$\mathcal{T}$-symmetric monoidal ∞-categories* and write $\mathbf{Cat}_{\mathcal{T}}^{\otimes} := \mathbf{Cat}_{\mathbb{F}_{\mathcal{T}}}^{\otimes}$.

Notation 1.57. Suppose $S \in \underline{\mathbb{F}}_I$. Associated with the structure map $\mathrm{Ind}_V^{\mathcal{T}} S \to V$ we have functors

$$\bigotimes_U^S : \mathcal{C}_S \to \mathcal{C}_V, \qquad \Delta^S : \mathcal{C}_V \to \mathcal{C}_S$$

called the *S-indexed tensor product and S-indexed diagonal.* We refer to the composite $(-)^{\otimes S} : \mathcal{C}_V \xrightarrow{\Delta^S} \mathcal{C}_S \xrightarrow{\otimes_U^S} \mathcal{C}_V$ as the *S-indexed tensor power.* In the case $\mathrm{Ind}_V^{\mathcal{T}} S = W$ is an orbit (i.e. S is a *transitive V-set*), we write

$$N_W^V := \bigotimes_U^W : \mathcal{C}_W \to \mathcal{C}_V. \qquad ◁$$

Observation 1.58. Suppose S, $|\mathrm{Orb}(S)| \cdot *_V$, and all orbits of S are is I-admissible V-sets. Then, the following path lies in I:

$$\mathrm{Ind}_V^{\mathcal{T}} S \to |\mathrm{Orb}(S)| \cdot V \to V.$$

In algebra, this yields the formula

$$\begin{tikzcd} \mathcal{C}_S \arrow[rr, "\bigotimes_U^S"] \arrow[dr, "(N_U^V -)"'] & & \mathcal{C}_V \\ & \mathcal{C}_V^{\times \mathrm{Orb}(S)} \arrow[ur, "\otimes"'] & \end{tikzcd}$$

i.e. $\bigotimes_U^S X_U \simeq \bigotimes_{U \in \mathrm{Orb}(S)} N_U^V X_U$. Thus, when I is an indexing category, the indexed tensor products in an I-symmetric monoidal ∞-category is are determined by their binary tensor products and norms. ◁

Construction 1.59. By **??**, the *orbits* functor $F_{\mathcal{T}} : \mathbb{F}_{\mathcal{T}} \to \mathbb{F}$ induces a right Kan extension functor

$$\Gamma := \mathrm{Span}(F_{\mathcal{T}})_* : \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_{\mathcal{T}}),\mathcal{C}) \to \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}),\mathcal{C}).$$

In particular, Γ is right adjoint to $\mathrm{Infl}_e^{\mathcal{T}} := \mathrm{Span}(F_{\mathcal{T}})^*$. When $\mathcal{C} = \mathbf{Cat}$, the counit of this adjunction is a natural $\mathcal{T}$-symmetric monoidal functor.

$$\mathrm{Infl}_e^{\mathcal{T}} \Gamma \mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}$$

We refer to the (symmetric monoidal) V-value of this as the *symmetric monoidal V-evaluation*

$$\mathrm{ev}_V : \Gamma \mathcal{C}^{\otimes} \to \mathcal{C}_V^{\otimes}. \qquad ◁$$

1.2.6. *Symmetric monoidal $\mathcal{T}$-∞-categories.* The ∞-category of *symmetric monoidal $\mathcal{T}$-∞-categories* is

$$\mathbf{Cat}^{\otimes}_{\nabla,\mathcal{T}} \simeq \mathrm{Coeff}^{\mathcal{T}}\mathbf{Cat}^{\otimes}_{\infty} \simeq \mathrm{CMon}\mathbf{Cat}_{\mathcal{T}}.$$

Definition 1.60. Suppose $L\mathcal{C} \subset \mathcal{C}$ is a localizing $\mathcal{T}$-subcategory of a symmetric monoidal $\mathcal{T}$-∞-category. We say that L is *compatible with the symmetric monoidal structure* if for each $V \in \mathcal{T}$, the localization L_V is compatible with the symmetric monoidal structure on $\mathcal{C}_V$ in the sense of [HA, Def 2.2.1.6]. ◃

Proposition 1.61. *If L is compatible with the symmetric monoidal structure, there exists a commutative diagram of $\mathcal{T}$-∞-categories*

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "L^{\otimes}"] \arrow[dr, "p"'] & & L\mathcal{C}^{\otimes} \arrow[dl] \\ & (\mathbb{F}_*)_{\mathrm{triv}} & \end{tikzcd}$$

satisfying the following conditions:

(1) *$L\mathcal{C}^{\otimes}$ is a symmetric monoidal $\mathcal{T}$-∞-category and $L^{\otimes}$ is a symmetric monoidal $\mathcal{T}$-functor,*
(2) *the underlying $\mathcal{T}$-functor of $L^{\otimes}$ is $L : \mathcal{C} \to L\mathcal{C}$, and*
(3) *$L^{\otimes}$ possesses a fully faithful and lax $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-adjoint extending the inclusion $L\mathcal{C} \subset \mathcal{C}$.*

Proof. This follows immediately from [NS22, Thm 2.9.2], which we summarize in Theorem 1.87. □

1.3. **The canonical symmetric monoidal structure on I-commutative monoids.** This section is dedicated to the observation that the parameterized presentability results of [Hil24] are sufficiently strong to repeat non-indexed rudiments of [GGN15] in the I-semiadditive setting.

Definition 1.62 (c.f. [Hil24, Thm 6.1.2])**.** A $\mathcal{T}$-∞-category $\mathcal{C}$ is *$\mathcal{T}$-presentable* if it strongly admits finite $\mathcal{T}$-coproducts and the straightening factors as

$$\mathcal{C} : \mathcal{T}^{\mathrm{op}} \to \mathrm{Pr}^{L,\kappa} \to \mathbf{Cat}$$

for some regular cardinal κ. The (nonfull) subcategory

$$\mathrm{Pr}^L_{\mathcal{T}} \subset \mathbf{Cat}_{\mathcal{T}}$$

has objects given by $\mathcal{T}$-presentable ∞-categories and morphisms given by $\mathcal{T}$-left adjoints. ◃

Observation 1.63. The conditions of factoring through $\mathrm{Pr}^{L,\kappa}$, of strongly admitting finite $\mathcal{T}$-coproducts, and of being $\mathcal{T}$-left adjoints are preserved by restriction; hence $\mathrm{Pr}^L_{\mathcal{T}}$ canonically lifts to a (nonfull) $\mathcal{T}$-subcategory

$$\underline{\mathrm{Pr}}^L_{\mathcal{T}} \subset \underline{\mathbf{Cat}}_{\mathcal{T}}$$ ◃

These satisfy an adjoint functor theorem [Hil24, Thm 6.2.1] and have analogous characterizations to the non-equivariant case; in particular, $\mathrm{Pr}^L_{\mathcal{T}} \subset \mathbf{Cat}_{\mathcal{T}}$ is closed under functor categories from small categories [Hil24, Lem 6.7.1] and by Definition 1.62, $\mathrm{Pr}^L_{\mathcal{T}}$ is closed under fiberwise-accessible $\mathcal{T}$-localizations. Hence $\underline{\mathrm{CMon}}_I(\mathcal{C})$ is $\mathcal{T}$-presentable when $\mathcal{C}$ is $\mathcal{T}$-presentable.

Additionally, in [Nar17], a $\mathcal{T}$-symmetric monoidal structure was constructed on $\underline{\mathrm{Pr}}^L_{\mathcal{T}}$, with V unit $\mathcal{S}_V$, with binary tensor products characterized by

$$\mathrm{Fun}^L_{\mathcal{T}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{L,L}_{\mathcal{T}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}),$$

the latter denoting functors which are $\mathcal{T}$-cocontinuous in each variable, and with norms characterized by

$$\mathrm{Fun}^L_W(N^W_V \mathcal{C}, \mathcal{E}) \simeq \mathrm{Fun}^L_V(\mathrm{CoInd}^W_V \mathcal{C}, \mathcal{E}).$$

Definition 1.64. The ∞-category of *presentably $\mathcal{T}$-symmetric ∞ monoidal categories* is the (non-full) subcategory $\mathrm{CAlg}_{\mathcal{T}}(\underline{\mathrm{Pr}}^{L,\otimes}_{\mathcal{T}}) \subset \mathbf{Cat}^{\otimes}_{\mathcal{T}}$; the ∞-category of *presentably symmetric monoidal $\mathcal{T}$-∞-categories* is the (non-full) subcategory $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{T}}) \subset \mathrm{CMon}(\mathbf{Cat}_{\mathcal{T}})$. ◃

Observation 1.65. Let $\mathrm{Fun}^{\delta}_V(\mathcal{C}^{\otimes S}, \mathcal{D})$ be the category of *distributive V-functors* as in [NS22]. Then, by the description in [Nar17], a $\mathcal{T}$-symmetric monoidal ∞-category whose underlying $\mathcal{T}$-category is presentable factors through the inclusion $\underline{\mathrm{Pr}}^L_{\mathcal{T}} \subset \underline{\mathbf{Cat}}_{\mathcal{T}}$ if and only if its structure maps $\mathcal{C}_V^{\otimes S} \to \mathcal{C}_V$ are in $\mathrm{Fun}^{\delta}_V(\mathcal{C}_V^{\otimes S}, \mathcal{C}_V)$; in the language of [NS22], a presentably $\mathcal{T}$-symmetric monoidal ∞-category is precisely a *distributive* $\mathcal{T}$-symmetric monoidal ∞-category whose underlying $\mathcal{T}$-∞-category is presentable. ◃

Hilman used the universal property of $\otimes$ in [Hil24, Prop 6.7.5] to prove the formula

$$\mathcal{C} \otimes \mathcal{D} \simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}). \tag{8}$$

Using this, for any presentable $\mathcal{C}$, we have

$$\begin{aligned}\underline{\mathrm{CMon}}_I(\mathcal{C}) &\simeq \underline{\mathrm{Fun}}^{\times}_{\mathcal{T}}(\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}}), \mathcal{C}) \\ &\simeq \underline{\mathrm{Fun}}^{\times}_{\mathcal{T}}(\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}}), \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\mathcal{C}, \underline{\mathcal{S}}_{\mathcal{T}})) \\ &\simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Fun}}^{\times}_{\mathcal{T}}(\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}}), \underline{\mathcal{S}}_{\mathcal{T}})) \\ &\simeq \mathcal{C} \otimes \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}}).\end{aligned}$$

In particular, this implies that the functor $\mathcal{C} \mapsto \underline{\mathrm{CMon}}_I(\mathcal{C})$ is *smashing*. In fact, we can say more.

Notation 1.66. We say that a presentable $\mathcal{T}$-∞category is *I-semiadditive* if its underlying $\mathcal{T}$-∞-category is I-semiadditive, and we let $\mathrm{Pr}^{L,I-\oplus}_{\mathcal{T}} \subset \mathrm{Pr}^L_{\mathcal{T}}$ be the full subcategory spanned by I-semiadditive presentable $\mathcal{T}$-categories. ◁

It follows from Cnossen-Lenz-Linsken's semiadditive closure theorem [CLL24, Thm B] that $\underline{\mathrm{CMon}}_I(-)$ implements the localization functor

$$\mathrm{Pr}^L_{\mathcal{T}} \to \mathrm{Pr}^{L,I-\oplus}_{\mathcal{T}}$$

left adjoint to the evident inclusion; by the above argument, we find that $\underline{\mathrm{CMon}}_I(-)$ is a *smashing localization*, hence a symmetric monoidal localization; by [GGN15, Lemma 3.6], this implies that given $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{T}})$, there is a unique compatible commutative algebra structure on its localization $\underline{\mathrm{CMon}}_I(\mathcal{C})$. In other words, we've shown the following.

Theorem A'. *The localizing subcategory*

$$\underline{\mathrm{CMon}}_I : \mathrm{Pr}^L_{\mathcal{T}} \rightleftarrows \mathrm{Pr}^{L,I-\oplus}_{\mathcal{T}} : \iota$$

is smashing; in particular, if $\mathcal{C}^{\otimes}$ is a presentably symmetric monoidal $\mathcal{T}$-category, then there is a unique presentably symmetric monoidal $\mathcal{T}$-∞-category $\underline{\mathrm{CMon}}^{\otimes-\mathrm{mode}}_I(\mathcal{C})$ possessing a (necessarily unique) symmetric monoidal lift

$$\mathrm{Fr}^{\otimes} : \mathcal{C}^{\otimes} \to \underline{\mathrm{CMon}}^{\otimes-\mathrm{mode}}_I(\mathcal{C})$$

of $\mathrm{Fr}^{\otimes} : \mathcal{C} \to \underline{\mathrm{CMon}}_I(\mathcal{C})$.

Warning 1.67. Theorem A' is not as *genuinely equivariant* as the user will often want, as it constructs *symmetric monoidal structures*, but never norm maps. The author is content with this for the purposes of this paper, as the algebraic interpretation of indexed tensor products of $\mathcal{T}$-operads is unclear. ◁

Observation 1.68. The $\mathcal{T}$-category $\underline{\mathcal{S}}_{\mathcal{T}}$ is freely generated under $\mathcal{T}$-colimits by one $\mathcal{T}$-point, in the sense that evaluation at $*_{\mathcal{T}}$ yields an equivalence

$$\mathrm{Fun}^L_{\mathcal{T}}(\underline{\mathcal{S}}_{\mathcal{T}}, \mathcal{C}) \simeq \Gamma\mathcal{C}.$$

In particular, every symmetric monoidal $\mathcal{T}$-∞-category receives at most one symmetric monoidal $\mathcal{T}$-left adjoint from $\underline{\mathcal{S}}_{\mathcal{T}}$; in the case $\mathcal{C} = \underline{\mathcal{S}}^{\times}_{\mathcal{T}}$ the condition of Theorem A' then may be read as saying that there is a unique presentably symmetric monoidal structure on $\underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}})$ whose V-value has unit $1^{\mathrm{mode}}_V = \mathrm{Fr}(*_V)$.

Furthermore, by Yoneda's lemma, this unit is characterized by the property that

$$\mathrm{Map}_V(1^{\mathrm{mode}}_V, X_V) \simeq \mathrm{Map}(*_V, X(V)) \simeq X(V).$$ ◁

We'd like to identify this symmetric monoidal structure via a familiar formula. We have a candidate:

Proposition 1.69 ([BS24b, Prop 4.24], via [CHLL24a, Prop 3.3.4])**.** *If $\mathcal{C}$ is presentably symmetric monoidal, then the Day convolution structure on $\mathrm{Fun}(\mathrm{Span}(\mathbb{F}_{\mathcal{T}}), \mathcal{C})$ with respect to the smash product on $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is compatible with the localization*

$$L_{\mathrm{Seg}} : \mathrm{PCMon}_{\mathcal{T}}(\mathcal{C}) \to \mathrm{CMon}_{\mathcal{T}}(\mathcal{C})$$

Proof. By the general criterion [CHLL24a, Prop 3.3.4], it suffices to verify that $A_+ \wedge - : \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is product-preserving, which follows by the fact that it is colimit preserving and $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is semiadditive. □

The rest of this subsection is dedicated to showing that this models the mode symmetric monoidal structure.

Theorem 1.70. *The localization $\mathcal{T}$-functor*
$$L_{\mathrm{Seg}} : \underline{\mathrm{PCMon}}_{\mathcal{T}}^{\mathrm{Day}}(\mathcal{C}) \to \underline{\mathrm{CMon}}_{\mathcal{T}}^{\mathrm{mode}}(\mathcal{C})$$
possesses a unique symmetric monoidal structure, i.e. $\underline{\mathrm{CMon}}_{\mathcal{T}}^{\mathrm{mode}}(\mathcal{C})$ is the localized Day convolution symmetric monoidal structure prescribed by Proposition 1.69.

We explicitly import the following lemma from the proof of [CHLL24a, Prop 3.3.4].

Lemma 1.71. *If $y : \mathrm{Span}(\underline{\mathbb{F}}_{\mathcal{T}}) \to \underline{\mathrm{PCMon}}_{\mathcal{T}}(\mathcal{S})$ denotes the Yoneda embedding, then there is a $\mathcal{T}$-natural equivalence*
$$\hom_V^{\mathrm{PCMon}_I(\mathcal{C})}(y(A), X) \simeq X(A_+ \wedge -).$$

Proof of Theorem 1.70. Applying the diagram
$$\begin{array}{ccc} \underline{\mathrm{PCMon}}_I(\mathcal{C}) & \simeq & \underline{\mathrm{PCMon}}_I(\mathcal{S}) \otimes \mathcal{C} \\ \downarrow{\scriptstyle L_{\mathrm{Seg}}} & & \downarrow{\scriptstyle L_{\mathrm{Seg}}} \\ \underline{\mathrm{CMon}}_I(\mathcal{C}) & \simeq & \underline{\mathrm{CMon}}_I(\mathcal{S}) \otimes \mathcal{C} \end{array}$$
of [CHLL24a, Prop 3.3.4], we find that it suffices to prove this in the case $\mathcal{C} = \underline{\mathcal{S}}_{\mathcal{T}}$.

The $\mathcal{T}$-Yoneda embedding is $\mathcal{T}$-symmetric monoidal for the $\mathcal{T}$-Day convolution by [NS22, Thm 6.0.12], so $1_V^{\mathrm{Day}} \simeq y(*_V)$. Hence Yoneda's lemma and Lemma 1.71 yields that
$$\mathrm{Map}_V(1_V^{\mathrm{Day}}, X_V) \simeq \mathrm{Map}(y(*_V), X) \simeq X(V),$$
which implies that $1^{\mathrm{Day}} \simeq 1^{\mathrm{mode}}$ and hence the theorem by Observation 1.68. □

Remark 1.72. It is not likely that it is necessary for $\mathcal{T}$ to be atomic orbital in the above argument; indeed, for $\underline{\mathrm{CMon}}_I(\mathcal{C}) := \underline{\mathrm{Fun}}_{\mathcal{T}}^{\times}(\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}}), \mathcal{C})$ to implement I-*semiadditivization*, it suffices to assume that I is a weak indexing category with respect to an implicit *atomic orbital subcategory* $\mathcal{P} \subset \mathcal{T}$ (c.f. [CLL23b; CLL24]). Unfortunately, the author is not aware of a symmetric monodial structure on partially presentable $\mathcal{T}$-categories, and developing such a thing would lead us far afield from our current operadic goals. ◁

1.4. **The homotopy I-symmetric monoidal d-category.** Recall that, for $d \geq -1$, a space X is d-*truncated* if $\tilde{\pi}_m X = 0$ for all $m > d$. We say that X is (-2)-*truncated* if it is empty.

Recall that a d-truncated ∞-category (i.e. a $(d+1)$-category) is an ∞-category $\mathcal{C}$ such that $\mathrm{Map}(X, Y)$ is d-truncated for all $X, Y \in \mathcal{C}$. We say that an ∞-category is a -1-*category* if it is either $*$ or empty. In general, we write $\mathbf{Cat}_d \subset \mathbf{Cat}$ for the full subcategory spanned by the ∞-categories with the property that they are d-categories.

Lemma 1.73 ([HTT, Cor 2.3.4.8, Prop 2.3.4.12, Cor 2.3.4.19])**.** *$\mathbf{Cat}_d$ is a $(d+1)$-category; additionally, the inclusion*
$$\mathbf{Cat}_d \hookrightarrow \mathbf{Cat}$$
has a right adjoint $h_d : \mathbf{Cat} \to \mathbf{Cat}_d$.

Definition 1.74. The $\mathcal{T}$-∞-category of small $\mathcal{T}$-d-categories is
$$\underline{\mathbf{Cat}}_{\mathcal{T},d} := \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathbf{Cat}_d.$$
A $\mathcal{T}$-*poset* is a $\mathcal{T}$-0-category. If $I \subset \mathbb{F}_{\mathcal{T}}$ is pullback-stable, the $\mathcal{T}$-∞-*category of small I-symmetric monoidal d-categories* is
$$\underline{\mathbf{Cat}}_{I,d}^{\otimes} := \underline{\mathrm{CMon}}_I \mathbf{Cat}_d. \quad ◁$$

Construction 1.75. By Lemmas 1.33 and 1.73 the functor $\underline{\mathbf{Cat}}_{\mathcal{T},d} \hookrightarrow \underline{\mathbf{Cat}}_{\mathcal{T}}$ is an inclusion of a localizing subcategory; let $h_{\mathcal{T},d} : \underline{\mathbf{Cat}}_{\mathcal{T}} \to \underline{\mathbf{Cat}}_{\mathcal{T},d}$ be the associated left adjoint.

The mapping spaces in a product of categories is the product of the mapping spaces; in particular, the inclusion $\mathbf{Cat}_d \hookrightarrow \mathbf{Cat}$ is product-preserving. Hence Lemmas 1.53 and 1.73 construct an adjunction
$$h_{\mathcal{T},d} : \mathbf{Cat}_I^{\otimes} \rightleftarrows \mathbf{Cat}_{I,d}^{\otimes} : \iota.$$
whose right adjoint is fully faithful. We refer to $h_{\mathcal{T},d}$ as the *homotopy I-symmetric monoidal d-category.* ◁

The remainder of this subsection will be dedicated to constructing recognition results for $\mathcal{T}$-symmetric monoidal d-categories, which will be useful throughout the remainder of the paper. We first reduce this consideration to that of plain $\mathcal{T}$-∞-categories; the following proposition follows by unwinding definitions and noting that $\mathbf{Cat}_d \hookrightarrow \mathbf{Cat}$ is closed under products.

Proposition 1.76. *If $I \subset \mathbb{F}_{\mathcal{T}}$ is a pullback-stable wide subcategory, then $\mathcal{C}^{\otimes} \in \mathbf{Cat}_I^{\otimes}$ is a I-symmetric monoidal d-category if and only if its underlying $\mathcal{T}$-∞-category $\mathcal{C}$ is a $\mathcal{T}$-d-category.*

Often in equivariant higher algebra, we will find that our objects come with natural maps to $\mathcal{T}$-1-categories, and we'd like to develop a recognition theorem in this case in terms of mapping spaces.

Proposition 1.77. *A $\mathcal{T}$-∞-category $\mathcal{C}$ is a $\mathcal{T}$-d-category if and only if*

$$\underline{\mathrm{Mor}}(\mathcal{C})_V := \mathrm{Fun}(\Delta^1, \mathcal{C}_V)^{\simeq}$$

is $(d-2)$-truncated for all $V \in \mathcal{T}$.

Proof. By definition, it suffices to prove this in the case $\mathcal{T} = *$. Fix $f, g \in \mathrm{Mor}(\mathcal{C})$. Then, we may present $\mathrm{Map}(f, g)$ as a disjoint union over a, b of homotopies

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "a"'] & X \arrow[d, "b"] \\ Y \arrow[r, "g"'] \arrow[ur, phantom, "\Nearrow"] & Z \end{tikzcd}$$

For fixed a, b, this is either empty or equivalent to the component of the space $\mathrm{Map}(S^1, \mathrm{Map}(W, Z))$ whose underlying map is homotopic to bf. If $\mathcal{C}$ is a d-category, then this is $(d-2)$-truncated; conversely, choosing $a, b = \mathrm{id}$ and $f = g$, if this is $(d-2)$-truncated for all f, then the mapping spaces of $\mathcal{C}$ are $(d-1)$-truncated, i.e. $\mathcal{C}$ is a d-category. □

Given $\psi : \Delta^1 \to \mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$, define the pullback $\mathcal{T}$-space

$$\begin{tikzcd} \underline{\mathrm{Mor}}_F^{\psi}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Mor}}(\mathcal{C}) \arrow[d] \\ B\underline{\mathrm{Aut}}_{\psi} \arrow[r, hook] & \underline{\mathrm{Mor}}(\mathcal{D}) \end{tikzcd}$$

We say that F has $(d-1)$-*truncated mapping fibers* if $\underline{\mathrm{Mor}}_F^{\psi}(\mathcal{C})$ is $(d-2)$-truncated for all $\psi \in \mathrm{Mor}(\mathcal{C})$.

Corollary 1.78. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a $\mathcal{T}$-functor and $\mathcal{D}$ is a $\mathcal{T}$-1-category. Then, the following are equivalent for $d \geq 1$:*

(1) F has $(d-1)$-truncated mapping fibers.
(2) $\mathcal{C}$ is a $\mathcal{T}$-d-category.

Additionally, the following are equivalent.

(1') F has (-1)-truncated mapping fibers.
(2') F includes $\mathcal{C}$ as a $\mathcal{T}$-subcategory of $\mathcal{D}$.

Proof. The only nontrivial part is the equivalence between (1') and (2'), which follows quickly from Observation 1.12 after unwinding definitions. □

1.5. **Examples of I-symmetric monoidal ∞-categories.** Throughout the following section, we will occasionally use the technology of $\mathcal{T}$-operads developed in [NS22], which we will go on to review in Section 2. Crucially, when I is a weak indexing category, we recognize I-symmetric monoidal ∞-categories as $\mathcal{T}$-operads cocartesian fibered over the weak $\mathcal{N}_\infty$-operad $\mathcal{N}_{I\infty}^{\otimes}$; we refer to maps between the underlying $\mathcal{T}$-operads of I-symmetric monoidal categories as *lax I-symmetric monoidal functors.*

We assure the skeptical reader that no results between this subsection and Section 4 reference the results herein, so the forward references do not create cyclic dependency. This subsection is placed here in order to encourage the reader to go into Section 2 with examples in mind; nevertheless, it would create no logical inconsistencies to read this section shortly before Section 4.

1.5.1. *(Co)cartesian I-symmetric monoidal ∞-categories.* Fix I a unital weak indexing system in the sense of [Ste24]. Denote by $\mathbf{Cat}_I^{\sqcup}, \mathbf{Cat}_I^{\times} \subset \mathbf{Cat}_{\mathcal{T}}$ the non-full subcategories with objects given by $\mathcal{T}$-categories attaining I-indexed coproducts (resp. products) and with morphisms given by $\mathcal{T}$-functors which preserve I-indexed coproducts (products). In Appendix B, we prove the following.

Theorem D'. *There are fully faithful embeddings $(-)^{I-\sqcup}, (-)^{I-\times}$ making the following commute:*

$$\begin{tikzcd} \mathbf{Cat}_I^{\sqcup} \arrow[r, hook, "(-)^{I-\sqcup}"] \arrow[dr, "U"'] & \mathbf{Cat}_I^{\otimes} \arrow[d, dashed, "U"] & \mathbf{Cat}_I^{\times} \arrow[l, hook', "(-)^{I-\times}"'] \arrow[dl, "U"] \\ & \mathbf{Cat}_{\mathcal{T}} & \end{tikzcd}$$

The image of $(-)^{I-\sqcup}$ is spanned by the I-symmetric monoidal ∞-categories whose indexed tensor products are indexed coproducts, and the image of $(-)^{I-\times}$ is spanned by those whose indexed tensor products are indexed products.

We call I-symmetric monoidal ∞-categories of the form $\mathcal{C}^{I-\sqcup}$ *cocartesian*, and $\mathcal{C}^{I-\times}$ *cartesian.*

Observation 1.79. Two opposing functors of ∞-categories $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ are adjoint if and only if the induced functors on their homotopy 1-categories are adjoint [RV17]; in particular, this applied to the pair $(f^*, f_\otimes)$ in a fixed I-symmetric monoidal ∞-category implies that an I-symmetric monoidal category is (co)cartesian if and only if its homotopy I-symmetric monoidal 1-category is (co)cartesian. ◁

Before characterizing the algebras in these, we point out that these are often presentable.

Proposition 1.80. *Suppose $\mathcal{C}$ is a presentable ∞-category*

(1) $\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}$ is I-presentably symmetric monoidal under the cocartesian structure.
(2) If finite products in $\mathcal{C}$ commute with colimits separately in each variable (i.e. it is Cartesian closed), then $\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}$ is I-presentably symmetric monoidal under the cartesian structure.

Proof. It follows from Hilman's characterization of parameterized presentability [Hil24, Thm 6.1.2] that $\underline{\mathrm{Coeff}}^{\mathcal{T}}$ is presentable. By Observation 1.65, in each case we're tasked with proving that the $\mathcal{T}$-symmetric monoidal structures are distributive. The first case is just commutativity of colimits with colimits, and the second is [NS22, Prop 3.2.5]. □

We would like to interpret algebras in $\mathcal{C}^{I-\times}$ purely in terms of $\mathcal{C}$ using the following definition.

Definition 1.81. Fix $\mathcal{O}^{\otimes}$ an I-operad interpreted as a $\mathcal{T}$-∞-category over $\underline{\mathbb{F}}_{I,*}$ (c.f. Appendix A.1) and let $\mathcal{C}$ be a $\mathcal{T}$-∞-category admitting I-indexed products. Then, an *$\mathcal{O}$-monoid in $\mathcal{C}$* is a functor $M : \mathcal{O}^{\otimes} \to \mathcal{C}$ satisfying the condition that, for each $X = (X_U) \in \mathcal{O}_S$, the canonical maps $M(X) \to \mathrm{CoInd}_U^V M(X_U)$ realize $M(X)$ as the indexed product

$$M(X) \simeq \prod_U^S M(X_U). \qquad ◁$$

In Appendix B, we prove the following equivariant lift to [HA, Prop 2.4.2.5].

Proposition 1.82. *The postcomposition functor*

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\times}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}^{\otimes}, \mathcal{C})$$

is fully faithful with image spanned by the $\mathcal{O}$-monoids.

The terminal I-operad is $\mathcal{N}_{I\infty}^{\otimes}$, so we set the notation $\underline{\mathrm{CAlg}}_I(\mathcal{C}) := \underline{\mathbf{Alg}}_{\mathcal{N}_{I\infty}}(\mathcal{C})$ for the $\mathcal{T}$-∞-category of *I-commutative algebras in $\mathcal{C}$*. Of fundamental importance is the following corollary to Proposition 1.82, which interprets I-commutative monoids as *operad algebras.*

Corollary 1.83 ("CMon = CAlg")**.** *There is a canonical equivalence $\underline{\mathrm{CMon}}_I(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_I(\mathcal{C}^{I-\times})$ over $\mathcal{C}$.*

Proof. By Proposition 1.82, I-commutative algebras in $\mathcal{C}^{I-\times}$ are I-semiadditive functors $\underline{\mathbb{F}}_{I,*} \to \mathcal{C}$. Our proof is similar to that of [Nar16, Thm 6.5]; There is a pullback square over $\mathcal{C}$

$$\begin{tikzcd}
\mathrm{CMon}_I(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{CAlg}_I(\mathcal{C}^{I-\times}) \arrow[r, phantom, "\simeq"] & \mathrm{Fun}^{I-\oplus}(\underline{\mathbb{F}}_{I,*}, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}})) \arrow[rr] & & \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{I-\oplus}(\underline{\mathbb{F}}_{I,*}, \underline{\mathcal{S}}_{\mathcal{T}}))
\end{tikzcd}$$

so it suffices to prove this in the case $\mathcal{C} = \underline{\mathcal{S}}_{\mathcal{T}}$. There, we simply compose equivalences as follows

$$\begin{tikzcd}
\mathrm{CMon}_I(\underline{\mathcal{S}}_{\mathcal{T}}) \arrow[rrr, "\sim"] \arrow[d, "1.51"'] & & & \mathrm{CAlg}_I(\mathcal{C}^{I-\times}) \\
\mathrm{CMon}_I(\mathcal{S}) \arrow[r, "A.6"] & \mathrm{Seg}_{\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}(\mathcal{S}) \arrow[r, "A.10"] & \mathrm{Seg}_{\underline{\mathbb{F}}_{I,*}} \arrow[r, "A.9"] & \mathrm{Fun}_{\mathcal{T}}^{I-\oplus}(\underline{\mathbb{F}}_{I,*}, \underline{\mathcal{S}}) \arrow[u, "1.82"]
\end{tikzcd}$$

noting that each arrow is marked with a reference proving that it's an equivalenc. □

Remark 1.84. As with much of the rest of this subsection, Corollary 1.83 possesses an alternative strategy where both are shown to furnish the I-semiadditive closure, the latter using [CLL24, Thm B]. The above argument was chosen for brevity, as its requisite parts are also needed elsewhere. ◁

Remark 1.85. In the case $\mathcal{C} \simeq \underline{\mathcal{S}}_G$, the analogous result was recently proved in [Mar24] for the ∞-category of algebras over the *graph G-operads* corresponding with indexing systems. To the knowledge of the author, this is one of the first concrete indications that the genuine operadic nerve of [Bon19] may induce equivalences between ∞-categories of algebras. ◁

Example 1.86. We briefly comment on why one may expect Corollary 1.83 in the context of of traditonal equivariant algebra. In order to set this up, recall that the $C_p = \mathbb{Z}/p\mathbb{Z}$-orbit category is the following:

$$\left\langle \begin{tikzcd} {[C_p/e]} \arrow[loop left, "\tau"] \arrow[r, "r"] & *_{C_p} \end{tikzcd} \;\middle|\; \tau^p = \mathrm{id}, \quad r = r\tau \right\rangle;$$

in particular, a C_p-coefficient system of sets is precisely a pair of sets X_e, X_{C_p}, an order-p-permutation of X_e, and a map $X_{C_p} \to X_e^{hC_p}$ which is C_p-equivariant for the trivial action on the codomain.[14] Coinduction in this setting is given by

$$\begin{tikzcd} X^p \arrow[loop left, "\tau^*"] & X \arrow[l, "\Delta"'] \end{tikzcd}$$

where τ^* permutes the factors. One can see this by noting that this presents $\mathrm{Map}(C_p/e, X)$, where C_p acts on the domain. If $Y \in \mathrm{Coeff}^{C_p}\mathbf{Set}$ is a C_p-coefficient system, then a map $Y_{C_p}^{a+b[C_p/e]} \to Y_{C_p}$ has signature

$$Y_{C_p}^a \times Y_e^b \to Y_{C_p}.$$

[14] The notation of homotopy fixed points were placed here to remind the viewer that they are computed as the fixed points of the Borel action on X_e, not due to any nontrivial homotopical considerations; following Elmendorf's theorem, some authors refer to X_{C_p} as the genuine C_p-fixed points of the coefficient system, which is a terminological collision we would like to avoid.

The applicable associativity law for I-commutative algebras (with C_p-set arities) dictates that, for all subdiagrams of the following that exist, they commute

$$\begin{tikzcd} Y_e^a \times Y_{C_p}^b \arrow[r] \arrow[d] \arrow[dr] & Y_e \times Y_{C_p}^b \arrow[r] \arrow[d] & Y_{C_p}^{b+1} \arrow[dl] \\ Y_e^a \times Y_{C_p} \arrow[r] \arrow[d] & Y_{C_p} & \\ Y_{C_p}^{a+1} \arrow[ur] & & \end{tikzcd}$$

By [Ste24], there are six unital C_p weak indexing systems. For variety, we describe $I = A\lambda$ for λ a nontrivial irredicible real orthogonal C_p-representation; thus given an $A\lambda$-commutative monoid we have maps $Y_e^n \to Y_e$ for all n, and maps $Y_{C_p}^a \times Y_e^b \to Y_{C_p}$ if and only if $a \le 1$.

Note that the data of a (strict) $A\lambda$-commutative algebra structure on Y is dictated by the unit elements $Y_e \leftarrow * \to Y_{C_p}$, the multiplication map $Y_e^2 \to Y_e$, the transfer map $Y_e \to Y_{C_p}$, and the action map $Y_{C_p} \times Y_e \to Y_{C_p}$. These are subject to the associativity/unitality condition that all maps $Y_{C_p}^a \times Y_e^b \to Y_{C_p}^{a'} \times Y_e^{b'}$ constructed out of composites of products of such maps agree; by closure of $\underline{\mathbb{F}}_{A\lambda}$ under self-indexed coproducts, maps occur in those arities if and only if the map of arities $a + b\left[C_p/e\right] \to a' + b'\left[C_p/e\right]$ is in $A\lambda$. Unwinding definitions, this is exactly the data of an $A\lambda$-commutative monoid. ◁

The cocartesian situation is more simple: the forgetful functor $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\sqcup}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O},\mathcal{C})$ is an *equivalence*. We study this more fully in Appendix B and Section 4.2.

1.5.2. *Constructing I-symmetric monoidal ∞-categories from other I-symmetric monoidal ∞-categories.* Fix I a one-object weak indexing system; that is, we assume that $*_V$ is I-admissible for all $V \in \mathcal{T}$, so that I-commutative monoids have underlying $\mathcal{T}$-objects. In this subsection, we review some known equivariant lifts to [HA, § 2.2.1].

When $\mathcal{C}^{\otimes} \subset \mathrm{Op}_I$ is an I-operad and $\mathcal{D} \subset \mathcal{C}$ is a full $\mathcal{T}$-subcategory. let $\mathcal{D}^{\otimes} \subset \mathcal{C}^{\otimes}$ be the full subcategory spanned by the objects belonging to

$$\mathcal{D}_S \coloneqq \prod_{U \in \mathrm{Orb}(S)} \mathcal{D}_U \subset \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_U \simeq \mathcal{C}_S.$$

Note that the composite map $\mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes} \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ presents an I-operad by construction.

Theorem 1.87 ([NS22, § 2.9])**.** *Let $L^{\otimes} : \mathcal{B}^{\otimes} \to \mathcal{C}^{\otimes}$ be an I-symmetric monoidal functor and let $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ be a full $\mathcal{T}$-subcategory. Then,*

(1) (Doctrinal adjunction) Suppose the underlying $\mathcal{T}$-functor L of $L^{\otimes}$ participates in a $\mathcal{T}$-adjunction

$$L : \mathcal{B} \rightleftarrows \mathcal{C} : R$$

Then, $L^{\otimes}$ has a unique lax I-symmetric monoidal right adjoint (i.e. map of I-operads) $R^{\otimes}$ prolonging R.

(2) (Full subcategories) Suppose that, for all $S \in \mathbb{F}_{I,V}$, the S-indexed tensor functor

$$^{\mathcal{C}}\bigotimes^{S} : \mathcal{C}_S \to \mathcal{S}_V$$

restricts to a functor $^{\mathcal{D}}\bigotimes^{S} : \mathcal{D}_S \to \mathcal{D}_V$. Then, the I-operad $\mathcal{D}^{\otimes}$ constructed above is an I-symmetric monoidal category, and the inclusion $\mathcal{D}^{\otimes} \hookrightarrow \mathcal{C}^{\otimes}$ is a symmetric monoidal functor prolonging i; furthermore, $\mathcal{D}^{\otimes}$ is the unique I-symmetric monoidal category over $\mathcal{C}^{\otimes}$ prolonging i.

(3) (Localization) Suppose ι has a left adjoint $L : \mathcal{C} \to \mathcal{D}$ such that $^{\mathcal{C}}\otimes^{S} : \mathcal{C}_S \to \mathcal{C}_V$ preserves L-equivalences. Then, $\mathcal{D}$ attains a I-symmetric monoidal structure together with an I-symmetric functor $L^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ prolonging L. Furthermore, the associated lax I-symmetric monoidal structure on i is symmetric monoidal if and only if $\mathcal{D}$ satisfies the conditions of part (2).

In particular, if $\mathcal{D}$ is an I-symmetric monoidal localization, then its indexed tensor functors are computed by

$$^{\mathcal{D}}\bigotimes_{U}^{S} X_U \simeq L\left({}^{\mathcal{C}}\bigotimes_{U}^{S} X_U \right).$$

Proof. (1) follows from [HA, Prop 7.3.2.6] on opposite categories. (2) is [NS22, Prop 2.9.1] and (3) is [NS22, Thm 2.9.2]. The final statement follows by noting that the composite $\mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is the identity, hence it is symmetric monoidal. □

1.5.3. *The pointwise $\mathcal{T}$-symmetric monoidal structure.* Once more fix I a one-object weak indexing system. In classical algebra, there are two well-known tensor products of functors $F, G : \mathcal{C} \to \mathcal{D}$: when $\mathcal{D}$ is monoidal, the *pointwise tensor product* sets $F \otimes G(C) := F(C) \otimes G(C)$, and when additionally $\mathcal{C}$ is monoidal, the *Day convolution product* sets $F \circledast G(-)$ to be the left Kan extension of the functor $F(-) \otimes G(-) : \mathcal{C}^2 \to \mathcal{D}$ along the tensor functor $\mathcal{C}^2 \to \mathcal{D}$.

[NS22] has equivariantly lifted of both structures. We first review pointwise indexed tensor products.

Theorem 1.88 ([NS22, Thm 3.3.1, 3.3.3])**.** *Let $\mathcal{K}$ be a $\mathcal{T}$-∞-category, and $\mathcal{C}^{\otimes}$ a $\mathcal{T}$-operad. Then, there exists a unique (functorial) I-operad structure $\mathrm{Fun}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}$ on $\mathrm{Fun}_{\mathcal{T}}(\mathcal{K},\mathcal{C})$ satisfying the universal property*

$$\mathbf{Alg}_{\mathcal{O}}(\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C}))$$

for $\mathcal{O} \in \mathrm{Op}_I$. Furthermore, when $\mathcal{C}^{\otimes}$ is I-symmetric monoidal, $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}$ is I-symmetric monoidal and satisfies the universal property

$$\mathrm{Fun}_{\mathcal{T}}^{I-\otimes}\left(\mathcal{D}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}\right) \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{K}, \underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})\right).$$

If S is I-admissible, then the S-indexed pointwise tensor product has values

$$\begin{tikzcd} \mathcal{D}_V \arrow[r, "\Delta^S"] \arrow[rrr, bend right, "\bigotimes_U^S F_U"'] & \mathcal{D}_S \arrow[r, "(F_U)"] & \mathcal{C}_S \arrow[r, "\bigotimes^S"] & \mathcal{C}_V \end{tikzcd}$$

Observation 1.89. Suppose $F : \mathcal{K}' \to \mathcal{K}$ is a functor. Then, the restriction and left Kan extension natural transformations

$$F_! : \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{K}', \underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})\right) \rightleftarrows \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{K}, \underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})\right) : F^*$$

yield I-symmetric monoidal functors $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}',\mathcal{C})^{\otimes-\mathrm{ptws}} \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}$ prolonging the left Kan extension and restriction functors between functor categories. In particular, give $X \in \Gamma^{\mathcal{T}}\mathcal{K}$ this yields a symmetric monoidal lift $\mathrm{ev}_X : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}} \to \mathcal{C}^{\otimes}$ of the ordinary evaluation $\mathcal{T}$-functor $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\{X\},\mathcal{C}) \simeq \mathcal{C}$. ◃

This establishes the pointwise structure as the *cotensor* in the ∞-category of I-symmetric monoidal ∞-categories. We reserve a similar name for the exponential objects in $\mathbf{Cat}_I^{\otimes}$

Definition 1.90. Fix $\mathcal{D}^{\otimes}, \mathcal{C}^{\otimes}$ a pair of I-symmetric monoidal categories. An I-symmetric monoidal category $\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})^{\otimes-\mathrm{ptws}}$ will be called the *pointwise I-symmetric monoidal ∞-category of I-symmetric monoidal functors from $\mathcal{D}$ to $\mathcal{C}$* if it satisfies the universal property

$$\mathrm{Fun}^{I-\otimes}(\mathcal{E}, \underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})^{\otimes-\mathrm{ptws}}) \simeq \mathrm{Fun}^{I-\otimes}(\mathcal{E} \times \mathcal{D}, \mathcal{C}).$$ ◃

We verify in Proposition 4.25 that the I-symmetric monoidal structure $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ of Theorem B is the pointwise I-symmetric monoidal ∞-category of I-symmetric monoidal functors from the *I-symmetric monoidal envelope* $\mathrm{Env}_I \mathcal{O}^{\otimes}$ to $\mathcal{C}$, the former being defined in Section 2.5.[15] This is useful in part because it is naturally compatible with the pointwise structure on functors:

[15] The author suspects that this structure exists for arbitrary I-symmetric monoidal ∞-categories $\mathcal{D}^{\otimes}, \mathcal{C}^{\otimes}$ by a similar construction to that of [NS22, § 3.3]. However, as can be anticipated by the fact that such structure does not appear in the nonequivariant case in [HA], this is not necessary for our present purposes, and would lead us too far afield to develop here.

Observation 1.91. Whenever $\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})^{\otimes-\mathrm{ptws}}$ exists and its underlying $\mathcal{T}$-∞-category is $\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})$, the forgetful natural transformation

$$\mathrm{Fun}_{\mathcal{T}}^{I-\otimes}\left(\mathcal{E},\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})^{\otimes-\mathrm{ptws}}\right)\to \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{E},\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})\right)$$

yields an I-symmetric monoidal functor $\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})^{\otimes-\mathrm{ptws}}\to\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{D},\mathcal{C})^{\otimes-\mathrm{ptws}}$ by Yoneda's lemma, and this prolongs the forgetful $\mathcal{T}$-functor $\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})\to\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{D},\mathcal{C})$. ◁

1.5.4. *Equivariant Day convolution.* Once again fix I a one-color weak indexing category. The other structure we recall is *Day convolution.*

Definition 1.92. Let $\mathcal{O}^\otimes,\mathcal{P}^\otimes$ be I-operads. Then, the *Day convolution I-operad* $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\otimes-\mathrm{Day}}$, if it exists, is the unique I-operad possessing a natural equivalence

$$\mathbf{Alg}_{\mathcal{Q}}\left(\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\otimes-\mathrm{Day}}\right)\simeq\mathbf{Alg}_{\mathcal{Q}\times\mathcal{P}}(\mathcal{O})$$

for all $\mathcal{Q}^\otimes\in\mathrm{Op}_I$. ◁

Remark 1.93. $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\otimes-\mathrm{Day}}$ is the *exponential object* in I-operads from $\mathcal{P}$ to $\mathcal{O}$; said another way, if $\mathcal{O}^\otimes,\mathcal{P}^\otimes$ are I-symmetric monoidal ∞-categories, then commutative algebras in $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\otimes-\mathrm{Day}}$ correspond with lax I-symmetric monoidal functors $\mathcal{P}^\otimes\to\mathcal{O}^\otimes$. ◁

Theorem 1.94 (Day convolution monoidal case)**.** *Suppose $\mathcal{O}^\otimes$ is an I-symmetric monoidal category and $\mathcal{C}^\otimes$ is a I-operad. Then, the I-operad $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\otimes-\mathrm{Day}}$ exists and satisfies the following properties:*

(1) The functor $\mathcal{C}\mapsto\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\otimes-\mathrm{Day}}$ is the right adjoint in an adjoint pair

$$(-)\times\mathcal{O}^\otimes:\mathrm{Op}_I \rightleftarrows \mathrm{Op}_I:\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},-)^{\otimes-\mathrm{Day}};$$

(2) the underlying $\mathcal{T}$-∞-category of $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\otimes-\mathrm{Day}}$ is $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})$;

(3) For all $S\in V$ and $\underline{V}$-functors $\mathcal{O}_{\underline{V}}\to\mathcal{C}_{\underline{V}}$, there exists a $\underline{V}$-left Kan extension diagram

$$\begin{tikzcd} \mathcal{O}_S \arrow[r, "(F_U)"] \arrow[d, "\otimes^S"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{C}_S \arrow[r, "\otimes^S"] & \mathcal{C}_V \\ \mathcal{O}_V \arrow[urr, bend right, "\circledast^S_U F_U"'] & & \end{tikzcd}$$

where $\circledast^S:\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})_S\to\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})_V$ is the S-indexed tensor functor.

(4) If $\mathcal{C}$ is a presentably I-symmetric monoidal ∞-category, then $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\otimes-\mathrm{Day}}$ is a presentably I-symmetric monoidal ∞-category.

(5) $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O},\mathcal{C})^{\otimes-\mathrm{Day}}$ is contravariantly functorial in $\mathcal{O}^\times$ and covariantly functorial in $\mathcal{C}$.

1.5.5. *The smash product of pointed $\mathcal{T}$-spaces.* Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category possessing a terminal object $*_{\mathcal{C}}$. Then, the slice category $\mathcal{C}_{/*}\simeq\mathcal{C}_*$ embeds as a localizing subcategory

$$\mathcal{C}_*\subset\underline{\mathrm{Fun}}_{\mathcal{T}}(\Delta^1\times\mathcal{T}^{\mathrm{op}},\mathcal{C}). \tag{9}$$

In [NS22, Ex 3.2.8], $\Delta^1\times\mathcal{T}^{\mathrm{op}}$ is given a $\mathcal{T}$-symmetric monoidal structure satisfying the condition that the associated Day convolution structure is compatible with the localization left adjoint to (9). Thus, $\mathcal{C}_*$ possesses a symmetric monoidal structure; we suspect that an analog of the argument of [GGN15] will show that this is uniquely determined by its unit.

In any case, the localization functor $\mathrm{Fun}_{\mathcal{T}}(\Delta^1\times\mathcal{T}^{\mathrm{op}},\mathcal{C})\to\mathcal{C}_*$ is computed by the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ *_{\mathcal{T}} \arrow[r, "Lf"'] & LY \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

In particular, we arrive at the formulas

$$X\wedge Y\simeq X\times Y/X\vee Y \qquad\qquad N_V^W X\simeq\mathrm{CoInd}_V^W X.$$

For instance, in the case $\mathcal{T} = \mathcal{O}_G$, we have pointed representation spheres S^V for all real orthogonal G-representations V; the above formulas compute the indexed tensor products

$$\bigwedge_{G/H_i}^{T} S^{V_i} \simeq S^{\bigoplus_{G/H_i}^{T} V_i}.$$

1.5.6. *The box product of I-commutative monoids and I-spectra.* The *spectral Mackey functor theorem* of [GM17] stipulates that

$$\mathrm{CMon}_G(\mathrm{Sp}) \simeq \lim\left(\cdots \xrightarrow{\Omega^\rho} \mathcal{S}_G \xrightarrow{\Omega^\rho} \mathcal{S}_G\right)$$

whenever G is a finite group. We refer to the result of this as Sp_G. It was noted in [Nar16] that this satisfies a universal property of G-*stability*, which we may generalize to $\mathcal{T}$.

Definition 1.95 (C.f. [CLL23a, Def 6.2.2])**.** Let I be an indexing system. Then, a $\mathcal{T}$-∞-category $\mathcal{C}$ is I-*stable* if it is I-semiadditive factors as

$$\mathcal{T}^{\mathrm{op}} \to \mathbf{Cat}^{\mathrm{St}} \hookrightarrow \mathbf{Cat},$$

i.e. it's *fiberwise-stable.* ◁

If $\mathcal{K}$ consists of I-product diagrams and finite fiberwise limits, then we denote by $\mathbf{Cat}_{\mathcal{T}}^{I-lex} := \mathbf{Cat}_{\mathcal{T}}^{\mathcal{K}-lex}$ the ∞-category of $\mathcal{T}$-∞-categories with finite fiberwise limits and I-products, and $\mathbf{Cat}_{\mathcal{T}}^{I-st} \subset \mathbf{Cat}_{\mathcal{T}}^{I-lex}$ the full subcategory spanned by I-stable $\mathcal{T}$-∞-categories.

We denote by $\mathrm{Sp}\otimes(-)$ the postcomposition functor

$$\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Cat}^{lex}) \to \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Cat}^{st}).$$

Proposition 1.96 ([CLL23a, Cor 6.2.6]; c.f. [BH21b, Def 1.5] or [Nar16, Thm 7.4])**.** *The fully faithful inclusion* $\mathbf{Cat}_{\mathcal{T}}^{I-st} \hookrightarrow \mathbf{Cat}_{\mathcal{T}}^{I-lex}$ *has a* right *adjoint given by* $\mathrm{CMon}_I(\mathrm{Sp}\otimes -)$.

In particular, this presents $\mathrm{Sp}_I := \mathrm{CMon}_I(\mathrm{Sp})$ as the I-*stabilization of $\mathcal{T}$-spaces.* We'll endow this with an equivariant symmetric monoidal structure, for which we need a definition.

By cite, the functor $\mathrm{Span}_{-,-}(-) : \mathrm{Trip}^{\mathrm{adeq}} \to \mathbf{Cat}$ is compatible with pullbacks. In particular, it sends triples of I-*symmetric monoidal categories* to I-symmetric monoidal categories. Recall the following definition.

Definition 1.97 ([BH22; Ste24])**.** A pair of $\mathcal{T}$-weak indexing category (I_a, I_m) is *compatible* if $\underline{\mathbb{F}}_{I_a} \subset \underline{\mathbb{F}}_{\mathcal{T}}^{\times}$ is an I_m-symmetric monoidal subcategory. ◁

Remark 1.98. By [Ste24], the subposet of weak indexing categories I_m such that (I_a, I_m) is compatible is a lower-ray $\mathrm{wIndex}_{\mathcal{T}, \leq m(I_a)}$ for the indexing category defined by

$$\underline{\mathbb{F}}_{m(I_a),V} = \left\{ S \in \mathbb{F}_V \mid \underline{\mathbb{F}}_{I_a} \subset \underline{\mathbb{F}}_{\mathcal{T}} \text{ closed under } S\text{-indexed products} \right\}. \qquad ◁$$

It follows from this that $(\underline{\mathbb{F}}_{\mathcal{T}}, \underline{\mathbb{F}}_{I_a}, \underline{\mathbb{F}}_{\mathcal{T}}) \subset (\underline{\mathbb{F}}_{\mathcal{T}}, \underline{\mathbb{F}}_{\mathcal{T}}, \underline{\mathbb{F}}_{\mathcal{T}})$ is an I_m-symmetric monoidal sub-adequate triple; hence $\mathrm{Span}_{-,-}(-)$ induces a map of I_m-symmetric monoidal categories.

$$\mathrm{Span}_{I_a}(\underline{\mathbb{F}}_{\mathcal{T}}) \subset \mathrm{Span}(\underline{\mathbb{F}}_{\mathcal{T}}).$$

Observation 1.99. Fix $\mathcal{C}$ a presentably I_m-symmetric monoidal ∞-category. Then, left Kan extension preserves product-preserving functors; hence the I_m-symmetric Day convolution structure preserves the full subcategory

$$\underline{\mathrm{CMon}}_{I_a}(\mathcal{C}) \subset \mathrm{Fun}(\mathrm{Span}_{I_a}(\underline{\mathbb{F}}_{\mathcal{T}}), \mathcal{C}),$$

yielding an I_m-symmetric monoidal Day convolution structure on $\underline{\mathrm{CMon}}_{I_a}(\mathcal{C})$. In analogy to Lewis' unpublished notes on the theory of Green functors [Lew81], we refer to this as the *indexed box product*, and write

$$\underline{\mathrm{CMon}}_{I_a}(\mathcal{C})^{\square} := \underline{\mathrm{CMon}}_{I_a}(\mathcal{C})^{\otimes-\mathrm{Day}}; \qquad \square^S := \circledast^S. \qquad ◁$$

Remark 1.100. When $\mathcal{C} = \underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{D})$, recall that Cnossen-Lenz-Linsken's result Theorem 1.51 yields an equivalence

$$\mathrm{CMon}_{I_a}(\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{D})) \simeq \mathrm{Fun}^{\times}(\mathrm{Span}_{I_a}(\mathbb{F}_{\mathcal{T}}), \mathcal{D});$$

in particular, unwinding definitions we find that there is a left Kan extension

$$\begin{tikzcd}
\prod\limits_{U \in \mathrm{Orb}(S)} \mathrm{Span}_{I_a}(\mathbb{F}_U) \arrow[r, "(M_U)"] \arrow[d, "(\mathrm{CoInd}_U^V)_{U \in \mathrm{Orb}(S)}"'] & \mathcal{D}^{\times \mathrm{Orb}(S)} \arrow[r, "\otimes"] \arrow[d, phantom, "\Rightarrow"] & \mathcal{D} \\
\mathrm{Span}_{I_a}(\mathbb{F}_V) \arrow[urr, bend right, "\boxtimes_U^S M_U"'] & &
\end{tikzcd}$$

Define the V-geometric fixed points of $M : \mathrm{Span}_{I_a}(\mathbb{F}_V) \to \mathcal{D}$ to be the left Kan extension of M along the "span of fixed points" functor $r : \mathrm{Span}_{I_a}(\mathbb{F}_V) \to \mathrm{Span}(\mathbb{F}_V)$. Composition of left Kan extensions then computes the geometric fixed points formulas

$$\begin{tikzcd}
\mathrm{Span}_{I_a}(\mathbb{F}_U) \arrow[rr, "M"] \arrow[d, "\mathrm{CoInd}_U^V"] \arrow[dd, bend right=60, "r_U"'] \arrow[dr, phantom, "\Rightarrow"] & & \mathcal{D} \\
\mathrm{Span}_{I_a}(\mathbb{F}_V) \arrow[urr, "N_U^V M"] \arrow[d, "r_V"'] \arrow[r, phantom, "\Rightarrow"] & {} & \\
\mathrm{Span}(\mathbb{F}) \arrow[uurr, bend right, "\Phi^V N_U^V M \simeq \Phi^U M"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Span}_{I_a}(\mathbb{F}_V)^{\times 2} \arrow[r, "(M,N)"] \arrow[d, "\wedge"'] \arrow[rrr, bend left, equals] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{D}^{\times 2} \arrow[r, "\otimes"] & \mathcal{D} \arrow[rrr, bend left, equals] & \mathrm{Span}_{I_a}(\mathbb{F}_V)^{\times 2} \arrow[r, "(M,N)"] \arrow[d, "(r_V, r_V)"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{D}^{\times 2} \arrow[r, "\otimes"] & \mathcal{D} \\
\mathrm{Span}_{I_a}(\mathbb{F}_V) \arrow[urr, "M \boxtimes N"] \arrow[d, "r_V"'] \arrow[r, phantom, "\Rightarrow"] & {} & & \mathrm{Span}(\mathbb{F})^{\times 2} \arrow[ur, "(\Phi^V M, \Phi^V N)"] \arrow[d, "\wedge"'] \arrow[r, phantom, "\Rightarrow"] & {} & \\
\mathrm{Span}(\mathbb{F}) \arrow[uurr, bend right, "\Phi^V(M \boxtimes N)"'] \arrow[rrr, bend right, equals] & & & \mathrm{Span}(\mathbb{F}) \arrow[uurr, bend right, "\Phi^V M \otimes \Phi^V N"'] & &
\end{tikzcd}$$

In particular, this yields the formula

$$\Phi^V \boxtimes_U^S M_U \simeq \bigotimes_{U \in \mathrm{Orb}(S)} \Phi^U M_U,$$

extending the formulae of [HHR16, Prop B.199, Prop B.209]. ◃

Notably, if (I_m, I_a) is a compatible pair of weak indexing systems, then Observation 1.99 constructs an I_m-symmetric monoidal structure on Sp_{I_a}; the author expects that this will satisfy an I_m-symmetric monoidal universal property akin to that of spectra developed in [GGN15]. Before then, we summarize another result from the literature.

Recollection 1.101. Let $I \subset \mathbb{F}_{\mathcal{T}}$ be a pullback-stable subcategory, so that $(\mathbb{F}_{\mathcal{T}}, I)$ is a span pair in the sense of [EH23]. Then, note that $\mathbb{F}_{\mathcal{T}}$ is an ∞-topos, so it is locally cartesian closed; hence [EH23, Rmk 2.4.7] implies that $(\mathbb{F}_{\mathcal{T}}, I, \mathbb{F}_{\mathcal{T}})$ is a bispan triple in the sense of [EH23, Def 2.4.3], i.e. it possesses an ∞-category

$$P_I^{\mathcal{T}} := \mathrm{Bispan}_{I,all}(\mathbb{F}_{\mathcal{T}})$$

whose core is that $\mathbb{F}_{\mathcal{T}}$, whose morphisms are bispans in $\mathbb{F}_{\mathcal{T}}$

$$S \leftarrow X \xrightarrow{f} Y \to T$$

with $f \in I$, and whose 2-cells are equivalences

$$\begin{tikzcd} & X \arrow[r] \arrow[d, "\sim"'] & Y \arrow[dr] \arrow[d, "\sim"'] & \\ S \arrow[ur] \arrow[dr] & & & T \\ & X' \arrow[r] & Y' \arrow[ur] & \end{tikzcd}$$

and whose composition is defined by outer bispan in the following diagram:

$$\begin{tikzcd} & & & V \times_S (Y \times_Z (g_* X \times_S Y)) \arrow[r] \arrow[dl] \arrow[dddlll, bend right] \arrow[rr, bend left] \arrow[d, phantom, "\lrcorner" rotate=-45] & Y \times_Z (g_* X \times_S Y) \arrow[r] \arrow[dl] \arrow[dd] \arrow[ddr, phantom, "\lrcorner"] & g_* X \times_S Y \arrow[dd, "g"] \arrow[dddr] & \\ & & V \times_S Y \arrow[r] \arrow[dl] \arrow[d, phantom, "\lrcorner" rotate=-45] & X \times_S Y \arrow[dl] \arrow[dr, "\pi"] \arrow[d, phantom, "\lrcorner" rotate=-45] & & & \\ & V \arrow[r] \arrow[dl] & X \arrow[dr] & & Y \arrow[r, "g"] \arrow[dl] & Z \arrow[dr] & \\ R & & & S & & & T \end{tikzcd}$$

(c.f. [EH23, Thm 2.5.1] and [BH18, Def 2.7]). Then, the ∞ category of *homotopical I-Tambara functors in $\mathcal{C}$* is the functor category

$$\mathrm{Tamb}_I(\mathcal{C}) := \mathrm{Fun}^{\times}(P_I^{\mathcal{T}}, \mathcal{C}).$$

Furthermore, there is a product-preserving functor $\iota : \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to P_I^{\mathcal{T}}$ which sends

$$\begin{tikzcd} S & X \arrow[l] \arrow[r] \arrow[d, mapsto] & T & \\ S & X \arrow[l] \arrow[r] & T \arrow[r, equal] & T \end{tikzcd}$$

In particular, this induces a forgetful functor $U = \iota^* : \mathrm{Tamb}_I(\mathcal{C}) \to \mathrm{CMon}_{\mathcal{T}}(\mathcal{C})$. $\triangleleft$

The following theorem was proved in the discrete setting for $\mathcal{T} = \mathcal{O}_G$ independently by Chan and Vekemens [Cha24; San23], and will appear for general $\mathcal{T}$ and I in upcomming work of Cnossen-Haugseng-Lenz-Linskens

Corollary 1.102. *There is a canonical equivalence*

$$\mathrm{CAlg}_I(\mathrm{CMon}_{\mathcal{T}}^{\square}(\mathcal{C})) \simeq \mathrm{Tamb}_I(\mathcal{C})$$

2. Equivariant operads and their Boardman-Vogt tensor products

In Section 2.1, we begin by recalling rudiments of the theory of *algebraic patterns and Segal objects* of [CH21] together with the theory of *fibrous patterns and the Segal envelope* of [BHS22]; in the case of $\mathfrak{O} = \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$, we show in Appendix A.1 that this recovers the theory of $\mathcal{T}$-symmetric monoidal ∞-categories, $\mathcal{T}$-∞-operads (henceforth $\mathcal{T}$-operads), and the $\mathcal{T}$-symmetric monoidal envelope of [NS22].

Using the language of fibrous patterns, in Section 2.2 we define the *Boardman Vogt tensor product*, and we show that it's closed and compatible with the Segal envelope in Propositions 2.35 and 2.38. Following this, in Section 2.3, we work out some basic structure and properties of I-operads, including the construction of the *weak $\mathcal{N}_\infty$-operads* of Theorem C. Then, in Section 2.4, we characterize the $\overset{\mathrm{BV}}{\otimes}$-unit of Op_I, and leverage this to compute the $\mathcal{T}$-categories underlying BV tensor products in general and operads of algebras in the unital case.

Following this, in Section 2.5, we finally use the Segal envelope of [BHS22] to lift the Boardman-Vogt tensor product to a symmetric monoidal structure on Op_I. This culminates in the proof of the following theorem.

Theorem B'. *There exists a unique symmetric monoidal structure $\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes}$ on $\underline{\mathrm{Op}}_{\mathcal{T}}$ attaining a (necessarily unique) symmetric monoidal structure on the fully faithful $\mathcal{T}$-functor*

$$\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\sqcup}} : \underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes} \to \underline{\mathbf{Cat}}^{\otimes-\mathrm{mode}}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}},$$

Furthermore, $\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes}$ satisfies the following properties.

(1) In the case $\mathcal{T} = *$, there is a canonical symmetric monoidal equivalence $\mathrm{Op}_e^\otimes \simeq \mathrm{Op}_\infty^\otimes$*, where the codomain has the symmetric monoidal structure of [BS24a]; in particular, the underlying tensor product is equivalent to that of [BV73; HM23; HA].*

(2) the underlying tensor functor $- \otimes^{\mathrm{BV}} \mathcal{O} : \mathrm{Op}_\mathcal{T} \to \mathrm{Op}_\mathcal{T}$ *possesses a right adjoint* $\underline{\mathbf{Alg}}^\otimes_\mathcal{O}(-)$*, whose underlying $\mathcal{T}$-∞-category is the $\mathcal{T}$-∞-category of algebras* $\underline{\mathbf{Alg}}_\mathcal{O}(-)$*; the associated ∞-category is the ∞-category of algebras* $\mathbf{Alg}_\mathcal{O}(-)$.

(3) The unit of $\mathrm{Op}_\mathcal{T}^\otimes$ *is the G-operad* $\mathrm{triv}_\mathcal{T}^\otimes$ *defined in [NS22]; hence* $\underline{\mathbf{Alg}}^\otimes_{\mathrm{triv}_\mathcal{T}}(\mathcal{O}) \simeq \mathcal{O}^\otimes$.

(4) When $\mathcal{C}^\otimes$ *is I-symmetric monoidal,* $\underline{\mathbf{Alg}}^\otimes_\mathcal{O}(\mathcal{C})$ *is I-symmteric monoidal; furthermore, when* $\mathcal{O}^\otimes \to \mathcal{P}^\otimes$ *is a map of $\mathcal{T}$-operads, the pullback $\mathcal{T}$-functor*

$$\underline{\mathbf{Alg}}^\otimes_\mathcal{P}(\mathcal{C}) \to \underline{\mathbf{Alg}}^\otimes_\mathcal{O}(\mathcal{C})$$

is I-symmetric monoidal; in particular, if $\mathcal{O}^\otimes$ *has one object, then pullback along the canonical map* $\mathrm{triv}^\otimes \to \mathcal{O}^\otimes$ *presents the unique forgetful natural transformation*

$$\underline{\mathbf{Alg}}^\otimes_\mathcal{P}(\mathcal{C}) \to \mathcal{C}^\otimes,$$

which is I-symmetric monoidal when $\mathcal{C}$ is I-symmetric monoidal.

(5) When $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ *is an I-symmetric monoidal functor, the induced lax I-symmetric monoidal functor*

$$\underline{\mathbf{Alg}}^\otimes_\mathcal{O}(\mathcal{C}) \to \underline{\mathbf{Alg}}^\otimes_\mathcal{O}(\mathcal{D})$$

is I-symmetric monoidal.

References. The initial statement is Proposition 2.70; statement (1) is Proposition 2.56. Statement (2) is Proposition 2.35 and Corollary 2.65. Statement (3) is Proposition 2.62. Statements (4) and (5) follow by translating Proposition 2.37 to I-operads using Definitions 2.39 and 2.40. □

After this, we go on to study the *underlying $\mathcal{T}$-symmetric monoidal envelope* functor in Section 2.6, showing in Corollary 2.77 that it forms a fiberwise-monadic $\mathcal{T}$-functor

$$\underline{\mathrm{sseq}}_\mathcal{T} : \underline{\mathrm{Op}}_\mathcal{T} \to \underline{\mathrm{Fun}}_\mathcal{T}(\underline{\Sigma}_\mathcal{T}, \underline{\mathcal{S}}_\mathcal{T});$$

in particular, we show that it is a conservative right $\mathcal{T}$-adjoint.

Last, in preparation for Section 3, we initiate in Section 2.7 the study of the localizing subcategory of $\mathcal{T}$-operads whose underlying $\mathcal{T}$-symmetric sequence is $(d-1)$-truncated, called *$\mathcal{T}$-d-operads*; we show in particular that the full $\mathcal{T}$-subcategory of $\underline{\mathrm{Op}}_\mathcal{T}$ spanned by $\mathcal{T}$-operads whose S-ary spaces are empty or contractible form a *$\mathcal{T}$-poset*.

2.1. **Recollections on algebraic patterns.** An algebraic pattern is a collection of data encoding *Segal conditions* for the purpose of homotopy-coherent algebra. This algebra is encoded in two constructions. First, given a pattern $\mathfrak{O}$ and a complete ∞-category $\mathcal{C}$, there is an ∞-category of *Segal $\mathfrak{O}$-objects in $\mathcal{C}$*, which we view as $\mathfrak{O}$-monoids in $\mathcal{C}$; these are presented as functors $\mathfrak{O} \to \mathcal{C}$ satisfying a Segal condition.

We may view $\mathfrak{O}$-objects in **Cat** (aka Segal $\mathfrak{O}$-∞-categories) as $\mathfrak{O}$-monoidal ∞-categories; these straighten to cocartesian fibrations over $\mathfrak{O}$ satisfying conditions. As in [HA, § 2], the condition of *being a cocartesian fibration* may be relaxed to construct a form of operads parameterized by $\mathfrak{O}$, called *fibrous $\mathfrak{O}$-patterns.*

In contrast to the categorical patterns of [HA, § B], these are manifestly ∞-categorical, and it is relatively easy to construct push-pull adjunctions between categories of fibrous patterns over different algebraic patterns; we found our theory of I-operads in this syntax for this reason, as the Boardman-Vogt tensor product is most easily defined in terms of pushforward along maps of algebraic patterns.

The author would like to emphasize that the program surrounding algebraic patterns has achieved many results not mentioned in this paper, as fibrous patterns only play a small role. For a significantly more thorough and elegant treatment, we recommend [BHS22; CH21; CH23].

2.1.1. *Algebraic patterns, Segal objects, and fibrous patterns.*

Definition 2.1. An *algebraic pattern* is a triple $(\mathfrak{B},(\mathfrak{B}^{\mathrm{in}},\mathfrak{B}^{\mathrm{act}}),\mathfrak{B}^{\mathrm{el}})$, where $(\mathfrak{B}^{\mathrm{in}},\mathfrak{B}^{\mathrm{act}})$ is a factorization system on $\mathfrak{B}$ and $\mathfrak{B}^{\mathrm{el}} \subset \mathfrak{B}^{\mathrm{in}}$ is a full subcategory.[16] The category $\mathrm{AlgPatt} \subset \mathrm{Fun}(\mathbf{Q},\mathbf{Cat})$ is the full subcategory spanned by algebraic patterns, where

$$\mathbf{Q} := \bullet \to \bullet \to \bullet \leftarrow \bullet. \tag{10}$$

◁

We refer to the morphisms in $\mathfrak{B}^{\mathrm{in}}$ as "inert morphisms," morphisms in $\mathfrak{B}^{\mathrm{act}}$ as "active morphisms," and objects in $\mathfrak{B}^{\mathrm{el}}$ as "elementary objects." When it is clear from context, we will abusively refer to the triple $\left(\mathfrak{B},(\mathfrak{B}^{\mathrm{in}},\mathfrak{B}^{\mathrm{act}}),\mathfrak{B}^{\mathrm{el}}\right)$ simply by the underlying ∞-category $\mathfrak{B}$. We have a primary source of examples as follows.

Construction 2.2. An *adequate quadruple* is the data of an adequate triple $\mathcal{X}_b,\mathcal{X}_f \subset \mathcal{X}$ in the sense of Section 1.2 together with a full subcategory $\mathcal{X}_0 \subset \mathcal{X}_b$; the ∞*-category of adequate quadruples* is the full subcategory

$$\mathbf{Quad}^{\mathrm{adeq}} \subset \mathrm{Fun}(\mathbf{Q},\mathbf{Cat})$$

spanned by adequate quadruples, where $\mathbf{Q}$ is defined by Eq. (10).

Given an adequate quadruple $\mathcal{X}_0 \subset \mathcal{X}_b \subset \mathcal{X} \supset \mathcal{X}_f$, the ∞-category $\mathsf{Span}_{b,f}(\mathcal{X})$ has a canonical factorization system by *backwards and forward maps*

$$\mathcal{X}_b^{\mathrm{op}} \hookrightarrow \mathsf{Span}_{b,f}(\mathcal{X}) \hookleftarrow \mathcal{X}_f.$$

We define the span pattern $\mathsf{Span}_{b,f}(\mathcal{X};\mathcal{X}_0)$ via the data

- underlying ∞-category $\mathsf{Span}_{b,f}(\mathcal{X})$,
- inert morphisms $\mathcal{X}_b^{\mathrm{op}} \subset \mathsf{Span}(\mathcal{X})$,
- active morphisms $\mathcal{X}_f \subset \mathsf{Span}(\mathcal{X})$, and
- elementary objects $\mathcal{X}_0^{\mathrm{el}} \subset \mathcal{X}_b^{\mathrm{op}}$.

Given a map of quadruples $\left(\mathcal{X},(\mathcal{X}_b,\mathcal{X}_f),\mathcal{X}_0\right) \to \left(\mathcal{Y},(\mathcal{Y}_b,\mathcal{Y}_f),\mathcal{Y}_0\right)$ the associated functor $\mathsf{Span}_{b,f}(\mathcal{X}) \to \mathsf{Span}_{b,f}(\mathcal{Y})$ preserves inert morphisms, active morphisms, and elementary objects by defintiion; hence the functor $\mathsf{Span}_{-,-}(-;-) : \mathbf{Quad}^{\mathrm{adeq}} \to \mathrm{Fun}(\mathbf{Q},\mathbf{Cat})$ descends to a functor

$$\mathsf{Span}_{-,-}(-;-) : \mathbf{Quad}^{\mathrm{adeq}} \to \mathrm{AlgPatt}.$$

◁

In particular, postcomposition yields a functor

$$\mathrm{Fun}(\mathcal{T}^{\mathrm{op}},\mathbf{Quad}^{\mathrm{adeq}}) \to \mathrm{Fun}(\mathcal{T}^{\mathrm{op}},\mathrm{AlgPatt}).$$

Example 2.3. When $\mathcal{T}$ is an ∞-category, and $I \subset \mathbb{F}_{\mathcal{T}}$ is a pullback-stable wide subcategory of a full subcategory $\mathbb{F}_{c(I)} \subset \mathbb{F}_{\mathcal{T}}$ (e.g. $I = \mathbb{F}_{\mathcal{T}}$ for $\mathcal{T}$ orbital), we define the *effective I-Burnside pattern*

$$\mathsf{Span}_I(\mathbb{F}_{\mathcal{T}}) := \mathsf{Span}_{\mathrm{all},I}\left(\mathbb{F}_{c(I)};\mathcal{T}^{\mathrm{op}} \cap \mathbb{F}_{c(I)}\right).$$

◁

Example 2.4. Given $\mathcal{T}$ an orbital ∞-category, we may define the ∞*-category of finite pointed $\mathcal{T}$-sets* as

$$\mathbb{F}_{\mathcal{T},*} := \mathsf{Span}_{\mathrm{si},\mathrm{all}}(\mathbb{F}_{\mathcal{T}}),$$

where $\mathbb{F}_{\mathcal{T}}^{\mathrm{si}} \subset \mathbb{F}_{\mathcal{T}}$ is the wide subcategory of summand inclusions. In fact, the class of summand inclusions is restriction-stable, so this lifts to an algebraic pattern

$$\mathrm{tot}\underline{\mathbb{F}}_{\mathcal{T},*} \simeq \mathsf{Span}_{\mathrm{si},\mathrm{all}}(\mathrm{tot}\underline{\mathbb{F}}_{\mathcal{T}};\mathcal{T}^{\mathrm{op}});$$

this possesses a canonical map of algebraic patterns

$$\varphi : \mathrm{tot}\underline{\mathbb{F}}_{\mathcal{T},*} \hookrightarrow \mathsf{Span}_{\mathrm{all},\mathrm{all}}(\mathrm{tot}\underline{\mathbb{F}}_{\mathcal{T}};\mathcal{T}^{\mathrm{op}}) \xrightarrow{U} \mathsf{Span}(\mathbb{F}_{\mathcal{T}}). \tag{11}$$

◁

Algebraic patterns provide a general framework for algebraic structures satisfying the associated *Segal conditions*, which are encoded in the notions of Segal objects.

[16] Throughout this paper, we adopt the definition of *factorization system* used in [CH21, Rmk 2.2], which does not assert any lifting properties; that is, a facorization system on $\mathcal{C}$ is a pair of wide subcategories $\mathcal{C}^L,\mathcal{C}^R \subset \mathcal{C}$ satisfying the condition that, for all maps $X \xrightarrow{f} X'$, the space of factorizations $X \xrightarrow{l} Y \xrightarrow{r} X'$ with $l \in \mathcal{C}^L$ and $r \in \mathcal{C}^R$ is contractible.

Definition 2.5. Let $\mathcal{C}$ be a complete ∞-category and let $\mathfrak{O}$ be an algebraic pattern. Then, the ∞-category of *Segal $\mathfrak{O}$-objects in $\mathcal{C}$* is the full subcategory $\mathrm{Seg}_{\mathfrak{O}}(\mathcal{C}) \subset \mathrm{Fun}(\mathfrak{O}, \mathcal{C})$ consisting of functors F such that, for every object $O \in \mathcal{O}$, the natural map

$$F(O) \to \lim_{E \in \mathfrak{O}^{\mathrm{el}}_{O/}} F(E)$$

is an equivalence, where $\mathfrak{O}^{\mathrm{el}}_{O/} := \mathfrak{O}^{\mathrm{el}} \times_{\mathfrak{O}^{\mathrm{in}}} \mathfrak{O}^{\mathrm{in}}_{O/}$ is the category of inert morphisms from O to an elementary object. ◃

Remark 2.6. By [CH21, Lem 2.9], a functor $F : \mathfrak{O} \to \mathcal{C}$ is a Segal $\mathfrak{O}$-object if and only if the associated functor $F|_{\mathfrak{O}^{\mathrm{int}}}$ is right Kan extended from $F|_{\mathfrak{O}^{\mathrm{el}}}$ along the inclusion $\mathfrak{O}^{\mathrm{el}} \to \mathfrak{O}^{\mathrm{int}}$. ◃

Example 2.7. We show in Lemma A.5 that, given $I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ a pullback-stable subcategory, $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})^{\mathrm{el}}_{Z/} = (\mathbb{F}_{\mathcal{T},/Z})^{\mathrm{op}}$ contains the set of orbits $\mathrm{Orb}(Z)$ as an initial subcategory. Hence there is an equivalence of full subcategories

$$\mathrm{Seg}_{\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}})}(\mathcal{C}) \simeq \mathrm{CMon}_I(\mathcal{C}) \qquad \subset \qquad \mathrm{Fun}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C}).$$ ◃

One benefit of the framework of Segal objects is their general monadicity result.

Proposition 2.8 ([CH21, Cor 8.2])**.** *if $\mathcal{O}$ is an algebraic pattern and $\mathcal{C}$ a presentable ∞-category, then the forgetful functor*

$$U : \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C})$$

is monadic; in particular, it is conservative.

Corollary 2.9. *A morphism of I-commutative monoids is an equivalence if and only if its underlying morphism of $c(I)$-objects is an equivalence; in particular, an I-symmetric monoidal functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is an equivalence if and only if the underlying $c(I)$-functor is an equivalence.*

Another benefit of Segal objects is a rich framework for functoriality.

Definition 2.10. A morphism of algebraic patterns $f : \mathfrak{P} \to \mathfrak{O}$ is a called a:

- *Segal morphism* if pullback $f^* : \mathrm{Fun}(\mathfrak{O}, \mathcal{C}) \to \mathrm{Fun}(\mathfrak{P}, \mathcal{C})$ preserves Segal objects in any complete ∞-category $\mathcal{C}$.
- *strong Segal morphism* if the associated functor $f^{\mathrm{el}}_{X/} : \mathfrak{P}^{\mathrm{el}}_{X/} \to \mathfrak{O}^{\mathrm{el}}_{f(X)/}$ is initial. ◃

Remark 2.11. [CH21, Lem 4.5] concludes that f is a Segal morphism if f^* preserves Segal objects in *spaces*. ◃

Example 2.12. We show in Proposition A.12 that, given any functor $\mathcal{T} \to \mathcal{T}'$ of orbital ∞-categories, the associated functor

$$\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}'})$$

is a Segal morphism.

Additionally, in Corollary A.8, we show that the map φ of Eq. (11) is a segal morphism, constructing a pullback map

$$\mathrm{CMon}_{\mathcal{T}}(\mathcal{C}) \simeq \mathrm{Seg}_{\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}(\mathcal{C}) \to \mathrm{Seg}_{\mathrm{tot}\underline{\mathbb{F}}_{\mathcal{T},*}}(\mathcal{C}).$$

In [Bar23, Cor 2.64], conditions for a strong Segal morphism were developed concerning when their pullback maps are equivalences, and these conditions were checked in [BHS22, Prop 5.2.14] in the case $\mathcal{T} = \mathcal{O}_G^{\mathrm{op}}$; we review their argument and extend it to arbitrary atomic orbital ∞-categories in Appendix A.1. The existence of such an equivalence (not induced by a pattern) is not new, and to the author's knowledge, first appeared as [Nar16, Thm 6.5]. ◃

Many examples of algebraic patterns come from modeling interchanging algebraic structures via compatible Segal conditions; these are corepresented by *product patterns*.

Lemma 2.13 ([CH21, Cor 5.5])**.** $\mathrm{AlgPatt} \subset \mathrm{Fun}(\mathbf{Q}, \mathbf{Cat})$ *is a localizing subcategory; in particular,* AlgPatt *has small limits.*

Example 2.14. In particular, AlgPatt has products. By [CH21, Ex 5.7], there is an equivalence

$$\mathrm{Seg}_{\mathfrak{B} \times \mathfrak{B}'}(\mathcal{C}) \simeq \mathrm{Seg}_{\mathfrak{B}} \mathrm{Seg}_{\mathfrak{B}'}(\mathcal{C}).$$

In particular, this combined with Example 2.7 gives a complete segal space model for I-symmetric monoidal categories; indeed, the pattern $\Delta^{\mathrm{op},\natural}$ of [CH21, Ex 5.8] has Segal $\Delta^{\mathrm{op},\natural}$-objects in $\mathcal{C}$ given by *complete Segal objects in* $\mathcal{C}$, specializing to the fact that

$$\mathrm{Seg}_{\Delta^{\mathrm{op},\natural}}(\mathcal{S}) \simeq \mathbf{Cat},$$

and hence

$$\mathrm{Seg}_{\Delta^{\mathrm{op},\natural}}(\mathcal{S}_{\mathcal{T}}) \simeq \mathrm{Seg}_{\mathcal{T}^{\mathrm{op,el}} \times \Delta^{\mathrm{op},\natural}}(\mathcal{S}) \simeq \mathrm{Seg}_{\mathcal{T}^{\mathrm{op,el}}}(\mathbf{Cat}) \simeq \mathbf{Cat}_{\mathcal{T}},$$

where $\mathcal{T}^{\mathrm{op,op,el}}$ is the algberaic pattern with $\left(\mathcal{T}^{\mathrm{op,el}}\right)^{\mathrm{el}} \simeq \left(\mathcal{T}^{\mathrm{op,el}}\right)^{\mathrm{int}} \simeq \mathcal{T}^{\mathrm{op}} \simeq \left(\mathcal{T}^{\mathrm{op,el}}\right)^{\mathrm{act}}$. Additionally,

$$\mathrm{Seg}_{\Delta^{\mathrm{op},\natural}}(\mathrm{CMon}_{\mathcal{T}}(\mathcal{S})) \simeq \mathrm{Seg}_{\Delta^{\mathrm{op},\natural} \times \mathrm{Span}(\mathbb{F}_{\mathcal{T}})}(\mathcal{S}) \simeq \mathrm{Seg}_{\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}(\mathbf{Cat}) \simeq \mathrm{CMon}_{\mathcal{T}}(\mathbf{Cat}). \qquad \triangleleft$$

Cartesian products of patterns play nicely with well-structured maps of patterns.

Lemma 2.15. *Suppose $f : \mathfrak{O} \to \mathfrak{P}$ and $f' : \mathfrak{O}' \to \mathfrak{P}'$ are (resp. strong) Segal morphisms. Then,*

$$f \times f' : \mathfrak{O} \times \mathfrak{O}' \to \mathfrak{P} \times \mathfrak{P}'$$

is a (strong) Segal morphism.

Proof. The case of Segal morphisms follows immediately from Example 2.14, so we assume that f, f' are strong Segal. Then, the induced map

$$f^{\mathrm{el}}_{X/} \times f'^{\mathrm{el}}_{X'/} = (f \times f')^{\mathrm{el}}_{(X,X')/} : (\mathfrak{O} \times \mathfrak{O}')^{\mathrm{el}}_{(X,X')/} \to (\mathfrak{P} \times \mathfrak{P}')^{\mathrm{el}}_{(fx,fx')/}$$

is a product of initial maps; it then follows that it is initial, since limits in product categories are computed pointwise. ☐

The unstraightening functor of [HTT] realizes $\mathrm{Seg}_{\mathfrak{O}}(\mathbf{Cat}_\infty)$ as a full subcategory of $\mathbf{Cat}_{\infty,/\mathfrak{O}}$ consisting of cocartesian fibrations satisfying Segal conditions; we relax this for the following definition, which is equivalent to the original definition stated in [BHS22, Def 4.1.2] by [BHS22, Prop 4.1.6].

Definition 2.16. Let $\mathfrak{B}$ be an algebraic pattern. A *fibrous $\mathfrak{B}$-pattern* is a map of algebraic patterns $\pi : \mathfrak{O} \to \mathfrak{B}$ such that

(1) (inert morphisms) $\mathfrak{O}$ has π-cocartesian lifts for inert morphisms of $\mathfrak{B}$,
(2) (Segal condition for colors) For every active morphism $\omega : V_0 \to V_1$ in $\mathfrak{B}$, the functor

$$\mathfrak{O}_{V_0} \to \lim_{\alpha \in \mathfrak{B}^{\mathrm{el}}_{V_1/}} \mathfrak{O}_{\omega_{\alpha,!} V_1}$$

induced by cocartesian transport along ω_α is an equivalence, where $\omega_{(-)} : \mathfrak{B}^{\mathrm{el}}_{Y/} \to \mathfrak{B}^{\mathrm{int}}_{X/}$ is the inert morphism appearing in the inert-active factorization of $\alpha \circ \omega$, and
(3) (Segal condition for multimorphisms) for every active morphism $\omega : V_0 \to V_1$ in $\mathfrak{B}$ and all objects $X_i \in \mathfrak{O}_{\mathfrak{B}_{V_i}}$, the commutative square

$$\begin{tikzcd} \mathrm{Map}_{\mathfrak{O}}(X_0, X_1) \arrow[r] \arrow[d] & \lim_{\alpha \in \mathfrak{B}^{\mathrm{el}}_{V_1/}} \mathrm{Map}_{\mathfrak{O}}(X_0, \omega_{\alpha,!} X_1) \arrow[d] \\ \mathrm{Map}_{\mathfrak{B}}(V_0, V_1) \arrow[r] & \lim_{\alpha \in \mathfrak{B}^{\mathrm{el}}_{V_1/}} \mathrm{Map}_{\mathfrak{B}}(V_0, \omega_{\alpha,!} V_1) \end{tikzcd}$$

is cartesian.

We denote by $\mathrm{Fbrs}(\mathfrak{B}) \subset \mathbf{Cat}^{\mathrm{Int-cocart}}_{/\mathfrak{B}}$ the full subcategory spanned by the fibrous $\mathfrak{B}$-patterns, where the latter category has objects the functors to $\mathfrak{B}$ possessing cocartesian lifts over inert morphisms and morphisms the functors preserving such cocartesian lifts. ◁

Remark 2.17. As noted in [BHS22, Rmk 4.1.8], in the presence of condition (3) above, condition (2) may be weakened to assrt that the functor $\mathfrak{O}_{V_0} \to \lim_{\alpha \in \mathfrak{B}^{\mathrm{el}}_{V_1/}} \mathfrak{O}_{\omega_{\alpha,!} V_1}$ is a π_0-equivalence. To match [BHS22, Prop 4.1.6], we may even take the intermediate assumption that this functor induces an equivalence on cores. ◁

Example 2.18. Fibrous $\mathbb{F}_*$-patterns are equivalent to ∞-operads (c.f. [HA]), and we will review in Appendix A.1 a proof due to [BHS22] that fibrous $\underline{\mathbb{F}}_{\mathcal{T},*}$-patterns are equivalent to the $\mathcal{T}$-∞-operads of [NS22]. ◁

A fibrous pattern $\pi : \mathfrak{O} \to \mathfrak{B}$ inherits a structure of an algebraic pattern whose inert morphisms consist of π-cocartesian lifts of inert morpisms in $\mathfrak{B}$, whose active morphisms are abitrary lifts of active morphisms in $\mathfrak{B}$, and whose elentary objects are spanned by lifts of elementary objects. This is canonical:

Proposition 2.19 ([BHS22, Cor 4.1.7])**.** *Fibrous patterns are closed under composition for the above pattern structure, inducing an equivalence*

$$\mathrm{Fbrs}(\mathfrak{O}) \simeq \mathrm{Fbrs}(\mathfrak{B})_{/\mathfrak{O}}.$$

Furthermore, fibrous $\mathfrak{B}$-patterns are well-behaved within $\mathbf{Cat}_{/\mathfrak{B}}$.

Proposition 2.20 ([BHS22, Cor 4.2.3])**.** *The fully faithful functor $U : \mathrm{Fbrs}(\mathfrak{B}) \to \mathrm{AlgPatt}_{/\mathfrak{B}}$ participates in an adjunction*

$$U : \mathrm{Fbrs}(\mathfrak{B}) \rightleftarrows \mathrm{AlgPatt}_{/\mathfrak{B}} : L_{\mathrm{Fbrs}}$$

We construct many Segal morphisms in Appendix A.3. Many more are constructed in the following lemma, which follows from [CH21, Lem 9.10] after noting that the *weak Segal fibrations* of [CH21, Def 9.6] are a generalization of Definition 2.16 (c.f. [BHS22, p. 31]).

Proposition 2.21 ([CH21, Lem 9.10])**.** *Fibrous patterns are strong Segal morphisms.*

2.1.2. *The Segal envelope.* In [BHS22, Lem 4.2.4] it was verified that a fibrous $\mathfrak{O}$-pattern is a cocartesian fibration if and only if it's the straightening of a Segal $\mathfrak{O}$-category; this lifts the fact that an operad $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category if and only if the corresponding functor $\mathcal{C}^{\otimes} \to \mathbb{F}_*$ is a cocartesian fibration. We would like to describe adjunctions relating fibrous patterns to Segal objects, but to do so, we need a few constructions.

Definition 2.22. Given $\mathfrak{O} \to \mathfrak{B}$ a map of algebraic patterns, the *Segal envelope of* $\mathfrak{O}$ *over* $\mathfrak{B}$ is the horizontal composite

$$\begin{tikzcd} \mathrm{Env}_{\mathfrak{B}}\mathfrak{O} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ar}_{\mathrm{act}}(\mathfrak{B}) \arrow[r, "t"] \arrow[d, "s"] & \mathfrak{B} \\ \mathfrak{O} \arrow[r] & \mathfrak{B} & \end{tikzcd}$$

Where $\mathrm{Ar}_{\mathrm{act}}(\mathfrak{B}) \subset \mathrm{Ar}(\mathfrak{B}) = \mathrm{Fun}(\Delta^1, \mathfrak{B})$ is the full subcategory spanned by active arrows. We denote the envelope of the identity as

$$\mathscr{A}_{\mathfrak{B}} := \mathrm{Ar}_{\mathrm{act}}(\mathfrak{B}) \xrightarrow{t} \mathfrak{B}. \quad \triangleleft$$

Let $\mathfrak{O}$ be an algebraic pattern and $\omega : X \to Y$ an active map. Define the pullback square

$$\begin{tikzcd} \mathfrak{O}^{\mathrm{el}}(\omega) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ar}(\mathfrak{O}^{\mathrm{int}}_{X/}) \arrow[d, "{(s,t)}"] \\ \mathfrak{O}^{\mathrm{el}}_{Y/} \times \mathfrak{O}^{\mathrm{el}}_{X/} \arrow[r, "{(\omega_{(-)}, \mathrm{id})}"] & \mathfrak{O}^{\mathrm{int}}_{X/} \times \mathfrak{O}^{\mathrm{int}}_{X/} \end{tikzcd}$$

where $\omega_{(-)} : \mathfrak{O}^{\mathrm{el}}_{Y/} \to \mathfrak{O}^{\mathrm{int}}_{X/}$ sends $\alpha : Y \to E$ to the inert map ω_a of the inert-active factorization of $X \xrightarrow{\omega} Y \xrightarrow{a} E$.

Definition 2.23. $\mathfrak{O}$ is *sound* if, for all $\omega : X \to Y$ active, the associated map $\mathfrak{O}^{\mathrm{el}}(\omega) \to \mathfrak{O}^{\mathrm{el}}_{X/}$ is initial. A sound pattern $\mathfrak{O}$ is *soundly extendable* if $\mathscr{A}_{\mathfrak{O}}$ is a Segal $\mathfrak{O}$-∞-category. $\triangleleft$

Soundness as a condition allows one to simplify Segal conditions, yielding functoriality results for the categories of Segal objects and fibrous patterns; sound extendibility reduces many instances of *relative Segal objects* of [BHS22, Def 3.1.8] to a morphism with Segal domain by [BHS22, Obs 3.1.9] in the setting of the Segal envelope. To that end, we prove the following in Proposition A.11 under the first assumption; the case with the second assumption is [BHS22, Lem 4.1.19], and we proceed by an analogous argument.

Proposition 2.24. *Suppose $f : \mathfrak{P} \to \mathfrak{O}$ is a Segal morphism and either $\mathfrak{O}$ is soundly extendable or f is strong Segal. Then, the pullback functor $f^* : \mathbf{Cat}_{/\mathfrak{P}} \to \mathbf{Cat}_{\mathfrak{O}}$ preserves fibrous patterns; in particular, the associated functor*

$$f^* : \mathrm{Fbrs}(\mathfrak{O}) \to \mathrm{Fbrs}(\mathfrak{P})$$

has a left adjoint given by $L_{\mathrm{Fbrs}} f_!$.

Example 2.25. We show in Lemma A.7 that $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is soundly extendable; hence Example 2.12 and Proposition 2.24 together yield a functor

$$\mathrm{Op}_{\mathcal{T}} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}});$$

we review a proof that this is an equivalence (originally due to [BHS22] when $\mathcal{T} = \mathcal{O}_G$) in Corollary A.8. ◁

Given $f : \mathfrak{P} \to \mathfrak{O}$ a Segal morphism between algebraic patterns, we then define the composite functor

$$f^{\circledast} : \mathrm{Seg}_{\mathfrak{O}}^{/\mathscr{A}_{\mathfrak{O}}} \xrightarrow{f^*} \mathrm{Seg}_{\mathfrak{O}}^{/f^*\mathscr{A}_{\mathfrak{O}}} \xrightarrow{q^*} \mathrm{Seg}_{\mathfrak{O}}^{/\mathscr{A}_{\mathfrak{P}}}$$

where q is the map fitting into the following diagram:

$$\begin{tikzcd}
\mathscr{A}_{\mathfrak{P}} \arrow[dr, "q"] \arrow[drr, "\mathscr{A}_f"] \arrow[ddr, "p"'] & & \\
& f^*\mathscr{A}_{\mathfrak{O}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{A}_{\mathfrak{O}} \arrow[d, "p"] \\
& \mathfrak{P} \arrow[r, "f"'] & \mathfrak{O}
\end{tikzcd}$$

This participates in the following theorem, which was proved under a *strong Segal* assumption which is rendered unnecessary by Proposition 2.24.

Theorem 2.26 ([BHS22, Prop 4.2.1, Prop 4.2.5, Thm 4.2.6, Rem 4.2.8])**.** *Let $\mathfrak{O}$ be a soundly extendable pattern. Then, $\mathrm{Env}_{\mathcal{O}}$ is the left adjoint in an adjoint pair*

$$\mathrm{Env}_{\mathfrak{O}} : \mathrm{Fbrs}(\mathfrak{O}) \rightleftarrows \mathrm{Seg}_{\mathfrak{O}}(\mathbf{Cat}_\infty) : \mathrm{Un}.$$

By taking slice categories, this induces an adjunction

$$\mathrm{Env}_{\mathfrak{O}}^{/\mathscr{A}_{\mathfrak{O}}} : \mathrm{Fbrs}(\mathfrak{O}) \rightleftarrows \mathrm{Seg}_{\mathfrak{O}}(\mathbf{Cat}_\infty)_{/\mathscr{A}_{\mathfrak{O}}}$$

whose left adjoint is fully faithful. Furthermore, if $f : \mathfrak{O} \to \mathfrak{P}$ is a Segal morphism between soundly extendable patterns, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Seg}_{\mathfrak{O}}(\mathbf{Cat}_\infty) \arrow[r, "\mathrm{Un}"] \arrow[d, "f^*"'] & \mathrm{Fbrs}(\mathfrak{O}) \arrow[r, hook, "\mathrm{Env}_{\mathfrak{O}}^{/\mathscr{A}_{\mathfrak{O}}}"] \arrow[d, "f^*"'] & \mathrm{Seg}_{\mathfrak{O}}(\mathbf{Cat}_\infty)_{/\mathscr{A}_{\mathfrak{O}}} \arrow[r, "\mathrm{Un}"] \arrow[d, "f^{\circledast}"'] & \mathrm{Fbrs}(\mathfrak{O}) \arrow[d, "f^*"] \\
\mathrm{Seg}_{\mathfrak{P}}(\mathbf{Cat}_\infty) \arrow[r, "\mathrm{Un}"'] & \mathrm{Fbrs}(\mathfrak{P}) \arrow[r, hook, "\mathrm{Env}_{\mathfrak{P}}^{/\mathscr{A}_{\mathfrak{P}}}"'] & \mathrm{Seg}_{\mathfrak{P}}(\mathbf{Cat}_\infty)_{/\mathscr{A}_{\mathfrak{P}}} \arrow[r, "\mathrm{Un}"'] & \mathrm{Fbrs}(\mathfrak{P})
\end{tikzcd}$$

We will make frequent use of product patterns, so we observe that they interact nicely with Segal envelopes.

Observation 2.27. If $\mathfrak{O}, \mathfrak{P}$ are fibrous $\mathfrak{B}$-patterns, then their Segal envelopes satisfy

$$\begin{aligned}
\mathrm{Env}_{\mathfrak{B}\times\mathfrak{B}}(\mathfrak{O}\times\mathfrak{P}) &\simeq (\mathfrak{O}\times\mathfrak{P}) \times_{\mathfrak{B}\times\mathfrak{B}} \mathrm{Ar}_{\mathrm{act}}(\mathfrak{B}\times\mathfrak{B}) \\
&\simeq (\mathfrak{O}\times_{\mathfrak{B}} \mathrm{Ar}_{\mathrm{act}}(\mathfrak{B})) \times (\mathfrak{P}\times_{\mathfrak{B}} \mathrm{Ar}_{\mathrm{act}}(\mathfrak{B})) \\
&\simeq \mathrm{Env}_{\mathfrak{B}}(\mathfrak{O}) \times \mathrm{Env}_{\mathfrak{B}}(\mathfrak{P})
\end{aligned}$$ ◁

2.1.3. *Algebraic patterns vs categorical patterns.* Adjacent to algebraic patterns is Lurie's notion of *categorical patterns*, as described in [HA, § B]. These make up a combinatorial model category capable of formalizing fibrous patterns and Segal $\mathcal{O}$-∞-categories.

Construction 2.28. Fix $\mathfrak{B}$ an algebraic pattern and let

$$\mathrm{CatPatt}(\mathfrak{B}) := \left(\mathrm{In}, All, \left\{\mathcal{O}^{\mathrm{act}}_{/V}\right\}_{V\in\mathcal{O}}\right)$$

$$\mathrm{CatPatt}^{\mathrm{Seg}}(\mathfrak{B}) := \left(All, All, \left\{\mathcal{O}^{\mathrm{act}}_{/V}\right\}_{V\in\mathcal{O}}\right)$$

Unwinding definitions using [HA, Def B.0.19], we find that we've constructed left proper combinatorial simplicial model structures for $\mathrm{Fbrs}(\mathfrak{B})$ and $\mathrm{Seg}_{\mathfrak{B}}(\mathbf{Cat})$:

$$\mathrm{Fbrs}(\mathfrak{B}) \simeq \left(\mathbf{Set}_\Delta^+\right)_{/\mathrm{CatPatt}(\mathfrak{B})}$$

$$\mathrm{Seg}(\mathfrak{B}) \simeq \left(\mathbf{Set}_\Delta^+\right)_{/\mathrm{CatPatt}^{\mathrm{Seg}}(\mathfrak{B})} \qquad \triangleleft$$

Furthermore, this recovers Nardin-Shah's model in the case $\mathfrak{B} = \underline{\mathbb{F}}_{\mathcal{T},*}$ [NS22, § 2.6].

Corollary 2.29 ([HA, Rmk B.2.5])**.** *The projection map* $p : \mathfrak{B} \times \mathfrak{B}' \to \mathfrak{B}$ *induces adjunctions*

$$\begin{tikzcd} \mathrm{Fbrs}(\mathfrak{B}) \arrow[rr, bend left, "p^*"] \arrow[rr, phantom, "\perp"] && \mathrm{Fbrs}(\mathfrak{B} \times \mathfrak{B}') \arrow[ll, bend left, "p_*"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Seg}_{\mathfrak{B}}(\mathbf{Cat}) \arrow[rr, bend left, "p^*"] \arrow[rr, phantom, "\perp"] && \mathrm{Seg}_{\mathfrak{B}\times\mathfrak{B}'}(\mathbf{Cat}) \arrow[ll, bend left, "p_*"] \end{tikzcd}$$

2.2. **Boardman-Vogt tensor products of fibrous patterns.** If $\mathcal{C}$ is an ∞-category, we refer to objects in the ∞-category $\mathrm{Magma}(\mathcal{C}) \subset \mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by arrows $X \times X \to X$ as *Magmas*. Writing $\mathrm{AlgPatt}^{\mathrm{Seg}} \subset \mathrm{AlgPatt}$ for the wide subcategory whose morphisms are Segal morphisms, we refer to elements of $\mathrm{Magma}(\mathrm{AlgPatt}^{\mathrm{Seg}})$ as *Magmatic patterns*.

Construction 2.30. Let $\mathfrak{B}$ be a magmatic pattern. Then, the $\mathfrak{B}$-*Boardman-Vogt tensor product* is the bifunctor $\overset{\mathrm{BV}}{\otimes} : \mathrm{Fbrs}(\mathfrak{B}) \times \mathrm{Fbrs}(\mathfrak{B}) \to \mathrm{Fbrs}(\mathfrak{B})$ defined by

$$\mathfrak{O} \overset{\mathrm{BV}}{\otimes} \mathfrak{P} := L_{\mathrm{Fbrs}}(\mathfrak{O} \times \mathfrak{P} \to \mathfrak{B} \times \mathfrak{B} \xrightarrow{\wedge} \mathfrak{B}). \qquad \triangleleft$$

We define this in order to have a mapping out property with respect to the following construction.

Definition 2.31. Let $\mathfrak{B}$ be a magmatic pattern and $\mathfrak{O}, \mathfrak{P}, \mathfrak{Q}$ fibrous $\mathfrak{B}$-patterns. Then, a *bifunctor of fibrous* $\mathfrak{B}$ *patterns* $\mathfrak{O} \times \mathfrak{P} \to \mathfrak{Q}$ is a diagram in AlgPatt

$$\begin{tikzcd} \mathfrak{O} \times \mathfrak{P} \arrow[r] \arrow[d] & \mathfrak{Q} \arrow[d] \\ \mathfrak{B} \times \mathfrak{B} \arrow[r, "\wedge"] & \mathfrak{B} \end{tikzcd}$$

$\triangleleft$

The collection of bifunctors fits into a full subcategory

$$\mathrm{BiFun}_{\mathfrak{B}}(\mathfrak{O}, \mathfrak{P}; \mathfrak{Q}) \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{AlgPatt})$$

Example 2.32. Let $\mathfrak{P}$ be a fibrous $\mathfrak{B}$-pattern, and consider $\mathfrak{B}$ to be a fibrous $\mathfrak{B}$-pattern via the identity. Then, the category of fibrous $\mathfrak{B}$-patterns $\mathfrak{B} \times \mathfrak{P} \to \mathfrak{B}$ is contractible, as it is equivalent to composite arrows $\mathfrak{B} \times \mathfrak{P} \to \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$. $\triangleleft$

Observation 2.33. There are natural equivalences

$$\begin{aligned} \mathrm{BiFun}_{\mathfrak{B}}(\mathfrak{O}, \mathfrak{P}; \mathfrak{Q}) &\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathfrak{B}\times\mathfrak{B}}(\mathfrak{O} \times \mathfrak{P}, \mu^* \mathfrak{Q}) \\ &\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathfrak{B}}(\mu_!(\mathfrak{O} \times \mathfrak{P}), \mathfrak{Q}) \\ &\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathfrak{B}}(\mathfrak{O} \overset{\mathrm{BV}}{\otimes} \mathfrak{P}, \mathfrak{Q}). \end{aligned} \qquad \triangleleft$$

Following in the tradition started by the namesake [BV73, § 2.3], in Observation 5.21 we interpret $\mathrm{BiFun}_{\mathfrak{B}}(\mathfrak{O}, \mathfrak{P}; \mathfrak{Q})$ in the context of equivariant operads as *interchanging* $\mathfrak{O}$ *and* $\mathfrak{P}$*-algebra structures*; as in [BV73, Prop 2.19] and the variety of recontextualizations of their ideas (e.g. [HA; Wei11], we additionally recognize this as $\mathfrak{O}$*-algebras in* $\mathfrak{P}$*-algebras*, making $\overset{\mathrm{BV}}{\otimes}$ into a closed tensor product.

Construction 2.34. Fix $\mathfrak{B}$ a sound magmatic pattern, let $F : \mathfrak{O} \times \mathfrak{P} \to \mathfrak{Q}$ be a bifunctor of fibrous $\mathfrak{B}$-patterns, and let $\mathfrak{C}$ be a fibrous $\mathfrak{Q}$-pattern. We have a diagram

$$\mathfrak{O} \xleftarrow{p} \mathfrak{O} \times \mathfrak{P} \xrightarrow{F} \mathfrak{Q};$$

admitting push-pull adjunctions $p_* \dashv p^*$ and $F_! \dashv F^*$ on Segal objects and fibrous patterns by Propositions 2.19 and 2.21 and Corollary 2.29. We define the pattern

$$\underline{\mathbf{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{C}) := p_* F^* \mathfrak{C} \in \mathrm{Fbrs}(\mathfrak{O});$$

this is the *fibrous $\mathfrak{O}$-pattern of $\mathfrak{P}$-algebras in $\mathfrak{C}$ over $\mathfrak{Q}$*. In most cases, we will have $\mathfrak{Q} = \mathfrak{O} = \mathfrak{B}$, in which case the information of a bifunctor $\mathfrak{B} \times \mathfrak{P} \to \mathfrak{B}$ is simply that of a fibrous $\mathfrak{B}$-pattern $\mathfrak{P}$. In this case, we simply write

$$\underline{\mathbf{Alg}}^{\otimes}_{\mathfrak{P}}(\mathfrak{C}) := \underline{\mathbf{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{B}}(\mathfrak{C}) \in \mathrm{Fbrs}(\mathfrak{B});$$

this is the *fibrous $\mathfrak{B}$-pattern of $\mathfrak{P}$-algebras in $\mathfrak{C}$*. ◁

In the case $\mathfrak{Q} = \mathfrak{O} = \mathfrak{B}$, the above diagram refines to

$$\mathfrak{B} \xleftarrow{p} \mathfrak{B} \times \mathfrak{P} \xrightarrow{\mathrm{id} \times \pi} \mathfrak{B} \times \mathfrak{B} \xrightarrow{\wedge} \mathfrak{B},$$

so the functor $\mathfrak{P} \mapsto \underline{\mathbf{Alg}}^{\otimes}_{\mathfrak{P}}(\mathfrak{C})$ has a left adjoint computed by $L_{\mathrm{Fbrs}} \mu_! (\mathrm{id} \times \pi)_! p^*$; explicitly, this is computed on $\mathfrak{P}'$ by the fibrous localization of the diagonal composite

$$\begin{tikzcd}
\mathfrak{P}' \times \mathfrak{P} \arrow[r, phantom, "\simeq"] \arrow[dr, dashed, "\pi_{\mathfrak{Q}} \times \mathrm{id}"] \arrow[ddr, bend right, "\pi_{\mathfrak{Q}} \times \pi_{\mathfrak{P}}"'] & \pi^* \mathfrak{P}' \arrow[d] & \\
& \mathfrak{B} \times \mathfrak{P} \arrow[d, dashed, "\mathrm{id} \times \pi_{\mathfrak{P}}"] \arrow[dr] & \\
& \mathfrak{B} \times \mathfrak{B} \arrow[r, "\wedge"'] & \mathfrak{B}
\end{tikzcd}$$

By definition, this is precisely $\mathfrak{P}' \otimes^{\mathrm{BV}} \mathfrak{P}$, so we've proved the following.

Proposition 2.35. *The functor $(-) \overset{BV}{\otimes} \mathfrak{O} : \mathrm{Fbrs}(\mathfrak{B}) \to \mathrm{Fbrs}(\mathfrak{B})$ is left adjoint to $\underline{\mathbf{Alg}}^{\otimes}_{\mathfrak{O}}(-)$.*

We additionally spell out a few useful characteristics of $\overset{\mathrm{BV}}{\otimes}$ here. First, we describe functoriality.

Observation 2.36. Fix the fibrous $\mathfrak{B}$-pattern $\mathfrak{Q}$. Suppose we have bifunctors of fibrous $\mathfrak{B}$-patterns

$$F : \mathfrak{O} \times \mathfrak{P} \to \mathfrak{Q} \leftarrow \mathfrak{O}' \times \mathfrak{P}' : G$$

together with a morphism of fibrous $\mathfrak{B}$-patterns $\varphi : \mathfrak{P} \to \mathfrak{P}'$ making the following diagram commute:

$$\begin{tikzcd}
& \mathfrak{O} \times \mathfrak{P} \arrow[dl, "\pi"] \arrow[dd] \arrow[dr, "F"] & \\
\mathfrak{O} & & \mathfrak{Q} \\
& \mathfrak{O} \times \mathfrak{P}' \arrow[ul, "\pi'"] \arrow[ur, "G"'] &
\end{tikzcd}$$

The right triangle possesses a Beck-Chevalley transformation

$$\pi^* \varphi_! \implies \mathrm{id}_! \pi'^* = \pi'^*,$$

which possesses a mate natural transformation $\pi'_* \implies \pi_* \varphi^*$, i.e. a "pullback" natural transformation

$$\mathbf{Alg}^{\otimes}_{\mathfrak{P}'/\mathfrak{Q}}(-) \implies \mathbf{Alg}^{\otimes}_{\mathfrak{P}/\mathfrak{Q}}(-).$$ ◁

We observe that, in all of the work above, we may have instead assumed that $\mathfrak{C} \in \mathrm{Seg}_{\mathfrak{B}}(\mathbf{Cat})$, in which case all of our constructions land in $\mathrm{Seg}_{\mathfrak{B}}(\mathbf{Cat})$. Spelled out, this yields the following.

Proposition 2.37. *Fix $\mathfrak{O}, \mathfrak{P}, \mathfrak{Q}, \mathfrak{C}$ as in Construction 2.34. Then*

(1) If $\mathfrak{C}$ is a Segal $\mathfrak{Q}$-∞-category, then $\underline{\mathbf{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{C})$ is a Segal $\mathfrak{O}$-∞-category

(2) if $\mathfrak{C} \to \mathfrak{D}$ is a morphism of Segal $\mathfrak{Q}$-∞-categories, then the induced map $\underline{\mathbf{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{C}) \to \underline{\mathbf{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{D})$ is a morphism of Segal $\mathfrak{O}$-∞-categories.

(3) If $\mathfrak{P} \to \mathfrak{P}'$ is a morphism of fibrous $\mathfrak{B}$-patterns and $\mathfrak{C}$ is a Segal $\mathfrak{Q}$-∞-category, then the induced map of fibrous patterns

$$\underline{\mathbf{Alg}}^{\otimes}_{\mathfrak{P}'/\mathfrak{Q}}(\mathfrak{C}) \to \underline{\mathbf{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{C})$$

is a functor of Segal $\mathfrak{O}$-∞-categories, i.e. it preserves cocartesian lifts for inert morphisms.

Finally, in analogy to [BS24a] we show that this tensor product is compatible with Segal envelopes.

Proposition 2.38. *The following diagram commutes*

$$\begin{tikzcd} \mathrm{Fbrs}(\mathfrak{B})^2 \arrow[rr, "\overset{BV}{\otimes}"] \arrow[d, "\mathrm{Env}"] & & \mathrm{Fbrs}(\mathfrak{B}) \arrow[d, "\mathrm{Env}"] \\ \mathrm{Fun}(\mathfrak{B}, \mathbf{Cat})^2 \arrow[r, "\circledast"] & \mathrm{Fun}(\mathfrak{B}, \mathbf{Cat}) \arrow[r, "L_{\mathrm{Seg}}"] & \mathrm{Seg}_{\mathfrak{B}}(\mathbf{Cat}) \end{tikzcd}$$

Proof. Fix $\mathfrak{C}$ a Segal $\mathfrak{B}$-∞-category. Then, there are natural equivalences

$$\begin{aligned} \mathrm{Fun}_{\mathrm{Seg}_{\mathfrak{B}}(\mathbf{Cat})}\left(\mathrm{Env}(\mathfrak{O} \overset{\mathrm{BV}}{\otimes} \mathfrak{P}), \mathfrak{C}\right) &\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathfrak{B}\times\mathfrak{B}}(\mathfrak{O} \times \mathfrak{P}, \mu^*\mathfrak{C}) \\ &\simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{B}\times\mathfrak{B}}(\mathrm{Env}_{\mathfrak{B}\times\mathfrak{B}}(\mathfrak{O} \times \mathfrak{P}), \mu^*\mathfrak{C}) \\ &\simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{B}\times\mathfrak{B}}(\mathrm{Env}_{\mathfrak{B}}(\mathfrak{O}) \times \mathrm{Env}_{\mathfrak{B}}(\mathfrak{P}), \mu^*\mathfrak{C}) \qquad (12) \\ &\simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{B}}\left(L_{\mathrm{Seg}}\mu_!(\mathrm{Env}_{\mathfrak{B}}(\mathfrak{O}) \times \mathrm{Env}_{\mathfrak{B}}(\mathfrak{P})), \mathfrak{C}\right) \\ &\simeq \mathrm{Fun}_{\mathrm{Seg}_{\mathfrak{B}}(\mathbf{Cat})}\left(L_{\mathrm{Seg}}\mathrm{Env}_{\mathfrak{B}}(\mathfrak{O}) \circledast \mathrm{Env}(\mathfrak{P}), \mathfrak{C}\right) \qquad (13) \end{aligned}$$

Equivalence Eq. (12) is Observation 2.27; Eq. (13) follows by symmetric monoidality of the Grothendieck construction [Ram22]. The result then follows by Yoneda's lemma. □

2.3. $\mathcal{T}$-operads and I-operads.

Definition 2.39. The ∞-category of $\mathcal{T}$-operads is

$$\mathrm{Op}_{\mathcal{T}} := \mathrm{Fbrs}(\mathrm{Span}(\mathbb{F}_{\mathcal{T}})).$$

More generally, when $I \subset \mathbb{F}_{\mathcal{T}}$ is pullback-stable, the *∞-category of I-operads* is

$$\mathrm{Op}_I := \mathrm{Fbrs}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})).$$ ◁

By Proposition 2.19, if $\mathcal{O}^{\otimes}$ is an I-operad, then it has a natural pattern structure s.t. $\mathcal{O}^{\otimes} \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ is a morphism of patterns; the inert morphisms are cocartesian lifts of backwards maps, and the active maps are *arbitrary* lifts of forwards maps.

Definition 2.40. The *∞-category of $\mathcal{O}$-monoidal ∞-categories is*

$$\mathbf{Cat}^{\otimes}_{\mathcal{O},I} := \mathrm{Seg}_{\mathcal{O}^{\otimes}}(\mathbf{Cat}).$$ ◁

When $\mathcal{O}^{\otimes} \in \mathrm{Op}_I$ is terminal, we write $\mathbf{Cat}^{\otimes}_I := \mathbf{Cat}^{\otimes}_{\mathcal{O},I}$; Corollary A.6 yields an equivalence

$$\mathbf{Cat}^{\otimes}_I \simeq \mathrm{CMon}_I(\mathbf{Cat}).$$

when I is clear from context, we will frequently simply write $\mathbf{Cat}^{\otimes}_{\mathcal{O}}$ for $\mathbf{Cat}^{\otimes}_{\mathcal{O},\mathcal{T}}$.

Construction 2.41. We show in Proposition A.15 that the Cartesian product in $\mathbb{F}_{\mathcal{T}}$ endows $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ with the structure of a magmatic pattern in the sense of Section 2.2 via the *smash product*; we refer to the resulting bifunctor as the *Boardman-Vogt tensor product*

$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} := L_{\mathrm{Fbrs}}\left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \times \mathrm{Span}(\mathbb{F}_{\mathcal{T}})\right).$$ ◁

Definition 2.42. If $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are I-operads, then an *$\mathcal{O}$-algebra in $\mathcal{P}$* is a map of I-operads $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$; the ∞-category of $\mathcal{O}$-algebras in $\mathcal{P}$ is written

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{P}) := \mathrm{Fun}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}).$$

The *$\mathcal{T}$-operad of $\mathcal{O}$-algebras in $\mathcal{P}$* is given by the right adjoint $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \in \mathrm{Op}_{\mathcal{T}}$ to the Boardman-Vogt tensor product (see Proposition 2.35). ◁

For us, the appropriate degree of generality for I will be that for which the pushforward functor $\mathrm{Op}^{\otimes}_I \to \mathrm{Op}^{\otimes}_{\mathcal{T}}$ is simply given by postcomposition along the canonical functor $\iota_I^{\mathcal{T}} : \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$; this turns out to be a familiar setting.

Proposition 2.43 ([NS22, Ex 2.4.7])**.** *Let $I \subset \mathbb{F}_{\mathcal{T}}$ be a core-full subcategory. Then, the functor*

$$\mathcal{N}_{I\infty}^{\otimes} := \left(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \xrightarrow{\pi_I} \mathrm{Span}(\mathbb{F}_{\mathcal{T}})\right)$$

is a $\mathcal{T}$-operad if and only if I is a weak indexing category in the sense of Definition 1.38.

If $\mathcal{O}^{\otimes} \simeq \mathcal{N}_{I\infty}^{\otimes}$ arises from Proposition 2.43, we say that $\mathcal{O}^{\otimes}$ is a *weak $\mathcal{N}_\infty$ $\mathcal{T}$-operad,* and we write

$$\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{C}) := \underline{\mathbf{Alg}}_{\mathcal{N}_{I\infty}}^{\otimes}(\mathcal{C})$$

for the $\mathcal{T}$-operad of *I-commutative algebras in $\mathcal{C}$*. We delay the proof of Proposition 2.43 until Page 46, first developing some hands-on structural knowledge of $\mathcal{T}$-operads and I-operads.

Fix I a weak indexing system. If $\mathcal{C}, \mathcal{D} \in \mathbf{Cat}_I^{\otimes}$ are I-symmetric monoidal categories, we say that a *lax I-symmetric monoidal functor* $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a map of their underlying $\mathcal{T}$-operads; this is an I-symmetric monoidal functor if and only if it lands in $\mathbf{Cat}_I^{\otimes}$, i.e. if and only if it preserves cocartesian lifts for arbitrary maps in $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$. Then, Proposition 2.37 immediately implies the following.

Corollary 2.44. *Fix $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ a map of $\mathcal{T}$-operads and $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a map of $\mathcal{T}$-symmetric monoidal ∞-categories. Then, $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is a $\mathcal{T}$-symmetric monoidal category, and the canonical lax $\mathcal{T}$-symmetric monoidal functors*

$$\underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \to \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}), \qquad \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{D})$$

are $\mathcal{T}$-symmetric monoidal.

Example 2.45. The terminal $\mathcal{T}$-operad is presented by $\mathrm{Comm}_{\mathcal{T}}^{\otimes} = \left(\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \xrightarrow{\mathrm{id}} \mathrm{Span}(\mathbb{F}_{\mathcal{T}})\right)$, and hence it is a weak $\mathcal{N}_\infty$-operad; we write $\underline{\mathrm{CAlg}}_{\mathcal{T}}^{\otimes}(\mathcal{C}) := \underline{\mathrm{CAlg}}_{\mathbb{F}_{\mathcal{T}}}^{\otimes}(\mathcal{C})$, and call these *$\mathcal{T}$-commutative algebras.* For any $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$, pullback along the unique map $\mathcal{O}^{\otimes} \to \mathrm{Comm}_{\mathcal{T}}^{\otimes}$ determines a unique natural $\mathcal{T}$-symmetric monoidal functor

$$\underline{\mathrm{CAlg}}_{\mathcal{T}}^{\otimes}(\mathcal{C}) \to \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}),$$

so we view $\mathcal{T}$-commutative algbras as a *universal $\mathcal{T}$-equivariant algebraic structure.* ◁

2.3.1. *The structure of $\mathcal{T}$-operads.* The Segal conditions for fibrous $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$-patterns were characterized in [BHS22] in the case $\mathcal{T} = \mathcal{O}_G$; we generalize this to weak indexing systems over general atomic orbital ∞-categories in Lemma A.5, and summarize the results here.

Construction 2.46. Given $\pi_{\mathcal{O}} : \mathcal{O}^{\otimes} \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ an I-operad and $S \in \mathbb{F}_{\mathcal{T}}$, we define

$$\mathcal{O}_S := \pi_{\mathcal{O}}^{-1}(S).$$

Then, inert cocartesian lifts endow on $(\mathcal{O}_V)_{V \in \mathcal{T}}$ the structure of a $\mathcal{T}$-category, formally given by the pullback

$$\begin{tikzcd}
U(\mathcal{O}^{\otimes}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{O}^{\otimes} \arrow[d] \\
\mathcal{T}^{\mathrm{op}} \arrow[r, hook] & \mathrm{Span}(\mathbb{F}_{\mathcal{T}})
\end{tikzcd}$$

We call this the *underlying $\mathcal{T}$-category*, and refer to it as $\mathcal{O}$ when this doesn't cause confusion. ◁

Proposition 2.47. *A functor $\pi : \mathcal{O}^{\otimes} \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ is an I-operad if and only if the following are satisfied:*

(a) $\mathcal{O}^{\otimes}$ has π-cocartesian lifts for backwards maps in $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$;

(b) (Segal condition for colors) For every $S \in \mathbb{F}_{\mathcal{T}}$, cocartesian transport along the π-cocartesian lifts lying over the inclusions $(S \leftarrow U = U \mid U \in \mathrm{Orb}(S))$ together induce an equivalence

$$\mathcal{O}_S \simeq \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}_U.$$

where the category of S-colors *is $\mathcal{O}_S := \pi^{-1}(S)$; and*

(c) *(Segal condition for multimorphisms) For every map of orbits $T \to S$ in I and pair of objects $(\mathbf{C}, \mathbf{D}) \in \mathcal{O}_T \times \mathcal{O}_U$, postcomposition with the π-cocartesian lifts $\mathbf{D} \to D_U$ lying over the inclusions $(S \leftarrow U = U \mid U \in \mathrm{Orb}(S))$ induces an equivalence*

$$\mathrm{Map}_{\mathcal{O}^\otimes}^{T \to S}(\mathbf{C}, \mathbf{D}) \simeq \prod_{U \in \mathrm{Orb}(S)} \mathrm{Map}_{\mathcal{O}^\otimes}^{T \leftarrow T_U \to U}(\mathbf{C}, D_U).$$

where $T_U \coloneqq T \times_S U$.

Furthermore, a cocartesian fibration $\pi : \mathcal{O}^\otimes \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ is an I-operad if and only if its unstraightening $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathbf{Cat}$ is an I-symmetric monoidal category.

Proof. Each of our conditions nearly matches with that of Definition 2.16, with the exception being that we evaluate the limits on the sub-diagram $\mathrm{Orb}(S) \subset \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})^{\mathrm{el}}_{S/}$; in fact, we show in Lemma A.2 that tgis is an initial subcategory, implying the proposition. $\square$

Remark 2.48. The existence of cocartesian lifts for backwards maps furnishes an equivlanece

$$\mathrm{Map}_{\mathcal{O}^\otimes}^{T \leftarrow T_U \to U}(\mathbf{C}, D_U) \simeq \mathrm{Map}_{\mathcal{O}^\otimes}^{T_U \to U}(\mathbf{C}_{T_U}, D_U),$$

where $\mathbf{C}_{T_U} \in \mathcal{O}_{T_U}$ is the T_U-tuple of colors underlying $\mathbf{C}$. Hence in the presence of Conditions (a) and (b), Condition (c) may equivalently stipulate that the map

$$\mathrm{Map}_{\mathcal{O}^\otimes}^{T \to S}(\mathbf{C}, \mathbf{D}) \to \prod_{U \in \mathrm{Orb}(S)} \mathrm{Map}_{\mathcal{O}^\otimes}^{T_U \to U}(\mathbf{C}_{T_U}, D_U)$$

is an equivalence. We will generally prefer this version, as the data of a $\mathcal{T}$-operad is most naturally viewed as living over the *active* (i.e. forward) maps. ◃

Remark 2.49. Practicioners of [HA, Def 2.1.10] should note that, by Remark 2.17, we may weaken Condition (b) to assert only that cocartesian transport induces a π_0-surjection $\mathcal{O}_S \to \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}_U$; with this modification, Proposition 2.47 recovers Lurie's definition of ∞-operads when $\mathcal{T} = *$. ◃

Using Proposition 2.47, we gain access to the *structure spaces* of $\mathcal{T}$-operads.

Construction 2.50. Let $\mathcal{O}^\otimes$ be a $\mathcal{T}$-operad. When $\mathbf{C}, \mathbf{D} \in \mathcal{O}^\otimes$ are objects, define

$$\mathrm{Mul}_{\mathcal{O}}(\mathbf{C}, \mathbf{D}) \coloneqq \coprod_{\substack{\psi : \pi(C) \to \pi(D) \\ \text{active}}} \mathrm{Map}_{\pi_{\mathcal{O}}}^{\psi}(\mathbf{C}, \mathbf{D}).$$

In the case $D \in \mathcal{O}_V^\otimes$, $S \in \mathbb{F}_V$, and $\mathbf{C} \in \mathcal{O}_S^\otimes$, we write

$$\mathcal{O}(\mathbf{C}; D) \coloneqq \mathrm{Map}_{\mathcal{O}}^{\mathrm{Ind}_V^{\mathcal{T}} S \to V}(\mathbf{C}; D).$$

Similarly, given $S \in \mathbb{F}_V$, with corresponding map $\psi : \mathrm{Ind}_V^{\mathcal{T}} S \to V$, we define

$$\mathcal{O}(S) \coloneqq \coprod_{(\mathbf{C}, D) \in \mathcal{O}_S \times \mathcal{O}_V} \mathcal{O}(\mathbf{C}; D);$$

we will refer to this is the *space of S-ary operations in $\mathcal{O}$*. ◃

We use this to define a litany of useful full subcategories of $\mathrm{Op}_{\mathcal{T}}$.

Definition 2.51. A $\mathcal{T}$-operad $\mathcal{O}^\otimes$ is:

- *at most one-colored* if $\mathcal{O}_V \in \{\varnothing, *\}$ for all $V \in \mathcal{T}$, i.e. $\mathcal{O}(*_V) \in \{\varnothing, *\}$ for all $V \in \mathcal{T}$,
- *at least one-colored* if $\mathcal{O}_V \neq \varnothing$ for all $V \in \mathcal{T}$, i.e. $\mathcal{O}(*_V) \neq \varnothing$ for all $V \in \mathcal{T}$,
- *one-colored* if $\mathcal{O}^\otimes$ is at least one-colored and at-most one colored,
- *almost E-unital* if $\mathcal{O}(\varnothing_V) = *$ whenever there exists some $S \neq *_V \in \mathbb{F}_V$ such that $\mathcal{O}(S) \neq \varnothing$.
- *E-unital* if $\mathcal{O}(\varnothing_V) = *$ whenever $\mathcal{O}(*_V) \neq \varnothing$.
- *almost-unital* if $\mathcal{O}^\otimes$ is almost-E-unital and at least one-colored,
- *unital* if $\mathcal{O}^\otimes$ is E-unital and at least one-colored,
- *almost-E-reduced* if $\mathcal{O}^\otimes$ is almost-E-unital and at-most one colored,
- *E-reduced* if $\mathcal{O}^\otimes$ is E-unital and at-most one colored,
- *almost-reduced* if $\mathcal{O}^\otimes$ is almost-unital and one-colored, and

- *reduced* if $\mathcal{O}^\otimes$ is unital and one-colored. ◁

Construction 2.52. Given $\mathcal{O}^\otimes$ a one-colored $\mathcal{T}$-operad, for any $T \leftarrow \mathrm{Ind}_V^{\mathcal{T}} S$, we have an equivalence

$$\mathcal{O}(S) \simeq \mathrm{Map}_{\pi_{\mathcal{O}}}^{T \leftarrow \mathrm{Ind}_V^{\mathcal{T}} S \to V}(\mathrm{Ind}_V^{\mathcal{T}} S; V)$$

due to the existence of cocartesian lifts for inert morphisms. Hence, given a map $U \to V$ in $\mathcal{T}$, composition in $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ induces a restriction map

$$\begin{tikzcd}
\mathcal{O}(S) \arrow[rr, "\mathrm{Res}^V_U"] \arrow[d, phantom, "\wr"] & & \mathcal{O}(\mathrm{Res}^V_U S) \arrow[d, phantom, "\wr"] \\
\mathrm{Map}_{\pi_{\mathcal{O}}}^{\mathrm{Ind}_V^{\mathcal{T}} S \to V}(\mathrm{Ind}_V^{\mathcal{T}} S; V) \arrow[rr] & & \mathrm{Map}_{\pi_{\mathcal{O}}}^{\mathrm{Ind}_V^{\mathcal{T}} S \leftarrow \mathrm{Ind}_V^{\mathcal{T}} S \times_V U \to U}(\mathrm{Ind}_U^{\mathcal{T}} \mathrm{Res}^V_U S; U)
\end{tikzcd} \tag{14}$$

Furthermore, given a map of V-sets $\varphi_{TS} : T \to S$, write $T_U \simeq T_U \times_S U \to U$ for the pullback, write $iT \in \mathcal{O}_T$ for the object in $\mathcal{O}^\otimes$ corresponding with $\mathrm{Ind}_V^{\mathcal{T}} T$, and write $\varphi_{TV} : iT \to iV$ for the structure map of T. The composition map in $\mathcal{O}^\otimes$ restricts to fibers to yield a structure map

$$\begin{tikzcd}
\mathcal{O}(S) \times \prod\limits_{U \in \mathrm{Orb}(S)} \mathcal{O}(T_U) \arrow[r, "\gamma"] \arrow[d, phantom, "\wr"] & \mathcal{O}(T) \arrow[d, phantom, "\wr"] \\
\mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{SV}}(iS; V) \times \mathrm{Map}_{\pi_{\mathcal{O}}}^{\varphi_{TS}}(iT, iS) \arrow[r] & \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{TV}}(iT; iV)
\end{tikzcd} \tag{15}$$

Lastly, note that every V-equivariant automorphism of S yields an automorphism of $\mathrm{Ind}_V^{\mathcal{T}} S$ over V, leading to an action

$$\rho_S : \mathrm{Aut}_V(S) \times \mathcal{O}(S) \longrightarrow \mathcal{O}(S). \tag{16}$$

We refer to Res^V_U as *restriction*, γ as the *composition*, and ρ_S as Σ-*action*. ◁

Proof of Proposition 2.43. Note that Conditions (IC-a) and (IC-c) are true by assumption (they were forced on us in order to make $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ definable). We verify the conditions of Proposition 2.47 for $I = \mathbb{F}_{\mathcal{T}}$.

Note that $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ has *unique* lifts for backwards maps, so condition (a) follows always. Furthermore, $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ always satisfies condition (b) by construction. Lastly, by unwiding definitions and noting that there exists a map of spaces $X \to Y \times \varnothing = \varnothing$ if and only if X is empty, Observation 1.39 implies that (c) is equivalent to Condition (IC-b). □

Remark 2.53. The structures of Eqs. (14) to (16) are compatible in the following ways:

- The restriction maps are Borel equivariant:

$$\begin{tikzcd}
\{\text{cocart lifts of } \mathrm{Aut}_V(S)\} \times \mathrm{Map}_{\mathcal{O}^\otimes}(S,V) \arrow[rrr, "\circ"] \arrow[dr, equal] \arrow[ddd, bend right, "\mathrm{Res}^V_W"'] & & & \{\text{cocart lifts of } \mathrm{Aut}_V(S)\} \times \mathrm{Map}_{\mathcal{O}^\otimes}(S,V) \arrow[dl, equal] \arrow[ddd, bend left, "\mathrm{Res}^V_W"] \\
& \mathrm{Aut}_V(S) \times \mathcal{O}(S) \arrow[r, "\rho"] \arrow[d, "\mathrm{Res}^V_W"] & \mathcal{O}(S) \arrow[d, "\mathrm{Res}^V_W"] & \\
& \mathrm{Aut}_W(\mathrm{Res}^V_W S) \times \mathcal{O}(\mathrm{Res}^V_W S) \arrow[r, "\rho"] \arrow[dl, equal] & \mathcal{O}(\mathrm{Res}^V_W S) \arrow[dr, equal] & \\
\left\{\text{cocart lifts of } \mathrm{Aut}_W(\mathrm{Res}^V_W S)\right\} \times \mathrm{Map}_{\mathcal{O}^\otimes}(\mathrm{Res}^V_W S, U) \arrow[rrr, "\circ"] & & & \mathrm{Map}_{\mathcal{O}^\otimes}(\mathrm{Res}^V_W S, W)
\end{tikzcd}$$

- The composition maps are Borel $\mathrm{Aut}_V(S) \times \prod\limits_{U \in \mathrm{Orb} S} \mathrm{Aut}_U(T_U)$-equivariant in an analogous way.
- The identity map on $*_V$ yields an element $1_V \in *_V$ which is taken to 1_V by Res^V_U.

- The map γ is unital, i.e. for all $\varphi_{SV} : \mathrm{Ind}_V^{\mathcal{T}} S \to V$, writing iS and iV for the associated objects of $\mathcal{O}^{\otimes}$, the following commutes.

$$\begin{tikzcd}
\mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^{\otimes}}(iS,iV) \arrow[rrr, "{(\mathrm{id},\{\mathrm{id}\})}"] \arrow[dr, equals] \arrow[ddd, bend right, "{(\{\mathrm{id}\},\mathrm{id})}"'] & & & \mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^{\otimes}}(iS,iV) \times \mathrm{Map}^{\mathrm{id}}_{\mathcal{O}^{\otimes}}(iS,iS) \arrow[dl, equals] \arrow[ddd, bend left, "\circ"] \\
& \mathcal{O}(S) \arrow[r, "{(\mathrm{id},(\{1_U\}))}"] \arrow[d, "{(\{1_V\},\mathrm{id})}"'] \arrow[dr, equals] & \mathcal{O}(S) \otimes \bigotimes\limits_{U \in \mathrm{Orb}(S)} \mathcal{O}(*_U) \arrow[d, dashed, "\gamma"] & \\
& \mathcal{O}(*_V) \otimes \mathcal{O}(S) \arrow[r, "\gamma"] \arrow[dl, equals] & \mathcal{O}(S) \arrow[dr, equals] & \\
\mathrm{Map}^{\mathrm{id}}_{\mathcal{O}^{\otimes}}(iV,iV) \times \mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^{\otimes}}(iS,iV) \arrow[rrr, "\circ"] & & & \mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^{\otimes}}(iS,iV)
\end{tikzcd}$$

- The map γ is compatible with restriction; given a composable pair of morphisms

$$\begin{tikzcd}
& \mathrm{Ind}_V^{\mathcal{T}} S \arrow[dr, "\varphi_{SV}"] & \\
\mathrm{Ind}_V^{\mathcal{T}} T \arrow[ur, "\varphi_{TS}"] \arrow[rr, "\varphi_{TV}"] & & V,
\end{tikzcd}$$

and $U \to V$ a map in $\mathcal{T}$,the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^{\otimes}}(iS,iV) \times \mathrm{Map}^{\varphi_{TS}}_{\mathcal{O}^{\otimes}}(iT,iS) \arrow[rrr, "\circ"] \arrow[dr, equals] \arrow[ddd, bend right, "\mathrm{Res}^V_W"'] & & & \mathrm{Map}^{\varphi_{TV}}_{\mathcal{O}^{\otimes}}(iT,iV) \arrow[dl, equals] \arrow[ddd, bend left, "\mathrm{Res}^V_W"] \\
& \mathcal{O}(S) \times \prod\limits_{U \in \mathrm{Orb}(s)} \mathcal{O}(T_U) \arrow[r, "\gamma"] \arrow[d, "\mathrm{Res}^W_V"] & \mathcal{O}(T) \arrow[d, "\mathrm{Res}^W_V"] & \\
& \mathcal{O}\left(\mathrm{Res}^V_W S\right) \times \prod\limits_{U' \in \mathrm{Orb}(S)} \mathcal{O}(T_{U'}) \arrow[r, "\gamma"] \arrow[dl, equals] & \mathcal{O}\left(\mathrm{Res}^V_W S\right) \arrow[dr, equals] & \\
\mathrm{Map}^{\mathrm{Res}^V_W \varphi_{SV}}_{\mathcal{O}^{\otimes}}\left(i\,\mathrm{Res}^V_W S, iW\right) \times \mathrm{Map}^{\mathrm{Res}^V_W \varphi_{TS}}_{\mathcal{O}^{\otimes}}(i\,\mathrm{Res}^V_W T, i\,\mathrm{Res}^V_W S) \arrow[rrr, "\circ"] & & & \mathrm{Map}^{\mathrm{Res}^V_W \varphi_{TV}}_{\mathcal{O}^{\otimes}}(i\,\mathrm{Res}^V_W T, iW)
\end{tikzcd}$$

- The map γ is associative; given a collection of maps and composites

$$\begin{tikzcd}
\mathrm{Ind}_V^{\mathcal{T}} R \arrow[r, "\varphi_{RT}"] \arrow[rrr, bend left, "\varphi_{RV}"] \arrow[rr, bend right, "\varphi_{RS}"'] & \mathrm{Ind}_V^{\mathcal{T}} T \arrow[r, "\varphi_{TS}"] \arrow[rr, bend left, "\varphi_{TV}"] & \mathrm{Ind}_V^{\mathcal{T}} S \arrow[r, "\varphi_{SV}"] & V,
\end{tikzcd}$$

writing $i(-) \coloneqq \mathrm{Ind}_V^{\mathcal{T}} -$ for the associated object of $\mathcal{O}^{\otimes}$, we have

$$\begin{tikzcd}
\mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^{\otimes}}(iS,iV) \times \mathrm{Map}^{\varphi_{TS}}_{\mathcal{O}^{\otimes}}(iT,iS) \times \mathrm{Map}^{\varphi_{RT}}_{\mathcal{O}^{\otimes}}(iR,iT) \arrow[r, "\circ"] \arrow[d, equals] \arrow[dddd, bend right, "\circ"'] & \mathrm{Map}^{\varphi_{TV}}_{\mathcal{O}^{\otimes}}(iT,iV) \times \mathrm{Map}^{\varphi_{RT}}_{\mathcal{O}^{\otimes}}(iR,iT) \arrow[d, equals] \arrow[dddd, bend left, "\circ"] \\
\left(\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S_U)} \mathcal{O}(T_U)\right) \times \prod_{\substack{U \in \mathrm{Orb}(S) \\ W \in \mathrm{Orb}(T_U)}} \mathcal{O}(R_W) \arrow[r, "\gamma"] \arrow[d, equals] & \mathcal{O}(T) \times \prod_{W \in \mathrm{Orb}(T)} \mathcal{O}(R_W) \arrow[d, "\gamma"] \\
\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S)} \left(\mathcal{O}(T_U) \times \prod_{W \in \mathrm{Orb}(T_U)} \mathcal{O}(R_W)\right) \arrow[d, "\gamma"] & \mathcal{O}\left(\coprod_W^T R_W\right) \arrow[d, equals] \\
\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}\left(\coprod_W^{T_U} R_W\right) \arrow[r, "\gamma"] \arrow[d, equals] & \mathcal{O}(R) \arrow[d, equals] \\
\mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^{\otimes}}(iS,iV) \times \mathrm{Map}^{\varphi_{RS}}_{\mathcal{O}^{\otimes}}(iR,iS) \arrow[r, "\circ"] & \mathrm{Map}^{\varphi_{RV}}_{\mathcal{O}^{\otimes}}(iR,iV)
\end{tikzcd}$$

Thus, passing to the homotopy category, the data of a $\mathcal{T}$-operad supplies a discrete genuine $\mathcal{T}$-operad in $\mathrm{ho}\mathcal{S}$ in the sense of Definition 5.4. ◁

2.3.2. *The $\mathcal{T}$-∞-category of $\mathcal{T}$-operads.* In fact, we may lift this to a $\mathcal{T}$-∞-*category* by the following.

Definition 2.54. We show in Proposition A.13 that $\mathrm{Ind}_U^V : \mathrm{Span}(\mathbb{F}_U) \to \mathrm{Span}(\mathbb{F}_V)$ is a Segal morphism for all maps $U \to V$ in $\mathcal{T}$. We refer to the resulting $\mathcal{T}$-∞-category

$$\underline{\mathrm{Op}}_{\mathcal{T}} : \mathcal{T}^{\mathrm{op}} \xrightarrow{(\mathbb{F}_{(-)})} \mathbf{Quad}^{\mathrm{adeq,op}} \xrightarrow{\mathrm{Span}} \mathrm{AlgPatt}^{\mathrm{Seg,op}} \xrightarrow{\mathrm{Fbrs}} \mathbf{Cat}.$$

as the *$\mathcal{T}$-∞-category of $\mathcal{T}$-operads.* ◁

Observation 2.55. The V-value of $\underline{\mathrm{Op}}_{\mathcal{T}}$ is $\mathrm{Op}_V \coloneqq \mathrm{Op}_{\underline{V}}$; the restriction functor $\mathrm{Res}_U^V : \mathrm{Op}_V \to \mathrm{Op}_U$ is implemented by the pullback

$$\begin{tikzcd}
\mathrm{Res}_U^V \mathcal{O}^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes} \arrow[d] \\
\mathrm{Span}(\mathbb{F}_U) \arrow[r] & \mathrm{Span}(\mathbb{F}_V).
\end{tikzcd}$$

◁

2.3.3. *Comparison with Nardin-Shah $\mathcal{T}$-∞-operads.* In Proposition A.1 and Corollary A.8, we prove the following generalization of the contents of [BHS22, §5.2], which identifies our $\mathcal{T}$-operads with those of [NS22].

Proposition 2.56. *Suppose $\mathcal{T}$ is an atomic orbital ∞-category. Then, $s : \underline{\mathbb{F}}_{\mathcal{T},*} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ implements equivalences of categories*

$$\mathbf{Cat}_{\mathcal{T}} \simeq \mathrm{Seg}_{\underline{\mathbb{F}}_{\mathcal{T},*}}(\mathcal{C});$$
$$\mathrm{Op}_{\mathcal{T}} \simeq \mathrm{Fbrs}(\underline{\mathbb{F}}_{\mathcal{T},*}).$$

Remark 2.57. By assumption, if $\mathcal{O}^{\otimes}$ is a fibrous $\underline{\mathbb{F}}_{\mathcal{T},*}$-pattern, it possesses cocartesian lifts over *all* morphisms in the composite $\mathcal{O}^{\otimes} \to \underline{\mathbb{F}}_{\mathcal{T},*} \to \mathcal{T}^{\mathrm{op}}$. Thus, fibrous $\underline{\mathbb{F}}_{\mathcal{T},*}$-patterns possess total $\mathcal{T}$-∞-*categories*, a fact which we will use from time to time. ◁

Definition 2.58. Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ be $\mathcal{T}$-operads, Then, the *$\mathcal{T}$-∞-category of $\mathcal{O}$-algebras in $\mathcal{P}$* is

$$\underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{P}) \coloneqq \underline{\mathrm{Fun}}^{\mathrm{int-cocart}}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T},*}}(s^*\mathcal{O}^{\otimes}, s^*\mathcal{P}^{\otimes}). \qquad ◁$$

Observation 2.59. Via Proposition 2.56, we find that $\Gamma^{\mathcal{T}} \underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{P}) \simeq \mathbf{Alg}_{\mathcal{O}}(\mathcal{P})$. Furthermore, we find that

$$\underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{P})_V \simeq \underline{\mathrm{Fun}}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_V)}(\mathrm{Res}^{\mathcal{T}}_V \mathcal{O}^{\otimes}, \mathrm{Res}^{\mathcal{T}}_V \mathcal{P}^{\otimes}) \simeq \mathbf{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{O}}(\mathrm{Res}^{\mathcal{T}}_V \mathcal{O})$$

with restriction functors induced by functoriality of Res^V_U. ◃

2.4. **$\mathcal{T}$-∞-categories underlying $\mathcal{T}$-operads of algebras.** Recall the *underlying $\mathcal{T}$-category* functor $U : \mathrm{Op}_{\mathcal{T}} \to \mathbf{Cat}_{\mathcal{T}}$ of **??**. In this subsection, we characterize the underlying $\mathcal{T}$-∞-category functor and its relationship with $\overset{\mathrm{BV}}{\otimes}$ and $\underline{\mathbf{Alg}}^{\otimes}_{-}(-)$. One significant reason to study the underlying $\mathcal{T}$-∞-category is the following.

Observation 2.60. In the case $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal category, U is a Segal $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$-pattern and $U(\mathcal{C}^{\otimes})$ its underlying $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})^{\mathrm{el}}$-pattern. Hence the composite map

$$\mathbf{Cat}^{\otimes}_I \to \mathrm{Op}_{\mathcal{T}} \to \mathbf{Cat}_{\mathcal{T}}$$

is conservative by Proposition 2.8. ◃

Warning 2.61. The functor U is *not* conservative on $\mathrm{Op}_{\mathcal{T}}$; indeed, users of ($\mathcal{T}$-)operads will find that they are often describing distinct algebraic theories as corepresented by *one-object* $\mathcal{T}$-operads, yet every map between one-object $\mathcal{T}$-operads is a U-equivalence. ◃

2.4.1. *The $\mathcal{T}$-∞-category underlying $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{T}}(-)$.* Let $\mathrm{triv}^{\otimes}_{\mathcal{T}} := \mathcal{N}^{\otimes}_{\mathbb{F}^{\simeq}_{\mathcal{T}}\infty}$. The following proposition was originally proved as [NS22, Cor 2.4.5], although it will eventually follow as an obvious special case of Proposition 3.8.

Proposition 2.62 ([NS22, Cor 2.4.5])**.** *U implements an equivalence*

$$\mathrm{Op}_{\mathcal{T},/\mathrm{triv}^{\otimes}} \simeq \mathbf{Cat}_{\mathcal{T}};$$

writing $\mathrm{triv}^{\otimes}(\mathcal{C}) := U^{-1}_{/\mathrm{triv}^{\otimes}}(\mathcal{C})$*, these are identified by the property*

$$\underline{\mathbf{Alg}}_{\mathrm{triv}^{\otimes}(\mathcal{C})}(\mathcal{P}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, U(\mathcal{P}^{\otimes}));$$

in particular, $\mathrm{triv}^{\otimes}(-) : \mathbf{Cat}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T}}$ *is a fully faithful left adjoint to the underlying $\mathcal{T}$-category.*

These correspond with operads constructed in Proposition 2.43 if and only if $\mathcal{C}$ has at most one V-object for each V, i.e. $\mathcal{C} = *_{\mathcal{F}} \subset *_{\mathcal{T}}$ for a $\mathcal{T}$-family $\mathcal{F}$. In this case, we write

$$\mathrm{triv}^{\otimes}_{\mathcal{F}} := \mathrm{triv}^{\otimes}(*_{\mathcal{F}}) \simeq \mathcal{N}^{\otimes}_{\mathbb{F}^{\simeq}_{\mathcal{F}}\infty}.$$

Observation 2.63. Proposition 2.62 directly implies that

$$\mathrm{triv}^{\otimes}(\mathcal{C}) \simeq L_{\mathrm{Fbrs}}(\mathcal{C} \to \mathcal{T}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}}));$$

furthermore, if $\mathcal{T}$ posseses a terminal object V, then we have

$$\mathrm{triv}^{\otimes}_{\mathcal{T}} \simeq L_{\mathrm{Fbrs}}(* \to \{V\} \hookrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}}). \quad ◃$$

In Corollary 3.14, we will show that $\mathrm{triv}^{\otimes}_{\mathcal{F}}$ is *idempotent* with respect to the Boardman-Vogt tensor product, and the associated smashing localization implements *$\mathcal{F}$-Borelification.* First, we show that $\mathrm{triv}^{\otimes}_{\mathcal{T}}$ is the $\overset{\mathrm{BV}}{\otimes}$-unit.

Proposition 2.64. *For all $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$, we have $\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathrm{triv}^{\otimes}_{\mathcal{T}}$; hence there exists a natural equivalence*

$$\underline{\mathbf{Alg}}^{\otimes}_{\mathrm{triv}_{\mathcal{T}}}(\mathcal{O}) \to \mathcal{O}^{\otimes}.$$

Proof. By Observation 2.63, the collection of bifunctors $\mathrm{triv}^{\otimes}_{\mathcal{T}} \times \mathcal{O} \to \mathcal{P}$ are precisely the functors of $\mathcal{T}$-operads $\mathcal{O} \to \mathcal{P}$; put another way, this demonstrates that the forgetful natural transformation

$$\mathbf{Alg}_{\mathcal{O} \otimes^{\mathrm{BV}} \mathrm{triv}}(\mathcal{P}) \to \mathbf{Alg}_{\mathcal{O}}(\mathcal{P})$$

is a natural equivalence; Yoneda's lemma then demonstrates that $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}^{\otimes}_{\mathcal{T}} \simeq \mathcal{O}^{\otimes}$.

For the remaining statement, we recite the folklore argument that the unit of a closed symmetric monoidal structure corepresents the identity:

$$\begin{aligned}\mathrm{Map}(\mathcal{O}^{\otimes}, \underline{\mathbf{Alg}}^{\otimes}_{\mathrm{triv}_{\mathcal{T}}}(\mathcal{P})) &\simeq \mathrm{Map}\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}^{\otimes}_{\mathcal{T}}, \mathcal{P}^{\otimes}\right) \\ &\simeq \mathrm{Map}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes})\end{aligned}$$

so Yoneda's lemma yields a natural equivalence $\underline{\mathbf{Alg}}^{\otimes}_{\mathrm{triv}_{\mathcal{T}}}(\mathcal{P}) \simeq \mathcal{P}^{\otimes}$. □

Using this, we have a sequence of natural equivalences

$$\begin{aligned}U\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) &\simeq \underline{\mathbf{Alg}}_{\mathrm{triv}_{\mathcal{T}}}\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) \\ &\simeq \underline{\mathbf{Alg}}_{\mathcal{O}}\underline{\mathbf{Alg}}^{\otimes}_{\mathrm{triv}_{\mathcal{T}}}(\mathcal{P}) \\ &\simeq \underline{\mathbf{Alg}}_{\mathcal{O}\otimes\mathrm{triv}_{\mathcal{T}}}(\mathcal{P}) \\ &\simeq \underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{P});\end{aligned}$$

in particular, we've proved the following corollary.

Corollary 2.65. *There exists a natural equivalence*

$$U\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) \simeq \underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{P}).$$

2.5. **Envelopes and coherences for the $\mathcal{T}$-BV tensor product.** In [NS22], a left adjoint to the inclusion $U : \mathrm{CMon}_{\mathcal{T}}\mathbf{Cat} \to \mathrm{Op}_{\mathcal{T}}$ was constructed, called the *$\mathcal{T}$-symmetric monoidal envelope*. This was greatly generalized by Theorem 2.26 in view of Propositions 2.47 and 2.56. For convenience, we spell this out here.

Corollary 2.66. *If $\mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a map of $\mathcal{T}$-operads, then the following diagram consists of maps of $\mathcal{T}$-operads*

$$\begin{tikzcd} \mathrm{Env}_{\mathcal{O}}\mathcal{P}^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ar}^{\mathrm{act}}(\mathcal{O}^{\otimes}) \arrow[r, "t"] \arrow[d, "s"] & \mathcal{O}^{\otimes} \\ \mathcal{P}^{\otimes} \arrow[r] & \mathcal{O}^{\otimes} & \end{tikzcd}$$

and the top horizontal composition is an $\mathcal{O}$-monoidal ∞-category.

When $\mathcal{O}^{\otimes} \simeq \mathcal{N}^{\otimes}_{I\infty}$, we simply write $\mathrm{Env}_I(-) := \mathrm{Env}_{\mathcal{N}_{I\infty}}(-)$; when $\mathcal{O}^{\otimes} \simeq \mathrm{Comm}^{\otimes}_{\mathcal{T}}$, we write $\mathrm{Env}(-) := \mathrm{Env}_{\mathrm{Comm}_{\mathcal{T}}}(-)$. Record a convenient property of the $\mathcal{T}$-symmetric monoidal envelope here, which follows by unwinding definitions, and allows us to reduce some structural questions about I-operads to equivariant higher category theory.

Lemma 2.67. *If $\mathcal{O} \in \mathrm{Op}^{\mathrm{oc}}_I$, the mapping fiber of $\mathrm{Env}_I(\mathcal{O})$ over a map $\psi : S \to T$ is*

$$\mathrm{Map}^{\psi}_{\mathrm{Env}_I(\mathcal{O}) \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}(iS; iT) \simeq \mathrm{Map}^{\psi}_{\mathcal{O}^{\otimes}}(iS; iT)$$

Example 2.68. Let I be a weak indexing category. Then, unwinding definitions, we find that

$$\mathrm{Env}_I \mathcal{N}^{\otimes}_{I\infty} \simeq \underline{\mathbb{F}}^{I-\sqcup}_{I},$$

where $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is the full $\mathcal{T}$-subcategory defined in Section 1.2, i.e. it is the I-symmetric monoidal subcategory generated by $\{*_V \mid V \in c(I)\}$. ◁

Example 2.69. In the case of $\mathcal{N}^{\otimes}_{I\infty} \in \mathrm{Op}_{\mathcal{T}}$, unwinding definitions, we find that

$$\mathrm{Ob}(\mathrm{Env}\mathcal{N}_{I\infty})_V = \begin{cases} \mathbb{F}_V & V \in c(I) \\ \emptyset & \text{otherwise.} \end{cases} \mathbb{F}_{c(I)};$$

furthermore, applying Lemma 2.67, we find that

$$\mathrm{Map}^{\psi}_{\mathrm{Env}(\mathcal{O})_V \to \mathbb{F}_V}(iS; iT) = \mathrm{Map}^{\psi}_{\mathcal{O}^{\otimes}}(iS; iT) = \begin{cases} * & \mathrm{Ind}^{\mathcal{T}}_V S \to \mathrm{Ind}^{\mathcal{T}}_V T \in I; \\ \emptyset & \text{otherwise.} \end{cases}$$

In particular, this is a $\mathcal{T}$-symmetric monoidal (non-full) subcategory of $\mathrm{EnvComm}_{\mathcal{T}}^{\otimes} \simeq \mathbb{F}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$, which we denote by

$$\underline{\mathbb{F}}_{I-\mathrm{wide}}^{\mathcal{T}-\sqcup} \subset \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}.$$

By inspection, $\mathrm{Env}_{\mathcal{T}} : \mathrm{Op}_{\mathcal{T}} \to \mathbf{Cat}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}^{\sqcup}}$ retricts to an embedding of posets $\mathrm{wIndex}_{\mathcal{T}} \hookrightarrow \mathrm{Sub}_{\mathbf{Cat}_{\mathcal{T}}^{\times}}(\underline{\mathbb{F}}_{\mathcal{T}}^{\sqcup})$ with image the subcategories which are equifibered in the sense of [BHS22]. ◁

We proved that $\mathrm{Env} : \underline{\mathrm{Op}}_{\mathcal{T}} \to \underline{\mathrm{CMon}}_{\mathcal{T}}$ was compatible with the localized Day convolution tensor product as Proposition 2.38, and that these are the binary tensor products in the mode symmetric monoidal structure on $\underline{\mathbf{Cat}}_{\mathcal{T}}^{\otimes}$ in Theorem 1.70, i.e.

$$\mathrm{Env}\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right) \simeq \mathrm{Env}\mathcal{O}^{\otimes} \otimes^{\mathrm{mode}} \mathrm{Env}\mathcal{P}^{\otimes}. \tag{17}$$

In particular, we prove in Corollary F without using coherences that $\mathrm{Comm}_{\mathcal{T}}^{\otimes}$ is $\overset{\mathrm{BV}}{\otimes}$-idempotent, so Eq. (17) implies that $\mathrm{EnvComm}_{\mathcal{T}}^{\otimes} \simeq \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$ is $\otimes^{\mathrm{mode}}$-idempotent. We use this in the following to Eq. (17) to a sliced statement, canonically lifting $\overset{\mathrm{BV}}{\otimes}$ to a symmetric monoidal structure.

Proposition 2.70. *$\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes} \subset \underline{\mathbf{Cat}}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$ is $\otimes$-closed, and $\otimes$ acts on $\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes}$ as $\overset{BV}{\otimes}$; hence there exists a unique symmetric monoidal $\mathcal{T}$-∞-category lifting $\overset{BV}{\otimes}$ such that the composite $\mathcal{T}$-functor*

$$\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes} \to \underline{\mathbf{Cat}}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}} \to \underline{\mathbf{Cat}}_{\mathcal{T}}^{\otimes}$$

is symmetric monoidal.

Proof. Eq. (17) yields a commutative diagram

$$\begin{tikzcd}
\mathrm{Env}\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right) \arrow[r, "\sim"] \arrow[d] & \mathrm{Env}(\mathcal{O}^{\otimes}) \otimes^{\mathrm{Mode}} \mathrm{Env}(\mathcal{P}^{\otimes}) \arrow[d] \\
\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \arrow[r, "\sim", "\eta"'] & \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \otimes^{\mathrm{Mode}} \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}
\end{tikzcd}$$

Inverting the bottom map, we find that we've constructed an equivalence

$$\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\left(\mathcal{O}^{\otimes} \otimes^{\mathrm{BV}} \mathcal{P}^{\otimes}\right) \simeq \wedge_!\left(\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\left(\mathcal{O}^{\otimes}\right) \otimes^{\mathrm{mode}} \mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\left(\mathcal{P}^{\otimes}\right)\right) \simeq \mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\left(\mathcal{O}^{\otimes}\right) \otimes^{\mathrm{mode}}_{/\underline{\mathbb{F}}_{\mathcal{T}}} \mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\left(\mathcal{P}^{\otimes}\right),$$

i.e. full $\mathcal{T}$-subcategory $\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}} : \underline{\mathrm{Op}}_{\mathcal{T}} \subset \underline{\mathbf{Cat}}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$ is $\otimes$-closed and the induced symmetric monoidal structure has bifunctor $\overset{\mathrm{BV}}{\otimes}$, as desired. □

Corollary 2.71. *When $\mathcal{T} = *$, there is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Op}_*^{\otimes} \simeq \mathrm{Op}_{\infty}^{\otimes},$$

where the latter is the Boardman-Vogt symmetric monoidal ∞-category of [BS24a]. In particular, this takes $\overset{BV}{\otimes}$ to the Boardman-Vogt tensor product of [BV73; HM23; HA].

Proof. After Proposition A.1 and Corollary A.8, what remains is to produce a symmetric monoidal structure on the equivalence $\mathrm{Op}_* \simeq \mathrm{Op}_{\infty}$ over $\mathbf{Cat}_{\infty}^{\otimes}$. In fact, the forgetful functor $\mathbf{Cat}^{\otimes}_{\infty,\mathbb{F}^{\sqcup}} \to \mathbf{Cat}_{\infty}^{\otimes}$ is symmetric monoidal (as all "unslicing" forgetful functors are), so Theorem A constructs a symmetric monoidal structure on the composite induced $\mathrm{Op}_*^{\otimes} \to \mathbf{Cat}_{\infty}^{\otimes}$, the latter having the mode symmetric monoidal structure. In fact, by [BS24a, Thm E], there is a *unique* such structure, so the equivalence is symmetric monoidal, and $\overset{\mathrm{BV}}{\otimes}$ is taken to the tensor functor in $\mathrm{Op}_{\infty}^{\otimes}$, which is the tensor product of [HA]. □

2.6. **The underlying $\mathcal{T}$-symmetric sequence.** Set the notation $\underline{\Sigma}_{\mathcal{T}} := \underline{\mathbb{F}}_{\mathcal{T},*}^{\simeq}$, where the latter is the $\mathcal{T}$-space core of Example 1.35. We refer to this as the *$\mathcal{T}$-symmetric $\mathcal{T}$-category*, and we refer to $\mathrm{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}},\mathcal{C})$ as the *∞-category of $\mathcal{T}$-symmetric sequences in $\mathcal{C}$*; in the case $\mathcal{C} = \underline{\mathcal{S}}_{\mathcal{T}}$, we refer to $\mathrm{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}},\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Fun}(\mathrm{tot}\underline{\Sigma}_{\mathcal{T}},\mathcal{S})$ simply as the *∞-category of $\mathcal{T}$-symmetric sequences*.

Observation 2.72. For any adequate triple $(\mathcal{X},\mathcal{X}_b,\mathcal{X}_f)$, the inclusion

$$\mathcal{X} \hookrightarrow \mathrm{Span}_{b,f}(\mathcal{X})$$

induces an equivalence on cores. In particular, choosing $(\underline{\mathbb{F}}_{\mathcal{T}},\underline{\mathbb{F}}_{\mathcal{T}}^{s.i.},\underline{\mathbb{F}}_{\mathcal{T}})$ (c.f. **??**), we find that the inclusion $(-)_+ : \underline{\mathbb{F}}_{\mathcal{T}} \to \underline{\mathbb{F}}_{\mathcal{T},*}$ induces an equivalence

$$\underline{\mathbb{F}}_{\mathcal{T}}^{\simeq} \simeq \underline{\mathbb{F}}_{\mathcal{T},*}^{\simeq} \simeq \underline{\Sigma}_{\mathcal{T}}.$$

In particular, unwinding definitions, we have the computation

$$\underline{\Sigma}_{\mathcal{T},/V} \simeq \mathbb{F}_V^{\simeq} \simeq \coprod_{S\in\mathbb{F}_V} B\,\mathrm{Aut}_V\, S$$

and that the restriction map $\underline{\Sigma}_{\mathcal{T},/V} \to \underline{\Sigma}_{\mathcal{T},/W}$ is induced by the forgetful maps $B\,\mathrm{Aut}_V\,S \to B\,\mathrm{Aut}_W\,S$. ◁

Observation 2.73. Under the equivalence $\mathrm{Op}_{\mathcal{T}} \simeq \mathrm{Fbrs}(\underline{\mathbb{F}}_{\mathcal{T},*})$, by Proposition 2.62, $\mathrm{triv}_{\mathcal{T}}^{\otimes}$ is modeled by the inclusion $\underline{\Sigma}_{\mathcal{T}} \hookrightarrow \underline{\mathbb{F}}_{\mathcal{T},*}$. Every morphism in the associated factorization system on $\underline{\Sigma}_{\mathcal{T}}$ is equivalent to an inert morphism; hence there exist equivalences

$$\mathbf{Cat}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathrm{tot}\underline{\Sigma}_{\mathcal{T}}} \simeq \mathrm{Fun}(\mathrm{tot}\underline{\Sigma}_{\mathcal{T}},\mathbf{Cat}) \simeq \mathrm{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}},\underline{\mathbf{Cat}}_{\mathcal{T}}). \qquad ◁$$

Construction 2.74. Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}^{\mathrm{red}}$, there is a structure map

$$\mathrm{Env}_{\mathcal{O}}\mathrm{triv}_{\mathcal{T}} \simeq \mathrm{triv}_{\mathcal{T}}^{\otimes} \times_{\mathrm{Comm}_{\mathcal{T}}^{\otimes}} \mathrm{Ar}^{\mathrm{act},/\mathrm{el}}(\mathcal{O}) \to \mathrm{triv}_{\mathcal{T}}^{\otimes}$$

which is an inert-cocartesian fibration by pullback-stability of inert-cocartesian fibrations [BHS22, Obs 2.1.7]. The *underlying $\mathcal{T}$-symmetric sequence of $\mathcal{O}^{\otimes}$* is

$$\mathcal{O}^{\otimes}_{\mathrm{sseq}} := \mathrm{Un}_{\mathrm{triv}_{\mathcal{T}}}\mathrm{Env}_{\mathcal{O}}\mathrm{triv}_{\mathcal{T}} \in \mathrm{Fun}(\mathrm{tot}\underline{\Sigma}_{\mathcal{T}},\mathbf{Cat}).$$

Unwinding definitions, we find that there exists a cartesian square

$$\begin{tikzcd}
\mathcal{O}(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Env}_{\mathcal{O}}\mathrm{triv} \arrow[r, equal] \arrow[d] & \mathrm{tot}\underline{\Sigma}_{\mathcal{T}} \times_{\underline{\mathbb{F}}_{\mathcal{T}}} \mathrm{Ar}^{\mathrm{act},/\mathrm{el}}(\mathcal{O}) \arrow[d] \\
* \arrow[r, "S"] & \mathrm{triv}^{\otimes} \arrow[r, equal] & \mathrm{tot}\underline{\Sigma}_{\mathcal{T}}
\end{tikzcd}$$

so that $\mathcal{O}^{\otimes}_{\mathrm{sseq}}$ is indeed a $\mathcal{T}$-symmetric sequence. The associated functor is denoted

$$\mathrm{sseq} : \mathrm{Op}_{\mathcal{T}} \to \mathrm{Fun}(\mathrm{tot}\underline{\Sigma}_{\mathcal{T}},\mathcal{S}). \qquad ◁$$

We will often use the following to reduce questions about $\mathcal{T}$-operads to $\mathcal{T}$-symmetric sequences.

Proposition 2.75. *Suppose a functor of $\mathcal{T}$-operads $\varphi : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ satisfies the following conditions:*

(a) φ induces surjective maps $\pi_0\mathcal{O}_V \to \pi_0\mathcal{P}_V$ for all $V \in \mathcal{T}$, and

(b) for all $V \in \mathcal{T}$, all $S \in \mathbb{F}_V$, all $\mathbf{C} \in \mathcal{O}_S$, and all $D \in \mathcal{O}_V$, the map φ induces equivalences $\varphi : \mathcal{O}(\mathbf{C};D) \xrightarrow{\sim} \mathcal{P}(\varphi\mathbf{C};\varphi D)$.

Then φ is an equivalence of $\mathcal{T}$-operads; in particular, the restricted functor

$$\mathrm{sseq} : \mathrm{Op}_{\mathcal{T}}^{\mathrm{oc}} \to \mathrm{Fun}(\mathrm{tot}\underline{\Sigma}_{\mathcal{T}},\mathcal{S})$$

is conservative.

To prove this, we proceed by reduction to the following observation.

Observation 2.76. If $\mathcal{C} \to \mathcal{D}$ is an equivalence of categories over $\mathcal{E}$, then it preserves and reflects cocartesian lifts of arrows in $\mathcal{E}$; in particular, if $\varphi : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a morphism of $\mathcal{T}$-operads who induces an equivalence $\mathrm{tot}\varphi : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ between the total ∞-categories of the associated functors to $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$, then its inverse is also a morphism of $\mathcal{T}$-operads. Said another way, we've observed that the functor $\mathrm{tot} : \mathrm{Op}_{\mathcal{T}} \to \mathbf{Cat}$ is conservative. ◁

Proof of Proposition 2.75. In view of Construction 2.74, the second statement follows immediately from the first, since morphisms of reduced $\mathcal{T}$-operads are automatically π_0-isomorphisms by two-out-of-three. Fixing φ satisfying (a) and (b), we will prove that φ is an equivalence of $\mathcal{T}$-operads. Using Observation 2.76, it suffices to prove that $\mathrm{tot}\varphi$ is an equivalence of ∞-categories.

By the Segal condition for colors, we have an equivalence of arrows

$$\begin{tikzcd} \pi_0\mathcal{O}_S \arrow[r, phantom, "\simeq"] \arrow[d, "\varphi_S"] & \prod_{V\in \mathrm{Orb}(S)} \pi_0\mathcal{O}_V \arrow[d, "\prod \varphi_V"] \\ \pi_0\mathcal{P}_S \arrow[r, phantom, "\simeq"] & \prod_{V\in \mathrm{Orb}(S)} \pi_0\mathcal{P}_V \end{tikzcd}$$

Since $\pi_0\mathcal{O} \simeq \coprod_S \pi_0\mathcal{O}_S$, (a) implies that φ is essentially surjective. Furthermore, the Segal condition for multimorphisms yields isomorphisms of arrows

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{O}^\otimes}(\mathbf{C},\mathbf{D}) \arrow[r, phantom, "\simeq"] \arrow[d, "\varphi"] & \coprod\limits_{f:\pi C\rightarrow \pi D} \mathrm{Map}^f_{\mathcal{O}}(\mathbf{C};\mathbf{D}) \arrow[r, phantom, "\simeq"] \arrow[d, "\coprod\varphi"] & \coprod\limits_{f} \prod\limits_{V\in \mathrm{Orb}(\pi(D))} \mathrm{Map}^{f_V}_{\mathcal{O}}(\mathbf{C}_{f_V^{-1}};D_V) \arrow[r, phantom, "\simeq"] \arrow[d, "\coprod\prod\varphi"] & \coprod\limits_f\prod\limits_V \mathcal{O}(\mathbf{C}_{f^{-1}V};D_V) \arrow[d, "\coprod\prod\varphi(T_V)"] \\ \mathrm{Map}_{\mathcal{P}^\otimes}(\varphi\mathbf{C},\varphi\mathbf{D}) \arrow[r, phantom, "\simeq"] & \coprod\limits_{f:\pi\mathbf{C}\rightarrow \pi\mathbf{D}} \mathrm{Map}^f_{\mathcal{P}}(\varphi\mathbf{C};\varphi\mathbf{D}) \arrow[r, phantom, "\simeq"] & \coprod\limits_{f} \prod\limits_{V\in \mathrm{Orb}(S)} \mathrm{Map}^{f'}_{\mathcal{P}}(\varphi\mathbf{C}_{f^{-1}V},\varphi D_V) \arrow[r, phantom, "\simeq"] & \coprod\limits_f\prod\limits_V \mathcal{P}(\varphi\mathbf{C}_{f^{-1}V};\varphi D_V). \end{tikzcd}$$

the right arrow is an equivalence by (b), so the leftmost arrow is an equivalence, hence φ is fully faithful. $\square$

The author learned the $U_\circ$ portion of the following argument from Thomas Blom.

Corollary 2.77. *The functor* $\mathrm{sseq}_{\mathcal{T}} : \mathrm{Op}^{oc}_{\mathcal{T}} \rightarrow \mathrm{Fun}(\mathrm{tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ *is monadic and preserves sifted colimits.*

Proof. By [BHS22, Cor 4.2.2], $\mathrm{Op}^{\mathrm{red}}_{\mathcal{T}}$ and $\mathrm{Fun}(\mathrm{tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ are presentable, so by Barr-Beck [HA, Thm 4.7.3.5] and the adjoint functor theorem [HTT, Cor 5.5.2.9], it suffices to prove that sseq is conservative and preesrves limits and sifted colimits. Conservativity is Proposition 2.75, and (co)limits in functor categories are computed pointwise by [HTT, Prop 5.1.2.2], so it suffices to prove that $\mathcal{O} \mapsto \mathcal{O}(S)$ preseres limits and sifted colimits. We separate this into manageable chunks via the following diagram:

$$\begin{tikzcd} \mathrm{Op}^{\mathrm{oc}}_{\mathcal{T}} \arrow[rr, "\mathcal{O}\mapsto\mathcal{O}(S)"] \arrow[d, "U_{\mathrm{Seg}}"] & & \mathcal{S} & & \mathcal{S}^{\pi_0\mathrm{Map}(\mathrm{Ind}_V^{\mathcal{T}}S,V)} \arrow[ll, "\pi"'] \\ \mathbf{Cat}^{\mathrm{Int-cocart,core-iso}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})} \arrow[r, "U_{\mathrm{cocart}}"] & \mathbf{Cat}^{\mathrm{core-iso}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})} \arrow[rrr, "U_\circ"] & & & \mathrm{Fun}\left((\mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\simeq})^{\times 2}, \mathcal{S}\right) \arrow[u, "\mathrm{ev}_{\mathrm{Ind}_V^{\mathcal{T}}S,V}"] \end{tikzcd}$$

π and $\mathrm{ev}_{\mathrm{Ind}_V^{\mathcal{T}}S,V}$ preserve (co)limits since they are evaluation of functor categories [HTT, Prop 5.1.2.2]. U_{Cocart} preserves limits and sifted colimits by [BHS22, Cor 2.1.5]. U_{Seg} preserves limits and sifted colimits, as each commute with finite products.

By [Hau20, Prop 3.12], $U_\circ$ is equivalent to the forgetful functor

$$\mathbf{Alg}(\mathcal{S}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\simeq},\mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\simeq}}) \rightarrow \mathcal{S}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}}),\mathrm{Span}(\mathbb{F}_{\mathcal{T}})},$$

where $\mathcal{S}^{\otimes}_{/Y,Y}$ is a symmetric monoidal structure on $\mathcal{S}_{/Y,Y} \simeq \mathcal{S}_{Y\times Y} \simeq \mathrm{Fun}(Y\times Y, \mathcal{S})$. This functor preserves limits and sifted colimits by [HA, Prop 3.2.3.1], completing the argument. $\square$

In particular, this constructs a left adjoint

$$\mathrm{Fr} : \mathrm{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \underline{\mathcal{S}}_{\mathcal{T}}) = \mathrm{Fun}(\mathrm{tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S}) \rightarrow \mathrm{Op}^{\mathrm{oc}}_{\mathcal{T}}$$

to sseq. We lift this to a $\mathcal{T}$-adjunction in the following construction.

Construction 2.78. The functor sseq is associated with a $\mathcal{T}$-functor $\underline{\mathrm{sseq}}$ as in the following diagram

$$\begin{tikzcd}
 & & & & \mathrm{Ar}^{\mathrm{act,/el}}(\mathcal{O}^{\otimes}) \arrow[d, dashed, "s"] \\
\mathcal{O}^{\otimes} \arrow[r, mapsto, bend left] & \mathrm{triv}_{\mathcal{T}}^{\otimes} \arrow[r] & \mathcal{O}^{\otimes} \arrow[rr, mapsto, bend left] & \mathrm{triv}_{\mathcal{T}}^{\otimes} \arrow[r] & \mathcal{O}^{\otimes} \arrow[ddddd, mapsto, bend left] \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\
\underline{\mathrm{Op}}^{\mathrm{oc}}_{\mathcal{T}} \arrow[r, hook] \arrow[d, "\underline{\mathrm{sseq}}"'] & \underline{\mathrm{Op}}_{\mathcal{T},\mathrm{triv}^{\otimes}_{\mathcal{T}}/} \arrow[rr] & & \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}}\Lambda^2_2, \mathrm{Op}_{\mathcal{T}}\right) \times_{\underline{\mathrm{Op}}_{\mathcal{T}}} \left\{\mathrm{triv}_{\mathcal{T}}^{\otimes}\right\} \arrow[d] & \\
\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\underline{\Sigma}_{\mathcal{T}}, \underline{\mathcal{S}}_{\mathcal{T}}\right) \arrow[r, equal] & \mathrm{Op}_{\mathcal{T},/\mathrm{triv}^{\otimes}_{\mathcal{T}}} & & \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}}\Delta^2_1, \underline{\mathrm{Op}}_{\mathcal{T}}\right) \times_{\underline{\mathrm{Op}}_{\mathcal{T}}} \left\{\mathrm{triv}_{\mathcal{T}}^{\otimes}\right\} \arrow[ll, "U"] & \\
\rotatebox{90}{$\ni$} & \rotatebox{90}{$\ni$} & & \rotatebox{90}{$\ni$} & \\
\mathrm{sseq}\,\mathcal{O}^{\otimes} & \mathrm{Env}_{\mathcal{O}}\mathrm{triv} \arrow[l, mapsto] \arrow[d] & & \mathrm{Env}_{\mathcal{O}}\mathrm{triv} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[ll, mapsto, shift right=4] & \mathrm{Ar}^{\mathrm{act,/el}}(\mathcal{O}^{\otimes}) \arrow[d, dashed, "s"] \\
 & \mathrm{triv}_{\mathcal{T}}^{\otimes} & & \mathrm{triv}_{\mathcal{T}}^{\otimes} \arrow[r] & \mathcal{O}^{\otimes}
\end{tikzcd}$$

By [HA, Prop 7.3.2.1], the pointwise left adjoints Fr lifts to a $\mathcal{T}$-adjunction

$$\underline{\mathrm{sseq}} : \underline{\mathrm{Op}}^{\mathrm{red}}_{\mathcal{T}} \leftrightarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \underline{\mathcal{S}}_{\mathcal{T}}) : \underline{\mathrm{Fr}},$$

i.e. $\underline{\mathrm{Fr}}$ is compatible with restriction. ◁

2.7. **$\mathcal{O}$-algebras in I-symmetric monoidal d-categories.** Recall that a space X is said to be *d-truncated* if it is empty or $\pi_n(X,x) = *$ for all $x \in X$ and $n > 0$; in particular, X is (-1)-truncated precisely if it is either empty or contractible. In Section 1.4, we applied this to mapping spaces to define *$\mathcal{T}$-symmetric monoidal d-categories*. In this section, we define a compatible notion of *$\mathcal{T}$-d-operads*, centered on the following result.

Proposition 2.79. *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-operad and let $d \geq -1$. Then, the following conditions are equivalent:*

- *(a) $\mathcal{O}(S)$ is d-truncated for all $S \in \mathbb{F}_V$.*
- *(b) The $\mathcal{T}$-functor $\mathrm{Env}\mathcal{O} \to \underline{\mathbb{F}}_{\mathcal{T}}$ has d-truncated mapping fibers.*

Proof. Let $\psi : T \to S$ be a map of $\mathcal{T}$-sets over V. Then, by Lemma 2.67, we have an equivalence

$$\begin{aligned}
\mathrm{Map}^{\psi}_{\mathrm{Env}\mathcal{O} \to \underline{\mathbb{F}}_{\mathcal{T}}}(iT, iS) &\simeq \mathrm{Map}^{\psi}_{\mathcal{O}^{\otimes}}(iT, iS) \\
&\simeq \prod_{U \in \mathrm{Orb}(S))} O(T_U).
\end{aligned}$$

Given $S \in \mathbb{F}_V$, choosing $\psi : S \to *_V$ shows (b) implies (a). Conversely, since a product of spaces is (d)-truncated precisely when its factors are, (a) implies (b). □

We define the full subcategory of *d-operads*

$$\iota_d : \mathrm{Op}_{\mathcal{T},d} \hookrightarrow \mathrm{Op}_{\mathcal{T}}$$

to be spanned by $\mathcal{T}$-operads satisfying the condition that $\mathcal{O}(S)$ is $(d-1)$-truncated for all $S \in \mathbb{F}_V$ as in Proposition 2.79. The following corollary then immediately follows from Proposition 2.79 and the mapping fiber truncation characterizations of Corollary 1.78.

Corollary 2.80. *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-operad and let $d \geq 1$. The following conditions are equivalent:*

- *(a) $\mathcal{O}$ is a d-operad, and*
- *(b) $\mathrm{Env}\mathcal{O}^{\otimes}$ is a $\mathcal{T}$-symmetric monoidal d-category.*

Furthermore, the following conditions are equivalent:

- *(a') $\mathcal{O}$ is a 0-operad, and*
- *(b') the $\mathcal{T}$-symmetric monoidal functor $\mathrm{Env}\mathcal{O}^{\otimes} \to \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$ is a $\mathcal{T}$-symmetric monoidal subcategory inclusion.*

Corollary 2.81. *The inclusion* $\mathrm{Op}_{\mathcal{T},d} \hookrightarrow \mathrm{Op}_{\mathcal{T}}$ *has a left adjoint* $h_{\mathcal{T},d}$ *satisfying*

$$(h_{\mathcal{T},d}\mathcal{O})(S) \simeq \tau_{\leq d}\mathcal{O}(S).$$

Furthermore, when $d \geq 1$*, this fits into the following diagram*

$$\begin{tikzcd} \mathrm{Op}_{\mathcal{T}} \arrow[r, "h_{\mathcal{T},d}"] \arrow[d] & \mathrm{Op}_{\mathcal{T},d} \arrow[d] \\ \mathbf{Cat}^{\otimes}_{\mathcal{T}} \arrow[r, "h_{\mathcal{T},d}"] & \mathbf{Cat}^{\otimes}_{\mathcal{T},d} \end{tikzcd}$$

In particular, when $\mathcal{C}^{\otimes}$ *is a* $\mathcal{T}$*-symmetric monoidal* d*-category, the canonical map* $\mathcal{O}^{\otimes} \to h_{\mathcal{T},d}\mathcal{O}^{\otimes}$ *induces an equivalence*

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}) \simeq \mathbf{Alg}_{h_{\mathcal{T},d}\mathcal{O}}(\mathcal{C}).$$

Proof. By [BHS22, Prop 4.2.1], the image of the fully faithful functor $\mathrm{Op}_{\mathcal{T}} \hookrightarrow \mathbf{Cat}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$ is spanned by the *equifibered* $\mathcal{T}$-symmetric monoidal ∞-categories, i.e. $\mathcal{C}^{\otimes}$ such that, given $T \to S$ a map of finite $\mathcal{T}$-sets, the associated diagram

$$\begin{tikzcd} \mathcal{C}_T \arrow[r] \arrow[d] & \mathcal{C}_S \arrow[d] \\ \mathbb{F}_T \arrow[r] & \mathbb{F}_S \end{tikzcd}$$

is cartesian. We separately argue in the case $d \geq 1$ and $d = 0$ that the image of this is closed under $h_{\mathcal{T},d}$; this will imply that $h_{\mathcal{T},d}\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\mathcal{O}^{\otimes}$ corresponds with a $\mathcal{T}$-d-operad $h_{\mathcal{T},d}\mathcal{O}^{\otimes}$, which computes the left adjoint to the inclusios $\mathrm{Op}_{\mathcal{T},d} \subset \mathrm{Op}_{\mathcal{T}}$ by fully faithfulness of $\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\mathcal{O}^{\otimes}$.

We first consider the case $d \geq 1$. In this case, since $h_{\mathcal{T},d} : \mathbf{Cat}^{\otimes}_{\mathcal{T}} \to \mathbf{Cat}^{\otimes}_{\mathcal{T},d}$ is applied pointwise, it preserves equifibrations, so $h_{\mathcal{T},d}\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\mathcal{O}^{\otimes}$ corresponds with a d-operad $h_{\mathcal{T},d}\mathcal{O}^{\otimes}$.

The case $d = 0$ is similar, except that we are tasked with replacing equifibered $\mathcal{T}$-symmetric monoidal functors with an equifibered subcategory. In fact, subcategories are precisely (-1)-truncated maps in **Cat**, so we may do this by taking the pointwise (-1)-truncation functor and applying [HTT, Prop 5.5.6.5] to see that the result is equifibered. □

Corollary 2.82. *Let* $\mathcal{O}^{\otimes}$ *be a* $\mathcal{T}$*-*d*-operad.*

(1) if $d \geq 1$*, then* $\mathbf{Alg}_{\mathcal{O}}(\mathcal{P})$ *is a* d*-category; hence* $\mathrm{Op}_{\mathcal{T},d}$ *is a* $(d+1)$*-category.*
(2) if $d = 0$*, then* $\mathbf{Alg}_{\mathcal{O}}(\mathcal{P})$ *is either empty or contractible; hence* $\mathrm{Op}_{\mathcal{T},0}$ *is a poset.*

Proof. In each case, the second statement follows from the first by noting that the mapping spaces in $\mathrm{Op}_{\mathcal{T}}$ are $\mathbf{Alg}_{\mathcal{O}}(\mathcal{P})^{\simeq}$. For the first statements, note that

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{P}) \simeq \mathbf{Alg}_{h_d\mathcal{O}}(\mathcal{P}) \simeq \mathrm{Fun}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}(\mathrm{Env}h_d\mathcal{O}^{\otimes}, \mathrm{Env}\mathcal{P}^{\otimes});$$

if $d \geq 1$, then this is a subcategory of a d-category, so it's a d-category. If $d = 0$, then this category is either empty or contractible since we verified that the map $\mathrm{Env}\mathcal{O}^{\otimes} \to \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$ is monic. □

3. Equivariant arities and support

Indexing systems were first defined in [BH15], and conjectured to classify the $\mathcal{N}_{I\infty}$-operads. This was separately verified in [BP21; GW18; NS22; Rub21a], each time introducing a different combinatorial expression for indexing systems. These have seen extensive combinatorial study in e.g. [BBR21; BHKKNOPST23; FOOQW22; HMOO22], which we do not repeat here. Instead, we carry out this program for the class of *arbitrary* suboperads of $\mathrm{Comm}^{\otimes}_{\mathcal{T}}$, who may not contain colors above all orbits or cotain fold maps for all of its colors; these will be called *weak* $\mathcal{N}_{\infty}$*-operads.*

In Section 3.1, we finally define the *arity support* functor $A : \mathrm{Op}_{\mathcal{T}} \to \mathrm{wIndex}_{\mathcal{T}}$. We go on in to finally define weak $\mathcal{N}_{\infty}$-operads, initially as the class of $\mathcal{T}$-0-operads; we show that they are the image of a fully faithful right adjoint to A in Corollary 3.10. Following these, in Section 3.2 we construct and characterize the *arity-Borelification* and *restriction* adjunctions

$$\begin{tikzcd} \mathrm{Op}_I \arrow[r, bend left, "E_I^J"] & \mathrm{Op}_J \arrow[l, bend left, "\mathrm{Bor}_I^J"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Op}_V \arrow[r, bend left, "\mathrm{Ind}_V^W"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, bend right, "\mathrm{CoInd}_V^W"'] & \mathrm{Op}_W; \arrow[l, "\mathrm{Res}_V^W"'] \end{tikzcd}$$

along the way, in Proposition 3.17, we compute the arity support of BV tensor products. Finally, we finish the section in Section 3.3 by defining and characterizing a wide variety of I-operads of algebraic interest in equivariant homotopy theory.

3.1. Arity support and weak $\mathcal{N}_\infty$-$\mathcal{T}$-operads.

Construction 3.1. Given $\mathcal{O} \in \mathrm{Op}_{\mathcal{T}}$, the *arity support of* $\mathcal{O}$ is the subcategory $A\mathcal{O} \subset \mathbb{F}_{\mathcal{T}}$ defined by

$$A\mathcal{O} := \left\{ \psi : T \to S \mid \mathrm{Mul}_{\mathcal{O}}^{\psi}(T; S) \neq \emptyset \right\} \subset \mathbb{F}_{\mathcal{T}} \qquad \triangleleft$$

In particular, maps of operads $\mathcal{O} \to \mathcal{P}$ are functors over $\mathsf{Span}(\mathbb{F}_{\mathcal{T}})$, hence they induce maps $\mathcal{O}(S) \to \mathcal{O}(P)$; this endows A with the structure of a functor

$$A : \mathrm{Op}_{\mathcal{T}} \to \mathrm{Sub}(\mathbb{F}_{\mathcal{T}}),$$

where the codomain is the poset of subcategories of $\mathbb{F}_{\mathcal{T}}$.

Remark 3.2. A product is empty if and only if one of its factors is empty, so $A\mathcal{O}$ is equal to

$$A\mathcal{O} = \left\{ \coprod_i \mathrm{Ind}_V^{\mathcal{T}} T_i \to V_i \;\middle|\; \forall i, \mathcal{O}(T_i) \neq \emptyset \right\} \subset \mathbb{F}_{\mathcal{T}}.$$

as a subcategory of $\mathbb{F}_{\mathcal{T}}$; in particular, this implies that A factors as

$$\mathrm{Op}_{\mathcal{T}} \xrightarrow{\mathrm{sseq}_{\mathcal{T}}} \mathrm{Fun}(\mathrm{tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S}) \to \mathrm{Sub}(\mathbb{F}_{\mathcal{T}}).$$

However, we will see that A has smaller image than the right functor in Proposition 3.4, so the associated essentially surjective functor will only factor through the essential image of $\mathrm{sseq}_{\mathcal{T}}$, rather than the full ∞-category of $\mathcal{T}$-symmetric sequences. $\triangleleft$

Example 3.3. For all $I \in \mathrm{wIndexCat}_{\mathcal{T}}$, we have $A\mathcal{N}_{I\infty} = I$, so $\mathrm{wIndexCat}_{\mathcal{T}} \subset A(\mathrm{Op}_{\mathcal{T}})$. $\triangleleft$

Proposition 3.4. *For all $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$, the subcategory $A\mathcal{O} \subset \mathbb{F}_{\mathcal{T}}$ is a weak indexing category; hence*

$$A(\mathrm{Op}_{\mathcal{T}}) = \mathrm{wIndexCat}_{\mathcal{T}} \subset \mathrm{Sub}(\mathbb{F}_{\mathcal{T}}).$$

Proof. The second statement follows from the first by Example 3.3, so it suffices to prove that $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$ satisfies Conditions (IC-a) to (IC-c).

Our main trick in characterizing $A\mathcal{O}$ is to leverage Construction 2.52 to transfer *nonemptyness* of the structure spaces of $\mathcal{O}^{\otimes}$ backwards along the $\mathcal{T}$-operad structure maps; indeed, there exists no map of spaces $X_1 \times X_2 \to Y_1 \times Y_2$ if and only if $X_1, X_2 \neq \emptyset$ and $Y_i = \emptyset$ for some i.

Using this, Condition (IC-a) follows by unwinding definitions using existence of the arity restriction map of Eq. (14). Similarly, Condition (IC-b) follows by unwinding definitions using the existence of the operadic composition map of Eq. (15). Lastly, Condition (IC-c) follows by existence of the $\mathrm{Aut}_V(S)$-action of Eq. (16). $\square$

Definition 3.5. A $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ is a *weak $\mathcal{N}_\infty$-operad* if it is a $\mathcal{T}$-0-operad, i.e. for all $S \in \underline{\mathbb{F}}_{\mathcal{T}}$, it satisfies

$$\mathcal{O}(S) \in \{*, \emptyset\} \qquad \forall S \in \mathbb{F}_V.$$

A weak $\mathcal{N}_\infty$-operad $\mathcal{O}^{\otimes}$ is an *$\mathcal{N}_\infty$-operad* if it has $n*_V$-ary operations for each V, i.e.

$$\mathcal{O}(n*_V) = * \qquad \forall V \in \mathcal{T},\ n \in \mathbb{N}. \qquad \triangleleft$$

Remark 3.6. Unwinding definitions, a weak $\mathcal{N}_\infty$-operad $\mathcal{O}^{\otimes}$ is an $\mathcal{N}_\infty$-operad if and only if its arity support $A\mathcal{O}$ is an indexing category. $\triangleleft$

Recall that the mapping fibers of $\mathcal{P}^{\otimes}$ a reduced $\mathcal{T}$-operad over backwards maps of $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ are contractible; the condition that $\mathcal{P}^{\otimes}$ is a 0-operad (i.e. $\mathrm{Mul}_{\mathcal{P}}(S;T)$ is (-1)-truncated) is equivalent by Corollary 1.78 to the statement that the map $\mathcal{P}^{\otimes} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is a subcategory inclusion; by inspecting mapping fibers, we find that $\mathcal{P}^{\otimes} = \mathrm{Span}_{A\mathcal{P}}(\mathbb{F}_{\mathcal{T}})$ as subcategories. We've proved the following.

Proposition 3.7. *If $\mathcal{P}^{\otimes}$ is a weak $\mathcal{N}_\infty$-operad, then there is a unique equivalence $\mathcal{P}^{\otimes} \simeq \mathcal{N}^{\otimes}_{A\mathcal{P}\infty}$.*

We use this to recognize weak $\mathcal{N}_\infty$-operads as *sub-terminal objects.*

Proposition 3.8. *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-operad and I a weak indexing system. Then there is an equivalence*

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{N}_{I\infty}) \simeq \begin{cases} * & A\mathcal{O} \leq I, \\ \varnothing & \text{otherwise.} \end{cases} \tag{18}$$

In particular, there is a unique map $\mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{A\mathcal{O}}$ witnessing a unique equivalence $h_{0,\mathcal{T}}\mathcal{O}^{\otimes} \simeq \mathcal{N}^{\otimes}_{A\mathcal{O}}$.

Proof. All statements of this proposition follow immediately from Eq. (18), so it suffices to prove that statement. By Corollaries 2.81 and 2.82, $\mathrm{Op}_{\mathcal{T},0}$ is a poset; the proof shows

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{N}^{\otimes}_{I\infty}) \simeq \mathbf{Alg}_{h_{0,\mathcal{T}}\mathcal{O}}(\mathcal{N}^{\otimes}_{I\infty}) \in \{\varnothing, *\}.$$

By Proposition 3.7 it suffices to characterize precisely when there exist maps $\mathcal{N}^{\otimes}_{I\infty} \to \mathcal{N}^{\otimes}_{J\infty}$.

In fact, unwinding definitions, we are asking for factorizations of subcategory inclusions

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \subset \mathrm{Span}_J(\mathbb{F}_{\mathcal{T}}) \subset \mathrm{Span}(\mathbb{F}_{\mathcal{T}});$$

this occurs if and only if $I \leq J$. □

Remark 3.9. By Corollary 2.77, the functor $\mathrm{ev}_S : \mathcal{O}^{\otimes} \mapsto \mathcal{O}(S)$ has a left adjoint $\mathrm{Fr}_S(-) : \mathcal{S} \to \mathrm{Op}_{\mathcal{T}}$; applying this to $* \in \mathcal{S}$, we find that $\mathcal{O}(S) \simeq \mathbf{Alg}_{\mathrm{Fr}_S(*)}(\mathcal{O})^{\simeq}$; in particular, if $\mathcal{P}^{\otimes}$ has the property that $\mathbf{Alg}_{\mathcal{O}}(\mathcal{P}) \in \{*, \varnothing\}$ for all $\mathcal{O}^{\otimes}$, then $\mathcal{P}^{\otimes}$ must be a weak $\mathcal{N}_\infty$-operad.

By [HTT, Rem 5.5.6.12], this demonstrates that the poset of *sub-terminal objects* $\mathrm{Sub}_{\mathrm{Op}_{\mathcal{T}}}(\mathrm{Comm}^{\otimes}_{\mathcal{T}})$ is spanned by the weak $\mathcal{N}_\infty$-operads, by Proposition 3.8, we then find that

$$\mathrm{Sub}_{\mathrm{Op}_{\mathcal{T}}}\left(\mathrm{Comm}^{\otimes}_{\mathcal{T}}\right) \simeq \mathrm{wIndex}_{\mathcal{T}}.$$ ◁

The following generalization of the indexing systems theorems of [BP21; GW18; NS22; Rub21a] then immediately follows from Propositions 3.4 and 3.8.

Corollary 3.10. *The functor of admissible maps admits a fully faithful right adjoint*

$$\begin{tikzcd} \mathrm{Op}_{\mathcal{T}} \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathrm{wIndex}_{\mathcal{T}} \arrow[l, bend left, "\mathcal{N}^{\otimes}_{(-)\infty}"] \end{tikzcd} \tag{19}$$

whose image consists of the weak $\mathcal{N}_\infty$-operads; furthermore, the following are equal full subcategories of $\mathrm{Op}_{\mathcal{T}}$:

$$\mathrm{Op}_I = \mathrm{Op}_{\mathcal{T},/\mathcal{N}_{I\infty}} = A^{-1}(\mathrm{wIndexCat}_{\mathcal{T},\leq I}).$$

Observation 3.11. Let P be a property in $\{\text{one-color}, \text{aE-unital}, \text{E-unital}, \text{almost-unital}, \text{unital}, \text{has finite fold maps}\}$. Then, note that

$$\mathcal{O}^{\otimes} \text{ has property } P \qquad \iff \qquad A\mathcal{O}^{\otimes} \text{ has property } P.$$

In particular, Corollary 3.10 restricts to an adjunction

$$\begin{tikzcd} \mathrm{Op}^P_{\mathcal{T}} \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathrm{wIndex}^P_{\mathcal{T}} \arrow[l, bend left, "\mathcal{N}^{\otimes}_{(-)\infty}"] \end{tikzcd}$$

◁

3.2. **Operadic restriction and arity-borelification.** Given $\varphi : \mathcal{T}' \to \mathcal{T}$ a functor of atomic orbital ∞-categories, we show in Proposition A.12 that the associated map of Burnside algeraic patterns $\mathrm{Span}(\mathbb{F}_{\mathcal{T}'}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is a Segal morphism. In this section, we use this to to define various adjunctions between categories of I-operads.

3.2.1. *Arity borelification and its left adjoint.*

Construction 3.12. Given a pair of related weak indexing systems $I \le J$, we may write the composite map of patterns

$$\iota_I^J : \mathrm{Span}_I(\mathbb{F}_{c(I)}) \to \mathrm{Span}_{J \cap \mathbb{F}_{c(I)}}\left(\mathbb{F}_{c(I)}\right) \to \mathrm{Span}_J\left(\mathbb{F}_{c(J)}\right),$$

which is a Segal morphism by Propositions A.12 and A.14, and a J-operad by Proposition 2.43. We set the notation $E_I^J := \iota_{I!}^J$ and $\mathrm{Bor}_I^J := \iota_I^{J*}$; in the case $J = \mathbb{F}_{\mathcal{T}}$ and $I = \mathbb{F}_{\mathcal{F}}$ for a $\mathcal{F} \subset \mathcal{T}$ a $\mathcal{T}$-family, we set $E_{\mathcal{F}}^{\mathcal{T}} := E_{\mathbb{F}_{\mathcal{F}}}^{\mathbb{F}_{\mathcal{T}}}$ and $\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} := \mathrm{Bor}_{\mathbb{F}_{\mathcal{F}}}^{\mathbb{F}_{\mathcal{T}}}$. ◁

Similarly, let $E_I^J : \mathrm{wIndexCat}_{\mathcal{T},\le I} \to \mathrm{wIndexCat}_{\mathcal{T},\le J}$ be the evident inclusion, with right adjoint $\mathrm{Bor}_I^J = (-) \cap \mathbb{F}_I : \mathrm{wIndexCat}_{\mathcal{T},\le J} \to \mathrm{wIndexCat}_{\mathcal{T},\le I}$. Note that these intertwine with A, i.e.

$$E_I^J A\mathcal{O} = A E_I^J \mathcal{O}; \qquad\qquad \mathrm{Bor}_I^J A\mathcal{O} = A\mathrm{Bor}_I^J \mathcal{O}.$$

Corollary 3.13. *For $I \le J$ weak indexing systems, $E_I^J := \iota_{I!}^J$ is an inclusion of a colocalizing $\mathcal{T}$-subcategory*

$$\begin{tikzcd} \underline{\mathrm{Op}}_I^{\otimes} \arrow[r, bend left, "E_I^J"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Op}}_J^{\otimes} \arrow[l, bend left, "\mathrm{Bor}_I^J"] \end{tikzcd}$$

whose terminal object is $\mathcal{N}_{I\infty}^{\otimes}$. Furthermore, there is are equivalences

$$E_I^{I'} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{E_I^{I'} J\infty}^{\otimes}$$
$$\mathrm{Bor}_I^{I'} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{\mathrm{Bor}_I^{I'} J\infty}^{\otimes}.$$

Proof. The first sentence follows by the above argument. The computations follow by examining the structure spaces of the resulting $\mathcal{T}$-operads. □

Corollary 3.14 (Color-borelification)**.** *Given $\mathcal{F} \in \mathrm{Fam}_{\mathcal{T}}$ is a $\mathcal{T}$-family, there is a natural equivalence*

$$\mathbf{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathcal{O}) \simeq \Gamma^{\mathcal{F}} \mathcal{O};$$

hence there is a natural equivalence

$$\mathrm{triv}_{\mathcal{F}}^{\otimes} \overset{BV}{\otimes} \mathcal{O}^{\otimes} \simeq E_{\mathcal{F}}^{\mathcal{T}} \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{O}^{\otimes}.$$

Proof. The first statement follows by noting that $\mathrm{triv}_{\mathcal{F}}^{\otimes} \simeq E_{\mathcal{F}}^{\mathcal{T}} \mathrm{triv}_{\mathcal{F}}^{\otimes}$, so that

$$\mathbf{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathcal{O}) \simeq \mathbf{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}}(\mathcal{O})) \simeq \Gamma^{\mathcal{F}} \mathcal{O}$$

by Proposition 2.64. The second statement then follows by Yoneda's lemma, noting that

$$\begin{aligned} \mathbf{Alg}_{\mathrm{triv}_{\mathcal{F}} \otimes \mathcal{O}}(\mathcal{P}) &\simeq \mathbf{Alg}_{\mathrm{triv}_{\mathcal{F}}} \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{P}) \\ &\simeq \Gamma^{\mathcal{F}} \mathbf{Alg}_{\mathcal{O}}(\mathcal{P}) \\ &\simeq \mathbf{Alg}_{\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{O}}(\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{P}) \\ &\simeq \mathbf{Alg}_{E_{\mathcal{F}}^{\mathcal{T}} \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{O}}(\mathcal{P}). \end{aligned}$$ □

Given $\mathcal{O} \in \mathrm{Op}_{\mathcal{T}}$, we set $c(\mathcal{O}) := c(A\mathcal{O}) = \{V \mid \mathcal{O}(*_V) \neq \emptyset\}$.

Remark 3.15. As with all smashing localizations, Corollary 3.14 implies that $\mathrm{Im} E_{\mathcal{F}}^{\mathcal{T}} = \left\{\mathcal{O}^{\otimes} \in \mathrm{Op}^{\mathcal{T}} \mid c(\mathcal{O}) \subset \mathcal{F}\right\}$ is a $\otimes$-*ideal*, i.e. if $c(\mathcal{O}) \subset \mathcal{F}$, and $\mathcal{P}^{\otimes}$ is arbitrary, then $c\left(\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{P}\right) \subset \mathcal{F}$. In particular, $\underline{\mathrm{Op}}_I^{\otimes}$ is a *nonunital* symmetric monoidal full subcategory of $\underline{\mathrm{Op}}_J^{\otimes}$. ◁

Observation 3.16. There are natural equivalences

$$\begin{aligned}\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes &\simeq \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathrm{triv}^\otimes_{c\mathcal{O}} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}^\otimes_{c\mathcal{P}} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes,\\ &\simeq \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathrm{triv}^\otimes_{c\mathcal{O}\cap c\mathcal{P}} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes,\\ &\simeq \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathrm{triv}^\otimes_{c\mathcal{O}\cap c\mathcal{P}} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}^\otimes_{c\mathcal{O}\cap c\mathcal{P}} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes,\\ &\simeq E^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}} \mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}}\left(\mathcal{O}^\otimes\right) \overset{\mathrm{BV}}{\otimes} E^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}} \mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}}\left(\mathcal{P}^\otimes\right),\\ &\simeq E^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}}\left(\mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}}\left(\mathcal{O}^\otimes\right) \overset{\mathrm{BV}}{\otimes} \mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}}\left(\mathcal{P}^\otimes\right)\right).\end{aligned}$$

The $c\mathcal{O}\cap c\mathcal{P}$-operads $\mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}}(\mathcal{O}^\otimes)$ and $\mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}}(\mathcal{P}^\otimes)$ both have at least one color; hence we may compute arbitrary tensor products of $\mathcal{T}$-operads via tensor products of equivariant operads with at least one color. ◁

Having done this, we may compute supports of arbitrary tensor products of $\mathcal{T}$-operads.

Proposition 3.17. *Suppose $\mathcal{O}^\otimes, \mathcal{P}^\otimes$ are $\mathcal{T}$-operads. Then,*

$$A\left(\mathcal{O} \overset{BV}{\otimes} \mathcal{P}\right) = E^{\mathcal{T}}_{\mathcal{F}} \mathrm{Bor}^{\mathcal{T}}_{\mathcal{F}}\left(A\mathcal{O} \vee A\mathcal{P}\right).$$

Proof. By Observation 3.16, we have equivalences

$$A\left(\mathcal{O}^\otimes \otimes \mathcal{P}^\otimes\right) \simeq E^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}} A\left(\mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}}\left(\mathcal{O}^\otimes\right) \overset{\mathrm{BV}}{\otimes} \mathrm{Bor}^{\mathcal{T}}_{c\mathcal{O}\cap c\mathcal{P}}\left(\mathcal{P}^\otimes\right)\right),$$

so it suffices to prove the proposition in the case that $\mathcal{O}^\otimes$ and $\mathcal{P}^\otimes$ have at least one color.

In this case, first note that there exist maps

$$\mathcal{O}^\otimes \otimes \mathrm{triv}^\otimes_{\mathcal{T}}, \mathrm{triv}^\otimes_{\mathcal{T}} \otimes \mathcal{P}^\otimes \to \mathcal{O}^\otimes \otimes \mathcal{P}^\otimes,$$

so that

$$A\mathcal{O} \vee A\mathcal{P} \le A(\mathcal{O} \vee \mathcal{P}).$$

On the other hand, there exists a composite map

$$\mathcal{O}^\otimes \otimes \mathcal{P}^\otimes \to \mathcal{N}^\otimes_{A\mathcal{O}\infty} \otimes \mathcal{N}^\otimes_{A\mathcal{P}\infty} \to \mathcal{N}^\otimes_{A\mathcal{O}\vee A\mathcal{P}\infty} \otimes \mathcal{N}^\otimes_{A\mathcal{O}\vee A\mathcal{P}\infty} \to \mathcal{N}^\otimes_{A\mathcal{O}\vee A\mathcal{P}\infty},$$

hence $A(\mathcal{O} \vee \mathcal{P}) \le A\mathcal{O} \vee A\mathcal{P}$. □

3.2.2. *Results about reduced $\mathcal{T}$-operads extend to the aE-reduced setting.* Given I an aE-unital weak indexing system, set the notation $\overline{I} := \mathrm{Bor}^{\mathcal{T}}_{v(I)} I$, where $v(I) = \{V \mid \varnothing \to V \in I\}$ is the family of *units* of I (c.f. [Ste24]).

Observation 3.18. For $\mathcal{P}$ an aE-unital $\mathcal{T}$-operad, the following is a pushout diagram:

$$\begin{tikzcd} E^{\mathcal{T}}_{v(\mathcal{P})}\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{P}^\otimes \arrow[r] & \mathcal{P}^\otimes \\ E^{\mathcal{T}}_{v(\mathcal{P})}\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathrm{triv}(\mathcal{P})^\otimes \arrow[r] \arrow[u] \arrow[ur, phantom, "\ulcorner"] & \mathrm{triv}(\mathcal{P})^\otimes \arrow[u] \end{tikzcd}$$

Applying this for $\mathcal{P}^\otimes := \mathcal{N}^\otimes_{I\infty}$, we have a diagram

$$\begin{tikzcd} \mathcal{N}^\otimes_{\overline{I}\infty} \arrow[r] & \mathcal{N}^\otimes_{I\infty} \\ \mathrm{triv}_{v(I)} \arrow[r] \arrow[u] \arrow[ur, phantom, "\ulcorner"] & \mathrm{triv}_{c(I)} \arrow[u] \end{tikzcd} \tag{20}$$

Unwinding definitions, this constructs a pullback diagram

$$\begin{tikzcd} \mathbf{Alg}_{\mathcal{P}}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}_{\mathcal{T}}(\mathcal{P}, \mathcal{C}) \arrow[d] \\ \mathbf{Alg}_{\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{P}}\left(\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{C}\right) \arrow[r] & \mathrm{Fun}_{v(\mathcal{P})}\left(\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{P}, \mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{C}\right) \end{tikzcd}$$

for all $\mathcal{T}$-operads (hence $\mathcal{T}$-symmetric monoidal categories) $\mathcal{C}$. In particular, if $\mathcal{P}$ has at most one object (i.e. it is *aE-reduced*, then the above diagram reads as

$$\begin{tikzcd} \mathbf{Alg}_{\mathcal{P}}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Gamma^{\mathcal{T}}\mathcal{C} \arrow[d] \\ \mathbf{Alg}_{\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{P}}\left(\mathrm{Bor}^{\mathcal{T}}_{v(\mathcal{P})}\mathcal{C}\right) \arrow[r] & \Gamma^{v(\mathcal{P})}\mathcal{C} \end{tikzcd}$$

In particular, a $\mathcal{P}^{\otimes}$-algebra structure is seen as simply a $\mathcal{T}$-object together with a (reduced) $\mathrm{Bor}^{\otimes}_{\nu(\mathcal{P})}\mathcal{P}^{\otimes}$-algebra structure on its $\nu(\mathcal{P})$-Borelification. ◁

Proposition 3.19. *Suppose I is an almost-E-unital weak indexing system. Then, for a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$, the map $\mathrm{Bor}^{\mathcal{T}}_{c(I)}\mathcal{O}^{\otimes} \rightarrow \mathcal{N}^{\otimes}_{\overline{I}\infty} \overset{BV}{\otimes} \mathcal{O}^{\otimes}$ is an equivalence if and only if the map*

$$\mathrm{Bor}(f) : \mathrm{Bor}^{\mathcal{T}}_{v(I)}\mathcal{O}^{\otimes} \rightarrow \mathcal{N}^{\otimes}_{\overline{I}\infty} \overset{BV}{\otimes} \mathrm{Bor}^{\mathcal{T}}_{v(I)}\mathcal{O}^{\otimes}$$

is an equivalence.

Proof. Tensoring Eq. (20) with $\mathcal{O}^{\otimes}$ yields the following.

$$\begin{tikzcd} E^{\mathcal{T}}_{v(I)}\left(\mathcal{N}^{\otimes}_{\overline{I}\infty} \overset{\mathrm{BV}}{\otimes} \mathrm{Bor}^{\mathcal{T}}_{v(I)}\mathcal{O}^{\otimes}\right) \arrow[r, phantom, "\simeq"] & E^{\mathcal{T}}_{\nu(I)}\mathrm{Bor}_{\nu(I)}\left(\mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes}\right) \arrow[r] & \mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes} \\ E^{\mathcal{T}}_{v(I)}\mathrm{Bor}^{\mathcal{T}}_{v(I)}\mathcal{O}^{\otimes} \arrow[u, "E\mathrm{Bor}(f)"] \arrow[r, phantom, "\simeq"] & E^{\mathcal{T}}_{v(I)}\mathrm{Bor}^{\mathcal{T}}_{v(I)}\mathrm{Bor}^{\mathcal{T}}_{c(I)}\mathcal{O}^{\otimes} \arrow[u] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Bor}_{c(I)}\mathcal{O}^{\otimes} \arrow[u, "f"'] \end{tikzcd}$$

In particular, we find that if f is an equivalence, then $\mathrm{Bor}(f)$ is an equivalence, and if $\mathrm{Bor}(f)$ is an equivalence, then $E\mathrm{Bor}(F)$ is an equivalence, so pushout stability of equivalences implies that f is an equivalence. □

3.2.3. *Operadic restriction and (co)induction.* Recall from Construction 2.78 that the underlying $\mathcal{T}$-symmetric sequence forms a $\mathcal{T}$-functor $\underline{\mathrm{sseq}} : \underline{\mathrm{Op}}^{\mathrm{red}}_{\mathcal{T}} \rightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \underline{\mathcal{S}}_{\mathcal{T}})$; in particular, restrictions of $\underline{V}$-operads correspond with restrictions of $\underline{V}$-symmetric sequences; We may use this to upgrade Corollary 3.10 to an adjunction of $\mathcal{T}$-categories.

Proposition 3.20. $\mathrm{Res}^W_V \mathcal{N}^{\otimes}_{I\infty} \simeq \mathcal{N}^{\otimes}_{\mathrm{Res}^W_V I\infty}$; *more generally, Eq. (19) lifts to a $\mathcal{T}$-adjunction*

$$\begin{tikzcd} \underline{\mathrm{Op}}_{\mathcal{T}} \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{wIndex}}_{\mathcal{T}} \arrow[l, bend left, "\mathcal{N}^{\otimes}_{(-)\infty}"] \end{tikzcd}$$

Proof. Restriction compatiblility of the underlying symmetric sequence implies that $\mathrm{Res}^W_V A\mathcal{O} = A\,\mathrm{Res}^W_V \mathcal{O}$, lifting A to a $\mathcal{T}$-functor $\underline{\mathrm{Op}}_{\mathcal{T}} \rightarrow \underline{\mathrm{wIndex}}_{\mathcal{T}}$ whose V-value is $A : \mathrm{Op}_V \rightarrow \mathrm{wIndex}_V$. The right adjoints $\mathcal{N}^{\otimes}_{(-)\infty}$ uniquely lift to a right $\mathcal{T}$-adjoint to $\mathcal{N}^{\otimes}_{(-)\infty}$ by [HA, Prop 7.3.2.1], completing the proposition. □

Since A is a $\mathcal{T}$-left adjoint, it is compatible with $\mathcal{T}$-colimits. Applying this for indexed coproducts, we immediately acquire the following properties of A.

Corollary 3.21. *If $\mathcal{O}, \mathcal{P}$ are $\mathcal{T}$-operads, then we have*

$$A(\mathcal{O} \sqcup \mathcal{P}) = A\mathcal{O} \vee A\mathcal{P}.$$

If $\mathcal{Q}$ is a V-operad, then we have

$$A\mathrm{Ind}^{\mathcal{T}}_V \mathcal{Q} = \mathrm{Ind}^{\mathcal{T}}_V A\mathcal{Q}.$$

We may compute use an analogous argument to that of [BHS22, Lem 4.1.13] to show that $\underline{\mathrm{Op}}_{\mathcal{T}}$ strongly admits $\mathcal{T}$-limits; since the fully faithful $\mathcal{T}$-functor $\underline{\mathrm{Op}}_{\mathcal{T}} \rightarrow \underline{\mathbf{Cat}}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$ possesses pointwise left adjoints (given by L_{Fbrs}), it possesses a $\mathcal{T}$-left adjoint; in particular, we may compute $\mathcal{T}$-limits of $\mathcal{T}$-operads in $\underline{\mathbf{Cat}}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$. Then, an analogous argument using [BHS22, Prop 2.3.7] constructs $\mathcal{T}$-limits in

$\underline{\mathbf{Cat}}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}}}$ in $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathrm{Span}(\mathbb{F}_{\mathcal{T}}), \underline{\mathbf{Cat}}_{\mathcal{T}})_{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$, which strongly admits $\mathcal{T}$-limits, as its a slice $\mathcal{T}$-∞-category of a functor $\mathcal{T}$-∞-category into a $\mathcal{T}$-∞-category which strongly admits $\mathcal{T}$-limits. In particular, this implies that $\mathrm{Res}_U^V : \mathrm{Op}_V \to \mathrm{Op}_U$ has a right adjoint, which we call $\mathrm{CoInd}_U^V : \mathrm{Op}_U \to \mathrm{Op}_V$.

Proposition 3.22. *If $\mathcal{O}^{\otimes}$ is a d-truncated V-operad, then $\mathrm{CoInd}_V^W \mathcal{O}^{\otimes}$ is d-truncated.*

Proof. This follows simply by taking right adjoints within the following diagram

$$\begin{tikzcd} \mathrm{Op}_{W,d} \arrow[r, "\mathrm{Res}_V^W"] \arrow[d, hook] & \mathrm{Op}_{V,d} \arrow[d, hook] \\ \mathrm{Op}_W \arrow[r, "\mathrm{Res}_V^W"] & \mathrm{Op}_V \end{tikzcd}$$

□

Corollary 3.23. *If $\iota_V^{\mathcal{T}} : \mathrm{tot}\underline{\Sigma}_V \to \mathrm{tot}\underline{\Sigma}_{\mathcal{T}}$ is the inclusion, then*

$$\mathrm{sseq}\,\mathrm{CoInd}_V^W \mathcal{O}^{\otimes} \simeq \mathrm{CoInd}_V^W \,\mathrm{sseq}\,\mathcal{O}^{\otimes};$$

in particular, we have

$$A\mathrm{CoInd}_V^W \mathcal{O} = \mathrm{CoInd}_V^W A\mathcal{O}.$$

Proof. The first statement follows by noting that $\mathrm{Fr}\,\mathrm{Res}_V^W = \iota_V^{W*}\mathrm{Fr}$ and taking right adjoints. For the second statement, fix some $S \in \mathbb{F}_U$ for $U \to W$. In view of [Ste24], we're tasked with proving that $\mathcal{O}(S) \neq \emptyset$ if and only if for all $U' \to W$, we have $\mathcal{O}(\mathrm{Res}_{U'}^W \mathrm{Ind}_U^W S) \neq \emptyset$.

The pointwise formula for right Kan extension along $\Sigma_V \to \Sigma_W$ yields

$$\mathcal{O}(S) \simeq \lim_{\mathrm{Ind}_V^W S \xleftarrow{\Sigma_W} T} \mathcal{O}(T) \simeq \lim_{\mathrm{Res}_{U'}^W \mathrm{Ind}_U^W S \simeq T} \mathcal{O}(T) \tag{21}$$

Note that a limit of spaces is nonempty if and only if its factors are nonempty; thus this limit is nonempty if and only if $\mathcal{O}(\mathrm{Res}_{U'}^W \mathrm{Ind}_U^W S)$ is nonempty for all $U' \to W$, as desired. □

We care about $\mathrm{CoInd}_V^{\mathcal{T}} \mathcal{O}^{\otimes}$ because it is a structure borne by *norms of algebras.*

Construction 3.24. Let $\mathcal{P}^{\otimes} \to \mathrm{CoInd}_V^W \mathcal{O}^{\otimes}$ be a functor of one-object I-operads, let $\mathcal{C}$ be a I-symmetric monoidal ∞-category, and let $V \to W$ be a transfer in I. Then, the adjunct map $\varphi : \mathrm{Res}_V^W \mathcal{P} \to \mathcal{O}^{\otimes}$ participates in a commutative diagram of symmetric monoidal functors

$$\begin{tikzcd} \mathbf{Alg}_{\mathcal{O}}(\mathrm{Res}_V^W \mathcal{C}) \arrow[r, "\varphi^*"] \arrow[d, "U_V"] & \mathbf{Alg}_{\mathrm{Res}_V^W \mathcal{P}}(\mathrm{Res}_V^W \mathcal{C}) \arrow[r, "N_V^W"] \arrow[d, "U_V"] & \mathbf{Alg}_{\mathcal{P}}(\mathcal{C}) \arrow[d, "U_W"] \\ \mathcal{C}_V \arrow[r, equal] & \mathcal{C}_V \arrow[r, "N_V^W"] & \mathcal{C}_W \end{tikzcd}$$

Intuitively, we view this situation as saying that $\mathrm{CoInd}_V^W \mathcal{O}^{\otimes}$ bears the *universal* structure which is naturally endowed on $N_V^W X$ ranging across $X \in \mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$. ◁

3.3. **Examples of I-operads.** In this subsection, we survey various examples of I-operads which corepresent notable algebraic theories.

3.3.1. *Basic examples of $\mathcal{N}_{I\infty}^{\otimes}$ operads.* Fix $\mathcal{F} \subset \mathcal{T}$ be a $\mathcal{T}$-family. In Section 2.4, we introduced the example $\mathrm{triv}_{\mathcal{F}}^{\otimes} := \mathcal{N}_{I_{\mathrm{triv},\mathcal{F}}}^{\otimes} \simeq E_{\mathcal{F}}^{\mathcal{T}} \mathrm{triv}_{\mathcal{F}}^{\otimes}$. It was verified in [NS22, Cor 2.4.5] that this is characterized by the algebras

$$\mathbf{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathcal{C}) \simeq \Gamma^{\mathcal{F}} \mathcal{C};$$

i.e. its algebras are *$\mathcal{F}$-objects*. Furthermore, we used this in Corollary 3.14 to verify that $\mathrm{triv}_{\mathcal{F}}$ is $\overset{\mathrm{BV}}{\otimes}$-idempotent, with corresponding localizing subcategory consisting of the image of $E_{\mathcal{F}}^{\mathcal{T}}$ (i.e. $(-) \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{\mathcal{F}}$ implements *color-borelification*).

Example 3.25. Let $\mathcal{F} \subset \mathcal{T}$ be a $\mathcal{T}$-family, and denote by $\underline{\mathbb{F}}_{I^0_{\mathcal{F}}}$ the weak indexing system satisfying

$$S \in \underline{\mathbb{F}}_{I^0_{\mathcal{F}}} \qquad \iff \qquad S = *_V \text{ or } S \in \{\emptyset_V \mid V \in \mathcal{F}\}.$$

We set the notation $\mathbb{E}^{\otimes}_{0,\mathcal{F}} := \mathcal{N}^{\otimes}_{I^0_{\mathcal{F}}\infty}$. Note that Eq. (20) specializes to a pushout presentation

$$\begin{tikzcd} E^{\mathcal{T}}_{\mathcal{F}} \mathbb{E}^{\otimes}_0 \arrow[r] & \mathbb{E}^{\otimes}_{0,\mathcal{F}} \\ \mathrm{triv}^{\otimes}_{\mathcal{F}} \arrow[u] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{triv}^{\otimes}_{\mathcal{T}} \arrow[u] \end{tikzcd} \tag{22}$$

Intuitively, this presents $\mathbb{E}_{0,\mathcal{F}}$-algebras as $\mathcal{T}$-objects together with a distinguished "1-shaped element" of their underlying $\mathcal{F}$-objects; more precisely, the universal property for pushouts yields

$$\mathbf{Alg}_{\mathbb{E}_{0,\mathcal{F}}}(\mathcal{C}) \simeq \Gamma^{\mathcal{T}} \mathcal{C} \times_{\Gamma_{\mathcal{F}}\mathcal{C}} \left(\Gamma^{\mathcal{F}} \mathcal{C}\right)^{1/}.$$

Furthermore, we prove a generalization of the following in Proposition 4.16.

Corollary 3.26. *$\mathbb{E}_{0,\mathcal{F}}$ is $\overset{BV}{\otimes}$-idempotent, whose corresponding (smashing-)localizing subcategory of $\mathrm{Op}_{\mathcal{T}}$ consists of those whose $\mathcal{F}$-borelification is E-unital. Furthermore, $\mathbb{E}_{0,\mathcal{F}}$ is initial among almost-reduced operads whose $\mathcal{F}$-borelifications are unital.*

◁

Example 3.27. Let $\mathbb{F}^{\infty}_{\mathcal{T}}$ be the minimal indexing system and I^{∞} the corresponding indexing category, as introduced in Section 1.2.3. We write $\mathbb{E}^{\otimes}_{\infty} := \mathcal{N}^{\otimes}_{I^{\infty}}$. ◁

$\mathbb{E}_{\infty}$ paramterizes no transfers; we would like to use this to develop a naive model for $\mathbb{E}_{\infty}$-algebras.

Construction 3.28. Given $\mathcal{O}^{\otimes}$ a $\mathcal{T}$-operad, and $V \in \mathcal{T}$, we may form the *V-value operad*

$$\Gamma^V \mathcal{O}^{\otimes} := i_V^* \mathcal{O}^{\otimes},$$

where $i_V : \mathrm{Span}(\mathbb{F}) \hookrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is the map of patterns extending the coproduct preserving functor $\mathbb{F} \hookrightarrow \mathbb{F}_{\mathcal{T}}$ sending $* \mapsto *_V$. Using this, we may set

$$\Gamma^{\mathcal{T}} \mathcal{O}^{\otimes} := \lim_{V \in \mathcal{T}} \mathcal{O}^{\otimes},$$

noting that this recovers Γ^V if V is terminal in $\mathcal{T}$. ◁

Unwinding definitions, we find that Corollary 1.50 implies that the map of patterns $\mathcal{T}^{\mathrm{op}} \times \mathrm{Span}(\mathbb{F}) \to \mathrm{Span}_{I^{\infty}}(\mathbb{F}_{\mathcal{T}})$ induces equivalences on Segal objects, hence on fibrous patterns. Further unwinding definitions, this yields an equivalence

$$\mathrm{Op}_{I^{\infty}} \simeq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Op}).$$

In particular, this yields the following.

Proposition 3.29. *The functor $\Gamma^{\mathcal{T}} : \mathrm{Op}_{\mathcal{T}} \to \mathrm{Op}$ has a fully faithful left adjoint $\mathrm{Infl}^{\mathcal{T}} : \mathrm{Op} \to \mathrm{Op}_{\mathcal{T}}$ whose image is spanned by the I^{∞}-operads whose corresponding functors $\mathcal{T}^{\mathrm{op}} \to \mathrm{Op}$ are constant.*

In particular, we find that $\mathbb{E}^{\otimes}_{\infty} \simeq \mathrm{Infl}^{\mathcal{T}} \mathbb{E}^{\otimes}_{\infty}$, i.e.

$$\mathbf{Alg}_{\mathbb{E}_{\infty}}(\mathcal{C}) \simeq \mathbf{Alg}_{\mathbb{E}_{\infty}}(\Gamma^{\mathcal{T}} \mathcal{C});$$

3.3.2. *Equivariant little disks/steiner operads.* In [Bon19], a *genuine operadic nerve* 1-categorical functor was constructed between a model of graph-G operads and a model for G-operads. Later, in Section 5.1, we lift this to a conservative functor of G-∞-categories

$$N^{\otimes} : g\underline{\mathrm{Op}}_G \to \underline{\mathrm{Op}}_G.$$

where $g\mathrm{Op}_H = \left(g\mathrm{Op}_G\right)_H$ is the ∞-category presented by any of the Quillen-equivalent model categories of dendroidal Segal H-operads, graph H-operads, or genuine H-operads (c.f. [BP22]) with their evident restriction functors. Guillou and May construct the following.

Proposition 3.30. *The* little V-disks graph H-operads *form a functor of G-1-categories*

$$\underline{\mathbf{Rep}}_{\mathbb{R}}^{\mathrm{Orth,Emb}}(G) \to \mathrm{graph}\text{–}\underline{\mathrm{Op}}_G,$$

the latter denoting the G-category whose H-value underlies the graph model structure *for G-operads.*

Using this, we define the G-operad

$$\mathbb{E}_V := N^{\otimes} D_V,$$

where D_V is the *little V-disks graph G-operad* of [GM17], whose n-ary $G \times \Sigma_n$ space has

$$D_V(n) := \mathrm{Emb}^{\mathrm{Rect.lin.}}(D(V) \times n, D(V)) \simeq \mathrm{Conf}_n(V)$$

by [GM17, Lem 1.2]. The resulting unital G-operad $\mathbb{E}_V$ was studied in [Hor19], who showed for instance that

$$\mathbb{E}_V(S) \simeq \mathrm{Emb}^{\mathrm{Rec.lin}}(D(V) \times S, D(V))^H \simeq \mathrm{Conf}_S^H(V),$$

where

$$\mathrm{Conf}_S^H(V) := \operatorname*{colim}_{\substack{W \subset V \\ \text{fin.dim}}} \mathrm{Conf}_S^H(W)$$

in view of Corollary 2.77.

Given V a real orthogonal G-representation, we let $AV := A\mathbb{E}_V$, i.e. AV corresponds with the weak indexing system $\mathbb{F}_{AV}$ of finite H-sets admitting an embedding into V.

Proposition 3.31. *Let G be a topological group, $H \subset G$ a closed subgroup, $S \in \mathbb{F}_H$ a finite H-set admitting an configuration $\iota : S \hookrightarrow W$, and V, W real orthogonal G-representations whose associated map*

$$\mathrm{Conf}_S^H(V) \hookrightarrow \mathrm{Conf}_S^H(V \oplus W)$$

is an equivalence. Then, $\mathrm{Conf}_S^H(V)$ is contractible.

Proof. Note that linear interpolation to ι yields a deformation of $\mathrm{Map}^H(S, V \oplus W)$ onto the subspace $\mathrm{Map}^H(S, W)$ consisting of maps whose image has zero projection to V. The path of a point beginning in the subspace $\mathrm{Conf}_S^H(V) \subset \mathrm{Conf}_S^H(V \oplus W)$ consisting of configurations with zero projection to W lands within $\mathrm{Conf}_S^H(V \oplus W)$ at all times; composing this deformation after the deformation retract $\mathrm{Conf}_S^H(V \oplus W) \xrightarrow{\sim} \mathrm{Conf}_H^S(V)$ thus yields a deformation retract of $\mathrm{Conf}_S^H(V \oplus W)$ onto $\{\iota\}$, so it is contractible.[17] By the equivalence $\mathrm{Conf}_S^H(V) \simeq \mathrm{Conf}_S^H(V \oplus W)$, the space $\mathrm{Conf}_S^H(V)$ is contractible as well. □

Remark 3.32. This is unsatisfying for a prominent reason; Fadell-Neuwirth's original strategy for proving the nonequivariant version of this benefits from significantly greater generality. In forthcoming work, the author hopes to demonstrate using an equivariant lift of Fadell-Neuwirth's homotopy fiber sequence to demonstrate for instance that $\mathbb{E}_{dV}^{\otimes}$ is $(d-2)$-connected for all d and V. ◁

We say that V is a *weak universe* if it is a direct sum of infinitely many copies of a collection of irreducible real orthogonal G-representations; equivalently, there is an equivalence $V \simeq V \oplus V$.

Corollary 3.33. *If there exists an equivalence $\mathbb{E}_V^{\otimes} \simeq \mathbb{E}_{V \oplus W}^{\otimes}$, then the canonical map $\mathrm{Bor}_{AW}^{\mathcal{T}} \mathbb{E}_V^{\otimes} \to \mathcal{N}_{AW}^{\otimes}$ is an equivalence; in particular, if V is a weak universe, then the canonical map*

$$\mathbb{E}_V^{\otimes} \to \mathcal{N}_{AV}^{\otimes}$$

is an equivalence.

Observation 3.34. If V is a *universe* (i.e. it is a weak universe admitting a positive-dimensional fixed point locus), then it admits embeddings of all finite sets; hence it is not just a weak $\mathcal{N}_\infty$-operad, but an $\mathcal{N}_\infty$-operad. ◁

---

[17] Said explicitly, let $h : [0,1] \to \mathrm{Conf}_S^H(V \oplus W)$ be the deformation retract onto those configurations with zero projection to W. Then, our deformation retract h' onto $\iota(w)$ is computed by

$$h'(t) = \begin{cases} h(2t) & t \leq \frac{1}{2}, \\ (2-2t) \cdot h(1) + (2t-1)\iota & t \geq \frac{1}{2}. \end{cases}$$

Because of the above observation, much study has been dedicated to the less general setting of *universes*; here, Rubin has given a complete and simple characterization of those indexing systems (equivalently, transfer systems) occurring as the arity-support of an $\mathbb{E}_V$-operad in [Rub21a], where they are modelled via *Steiner operads*. Nevertheless, we do not need this assumption to work with $\mathbb{E}_V$.

Corollary H (Equivariant infinitary Dunn additivity)**.** *Let G be a finite group and V, W real orthogonal G-representations satisfying at least one of the following conditions:*

(a) V, W are weak G-universes, or
(b) the canonical map $\mathbb{E}_V^{\otimes} \simeq \mathbb{E}_{V\oplus W}^{\otimes}$ is an equivalence.

Then the canonical map

$$\mathbb{E}_V^{\otimes} \overset{BV}{\otimes} \mathbb{E}_W^{\otimes} \to \mathbb{E}_{V\oplus W}^{\otimes}$$

is an equivalence; equivalently, for any G-symmetric monoidal category $\mathcal{C}$, the pullback functors

$$\mathbf{Alg}_{\mathbb{E}_V}\underline{\mathbf{Alg}}^{\otimes}_{\mathbb{E}_W}(\mathcal{C}) \leftarrow \mathbf{Alg}_{\mathbb{E}_{V\oplus W}}(\mathcal{C}) \to \mathbf{Alg}_{\mathbb{E}_W}\underline{\mathbf{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C})$$

are equivalences.

Proof. Given Corollary 3.33, case (a) follows from Theorem G and case (b) follows from **??**. □

Example 3.35. Let p be prime and let λ be an irreducible real orthogonal C_p-representation given by rotating the plane (or line if $p = 2$) by a primitive pth root of unity. Then, we may explicitly describe $A\infty\lambda = A\lambda$ by noting that it has infinitely many orbits of type $\left[C_p/e\right]$ and exactly one orbit of type $*_{C_p}$; this implies that it admits a C_p-equivariant embeddings of the C_p-set $a *_{C_p} + b\left[C_p/e\right]$ if and only if $a \le 1$.

Moreover, the underlying vector space of λ is positive-dimensional, so it admits embeddings of $a*_e$ for all a. Hence we've completely characterized the weak indexing system, and it matches windex. ◁

3.3.3. *Equivariant linear isometries.* Let V be a real orthogonal G-representation. The nth space of the *linear isometries operad* $\mathcal{L}(V)$, given by the linear isometries $\mathcal{L}(V^n, V)$, canonically acquires an action of $G \times \Sigma_n$, where G acts on V. Hence it presents a graph G-operad.

Proposition 3.36. *The* V-linear isometries H-operads *form a functor of G-1-categories*

$$\underline{\mathbf{Rep}}_{\mathbb{R}}^{\mathrm{Orth,Emb}}(G) \to sg\underline{\mathrm{Op}}_G$$

We refer to the associated G-operad simply as $\mathcal{L}_V$. The following result is claimed frequently in the literature, but the author was not able to find a proof; instead, she could only references recursively claiming it to be analogous to the nonequivariant case. We find this to be true, but spell it out regardless.

Proposition 3.37. *For any weak G-universe V, $\mathcal{L}_V$ is an $\mathcal{N}_\infty$-operad.*

Note that V being a weak G-universe is equivalent to existence of an equivalence

$$V \simeq V \oplus V;$$

hence it suffices to prove that $\mathcal{L}_V$ is a weak $\mathcal{N}_\infty$-operad. Unwinding definitions, we find that its space of S-ary operations are given by the Γ_S-fixed points

$$\mathcal{L}_V(S) \simeq \mathcal{L}(V^{\oplus|S|}, V)^{\Gamma_S} \simeq \mathcal{L}(V^{\oplus S}, V),$$

where V^S is the S-fold direct sum $V^{\oplus S} \simeq \bigoplus_{G/H\in \mathrm{Orb}(S)} \mathrm{Ind}_H^G \mathrm{Res}_H^G V$. Thus, it suffices to prove the following.

Lemma 3.38. *If V is a weak G-universe and W a real orthogonal G-representation, then the space of equivariant linear isometric embeddings $\mathcal{L}(W, V)$ is either empty or contractible.*

Proof. Assume W embeds into V, and fix ι one such embedding. Unsurprisingly, we perform an analogous swindle to [May77]

Indeed, we write a decomposition $V \simeq V \oplus V$, and we perform a sequence of linear deformation retracts of $\mathcal{L}(W, V) \simeq \mathcal{L}(W, V \oplus V)$; the first deforms linearly onto those linear isometries intersecting trivially with the first summand, and the second deforms linearly onto $\iota \oplus 0$. □

Thus, Theorem G will imply the following.

Corollary 3.39. *Given U, V weak universes, $\mathcal{L}_U^{\otimes} \otimes \mathcal{L}_V^{\otimes}$ is an $\mathcal{N}_\infty$-operad.*

Example 3.40. If V is a weak G-universe with 0-dimensional fixed points, then it only embeds its self-induction from subgroups $H \subset G$ such that $V^H = 0$; indeed, we have $\left(\mathrm{Ind}_H^G \mathrm{Res}_H^G V\right)^G \simeq V^H$.

In particular, if λ is an irreducible C_p representation rotating the plane (or line when $p = 2$) by a primitive pth root of unity, the above argument shows that the canonical map $\mathbb{E}_\infty^{\otimes} \to \mathcal{L}_{\infty\lambda}^{\otimes}$ is an equivalence. $\triangleleft$

Example 3.41 ([Rub21a, Prop 5.2, Cor 5.4])**.** The following conditions are equivalent:

(a) V is a complete G-universe;
(b) $A\mathcal{L}_V$ contains the transfer $e \subset G$;
(c) $\mathcal{L}_V^{\otimes} \simeq \mathrm{Comm}_G^{\otimes}$. $\triangleleft$

The author is not aware of how to compute $A\mathcal{L}_U$ in general. In fact, we can't even reduce to irreducibles in the obvious way, as shown by the following disturbing fact.

Remark 3.42. We do not attain an equivalence $\mathcal{L}_U^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{L}_V^{\otimes} \simeq \mathcal{L}_{U \oplus V}^{\otimes}$. We see this from Example 3.41, since there are exactly 2 C_p-indexing systems, given by $\underline{\mathbb{F}}_{C_p}^{\infty}$ and $\underline{\mathbb{F}}_{C_p}$. This directly implies that

$$\mathcal{L}_{\lambda_p(i)}^{\otimes} \simeq \mathbb{E}_\infty^{\otimes}$$

for all i, where $\lambda_p(i)$ is the 2-dimensional real orthogonal C_p-representation on which a fixed generator acts by rotating by $2\pi/i$; hence the canonical map

$$\mathbb{E}_\infty^{\otimes} \simeq \bigotimes_{i=0}^{p} \mathcal{L}_{\lambda_p(i)}^{\otimes} \to \mathcal{L}_{\bigoplus_i \lambda_p(i)} = \underline{\mathbb{F}}_{C_p}$$

is not an equivalence. $\triangleleft$

3.3.4. *Rubin's combinatorial free and associative G-operads.* We will see in Section 5.2 that *discrete genuine G-operads* are equivalent to G-1-operads over G-symmetric sequences. A rich source of these is Rubin's N *operads* [Rub21a].

In particular, in the setting of *graph operads*, much is said in [Rub21a, § 4] concerning free and assiative graph G-operads on symmetric sequences in G-sets; for instance, he realizes arbitrary indexing systems via free and associative graph G-operads. Unforunately, the author is not aware of a uniform scheme to translate between G-symmetric sequences of sets and symmetric sequences of G-sets, so we do not comment at depth about these; the author believes that it is likely that Rubin's characterizations carry over, and can form a basis for a discrete notion of *equivariantly associative algebraic structures* extending the $\mathbb{E}_V$ family.

4. Equivariant algebras

Philosophical remark 4.1. The restricted coYoneda embedding

$$\begin{tikzcd}
\mathrm{Op}_{\mathcal{T}} \arrow[r] & \mathbf{Cat}^{\otimes}_{\mathcal{T}, /\underline{\mathbb{F}}_{\mathcal{T}}^{\sqcup}} \arrow[r] & \mathbf{Cat}^{\otimes}_{\mathcal{T}} \arrow[r] & \mathrm{Fun}(\mathbf{Cat}^{\otimes}_{\mathcal{T}}, \mathbf{Cat}) \\
\mathcal{O}^{\otimes} \arrow[u, phantom, "\Cup" rotate=90] \arrow[r, mapsto] & \mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\sqcup}} \mathcal{O}^{\otimes} \arrow[u, phantom, "\Cup"] \arrow[r, mapsto] & \mathrm{Env}\mathcal{O}^{\otimes} \arrow[u, phantom, "\Cup"] \arrow[r, mapsto] & \mathbf{Alg}_{\mathcal{O}}(-) \arrow[u, phantom, "\Cup"]
\end{tikzcd}$$

is conservative; indeed, the first and last arrow are fully faithful, and the middle is conservative as it simply forgets the structure map to $\underline{\mathbb{F}}_{\mathcal{T}}^{\sqcup}$. hence $\mathcal{T}$-operads are determined conservatively by their theories of *algebras on $\mathcal{T}$-symmetric monoidal categories*

On the other hand, the right adjoint $\mathbf{Cat}_{\mathcal{T}}^{\otimes} \to \mathrm{Op}_{\mathcal{T}}$ is full on cores, since automorphisms over $\mathfrak{B}$ automatically preserve cocartesian lifts. Hence the associated map of spaces

$$\begin{tikzcd}
\mathbf{Cat}^{\otimes, \simeq}_{\mathcal{T}} \arrow[r] & \mathrm{Op}^{\simeq}_{\mathcal{T}} \arrow[r] & \mathrm{Fun}(\mathrm{Op}_{\mathcal{T}}, \mathbf{Cat})^{\simeq} \\
\mathcal{C}^{\otimes} \arrow[u, phantom, "\Cup"] \arrow[rr, mapsto] & & \mathbf{Alg}_{(-)}(\mathcal{C}) \arrow[u, phantom, "\Cup"]
\end{tikzcd}$$

is a summand inclusion. That is, a $\mathcal{T}$-symmetric monoidal category is determined (functorially on equivalences) by its categories of $\mathcal{O}$-algebras for each $\mathcal{O} \in \mathrm{Op}_{\mathcal{T}}$. ◃

Following along these lines, we further restrict our view from $\mathcal{O}$-algebras in $\mathcal{T}$-symmetric monoidal categories to a universal case; on one hand, in Section 4.1 we prove that the functor $\mathbf{Alg}_{(-)}(\underline{\mathcal{S}}_{\mathcal{T}}) : \mathrm{Op}_{\mathcal{T}}^{\mathrm{uni}} \to \mathbf{Cat}$ is conservative by explicitly computing its monad. On the other hand, $\underline{\mathcal{S}}_{\mathcal{T}}$ is cartesian in the sense of Theorem D, so Proposition 1.82 expresses its algebras category-theoretically as *$\mathcal{O}$-monoids*.

We show in Corollary 4.5 that $\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{C}^{I-\times})$ is a cartesian; by Theorem 1.48 and Corollary 1.83, its underlying $\mathcal{T}$-symmetric monoidal category $\underline{\mathbf{Cat}}_I(\mathcal{C}^{I-\times}) \simeq \underline{\mathrm{CMon}}_I(\mathcal{C})$ is I-semiadditive so $\underline{\mathrm{CAlg}}_I^{\otimes}(\underline{\mathcal{S}}_{\mathcal{T}})$ is a cocartesian I-symmetric monoidal category. We use this in Section 4.2 to bootstrap to the general case, proving that $\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{C})$ is I-cocartesian for all $\mathcal{C}^{\otimes}$, i.e. *I-indexed tensor products of I-commutative algebras are I-indexed coproducts*. Using work from Appendix B, we use this to conclude lifts of Theorem E and Corollary F.

We take this to its logical extreme in Section 4.3, using this to completely characterize the smashing localizations associated with $\otimes$-idempotent weak $\mathcal{N}_\infty$-operads. As promised in the introduction, we use this classification to prove a generalization of Theorem G. Following this, we demonstrate the strength of our results in Section 4.4 by using them to clarify properties of Real topological Hochschild homology.

4.1. **The monad for $\mathcal{O}$-algebras.** Fix $\mathcal{O}^{\otimes}$ a one-object $\mathcal{T}$-operad, fix $\mathcal{C}^{\otimes}$ a distributive $\mathcal{O}$-monoidal category in the sense of [NS22] (e.g. it may be presentably $\mathcal{O}$-monoidal) and let $\mathrm{triv}_{\mathcal{T}}^{\otimes} \to \mathcal{C}^{\otimes}$ be the functor of operads associated with a $\mathcal{T}$-object $X \in \Gamma\mathcal{C}$. Denote by $X^{\otimes} : \mathrm{Env}_{\mathcal{O}}\mathrm{triv}_{\mathcal{T}}^{\otimes} \to \mathcal{C}^{\otimes}$ the associated $\mathcal{O}$-symmetric monoidal functor, and denote by

$$\mathcal{O}_{\mathrm{sseq}}(X) : \mathrm{Env}_{\mathcal{O}}\mathrm{triv}_{\mathcal{T}} \to \mathcal{C}$$

the underlying $\mathcal{T}$-functor. Recall that

$$X^{\otimes S} \simeq \bigotimes_{V \in \mathrm{Orb}(S)} N_V^{\mathcal{T}} X_V \in \Gamma\mathcal{C}.$$

Proposition 4.2 ("Equivariant [SY19, Lem 2.4.2]"). *The forgetful functor $U : \underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ is monadic, and the associated monad $T_{\mathcal{O}}$ acts on $X \in \mathcal{C}$ by the indexed colimit*

$$T_{\mathcal{O}} X := \underline{\mathrm{colim}}\, \mathcal{O}_{\mathrm{sseq}}(X).$$

In particular, we have

$$(T_{\mathcal{O}} X)_V \simeq \coprod_{S \in \mathbb{F}_V} \left(\mathcal{O}(S) \cdot X^{\otimes S} \right)_{h \mathrm{Aut}_V S}. \tag{23}$$

Proof. Monadicity is precisely [NS22, Cor 5.1.5], so it suffices to compute the associated monad.

By [NS22, Rem 4.3.6], the left adjoint $\mathrm{Fr} : \mathcal{C} \to \mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$ is computed on X by $\mathcal{T}$-operadic left Kan extension of the corresponding map $\mathrm{triv}^{\otimes} \xrightarrow{X} \mathcal{C}^{\otimes}$ along the canonical inclusion $\mathrm{triv}^{\otimes} \to \mathcal{O}^{\otimes}$, and the underlying $\mathcal{T}$-functor of this is computed by the $\mathcal{T}$-left Kan extension

$$\begin{tikzcd}
\mathrm{Env}_{\mathcal{O}}\mathrm{triv} \arrow[r, equal] \arrow[dd] & \underline{\Sigma}_{\mathcal{T}} \times_{\underline{\mathbb{F}}_{\mathcal{T}}} \mathrm{Ar}^{\mathrm{act},/\mathrm{el}}(\mathcal{O}) \arrow[r, "X"] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \mathcal{C} \\
& \underline{\Sigma}_{\mathcal{T}} \arrow[d] \arrow[ur, dashed, "\widetilde{T}_{\mathcal{O}} X"'] \arrow[dr, phantom, "\Downarrow"] & \\
\mathcal{O} \arrow[r, equal] & *_{\mathcal{T}} \arrow[uur, dashed, "T_{\mathcal{O}} X"'] &
\end{tikzcd}$$

$\mathcal{T}$-left Kan extension diagrams to $\underline{*}_{\mathcal{T}}$ are $\mathcal{T}$-colimit diagrams by definition (see [Sha23, Def 10.1] when $D = \underline{*}_{\mathcal{T}}$), so the underlying $\mathcal{T}$-object is

$$T_{\mathcal{O}} X \simeq \underline{\mathrm{colim}}\, \mathcal{O}_{\mathrm{sseq}}(X).$$

More generally, the $\mathcal{T}$-left Kan extension $\widetilde{T}_{\mathcal{O}} X$ has values

$$\begin{aligned}\widetilde{T}_{\mathcal{O}} X(S) &\simeq \operatorname*{colim}_{\{S\} \times_{\underline{\mathbb{F}}_{\mathcal{T}}} \mathrm{Ar}^{\mathrm{act}/\mathrm{el}}(\mathcal{O})} X^{\otimes} \\ &\simeq \operatorname*{colim}_{\pi_{\mathcal{O}}^{-1}(S)} X^{\otimes S} \\ &\simeq \mathcal{O}(S) \cdot X^{\otimes S}.\end{aligned}$$

By composition of left Kan extensions and [Sha23, Prop 5.5], we then have

$$\begin{aligned}(T_{\mathcal{O}} X)_V &\simeq \operatorname*{colim}_{S \in \mathbb{F}_V^{\simeq}} \widetilde{T}_{\mathcal{O}} X^{\otimes S} \\ &\simeq \operatorname*{colim}_{S \in \mathbb{F}_V^{\simeq}} \mathcal{O}(S) \cdot X^{\otimes S} \\ &\simeq \coprod_{S \in \mathbb{F}_V} \left(\mathcal{O}(S) \cdot X^{\otimes S}\right)_{h \operatorname{Aut}_V S}.\end{aligned}$$

□

Remark 4.3. Let $\mathcal{O}_{G \times \Sigma_n, \Gamma_n} \subset \mathcal{O}_{G \times \Sigma_n}$ be the full subcategory spanned by $G \times \Sigma_n / \Gamma_S$ for $\phi_S : H \to \Sigma_n$ with associated graph subgroup $\Gamma_S = \{(h, \phi_S(h)) \mid h \in H\} \subset H \times \Sigma_{|S|}$. Then, a G-equivalence

$$\coprod_{n \in \mathbb{N}} \mathcal{O}_{G \times \Sigma_n, \Gamma_n} \simeq \underline{\Sigma}_G$$

was constructed in [NS22, Ex 4.3.7], and in particular, this provides a formula akin to Eq. (23) in the language of *graph families.* ◁

By [NS22, Prop 3.2.5] (noting that all colimits involved are finite), the Cartesian $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{C})$ is distributive whenever $\mathcal{C}$ is a cocomplete Cartesian closed category. We apply this to $\underline{\mathcal{S}}_{\mathcal{T}} := \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{S}$.

Corollary 4.4. *The functor* $\mathbf{Alg}_{(-)}(\underline{\mathcal{S}}_{\mathcal{T}}) : \mathrm{Op}_{\mathcal{T}}^{\mathrm{oc}} \to \mathbf{Cat}$ *is conservative.*

Proof. Suppose $\varphi : \mathcal{O} \to \mathcal{P}$ induces an equivalence $\mathbf{Alg}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \xrightarrow{\sim} \mathbf{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$. Then φ induces a natural equivalence $T_{\mathcal{O}} \implies T_{\mathcal{P}}$ respecting the summand decomposition in Proposition 4.2. Choosing $X = S \in \mathbb{F}_V$, there is a natural coproduct decomposition

$$\begin{aligned}\left(\mathcal{O}(S) \times S^{\times S}\right)_{h \operatorname{Aut}_V S} &\simeq (\mathcal{O}(S) \times \operatorname{Aut}_V S)_{h \operatorname{Aug}_V S} \sqcup J_{\mathcal{O},S} \\ &\simeq \mathcal{O}(S) \sqcup J_{\mathcal{O},S},\end{aligned}$$

for some $J_{\mathcal{O},S}$; hence the summand-preserving equivalence $T_\varphi : T_{\mathcal{O}} S \implies T_{\mathcal{P}} S$ implies that $\varphi(S) : \mathcal{O}(S) \to \mathcal{P}(S)$ is an equivalence for all S, i.e. $\operatorname{sseq} \varphi : \operatorname{sseq} \mathcal{O} \to \operatorname{sseq} \mathcal{P}$ is an equivalence of $\mathcal{T}$-symmetric sequences. Thus Proposition 2.75 implies that φ is an equivalence. □

We also point out a straightforward consequence of the fact that the forgetful functor is a right $\mathcal{T}$-adjoint.

Corollary 4.5. *The I-symmetric monoidal ∞-category* $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}\left(\mathcal{C}^{I-\times}\right)$ *is cartesian.*

Proof. The forgetful functor $U : \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}\left(\mathcal{C}^{I-\times}\right) \to \mathcal{C}$ is conservative, preserves $\mathcal{T}$-limits, and preserves tensor products; for all $(X_W) \in \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}\left(\mathcal{C}^{I-\times}\right)_S$, the canonical map

$$U\left(\bigotimes_W^S X_W\right) \simeq \bigotimes_W^S U(X_W) \to \prod_W^S U(X_W) \simeq U\left(\prod_W^S X_W\right)$$

is an equivalence, so $\bigotimes_W^S X_W \to \prod_W^S X_W$. □

To finish the section, we repeat the above work without the one-color assumption.,

Observation 4.6. By either [NS22, Lem 2.4.4] or [CH21, Lem 2.9], we find that $\underline{\Sigma}_{\mathcal{T}}$-fibrous patterns are right Kan extended from their underlying $\mathcal{T}^{\mathrm{op}}$-fibrous patterns. Unwinding definitions, this expresses

$$\pi_0 \mathrm{triv}(\mathcal{O})_V \simeq \{(\mathbf{C}, D) \in \mathcal{O}_S \times \mathcal{O}_V \mid S \in \mathbb{F}_V\}$$

$\triangleleft$

Observation 4.7. Analogously to the above, for $\mathcal{O}^{\otimes}$ an *arbitrary* $\mathcal{T}$-operad, the operadic left kan extension formula of [NS22, Rmk 4.3.6] expresses the values of the associated monad as the left Kan extension

$$\begin{tikzcd}
\mathrm{Env}_{\mathcal{O}}\mathrm{triv}(\mathcal{O}) \arrow[r, equal] \arrow[dd] & \mathrm{tottriv}(\mathcal{O})^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{\mathrm{act},/\mathrm{el}}(\mathcal{O}) \arrow[rr, "X"] \arrow[d] \arrow[r, phantom, "\Rightarrow"] & & \mathcal{C} \\
& \mathrm{tottriv}^{\otimes}_{(}\mathcal{O})^{\otimes} \arrow[d] \arrow[urr, dashed, "\widetilde{T}_{\mathcal{O}}X"] \arrow[r, phantom, "\Rightarrow"] & & \\
\mathcal{O} \arrow[r, equal] & \mathcal{O} \arrow[uurr, dashed, "T_{\mathcal{O}}X"'] & &
\end{tikzcd}$$

The $\mathcal{T}$-functor $\tilde{T}_{\mathcal{O}}(X)$ sends

$$(\mathbf{C}, D) \mapsto \left(\mathcal{O}(\mathbf{C}; D) \otimes \bigotimes_{U}^{S} X_U \right)_{h\mathrm{Aut}_V S} \tag{24}$$

$\triangleleft$

Corollary 4.8 ("Equivariant [HM23, Thm 4.1.1]")**.** *A map of $\mathcal{T}$-operads $\varphi : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is an equivalence if and only if it satisfies the following conditions:*

(a) $U(\varphi) : \mathcal{O} \to \mathcal{P}$ is $\mathcal{T}$-essentially surjective, and
(b) the pullback functor $\varphi^* : \mathbf{Alg}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \to \mathbf{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is an equivalence of ∞-categories.

Proof. The fact that φ being an equivalence implies the above conditions is obvious, so assume the above conditions. The result follows by using an identical argument to Corollary 4.4, using Eq. (24) instead of Eq. (23) to show that $\varphi : \mathcal{O}(\mathbf{C}; D) \to \mathcal{P}(\varphi \mathbf{C}; \varphi D)$ is an equivalence for all C, concluding the equivalence from Proposition 2.75. $\square$

4.2. **Indexed tensor products of I-commutative algebras.** In Lemma B.4, we show that every object in a cocartesian I-symmetric monoidal structure bears a canonical I-commutative algebra algebra structure, i.e. $\underline{\mathrm{CAlg}}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence. In this subsection, we demonstrate the converse, or equivalently, we demonstrate that I-indexed tensor products of I-commutative algebras are indexed coproducts. We go on to use this to completely characterize the smashing localization on $\mathrm{Op}_{\mathcal{T}}$ associated with aE-unital weak $\mathcal{N}_\infty$-operads.

First, we need some prerequisites on unital $\mathcal{T}$-operads, beginning with the following.

Observation 4.9. If $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal category with unit $\mathcal{T}$-object $1_\bullet$ and $X \in \mathcal{C}_V$, then $\mathrm{Map}_{\mathcal{O}^{\otimes}}(\varnothing, X) \simeq \mathrm{Map}_{\mathcal{C}_V}(1_V, X)$, so $\mathcal{C}^{\otimes}$ is unital if and only if $1_\bullet$ is initial; in particular, if $\mathcal{C}^{\otimes}$ is cartesian, then it is unital if and only if it is pointed. $\triangleleft$

Using this, in [NS22] unitality was shown to be compatible with algebras, which we recall here.

Proposition 4.10 ([NS22, Thm 5.2.11])**.** *If $\mathcal{O}^{\otimes}$ is a unital $\mathcal{T}$-operad and $\mathcal{C}^{\otimes}$ an $\mathcal{O}$-monoidal ∞-category, then $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ is unital.*

Thus Yoneda's lemma characterizes $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$ by its algebras over *unital* $\mathcal{T}$-operads.

Theorem 4.11 (Indexed tensor products of $\mathcal{N}_\infty$-algebras)**.** *The following are equivalent for $\mathcal{C}^{\otimes} \in \mathbf{Cat}_I^{\otimes}$.*

(a) For all morphisms $f : S \to T$ in $\mathcal{I}$, the action map $f_{\otimes} : \mathcal{C}_S \to \mathcal{C}_T$ is left adjoint to $f^* : \mathcal{C}_T \to \mathcal{C}_S$.
(b) There is an I-symmetric monoidal equivalence $\mathcal{C}^{\otimes} \simeq \mathcal{C}^{I-\sqcup}$ extending the identity on $\mathcal{C}$.
(c) For all unital I-operads $\mathcal{O}^{\otimes}$, the forgetful functor $\underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O}, \mathcal{C})$ is an equivalence.
(d) The forgetful functor $\underline{\mathrm{CAlg}}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence.

In order to prove Theorem 4.11, we introduce yet another condition:

(b') There is an I-symmetric monoidal equivalence $\mathcal{C}^{\otimes} \simeq \mathcal{C}^{I-\sqcup}$.

The implication (b') $\implies$ (c) is precisely the computation Lemma B.4. For the implication (c) $\implies$ (b'), note that $\mathcal{C}^{1/} \simeq \mathbf{Alg}_{\mathbb{E}_0}(\mathcal{C}) \to \mathcal{C}$ an equivalence implies that $\mathcal{C}^{\otimes}$ is unital by Proposition 4.10; hence Yoneda's lemma applied to $\mathrm{Op}_I^{\mathrm{uni}}$ constructs an I-operad equivalence $\mathcal{C}^{\otimes} \simeq \mathcal{C}^{I-\sqcup}$, which is an I-symmetric monoidal equivalence by Philosophical remark 4.1.

Furthermore, the implication (b') $\implies$ (a) follows by definition, (a) $\implies$ (b) is precisely Theorem D', and the statements (b) $\implies$ (b') and (c) $\implies$ (d) follow by neglect of assumptions. To summarize, we've arrived at the implications

$$
\begin{tikzcd}
 & \text{(b)} \arrow[d, Rightarrow] & & \\
\text{(a)} \arrow[ur, Rightarrow] & & \text{(c)} \arrow[r, Rightarrow] \arrow[dl, Leftrightarrow] & \text{(d)} \\
 & \text{(b')} \arrow[ul, Rightarrow] & &
\end{tikzcd}
\tag{25}
$$

Our workhorse lemma for closing the gap is the following.

Lemma 4.12. *The following are equivalent for $\mathcal{P}^{\otimes} \in \mathrm{Op}_I$:*

(e) *The $\mathcal{T}$-∞-category $\underline{\mathbf{Alg}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive.*

(f) *For all $\mathcal{O}^{\otimes} \in \mathrm{Op}_I^{\mathrm{uni}}$, the forgetful functor*
$$\mathbf{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathbf{Alg}_{\mathcal{O}}\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \underline{\mathcal{S}}_{\mathcal{T}})$$
is an equivalence.

(g) *For all $\mathcal{O}^{\otimes} \in \mathrm{Op}_I^{\mathrm{uni}}$, the map $\mathrm{triv}^{\otimes}_{\mathcal{O}} \otimes^{\mathrm{BV}} \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes} \otimes^{\mathrm{BV}} \mathcal{P}^{\otimes}$ is an equivalence.*

(h) *For all $\mathcal{O}^{\otimes} \in \mathrm{Op}_I^{\mathrm{uni}}$ and $\mathcal{C} \in \mathbf{Cat}_I^{\otimes}$, the forgetful functor*
$$\mathbf{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\mathcal{C}) \simeq \mathbf{Alg}_{\mathcal{O}}\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \mathcal{C})$$
is an equivalence.

Proof. Since Corollary 4.5 shows that $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is cartesian, the equivalence between (e) $\iff$ (f) is just (a) $\iff$ (c) applied to $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$. (f) $\implies$ (g) follows from Corollary 4.8, and the implications (g) $\implies$ (h) $\implies$ (f) are obvious. □

Proof of Theorem 4.11. After the implications illustrated in Eq. (25), it suffices to prove that $\underline{\mathrm{CAlg}}_I(\mathcal{C})$ satisfies (c) for all $\mathcal{C}^{\otimes} \in \mathbf{Cat}_I^{\otimes}$; by Lemma 4.12, it suffices to prove that $\underline{\mathrm{CAlg}}_I(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive. But in fact, by Corollary 1.83 there is an equivalence $\underline{\mathrm{CAlg}}_I(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}})$ and the latter is I-semiadditive by Cnossen-Lenz-Linsken's semiadditive closure theorem Theorem 1.48. □

Rephrasing things somewhat, we've arrive at the following theorem.

Theorem E'. *Let $\mathcal{O}^{\otimes}$ be an almost-E-reduced $\mathcal{T}$-operad. Then, the following properties are equivalenent.*

(a) *The $\mathcal{T}$-∞-category $\underline{\mathbf{Alg}}_{\mathcal{O}}\underline{\mathcal{S}}_{\mathcal{T}}$ is $A\mathcal{O}$-semiadditive.*

(b) *The unique map $\mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{A\mathcal{O}\infty}$ is an equivalence.*

Furthermore, for any almost-E-unital weak indexing system I and I-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, the I-symmetric monoidal ∞-category $\underline{\mathrm{CAlg}}^{\otimes}_I\mathcal{C}$ is cocartesian.

Proof. By Lemma 4.12 and Theorem 4.11, Condition (a) is equivalent to the condition that $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ is $A\mathcal{O}$-cocartesian for all $\mathcal{C}$. In fact by Theorem 4.11, this is equivalent to existence of the first equivalence in
$$\mathrm{CAlg}^{\otimes}_{A\mathcal{O}}\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \simeq \mathbf{Alg}_{\mathcal{O}}\underline{\mathrm{CAlg}}_{A\mathcal{O}}(\mathcal{C}) \simeq \mathbf{Alg}_{\mathcal{O}}(\mathcal{C}),$$
which by Yoneda's lemma is equivalent to the unique map $\mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{A\mathcal{O}\infty}$ being an equivalence, i.e. Condition (b). The remaining statement follows immediately from Theorem 4.11. □

Corollary 4.13. *Let $\mathcal{O}^{\otimes}$ be a reduced I-operad. Then, the canonical map $F : \mathcal{N}^{\otimes}_{I\infty} \to \mathcal{N}^{\otimes}_{I\infty} \otimes \mathcal{O}^{\otimes}$ is an equivalence.*

Proof. By Theorem 4.11, the forgetful map

$$F^*: \mathbf{Alg}_{\mathcal{O}\otimes\mathcal{N}_{I\infty}}(\mathcal{C}) \simeq \mathbf{Alg}_{\mathcal{O}}\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{N}_{I\infty}}(\mathcal{C}) \to \underline{\mathbf{Alg}}_{\mathcal{N}_{I\infty}}(\mathcal{C})$$

is an equivalence for all distributive G-symmetric monoidal categories $\mathcal{C}$; the statement follows by specializing to $\mathcal{C} := \underline{\mathcal{S}}_G$ and applying Corollary 4.4. □

4.3. **The smashing localization for $\mathcal{N}^{\otimes}_{I\infty}$ and the main theorem.**

4.3.1. *The smashing localization classified by $\mathcal{N}^{\otimes}_{I\infty}$.* We would like to prove the following.

Theorem 4.14. *Let I be an aE-unital weak indexing system. Then, a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ satisfies $\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \simeq \mathcal{O}^{\otimes}$ if and only if the following conditions are satisfied:*

(a) $c(\mathcal{O}) \subset c(I)$.
(b) $v(\mathcal{O}) \supset v(I)$.
(c) The canonical map $\mathrm{Bor}^{\mathcal{T}}_{I\cap c(O)}\mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{I\cap c(\mathcal{O})}$ is an equivalence.

Remark 4.15. Condition (c) of Theorem 4.14 is equivalent to the condition that, for all $\mathcal{P}^{\otimes} \in \mathrm{Op}_{I\cap c(\mathcal{O})}$ and $\mathcal{C} \in \mathbf{Cat}^{\otimes}_{\mathcal{T}}$, the forgetful map $\mathbf{Alg}_{\mathcal{P}}\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \to \mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$ is an equivalence; by Theorem 4.11, this in turn is equivalent to the condition that, for all $\mathcal{C}$ (or just $\mathcal{C} = \underline{\mathcal{S}}_{\mathcal{T}}$) and all I-admissible $c(\mathcal{O})$-sets S, the S-indexed tensor products in $\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ are indexed coproducts. ◁

Note that $c\left(\mathcal{O} \otimes \mathcal{N}^{\otimes}_{I\infty}\right) \simeq c(\mathcal{O}) \cap c(I)$, so (a) is necessary; in fact, assuming (a), we may apply Proposition 3.19. This reduces Theorem 4.14 to the following proposition.

Proposition 4.16. *Let I be a unital weak indexing system. Then, an at-least one color $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ satisfies $\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \simeq \mathcal{O}^{\otimes}$ if and only if the following are true:*

(b) $\mathcal{O}^{\otimes}$ is unital.
(c) The canonical map $\mathrm{Bor}^{\mathcal{T}}_I \mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{I\infty}$ is an equivalence.

The hard step of this is the following lemma, whose proof we slightly postpone.

Lemma 4.17. *$\mathcal{O}^{\otimes} \in \mathrm{Op}^{\mathrm{oc}}_{\mathcal{T}}$ satisfies $\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathbb{E}^{\otimes}_0$ if and only if it is unital.*

Proof of Proposition 4.16. First assume that $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \simeq \mathcal{O}^{\otimes}$. By Lemma 4.17, we have

$$\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathbb{E}^{\otimes}_0 \simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}^{\otimes}_0,$$

so $\mathcal{O}^{\otimes}$ is unital. To prove (c), in light of Remark 4.15, it suffices to note that the equivalence $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \simeq \mathcal{O}^{\otimes}$ demonstrates that the canonical map

$$\begin{aligned} \mathrm{CAlg}_I(\underline{\mathcal{S}}_{\mathcal{T}}) &\xleftarrow{\sim} \mathbf{Alg}_{\mathrm{Bor}^{\mathcal{T}}_I\mathcal{O}}\underline{\mathrm{CAlg}}^{\otimes}_I(\underline{\mathcal{S}}_{\mathcal{T}}) \\ &\simeq \mathrm{CAlg}^{\otimes}_I \underline{\mathbf{Alg}}^{\otimes}_{\mathrm{Bor}^{\mathcal{T}}_I\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}}) \\ &\to \mathbf{Alg}_{\mathrm{Bor}^{\mathcal{T}}_I\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}}) \end{aligned}$$

is an equivalence, so Corollary 4.8 proves that $\mathrm{Bor}^{\mathcal{T}}_I \mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{I\infty}$ is an equivalence. The converse follows by noting that each of the above arguments works in reverse. □

4.3.2. *The proof of the main theorem.* We are finally ready for Theorem G. We start with the unital case.

Proposition 4.18. *When I and J are unital, there is an equivalence $\mathcal{N}^{\otimes}_{I\infty} \overset{BV}{\otimes} \mathcal{N}^{\otimes}_{J\infty} \simeq \mathcal{N}^{\otimes}_{I\vee J\infty}$.*

Proof. By [CSY20, Prop 5.1.8], $\mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{J\infty}$ is an $\overset{\mathrm{BV}}{\otimes}$-idempotent classifying the conjunction of the properties which are classified by $\mathcal{N}^{\otimes}_{I\infty}$ and $\mathcal{N}^{\otimes}_{\infty}$; that is, a unital $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ is fixed by $(-) \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{J\infty}$ if and only if $\underline{\mathbf{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive and J-semiadditive; By Corollary 1.44, this is equivalent to the property that $\underline{\mathbf{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is $I \vee J$-semiadditive , i.e. $\mathcal{O}^{\otimes}$ is fixed by $(-) \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I \vee J}$. Thus, we have

$$\mathcal{N}^{\otimes}_{I \vee J} \simeq \mathcal{N}^{\otimes}_{I \vee J} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\overset{\mathrm{BV}}{\otimes}}_{J\infty} \simeq \mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}.$$

□

We may now conclude the full theorem, which we restate in the orbital case.

Theorem G'. *The functor $\mathcal{N}^{\otimes}_{(-)\infty} : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T}}$ lifts to a fully faithful $\mathcal{T}$-right adjoint*

$$\begin{tikzcd} \underline{\mathrm{wIndex}}_{\mathcal{T}} \arrow[rr, hook, bend right, "\mathcal{N}^{\otimes}_{(-)\infty}"'] \arrow[rr, phantom, "\top"] & & \underline{\mathrm{Op}}_{\mathcal{T}} \arrow[ll, bend right, "A"'] \end{tikzcd}$$

whose restriction $\underline{\mathrm{wIndex}}^{a E\mathrm{uni}}_{\mathcal{T}} \subset \underline{\mathrm{Op}}_{\mathcal{T}}$ is symmetric monoidal. Furthermore, the resulting tensor product on $\mathrm{wIndex}^{a E\mathrm{uni},\otimes}_{\mathcal{T}}$ *is computed by the Borelified join*

$$I \otimes J = \mathrm{Bor}^{\mathcal{T}}_{\mathrm{cSupp}(I \cap J)}(I \vee J);$$

in particular, when I and J are almost-E-unital weak indexing systems, we have

$$\begin{aligned} \mathcal{N}^{\otimes}_{I\infty} \otimes \mathcal{N}^{\otimes}_{J\infty} &\simeq \mathcal{N}^{\otimes}_{(I \vee J)\infty} \otimes \mathrm{triv}^{\otimes}_{c(I \cap J)} \\ \mathcal{N}^{\otimes}_{I\infty} \times \mathcal{N}^{\otimes}_{J\infty} &\simeq \mathcal{N}^{\otimes}_{(I \cap J)\infty} \\ \mathrm{Res}^{W}_{V} \mathcal{N}^{\otimes}_{I\infty} &\simeq \mathcal{N}^{\otimes}_{\mathrm{Res}^{W}_{V} I\infty} \\ \mathrm{CoInd}^{W}_{V} \mathcal{N}^{\otimes}_{I\infty} &\simeq \mathcal{N}^{\otimes}_{\mathrm{CoInd}^{W}_{V} I\infty}. \end{aligned}$$

Hence W-norms of I-commutative algebras are CoInd^{W}_{V} *I-commutative algebras, and when I, J are almost-unital, we have*

$$\underline{\mathrm{CAlg}}^{\otimes}_{I}\underline{\mathrm{CAlg}}^{\otimes}_{J}(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_{I \vee J}(\mathcal{C}). \tag{26}$$

Proof of Theorem G'. The $\mathcal{T}$-adjunction is precisely Proposition 3.20, the equations are immediate from the symmetric monoidal adjunction, the statement about norms of I-commutative algebras is Construction 3.24, and Eq. (26) follows immediately from symmetric monoidality of $\mathcal{N}^{\otimes}_{(-)\infty}$. We are left with proving that the adjunction is symmetric monoidal in the aE-unital case.

In view of Proposition 3.17, to prove that this is a $\mathcal{T}$-symmetric monoidal adjunction with the prescribed tensor product, it suffices to prove that the collection of aE-unital weak $\mathcal{N}_{\infty}$-operads is $\overset{\mathrm{BV}}{\otimes}$-closed, for which it suffices to prove that for all aE-unital weak indexing systems I and J, the unique map $\varphi : \mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{J\infty} \to \mathcal{N}^{\otimes}_{I \vee J}$ is an equivalence. In fact, by Proposition 3.19, it suffices to prove that $\mathrm{Bor}^{\mathcal{T}}_{v(I \cap J)}(\varphi)$ is an equivalence, i.e. we may assume that I and J are unital. Then, the statement is precisely Proposition 4.18. □

4.3.3. *Unitalization.* We now focus on Lemma 4.17, beginning by recalling a result of Nardin-Shah.

Proposition 4.19 ([NS22, Thm 5.2.10])**.** *If $\mathcal{C}$ is a $\mathcal{T}$-symmetric monoidal ∞-category with unit $\mathcal{T}$-object 1, then there is a canonical equivalence $\underline{\mathbf{Alg}}_{\mathbb{E}_0}(\mathcal{C}) \simeq \mathcal{C}^{1/}$.*

In the case that $\mathcal{C}$ is a cartesian I_0-symmetric monoidal category (i.e. the unit is terminal, e.g. it is pulled back from a cartesian $\mathcal{T}$-symmetric monoidal category), this has a more familiar form, as

$$\mathbf{Alg}_{\mathbb{E}_0}(\mathcal{C}^{\times}) \simeq \left(\Gamma^{\mathcal{T}} \mathcal{C}\right)_{*}.$$

We use this to prove the following strengthening of Lemma 4.17.

Proposition 4.20. *Given a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ with at least one color, the following are equivalent:*

(a) $\mathrm{Bor}^{\mathcal{T}}_{I_0}\mathcal{O}^{\otimes}$ *is unital.*

(b) $\mathcal{O}^\otimes$ *is unital.*
(c) The ∞*-category* $\mathbf{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ *is pointed.*
(d) $\mathcal{O}^\otimes \simeq \mathcal{O}^\otimes \overset{BV}{\otimes} \mathbb{E}_0^\otimes$.
(e) $\mathrm{Bor}_{I_0}^{\mathcal{T}} \mathcal{O}^\otimes \simeq \mathbb{E}_0^\otimes \overset{BV}{\otimes} \mathrm{Bor}_{I_0}^{\mathcal{T}} \mathcal{O}^\otimes$.
(f) The ∞*-category* $\mathbf{Alg}_{\mathrm{Bor}_{I_0}^{\mathcal{T}}\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ *is pointed*

Proof. (a) $\implies$ (b) follows immediately by definition; (b) $\implies$ (c) follows immeditely by [NS22, Thm 5.2.11]. (c) $\implies$ (d) and (e) $\implies$ (f), since $\mathbf{Alg}_{\mathcal{O}\otimes\mathbb{E}_0}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Mon}_{\mathbb{E}_0}\mathbf{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathbf{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})_*$ over $\mathbf{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$. (d) $\implies$ (e) follows by applying Borelification.

What's left is to prove that (f) $\implies$ (a). We argue the contrapositive, writing $\mathcal{P}^\otimes := \mathrm{Bor}_{I_0}^{\mathcal{T}} \mathcal{O}^\otimes$, assuming that $\mathcal{P}^\otimes$ is not unital, and fixing $C \in \mathcal{P}_V$ such that $\mathcal{P}(\varnothing_V; C) \neq *$. We choose the "skyscraper" $\mathcal{P}$-algebra M, with values

$$M(D) = \begin{cases} \mathcal{P}(\varnothing_V, C) & D = C \\ * & \text{otherwise,} \end{cases}$$

gotten by truncating the functor corepresented by $\varnothing$. Then, note that

$$\mathrm{Map}(*_{\mathcal{P}}, M) \simeq \mathcal{P}(\varnothing; C) \not\simeq *,$$

so the unit $*_{\mathcal{P}} \in \mathbf{Alg}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is not initial. By [NS22, Thm 5.2.11] it is terminal, so by contrapositgion we have shown (f) $\implies$ (a). ☐

Last, we point out a corollary. In Appendix B, given $\mathcal{C}$ a $\mathcal{T}$-category (which may not admit I-indexed coproducts), we construct an I-operad $\mathcal{C}^{I-\sqcup}$ together with an equivalence

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\sqcup}) \simeq \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \tag{27}$$

for all unital I-operads $\mathcal{O}$. In particular, this proves the following.

Corollary 4.21. *The restriction* $U_{\mathrm{uni}} : \underline{\mathrm{Op}}_{\mathcal{T}}^{\mathrm{uni}} \to \underline{\mathbf{Cat}}_{\mathcal{T}}$ *is left* $\mathcal{T}$*-adjoint to* $(-)^{I-\sqcup}$*.*

Warning 4.22. Corollary 4.21 shows that no nontrivial $\mathcal{T}$-colimit of one-color $\mathcal{T}$-operads has one color; in particular, no one-color $\mathcal{T}$-operads are the result of a nontrivial induction. ◁

Furthermore, note that Theorem 4.11 yields equivalences

$$\begin{aligned} \mathrm{CAlg}_{\mathcal{T}} \underline{\mathbf{Alg}}_{\mathcal{O}}^\otimes(\mathcal{C}^{I-\sqcup}) &\simeq \mathbf{Alg}_{\mathcal{O}} \underline{\mathrm{CAlg}}_{\mathcal{T}}^\otimes(\mathcal{C}^{I-\sqcup}) \\ &\simeq \mathbf{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\sqcup}), \end{aligned}$$

for all $\mathcal{O}^\otimes \in \mathrm{Op}_{\mathcal{T}}^{\mathrm{uni}}$. Hence Theorem 4.11 implies the following.

Corollary 4.23. *Suppose* $\mathcal{O}^\otimes$ *is a unital* I*-operad and* $\mathcal{C}$ *admits* I*-indexed coproducts. Then, the* I*-symmetric monoidal category* $\underline{\mathbf{Alg}}_{\mathcal{O}}^\otimes\left(\mathcal{C}^{I-\sqcup}\right)$ *is cocartesian.*

We use this to compute the $\mathcal{T}$-category underlying BV tensor products.

Proposition 4.24. *The underlying category* $U|_{\mathrm{uni}} : \mathrm{Op}_{\mathcal{T}}^{\mathrm{uni}} \to \mathbf{Cat}_{\mathcal{T}}$ *functor sends*

$$U\left(\mathcal{O}^\otimes \overset{BV}{\otimes} \mathcal{P}^\otimes\right) \simeq U^{\mathrm{uni}}(\mathcal{O}^\otimes) \times U^{\mathrm{uni}}(\mathcal{P}^\otimes).$$

in particular, $\underline{\mathrm{Op}}_{\mathcal{T}}^{\mathrm{red}} \subset \underline{\mathrm{Op}}_{\mathcal{T}}$ *is a* $\overset{BV}{\otimes}$*-closed* $\mathcal{T}$*-subcategory.*

Proof. Corollaries 4.21 and 4.23 together yield a string of equivalences

$$\begin{aligned} \mathrm{Fun}_{\mathcal{T}}\left(U\left(\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes\right), \mathcal{C}\right) &\simeq \mathbf{Alg}_{\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{P}}\left(\mathcal{C}^{I-\sqcup}\right) \\ &\simeq \mathbf{Alg}_{\mathcal{O}} \underline{\mathbf{Alg}}_{\mathcal{P}}^\otimes\left(\mathcal{C}^{I-\sqcup}\right) \\ &\simeq \mathbf{Alg}_{\mathcal{O}} \underline{\mathrm{Fun}}_{\mathcal{T}}\left(U(\mathcal{P}^\otimes), \mathcal{C}\right)^{I-\sqcup} \\ &\simeq \mathrm{Fun}_{\mathcal{T}}\left(U(\mathcal{O}^\otimes), \underline{\mathrm{Fun}}_{\mathcal{T}}\left(U(\mathcal{P}^\otimes), \mathcal{C}\right)\right) \\ &\simeq \mathrm{Fun}_{\mathcal{T}}\left((U(\mathcal{O}^\otimes) \times U(\mathcal{P}^\otimes), \mathcal{C}\right), \end{aligned}$$

so the result follows by Yoneda's lemma. $\square$

4.4. **Application: iterated Real topological Hochschild homology.** Let $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$ be an *arbitrary* $\mathcal{T}$-operad. Then, for all $\mathcal{P}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$ and $\mathcal{C} \in \mathbf{Cat}_{\mathcal{T}}^{\otimes}$, we have an equivalence

$$\begin{aligned}\mathrm{Fun}_{\mathcal{T}}^{\otimes}\left(\mathrm{Env}\mathcal{P}^{\otimes}, \underline{\widetilde{\mathrm{Fun}}}_{\mathcal{T}}^{\otimes}\left(\mathrm{Env}\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes}\right)\right) &\simeq \mathrm{Fun}_{\mathcal{T}}^{\otimes}\left(\mathrm{Env}\mathcal{P}^{\otimes} \times \mathrm{Env}\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes}\right)\\ &\simeq \mathrm{Fun}_{\mathcal{T}}^{\otimes}\left(\mathrm{Env}\left(\mathcal{P}^{\otimes} \otimes \mathcal{O}^{\otimes}\right), \mathcal{C}^{\otimes}\right)\\ &\simeq \mathbf{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\mathcal{C})\\ &\simeq \mathbf{Alg}_{\mathcal{P}}\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C});\end{aligned}$$

Thus, we've proved the following.

Proposition 4.25. *Under the equivalence $\underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}^{\otimes}(\mathrm{Env}_{\mathcal{T}}\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$, the $\mathcal{T}$-symmetric monoidal structure on the latter is the (restricted) pointwise structure.*

In particular, this constructs a G-symmetric monoidal lift for *genuine equivariant factorization homology.*

Corollary 4.26. *Given M a V-framed smooth G-manifold, M-factorization homology lifts to a G-symmetric monoidal functor*

$$\int_M : \underline{\mathbf{Alg}}_{\mathbb{E}_V}^{\otimes}(\mathcal{C}) \to \mathcal{C}^{\otimes};$$

in particular, it further lifts to a G-symmetric monoidal functor

$$\int_M : \underline{\mathrm{CAlg}}_{AV}^{\otimes}(\mathcal{C}) \to \underline{\mathrm{CAlg}}_{AV}^{\otimes}(\mathcal{C}).$$

Proof. In the notation of [Hor19], let $\iota^{\otimes} : \underline{\mathrm{Disk}}^{G,V-fr,\sqcup} \to \underline{\mathrm{Mfld}}^{G,V-fr,\sqcup}$ be the symmetric monoidal inclusion of V-framed G-disks into V-framed G-manifolds. Then, $\int_M$ may be presented as a composition

$$\begin{tikzcd}
\mathrm{Fun}_G\left(\underline{\mathrm{Disk}}^{G,V-fr}, \mathcal{C}\right) \arrow[r, "\iota_!"] & \mathrm{Fun}_G\left(\underline{\mathrm{Mfld}}^{G,V-fr}, \mathcal{C}\right) \arrow[ddr, bend left, "\mathrm{ev}_M"] \\
\mathrm{Fun}_G^{\otimes}\left(\underline{\mathrm{Disk}}^{G,V-fr}, \mathcal{C}\right) \arrow[u, "U"] \arrow[r] & \mathrm{Fun}_G^{\otimes}\left(\underline{\mathrm{Mfld}}^{G,V-fr}, \mathcal{C}\right) \arrow[u] \arrow[d] \\
\mathbf{Alg}_{\mathbb{E}_V}(\mathcal{C}) \arrow[u, phantom, "\wr"] \arrow[r, "\int_M"] & \mathcal{C}
\end{tikzcd}$$

To construct the lift of $\int_M$, we may compose G-symmetric monoidal lifts of U, $\iota_!$, and ev_M; these are given by Observations 1.89 and 1.91. $\square$

Corollary 4.27. *Real topological Hochschild homology lifts to a C_2-symmetric monoidal functor*

$$\mathrm{THR} : \underline{\mathbf{Alg}}_{\mathbb{E}_\sigma}^{\otimes}(\mathrm{Sp}) \to \underline{\mathrm{Sp}}_{C_2};$$

in particular, if V contains infinitely many copies of σ, then THR *lifts to a C_2-symmetric monoidal functor*

$$\mathrm{THR} : \underline{\mathbf{Alg}}_{\mathbb{E}_V}^{\otimes}(\mathcal{C}) \to \underline{\mathbf{Alg}}_{\mathbb{E}_V}^{\otimes}(\mathcal{C}).$$

Furthermore, given $A \in \mathrm{CAlg}_{C_2}(\mathcal{C})$, there is an equivalence

$$\mathrm{THR}(A) \simeq \mathrm{colim}_{S^\sigma} A,$$

with colimit taken in $\mathrm{CAlg}_{C_2}(\mathcal{C})$.

Proof. The last sentence is the only part which does not follow immediately from combining Horev's facotization homology formula [Hor19, Rmk 7.1.2] with Corollary 4.26 in view of the equivariant infinitary

Dunn additivity of Corollary H. In fact, the collar decomposition formula of [Hor19, Prop 7.1.1] yields a coequalizer diagram

$$\begin{tikzcd} N_e^{C_2} A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, phantom, "\wr"] & A \otimes A \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{THR}(A) \arrow[d, equal] \\ \mathrm{CoInd}_e^{C_2} \mathrm{Res}_e^{C_2} A \arrow[r, shift left] \arrow[r, shift right] & A \oplus A \arrow[r] & \mathrm{THR}(A) \end{tikzcd}$$

Pulling A out of the bottom expression, we find that $\mathrm{THR}(A) \simeq \mathrm{colim}_X A$, where X is the C_2-space $\mathrm{CoEq}([C_2/e] \rightrightarrows 2*_{C_2}) \xrightarrow{\sim} X$; this is just the standard C_2-cell presentation of $X = S^\sigma$. □

5. Equivariant discrete algebras and the surrounding literature

In Section 5.1, we repay the debt incurred in Section 3.3, and we prove that the total right derived functor of [Bon19]'s *genuine operadic nerve* exists and is conservative. Following this, in **??** we study this nerve at greater depth in the discrete setting, verifying that all models for discrete G-operads agree, and producing a new concrete model for $\mathcal{T}$-1-operads. In Section 5.3 we leverage this new model to construct explicit counterexamples, demonstrating that outside of the aE-unital setting, $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ fails to even have connected structure spaces; thus we finally conclude Theorem E and Corollary F. Finally, we finish the section in Section 5.4 with an attempt to stimulate discussion around equivariant higher algebra, listing a wide variety of basic (though not easy) questions and conjectures.

5.1. **A conservative ∞-categorical genuine operadic nerve.** In [BP21], a model category $s\mathrm{Op}_G$ of *colored genuine G-equivariant operads* was constructed, and later shown to be quillen equivalent to several other variations on *G-operads* (e.g. [BP20a, Tab 1]). We refer to the resulting ∞-category as $g\mathrm{Op}_G$, and its one-color variant as $g\mathrm{Op}_G^{\mathrm{oc}}$.

Generalizing [HA, Def 2.1.1.3], Bonventre went on to construct a *genuine operadic nerve* 1-categorical functor sending objects in the model category of genuine G-operads to objects in a model category presenting G-operads (c.f. [NS22, § 2.6]):

$$N^{\otimes} : s\mathrm{Op}_G \to s\mathbf{Set}^+_{\underline{\mathbb{F}}_{G,*}, Ne}.$$

The functor $N^{\otimes}$ was shown to preserve the respective classes of fibrant objects in [Bon19, Thm 4.10]. We go on to endow $N^{\otimes}$ with homotopical structure in the following result.

Proposition 5.1. *$N^{\otimes}$ preserves and reflects weak equivalences between one-color locally fibrant genuine equivariant G-operads.*

Proof. By [BP21, Thm II, Prop 4.31], the functor $U : s\mathrm{Op}_G^{\mathrm{oc}} \to \mathrm{Fun}(\underline{\Sigma}_G, \mathrm{sSet})$ is monadic and $g\mathrm{Op}_G^{\mathrm{oc}}$ possesses the (right-)transferred model structure from the projective model structure on $\mathrm{Fun}\left(\underline{\Sigma}_G, \mathrm{sSet}_{\mathrm{Quillen}}\right)$; in particular, U preserves and reflects weak equivalences.

It is not hard to see that the *underlying symmetric sequence* functor sseq of Section 4.1 may be presented as total right-derived from a functor

$$s\mathrm{sseq} : \mathrm{sSet}^{+,\mathrm{oc}}_{/(\underline{\mathbb{F}}_{\mathcal{T}}, Ne)} \to \mathrm{Fun}\left(\underline{\Sigma}_G, \mathrm{sSet}_{\mathrm{Quillen}}\right)_{\mathrm{Proj}}$$

setting $\mathcal{O}_{\mathrm{sseq}}(S) := \pi_{\mathcal{O}}^{-1}(\mathrm{Ind}_H^G S \to G/H)$; by Proposition 2.75 sseq is conservative, so $s$sseq preserves and reflects weak equivalences between fibrant objects. Hence it suffices unwind definitions and note that the following functor commutes

$$\begin{tikzcd} s\mathrm{Op}_G^{\mathrm{oc}} \arrow[r, "N^{\otimes}"] \arrow[dr, "U"'] & \mathrm{sSet}^{+,\mathrm{oc}}_{/(\underline{\mathbb{F}}_G, Ne)} \arrow[d, "s\mathrm{sseq}"] \\ & \mathrm{Fun}(\underline{\Sigma}_G, \mathrm{sSet}) \end{tikzcd}$$

□

In fact, the one-color assumption was not necessary.

Proposition 5.2. *$N^{\otimes}$ preserves and reflects weak equivalences between arbitrary locally fibrant genuine equivariant G-operads.*

Proof. It is not too hard to see that $N^\otimes$ preserves and reflects the property of *inducing bijections on sets of colors*, so we may fix a coefficient system of sets of colors $\mathfrak{C}$. Then, we are tasked with proving that $N^\otimes_{\mathfrak{C}} : s\mathrm{Op}_{G,\mathfrak{C}} \to \mathrm{Op}_{G,\mathfrak{C}} := (\pi_0 U)^{-1}(\mathfrak{C})$ preserves and reflects weak equivalences between fibrant objects. Thankfully, we have the same tools as in the one-color case; writing $\underline{\Sigma}_{\mathfrak{C}}$ as in [BP22, Def 3.1], $s\mathrm{Op}_{G,\mathfrak{C}}$ possesses the right-transfered model structure from along a monadic functor $U : s\mathrm{Op}_{G,\mathfrak{C}} \to \mathrm{Fun}\left(\underline{\Sigma}_{\mathfrak{C}}, s\mathbf{Set}_{\mathrm{Quillen}}\right)$ by [BP22, § 5.2]. Furthermore, Proposition 2.75 constructs a functor $s\,\mathrm{sseq} : \mathrm{sSet}^{+,\mathfrak{C}}_{/(\underline{\mathbb{F}}_{\mathcal{T}}, Ne)} \to \mathrm{Fun}\left(\underline{\Sigma}_{\mathfrak{C}}, \mathrm{sSet}_{\mathrm{Quillen}}\right)$ which preserves and reflects weak equivalences between fibrant objects, and such that $N^\otimes$ is a functor over $\mathrm{Fun}\left(\underline{\Sigma}_{\mathfrak{C}}, \mathrm{sSet}_{\mathrm{Quillen}}\right)$; by two-out-of-three for weak equivalences, $N^\otimes$ preserves and reflects weak equivalences between fibrant objects. □

The theory of total right derived functors (e.g. [Rie14, § 2]) then immediately yields the following corollary.

Corollary 5.3. *$N^\otimes$ presents a conservative functor of ∞-categories $N^\otimes : g\mathrm{Op}_G \to \mathrm{Op}_G$, whose restriction participates in a commutative diagram*

$$\begin{tikzcd} g\mathrm{Op}_G^{\mathrm{oc}} \arrow[r, "N^\otimes"] \arrow[dr, "U"'] & \mathrm{Op}_G^{\mathrm{oc}} \arrow[d, "\mathrm{sseq}"] \\ & \mathrm{Fun}(\underline{\Sigma}_G, \mathcal{S}) \end{tikzcd}$$

5.2. **The discrete genuine operadic nerve is an equivalence.** Recall that whenever $\mathcal{O}^\otimes$ is a $\mathcal{T}$-operad and $\mathcal{C}^\otimes$ is a $\mathcal{T}$-1-category, there is an equivalence of $\mathcal{T}$-1-categories

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}) \simeq \mathbf{Alg}_{h_1\mathcal{O}}(\mathcal{C});$$

because of this, for the rest of this subsection, we assume all $\mathcal{T}$-operads are $\mathcal{T}$-1-operads.

Definition 5.4. A discrete genuine $\mathcal{T}$-operad in a symmetric monoidal 1-category $\mathcal{V}$ the data of:

(1) a $\mathcal{T}$-symmetric sequence $\mathcal{O}(-) : \mathrm{tot}\underline{\Sigma}_{\mathcal{T}} \to \mathcal{V}$,
(2) for all $V \in \mathcal{T}$, a distinguished "identity" elements $1_V \in \mathcal{O}(*_V)$, and
(3) for all $S \in \mathbb{F}_V$ and $U \in \mathbb{F}_S$, a Borel $\Sigma_S \times \prod_{U \in \mathrm{Orb}(S)} \Sigma_{T_U}$-equivariant "composition" map

$$\gamma : \mathcal{O}(S) \otimes \bigotimes_{U \in \mathrm{Orb}(S)} (T_U) \to \mathcal{O}\left(\coprod_U^S T_U\right)$$

subject to the following compatibilities for all :

(a) (restriction-stability of the identity) for all $U \to V$, the map $\mathrm{Res}_U^V : \mathcal{O}(*_V) \to \mathcal{O}(*_U)$ sends 1_V to 1_U;
(b) (restriction-stability of composition) for all $U \to V$, the following commutes

$$\begin{tikzcd} \mathcal{O}(S) \times \prod\limits_{U \in \mathrm{Orb}(s)} \mathcal{O}(T_U) \arrow[r, "\gamma"] \arrow[d, "\mathrm{Res}_V^W"] & \mathcal{O}(T) \arrow[d, "\mathrm{Res}_V^W"] \\ \mathcal{O}\left(\mathrm{Res}_W^V S\right) \times \prod\limits_{U' \in \mathrm{Orb}(S)} \mathcal{O}(T_{U'}) \arrow[r, "\gamma"] & \mathcal{O}\left(\mathrm{Res}_W^V S\right) \end{tikzcd}$$

(c) (unitality) for all $S \in \mathbb{F}_V$, the following diagram commutes

$$\begin{tikzcd} \mathcal{O}(S) \arrow[r, "(\mathrm{id}, (\{1_U\}))"] \arrow[d, "(1_V, \mathrm{id})"'] \arrow[dr, equal] & \mathcal{O}(S) \otimes \bigotimes\limits_{U \in \mathrm{Orb}(S)} \mathcal{O}(*_U) \arrow[d, "\gamma"] \\ \mathcal{O}(*_V) \otimes \mathcal{O}(S) \arrow[r, "\gamma"] & \mathcal{O}(S) \end{tikzcd}$$

(d) (associativity) For all $S \in \mathbb{F}_V$, $(T_U) \in \mathbb{F}_S$ writing $T := \coprod_U^S T_U$, and $(R_W) \in \mathbb{F}_T$ writing $R := \coprod_W^T R_W$, the following diagram commutes

$$\begin{tikzcd}
\left(\mathcal{O}(S) \otimes \bigotimes\limits_{U \in \mathrm{Orb}(S_U)} \mathcal{O}(T_U)\right) \otimes \bigotimes\limits_{\substack{U \in \mathrm{Orb}(S) \\ W \in \mathrm{Orb}(T_U)}} \mathcal{O}(T_U) \arrow[r, "\gamma"] \arrow[d, equal] & \mathcal{O}(T) \otimes \bigotimes\limits_{W \in \mathrm{Orb}(T)} \mathcal{O}(R_W) \arrow[d, "\gamma"] \\
\mathcal{O}(S) \otimes \bigotimes\limits_{U \in \mathrm{Orb}(S)} \left(\mathcal{O}(T_U) \otimes \bigotimes\limits_{W \in \mathrm{Orb}(T_U)} \mathcal{O}(R_U)\right) \arrow[d, "\gamma"] & \mathcal{O}\left(\coprod\limits_W^T R_W\right) \arrow[d, equal] \\
\mathcal{O}(S) \otimes \bigotimes\limits_{U \in \mathrm{Orb}(S)} \mathcal{O}\left(\coprod\limits_W^{T_U} R_W\right) \arrow[r, "\gamma"] & \mathcal{O}(R)
\end{tikzcd}$$

A morphism of discrete $\mathcal{T}$-operads in $\mathcal{V}$ is a map of $\mathcal{T}$-symmetric sequences in $\mathcal{V}$ preserving 1_V and intertwining γ; we refer to the resulting 1-category as $g\mathrm{Op}_{\mathcal{T}}(\mathcal{V})$. ◁

Remark 5.5. By inspection, we see that $g\mathrm{Op}_{\mathcal{O}_G}(\mathrm{sSet}) \simeq s\mathrm{Op}_{G,*_G}$ in the sense of [Bon19, Def 3.22]. In particular, the natural fully faithful embedding $g\mathrm{Op}_{\mathcal{O}_G}(\mathbf{Set}) \hookrightarrow g\mathrm{Op}_{\mathcal{O}_G}(s\mathbf{Set}) \simeq s\mathrm{Op}_{G,*_G}$ has image spanned by those genuine G-operads whose structure simplicial sets are discrete. ◁

We henceforth specialize to discrete genuine $\mathcal{T}$-operads in **Set**, which we refer to simply as *discrete genuine $\mathcal{T}$-operads.* From the data of a discrete genuine $\mathcal{T}$-operad $\mathcal{O}$, we construct a 1-category $N^{\otimes}\mathcal{O}$ with a functor $\mathcal{O}^{\otimes} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ using the recipe

$$\mathrm{Hom}_{N^{\otimes}\mathcal{O}}(T,S) := \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T_U)$$

with composition maps given by γ and identity arrow on T given by $(1_U)_{\mathrm{Orb}(T)}$. This is a specialization of the genuine operadic nerve of [Bon19] in the case $\mathcal{T} = \mathcal{O}_G$, and of the $\mathcal{T}$-operadic nerve of [NS22, § 2.5] in the case that $\mathcal{T}$ is a 1-category. Conversely, from the data of a $\mathcal{T}$-1-operad $\mathcal{O}$, the data of a discrete genuine $\mathcal{T}$-operad $\mathcal{O}(-)$ is supplied by Remark 2.53.

Proposition 5.6. *$N^{\otimes}$ descends to a functor $g\mathrm{Op}_{\mathcal{T}}(\mathbf{Set}) \to \mathrm{Op}^{\mathrm{oc}}_{\mathcal{T},0}$ with quasi-inverse $\mathcal{O}(-)$.*

Proof. Since $N^{\otimes}$ is compatible with restrictions, we may replace $\mathcal{T}$ with $\underline{V}$, and hence we may assume that $\mathcal{T}$ is a 1-category. In this case, [NS22, Prop 2.5.6] implies that $N^{\otimes}$ is a $\mathcal{T}$-1-operad. Thus it suffices to prove that the compositions $g\mathrm{Op}_{\mathcal{T}}(\mathbf{Set}) \to g\mathrm{Op}_{\mathcal{T}}(\mathbf{Set})$ and $\mathrm{Op}^{oc}_{\mathcal{T},0}$ are homotopic to the identity; this follows immediately after unwinding definitions. □

Now having an explicit combinatorial model for $\mathcal{T}$-1-operads, we focus on algebras.

Definition 5.7. If $\mathcal{C}^{\otimes}$ is a $\mathcal{T}$-symmetric monoidal ∞-category and $X \in \Gamma^{\mathcal{T}}\mathcal{C}$ a $\mathcal{T}$-object, then the *endomorphism operad of* X is the full sub-operad $\mathrm{End}_X^{\otimes} \subset \mathcal{C}^{\otimes}$ spanned by X (c.f. Theorem 1.87). ◁

Observation 5.8. End_X has underlying symmetric sequence $\mathrm{End}_X(S) \simeq \mathrm{Map}(X_V^{\otimes S}, X_V)$, identity element $1_V = \mathrm{id}_{X_V}$, and composition map given by composition of maps. ◁

In general, an $\mathcal{O}$-algebra in $\mathcal{C}^{\otimes}$ may be viewed as the information of its underlying object X together with the factored map $\mathcal{O}^{\otimes} \to \mathrm{End}_X^{\otimes} \hookrightarrow \mathcal{C}^{\otimes}$. The following proposition follows by unwinding definitions.

Proposition 5.9. *If $\mathcal{C}^\otimes$ is a $\mathcal{T}$-1-category and X, Y are $\mathcal{O}$-algebras in $\mathcal{C}^\otimes$, then the hom set $\mathrm{Hom}_{\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})}(X,Y) \subset \mathrm{Hom}_{\mathcal{C}}(X,Y)$ consists of those maps such that the following diagram of operads commutes:*

$$\begin{tikzcd} & \mathrm{End}_X^\otimes \arrow[d] \\ \mathcal{O}^\otimes \arrow[ur] \arrow[dr] & \\ & \mathrm{End}_Y^\otimes \end{tikzcd}$$

For the sake of comparison, we will propose one more model for discrete I-commutative algebras.

Definition 5.10. Let I be a one-color weak indexing category. Then, a *strict I-commutative algebra in $\mathcal{C}$* is the data of a $\mathcal{T}$-object X together with $\mathrm{Aut}_V S$-equivariant maps $\mu_S : X_V^{\otimes S} \to X_V$ for all $S \in \mathbb{F}_{I,V}$ subject to the following conditions:

(1) (restriction-stability) The functor Res_U^V takes μ_S to $\mu_{\mathrm{Res}_U^V S}$.
(2) (unitality) for all maps $S \sqcup *_V \in \mathbb{F}_{I,V}$, the following diagram commutes:

$$\begin{tikzcd} & X_V^{\otimes S \sqcup *_V} \arrow[dr] & \\ X_V \arrow[ur] \arrow[rr, equal] & & X_V \end{tikzcd}$$

(3) (associativity) for all S-tuples $(T_U) \in \mathbb{F}_{I,S}$, writing $T = \coprod_U^S T_U$, the following diagram commutes:

$$\begin{tikzcd} \bigotimes_U^S X_U^{\otimes T_U} \arrow[r, "(\mu_{T_U})"] \arrow[d, phantom, "\wr"] & X_V^{\otimes S} \arrow[d, "\mu_S"] \\ X_V^{\otimes T} \arrow[r, "\mu_T"] & X_V \end{tikzcd}$$

◁

Proposition 5.11. *If $\mathcal{C}^\otimes$ is a $\mathcal{T}$-symmetric monoidal 1-category, then the categories of I-commutative algebras and strict I-commutative algebras in $\mathcal{C}$ agree.*

Proof. This follows from Observation 5.8, noting that $\mathrm{Map}(\mathcal{N}_{I\infty}^\otimes, \mathrm{End}_X^\otimes) \simeq \mathrm{Map}(\mathcal{N}_{I\infty}^\otimes, \mathrm{Bor}_I^{\mathcal{T}} \mathrm{End}_X^\otimes)$ and unwinding definitions using Proposition 5.6. □

Corollary 5.12. *If $\mathcal{C}$ is a G-symmetric monoidal 1-category and I is an indexing system, then I-commutative algebras in $\mathcal{C}$ are equivalent to [Cha24, Def 5.6]'s "I-commutative monoids" over $\mathcal{C}$.*

Proof. This follows by matching Definition 5.10 with [Cha24, Def 5.6], noting (e.g. by [Ste24]) that it suffices to check the associativity and unitality conditions of Definition 5.10 for $S = 2*_H$ or an orbit, since indexing systems are generated under binary coproducts and self-inductions by $\{2*_H\}$ and transitive H-sets. □

5.3. **Failure of the non-aE-unital equivariant Eckmann-Hilton argument.**

Observation 5.13. Fix I a weak indexing system. By Propositions 3.8 and 3.17, there is a contractible space of diagrams of the following form:

$$\mathcal{N}_{I\infty}^\otimes \simeq \mathcal{N}_{I\infty}^\otimes \otimes^{\mathrm{BV}} \mathrm{triv}_{\mathrm{cSupp}(I)}^\otimes \xrightarrow{\mathrm{id} \otimes^{\mathrm{BV}} \mathrm{can}} \mathcal{N}_{I\infty}^\otimes \otimes^{\mathrm{BV}} \mathcal{N}_{I\infty}^\otimes \to \mathcal{N}_{I\infty}^\otimes;$$

furthermore, the composite $\mathcal{N}_{I\infty}^\otimes \to \mathcal{N}_{I\infty}^\otimes$ is homotopic to the identity by Proposition 3.8.

In particular, this implies that there is a canonical natural *split codiagonal* diagram

$$\begin{tikzcd} & \mathrm{CAlg}_I \underline{\mathrm{CAlg}}_I^\otimes(-) \arrow[dr, "U"] & \\ \mathrm{CAlg}_I(-) \arrow[ur, "\delta"] \arrow[rr, equal] & & \mathrm{CAlg}_I(-) \end{tikzcd}$$

◁

We will interpret $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$-algebras as pairs of interchanging I-commutative algebra structures in Observation 5.21, thus δ will take a structure to two interchanging copies of itself, and U will simply forget one of the structures. Hence a weak form of the *Eckmann-Hilton argument* states that the functor U is an equivalence, or equivalently, δ is an equivalence.

Unfortunately, this does not hold for all weak indexing systems I. The following counterexample to nonunital Eckmann-Hilton was pointed out to the author by Piotr Pstrągowski.

Example 5.14. Let R be a nonzero commutative ring. Then, the Abelian group underlying R sports a $\mathrm{Comm}_{nu}^{\otimes} \otimes \mathrm{Comm}_{nu}^{\otimes}$ structure given by the two multiplications

$$\mu(r,s) = rs, \qquad\qquad \mu_0(r,s) = 0,$$

which are easily seen to satisfy interchange but be distinct. In particular, the associated $\mathrm{Comm}_{nu}^{\otimes} \otimes \mathrm{Comm}_{nu}^{\otimes}$-algebra is not in the essential image of the codiagonal

$$\mathbf{Alg}_{\mathrm{Comm}_{nu}}(\mathbf{Ab}) \to \mathbf{Alg}_{\mathrm{Comm}_{nu}} \underline{\mathbf{Alg}}_{\mathrm{Comm}_{nu}}(\mathbf{Ab}),$$

so δ is not an equivalence. ◁

An analogous weak form of the ∞-categorical Eckmann-Hilton argument of [SY19] yields a classification of $\otimes^{\mathrm{BV}}$-idempotent algebras in *reduced* ∞-operads. In fact, Example 5.14 shows that the associated unitality assumption only misses one example among nonequivariant weak $\mathcal{N}_\infty$-operad.

Corollary 5.15. *A weak $\mathcal{N}_\infty$-$*$-operad $\mathcal{O}^{\otimes}$ possesses a map $\mathrm{triv}^{\otimes} \to \mathcal{O}^{\otimes}$ inducing an equivalence*

$$\mathcal{O}^{\otimes} \xrightarrow{\sim} \mathcal{O}^{\otimes} \otimes^{\mathrm{BV}} \mathcal{O}^{\otimes}$$

if and only if $\mathcal{O}^{\otimes}$ is equivalent to $\mathrm{triv}^{\otimes}$, $\mathbb{E}_0^{\otimes}$, or $\mathbb{E}_\infty^{\otimes}$.

Proof. [SY19, Cor 5.3.4] covers the unital case, so it suffices to assume that $\mathcal{O}(\varnothing) = \varnothing$ and show that $\mathcal{O}^{\otimes} \simeq \mathrm{triv}^{\otimes}$. Note that Comm_{nu} is the terminal nonunital $\mathcal{N}_\infty$-$*$-operad, i.e. there exists a map $\mathcal{O}^{\otimes} \to \mathrm{Comm}_{nu}$, yielding a diagram

$$\begin{tikzcd} \mathcal{O}^{\otimes} \otimes \mathcal{O}^{\otimes} \arrow[r] & \mathrm{Comm}_{nu}^{\otimes} \otimes \mathrm{Comm}_{nu}^{\otimes} \\ \mathcal{O}^{\otimes} \arrow[r] \arrow[u] & \mathrm{Comm}_{nu}^{\otimes} \arrow[u] \end{tikzcd}$$

Pulling back the example of Example 5.14, we find that if $\mathcal{O}(n) = *$ for any $n \neq 1$, then R has a $\mathcal{O}^{\otimes} \otimes \mathcal{O}^{\otimes}$-structure that is not in the image of the diagonal; hence $\mathcal{O}(n) = \varnothing$ when $n \neq 1$, i.e. it's equivalent to $\mathrm{triv}^{\otimes}$. □

By [Ste24], this is precisely the list of nonempty aE-unital weak indexing systems for $*$. In this section, we introduce an equivariant analogue to this argument in order to prove the following proposition; in order to do so, we say that $\mathcal{O}^{\otimes}$ is *n-connected* if $\mathcal{O}(S)$ is n-connected for all n, and we say that $\mathcal{O}^{\otimes}$ is *connected* if it is 0-connected.

Proposition 5.16. *Suppose $\mathcal{N}_{I\infty}^{\otimes} \overset{BV}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ is connected. Then, I aE-unital.*

Our strategy for proving this centers on the following proposition.

Proposition 5.17. *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-operad. Then, the following are equivalent:*

(a) $\mathcal{O}^{\otimes}$ is $(n-1)$-conncted.
(b) The canonical map $h_n\mathcal{O}^{\otimes} \to \mathcal{N}_{A\mathcal{O}\infty}^{\otimes}$ is an equivalence.
(c) For all $\mathcal{T}$-symmetric monoidal n-categories $\mathcal{C}$, the canonical $\mathcal{T}$-symmetric monoidal functor

$$\mathrm{CAlg}_{A\mathcal{O}}(\mathcal{C}) \to \mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$$

is an equivalence.
(d) The canonical $\mathcal{T}$-symmetric monoidal functor

$$\mathrm{CAlg}_{A\mathcal{O}}(\mathcal{S}_{\leq n-1}) \to \mathbf{Alg}_{\mathcal{O}}(\mathcal{S}_{\leq n-1})$$

is an equivalence.

Proof. (a) $\implies$ (b) follows immediately from Corollary 2.81. Similarly, using the adjunction, we find that (b) implies that $\mathrm{CAlg}_{A\mathcal{O}}(\mathcal{C}) \to \mathbf{Alg}_{h_n\mathcal{O}}(\mathcal{C}) \simeq \mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$ is an equivalence for all $\mathcal{C} \in \mathbf{Cat}^{\otimes}_{\mathcal{T},d} \subset \mathrm{Op}_{\mathcal{T},d}$, implying (c). (c) obviously implies (d). The remaining implication follows by the same argument as Proposition 4.2; we find that, for all S such that $\mathcal{O}(S) \neq \varnothing$, the map

$$\tau_{\leq(n-1)}\mathcal{O}(S) \simeq (h_n\mathcal{O})(S) \to \mathcal{N}_{A\mathcal{O}}(S) \simeq *$$

is an equivalence, implying (a). $\square$

Thus, given a non-aE-unital weak indexing category I, it will suffices to construct two distinct interchanging I-commutative algebra structures in some $\mathcal{T}$-symmetric monoidal 1-category. We do so by passing to a universal case.

Construction 5.18. Let $\mathcal{F}^\perp \subset \mathcal{T}$ be a $\mathcal{T}$-cofamily Then, define the full subcategory

$$\mathbb{F}_V \supset \mathbb{F}_{\mathcal{F}^\perp-nu,V} = \begin{cases} \mathbb{F}_V - \{\varnothing_V\} & V \in \mathcal{F}^\perp; \\ \mathbb{F}_V & \text{otherwise.} \end{cases}$$

This is evidently closed under restriction, so it defines a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_{\mathcal{F}^\perp-nu} \subset \underline{\mathbb{F}}_{\mathcal{T}}$. Furthermore, it has contractible V-sets and is closed under self-indexed coproducts by inspection. Hence it is a weak indexing system. ◁

Observation 5.19. $\underline{\mathbb{F}}_{\mathcal{F}^\perp-nu}$ is the terminal weak indexing system possessing unit-family $v(I) = \mathcal{F}$; $\underline{\mathbb{F}}_I$ is non-aE-unital if and only if it shares a non-contractible V-set with $\underline{\mathbb{F}}_{v(I)^\perp-nu}$ for some $V \in v(I)^\perp$; thus, to prove Proposition 5.16, it suffices to construct two interchanging $\mathcal{N}_I^{\mathcal{F}^\perp}$-algebra structures who differ in $v(I)^\perp$-arities and apply the analogous argument to Corollary 5.15. ◁

Construction 5.20. Let M be a nontrivial commutative monoid and le $F : \mathsf{Span}(\mathbb{F}_{\mathcal{T}}) \to \mathbf{Set}$ be the functor

$$F(S) := M^{|S|}$$

with functoriality induced by the action maps in M; this is evidently product-preserving, i.e. it's a $\mathcal{T}$-commutative monoid in $\mathbf{Set}$. In particular, since $\mathrm{Comm}^{\otimes}_{\mathcal{T}} \otimes \mathbb{E}_0^{\otimes} \simeq \mathrm{Comm}^{\otimes}_{\mathcal{T}}$, this is in the image of the forgetful functor $\mathrm{CAlg}_{\mathcal{T}}(\mathbf{Set}_*) \to \mathrm{CMon}_{\mathcal{T}}(\mathbf{Set})$, so we replace F with a product preserving functor $F' : \mathsf{Span}(\mathbb{F}_{\mathcal{T}}) \to \mathbf{Set}_*$.

We furthermore modify this, constructing a new functor $G : \mathsf{Span}_{I_{\mathcal{F}^\perp-nu}}(\mathbb{F}_{\mathcal{T}}) \to \mathbf{Set}_*$ via

$$G(S) := \prod_{U \in \mathrm{Orb}(S) \cap \mathcal{F}^\perp} F'(U).$$

This is product-preserving, so it yields an $I_{\mathcal{F}^\perp-nu}$-commutative monoid in $\mathbf{Set}_*$. Last, we let G_0 be the $I_{\mathcal{F}^\perp-nu}$ on the underlying G-coefficient system of pointed sets whose action maps are all zero. ◁

We would like to show that G and G_0 interchange, for which we make the following observation.

Observation 5.21. Let $\mathcal{C}^{\otimes}$ be a $\mathcal{T}$-symmetric monoidal 1-category, and let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ be 1-object $\mathcal{T}$-1-operads. The data of a bifunctor of $\mathcal{T}$-operads $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathcal{C}^{\otimes}$ maybe viewed as an object $X \in \Gamma^{\mathcal{T}}\mathcal{C}$ (which is the image of intert morphisms of $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}$) together with action maps

$$X_H^{\otimes S} \otimes \mathcal{O}(S) \to X_H \qquad\qquad X_H^{\otimes S} \otimes \mathcal{P}(S) \to X_H$$

subject to the functoriality condition that these structures yield an $\mathcal{O}$-algebra, a $\mathcal{P}$-algebra, and these structures satisfy the interchange law

$$\begin{tikzcd}
\bigotimes_U^S X_V^{\otimes \mathrm{Res}_U^V T} \arrow[d, "\bigotimes_U^S \mathrm{Res}_U^V \mu_T"'] & \simeq & X_V^{\otimes S \times T} & \simeq & \bigotimes_W^T X_V^{\otimes \mathrm{Res}_W^V S} \arrow[r, "\bigotimes_W^T \mathrm{Res}_W^V \mu_S"] & X_V^{\otimes T} \arrow[d, "\mu_T"] \\
X_V^{\otimes S} \arrow[rrrrr, "\mu_S"] & & & & & X_V
\end{tikzcd}$$

for all pairs $\mu_S \in \mathcal{O}(S)$ and $\mu_T \in \mathcal{P}(T)$. A morphism of $\mathcal{O}\otimes\mathcal{P}$-algebras is a natural transformation of bifunctors, i.e. a morphism of $\mathcal{T}$-objects $X \to Y$ which is both a $\mathcal{O}$-algebra map and a $\mathcal{P}$-algebra map.

In particular, an $\mathcal{N}_{I\infty} \otimes \mathcal{N}_{I\infty}$-algebra is equivalently a pair of collections of maps $\mu, \mu' : X^{\otimes T} \to X^{\otimes R}$ for all $T \to R$ in I which are separately $\mathcal{N}_{I\infty}$-algebra structures and which satisfy the interchange law

$$\begin{tikzcd}
\bigotimes\limits_U^S X_V^{\otimes \mathrm{Res}_U^V T} \arrow[d, dashed, "\otimes \mu"] & \simeq & X_V^{\otimes S \times T} & \simeq & \bigotimes\limits_W^T X_V^{\otimes \mathrm{Res}_W^V S} \arrow[r, "\otimes \mu'"] & X_V^{\otimes T} \arrow[d, "\mu"] \\
X_V^{\otimes S} \arrow[rrrrr, "\mu'"] & & & & & X_V
\end{tikzcd}$$

◁

Lemma 5.22. *G and G_0 interchange.*

Proof. It suffices to note that all of the compositions in Observation 5.21 factor through a zero map, and hence they are all zero, making the diagram commute. □

Corollary 5.23. *If $\mathcal{N}_{I\infty}^{\otimes}$ is not aE-unital, then $\mathcal{N}_{I\infty}^{\otimes} \overset{BV}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ is not connected.*

Proof. Note that

$$\begin{aligned}
\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes} \text{ connected } &\iff \tau_{\leq 1} \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{N}_{A\mathcal{N}_{I\infty} \otimes \mathcal{N}_{I\infty}} \simeq \mathcal{N}_{I\infty}^{\otimes} \\
&\implies \mathrm{CAlg}_I(\mathbf{Set}_*) \to \mathbf{Alg}_{\mathcal{N}_{I\infty} \otimes \mathcal{N}_{I\infty}}(\mathbf{Set}_*) \text{ essentially surjective.}
\end{aligned}$$

Furthermore, Lemma 5.22 constructs an $\mathcal{N}_{v(I)^{\perp}-nu\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{v(I)^{\perp}-nu\infty}^{\otimes}$ satisfying the condition that its two individual structure maps $G(S) \to G(*_V)$ differ whenever $V \in v(I)^{\perp}$ and $S \neq *_V$. Since I is not aE-unital, it must have some noncontractible $S \in \mathbb{F}_{I,V}$ for $V v(I)^{\perp}$, so the pullback $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ structure on (G, G_0) has two distinct underlying I-algebra structures, implying it is outside of this essential image. The contrapositive shows that $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\overset{\mathrm{BV}}{\otimes}}$ is not connected. □

By combining Corollaries 4.13 and 5.23, we have the following.

Corollary F'. *$\mathcal{N}_{I\infty}^{\otimes} \otimes \mathcal{N}_{I\infty}^{\otimes}$ is a weak $\mathcal{N}_{\infty}$-operad if and only if I is almost-E-unital. in this case, if $\mathcal{O}^{\otimes}$ is a reduced I-operad, then the unique map*

$$\mathcal{O}^{\otimes} \otimes \mathcal{N}_{I\infty}^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$$

is an equivalence.

Remark 5.24. Using the above argument, one can show that if $\mathcal{O}^{\otimes}$ is a $\overset{\mathrm{BV}}{\otimes}$-idempotent $\mathcal{T}$-operad, then its nullary spaces $\mathcal{O}(\varnothing_V)$ are nonempty. If additionally $\mathcal{O}(\varnothing_V)$ are assumed to be contractible (i.e. $\mathcal{O}^{\otimes}$ is aE-unital), then Proposition 4.24 shows that the underlying fixed point catgeories $\mathcal{O}_V$ are all $\times$-idempotent algebras, i.e. they are contractible or empty. Hence $\mathcal{O}^{\otimes}$ will be shown to be aE-reduced. It is likely that the equivariant analog to [SY19] will demonstrate that such idempotents are all infinitely connected; hence the author believes that the aE-unital weak $\mathcal{N}_{\infty}$-operads are likely to completely enumerate the $\overset{\mathrm{BV}}{\otimes}$-idempotent algebras in $\mathrm{Op}_{\mathcal{T}}$. ◁

5.4. Conjectures and future directions.

5.4.1. *Closing the gap between models.* Furthermore, several papers such as [BH15; Rub21b; Szc23] have characterized the behaviour of various "Boardman-Vogt" tensor products on examples in various models. We propose means to close the loop.

Conjecture 5.25. *The Boardman-Vogt tensor products of [BH15; Rub21b; Szc23] lift to a common symmetric monoidal ∞-category $g\mathrm{Op}_G^{\otimes}$ possessing a G-symmetric monoidal equivalence*

$$g\mathrm{Op}_G^{\otimes} \simeq \mathrm{Op}_G^{\otimes}.$$

We are interested in this conjecture for two reasons; on one hand, some tensor products of G-operads have been computed in models, such as tensor products of models for N_∞-operads in [Rub21b] and tensor products of models for $\mathbb{E}_V$ operads in [Szc23]. On the other, the model categories are hard to work with, and to the author's knowledge, no BV tensor product on models has been lifted to a homotopical symmetric monoidal closed structure, so these results are difficult to apply to constructions of algebras.

We suggest two possible lines of argumentation for the equivalence of ∞-categories. First, note that $N^\otimes$ is a conservative functor between two ∞-categories who are each monadic over $\mathrm{Fun}(\underline{\Sigma}_G, \mathcal{S})$; To compare our notions, it suffices to characterize the *free G-operad on a G-symmetric sequence* and provide an explicit comparison between it and the genuine equivariant operad monad of [BP21, § 4.2]. If these monads are shown to be equivalent via $N^\otimes$, then $N^\otimes$ itself with be an equivalence.

Another line of argumentation is to generalize the non-equivariant case; for instance, we conjecture that [Bar18, § 10] applied to the perfect operator category $\underline{\mathbb{F}}_G$ will provide an equivalence between G-operads and $\mathrm{Seg}_{\Delta^{\mathrm{op}}_{\underline{\mathbb{F}}_G}}(\mathcal{S})$, the latter being comparable to the equivariant dendroidal Segal spaces of [BP20b; Per18] by an equivariant lift of the argument of [CHH18] in the language of algebraic patterns and using the recognition principle for Morita equivalences of patterns due to [Bar23, Thm 2.63].

The underlying tensor products and norms seem amenable to argumentation once pushed to structures on a common ∞-category; for instance, the universal property of BV tensor products in [Szc23, Def 1.7.2] bears resemblance to the fact that our BV tensor product corepresents bifunctors of G-operads.

5.4.2. *The equivariant homotopical Eckmannn-Hilton argument.* We conjecture a strengthening of Corollary F.

Conjecture 5.26. *Suppose I is an aE-unital weak indexing system and $\mathcal{O},\mathcal{P}$ are d_1, d_2-connected reduced I-operads with $A\mathcal{O} = A\mathcal{P}$. Then, $\mathcal{O} \otimes \mathcal{P}$ is $(d_1 + d_2 - 2)$-connected.*

Note that this immediately implies a weak form of *infinite loop space theory*, i.e. the map

$$\mathrm{colim}_n \left(\mathcal{O}^\otimes\right)^{\otimes n} \to \mathcal{N}_{A\mathcal{O}\infty}$$

is an equivalence for all aE-reduced $\mathcal{O}$, or equivalently, letting $\underline{\mathbf{Alg}}^\otimes_{\mathcal{O},n}(\mathcal{C}) := \underline{\mathbf{Alg}}^\otimes_{\mathcal{O},n-1}(\mathcal{C})$ with $\underline{\mathbf{Alg}}^\otimes_{\mathcal{O},0}(\mathcal{C}) = \mathcal{C}$,

$$\lim_n \underline{\mathbf{Alg}}^\otimes_{\mathcal{O},n}(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}^\otimes_{A\mathcal{O}}(\mathcal{C}).$$

The author hopes to fulfill this in upcoming work bearing similarity to [SY19]. In view of Proposition 3.4, we will acquire an inductive strategy to construct algebras over *any* aE-unital weak N_∞ operad, using at each step e.g. the associative or free I-operads of [Rub21a].

We also would immediately acquire an intrinsic characterization of almost-unital weak N_∞-operads, and hence of A; since infinite tensor products of almost-reduced $\mathcal{T}$ operads are weak $\mathcal{N}_\infty$-operads, and weak $\mathcal{N}_\infty$-operads are idempotent by Theorem G, the argument of Remark 5.24 will immediately show that the $\overset{\mathrm{BV}}{\otimes}$-idempotent algebras in $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}$ are precisely the almost-unital weak $\mathcal{N}_\infty$-operads.

5.4.3. *Equivariant Dunn additivity.* In the thesis [Szc23], the non-homotopical graph-operad equivalent to the following conjecture was proved.

Conjecture 5.27. *The map $\mu : \mathbb{E}_V^\otimes \otimes \mathbb{E}_W^\otimes \to \mathbb{E}_{V\oplus W}^\otimes$ is an equivalence of G-operads.*

In forthcoming work, the author plans to prove this theorem after stabilizing to spectral G-operads.

5.4.4. *Discrete models for G-operads.* Much of the strategy employed in sources such as [HA] which characterize $\mathbb{E}_n$-algebras consists of reduction to the $\mathbb{E}_1$-case via Dunn's additivity theorem; $\mathbb{E}_1$ is a discrete operad, and hence it is amenable to combinatorial study. Unfortunately, Conjecture 5.27 does not predict such a luxury in the equivariant setting; for instance, if $|G|$ is odd, then G admits no nontrivial 1-dimensional real orthogonal G-representation. Given V of finite dimension at least 2, $\mathbb{E}_V(2*_e) \simeq \mathrm{Conf}^e_{[2]}(V) \simeq S(V)^e$, which is *not* discrete, as it has nonvanishing $\dim V$th homotopy group. Thus we are inspired to ask the following difficult question.

Question 5.28. Does there exist a family of G-operads $\mathbb{O}$ such that $\mathbb{E}_V \in \mathbb{O}$ for all V and such that $\mathbb{O}$ is generated under $\overset{\mathrm{BV}}{\otimes}$ by discrete G-operads? ◁

One potentially fruitful source of examples is the subject of the next set of questions.

5.4.5. *Coinduced V-operads and free equivariant symmetric sequences.*

Question 5.29. Let $\mathcal{O}$ be a $\underline{V}$ operad and $U \to W$ a map. What structure does a $\mathrm{CoInd}_U^V \mathcal{O}$-algebra have? ◁

This is nontrivial, as coinduced operads are characterized by *mapping-in properties*, but their algebras are maps *out*. It is useful, as Construction 3.24 uses this mapping-in property to argue that $\mathrm{CoInd}_U^V \mathcal{O}^{\otimes}$ is the universal structure borne by V-norms of $\mathcal{O}^{\otimes}$-algebras. It is old, as coinduced operads appear in the graph model structure as early as [BH15, § 6.2.1]

For instance, Proposition 3.22 leads to the following perplexing observations:

Observation 5.30. $\mathrm{CoInd}_e^G \mathbb{E}_1$ is a discrete G-operad whose underlying weak indexing system is complete; $\mathrm{CoInd}_e^G \mathbb{E}_2$ is a 1-truncated G-operad whose underlying weak indexing system is complete. ◁

The author is frustrated to report that she has guesses as to what $\mathrm{CoInd}_e^G \mathbb{E}_n$ is when $1 < n < \infty$ despite its structure being borne by HHR norms of all $\mathbb{E}_n$-rings.

Observation 5.31. Let $X_\bullet$ be a $\underline{V}$-symmetric sequence. Then,

$$\begin{aligned}\mathrm{Map}_{\mathrm{sseq}}(X_\bullet, \mathrm{sseq}\,\mathrm{CoInd}_U^V \mathcal{O}) &\simeq \mathrm{Map}(\mathrm{Fr}(X_\bullet)^{\otimes}, \mathrm{CoInd}_U^V \mathcal{O}^{\otimes}) \\ &\simeq \mathrm{Map}(\mathrm{Res}_U^V \mathrm{Fr}(X_\bullet)^{\otimes}, \mathcal{O}^{\otimes}) \\ &\simeq \mathrm{Map}(\mathrm{Fr}(\mathrm{Res}_U^V X_\bullet)^{\otimes}, \mathcal{O}^{\otimes}) \\ &\simeq \mathrm{Map}_{\mathrm{sseq}}(\mathrm{Res}_U^V X_\bullet, \mathrm{sseq}\,\mathcal{O}).\end{aligned}$$

In particular, if $\mathrm{Fr}(S)$ is the free $\underline{V}$-symmetric sequence on $S \in \underline{\mathbb{F}}_V$, this demonstrates that

$$\mathrm{CoInd}_U^V \mathcal{O}(S) \simeq \mathrm{Map}_{\mathrm{sseq}}(\mathrm{Res}_U^V \mathrm{Fr}(S), \mathrm{sseq}\,\mathcal{O});$$

thus, combinatorial control of free $\underline{V}$-symmetric sequences is likely to yield information about the equivariant symmetric sequence underlying coinduced V-operads; in particular, since the underlying V-symmetric sequence functor is conservative, this is a potential avenue by which to “guess and check” the identity of coinduced V-operads, giving intrinsic characterization of the structure of HHR norms of $\mathcal{O}$-algebras. ◁

5.4.6. *On developing global operads.*

Definition 5.32. Let $\mathcal{T}$ be an ∞-category. Then, a *weak indexing datum of $\mathcal{T}$* is a pair (P, I_P), where P is an atomic orbital subcategory and I_P is a P-weak indexing category. ◁

There is a cartesian symmetric monoidal subcategory $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \subset \mathrm{Span}_{\mathcal{P}}(\mathbb{F}_{\mathcal{T}})$, yielding on this category the structure of a symmetric monoidal algebraic pattern, allowing one to define the Boardman-Vogt tensor product.

Definition 5.33. Let $\mathcal{T}$ be an ∞-category. Then, the *$\otimes$-category of $\mathcal{T}$-I-operads* is

$$\mathrm{Op}_{\mathcal{T},I} := \left(\mathrm{Fbrs}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})), \overset{\mathrm{BV}}{\otimes}\right). \qquad ◁$$

Question 5.34. Does the work of this paper and [NS22; Ste24] extend to $\mathrm{Op}_{\mathcal{T},I}$? ◁

Recollection 5.35. In [CLL23a, § 4.7], the free $\mathcal{T}$-∞-category $\underline{\mathbb{F}}_{P,*} := \underline{\mathbb{F}}_{\mathcal{T},*}^{P}$ admitting P-coproducts on a point was constructed; in particular, since $\mathrm{Span}_P(\underline{\mathbb{F}}_{\mathcal{T}})$ admits finite P-products and is P-semiadditive, it admits finite P-coproducts, and hence admits a unique P-coproduct preserving $\mathcal{T}$-functor

$$\iota : \underline{\mathbb{F}}_{P,*} \to \mathrm{Span}_P(\underline{\mathbb{F}}_{\mathcal{T}})$$

sending $*_+ \mapsto *$. ◁

If one would like to repeat arguments from Appendix B and [NS22; Ste24] verbatim, one needs a [HA]-style pattern modelling $\mathrm{Op}_{\mathcal{T},I}$; this is especially important for Proposition 1.82, whose conclusion can't easily be formulated over effective Burnside patterns in the first place. Thus we formulate the following conjecture:

Conjecture 5.36. *$\underline{\mathbb{F}}_{P,*}$ admits a structure as a sound algebraic pattern such that the composite functor*

$$\underline{\mathbb{F}}_{P,*} \to \mathrm{Span}_P(\underline{\mathbb{F}}_{\mathcal{T}}) \to \mathrm{Span}_P(\mathbb{F}_{\mathcal{T}})$$

is a Morita equivalence.

Appendix A. Burnside algebraic patterns: the atomic orbital case

The following appendices are not written to be particularly original; most of their contents appear as straightforward technical extensions of beloved works in higher algebra, and they are included for the sake of mathematical completeness.

A.1. I-operads as fibrous patterns.

This subsection deviates only slightly from [BHS22, § 5.2], so we suggest that the reader first read their work. We're interested in proving Proposition 2.56, so we freely use its notation.

A.1.1. *The pattern $\underline{\mathbb{F}}_{\mathcal{T},*}$.*

Our first step is to prove the following proposition.

Proposition A.1. *There are equivalences of categories*

$$\mathrm{Seg}_{\underline{\mathbb{F}}_{\mathcal{T},*}}(\mathcal{C}) \simeq \mathrm{CMon}_{\mathcal{T}}(\underline{\mathcal{C}}),$$
$$\mathrm{Fbrs}(\underline{\mathbb{F}}_{\mathcal{T},*}) \simeq \mathrm{Op}_{\mathcal{T},\infty},$$

the latter denoting Nardin-Shah [NS22]'s ∞-category of $\mathcal{T}$-∞-categories.

To prove this, we must understand the associated Segal conditions. The following lemma characterizes their indexing category.

Lemma A.2 ([BHS22, Obs 5.2.9])**.** *Fix $[S \to U]$ an object in $\underline{\mathbb{F}}_{\mathcal{T},*}$. Then, there are equivalences*

$$\left(\left(\underline{\mathbb{F}}_{\mathcal{T},*}\right)^{\mathrm{el}}_{[S\to U]/}\right)^{\mathrm{op}} \simeq \mathcal{T} \times_{\underline{\mathbb{F}}_{\mathcal{T}}} \underline{\mathbb{F}}^{si}_{\mathcal{T},*,/[S\to U]} \tag{28}$$
$$\simeq \mathcal{T} \times_{\underline{\mathbb{F}}_{\mathcal{T}}} \underline{\mathbb{F}}_{\mathcal{T},*,/[S\to U]}. \tag{29}$$

Furthermore, the full subcategory of $\mathcal{T} \times_{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{T},*,/[S\to U]}$ consisting of morphisms $f : T \to S$ such that f is a summand inclusion is an initial subcategory equivalent to the set $\mathrm{Orb}(S)$.

Proof. (28) follows by definition. For (29), this follows by noting that whenever $[U = U] \to [S \to V]$ is a morphism in $\underline{\mathbb{F}}_{\mathcal{T}}$ out of an orbit, the associated morphism $U \to S \times_V U$ is a summand inclusion, as it's split by the projection $S \times_V U \to U$.

For the remaining statement, the inclusion $\mathrm{Orb}(S) \hookrightarrow \mathcal{T} \times_{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{T},*,/[S\to U]}$ has a right adjoint sending $f : T \to S$ to $f(T) \to S$, so it is initial. $\square$

Lemma A.3 ([BHS22, Footnote 6])**.** *The pattern $\underline{\mathbb{F}}_{\mathcal{T},*}$ is sound.*

Proof. We verify the conditions of [BHS22, Prop 3.3.23]. First, we must verify that $\left(\underline{\mathbb{F}}^{si}_{\mathcal{T}}\right)_{/S} \hookrightarrow \underline{\mathbb{F}}_{\mathcal{T},/S}$ is fully faithful, i.e. if there is a diagram

$$\begin{tikzcd} S_2 \arrow[r] \arrow[d] & S_1 \arrow[r] \arrow[d] & S_0 \arrow[d] \\ U_2 \arrow[r] & U_1 \arrow[r] & U_0 \end{tikzcd}$$

such that the associated maps $S_2 \to S_0 \times_{U_0} U_2$ and $S_1 \to S_0 \times_{U_0} U_1$ are summand inclusions, the map $S_2 \to S_1 \times_{U_1} U_2$ is a summand inclusion. In fact, the associated map $S_2 \to S_0 \times_{U_0} U_2$ may be decomposed as

$$S_2 \to S_1 \times_{U_1} U_2 \to S_0 \times_{U_0} U_1 \times_{U_1} U_2 \simeq S_0 \times_{U_0} U_2.$$

The composition and second map are each summand inclusions, or equivalently, split monomorphisms; this implies that the first map is a split monomorphism, so $S \to S_1 \times_{U_1} U_2$ must be a summand inclusion as well, i.e. $\left(\underline{\mathbb{F}}^{si}_{\mathcal{T}}\right)_{/S} \hookrightarrow \underline{\mathbb{F}}_{\mathcal{T},/S}$ is fully faithful.

Last, we must verify that

$$\underline{\mathbb{F}}^{si,\mathrm{el}}_{\mathcal{T},/[S\to U]} \hookrightarrow \underline{\mathbb{F}}^{\mathrm{el}}_{\mathcal{T},/[S\to U]}$$

is final for all $[S \to U] \in \underline{\mathbb{F}}_{\mathcal{T}}$; in fact, it is an equivalence by Lemma A.2. $\square$

Proof of Proposition A.1. For the first statement, note by Lemma A.2 that a Segal $\underline{\mathbb{F}}_{\mathcal{T},*}$-object in $\mathcal{C}$ is equivalent to a functor

$$M : \underline{\mathbb{F}}_{\mathcal{T},*} \to \mathcal{C}$$

satisfying $M(\prod_i U_i) \simeq \prod_i M(U_i)$; this is precisely the condition that M is product preserving, i.e. it is a $\mathcal{T}$-commutative monoid object.

For the second statement, Lemma A.3 together with [BHS22, Prop 4.1.7] reduce the Segal conditions of a fibrous pattern to precisely the conditions of [NS22, Def 2.1.7]. $\square$

We now turn to the remaining statements of Proposition 2.56 making use of the following theorem:

Theorem A.4 ([BHS22, Prop 3.1.16, Thm 5.1.1])**.** *Suppose $\mathcal{O} \to \mathcal{P}$ is a strong Segal morphism of algebraic patterns such that the following conditions hold:*

(1) $f^{\mathrm{el}} : \mathcal{O}^{\mathrm{el}} \to \mathcal{P}^{\mathrm{el}}$ is an equivalence, and
(2) for every $O \in \mathcal{O}$, the functor $\left(\mathcal{O}^{\mathrm{act}}_{/O}\right)^{\simeq} \to \left(\mathcal{P}^{\mathrm{act}}_{/f(O)}\right)^{\simeq}$ is an equivalence.

Then, the functor $f^* : \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ is an equivalence. Furthermore, if $\mathcal{P}$ is soundly extendable, then $f^* : \mathrm{Fbrs}(\mathcal{P}) \to \mathrm{Fbrs}(\mathcal{O})$ is an equivalence.

For posterity, we temporarily increase in generality.

A.1.2. *Global effective burnside patterns.* Let $\mathcal{T}$ be an ∞-category and $I \subset \mathbb{F}^{P}_{\mathcal{T}} \subset \mathbb{F}_{\mathcal{T}}$ a one-object weak indexing category of an atomic orbital subcategory of $\mathcal{T}$ in the sense of [CLL24]; write

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \coloneqq \mathrm{Span}_{all,I}(\mathbb{F}_{\mathcal{T}}; \mathcal{T}^{\mathrm{op}})$$

for the resulting pattern. There is a span pattern analog to Lemma A.2 which is proved identically.

Lemma A.5. *For $\mathcal{T}$ an* arbitrary *∞-category, the full subcategory of $\left(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})^{\mathrm{el}}_{/S}\right)^{\mathrm{op}} \simeq \mathcal{T} \times_{\mathbb{F}_{\mathcal{T}}} \mathbb{F}_{\mathcal{T},/S}$ consisting of morphisms $f : T \to S$ such that f is a summand inclusion is an initial subcategory equivalent to the set* $\mathrm{Orb}(S)$.

Unwinding definitions, this demonstrates the following.

Corollary A.6. *The forgetful functor*

$$\mathrm{Seg}_{\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}(\mathcal{C}) \to \mathrm{Fun}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C})$$

is fully faithful with image spanned by the product preserving functors.

Global effective Burnside patterns are generally well behaved:

Lemma A.7. *The pattern $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ is soundly extendable.*

Proof. It is sound by [BHS22, Cor 3.3.24]. To see that $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is extendable, it is equivalent to prove that $\mathscr{A}_{\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$ is a Segal $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$-$\infty$-category, i.e. for every $S \in \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$, the associated functor φ of

$$\begin{tikzcd}
\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})^{\mathrm{act}}_{/S} \arrow[r, "\sim"] \arrow[d] & I_{/S} \arrow[r, "\sim"] \arrow[d] & \prod\limits_{V \in \mathrm{Orb}(S)} I_{/V} \arrow[dl, "\varphi"] \\
\lim\limits_{V \in \mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\mathrm{el}}_{S/}} \mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\mathrm{act}}_{/V} \arrow[r, "\sim"] & \lim\limits_{V \in \mathcal{T} \times_{\mathbb{F}_{\mathcal{T}}} \mathbb{F}_{\mathcal{T},/S}} I_{/V} &
\end{tikzcd}$$

is an equivalence. In fact, it is an equivalence by Lemma A.5. $\square$

A.1.3. *The equivalence.* We resume our original assumption that $\mathcal{T}$ is atomic orbital.

Corollary A.8. *The source functor $s : \underline{\mathbb{F}}_{\mathcal{T},*} \hookrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ induces equivalences of categories*

$$\mathrm{Seg}_{\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}(\mathcal{C}) \simeq \mathrm{Seg}_{\underline{\mathbb{F}}_{\mathcal{T},*}}(\mathcal{C});$$
$$\mathrm{Fbrs}(\mathrm{Span}(\mathbb{F}_{\mathcal{T}})) \simeq \mathrm{Fbrs}(\underline{\mathbb{F}}_{\mathcal{T},*}).$$

Proof. It is clear that s is a morphism of algebraic patterns, as it is induced by a morphism of quadruples. The pattern $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is soundly extendable by Lemma A.7. In order to verify that s is a strong Segal morphism, we must verify that $s^{\mathrm{el}}_{[S\to V]/}$ is initial. In fact, by the following diagram,

$$\begin{tikzcd}
\underline{\mathbb{F}}^{\mathrm{el}}_{\mathcal{F},*,[S\to V]/} \arrow[r, "\sim"] \arrow[d, "\iota^{\mathrm{el}}_{[S\to V]/}"] & \left(\mathcal{F}\times_{\mathbb{F}_{\mathcal{F}}} \underline{\mathbb{F}}^{si}_{\mathcal{F},/[S\to V]}\right)^{\mathrm{op}} \arrow[r, "\sim"] \arrow[d] & \prod\limits_{U\in\mathrm{Orb}(S)} (B\mathrm{Aut}_{\mathcal{F}}(U))^{\mathrm{op}} \arrow[d, "\varphi"] \\
\mathrm{Span}(\mathbb{F}_{\mathcal{T}};\mathcal{F})^{\mathrm{el}}_{S/} \arrow[r, "\sim"] & \left(\mathcal{F}\times_{\mathbb{F}_{\mathcal{T}}} \mathbb{F}_{\mathcal{T},/S}\right)^{\mathrm{op}} \arrow[r] & \prod\limits_{U\in\mathrm{Orb}(S)} (\mathcal{F}_{/U})^{\mathrm{op}}
\end{tikzcd}$$

it suffices to verify that the functor φ is final. Indeed, since $\mathcal{T}$ is atomic, the subcategory $B\mathrm{Aut}_{\mathcal{T}}(U)\hookrightarrow\mathcal{T}_{/U}$ is downwards closed, i.e. initial. This implies φ is a product of opposites of initial functors, hence it is final.

It remains to check that s satisfies the conditions of Theorem A.4. We check this in parts. Condition 1 follows immediately by construction. Condition 2 follows by noting that the following diagram commutes:

$$\begin{tikzcd}
\underline{\mathbb{F}}^{\mathrm{act}}_{\mathcal{T},*,/[S\to V]} \arrow[r, "\sim"] \arrow[d] & \underline{\mathbb{F}}_{\mathcal{T},/[S\to V]} \arrow[r, "\sim"] \arrow[d] & \mathbb{F}_{\underline{V},/S} \arrow[r, "\sim"] \arrow[d] & \prod\limits_{U\in\mathrm{Orb}(S)} \underline{V}_{/U} \arrow[d, "\varphi"] \\
\mathrm{Span}(\mathbb{F}_{\mathcal{T}};\mathcal{F})^{\mathrm{act}}_{/S} \arrow[r, "\sim"] & \mathbb{F}_{\mathcal{T},/S} \arrow[r, equal] & \mathbb{F}_{\mathcal{T},/S} \arrow[r, "\sim"] & \prod\limits_{U\in\mathrm{Orb}(S)} \mathcal{T}_{/U}
\end{tikzcd}$$

and by noting that φ is an equivalence, since $\underline{V}\subset\mathcal{T}$ is a full subcategory containing any element attaining a map to V, and there exists a map $U\to S\to V$. $\square$

In fact, we may say something more general; define the pullback pattern

$$\begin{tikzcd}
\underline{\mathbb{F}}_{I,*} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathbb{F}}_{\mathcal{T},*} \arrow[d] \\
\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \arrow[r] & \mathrm{Span}(\mathbb{F}_{\mathcal{T}})
\end{tikzcd}$$

so that $\mathbb{F}_{I,V,*}$ corresponds with pointed I-admissible V-sets.

Observation A.9. By Lemma A.2, $\underline{\mathbb{F}}_{I,*}$-Segal objects in $\mathcal{C}$ are precisely I-semiadditive functors $\underline{\mathbb{F}}_{I,*}\to\mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$. ◃

The conditions of Theorem A.4 follow from the case $I=\mathcal{T}$, so we have the following.

Corollary A.10. *If I is a weak indexing category, then pullback along the map $\underline{\mathbb{F}}_{I,*}\simeq\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ induces an equivalence*

$$\mathrm{Op}_I\simeq\mathrm{Fbrs}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}))\simeq\mathrm{Fbrs}(\underline{\mathbb{F}}_{I,*})$$

A.2. **Pullback of fibrous patterns along Segal morphisms and sound extendability.**

Proposition A.11. *Suppose $\varphi:\mathfrak{O}\to\mathfrak{P}$ is morphism of algebraic patterns and $\mathfrak{P}$ is soundly extendable. Then,*

(1) *If the precomposition functor*

$$\varphi^*:\mathrm{Fun}(\mathfrak{P},\mathbf{Cat})\to\mathrm{Fun}(\mathfrak{O},\mathbf{Cat})$$

preserves Segal objects, then the pullback functor

$$\varphi^*:\mathbf{Cat}_{/\mathfrak{P}}\to\mathbf{Cat}_{/\mathfrak{O}}$$

preserves fibrous patterns.

(2) *If φ is an inert-cocartesian fibration and the left Kan extension functor*

$$\varphi_!:\mathrm{Fun}(\mathfrak{O},\mathbf{Cat})\to\mathrm{Fun}(\mathcal{P},\mathbf{Cat})$$

preserves Segal objects, then postcomposition

$$\varphi_!:\mathbf{Cat}_{/\mathfrak{O}}\to\mathbf{Cat}_{/\mathfrak{P}}$$

preserves fibrous patterns.

In particular, if φ is an inert-cocartesian Segal morphism between soundly extendable patterns whose left Kan extension preserves Segal categories, then pullback and postcomposition restrict to an adjunction on fibrous patterns

$$\varphi_! : \mathrm{Fbrs}(\mathfrak{O}) \rightleftarrows \mathrm{Fbrs}(\mathfrak{P}) : \varphi^*$$

Proof. Our argument mirrors that of [BHS22, Lem 4.1.19]. In either case, the property of being an inert-cocartesian fibration is always preserved, either by assumption or by [BHS22, Obs 2.2.6].

We prove (1) first. Fixing $\mathscr{F} \in \mathrm{Fbrs}(\mathfrak{P})$, by [BHS22, Obs 4.1.3], it suffices to prove that the left vertical arrow in the following pullback diagram is a relative Segal $\mathfrak{O}$-∞-category.

$$\begin{tikzcd} \mathrm{St}^{\mathrm{int}}_{\mathfrak{O}}(\varphi^*\mathscr{F}) \arrow[r] \arrow[d] & \varphi^*\mathrm{St}^{\mathrm{int}}_{\mathfrak{P}}\mathscr{F} \arrow[d] \\ \mathscr{A}_{\mathfrak{O}} \arrow[r] & \varphi^*\mathscr{A}_{\mathfrak{P}} \end{tikzcd}$$

By [BHS22, Lem 3.1.10], relative Segal $\mathfrak{O}$-∞-categories are pullback-stable, so it suffices to prove that the right vertical arrow is a relative Segal $\mathfrak{O}$-∞-category. By sound extendability $\mathscr{A}_{\mathfrak{P}}$ is a Segal $\mathfrak{P}$-∞-category, and since φ^* preserves Segal ∞-categories, $\varphi^*\mathscr{A}_{\mathfrak{P}}$ is a Segal $\mathfrak{O}$-∞-category; by [BHS22, Obs 3.1.8] it then suffices to prove that $\varphi^*\mathrm{St}^{\mathrm{int}}_{\mathfrak{P}}\mathscr{F}$ is a Segal $\mathfrak{O}$-∞-category. Since φ^* preserves Segal ∞-categories, it suffices to prove that $\mathrm{St}^{\mathrm{int}}_{\mathfrak{P}}\mathscr{F}$ is a Segal $\mathfrak{P}$-category, which follows by the assumption that $\mathscr{F}$ is a fibrous pattern.

(2) is similar; this time, by taking left adjoints to the commutative square of [BHS22, Prop 4.2.5], it suffices to prove that the composition

$$\varphi_!\mathrm{St}^{\mathrm{int}}_{\mathfrak{O}}\mathscr{F} \to \varphi_!\mathscr{A}_{\mathfrak{O}} \to \mathscr{A}_{\mathfrak{P}}$$

is relative Segal; since $\mathfrak{P}$ is soundly extendable, [BHS22, Obs 3.1.8] again reduces this to verifying that $\varphi_!\mathrm{St}^{\mathrm{int}}_{\mathfrak{O}}\mathscr{F}$ is Segal; this follows from the facts that $\mathscr{F}$ is a fibrous pattern and $\varphi_!$ preserves Segal ∞-categories. ☐

A.3. **Segal morphisms between effective Burnside patterns.** In this section, we fill our grab bag full of a wide variety of Segal morphisms between effective Burnside patterns.

Proposition A.12. *Suppose $F \subset F' \subset \mathbb{F}_{\mathcal{T}}$ are wide subcategories. Then, the inclusion*

$$\iota : \mathrm{Span}_F(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}_{F'}(\mathbb{F}_{\mathcal{T}})$$

is a Segal morphism.

Proof. We are tasked with verifying that precomposition with ι preserves product-preserving functors, i.e. that ι is a product-preserving functor. In fact, this is immediate, since a functor $\mathrm{Span}_F(\mathbb{F}_{\mathcal{T}}) \to \mathcal{C}$ is product-preserving if and only if the backwards maps $(S \leftarrow U)_{U \in \mathrm{Orb}(S)}$ together map to a product diagram, which is obviously true of ι. ☐

Proposition A.13. *Suppose $\varphi : V \to W$ is a morphism in $\mathcal{T}$. Then, the associated functor* $\mathrm{Span}(\mathrm{Ind}_V^W)$: $\mathrm{Span}(\mathbb{F}_V) \to \mathrm{Span}(\mathbb{F}_W)$ *is a Segal morphism.*

Proof. We're tasked with proving that precomposition along $\mathrm{Span}(\mathrm{Ind}_V^W)$ preserves product-preserving functors, i.e. it is a product-preserving functor. Since $\mathrm{Span}(\mathbb{F}_V)$ and $\mathrm{Span}(\mathbb{F}_W)$ are semiadditive, it is equivalent to prove that $\mathrm{Span}(\mathrm{Ind}_V^W)$ is coproduct-preserving; since coproducts in $\mathrm{Span}(\mathbb{F}_V)$ are computed in $\mathbb{F}_V$, it's equivalent to prove that $\mathrm{Ind}_V^W : \mathbb{F}_V \to \mathbb{F}_W$ is coproduct-preserving, which follows from the fact that it's a left adjoint. ☐

Proposition A.14. *If $f : \mathcal{T}' \to \mathcal{T}$ is a functor of atomic orbital ∞-categories, then the associated functor* $\mathrm{Span}(\mathbb{F}_{\mathcal{T}'}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ *is a Segal morphism.*

Proof. By [CH21, Rem 4.3], it suffices to verify that $f^{\mathrm{el}}_{X/}$ induces an equivalence on the left vertical arrow

$$\begin{tikzcd} \lim\limits_{\mathrm{Span}(\mathcal{T})^{\mathrm{el}}_{f(X)/}} F \arrow[r, "\sim"] \arrow[d] & \prod\limits_{U \in \mathrm{Orb}(f(X))} F(U) \arrow[d] \\ \lim\limits_{\mathrm{Span}(\mathcal{T}')^{\mathrm{el}}_{X/}} F \circ f^{\mathrm{el}} \arrow[r, "\sim"] & \prod\limits_{V \in \mathrm{Orb}(X)} Ff(V) \end{tikzcd}$$

whenever F is restricted from a Segal $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ space. This follows by noting that the horizontal arrows are equivalences by construction, and $\mathrm{Span}(f)$ sends the set of orbits of X bijectively onto the set of orbits of $f(X)$. $\square$

Proposition A.15. *The map* $\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \times \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \xrightarrow{\wedge} \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ *is a Segal morphism.*

Proof. By [CH21, Ex 5.7], a functor $\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \times \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \to \mathcal{C}$ is a Segal object if and only if it preserves products separately in each variable. Hence we're tasked with verifying that $\wedge^* F$ preserves products separately in each variable whenever F preserves products. In fact, this follows by distributivity of products and coproducs in $\mathbb{F}_{\mathcal{T}}$; indeed, we have

$$\begin{aligned} \wedge^* F\left((X_+ \oplus Z_+, Y_+)\right) &\simeq F\left((X \sqcup X') \times Y\right)_+ \\ &\simeq F\left((X \times Y) \sqcup (X' \times Y)\right)_+ \\ &\simeq F\left((X_+ \wedge Y_+) \oplus (X'_+ \wedge Y_+)\right) \\ &\simeq F\left(X_+ \wedge Y_+\right) \oplus F\left(X'_+ \wedge Y_+\right) \\ &\simeq \wedge^* F\left(X_+, Y_+\right) \oplus \wedge^* F\left(X'_+, Y_+\right). \end{aligned}$$ $\square$

Appendix B. Cartesian and cocartesian I-symmetric monoidal ∞-categories

Fix I a unital weak indexing category. This appendix can be understood as a lift of [HA, § 2.4.1-2.4.3] to the setting of (co)cartesian I-symmetric monoidal ∞-categories; we proceed by an essentially similar strategy, complicated only by less convenient combinatorics. In particular, we use the combinatorics of $\underline{\mathbb{F}}_{I,*}$-fibrous patterns throughout, so we will freely synonymize $\mathrm{Op}_{\mathcal{T}}$ and $\mathrm{Fbrs}(\underline{\mathbb{F}}_{I,*})$ throughout.

Define the $\mathcal{T}$-1-category $\underline{\Gamma}^*_I$ to have V-values

$$\Gamma^*_{I,V} \coloneqq \left\{ U_+ \xrightarrow{s.i.} S_+ \mid U \in \underline{V} \right\} \subset \mathrm{Ar}(\underline{\mathbb{F}}_{I,*})_V;$$

that is, the objects of $\Gamma_{I,V}$ are pointed I-admissible V-sets with a distinguished orbit, and the morphisms of $\Gamma_{I,V}$ preserve distinguished orbits. This possesses a tautological forgetful functor $\underline{\Gamma}^*_I \to \underline{\mathbb{F}}_{I,*}$. We use this to construct an ∞-category $\mathcal{C}$ over $\underline{\mathbb{F}}_{I,*}$ in Appendix B.2 satisfying the following universal property.

Proposition B.1. *Given $\mathcal{C}$ a $\mathcal{T}$-∞-category, there exists an ∞-category $\mathcal{C}^{I-\sqcup}$ over $\underline{\mathbb{F}}_{I,*}$ satisfying the universal property that there is a natural equivalence*

$$\mathrm{Fun}_{\underline{\mathbb{F}}_{I,*}}(\mathcal{D}, \mathcal{C}^{I-\sqcup}) \simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{D} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I, \mathcal{C});$$

that is, the functor $(-) \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I : \mathbf{Cat}_{\infty,/\underline{\mathbb{F}}_{I,*}} \to \mathbf{Cat}_{\mathcal{T}}$ possesses a right adjoint $(-)^{I-\sqcup}$.

An object of $\mathcal{C}^{I-\sqcup}$ may be viewed as S_+ a pointed V-set and $\mathbf{C} = (C_W) \in \mathcal{C}_S$ an S-tuple of elements of $\mathcal{C}$; a morphism $f : \mathbf{C} \to \mathbf{D}$ may be viewed as a $\underline{\mathbb{F}}_{I,*}$-map $(S_+ \to V_{S,+}) \xrightarrow{f} (T_+ \to V_{T,+})$ together with a collection of maps

$$\left\{ f_W : N_W^U C_W \to D_U \mid W \in f^{-1}(U) \right\}$$

for all $U \in \mathrm{Orb}(T)$. In particular, we have the following:

Lemma B.2. *$\mathcal{C}^{I-\sqcup}$ satisfies the Segal conditions (b) and (c) of [NS22, Def 2.1.7].*

Furthermore, unwinding definitions and applying [HTT, Cor 3.2.2.13], we find the following.

Proposition B.3. *A morphism $f : (\mathbf{C}, S) \to (\mathbf{D}, T)$ is π-cocartesian if and only if $\{f_W\}$ witnesses D_U as the coproduct*

$$\coprod_{W \in f^{-1}(U)} N_W^U C_W \simeq D_U$$

for all $U \in \mathrm{Orb}(T)$. In particular, f is inert if and only if the following conditions are satisfied:

(a) The projected morphism $\pi(f) : S \to T$ is inert.
(b) The associated map $C_{f^{-1}(U)} \to D_U$ is an equivalence for all $U \in \mathrm{Orb}(T)$.

Hence $\mathcal{C}^{I-\sqcup}$ is an I-operad, which is an I-symmetric monoidal ∞-category if and only if $\mathcal{C}$ admits I-indexed coproducts.

Thus, when $\mathcal{C}^{\otimes}$ admits I-indexed products, we've constructed an I-symmetric monoidal ∞-category whose indexed tensor products are coproducts; we will now compute its algebras, eventually forcing all other such I-symmetric monoidal structures to be equivalent to this one.

B.1. **$\mathcal{O}$-comonoids and (co)cartesian rigidity.** Define a diagram of Cartesian squares.

$$\begin{tikzcd}
\mathcal{O} \arrow[r, hook, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}_\Gamma^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes} \arrow[d] \\
\mathcal{T}^{\mathrm{op}} \arrow[r, hook] & \underline{\Gamma}_I^* \arrow[r] & \underline{\mathbb{F}}_{I,*}
\end{tikzcd}$$

Note that the objects of $\mathcal{O}_\Gamma^{\otimes}$ consist of triples $(S_+ \to V_+, U, X)$ where $U \in \mathrm{Orb}(S)$ and $X \in \mathcal{O}_S$, and the image of ι is equivalent to the triples where $S \in \underline{V}$, hence $U = S$.

Further tote that cocartesian transport along inert morphism $U_+ \hookrightarrow S_+$ induces an equivalence

$$\mathrm{Map}_{\mathcal{O}_\Gamma^{\otimes}}(Y, (S_+ \to V_+, U, X))) \simeq \mathrm{Map}_{\mathcal{O}_\Gamma^{\otimes}}(Y, (U_+ \to V_+, U, X_U)))$$

for all $Y \in \mathcal{O}$. In particular, ι witnesses $\mathcal{O}$ as a *colocalizing subcategory*, with colocalization functor

$$R(S_+ \to V_+, U, X) \simeq (U_+ \to V_+, U, X_U).$$

Lemma B.4. *Fix a functor $A : \mathcal{O}_\Gamma^{\otimes} \to \mathcal{C}$. Then, the following are equivalent*

(a) The corresponding map $\mathcal{O}^{\otimes} \to \mathcal{C}^{I-\sqcup}$ is a functor of I-operads.
(b) For all morphisms α in $\mathcal{O}_\Gamma^{\otimes}$ whose image in $\mathcal{O}^{\otimes}$ is inert, $A(\alpha)$ is an equivalence in $\mathcal{C}$.
(c) If $f : (S_+ \to V_+, U, X) \to (U_+ \to V_+, U, X_U)$ is a cocartesian lift of the corresponding inert morphism, then $A(f)$ is an equivalence.
(d) A is left Kan extended from $\mathcal{O}$.

Furthermore, every functor $F : \mathcal{O} \to \mathcal{C}$ admits a left Kan extension along $\mathcal{O} \hookrightarrow \mathcal{O}_\Gamma^{\otimes}$; in particular, the forgetful functor $\underline{\mathbf{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_G(\mathcal{O}, \mathcal{C})$ is an equivalence.

Proof. (a) $\iff$ (b) follows immediately from Proposition B.3. (b) $\iff$ (c) is immediate by definition. (c) $\iff$ (d) and the remining statement both follow by the more general observation that the left Kan extension of $F : \mathcal{C} \to \mathcal{D}$ along a functor $L : \mathcal{C} \to \mathcal{E}$ with right adjoint R is given by the composite $FR : \mathcal{E} \to \mathcal{C} \to \mathcal{D}$. ☐

We would additionally like to characterize I-symmetric monoidal functors into $\mathcal{C}^{I-\sqcup}$. The following lemma follows immediately from Proposition B.3.

Lemma B.5. *Assume $\mathcal{C}$ has I-indexed coproducts and $\mathcal{D}^{\otimes}$ is an I-symmetric monoidal ∞-category. Then, TFAE for a lax I-symmetric monoidal functor $\varphi : \mathcal{D}^{\otimes} \to \mathcal{C}^{I-\sqcup}$:*

(1) φ is a map of I-symmetric monoidal categories.
(2) The corresponding $\mathcal{T}$-functor $F : \mathcal{D}^{\otimes} \to \mathcal{C}$ satisfies the property that, for all $(X_U) \in \mathcal{D}_S$, the canonical maps $\mathrm{Ind}_U^V F(X_U) \to F(X)$ exhibit $F(X)$ as the indexed coproduct

$$\coprod_U^S F(X_U) \simeq F(X).$$

We use this for the following fundamental proposition underlying (co)cartesian rigidity.

Proposition B.6. *Suppose $\mathcal{D}^{\otimes}$ is an I-symmetric monoidal category satisfying the condition that its action maps $f_{\otimes}: \mathcal{D}_S \to \mathcal{D}_V$ are left adjoint to the restriction map $f^*: \mathcal{D}_V \to \mathcal{D}_S$. Then, the forgetful functor*

$$U: \mathrm{Fun}_I^{\otimes}(\mathcal{D}^{\otimes}, \mathcal{C}^{I-\sqcup}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{D}, \mathcal{C})$$

is fully faithful with image spanned by the I-coproduct preserving functors; dually, if $\mathcal{E}^{\otimes}$ is an I-symmetric monoidal category satisfying the condition that its action maps $f_{\otimes}: \mathcal{E}_S \to \mathcal{E}_V$ are right *adjoint to the restriction map $f^*: \mathcal{E}_V \to \mathcal{E}_S$, then the forgetful functor*

$$U: \mathrm{Fun}_I^{\otimes}(\mathcal{E}^{\otimes}, (\mathcal{C}^{I-\times})^{v\,\mathrm{op}}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{E}, \mathcal{C})$$

is fully faithful with image spanned by the I-product preserving functors, $(-)^{v\,\mathrm{op}}$ denoting the fiberwise opposite over $\underline{\mathbb{F}}_{I,*}$.

Proof. The first statement follows by noting that those $\mathcal{T}$-functors $\mathcal{D}^{\otimes} \to \mathcal{C}$ satisfying the conditions of Lemma B.5 are precisely those which are left Kan extended along the (fully faithful) $\mathcal{T}$-functor $\mathcal{D} \hookrightarrow \mathcal{D}^{\otimes}$ from I-coproduct preserving functors. The second follows by taking fiberwise opposites. □

We are now ready to prove our main generalization for Theorem D' (see p. 27).

Proof of Theorem D'. The two cases are dual, so we prove it for $(-)^{I-\sqcup}$. To see that it's fully faithful, it suffices to note that the action maps in $\mathcal{C}^{I-\sqcup}$ are left adjoint to restriction and apply Proposition B.6. The compatibility with U is obvious, and the description of the image follows immediately from Proposition B.6. □

B.2. **A quasicategory modeling $\mathcal{C}^{I-\sqcup}$.** Let $\mathcal{T}$ be a quasicategory and $\mathcal{C} \in \mathrm{sSet}^{\mathrm{cocart}}_{/\mathcal{T}}$ a cocartesian fibration to $\mathcal{T}$. There exists a simplicial set $\mathcal{C}^{I-\sqcup}$ satisfying the universal property

$$\mathrm{Hom}_{\underline{\mathbb{F}}_{I,*}}(K, \mathcal{C}^{I-\sqcup}) \simeq \mathrm{Hom}_{\mathcal{T}}(K \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I, \mathcal{C}). \tag{30}$$

Lemma B.7. *The map of simplicial sets $\mathcal{C}^{I-\sqcup} \to \underline{\mathbb{F}}_{I,*}$ is an inner fibration; hence $\mathcal{C}^{I-\sqcup}$ is a quasicategory.*

Proof. The proof is exactly analogous to the analogous part of [HA, Prop 2.4.3.3]; that is, we may apply the universal property

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f_0"] \arrow[d] & \mathcal{C}^{I-\sqcup} \arrow[d] \\ \Delta^n \arrow[r, "(S_{0,+}\to\cdots\to S_{n,+})"'] \arrow[ur, dashed] & \underline{\mathbb{F}}_{I,*} \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} \Lambda^n_i \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I \arrow[d] & \simeq & \coprod\limits_{\substack{U \in \mathrm{Orb}(S) \\ f(U) \in S^{\circ}_{n,+}}} \Lambda^n_i \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^n \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I & \simeq & \coprod\limits_{\substack{U \in \mathrm{Orb}(S) \\ f(U) \in S^{\circ}_{n,+}}} \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{T}^{\mathrm{op}} \end{tikzcd}$$

after which the lifting problem on the RHS has solutions in bijection with the tuples of solutions to the lifting problems made up of the summands, which exist by assumption that the functor $\mathcal{C} \to \mathcal{T}$ is a cocartesian fibration (hence an inner fibration).

The remaining claim follows by noting that $\underline{\mathbb{F}}_{I,*}$ is a quasicategory, so the composite map of simplicial sets $\mathcal{C}^{I-\sqcup} \to \underline{\mathbb{F}}_{I,*} \to *$ is an inner fibration. □

Proof of Proposition B.1. Unwinding the above work, we've verified that $\mathcal{C}^{I-\sqcup}$ is a quasicategory over $\underline{\mathbb{F}}_{I,*}$. Fixing some quasicategory $\mathcal{D}$ over $\underline{\mathbb{F}}_{I,*}$ and applying Eq. (30) for $K := \mathcal{D} \times \Delta^n$, we find that $\mathrm{Fun}(K, \mathcal{C}^{I-\sqcup}) \simeq \mathrm{Fun}_{\mathcal{T}}(K \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I, \mathcal{C})$. The result then follows by replacing "quasicategory" with "∞-category." □

B.3. **$\mathcal{O}$-monoids.** Recall that an *$\mathcal{O}$-monoid in $\mathcal{C}$* is a functor $\mathcal{O}^{\otimes} \to \mathcal{C}$ satisfying the condition that for all $X = (X_U) \in \mathcal{C}_S$, the canonical maps $F(X) \to F(X_U)$ witness $F(X)$ as the indexed product

$$F(X) \simeq \prod_U^S F(X_U).$$

We are tasked with proving the following.

Proposition 1.82. *Fix $\mathcal{C}$ a $\mathcal{T}$-category. Then, the postcomposition functor $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\times}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}^{\otimes}, \mathcal{C})$ is fully faithful with image spanned by the $\mathcal{O}$-monoids.*

In order to do so, we introduce a construction.

Construction B.8. The (non-full) $\mathcal{T}$-subcategory $\Gamma_I^{\times} \subset \mathrm{Ar}(\underline{\mathbb{F}}_{I,*})$ has V-objects given by summand inclusions of pointed V-sets $\overline{S} \hookrightarrow S$ and morphisms of V-objects given by maps $\alpha : S \to T$ with the property that $\alpha^{-1}(\overline{T}) \subset \overline{S}$. ◁

Recollection B.9 ([NS22, Def 2.1.2])**.** A morphism f in $\underline{\mathbb{F}}_{I,*}$ from $S_+ \in \mathbb{F}_{I,*,U}$ to $T_+ \in \mathbb{F}_{I,*,V}$ may be modelled as a morphism of spans

$$\begin{tikzcd}
S \arrow[dd] & & f^{-1}(T) \arrow[ll] \arrow[r, "f^\circ"] \arrow[dd] & T \arrow[dd] \\
& \mathrm{Res}_U^V S \arrow[ul] \arrow[ur, hook, "\iota_f"'] \arrow[dr] & & \\
U & & V \arrow[ll] \arrow[r, equal] & V
\end{tikzcd}$$

such that $f^\circ \in I$. Such a morphism is $\pi_{\underline{\mathbb{F}}_{I,*}}$-cocartesian if f° and ι_f are both equivalences, i.e. it witnesses an equivalence $\mathrm{Res}_U^V S_+ \xrightarrow{\sim} T_+$. ◁

Let $T_+ \to S_+$ be a map in $\underline{\mathbb{F}}_{I,*}$ lying over an orbit map $U \to V$ and let $\overline{S} \subset S$ be an element of $\underline{\Gamma}_I^{\times}$ lying over S_+. We would like to construct a Cartesian edge landing on $\overline{S} \subset S$; we do so by setting $\overline{T} := f^{-1}(\mathrm{Res}_U^V \overline{S}) \subset f^{-1}(S) \subset T$, and letting the associated map $t : \left(f^{-1}(\mathrm{Res}_U^V \overline{S}) \subset T\right) \to \left(\overline{S} \subset S\right)$ be the canonical one. The following lemma then follows by unwinding definitions, where $U : \underline{\Gamma}_I^{\times} \to \underline{\mathbb{F}}_{I,*}$ denotes the forgetful functor.

Lemma B.10. *t is a U-cartesian arrow; in particular, U is a cartesian fibration.*

Given $\mathcal{C}$ a $\mathcal{T}$-∞-category, modelled as a quasicategory cocartesian fibered over a fixed model for $\mathcal{T}^{\mathrm{op}}$, we may define a simplicial set $\tilde{\mathcal{C}}^{I-\times}$ over $\underline{\mathbb{F}}_{I,*}$ by the universal property

$$\mathrm{Hom}_{/\underline{\mathbb{F}}_{I,*}}(K, \tilde{\mathcal{C}}^{I-\times}) \simeq \mathrm{Hom}_{/\mathcal{T}^{\mathrm{op}}}(K \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_I^{\times}, \mathcal{C}).$$

For $S_+ \in \underline{\mathbb{F}}_{I,*}$, we view objects in $\tilde{\mathcal{C}}^{I-\times}_{S_+}$ over V as V-functors $\mathscr{P}_V(S)^{\mathrm{op}} \to \mathcal{C}_V$, where $\mathscr{P}_V(S)$ is the poset of V-subsets of S.

The following lemma is then immediately implied by [HTT, Cor 3.2.2.13].

Lemma B.11. *Let $\tilde{p} : \tilde{\mathcal{C}}^{I-\times} \to \underline{\mathbb{F}}_{I,*}$ be the projection, and let $\tilde{\alpha} : F \to G$ be a morphism lying over a morphism $\alpha : T \to S$ lying over an orbit map $U \to V$. Then, $\tilde{\alpha}$ is $\tilde{p}$-cocartesian in the sense of [HTT] if and only if, for all $T' \subset T$, the induced map $F(\alpha^{-1}(\mathrm{Res}_U^V T')) \to \mathrm{Res}_U^V G(T')$ is an equivalence; in particular, $\tilde{p}$ is a cocartesian fibration of simplicial sets*

Since $\tilde{\mathcal{C}}^{I-\times} \to \underline{\mathbb{F}}_{I,*}$ is a cocartesian fibration of simplicial sets, it is an inner fibration, so $\tilde{\mathcal{C}}^{I-\times}$ is a quasicategory. Using this, we henceforth treat $\tilde{\mathcal{C}}^{I-\times} \to \underline{\mathbb{F}}_{I,*}$ as a cocartesian fibration of ∞-categories. Let $\mathcal{C}^{I-\times} \subset \tilde{\mathcal{C}}^{I-\times}$ be the full subcategory spanned by those functors $\mathscr{P}(S)^{\mathrm{op}} \to \mathcal{C}_{\underline{V}}$ satisfying the property that, for all $T \subset S$, the maps

$$F(T) \to \mathrm{CoInd}_U^V \mathrm{Res}_U^V F(U)$$

exhibit $F(T)$ as the T-indexed product $F(T) \simeq \prod_U^T F(U)$ in $\mathcal{C}$. Once again, the following follows by definition.

Proposition B.12. *A morphism in $\mathcal{C}^{I-\times}$ is p-cocartesian if and only if it lifts to a $\tilde{p}$-cocartesian morphism of $\mathcal{C}^{I-\times}$. In particular, the projection $p : \mathcal{C}^{I-\times} \to \underline{\mathbb{F}}_{I,*}$ is an I-symmetric monoidal category if and only if $\mathcal{C}$ admits I-indexed products.*

Observation B.13. $\mathcal{C}^{I-\times}$ is a cartesian I-symmetric monoidal ∞ category with underlying $\mathcal{T}$-∞-category $\mathcal{C}$, so we have not created a clash in notation. ◁

Observation B.14. The structure map $\mathcal{O}^{\otimes} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_{I,*} \to \mathcal{O}^{\otimes}$ admits a left adjoint L sending $X \in \mathcal{O}^{\otimes}_{S_+}$ to $(X, S \subset S)$; the unit map of this adjunction is evidently an equivalence, so $L : \mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_{I,*}$ is fully faithful. ◁

Fix a $\mathcal{T}$ functor $A : \mathcal{O}^{\otimes} \times_{\underline{\mathbb{F}}_{I,*}} \Gamma^{\times} \to \mathcal{C}$ with corresponding functor $\varphi : \mathcal{O}^{\otimes} \to \tilde{\mathcal{C}}^{I-\times}$ and restricted functor $A' : \mathcal{O}^{\otimes} \to \mathcal{C}$. Lemma B.11 immediately implies the following.

Lemma B.15. *Suppose A' is a $\mathcal{T}$-functor. Then, the following conditions are equivalent:*

(a) *The map φ is a functor of I-operads.*
(b) *For all morphisms α in $\mathcal{O}^{\otimes} \times_{\mathbb{F}_{I,*}} \underline{\Gamma}_I^{\times}$ whose image in $\mathcal{O}^{\otimes}$ is inert $A(\alpha)$ is an equivalence in $\mathcal{C}$.*
(c) *If $f : (\overline{S}_+ \to V_+, \overline{S}, F, X) \to (S_+ \to V_+, \overline{S}, F, X)$ is a cocartesian lift of the corresponding inert morphism, then $A(f)$ is an equivalence.*
(d) *A is right Kan extended from A' along L.*

In this case, the composite map $\mathcal{O}^{\otimes} \to \tilde{\mathcal{C}}^{I-\times} \to \mathcal{C}$ is homotopic to A'.

We use this to finally identify Cartesian algebras in the following lemma, which also follows imeediately from Lemma B.11.

Lemma B.16. *Suppose φ is a functor of I-operads. Then, the following conditions are equivalent:*

(a) *φ factors through the inclusion $\mathcal{C}^{I-\times} \subset \tilde{\mathcal{C}}^{I-\times}$.*
(b) *A' is an $\mathcal{O}$-monoid.*

Proof of Proposition 1.82. $\mathcal{C}^{I-\times} \hookrightarrow \tilde{\mathcal{C}}^{I-\times}$ is fully faithful, and hence it is a monomorphism in **Cat**. This implies that the associated functor

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\times}) \hookrightarrow \mathrm{Fun}^{int-cocart}_{/\mathbb{F}_{I,*}}\left(\mathcal{O}^{\otimes}, \tilde{\mathcal{C}}^{I-\times}\right) \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{O}^{\otimes}, \mathcal{C}\right)$$

is fully faithful. By Lemma B.16, its image is the $\mathcal{O}$-monoids. ☐

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