ON TENSOR PRODUCTS OF EQUIVARIANT COMMUTATIVE OPERADS

NATALIE STEWART

Abstract. We affirm and generalize a conjecture of Blumberg and Hill: unital weak $\mathcal{N}_\infty$-operads are closed under ∞-categorical Boardman-Vogt tensor products and the resulting tensor products correspond with *joins of weak indexing systems*; in particular, we acquire a natural G-symmetric monoidal equivalence

$$\underline{\mathrm{CAlg}}_I^{\otimes} \underline{\mathrm{CAlg}}_J^{\otimes} \mathcal{C} \simeq \underline{\mathrm{CAlg}}_{I \vee J}^{\otimes} \mathcal{C}.$$

We accomplish this by showing that $\mathcal{N}_{I\infty}^{\otimes}$ is $\overset{\mathrm{BV}}{\otimes}$-idempotent and $\mathcal{O}^{\otimes}$ is local for the corresponding smashing localization if and only if $\underline{\mathrm{Mon}}_{\mathcal{O}}(\underline{\mathcal{S}}_G)$ has I-indexed Wirthmüller isomorphisms. This verifies homotopical additivity of the equivariant little disks operads in a number of infinitary cases.

Along the way, we acquire a number of structural results concerning G-operads, including a canonical lift of $\overset{\mathrm{BV}}{\otimes}$ to a presentably symmetric monoidal G-∞-structure and a general disintegration & assembly procedure for computing tensor products of non-reduced unital G-operads. Additionally, we achieve the expected corollaries for (iterated) Real topological Hochschild homology and construct a natural I-symmetric monoidal structure on right modules over an $\mathcal{N}_{I\infty}$-algebra. All such results are proved in the generality of atomic orbital ∞-categories.

Contents

Introduction

In this paper, we study the behavior of Blumberg-Hill's $\mathcal{N}_\infty$-operads under the equivariant Boardman-Vogt tensor product of [Ste25a]. We do so by means of a characterization of algebras (co)cartesian I-symmetric monoidal ∞-categories, where I is a (weak) indexing system in the sense of [BH15; Ste24]: cocartesian

Date: February 20, 2025.
Status: needs some reorganization.

I-symmetric monoidal structures are characterized by the property that their objects have contractible spaces of I-commutative algebra structures, and cartesian I-symmetric monoidal structures are characterized by an $\mathcal{O}$*-monoid* formula generalizing [HA, Prop 2.4.2.5].

In the unital case, we conclude that the unique map $\mathrm{triv}_G^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$ witnesses $\mathcal{N}_{I\infty}^{\otimes}$ as an idempotent object in Op_G in the sense of [HA] and we characterize its associated smashing localization in terms of indexed semiadditivity; we conclude that there is a unique equivalence $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{I\vee J\infty}^{\otimes}$ when I, J are unital weak indexing systems, confirming conjecture 6.27 of [BH15] in this setting. In particular, we acquire a unique natural equivalence

$$\mathrm{CAlg}_I \underline{\mathrm{CAlg}}_J^{\otimes}(\mathcal{C}) \simeq \mathrm{CAlg}_{I\vee J}(\mathcal{C}).$$

This is the third part of an ongoing project [Ste24; Ste25a] to develop the parameterized and equivariant higher algebra predicted in [BDGNS16; NS22] into simply usable foundations for equivariant homotopy theory and K-theory; as such, we spend the latter third of the paper developing higher algebraic corollaries to the equivalence $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{I\infty}^{\otimes}$.

These corollaries fall into two classes: the first class gives $\mathrm{Comm}_G^{\otimes} \in \mathrm{Op}_G$ a unique idempotent algebra structure, which determines a compatible idempotent algebra on its G-symmetric monoidal envelope, enabling *symmetric monoidality* of the equifibered perspective of [BHS22; BS24b]. From this, we lift Op_G with the Boardman-Vogt tensor product to a canonical presentably symmetric monoidal G-∞-category; for instance, we develop equivariant operadic disintegration and assembly, and the associated distributivity of $\overset{\mathrm{BV}}{\otimes}$ allows us to compute tensor products of unital G-operads whose underlying G-∞-categories are G-spaces in terms of tensor products of reduced G-operads.

The second class simply exploits the characterization of G-operads $\mathcal{O}^{\otimes}$ satisfying either $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes}$ or $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes} \simeq \mathcal{N}_{I\infty}^{\otimes}$. In the former case for $\mathcal{N}_{I\infty}^{\otimes} \simeq \mathbb{E}_{\infty}^{\otimes}$, we get an $\mathcal{O}$-symmetric monoidal structure on left modules over an $\mathcal{O}^{\otimes}$-algebra. In the latter case for $\mathcal{O}^{\otimes} \simeq \mathbb{E}_V$, we acquire an I-commutative algebra structure on (lax) I-symmetric monoidal $\mathbb{E}_V$-algebra invariants of I-commutative algebras; for instance, this constructs an I-commutative algebra structure on Real topological Hochschild homology of an I-commutative algebra whenever I contains C_2-set map $n \cdot [C_2/e] + \varepsilon \cdot *_{C_2} \longrightarrow *_{C_2}$ for all $n \in \mathbb{N}$ and $\varepsilon \in \{0,1\}$.

We now move to a more careful account of the background, motivation, and main results of this paper.

Background and motivation. Let $\mathcal{C}$ be a 1-category with finite products. Recall that a *commutative monoid in* $\mathcal{C}$ is the data

$$A \in \mathrm{Ob}(\mathcal{C}); \qquad \text{multiplication } \mu\colon A \times A \to A; \qquad \text{unit } \eta\colon * \to A,$$

subject to the usual unitality, associativity, and commutativity assumptions; more generally, if $\mathcal{C}$ is a symmetric monoidal 1-category, a *commutative algebra in* $\mathcal{C}$ is the data of

$$R \in \mathrm{Ob}(\mathcal{C}); \qquad \text{multiplication } \mu\colon R \otimes R \to R; \qquad \text{unit } \eta\colon 1 \to R,$$

satisfying analogous conditions. When $\mathcal{C} = \mathsf{Set}$, this recovers the traditional theory of commutative monoids, and when $\mathcal{C} = \mathrm{Mod}_k$ with the tensor product of k-modules, this recovers the traditional theory of commutative k-algebras. These have been the subject of a great deal of homotopy theory in three guises:

(1) We may define the $(2,1)$-category $\mathsf{Span}(\mathbb{F})$ to have objects the finite sets, morphisms from X to Y the spans of finite sets $X \leftarrow R \to Y$, 2-cells the isomorphisms of spans

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] \arrow[dd, "\sim"] & \\ X & & Y, \\ & R' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

and composition the pullback of spans

$$\begin{tikzcd} & & R_{XZ} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & R_{XY} \arrow[dl] \arrow[dr] & & R_{YZ} \arrow[dl] \arrow[dr] & \\ X & & Y & & Z. \end{tikzcd}$$

If $\mathcal{C}$ is an ∞-category, then we define the *∞-category of commutative monoids in $\mathcal{C}$* as the models of the associated Lawvere theory; that is, we define the product-preserving functor category

$$\mathrm{CMon}(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}), \mathcal{C}),$$

noting that products in $\mathrm{Span}(\mathbb{F})$ correspond with disjoint unions of finite sets. Indeed, if $\mathcal{C}$ is a 1-category and A a commutative monoid in $\mathcal{C}$, we flesh this out with the dictionary

$$\begin{aligned}
([2] = [2] \to [1]) &\qquad\longmapsto\qquad \mu\colon A^{\times 2} \to A;\\
(\varnothing = \varnothing \to [1]) &\qquad\longmapsto\qquad \eta\colon * \simeq A^{\times 0} \to A;\\
([1] \leftarrow [2] = [2]) &\qquad\longmapsto\qquad \Delta\colon A \to A^{\times 2}\\
([1] \leftarrow \varnothing = \varnothing) &\qquad\longmapsto\qquad !\colon A \to A^{\times 0} \simeq *.
\end{aligned}$$

Unitality, associativity, and commutativity are conveniently packaged by functoriality. This turns out to be equivalent to Graeme Segal's *special* Γ *spaces* [Seg74] when $\mathcal{C} = \mathcal{S}$, and for general $\mathcal{C}$, it recovers the anologously defined theory in $\mathcal{C}$ (c.f. [BHS22, Ex 3.1.6, Prop 3.1.16, Pf. of prop 5.2.14]).

(2) We say that an ∞-category is *semiadditive* if it has finite products and coproducts and for all finite sets S, the canonical natural transformation $\coprod_{s\in S}(-) \implies \prod_{s\in S}(-)$ is an equivalence. Then, the full subcategory $\mathrm{Pr}^{L,\oplus} \subset \mathrm{Pr}^{L}$ of *semiadditive presentable ∞-categories* possesses a localization functor $L_{\oplus}\colon \mathrm{Pr}^{L} \to \mathrm{Pr}^{L,\oplus}$, which we study.

(3) Let Op denote the ∞-category of operads.[1] Then, there is a terminal operad $\mathrm{Comm}^{\otimes} \simeq \mathbb{E}_{\infty}^{\otimes}$; given $\mathcal{C}$ a symmetric monoidal ∞-category, we may form the ∞-category of *commutative algebra objects*

$$\mathrm{CAlg}(\mathcal{C}) := \mathrm{Alg}_{\mathrm{Comm}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_{\infty}}(\mathcal{C}).$$

We study this and its specialization to the cartesian symmetric monoidal structure.

These perspectives each present the same ∞-category, i.e. [Cra11; GGN15] show that

$$\mathrm{CMon}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C}^{\times}) \simeq L_{\oplus}\mathcal{C}.$$

As a result, translating between these perspectives has proved invaluable; for instance, [GGN15] uses Perspectives 2 and 3 to construct an essentially unique symmetric monoidal structure on $\mathrm{CMon}(\mathcal{C})$ and [CHLL24a] uses Perspectives 1 and 3 to model commutative algebras in $\mathrm{CMon}(\mathcal{C})^{\otimes}$ as models for the Lawvere theory of *commutative semirings*.

Crucially, Perspective 3 may be used to construct homotopical lifts of the *Eckmann-Hilton argument*; for instance, in [HA], it is shown that for *any* reduced operad $\mathcal{O}^{\otimes}$, the forgetful functors

$$\mathrm{CAlg}\mathrm{Alg}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C}) \leftarrow \mathrm{Alg}_{\mathcal{O}}\mathrm{CAlg}^{\otimes}(\mathcal{C}),$$

are equivalences for the "pointwise" symmetric monoidal structure on algebras. Such a task may be accomplished by recognizing the far left and far right side each as algebras over the *Boardman-Vogt tensor product* $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes}$ and each arrow as pullback along the canonical map

$$\mathrm{Comm}^{\otimes} \simeq \mathrm{triv}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes} \xrightarrow{\mathrm{can}\otimes\mathrm{id}} \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes};$$

that the above maps are equivalences is then equivalent to the statement that the object $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^{\otimes} \in \mathrm{Op}$ is terminal, which is well-known.

This result is used ubiquitously to replace (lax) symmetric monoidal functors $\mathrm{Alg}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathcal{C}^{\otimes}$ with (lax) symmetric monodial endofunctors

$$\mathrm{CAlg}^{\otimes}(\mathcal{C}) \simeq \mathrm{CAlg}^{\otimes}\mathrm{Alg}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathrm{CAlg}^{\otimes}(\mathcal{C});$$

for instance, this underlies the symmetric monoidal structure on left-modules [HA] and the multiplicative structure on various invariants such as factorization homology [HA, Thm 5.5.3.2], THH, and TC [NS18, § IV.2].

This paper concerns the analog of Perspective 3 in the equivariant theory of algebra stemming from Hill-Hopkins-Ravanel's use of *norms of G-spectra* on the Kervarire invariant one problem, as well as the resulting theory of *indexed tensor products and (co)products* (c.f. [HH16]).

[1] This is unambiguous [HM23], but we will tend to model these as ∞-operads in the sense of [HA].

For the rest of this introduction, fix G a finite group. In G-equivariant homotopy theory, the point is replaced with elements of the *orbit category* $\mathcal{O}_G \subset \mathrm{Set}_G$, whose objects are homogeneous G-sets $[G/H]$; indeed, Elmendorf's theorem [Elm83] realizes G-spaces as coefficient systems $\mathcal{S}_G \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S})$.[2] In G-equivariant higher category theory, ∞-categories are thus replaced with G-∞-categories

$$\mathrm{Cat}_G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathrm{Cat}).$$

In G-equivariant higher algebra, following Perspective 1, we may form the effective Burnside 2-category $\mathrm{Span}(\mathbb{F}_G)$ whose objects are finite G-sets, whose morphisms are spans, whose 2-cells are isomorphisms of spans, and whose composition is pullback; the following central definition is the heart of this subject.

Definition. The *∞-category of G-commutative monoids in $\mathcal{C}$* is the product-preserving functor ∞-category

$$\mathrm{CMon}_G(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_G), \mathcal{C});$$

the *∞-category of small G-symmetric monoidal ∞-categories* is

$$\mathrm{Cat}_G^{\otimes} := \mathrm{CMon}_G(\mathrm{Cat}). \qquad \triangleleft$$

These are a homotopical lift of Dress' *Mackey functors* [Dre71] (c.f. [Lin76]). Indeed, given $\mathcal{C}^{\otimes} \in \mathrm{Cat}_G^{\otimes}$ a G-symmetric monoidal ∞-category, the product-preserving functor

$$\iota_H \colon \mathrm{Span}(\mathbb{F}) \xrightarrow{* \mapsto G/H} \mathrm{Span}(\mathbb{F}_G)$$

constructs a symmetric monoidal ∞-category $\mathcal{C}_H^{\otimes} := \iota_H^* \mathcal{C}^{\otimes}$ whose underlying ∞-category $\mathcal{C}_H$ is the *value* of $\mathcal{C}^{\otimes}$ on the orbit $[G/H]$. For all subgroups $K \subset H \subset G$, the covariant and contravariant functoriality of $\mathcal{C}^{\otimes}$ then yield symmetric monoidal *restriction* and *norm* functors

$$\mathrm{Res}_K^H \colon \mathcal{C}_H^{\otimes} \to \mathcal{C}_K^{\otimes},$$
$$N_K^H \colon \mathcal{C}_K^{\otimes} \to \mathcal{C}_H^{\otimes},$$

which satisfy a form of Mackey's *double coset formula*.

Example. By [BH21; CHLL24b], there exists a unique presentably G-symmetric monoidal ∞-category $\underline{\mathrm{Sp}}_G^{\otimes}$ such that:

- the H-value of $\underline{\mathrm{Sp}}_G^{\otimes}$ is the symmetric monoidal ∞-category $\left(\underline{\mathrm{Sp}}_G^{\otimes}\right)_H \simeq \mathrm{Sp}_H^{\otimes}$ of *genuine H-spectra* under the usual tensor product;
- the restriction functors $\mathrm{Res}_K^H \colon \mathrm{Sp}_H^{\otimes} \to \mathrm{Sp}_K^{\otimes}$ are the usual restriction functors; and
- the norm functors $N_K^H \colon \mathrm{Sp}_K^{\otimes} \to \mathrm{Sp}_H^{\otimes}$ are the *HHR norm* of [HHR16].

In fact, this symmetric monoidal structure is completely determined by its unit object $\mathbb{S}_G \in \mathrm{Sp}_G^{\otimes}$. $\triangleleft$

Fix $\mathcal{C}^{\otimes} \in \mathrm{Cat}_G^{\otimes}$. If $H \subset G$ is a subgroup and $S \in \mathbb{F}_H$ a finite H-set, we may form the induced G-set $\mathrm{Ind}_H^G S \to [G/H]$, and the covariant and contravariant functoriality then yield an *S-indexed tensor product* and *S-indexed diagonal*

$$\bigotimes_K^S \colon \mathcal{C}_S \to \mathcal{C}_H, \qquad\qquad \Delta^S \colon \mathcal{C}_H \to \mathcal{C}_S.$$

where $\mathcal{C}_S := \prod_{[H/K] \in \mathrm{Orb}(S)} \mathcal{C}_K$. Note that N_H^K is the $[H/K]$-indexed tensor product and Res_K^H the $[H/K]$-indexed diagonal. As explained in [Ste25a], functoriality applied to the "orbit collapse" factorization $\mathrm{Ind}_H^G S \to \coprod_{[H/K] \in \mathrm{Orb}(S)} [G/H] \to [G/H]$ yields equivalences

$$\bigotimes_K^S X_K \simeq \bigotimes_{[H/K] \in \mathrm{Orb}(S)} N_K^H X_K, \qquad\qquad \Delta^S(X) = \left(\mathrm{Res}_K^H X\right)_{[H/K] \in \mathrm{Orb}(S)},$$

[2] Maps $[G/K] \to [G/H]$ may equivalently be presented as elements of g such that $gKg^{-1} \subset H$, modulo K; see e.g. [Die09] for details.

so we may often reduce arguments about S-fold tensor products to to binary tensor products and norms. Similarly, we define the *S-fold tensor power*

$$X_H^{\otimes S} := \bigotimes_K^S \left(\Delta^S X_H\right) \simeq \bigotimes_K^S \operatorname{Res}_H^K X_H \simeq \bigotimes_{[H/K]\in \operatorname{Orb}(S)} N_K^H \operatorname{Res}_K^H X_H.$$

If it exists, the pointwise left-adjoint to Δ^S is the *indexed coproduct*

$$\coprod_K^S X_K \simeq \coprod_{[H/K]\in \operatorname{Orb}(S)} \operatorname{Ind}_K^H S,$$

where Ind_K^H is the left adjoint to the restriction map $\mathcal{C}_H \to \mathcal{C}_K$. The *indexed products* are defined analogously.

Given $H \subset G$ a subgroup, we say that $\mathcal{C}$ is *H-pointed* if $\mathcal{C}_K$ is pointed for all $K \subset H$. Given $S \in \mathbb{F}_H$, we say that S is *$\mathcal{C}$-ambidextrous* if $\mathcal{C}$ is H-pointed, $\mathcal{C}$ admits S-indexed products and coproducts, and *norm* natural transformation

$$\coprod_K^S (-) \Longrightarrow \prod_K^S (-) \colon \mathcal{C}_S \to \mathcal{C}_H$$

of [Nar16, § 5] is an equivalence (see [Ste25a]). We say that $\mathcal{C}$ is *G-semiadditive* if S is $\mathcal{C}$-ambidextrous for all $S \in \mathbb{F}_H$ and $H \subset G$. More generally, if $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_G$ is a weak indexing system corresponding with the weak indexing category I (see [Ste24] or our review in Section 1.2), we say that $\mathcal{C}$ is *I-semiadditive* if S is $\mathcal{C}$-ambidextrous whenever $S \in \mathbb{F}_{I,H}$.

In this level of generality, Perspectives 1 and 2 are known to present equivalent ∞-categories of I-commutative monoids; indeed, the *semiadditive closure* theorem of [CLL24, Thm B] demonstrates that $\operatorname{Pr}_G^{L,I-\oplus} \subset \operatorname{Pr}_G^L$ is a smashing localization implemented by

$$L_{I-\oplus}(\mathcal{C}) \simeq \underline{\operatorname{CMon}}_I(\mathcal{C}) := \underline{\operatorname{Fun}}_G^\times(\operatorname{Span}_I(\underline{\mathbb{F}}_G), \mathcal{C}),$$

and in particular, when $\mathcal{C}$ is a G-∞-category of coefficient systems

$$\underline{\operatorname{Coeff}}^G(\mathcal{D})_H := \operatorname{Fun}(\mathcal{O}_H^{\mathrm{op}}, \mathcal{D}),$$

[CLL24, Thm C] yields the formula

$$\underline{\operatorname{CMon}}_I(\underline{\operatorname{Coeff}}^G(\mathcal{D}))_H \simeq \operatorname{Fun}^\times(\operatorname{Span}_I(\mathbb{F}_H), \mathcal{D}),$$

where $\operatorname{Span}_I(\mathbb{F}_H) \subset \operatorname{Span}(\mathbb{F}_H)$ is the wide subcategory of spans whose forward maps lie in the restriction of I to $\mathbb{F}_H$. Thus, we set the notation $\operatorname{CMon}_I(\mathcal{D}) := \underline{\operatorname{CMon}}_I(\underline{\operatorname{Coeff}}^G(\mathcal{D}))_G \simeq \operatorname{Fun}^\times(\operatorname{Span}_I(\mathbb{F}_G), \mathcal{D})$ and make the following definition.

Definition. For I is a weak indexing category, the *∞-category of small I-symmetric monoidal ∞-categories* is

$$\operatorname{Cat}_I^\otimes := \operatorname{Fun}^\times(\operatorname{Span}_I(\mathbb{F}_G), \operatorname{Cat}). \qquad \triangleleft$$

Following through on Perspective 3, algebraic objects $X_\bullet$ in a G-symmetric monoidal ∞-category should possess collections of *S-ary operations* $X_H^{\otimes S} \to X_H$ subject to various conditions, controlled by a theory of *genuine equivariant operads*; we use Nardin-Shah's ∞-category Op_G (see [Ste25a]), whose objects we call *G-operads*. There, given $\mathcal{O}^\otimes \in \operatorname{Op}_G$ a G-operad, $K \subset H \subset G$ a pair of subgroups, $S \in \mathbb{F}_H$ a finite H-set, and T_i a finite K_i-set for all orbits $[H/K_i] \subset S$, we constructed in [Ste25a] a *space of S-ary operations* $\mathcal{O}(S)$, *operadic composition maps*

$$\gamma \colon \mathcal{O}(S) \otimes \bigotimes_{[H/K_i]\in \operatorname{Orb}(S)} \mathcal{O}(T_i) \to \mathcal{O}\left(\coprod_{[H/K_i]\in \operatorname{Orb}(S)} \operatorname{Ind}_{K_i}^H T_i\right), \tag{1}$$

operadic restriction maps

$$\operatorname{Res} \colon \mathcal{O}(S) \to \mathcal{O}\left(\operatorname{Res}_K^H S\right), \tag{2}$$

and *equivariant symmetric group action*

$$\rho \colon \operatorname{Aut}_H(S) \times \mathcal{O}(S) \to \mathcal{O}(S). \tag{3}$$

We made the following definitions, of which the reader may focus on *having one color* and *almost-unitality*.

Definition. A G-operad $\mathcal{O}^{\otimes}$

(a) *has at most one color* if $\mathcal{O}(*_H) \in \{\varnothing, *\}$ for all $H \subset G$,
(b) *has at least one color* if $\mathcal{O}(*_H) \neq \varnothing$ for all $H \subset G$,
(c) *has one color* if it has at most one color and at least one color,
(d) *is almost essentially unital* if $\mathcal{O}(\varnothing_H) = *$ whenever there exists some $S \in \mathbb{F}_H - \{*_H\}$ with $\mathcal{O}(S) \neq \varnothing$,
(e) *is almost-unital* if it's almost essentially unital and has at least one color,
(f) *is almost essentially reduced* if it is almost essentially unital and has at most color,
(g) *is almost-reduced* if it is almost-unital and has one color, and
(h) *is a G-d-operad* if $\mathcal{O}(S)$ is $(d-1)$-truncated for all $S \in \mathbb{F}_H$.[3]

The corresponding full subcategories are

$$\mathrm{Op}_G^{\geq \mathrm{oc}}, \mathrm{Op}_G^{\leq \mathrm{oc}}, \mathrm{Op}_G^{\mathrm{oc}}, \mathrm{Op}_G^{aE\mathrm{uni}}, \mathrm{Op}_G^{a\mathrm{uni}}, \mathrm{Op}_G^{aE\,\mathrm{red}}, \mathrm{Op}_G^{a\,\mathrm{red}}, \mathrm{Op}_{G,d} \subset \mathrm{Op}_G. \quad \triangleleft$$

We showed in [Ste25a] that Eqs. (2) and (3) lift to a monadic functor $\mathrm{Op}_G^{\mathrm{oc}} \to \mathrm{Fun}(\mathrm{Tot}\Sigma_G, \mathcal{S})$, i.e. one color G-operads are monadic over G-symmetric sequences.

When $\mathcal{O}^{\otimes}$ has one color, an $\mathcal{O}$-algebra in the G-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ can intuitively be viewed as a tuple $\left(X_H \in \mathcal{C}_H^{BW_G(H)}\right)_{G/H \in \mathcal{O}_G}$ satisfying $X_K \simeq \mathrm{Res}_K^H X_H$ for all $K \subset H \subset G$, together with $\mathcal{O}(S)$-actions

$$\mu_S : \mathcal{O}(S) \to \mathrm{Map}_{\mathcal{C}_H}\left(X_H^{\otimes S}, X_H\right) \tag{4}$$

for all $H \subset G$ and $S \in \mathbb{F}_H$, homotopy-coherently compatible with Eqs. (1) to (3).[4]

Example. There exists a terminal G-operad $\mathrm{Comm}_G^{\otimes}$, which is characterized up to (unique) equivalence by the property that $\mathrm{Comm}_G(S)$ is contractible for all $S \in \mathbb{F}_H$; its algebras are endowed with contractible spaces of maps $X_H^{\otimes S} \to X_H$ for all $S \in \mathbb{F}_H$, as well as coherent homotopies witnessing their compatibility. We call these *G-commutative algberas.*

On one hand, we saw in [Ste25a] that Comm_G-algebras present a homotopical lift of Hill-Hopkins' *G-commutative monoids* [HH16, § 4], though we prefer to reserve this name for the Cartesian case, following the convention of [HA]. On the other hand, our model agrees with that of [CHLL24b], so the recent *homotopical Tambara functor theorem* of Cnossen, Lenz, and Linskens [CHLL24b, Thm B] presents G-commutative algebra objects in $\underline{\mathrm{Sp}}_G^{\otimes}$ as a form of *homotopical G-Tambara functors.* $\triangleleft$

Example. Let V be a real orthogonal G-representation; then, there is a *little V-disks G-operad* $\mathbb{E}_V^{\otimes}$ whose structure spaces are *spaces of equivariant configurations*:

$$\mathbb{E}_V(S) \simeq \mathrm{Conf}_S^H(V)$$

(see [Hor19]). This is modelled by the *Steiner graph G-operad*, so e.g. pointed G-spaces of the form $X = \Omega^V Y := \mathrm{Map}_*\left(S^V, Y\right)$ lift to $\mathbb{E}_V$-spaces by composition of loops [GM11]; many $\mathbb{E}_V$-algebras will be able to be constructed in $\underline{\mathrm{Sp}}_G^{\otimes}$ as equivariant Thom spectra of V-fold loop spaces. $\triangleleft$

In this paper, we are primarily concerned with homotopy coherently interchanging $\mathcal{O}-$ and $\mathcal{P}$-algebra structures, which are implemented as algebras over *Boardman-Vogt tensor product* $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ of [Ste25a]. Our main theorem will concern the following.

Example. Given $I \subset \mathbb{F}_G$ a weak indexing category, in [Ste25a] we constructed a G-operad $\mathcal{N}_{I\infty}^{\otimes}$ which is characterized by its structure spaces

$$\mathcal{N}_{I\infty}(S) \simeq \begin{cases} * & S \in \underline{\mathbb{F}}_I \\ \varnothing & S \notin \underline{\mathbb{F}}_I \end{cases} \tag{5}$$

This recovers the notion from [BH15] when I is an indexing category, in which case they are called *$\mathcal{N}_\infty$-operads.* $\triangleleft$

[3] A space is -1-*truncated* if it is either empty or contractible; for all $k \geq 0$, a space X is *truncated* if it is a disjoint union of connected spaces $(X_\alpha)_{\alpha \in A}$ such that, for each $\ell > k$ and $\alpha \in A$, the ℓth homotopy group $\pi_\ell(X_\alpha)$ is trivial.

[4] Here, $W_G(H) = N_G(H)/H$ is the *Weyl group* of $H \subset G$, i.e. the automorphism group of the homogeneous G-set $[G/H]$.

For instance, we verify in Corollary 3.14 that the condition $V \oplus V \simeq V$ for an orthogonal G-representation V implies that $\mathbb{E}_V$ is a weak $\mathcal{N}_\infty$-operad, which is an $\mathcal{N}_\infty$-operad precisely when V^G is positive-dimensional. For instance, $\mathbb{E}_\infty$ is presents the initial $\mathcal{N}_\infty$-operad, and its algebras are *naive* commutative algebra objects [Ste25a]:

$$\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C}_G).$$

If I is an *indexing* category, the structure of an $\mathcal{N}_{I\infty}$-ring spectrum is intuitively viewed as commutative ring structures on each spectrum X_H, connected by multiplicative I-indexed norms, suitably compatible with the restriction and (additive) transfer structures inherent to G-spectra. We refer to $\mathcal{N}_{I\infty}$-algebras in general as *I-commutative algebras* and $\mathcal{N}_{I\infty}$-ring spectra as *I-commutative ring spectra*.

I-symmetric monoidal ∞-categories have underlying I-operads; for $\mathcal{C} \in \mathrm{Cat}_I^{\otimes}$, we define the ∞-category of *I-commutative algebras* in $\mathcal{C}$ as

$$\mathrm{CAlg}_I(\mathcal{C}) \coloneqq \mathrm{Alg}_{\mathcal{N}_{I\infty}}(\mathcal{C}).$$

The theory of arity support [Ste25a] constructs a unique pairing $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \to \mathcal{N}_{I\vee J\infty}$, where $I \vee J$ is the join in the poset of indexing categories; intuitively, given an algebra with $I \vee J$-indexed norms, we may separate these into I-indexed norms and J-indexed norms which satisfy an applicable interchange law. Moreover, the transfer system for $I \vee J$ consists of those inclusions $K \subset H$ which can be factored as

$$K \subset K_{I1} \subset K_{J1} \subset K_{I2} \subset \cdots \subset K_{Jn} \subset H$$

where $K_{I\ell} \subset K_{J\ell}$ is in I and $K_{J\ell} \subset K_{I(\ell+1)}$ is in J [Rub21, Prop 3.1]; intuition would then suggest that we may combine interchanging I- and J-commutative algebra structures to construct an $I \vee J$-commutative algebra structure. Thus Blumberg and Hill conjectured that there is an equivalence $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{I\vee J\infty}^{\otimes}$ [BH15, Conj 6.27]; the main theorem of this paper confirms their conjecture in Op_G.

Summary of main results. We begin by characterizing the *(co)cartesian* I-symmetric monoidal structure.

Theorem A. *When I is almost-unital, there are fully faithful embeddings $(-)^{I-\sqcup}$ and $(-)^{I-\times}$ making the following commute:*

$$\begin{tikzcd}
\mathrm{Cat}_I^{\sqcup} \arrow[r, hook, "(-)^{I-\sqcup}"] \arrow[dr, "U"'] & \mathrm{Cat}_I^{\otimes} \arrow[d, "U"] & \mathrm{Cat}_I^{\times} \arrow[l, hook', "(-)^{I-\times}"'] \arrow[dl, "U"] \\
& \mathrm{Cat}_G &
\end{tikzcd}$$

The image of $(-)^{I-\sqcup}$ is spanned by the I-symmetric monoidal ∞-categories whose I-indexed tensor products are indexed coproducts and the image of $(-)^{I-\times}$ is spanned by those whose I-indexed tensor products are indexed products.

Remark. After this introduction, we replace $\mathcal{O}_G$ with an atomic orbital ∞-category $\mathcal{T}$; we prove Theorem A as well as the other theorems in this introduction in this setting, greatly generalizing the stated results at the cost of ease of exposition. ◁

We refer to I-symmetric monoidal ∞-categories of the form $\mathcal{C}^{I-\times}$ as *cartesian*, and $\mathcal{C}^{I-\sqcup}$ *cocartesian*. In Corollary 1.39, we go on to characterize the *∞-category of I-commutative monoids* in $\mathcal{C}$ a complete ∞-category as an ∞-category of I-commutative algebras, integrating Perspectives 1 to 3:

$$\mathrm{CMon}_I(\mathcal{C}) \simeq \mathrm{CAlg}_I(\mathcal{C}^{I-\times}).$$

In Section 1.5 we verify that $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is cartesian when $\mathcal{C}$ is. Following this, in Section 2.1 we show that I-indexed tensor products in $\underline{\mathrm{CAlg}}_I^{\otimes}\mathcal{C}$ are indexed coproducts (i.e. its underlying I-symmetric monoidal ∞-category is *cocartesian*) and that this completely characterizes $\mathcal{N}_{I\infty}^{\otimes}$. The heart of our strategy will use the explicit monadic description of [Ste25a] to reduce this to the case $\mathcal{C}^{\otimes} \simeq \underline{\mathcal{S}}_G^{G-\times}$ is the *cartesian G-symmetric monoidal ∞-category of G-spaces*; in this case, we may easily see that the cartesian I-symmetric monoidal ∞-category $\underline{\mathrm{CAlg}}_I^{\otimes}(\underline{\mathcal{S}}_G^{G-\times}) \simeq \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_G)^{I-\times}$ is cocartesian, as its underlying G-∞-category is I-semiadditive by [CLL24, Thm B-C]. We conclude the following.

Theorem B. *Let $\mathcal{O}^{\otimes}$ be an almost essentially reduced G-operad. Then, the following conditions are equivalent.*

(a) The G-∞-category $\underline{\mathrm{Alg}}_{\mathcal{O}}\underline{\mathcal{S}}_G$ is $A\mathcal{O}$-semiadditive.

(b) The unique map $\mathcal{O}^\otimes \to \mathcal{N}^\otimes_{A\mathcal{O}\infty}$ is an equivalence.

Furthermore, for all almost essentially unital weak indexing categories I and I-symmetric monoidal ∞-categories $\mathcal{C}^\otimes$, the I-symmetric monoidal ∞-category $\underline{\mathrm{CAlg}}^\otimes_I \mathcal{C}$ is cocartesian.

For the following theorem, we say that an I-operad $\mathcal{O}^\otimes$ is *reduced* if, for all $S \in \mathbb{F}_H$ which is empty or contractible, the unique map $\mathcal{O}^\otimes \to \mathcal{N}_{I\infty}$ induces an equivalence

$$\mathcal{O}(S) \simeq \mathcal{N}_{I\infty}(S)$$

(c.f. Eq. (5)). We completely characterize algebras in cocartesian I-symmetric monoidal categories in Theorem 2.2, and from this Theorem B entirely characterizes the tensor products of reduced I-operads with $\mathcal{N}^\otimes_{I\infty}$ in the almost essentially reduced setting.

Corollary C. *$\mathcal{N}^\otimes_{I\infty} \overset{BV}{\otimes} \mathcal{N}^\otimes_{I\infty}$ is a weak $\mathcal{N}_\infty$-operad if and only if I is almost essentially unital. In this case, if $\mathcal{O}^\otimes$ is a reduced I-operad, then the unique map*

$$\mathcal{O}^\otimes \otimes \mathcal{N}^\otimes_{I\infty} \to \mathcal{N}^\otimes_{I\infty}$$

is an equivalence.

In particular, this implies that whenever I is almost unital, there exists a map $\mathrm{triv}^\otimes_G \to \mathcal{N}^\otimes_{I\infty}$ witnessing $\mathcal{N}^\otimes_{I\infty}$ as an idempotent algebra in Op_G. We verified in [Ste25a] that $\mathrm{Env}\colon \mathrm{Op}_G \to \mathrm{Cat}^\otimes_G$ is compatible with the unit and tensor products under the mode symmetric monoidal structure on $\mathrm{Cat}^\otimes_G$; this yields a $\otimes^{\mathrm{Mode}}$-idempotent algebra structure on $\underline{\mathbb{F}}_G^{G-\sqcup} = \mathrm{Env}(\mathrm{Comm}_G)$, and hence a symmetric monoidal structure on $\underline{\mathrm{Cat}}^\otimes_{G,/\underline{\mathbb{F}}_G^{G-\sqcup}}$. We acquire an equivariantization of a modification of [BS24a].

Corollary D. *There exists a unique symmetric monoidal structure $\underline{\mathrm{Op}}^\otimes_G$ on $\underline{\mathrm{Op}}_G$ attaining a (necessarily unique) symmetric monoidal structure on the fully faithful G-functor*

$$\mathrm{Env}^{/\underline{\mathbb{F}}_G^{G-\sqcup}} \colon \underline{\mathrm{Op}}^\otimes_G \to \underline{\mathrm{Cat}}^{\otimes-\mathrm{mode}}_{G,/\underline{\mathbb{F}}_G^{G-\sqcup}}$$

of [BHS22; NS22]; the tensor product of this structure is $\overset{BV}{\otimes}$ and the H-unit is $\mathrm{triv}^\otimes_H$.

Idempotent algebras correspond with smashing localizations, i.e. they classify $\otimes$-absorptive properties [HA, § 4.8.2]; in view of Corollary C, when $I \le J$ are almost unital, we would like to characterize the smashing localization that $\mathcal{N}^\otimes_{I\infty}$ induces on $\mathrm{Op}^{\mathrm{red}}_J$ using the adjunction $- \overset{\mathrm{BV}}{\otimes} \mathcal{O}^\otimes \dashv \underline{\mathrm{Alg}}^\otimes_{\mathcal{O}}(-)$. Namely, in [Ste25a], we constructed a right adjoint to the natural inclusion $E_I^J : \mathrm{Op}_I \to \mathrm{Op}_J$, called the *$I$-borelification* Bor_I^J; in Theorem 2.10, we conclude that the smashing localization corresponding with $\mathcal{N}^\otimes_{I\infty} \in \mathrm{Op}^{\mathrm{red}}_J$ classifies the property of *having commutative Borel I-type*:

$$\begin{aligned}
\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}^\otimes_{I\infty} \simeq \mathcal{O}^\otimes \quad &\iff \quad \mathrm{Bor}_I^J \mathcal{O}^\otimes \simeq \mathcal{N}^\otimes_{I\infty}, \\
&\iff \quad \forall \mathcal{C}^\otimes \in \mathrm{Cat}^\otimes_J, \quad \forall S \in \mathbb{F}_{I,V}, \quad \coprod_U^S \simeq \bigotimes_U^S : \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})_S \to \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})_V, \\
&\iff \quad \underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_G) \text{ is } I\text{-semiadditive.}
\end{aligned}$$

Tensor products of idempotents algebras are themselves idempotent algebras, corresponding with an intersection of the associated smashing localizations [CSY20, Prop 5.1.8]. This affirms Blumberg and Hill's conjecture.

Theorem E. *The functor $\mathcal{N}^\otimes_{(-)\infty} : \mathrm{wIndex}_G \to \mathrm{Op}_G$ restricts to a fully faithful symmetric monoidal G-right adjoint*

$$\begin{tikzcd}
\underline{\mathrm{wIndex}}^{aE\mathrm{uni}}_G \arrow[r, hook, bend right, "\mathcal{N}^\otimes_{(-)\infty}"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Op}}^{aE\mathrm{uni}}_G \arrow[l, bend right, "A"']
\end{tikzcd}$$

Furthermore, the resulting tensor product of weak $\mathcal{N}_\infty$-operads is computed by the Borelified join

$$\mathcal{N}_I^{\otimes} \overset{BV}{\otimes} \mathcal{N}_J^{\otimes} \simeq \mathcal{N}^{\otimes}_{\mathrm{Bor}^G_{c(I\cap J)}(I \vee J)}.$$

Hence when I, J are unital weak indexing categories and $\mathcal{C}^{\otimes}$ is an $I \vee J$-symmetric monoidal ∞-category, there is a canonical equivalence of $I \vee J$-symmetric monoidal ∞-categories

$$\underline{\mathrm{CAlg}}_I^{\otimes}\underline{\mathrm{CAlg}}_J^{\otimes}(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_{I\vee J}^{\otimes}(\mathcal{C}).$$

For instance, using [CHLL24b, Thm 4.3.6] to identify I-Tambara functors in an ∞-category $\mathcal{C}$ with I-commutative algebras in Mackey functors, this confirms that $I \vee J$-Tambara functors are equivalent to I-commutative algebras in J-Tambara functors.

Remark. The reader interesting in computing tensor products of G-operads may benefit from reading the combinatorial characterization of joins of weak indexing systems in terms of *closures* in [Ste24]; there, we prove that the join of weak indexing systems $\mathbb{F}_I \vee \mathbb{F}_J$ is computed by closing the union $\mathbb{F}_I \cup \mathbb{F}_J$ under iterated I and J-indexed coproducts. ◁

Along the way, we acquire various corollaries in equivariant higher algebra. For instance, in Section 3.4, we easily acquire an infinitary case of an equivariant homotopical lift of Dunn's additivity theorem [Dun88]. In Section 3.5 we use this to define iterated Real topological Hochschild homology for $\mathbb{E}_V$-algebras whenever V admits an $\infty\sigma$ summand and express them as $(S^\sigma)^n$-indexed colimits. Additionally, in Section 3.3 we use Corollary C to construct a natural I-symmetric monoidal structure on right modules over an $\mathcal{N}_{I\infty}$-algebra whenever I is an indexing category.

Notation and conventions. We assume that the reader is familiar with the technology of higher category theory and higher algebra as developed in [HTT] and [HA, § 2-3], though we encourage the reader to engage with such technologies via a "big picture" perspective akin to that of [Gep19, § 1-2] and [Hau23, § 1-3]. We additionally assume that the reader is familiar with *parameterized* higher category theory over an ∞-category as developed in [Sha22; Sha23]; the material reviewed in the prequel [Ste25a, § 1] will be enough.

Throughout this paper, we frequently describe conditions which may be satisfied by objects parameterized over some ∞-category $\mathcal{T}$. If P is a property, in the instance where there exists Borelification adjunctions

$$E_{\mathcal{F}}^{\mathcal{T}} : \mathcal{C}_{\mathcal{F}} \rightleftarrows \mathcal{C}_{\mathcal{T}} : \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}}$$

along family inclusions $\mathcal{F} \subset \mathcal{T}$, we say that $X \in \mathcal{C}_{\mathcal{T}}$ is E-P when there exists some $\overline{X} \in \mathcal{C}_{\mathcal{F}}$ which is P such that $X \simeq E_{\mathcal{F}}^{\mathcal{T}}\overline{X}$. We say that X is *almost* E-P (or aE-P) if $\mathcal{C}_{\mathcal{F}}$ has a terminal object $*_{\mathcal{F}}$ for all $\mathcal{F}$, and there is a pushout expression

$$X \simeq *_{\mathcal{F}'} \sqcup_{*_{\mathcal{F}}} *_{\mathcal{F}'}$$

for some $\mathcal{F}' \subset \mathcal{F}$; we say that X is *almost* P (or a-P) if it's almost E-P and $\mathcal{F}' = \mathcal{T}$ in the above.

Acknowledgements. I would like to thank Jeremy Hahn for suggesting the problem of *constructing equivariant multiplications on $BP_{\mathbb{R}}$*, whose (ongoing) work necessitated many of the results on equivariant Boardman-Vogt tensor products developed in this paper. I'm indebted to Maxime Ramzi for disillusioning me to a fatally flawed strategy on work related to this, leading me to the drawing board; it wasn't so clear at the time, but it was in subsequent conversation with him that the main idea of this paper emerged. I am additionally grateful to Piotr Pstrągowski, who pointed out a mistake in my early strategy in this paper, leading to the condition of *almost essential unitality* on the main theorem.

Additionally, I would like to thank Andy Senger, Clark Barwick, and Dhilan Lahoti, with whom I had many enlightening (to me) conversations about the topic of this paper. Of course, none of this work would be possible without the help of my advisor, Mike Hopkins, who I'd like to thank for many helpful conversations.

While developing this material, the author was supported by the NSF Grant No. DGE 2140743.

1. I-symmetric monoidal categories and I-operads

We begin in Section 1.1 by recalling results of [CLL24; Nar16; NS22; Ste24; Ste25a] concerning the theory of I-commutative monoids and I-symmetric monoidal ∞-categories. Moving on, in Section 1.2 we recall results of [NS22; Ste25a] concerning $\mathcal{T}$-operads; in either case, all reviewed information was used in the preceding article [Ste25a]. We finish the section in **??** with a tour through the gamut of existing examples of I-symmetric monoidal ∞-categories, including summarizing the results of Appendix A in the cartesian case.

1.1. **Recollections on I-commutative monoids and I-symmetric monoidal ∞-categories.**

1.1.1. *Weak indexing systems and semiadditivity.* We will use the following machinery of [Ste24].

Definition 1.1. A *$\mathcal{T}$-weak indexing category* is a subcategory $I \subset \mathbb{F}_{\mathcal{T}}$ satisfying the following conditions:

(IC-a) (restrictions) I is stable under arbitrary pullbacks in $\underline{\mathbb{F}}_{\mathcal{T}}$;
(IC-b) (segal condition) $T \to S$ and $T' \to S$ are both in I if and only if $T \sqcup T' \to S \sqcup S'$ is in I; and
(IC-c) ($\Sigma_{\mathcal{T}}$-action) if $S \in I$, then all automorphisms of S are in I.

A *$\mathcal{T}$-weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ satisfying the following conditions:

(IS-a) whenever the V-value $\mathbb{F}_{I,V} := (\underline{\mathbb{F}}_I)_V$ is nonempty, we have $*_V \in \underline{\mathbb{F}}_{I,V}$; and
(IS-b) $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is closed under $\underline{\mathbb{F}}_I$-indexed coproducts. ◁

We say that a $\mathcal{T}$-weak indexing system $\underline{\mathbb{F}}_I$:

(i) has one color if for all $V \in \mathcal{T}$, we have $\mathbb{F}_{I,V} \neq \varnothing$;
(ii) is almost essentially unital (or aE-unital) if $\underline{\mathbb{F}}_I$ has a non-contractible V-set, $\varnothing_V \in \mathbb{F}_{I,V}$;
(iii) is unital if $\varnothing_V \in \mathbb{F}_{I,V}$ for all $V \in \mathcal{T}$;
(iv) is an *indexing system* if the subcategory $\underline{\mathbb{F}}_{I,V} \subset \mathbb{F}_V$ is closed under finite coproducts for all $V \in \mathcal{T}$.

These occupy embedded sub-posets

$$\mathrm{Index}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}.$$

We denote the *I-admissible V-sets* by

$$\mathbb{F}_{I,V} := \left\{ S \in \mathbb{F}_{I,V} \mid \mathrm{Ind}_V^{\mathcal{T}} S \to V \in I \right\} \subset \mathbb{F}_V;$$

these assemble into a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$. In [Ste24, Thm A] we prove the following and express the conditions of Definition 1.1 in the language of weak indexing categories.

Proposition 1.2. *The assignment $I \mapsto \underline{\mathbb{F}}_I$ implements an equivalence between the posets of $\mathcal{T}$-weak indexing categories and $\mathcal{T}$-weak indexing systems.*

One reason for this is *indexed semiadditivity.* A $\mathcal{T}$-∞-category is said to be *V-pointed* if $\mathcal{C}_U$ is a pointed ∞-category for all $U \to V$. When $S \in \mathbb{F}_V$ is a finite V-set and $\mathcal{C}$ a V-pointed $\mathcal{T}$-∞-category which admits S-indexed products and coproducts, Nardin [Nar16] defined a *norm natural transformation*

$$\mathrm{Nm}_s \colon \coprod_U^S(-) \implies \prod_U^S(-).$$

We say that S is *$\mathcal{C}$-ambidextrous* if $\mathcal{C}$ is V-pointed and Nm_S is an equivalence; given $\underline{\mathbb{F}}_I$ a weak indexing system, we say that $\mathcal{C}$ is *I-semiadditive* if S is $\mathcal{C}$-ambidextrous for all $S \in \underline{\mathbb{F}}_I$. In [Ste25a] we proved that the collection of $\mathcal{C}$-ambidextrous finite V-sets form a weak indexing system and concluded the following important observation.

Proposition 1.3 ([Ste25a])**.** *Let $\vee$ denote the join in* $\mathrm{wIndexCat}_{\mathcal{T}}$*. Then, $\mathcal{C}$ is I-semiadditive and J-semiadditive if and only if $\mathcal{C}$ is $I \vee J$-semiadditive.*

1.1.2. *I-commutative monoids.* In [Bar14], the notion of *adequate* triple was defined, consisting of triples $(\mathcal{C}, \mathcal{C}_b, \mathcal{C}_f)$ with $\mathcal{C}_f, \mathcal{C}_B \subset \mathcal{C}$ a pair of core-preserving wide subcategories satisfying pullback-stability and distributivity conditions; if I is a weak indexing category, then $(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I)$ is an adequate triple.

Adequate triples form a full subcategory $\mathrm{Trip}^{\mathrm{Adeq}} \subset \mathrm{Fun}(\bullet \to \bullet \leftarrow \bullet, \mathrm{Cat})$; [Bar14] constructed a functor

$$\mathrm{Span}_{-,-}(-) \colon \mathrm{Trip}^{\mathrm{Adeq}} \to \mathrm{Cat},$$

called the *effective Burnside category.* In the case that $c(I)$ is a 1-category (e.g. $\mathcal{T}$ has a terminal object), $\mathbb{F}_{c(I)}$ is a 1-category, so the effective Burnside category

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) := \mathrm{Span}_{\mathbb{F}_{c(I)}, I}(\mathbb{F}_{c(I)})$$

is a $(2,1)$-category with objects agreeing with $\mathbb{F}_{c(I)}$, morphisms the spans $X \leftarrow R \xrightarrow{f} Y$ with f in I, 2-cells the isomorphisms of spans, and composition of morphisms computed by pullbacks in $\mathbb{F}_{c(I)}$ (which are guaranteed to be morphisms in $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ by pullback-stability of I).

Much of the technical work of [Bar14; BGS20] has been extended by [HHLN23], so we generally refer the reader there. At any rate, we recall this in order to define *homotopical incomplete Mackey functors for I*, which we call *I-commutative monoids.*

Definition 1.4. If $\mathcal{C}$ is an ∞-category with finite products, then an *I-commutative monoid in $\mathcal{C}$* is a product-preserving functor $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathcal{C}$ More generally, if $\mathcal{D}$ is a $\mathcal{T}$-∞-category with I-indexed products, then an *I-commutative monoid in $\mathcal{D}$* is an I-indexed product-preserving functor $\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}}) \to \mathcal{D}$. We write

$$\begin{aligned}
\underline{\mathrm{CMon}}(\mathcal{D}) &:= \underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\times}(\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}}), \mathcal{D}) \\
\mathrm{CMon}(\mathcal{D}) &:= \Gamma^{\mathcal{T}} \underline{\mathrm{CMon}}(\mathcal{D}) \\
\underline{\mathrm{CMon}}(\mathcal{C}) &:= \underline{\mathrm{CMon}}(\underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{C}) \\
\mathrm{CMon}(\mathcal{C}) &:= \mathrm{CMon}(\underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{C}).
\end{aligned}$$ ◁

An important result of Cnossen-Lenz-Linskens resolves the notational clash.

Proposition 1.5 ([CLL24, Thm C])**.** *When $\mathcal{C}$ is an ∞-category, restriction furnishes an equivalence*

$$\mathrm{CMon}(\mathcal{C}) \simeq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C}),$$

and more generally, we have $\underline{\mathrm{CMon}}(\mathcal{C})_V \simeq \mathrm{Fun}_V^{\times}(\mathrm{Span}_I(\mathbb{F}_V), \mathcal{C})$ with restriction given by pullback along $\mathrm{Span}_I(\mathbb{F}_V) \to \mathrm{Span}_I(\mathbb{F}_W)$.

Let I be a one-object weak indexing category and let $\mathrm{Cat}_{\mathcal{T}}^{I-\times} \subset \mathrm{Cat}_{\mathcal{T}}$ be the (non-full) subcategory whose objects are $\mathcal{T}$-categories admitting I-indexed products and functors preserving I-indexed products. Let $\mathrm{Cat}_I^{I-\oplus} \subset \mathrm{Cat}_{\mathcal{T}}^{I-\times}$ be the full subcategory spanned by I-semiadditive $\mathcal{T}$-∞-categories. The following result is fundamental in the theory of equivariant semiadditivity and equivariant higher algebra.

Theorem 1.6 ([CLL24, Thm B])**.** $\mathrm{Cat}_{\mathcal{T}}^{I-\oplus} \subset \mathrm{Cat}_{\mathcal{T}}^{I-\times}$ *is a* localizing *subcategory with localization functor* $\underline{\mathrm{CMon}}(-)$.

In addition, we verified the following corollary to [CH21, Cor 8.2].

Lemma 1.7. *If $\mathcal{C}$ is an ∞-category and I a one-object weak indexing category, then the* underlying coefficient system *functor* $\mathrm{CMon}_I(\mathcal{C}) \to \Gamma^{\mathcal{T}} \mathcal{C}$ *is conservative; in particular, if a $\mathcal{T}$-symmetric monoidal functor's underlying $\mathcal{T}$-functor is an equivalence, then it is a $\mathcal{T}$-symmetric monoidal equivalence.*

1.2. Recollections on $\mathcal{T}$-operads.

1.2.1. *$\mathcal{T}$-operads and $\mathcal{T}$-symmetric monoidal ∞-categories.* In [Ste25a], we made the following definition.

Definition 1.8. A *$\mathcal{T}$-operad* is a functor $\pi : \mathcal{O}^{\otimes} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ satisfying the following conditions.

(a) $\mathcal{O}^{\otimes}$ has π-cocartesian lifts for backwards maps in $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$;

(b) (Segal condition for colors) for every $S \in \mathbb{F}_{\mathcal{T}}$, cocartesian transport along the π-cocartesian lifts lying over the inclusions $(S \leftarrow U = U \mid U \in \mathrm{Orb}(S))$ together induce an equivalence

$$\mathcal{O}_S \simeq \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}_U;$$

(c) (Segal condition for multimorphisms) for every map of orbits $T \to S$ in I and pair of objects $(\mathbf{C}, \mathbf{D}) \in \mathcal{O}_T \times \mathcal{O}_U$, postcomposition with the π-cocartesian lifts $\mathbf{D} \to D_U$ lying over the inclusions $(S \leftarrow U = U \mid U \in \mathrm{Orb}(S))$ induces an equivalence

$$\mathrm{Map}_{\mathcal{O}^{\otimes}}^{T \to S}(\mathbf{C}, \mathbf{D}) \simeq \prod_{U \in \mathrm{Orb}(S)} \mathrm{Map}_{\mathcal{O}^{\otimes}}^{T \leftarrow T_U \to U}(\mathbf{C}, D_U).$$

where $T_U := T \times_S U$.

The corresponding category is a full subcategory $\mathrm{Op}_{\mathcal{T}} \to \mathrm{Cat}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$; that is, a morphism of $\mathcal{T}$-operads is a functor $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ sending $\pi_{\mathcal{O}}$-cocartesian morphisms to $\pi_{\mathcal{P}}$-cocartesian morphisms. We also call these *$\mathcal{O}$-algebras in $\mathcal{P}$* and we let

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) := \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}}}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}) \subset \mathrm{Fun}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes})$$

be the full subcategory spanned by $\mathcal{O}$-algebras in $\mathcal{P}$. ◁

Furthermore, let $\underline{\mathbb{F}}_{\mathcal{T},*} := \mathrm{Span}_{\text{summand inclusion},I}(\underline{\mathbb{F}}_{\mathcal{T}})$. There is an associated map

$$\varphi\colon \mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*} \to \mathrm{Span}_I(\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T}}) \xrightarrow{s} \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}).$$

Let $\mathrm{Op}_{\mathcal{T},\infty}$ be the $\mathcal{T}$-∞-operads of [NS22, Def 2.1.7]. In [Ste25a] we verified that the argument of [BHS22, § 5.2] lifts to show that pullback along φ furnishes an equivalence $\mathrm{Op}_{\mathcal{T}} \xrightarrow{\sim} \mathrm{Op}_{\mathcal{T},\infty}$. By doing so, we acquired a *conservative* functor

$$\mathrm{Tot}_{\mathcal{T}}\colon \mathrm{Op}_{\mathcal{T}} \simeq \mathrm{Op}_{\mathcal{T},\infty} \subset \mathrm{Cat}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T},*}} \xrightarrow{\mathrm{Cat}}_{\mathcal{T}}$$

taking a $\mathcal{T}$-operad to the total $\mathcal{T}$-∞-category of the pullback fibration of $\mathcal{T}$-∞-categories $\mathrm{Tot}_{\mathcal{T}}\varphi^*\mathcal{O}^{\otimes} \to \underline{\mathbb{F}}_{\mathcal{T},*}$.

Furthermore, we noted that a cocartesian fibration $\pi : \mathcal{O}^{\otimes} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is an I-operad if and only if its unstraightening $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Cat}$ is an I-symmetric monoidal category. [BHS22] and [NS22] thus independently construct an adjunction

$$\begin{tikzcd} \mathrm{Op}_{\mathcal{T}} \arrow[r, bend left, "\mathrm{Env}"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_{\mathcal{T}}^{\otimes} \arrow[l, bend left, "U"] \end{tikzcd}$$

In [Ste25a] we computed $\mathrm{Env}(\mathrm{Comm}_{\mathcal{T}}) \simeq \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$, i.e. it is the $\mathcal{T}$-∞-category of finite $\mathcal{T}$-sets with indexed tensor products given by indexed coproducts; [BHS22, Prop 4.21] then verifies that the *sliced* left adjoint $\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Cat}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$ is fully faithful and identify its image, i.e. $\mathrm{Op}_{\mathcal{T}}$ is a colocalizing subcategory of $\mathcal{T}$-symmetric monoidal ∞-categories over $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$ consisting of the *equifibrations.*

1.2.2. *The underlying $\mathcal{T}$-symmetric sequence.* From there, we defined an *underlying $\mathcal{T}$-symmetric sequence* functor and proved the following.

Theorem 1.9 ([Ste25a, Thm A])**.** *The underlying $\mathcal{T}$-symmetric sequence functor* $\mathrm{sseq}\colon \mathrm{Op}_{\mathcal{T}}^{\le oc} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ *is monadic.*

In particular, it is conservative. The V-objects in $\underline{\Sigma}_{\mathcal{T}} \simeq \underline{\mathbb{F}}_{\mathcal{T}}^{\simeq}$ are finite V-sets; given $S \in \Sigma_V \simeq \mathbb{F}_V^{\simeq}$, writing $\mathcal{O}(S)$ for $\mathrm{sseq}\,\mathcal{O}^{\otimes}(S)$, we remember this as saying that at most one color $\mathcal{T}$-operads are identified conservatively by their S-ary structure spaces. Using this, we define the full subcategory of *$\mathcal{T}$-d-operads* as those with $(d-1)$-truncated structure spaces:

$$\mathrm{Op}_{\mathcal{T},d} := \left\{\mathcal{O}^{\otimes} \mid \forall S,\, \mathcal{O}(S) \in \mathcal{S}_{\le(d-1)}\right\} \subset \mathrm{Op}_{\mathcal{T}}$$

We proved the following.

Proposition 1.10 ([Ste25a])**.** *The inclusion $\mathrm{Op}_{\mathcal{T},d} \subset \mathrm{Op}_{\mathcal{T}}$ has a left adjoint $h_d\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T},d}$, and given $\mathcal{P}^{\otimes} \in \mathrm{Op}_{\mathcal{T},d}$, the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ is a d-category; moreover, if $\mathcal{P}^{\otimes} \in \mathrm{Op}_{\mathcal{T},0}$, then $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ is either empty or contractible. In particular, $\mathrm{Op}_{\mathcal{T},d}$ is a $(d+1)$-category and $\mathrm{Op}_{\mathcal{T},0}$ is a poset.*

We call $h_d\mathcal{O}^{\otimes}$ the *homotopy $\mathcal{T}$-d-operad of $\mathcal{O}^{\otimes}$*. We went on to compute the *free $\mathcal{O}$-algebra monad*; for algebras in a cartesian structure on coefficient systems in a cocomplete cartesian closed ∞-category $\mathcal{C}$, this sends $X \in \mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ to the coefficient system $T_{\mathcal{O}}X$ with

$$(T_{\mathcal{O}}X)^V \simeq \coprod_{S \in \mathbb{F}_V} \left(\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S) X^U}\right)_{h\mathrm{Aut}_V(S)}.$$

In particular, given $S \in \mathbb{F}_V$, in [Ste25a] we found a natural splitting $\mathrm{Fr}_{\mathcal{C}}\mathcal{O}(S) \oplus J \simeq (T_{\mathcal{O}}S)_V$,where $\mathrm{Fr}_{\mathcal{C}}\colon \mathcal{S} \to \mathcal{C}$ is the unique symmetric monoidal left adjoint. Via a multiple-color version of this argument, we concluded the following.

Proposition 1.11 ([Ste25a])**.** *A map of $\mathcal{T}$-operads $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is an h_d-equivalence if and only if:*

(a) the underlying $\mathcal{T}$-functor $U(\varphi)\colon \mathcal{O} \to \mathcal{P}$ is essentially surjective, and

(b) the pullback functor $\varphi^*\colon \mathrm{Alg}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\le(d-1)}) \to \mathrm{Alg}_{\mathcal{O}}\left(\underline{\mathcal{S}}_{\mathcal{T},\le(d-1)}\right)$ is an equivalence.

In particular, φ is an equivalence if and only if it is U-essentially surjective and induces an equivalence on algebras in $\underline{\mathcal{S}}_{\mathcal{T}}$.

Given a map $U \to V$ in $\mathcal{T}$ and a finite V-set $S \in \mathbb{F}_V$, in [Ste25a] we defined used cocartesian transport to define a *restriction map*

$$\mathcal{O}(S) \to \mathcal{O}(\mathrm{Res}_U^V S) \tag{6}$$

Furthermore, given a finite S-set T, writing $T_U := T \times_S U$, we used composition to define a map

$$\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T_U) \to \mathcal{O}(T) \tag{7}$$

Last, we used cocartesian transport to define a Σ *action*

$$\rho_S : \mathrm{Aut}_V(S) \times \mathcal{O}(S) \longrightarrow \mathcal{O}(S). \tag{8}$$

1.2.3. *Rudiments of weak $\mathcal{N}_\infty$-operads.* In [Ste25a], we constructed a family of $\mathcal{T}$-operads:

Proposition 1.12 ([Ste25a])**.** *Let $I \subset \mathbb{F}_{\mathcal{T}}$ be a core-full and pullback-stable subcategory. Then,* $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ *presents a $\mathcal{T}$-operad if and only if I is a weak indexing category.*

These are called weak $\mathcal{N}_\infty$-operads; in the case that I is an indexing category, these are called $\mathcal{N}_\infty$-operads. To state their universal property, we defined the *arity support* subcategory

$$A(\mathcal{O}) = \left\{ T \to S \;\middle|\; \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T \times_S U) \neq \varnothing \right\} \subset \mathbb{F}_{\mathcal{T}},$$

Theorem 1.13 ([Ste25a])**.** *The arity support of a $\mathcal{T}$-operad is a weak indexing category, and the associated essential surjection has a fully faithful right adjoint*

$$\begin{tikzcd} \mathrm{Op}_{\mathcal{T}} \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathrm{wIndexCat}_{\mathcal{T}} \arrow[l, bend left, "\mathcal{N}_{-\infty}"] \end{tikzcd}$$

The essential image of $\mathcal{N}_{-\infty}$ is spanned by $\mathcal{T}$-operads $\mathcal{O}^{\otimes}$ satisfying any of the following equivalent conditions.

(a) $\mathcal{O}^{\otimes}$ is a weak $\mathcal{N}_\infty$-operad.
(b) $\mathcal{O}^{\otimes}$ is a $\mathcal{T}$-0-operad.
(c) The map of $\mathcal{T}$-operads $\mathcal{O}^{\otimes} \to \mathrm{Comm}_{\mathcal{T}}^{\otimes}$ is a monomorphism.

In particular, this isolates the weak $\mathcal{N}_\infty$-operads as those possessing a fully faithful unslicing functor

$$\mathrm{Op}_I := \mathrm{Op}_{\mathcal{T},/\mathcal{N}_{I\infty}} \hookrightarrow \mathrm{Op}_{\mathcal{T}}.$$

These posses an intrinsic characterization as *I-operads* (see [Ste25a]), leading to a collection of *arity-Borelification* adjunctions

$$\begin{tikzcd} \mathrm{Op}_I \arrow[r, bend left, "E_I^J"] \arrow[r, phantom, "\perp"] & \mathrm{Op}_J; \arrow[l, bend left, "\mathrm{Bor}_I^J"] \end{tikzcd}$$

given $I \le J$, the Borelification functor Bor_I^J is simply given by pullback along the unique map $\mathcal{N}_{I\infty}^{\otimes} \to \mathcal{N}_{J\infty}^{\otimes}$. It follows from Theorem 1.13 that $A\mathrm{Bor}_I^J \mathcal{O} \simeq J \cap A\mathcal{O}$ and $AE_I^J \mathcal{O} \simeq A\mathcal{O}$.

A particularly useful instance of this concerns the initial one-color indexing system I^{triv}; the corresponding weak $\mathcal{N}_\infty$-operad $\mathcal{N}_{I^{\mathrm{triv}}\infty}^{\otimes} \simeq \mathrm{triv}_{\mathcal{T}}^{\otimes}$ is called the *trivial $\mathcal{T}$-operad*, and it is characterized by its algebras [NS22; Ste25a]

$$\underline{\mathrm{CAlg}}_{I^{\mathrm{triv}}}(\mathcal{O}) \simeq U\mathcal{O};$$

in particular, the restriction of the underlying $\mathcal{T}$-∞-categories construction to I^{triv}-operads yields an equivalence $\mathrm{Op}_{I^{\mathrm{triv}}} \xrightarrow{\sim} \mathrm{Cat}_{\mathcal{T}}$ and $E_{I^{\mathrm{triv}}}^J$ is compatible with U [NS22]; that is, the adjunction $E_{I^{\mathrm{triv}}}^{\mathcal{T}} \vdash \mathrm{Bor}_I^J$ takes the form of an adjunction $\mathrm{triv}(-)^{\otimes} \vdash U$, i.e. given a $\mathcal{T}$-∞-category $\mathcal{C}$, it yields a $\mathcal{T}$-operad characterized by its algebras

$$\mathrm{Alg}_{\mathrm{triv}(\mathcal{C})}(\mathcal{O}) \simeq \mathrm{Fun}(\mathcal{C}, U\mathcal{O});$$

this formula allows for an alternative construction for $\mathrm{triv}(\mathcal{C})$ is as the operadic localization [Ste25a]

$$\mathrm{triv}(\mathcal{C})^{\otimes} \simeq L_{\mathrm{Op}_{\mathcal{T}}}(\mathcal{C} \to \mathcal{T}^{\mathrm{op}} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})).$$

1.2.4. *The Boardman-Vogt tensor product.* In [Ste25a, Thm D], in the case that $\mathcal{T}$ has a terminal object, we equipped $\mathrm{Op}_{\mathcal{T}}$ with a closed *Boardman-Vogt tensor product*

$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} := L_{\mathrm{Op}}\left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \longrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \times \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \xrightarrow{\wedge} \mathrm{Span}(\mathbb{F}_{\mathcal{T}})\right),$$

where $L_{\mathrm{Op}}\colon \mathrm{Cat}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})} \to \mathrm{Op}_{\mathcal{T}}$ is the left adjoint to the inclusion [BHS22, Cor 4.2.3]. Its internal hom is denoted $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P})$; its underlying $\mathcal{T}$-∞-category is denoted $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P})$, and it has values

$$\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P})_V \simeq \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{O}}(\mathrm{Res}^{\mathcal{T}}_V \mathcal{P}),$$

where given $V \in \mathcal{T}$, the morphism $\mathrm{Res}^{\mathcal{T}}_V\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Op}_V := \mathrm{Op}_{\mathcal{T}_{/V}}$ is given by pullback along the morphism of algebraic patterns $\mathrm{Span}(\mathbb{F}_V) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$. We verified several properties in that paper; for instance, $\underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C})$ is an I-symmetric monoidal ∞-category when $\mathcal{C}$ is, functorially for I-symmetric monoidal maps in $\mathcal{C}^{\otimes}$ and $\mathcal{T}$-operad maps in $\mathcal{P}^{\otimes}$. The following example drives us interpreting $\mathcal{O} \otimes \mathcal{P}$-algebras as *homotopy-coherently interchanging pairs of $\mathcal{O}$-algebra and $\mathcal{P}$-algebra structures.*

Recollection 1.14. Suppose $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal 1-category and $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are one-color $\mathcal{T}$-operads. We saw in [Ste25a] that an $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$-algebra structure on a $\mathcal{T}$-object $X \in \Gamma^{\mathcal{T}} \mathcal{C}$ is equivalently viewed as a pair of $\mathcal{O}$-algebra and $\mathcal{P}$-algebra structures subject to the *interchange relation* that, for all $\mu_S \in \mathcal{O}(S)$ and $\mu_T \in \mathcal{P}(T)$, the following diagram commutes.

$$\begin{tikzcd}
\bigotimes_U^S X_V^{\otimes \mathrm{Res}^V_U T} \arrow[r, phantom, "\simeq"] \arrow[d, "(\mathrm{Res}^V_U \mu_T)"'] & X_V^{\otimes S \times T} \arrow[r, phantom, "\simeq"] & \bigotimes_W^T X_V^{\otimes \mathrm{Res}^V_W S} \arrow[r, "(\mathrm{Res}^V_W \mu_S)"] & X_V^{\otimes T} \arrow[d, "\mu_T"] \\
X_V^{\otimes S} \arrow[rrr, "\mu_S"] & & & X_V
\end{tikzcd}$$

A morphism of $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$-algebras is simply a morphism of $\mathcal{T}$-objects which is simultaneously an $\mathcal{O}$-algebra map and a $\mathcal{P}$-algebra map. ◁

An important one is the following.

Proposition 1.15 ([Ste25a, Thm D.(3)]). $\mathrm{triv}^{\otimes}_{\mathcal{T}}$ *is the* $\overset{BV}{\otimes}$*-unit; hence there is an equivalence of $\mathcal{T}$-operads*

$$\underline{\mathrm{Alg}}_{\mathrm{triv}_{\mathcal{T}}}(\mathcal{O}) \simeq \mathcal{O}^{\otimes}$$

We additionally characterized the interaction of the Boardman-Vogt tensor product and unit with the $\mathcal{T}$-symmetric monoidal envelope.

Proposition 1.16 ([Ste25a, Thm D.(7)]). *The $\mathcal{T}$-symmetric monoidal envelope intertwines the mode symmetric monoidal structure with Boardman-Vogt tensor products, i.e.*

$$\mathrm{Env}\left(\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}\right) \simeq \mathrm{Env}\left(\mathcal{O}^{\otimes}\right) \otimes^{\mathrm{Mode}} \mathrm{Env}\left(\mathcal{P}^{\otimes}\right).$$

Furthermore, $\mathrm{Env}\left(\mathrm{triv}^{\otimes}_{\mathcal{T}}\right)$ *is the* $\otimes^{\mathrm{Mode}}$*-unit.*

For the remainder of Sections 1 and 2 we assume that we've constructed a symmetric monoidal structure on $\mathrm{Op}_{\mathcal{T}}$ satisfying [Ste25a, Thm D], and hence the results of this subsection; for general $\mathcal{T}$, this is true of $\mathcal{T}_{/V}$. In Section 3.1, we will establish such a structure on *arbitrary* $\mathcal{T}$, so the results of Sections 1 and 2 will apply for arbitrary $\mathcal{T}$.

1.2.5. *Inflation and fixed points.* In [Ste25a], we verified that the subcategory of I^∞-operads is equivalent to *operadic coefficient systems*:

$$\mathrm{Op}_{I^\infty} \simeq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Op}).$$

In particular, limits and diagonals yield a push-pull adjunction

$$\begin{tikzcd} \mathrm{Op} \arrow[r, bend left, "\mathrm{Infl}_e^{I^\infty}"] \arrow[r, phantom, "\perp"] & \mathrm{Op}_{I^\infty} \arrow[l, bend left, "\Gamma^{\mathcal{T}}"] \arrow[r, bend left, "E_{I^\infty}^{\mathcal{T}}"] \arrow[r, phantom, "\perp"] & \mathrm{Op}_{\mathcal{T}} \arrow[l, bend left, "\mathrm{Bor}_{I^\infty}^{\mathcal{T}}"] \end{tikzcd}$$

We also refer to the composite adjunction as $\mathrm{Infl}_e^{\mathcal{T}} : \mathrm{Op} \rightleftarrows \mathrm{Op}_{\mathcal{T}} : \Gamma^{\mathcal{T}}$. The left adjoint of this is fully faithful with essential image spanned by the I^∞-operads whose corresponding operadic coefficient system is constant. The main use of these is their compatibility with tensor products and algebras; in particular, in [Ste25a, Thm D.(6)] we showed

$$\Gamma^{\mathcal{T}} \underline{\mathrm{Alg}}^{\otimes}_{\mathrm{Infl}_e^{\mathcal{T}} \mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}^{\otimes}_{\mathcal{O}}\left(\Gamma^{\mathcal{T}} \mathcal{C}\right);$$

$$\mathrm{Infl}_e^{\mathcal{T}} \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Infl}_e^{\mathcal{T}} \mathcal{P}^{\otimes} \simeq \mathrm{Infl}_e^{\mathcal{T}} \left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right).$$

Example 1.17. The weak $\mathcal{N}_\infty$-operads $\mathrm{triv}_{\mathcal{T}}^{\otimes}$, $\mathbb{E}_0^{\otimes} := \mathcal{N}^{\otimes}_{I^0\infty}$ and $\mathbb{E}_\infty^{\otimes} := \mathcal{N}^{\otimes}_{I^\infty\infty}$ are inflated. In particular, taking algebras, we have that

$$\Gamma^{\mathcal{T}} \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_0}(\mathcal{C}) \simeq \left(\Gamma^{\mathcal{T}} \mathcal{C}\right)^{\otimes}_{1/};$$

$$\Gamma^{\mathcal{T}} \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_\infty}(\mathcal{C}) \simeq \mathrm{CAlg}^{\otimes}\left(\Gamma^{\mathcal{T}} \mathcal{C}\right).$$

This first result is not new, and has been generalized by [NS22, Thm 5.2.10]. ◁

1.3. More on restriction and arity-borelification.

1.3.1. *Color-borelification.* Let $\mathcal{F} \subset \mathcal{T}$ be a $\mathcal{T}$-family. There is a terminal $\mathcal{F}$-colored weak indexing category $I_{\mathcal{F}}$; we refer to $I_{\mathcal{F}}$-Borelification as $\mathcal{F}$*-Borelification* and write $\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} := \mathrm{Bor}_{I_{\mathcal{F}}}^{\mathcal{T}}$, noting that there is an equivalence

$$\mathrm{Op}_{\mathcal{F}} \simeq \mathrm{Op}_{I_{\mathcal{F}}}.$$

Let $\mathrm{triv}_{\mathcal{F}}^{\otimes} := \mathrm{triv}(*_{\mathcal{F}})$; this is the weak $\mathcal{N}_\infty$-operad for the initial $\mathcal{F}$-colored weak indexing system $I_{\mathcal{F}}^{\mathrm{triv}}$, and in particular, there is an equivalence

$$\mathrm{triv}_{\mathcal{F}}^{\otimes} \simeq E_{\mathcal{F}}^{\mathcal{T}} \mathrm{triv}_{\mathcal{F}}^{\otimes}.$$

This is our first example of a $\otimes$-idempotent $\mathcal{T}$-operad after $\mathrm{triv}_{\mathcal{T}}^{\otimes}$.

Proposition 1.18 (Color-borelification)**.** *Given $\mathcal{F} \in \mathrm{Fam}_{\mathcal{T}}$ is a $\mathcal{T}$-family, there is a natural equivalence*

$$\mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathcal{O}) \simeq \Gamma^{\mathcal{F}} \mathcal{O};$$

hence there is a natural equivalence

$$\mathrm{triv}_{\mathcal{F}}^{\otimes} \overset{BV}{\otimes} \mathcal{O}^{\otimes} \simeq E_{\mathcal{F}}^{\mathcal{T}} \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{O}^{\otimes}.$$

Proof. The first statement follows by noting that $\mathrm{triv}_{\mathcal{F}}^{\otimes} \simeq E_{\mathcal{F}}^{\mathcal{T}} \mathrm{triv}_{\mathcal{F}}^{\otimes}$, so that

$$\mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathcal{O}) \simeq \mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}}}(\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}}(\mathcal{O})) \simeq \Gamma^{\mathcal{F}} \mathcal{O}$$

by Proposition 1.15. The second statement then follows by Yoneda's lemma, noting that

$$\begin{aligned} \mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}} \otimes \mathcal{O}}(\mathcal{P}) &\simeq \mathrm{Alg}_{\mathrm{triv}_{\mathcal{F}}} \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) \\ &\simeq \Gamma^{\mathcal{F}} \mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \\ &\simeq \mathrm{Alg}_{\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{O}}(\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{P}) \\ &\simeq \mathrm{Alg}_{E_{\mathcal{F}}^{\mathcal{T}} \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{O}}(\mathcal{P}). \end{aligned}$$

□

Given $\mathcal{O} \in \mathrm{Op}_{\mathcal{T}}$, we set $c(\mathcal{O}) := c(A\mathcal{O}) = \{V \mid \mathcal{O}(*_V) \neq \varnothing\}$.

Remark 1.19. Proposition 1.18 implies that $\mathrm{Im} E_{\mathcal{F}}^{\mathcal{T}} = \left\{\mathcal{O}^{\otimes} \in \mathrm{Op}^{\mathcal{T}} \mid c(\mathcal{O}) \subset \mathcal{F}\right\}$ is a $\otimes$-*ideal*, i.e. if $c(\mathcal{O}) \subset \mathcal{F}$, and $\mathcal{P}^{\otimes}$ is arbitrary, then $c\left(\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{P}\right) \subset \mathcal{F}$. In particular, $\underline{\mathrm{Op}}_I^{\otimes}$ is a *nonunital* symmetric monoidal full subcategory of $\underline{\mathrm{Op}}_J^{\otimes}$. ◁

This is important because of the following observation.

Observation 1.20. There are natural equivalences

$$\begin{aligned}
\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} &\simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{c\mathcal{O}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{c\mathcal{P}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}, \\
&\simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{c\mathcal{O}\cap c\mathcal{P}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}, \\
&\simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{c\mathcal{O}\cap c\mathcal{P}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{c\mathcal{O}\cap c\mathcal{P}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}, \\
&\simeq E_{c\mathcal{O}\cap c\mathcal{P}}^{\mathcal{T}} \mathrm{Bor}_{c\mathcal{O}\cap c\mathcal{P}}^{\mathcal{T}}\left(\mathcal{O}^{\otimes}\right) \overset{\mathrm{BV}}{\otimes} E_{c\mathcal{O}\cap c\mathcal{P}}^{\mathcal{T}} \mathrm{Bor}_{c\mathcal{O}\cap c\mathcal{P}}^{\mathcal{T}}\left(\mathcal{P}^{\otimes}\right), \\
&\simeq E_{c\mathcal{O}\cap c\mathcal{P}}^{\mathcal{T}} \left(\mathrm{Bor}_{c\mathcal{O}\cap c\mathcal{P}}^{\mathcal{T}}\left(\mathcal{O}^{\otimes}\right) \overset{\mathrm{BV}}{\otimes} \mathrm{Bor}_{c\mathcal{O}\cap c\mathcal{P}}^{\mathcal{T}}\left(\mathcal{P}^{\otimes}\right)\right).
\end{aligned}$$

The $c\mathcal{O} \cap c\mathcal{P}$-operads $\mathrm{Bor}_{c\mathcal{O}\cap c\mathcal{P}}^{\mathcal{T}}(\mathcal{O}^{\otimes})$ and $\mathrm{Bor}_{c\mathcal{O}\cap c\mathcal{P}}^{\mathcal{T}}(\mathcal{P}^{\otimes})$ both have at least one color; hence we may compute arbitrary tensor products of $\mathcal{T}$-operads via tensor products of equivariant operads with at least one color. ◁

Having done this, we may compute arity-supports of arbitrary tensor products of $\mathcal{T}$-operads.

Proposition 1.21. *Suppose $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are $\mathcal{T}$-operads. Then,*

$$A\left(\mathcal{O} \overset{BV}{\otimes} \mathcal{P}\right) = E_{\mathcal{F}}^{\mathcal{T}} \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}}(A\mathcal{O} \vee A\mathcal{P}).$$

Proof. By Observation 1.20, we have equivalences

$$A\left(\mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes}\right) \simeq E_{c\mathcal{O}\cap c\mathcal{P}}^{\mathcal{T}} A\left(\mathrm{Bor}_{c\mathcal{O}\cap c\mathcal{P}}^{\mathcal{T}}\left(\mathcal{O}^{\otimes}\right) \overset{\mathrm{BV}}{\otimes} \mathrm{Bor}_{c\mathcal{O}\cap c\mathcal{P}}^{\mathcal{T}}\left(\mathcal{P}^{\otimes}\right)\right),$$

so it suffices to prove the proposition in the case that $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ have at least one color.

In this case, first note that there exist maps

$$\mathcal{O}^{\otimes} \otimes \mathrm{triv}_{\mathcal{T}}^{\otimes}, \mathrm{triv}_{\mathcal{T}}^{\otimes} \otimes \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes},$$

so that

$$A\mathcal{O} \vee A\mathcal{P} \leq A(\mathcal{O} \vee \mathcal{P}).$$

On the other hand, there exists a composite map

$$\mathcal{O}^{\otimes} \otimes \mathcal{P}^{\otimes} \to \mathcal{N}_{A\mathcal{O}\infty}^{\otimes} \otimes \mathcal{N}_{A\mathcal{P}\infty}^{\otimes} \to \mathcal{N}_{A\mathcal{O}\vee A\mathcal{P}\infty}^{\otimes} \otimes \mathcal{N}_{A\mathcal{O}\vee A\mathcal{P}\infty}^{\otimes} \to \mathcal{N}_{A\mathcal{O}\vee A\mathcal{P}\infty}^{\otimes},$$

whose last map is presented by the bifunctor

$$\begin{tikzcd}
\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \times \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \arrow[r, "\wedge"] \arrow[d] & \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \arrow[d] \\
\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \times \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \arrow[r, "\wedge"] & \mathrm{Span}(\mathbb{F}_{\mathcal{T}});
\end{tikzcd}$$

here, the top map is defined canonically by the fact that weak indexing categories $I \subset \mathbb{F}_{\mathcal{T}}$ are closed under cartesian products [Ste24]. This map implies that $A(\mathcal{O} \vee \mathcal{P}) \leq A\mathcal{O} \vee A\mathcal{P}$, as desired. □

The following corollary is immediate.

Corollary 1.22. $\mathrm{Op}_I \subset \mathrm{Op}_{\mathcal{T}}$ *is closed under binary tensor products; if I has one color, then* $\mathrm{triv}_{\mathcal{T}}^{\otimes} \in \mathrm{Op}_I$.

1.3.2. *Operadic restriction and (co)induction.* Recall from [Ste25a] that the underlying $\mathcal{T}$-symmetric sequence forms a $\mathcal{T}$-functor $\underline{\mathrm{sseq}} : \underline{\mathrm{Op}}_{\mathcal{T}}^{\mathrm{red}} \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \underline{\mathcal{S}}_{\mathcal{T}})$; in particular, restrictions of $\underline{V}$-operads correspond with restrictions of $\underline{V}$-symmetric sequences; We may use this to upgrade Theorem 1.13 to an adjunction of $\mathcal{T}$-categories.

Proposition 1.23. $\mathrm{Res}_V^W \mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{N}^{\otimes}_{\mathrm{Res}_V^W I\infty}$; *more generally,* $A \dashv \mathcal{N}^{\otimes}_{(-)\infty}$ *lifts to a* $\mathcal{T}$*-adjunction*

$$\begin{tikzcd} \underline{\mathrm{Op}}_{\mathcal{T}} \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{wIndex}}_{\mathcal{T}} \arrow[l, bend left, "\mathcal{N}^{\otimes}_{(-)\infty}"] \end{tikzcd}$$

Proof. Restriction compatiblility of the underlying symmetric sequence implies that $\mathrm{Res}_V^W A\mathcal{O} = A\,\mathrm{Res}_V^W \mathcal{O}$, lifting A to a $\mathcal{T}$-functor $\underline{\mathrm{Op}}_{\mathcal{T}} \to \underline{\mathrm{wIndex}}_{\mathcal{T}}$ whose V-value is $A : \mathrm{Op}_V \to \mathrm{wIndex}_V$. The right adjoints $\mathcal{N}^{\otimes}_{(-)\infty}$ uniquely lift to a right $\mathcal{T}$-adjoint to $\mathcal{N}^{\otimes}_{(-)\infty}$ by [HA, Prop 7.3.2.1], completing the proposition. □

Since A is a $\mathcal{T}$-left adjoint, it is compatible with $\mathcal{T}$-colimits. Applying this for indexed coproducts, we immediately acquire the following properties of A.

Corollary 1.24. *If* $\mathcal{O},\mathcal{P}$ *are* $\mathcal{T}$*-operads, then we have*

$$A(\mathcal{O} \sqcup \mathcal{P}) = A\mathcal{O} \vee A\mathcal{P}.$$

If $\mathcal{Q}$ *is a* V*-operad, then we have*

$$A\mathrm{Ind}_V^{\mathcal{T}} \mathcal{Q} = \mathrm{Ind}_V^{\mathcal{T}} A\mathcal{Q}.$$

We may compute use an analogous argument to that of [BHS22, Lem 4.1.13] to show that $\underline{\mathrm{Op}}_{\mathcal{T}}$ strongly admits $\mathcal{T}$-limits; since the fully faithful $\mathcal{T}$-functor $\underline{\mathrm{Op}}_{\mathcal{T}} \to \underline{\mathrm{Cat}}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$ possesses pointwise left adjoints (given by L_{Fbrs}), it possesses a $\mathcal{T}$-left adjoint; in particular, we may compute $\mathcal{T}$-limits of $\mathcal{T}$-operads in $\underline{\mathrm{Cat}}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$. Then, an analogous argument using [BHS22, Prop 2.3.7] constructs $\mathcal{T}$-limits in $\underline{\mathrm{Cat}}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$ in $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathrm{Span}(\mathbb{F}_{\mathcal{T}}), \underline{\mathrm{Cat}}_{\mathcal{T}})_{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$, which strongly admits $\mathcal{T}$-limits, as its a slice $\mathcal{T}$-∞-category of a functor $\mathcal{T}$-∞-category into a $\mathcal{T}$-∞-category which strongly admits $\mathcal{T}$-limits. In particular, this implies that $\mathrm{Res}_U^V : \mathrm{Op}_V \to \mathrm{Op}_U$ has a right adjoint, which we call $\mathrm{CoInd}_U^V : \mathrm{Op}_U \to \mathrm{Op}_V$.

Proposition 1.25. *If* $\mathcal{O}^{\otimes}$ *is a* V*-*d*-operad, then* $\mathrm{CoInd}_V^W \mathcal{O}^{\otimes}$ *is a* W*-*d*-operad.*

Proof. Flip the vertical arrows. This follows simply by taking right adjoints within the following diagram

$$\begin{tikzcd} \mathrm{Op}_{W,d} \arrow[r, "\mathrm{Res}_V^W"] \arrow[d, hook] & \mathrm{Op}_{V,d} \arrow[d, hook] \\ \mathrm{Op}_W \arrow[r, "\mathrm{Res}_V^W"] & \mathrm{Op}_V \end{tikzcd}$$

□

Corollary 1.26. *We have equivalences*

$$\mathrm{sseq}\,\mathrm{CoInd}_V^W \mathcal{O}^{\otimes} \simeq \mathrm{CoInd}_V^W\,\mathrm{sseq}\,\mathcal{O}^{\otimes};$$
$$A\mathrm{CoInd}_V^W \mathcal{O} = \mathrm{CoInd}_V^W A\mathcal{O}.$$

Proof. The first statement equivalence by noting that $\mathrm{Fr}\,\mathrm{Res}_V^W = \iota_V^{W*}\mathrm{Fr}$ and taking right adjoints. The second follows by the same diagram as in Proposition 1.25 together with the fact $\underline{\mathrm{Op}}_{\mathcal{T},0}$ is equivalent to $\underline{\mathrm{wIndex}}_{\mathcal{T}}$. □

We care about $\mathrm{CoInd}_V^{\mathcal{T}} \mathcal{O}^{\otimes}$ because it is a structure borne by *norms of algebras*.

Construction 1.27. Let $\mathcal{P}^{\otimes} \to \mathrm{CoInd}_V^W \mathcal{O}^{\otimes}$ be a functor of one-object I-operads, let $\mathcal{C}$ be a I-symmetric monoidal ∞-category, and let $V \to W$ be a transfer in I. Then, the adjunct map $\varphi : \mathrm{Res}_V^W \mathcal{P} \to \mathcal{O}^{\otimes}$ participates in a commutative diagram of symmetric monoidal functors

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Res}_V^W \mathcal{C}) \arrow[r, "\varphi^*"] \arrow[d, "U_V"] & \mathrm{Alg}_{\mathrm{Res}_V^W \mathcal{P}}(\mathrm{Res}_V^W \mathcal{C}) \arrow[r, "N_V^W"] \arrow[d, "U_V"] & \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \arrow[d, "U_W"] \\ \mathcal{C}_V \arrow[r, equal] & \mathcal{C}_V \arrow[r, "N_V^W"] & \mathcal{C}_W \end{tikzcd}$$

Intuitively, we view this situation as saying that $\mathrm{CoInd}_V^W \mathcal{O}^\otimes$ bears the *universal* structure which is naturally endowed on $N_V^W X$ ranging across $X \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. ◁

1.4. $\mathcal{F}$-unitality.

We begin with a definition.

Definition 1.28. We say that an I-operad $\mathcal{O}^\otimes$ is *unital* if $\mathcal{O}(\varnothing_V) = *$ for all $V \in v(I)$. More generally if $\mathcal{F} \subset v(I)$ is a family, we say that $\mathcal{O}^\otimes$ is *$\mathcal{F}$-unital* if $\mathcal{O}(\varnothing_V) \simeq *$ for all $V \in \mathcal{F}$, or equivalently, if $\mathrm{Bor}_{I \cap \mathbb{F}_{\mathcal{F}}}^I \mathcal{O}^\otimes$ is unital. ◁

Observation 1.29. If $\mathcal{C}^\otimes$ is an I-symmetric monoidal ∞-category with unit $\mathcal{T}$-object $1_\bullet$ and $X \in \mathcal{C}_V$, then $\mathrm{Map}_{\mathcal{O}^\otimes}(\varnothing_V, X) \simeq \mathrm{Map}_{\mathcal{C}_V}(1_V, X)$, so $\mathcal{C}^\otimes$ is $\mathcal{F}$-unital if and only if $1_\bullet \in \Gamma^{\mathcal{F}}\mathcal{C}$ is initial; in particular, if $\mathcal{C}^\otimes$ is cartesian, then it is $\mathcal{F}$-unital if and only if it is $\mathcal{F}$-pointed. ◁

Lemma 1.30. *If $\mathcal{C}^\otimes$ is an I-symmetric monoidal ∞-category, then $\mathcal{C}^\otimes$ is $\mathcal{F}$-unital if and only if the forgetful $\mathcal{T}$-functor $U \colon \underline{\mathrm{Alg}}_{\mathbb{E}_{0,\mathcal{F}}}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*

Proof. The forward implication follows from the computation Lemma A.10 in the case $I_{\mathcal{F}}^0$, so assume U is an equivalence. Then, for all $V \in \mathcal{F}$, $\mathcal{C}_V^{1_V/} \simeq \mathrm{Alg}_{\mathbb{E}_{0,\mathcal{F}}}(\mathcal{C})_V \to \mathcal{C}$ is an equivalence, so $1_V \in \mathcal{C}_V$ is initial. Thus Observation 1.29 implies the lemma. □

The main result of this subsection is the following proposition.

Proposition 1.31. *If there exists an equivalence $\mathcal{O}^\otimes \simeq \mathcal{O}^\otimes \overset{BV}{\otimes} \mathbb{E}_{0,\mathcal{F}}^\otimes$, then $\mathcal{O}^\otimes$ is $\mathcal{F}$-unital.*

We will establish the converse in ref in full generality. For this direction, we can quickly reduce to the case $\mathcal{F} = \mathcal{T}$, which we prove in the following.

Proposition 1.32. *Given a $\mathcal{T}$-operad $\mathcal{O}^\otimes$ with at least one color, the following are equivalent:*

(a) $\mathrm{Bor}_{I_0}^{\mathcal{T}} \mathcal{O}^\otimes$ *is unital.*
(b) $\mathcal{O}^\otimes$ *is unital.*
(c) The ∞-category $\mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is pointed.
(d) $\mathcal{O}^\otimes \simeq \mathcal{O}^\otimes \overset{BV}{\otimes} \mathbb{E}_0^\otimes$.
(e) $\mathrm{Bor}_{I_0}^{\mathcal{T}} \mathcal{O}^\otimes \simeq \mathbb{E}_0^\otimes \overset{BV}{\otimes} \mathrm{Bor}_{I_0}^{\mathcal{T}} \mathcal{O}^\otimes$.
(f) The ∞-category $\mathrm{Alg}_{\mathrm{Bor}_{I_0}^{\mathcal{T}} \mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is pointed

Proof. (a) $\implies$ (b) follows immediately by definition; (b) $\implies$ (c) follows immediately by Example 1.17 (c) $\implies$ (d) and (e) $\implies$ (f), since $\mathrm{Alg}_{\mathcal{O} \otimes \mathbb{E}_0}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Mon}_{\mathbb{E}_0} \underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})_*$ over $\mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$. (d) $\implies$ (e) follows by applying Borelification.

What's left is to prove that (f) $\implies$ (a). We argue the contrapositive, writing $\mathcal{P}^\otimes := \mathrm{Bor}_{I_0}^{\mathcal{T}} \mathcal{O}^\otimes$, assuming that $\mathcal{P}^\otimes$ is not unital, and fixing $C \in \mathcal{P}_V$ such that $\mathcal{P}(\varnothing_V; C) \neq *$. We choose the "skyscraper" $\mathcal{P}$-algebra M, with values

$$M(D) = \begin{cases} \mathcal{P}(\varnothing_V, C) & D = C \\ * & \text{otherwise,} \end{cases}$$

gotten by truncating the functor corepresented by $\varnothing$. Then, note that

$$\mathrm{Map}(*_{\mathcal{P}}, M) \simeq \mathcal{P}(\varnothing; C) \not\simeq *,$$

so the unit $*_{\mathcal{P}} \in \mathrm{Alg}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is not initial. By [NS22, Thm 5.2.11] it is terminal, so by contraposition we have shown (f) $\implies$ (a). □

Proof of Proposition 1.31. If $\mathcal{O}^\otimes \simeq \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{0,\mathcal{F}}^\otimes$, then setting $\mathcal{P}^\otimes := \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \mathcal{O}^\otimes$, $\mathcal{F}$-Boreleification yields $\mathcal{P}^\otimes \simeq \mathcal{P}^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{E}_0^\otimes$, so Proposition 1.32 implies that $\mathcal{P}^\otimes$ is unital; equivalently, $\mathcal{O}^\otimes$ is $\mathcal{F}$-unital. □

1.5. **Cartesian I-symmetric monoidal ∞-categories.** Fix I an almost unital weak indexing system in the sense of [Ste24]. Denote by $\mathrm{Cat}_I^{I-\sqcup}, \mathrm{Cat}_I^{I-\times} \subset \mathrm{Cat}_{\mathcal{T}}$ the non-full subcategories with objects given by $\mathcal{T}$-∞-categories attaining I-indexed coproducts (resp. products) and with morphisms given by $\mathcal{T}$-functors which preserve I-indexed coproducts (products). In Appendix A, we prove the following.

Theorem A'. *There are fully faithful embeddings $(-)^{I-\sqcup}, (-)^{I-\times}$ making the following commute:*

$$\begin{tikzcd} \mathrm{Cat}_I^{I-\sqcup} \arrow[r, hook, "(-)^{I-\sqcup}"] \arrow[dr, "U"'] & \mathrm{Cat}_I^{\otimes} \arrow[d, "U"] & \mathrm{Cat}_I^{I-\times} \arrow[l, hook', "(-)^{I-\times}"'] \arrow[dl, "U"] \\ & \mathrm{Cat}_{\mathcal{T}} & \end{tikzcd}$$

The image of $(-)^{I-\sqcup}$ is spanned by the I-symmetric monoidal ∞-categories whose I-admissible indexed tensor functors $\otimes^S : \mathcal{C}_S \to \mathcal{C}_V$ are left adjoint to the indexed diagonal $\Delta^S : \mathcal{C}_V \to \mathcal{C}_S$ (i.e. whose indexed tensor products are are indexed coproducts), and the image of $(-)^{I-\times}$ is spanned by those whose I-admissible indexed tensor functors $\otimes^S$ are right adjoint to Δ^S.

We call I-symmetric monoidal ∞-categories of the form $\mathcal{C}^{I-\sqcup}$ *cocartesian*, and $\mathcal{C}^{I-\times}$ *cartesian*. Before characterizing the algebras in these, we point out that these are often presentable.

Proposition 1.33. *Suppose $\mathcal{C}$ is a presentable ∞-category*

1. *$\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}$ is I-presentably symmetric monoidal under the cocartesian structure.*
2. *If finite products in $\mathcal{C}$ commute with colimits separately in each variable (i.e. it is Cartesian closed), then $\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}$ is I-presentably symmetric monoidal under the cartesian structure.*

Proof. It follows from Hilman's characterization of parameterized presentability [Hil24, Thm 6.1.2] that $\underline{\mathrm{Coeff}}^{\mathcal{T}}$ is presentable, so we're tasked with proving that the $\mathcal{T}$-symmetric monoidal structures are distributive. The first case is just commutativity of colimits with colimits, and the second is [NS22, Prop 3.2.5]. ☐

Additionally, I-indexed tensor products of algebras in cartesian I-symmetric monoidal ∞-categories are indexed products.

Proposition 1.34. *$\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}\left(\mathcal{C}^{I-\times}\right)$ is a cartesian I-symmetric monoidal ∞-category.*

Proof. The forgetful $\mathcal{T}$-functor $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}^{I-\times}) \to \mathcal{C}^{I-\times}$ is I-symmetric monoidal, product-preserving, and conservative. Better strategy: explicitly and naturally determine its $\mathcal{P}$-algebras in $\mathcal{T}$-spaces using the Segal object description. ☐

We would like to interpret algebras in $\mathcal{C}^{I-\times}$ purely in terms of $\mathcal{C}$ using the following definition.

Definition 1.35. Fix $\mathcal{O}^{\otimes}$ an I-operad. Then, an *$\mathcal{O}$-monoid in $\mathcal{C}$* is a $\mathcal{T}$-functor $M : \mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathcal{C}$ satisfying the condition that, for each orbit $V \in \mathcal{T}$, each finite V-set $S \in \mathbb{F}_V$, and each S-tuple $X = (X_U) \in \mathcal{O}_S$, the canonical maps $M(X) \to \mathrm{CoInd}_U^V M(X_U)$ realize $M(X)$ as the indexed product

$$M(X) \simeq \prod_U^S M(X_U). \qquad \triangleleft$$

In Appendix A, we prove the following equivariant lift of [HA, Prop 2.4.2.5].

Proposition 1.36. *The postcomposition functor*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\times}) \to \mathrm{Fun}_{\mathcal{T}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}, \mathcal{C})$$

is fully faithful with image spanned by the $\mathcal{O}$-monoids.

Corollary 1.37. *The postcomposition functor*

$$\mathrm{Alg}_{\mathcal{O}}(\underline{\mathrm{Coeff}}^G(\mathcal{D})^{I-\times}) \simeq \mathrm{Fun}(\mathrm{Tot}\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}, \mathcal{D})$$

is fully faithful with image spanned by $\mathrm{Seg}_{\mathrm{Tot}\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}}(\mathcal{D})$.

Proof. After Proposition 1.36, it suffices to characterize the image of $\mathcal{O}$-monoids under the equivalence

$$\mathrm{Fun}(\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes}, \mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{T}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}, \underline{\mathrm{Coeff}}_G(\mathcal{D})).$$

By [Nar17, Ex 1.17], given a finite V-set $S \in \mathbb{F}_V$ and writing $\mathrm{Tot}S \simeq \coprod_{U \in \mathrm{Orb}(S)} \mathcal{T}_{/U}$ for the total ∞-category of the associated V-category, the above identification turns S-indexed products into right Kan extensions:

$$\begin{tikzcd} \mathrm{Fun}_{\mathcal{T}}(S, \underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{D})) \arrow[r, "\prod^S"] \arrow[d, phantom, "\wr"] & \mathrm{Coeff}^{\mathcal{T}}(\mathcal{D}) \arrow[d, phantom, "\wr"] \\ \mathrm{Fun}(\mathrm{Tot}S, \mathcal{D}) \arrow[r, "\mathrm{RKE}"] & \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{D}) \end{tikzcd}$$

Thus the image of $\mathrm{Mon}_{\mathcal{O}}(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D})$ consists of the functors $\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathcal{D}$ whose image of an object $((X_U), S) \in \mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes}$ is right Kan extended along elementary maps which is what we want by the identification of fixed points of indexed limits $\square$

Corollary 1.38. *The postcomposition functor*

$$\mathrm{Alg}_{\mathcal{O}}(\underline{\mathrm{Coeff}}^{G}(\mathcal{D})^{I-\times}) \simeq \mathrm{Fun}(\mathrm{Tot}\mathcal{O}^{\otimes}, \mathcal{D})$$

is fully faithful with image spanned by $\mathrm{Seg}_{\mathrm{Tot}\mathcal{O}^{\otimes}}(\mathcal{D})$.

Proof. This follows from [Ste25a] and Corollary 1.37. $\square$

Of fundamental importance is the following corollary to Proposition 1.36, which interprets I-commutative monoids as *operad algebras*.

Corollary 1.39 ("CMon = CAlg"). *There is a canonical equivalence* $\underline{\mathrm{CMon}}_I(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_I(\mathcal{C}^{I-\times})$ *over* $\mathcal{C}$.

Proof. By Proposition 1.36, I-commutative algebras in $\mathcal{C}^{I-\times}$ are I-semiadditive functors $\underline{\mathbb{F}}_{I,*} \to \mathcal{C}$. Our proof is similar to that of [Nar16, Thm 6.5]; There is a pullback square over $\mathcal{C}$

$$\begin{tikzcd} \mathrm{CMon}_I(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{CAlg}_I(\mathcal{C}^{I-\times}) \arrow[r, phantom, "\simeq"] & \mathrm{Fun}^{I-\oplus}(\underline{\mathbb{F}}_{I,*}, \mathcal{C}) \arrow[d] \\ \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}})) \arrow[rr] & & \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{I-\oplus}(\underline{\mathbb{F}}_{I,*}, \underline{\mathcal{S}}_{\mathcal{T}})) \end{tikzcd}$$

so it suffices to prove this in the case $\mathcal{C} = \underline{\mathcal{S}}_{\mathcal{T}}$. There, we simply compose equivalences as follows

$$\begin{tikzcd} \mathrm{CMon}_I(\underline{\mathcal{S}}_{\mathcal{T}}) \arrow[rrr, "\sim"] \arrow[d, "1.5"'] & & & \mathrm{CAlg}_I(\underline{\mathcal{S}}_{\mathcal{T}}^{I-\times}) \\ \mathrm{CMon}_I(\mathcal{S}) \arrow[r] & \mathrm{Seg}_{\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}(\mathcal{S}) \arrow[r] & \mathrm{Seg}_{\underline{\mathbb{F}}_{I,*}} \arrow[r] & \mathrm{Fun}_{\mathcal{T}}^{I-\oplus}(\underline{\mathbb{F}}_{I,*}, \underline{\mathcal{S}}_{\mathcal{T}}) \arrow[u, "1.36"'] \end{tikzcd}$$

where each of the bottom arrows are shown to be equivalences in Appendix A of [Ste25a]. $\square$

Remark 1.40. As with much of the rest of this subsection, Corollary 1.39 possesses an alternative strategy where both are shown to furnish the I-semiadditive closure, the latter using [CLL24, Thm B]. The above argument was chosen for brevity, as its requisite parts are also needed elsewhere. $\triangleleft$

Remark 1.41. In the case $\mathcal{C} \simeq \underline{\mathcal{S}}_G$, the analogous result was recently proved in [Mar24] for the ∞-category of algebras over the *graph* G*-operads* corresponding with indexing systems. To the knowledge of the author, this is one of the first concrete indications that the genuine operadic nerve of [Bon19] may induce equivalences between ∞-categories of algebras. $\triangleleft$

The cocartesian situation is more simple: the forgetful functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\sqcup}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \mathcal{C})$ is an *equivalence*. We study this more fully in Appendix A and Section 2.1. In order to do so, we note the following.

Proposition 1.42 (Equivariant [GGN15, Prop 2.3]). *Suppose $\mathcal{C}$ is a $\mathcal{T}$-∞-category with I-indexed products and coproducts. Then, the following conditions are equivalent.*

(a) $\mathcal{C}$ is I-semiadditive.
(b) There exists an I-symmetric monoidal equivalence $\mathcal{C}^{I-\times} \simeq \mathcal{C}^{I-\sqcup}$ lifting the identity.
(c) The forgetful $\mathcal{T}$-functor $\underline{\mathrm{CMon}}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence.

Proof. Given (a), the I-admissible indexed product maps $\prod_U^S : \mathcal{C}_S \to \mathcal{C}_V$ are *left* adjoint to the restriction map $\Delta^S : \mathcal{C}_V \to \mathcal{C}_S$, so by Proposition A.12, the identity on $\mathcal{C}$ lifts to a symmetric monoidal functor $\mathcal{C}^{I-\times} \to \mathcal{C}^{I-\sqcup}$. We will see in Lemma 1.7 that an I-symmetric monoidal functor is an I-symmetric monoidal equivalence if and only if its underlying $\mathcal{T}$-functor is an equivalence, so this implies (b).

The implication (b) $\implies$ (c) is just Corollary 1.39 and Lemma A.10 and the implication (b) $\implies$ (c) follows from the fact that $\underline{\mathrm{CMon}}_I(\mathcal{C})$ is I-semiadditive [CLL24, Thm B]. □

2. I-commutative algebras

Philosophical remark 2.1. On one hand, it follows from Proposition 1.11 that $\mathcal{T}$-operads are determined conservatively by their theories of *algebras on $\mathcal{T}$-symmetric monoidal categories*; indeed, it suffices to characterize their algebras in the case $\underline{\mathcal{S}}_{\mathcal{T}}^{\mathcal{T}-\times}$.

On the other hand, the right adjoint $\mathrm{Cat}_{\mathcal{T}}^{\otimes} \to \mathrm{Op}_{\mathcal{T}}$ is full on cores, since automorphisms in a slice category $\mathrm{Cat}_{\mathcal{C}}$ automatically preserve cocartesian morphisms. Hence the associated map of spaces

$$\begin{tikzcd} \mathrm{Cat}_{\mathcal{T}}^{\otimes,\simeq} \arrow[r] & \mathrm{Op}_{\mathcal{T}}^{\simeq} \arrow[r] & \mathrm{Fun}(\mathrm{Op}_{\mathcal{T}}, \mathrm{Cat})^{\simeq} \\ \mathcal{C}^{\otimes} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[rr, mapsto] & & \mathrm{Alg}_{(-)}(\mathcal{C}) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

is a summand inclusion. That is, a $\mathcal{T}$-symmetric monoidal category is determined (functorially on equivalences) by its categories of $\mathcal{O}$-algebras for each $\mathcal{O} \in \mathrm{Op}_{\mathcal{T}}$. ◁

Following along these lines and using Proposition 1.36, we will generally characterize algebraic theories in *arbitrary* $\mathcal{T}$-symmetric monoidal ∞-categories by reducing to the universal case of $\underline{\mathcal{S}}_{\mathcal{T}}^{\mathcal{T}-\times}$, which we study using category theoretic means. Indeed, in Section 2.1 we use this to bootstrap I-semiadditivity of $\underline{\mathrm{CMon}}_I(\mathcal{C})$ to I-cocartesianness of $\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{C})$ for $\mathcal{C}^{\otimes}$ an arbitrary I-symmetric monoidal ∞-category. Using work from Appendix A, we use this to conclude lifts of Theorem B and Corollary C.

We take this to its logical extreme in Section 2.2, using this to completely characterize the smashing localizations associated with $\otimes$-idempotent weak $\mathcal{N}_\infty$-operads. As promised in the introduction, we use this classification to prove a generalization of Theorem E. Following this, in Section 2.3 we show that our results are sharp; if I is not almost essentially unital, then $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ fails to be connected, so $\mathcal{N}_{I\infty}^{\otimes}$ is idempotent under $\overset{\mathrm{BV}}{\otimes}$ if and only if I is almost essentially unital.

2.1. Indexed tensor products of I-commutative algebras.

Fix I an almost-unital weak indexing system. In Lemma A.10, we show that every object in a cocartesian I-symmetric monoidal structure bears a canonical I-commutative algebra algebra structure, i.e. $\underline{\mathrm{CAlg}}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence. In this subsection, we demonstrate the converse, or equivalently, we demonstrate that I-indexed tensor products of I-commutative algebras are indexed coproducts.

2.1.1. *The I-symmetric monoidal case.* The primary case of interest is the following.

Theorem 2.2 (Indexed tensor products of $\mathcal{N}_\infty$-algebras)**.** *The following are equivalent for $\mathcal{C}^{\otimes} \in \mathrm{Cat}_I^{\otimes}$.*

- *(a) For all morphisms $f : S \to T$ in $\mathcal{I}$, the action map $f_{\otimes} : \mathcal{C}_S \to \mathcal{C}_T$ is left adjoint to $f^* : \mathcal{C}_T \to \mathcal{C}_S$.*
- *(b) There is an I-symmetric monoidal equivalence $\mathcal{C}^{\otimes} \simeq \mathcal{C}^{I-\sqcup}$ extending the identity on $\mathcal{C}$.*
- *(c) For all unital I-operads $\mathcal{O}^{\otimes}$, the forgetful functor $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O}, \mathcal{C})$ is an equivalence.*
- *(d) The forgetful functor $\underline{\mathrm{CAlg}}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*

In order to prove Theorem 2.2, we introduce yet another condition:

- (b') There is an I-symmetric monoidal equivalence $\mathcal{C}^{\otimes} \simeq \mathcal{C}^{I-\sqcup}$.

The implication (b') $\implies$ (c) is precisely the computation Lemma A.10. For the implication (c) $\implies$ (b'), note that Lemma 1.30 implies that $\mathcal{C}^{\otimes}$ is unital; hence Yoneda's lemma applied to $\mathrm{Op}_I^{\mathrm{uni}}$ constructs an I-operad equivalence $\mathcal{C}^{\otimes} \simeq \mathcal{C}^{I-\sqcup}$, which is an I-symmetric monoidal equivalence by Philosophical remark 2.1.

Furthermore, the implication (b') $\implies$ (a) follows by definition, (a) $\implies$ (b) is precisely Theorem A', and the statements (b) $\implies$ (b') and (c) $\implies$ (d) follow by neglect of assumptions. To summarize, we've arrived at the implications

$$\begin{tikzcd} & \text{(b)} \arrow[d, Rightarrow] & & \\ \text{(a)} \arrow[ur, Rightarrow] & & \text{(c)} \arrow[r, Rightarrow] & \text{(d)} \\ & \text{(b')} \arrow[ul, Rightarrow] \arrow[ur, Leftrightarrow] & & \end{tikzcd} \tag{9}$$

Our workhorse lemma for closing the gap is the following.

Lemma 2.3. *The following are equivalent for $\mathcal{P}^\otimes \in \mathrm{Op}_I$:*

(e) The $\mathcal{T}$-∞-category $\underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive.

(f) For all $\mathcal{O}^\otimes \in \mathrm{Op}_I^{\mathrm{uni}}$, the forgetful functor

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}^\otimes_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \underline{\mathcal{S}}_{\mathcal{T}})$$

is an equivalence.

(g) For all $\mathcal{O}^\otimes \in \mathrm{Op}_I^{\mathrm{uni}}$, the map $\mathrm{triv}^\otimes_{\mathcal{O}} \otimes^{\mathrm{BV}} \mathcal{P}^\otimes \to \mathcal{O}^\otimes \otimes^{\mathrm{BV}} \mathcal{P}^\otimes$ is an equivalence.

(h) For all $\mathcal{O}^\otimes \in \mathrm{Op}_I^{\mathrm{uni}}$ and $\mathcal{C} \in \mathrm{Cat}_I^\otimes$, the forgetful functor

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}^\otimes_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \mathcal{C})$$

is an equivalence.

Proof. Since Proposition 1.34 shows that $\underline{\mathrm{Alg}}^\otimes_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is cartesian, the equivalence between (e) $\iff$ (f) is just (a) $\iff$ (c) applied to $\underline{\mathrm{Alg}}^\otimes_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$. (f) $\implies$ (g) follows from Proposition 1.11, and the implications (g) $\implies$ (h) $\implies$ (f) are obvious. $\square$

Proof of Theorem 2.2. After the implications illustrated in Eq. (9), it suffices to prove that $\underline{\mathrm{CAlg}}_I(\mathcal{C})$ satisfies (c) for all $\mathcal{C}^\otimes \in \mathrm{Cat}_I^\otimes$; by Lemma 2.3, it suffices to prove that $\underline{\mathrm{CAlg}}_I(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive. But in fact, by Corollary 1.39 there is an equivalence $\underline{\mathrm{CAlg}}_I(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}})$ and the latter is I-semiadditive by Cnossen-Lenz-Linsken's semiadditive closure theorem Theorem 1.6. $\square$

Rephrasing things somewhat, we've arrive at the following theorem.

Theorem B'. *Let $\mathcal{O}^\otimes$ be an almost-E-reduced $\mathcal{T}$-operad. Then, the following properties are equivalenent.*

(a) The $\mathcal{T}$-∞-category $\underline{\mathrm{Alg}}_{\mathcal{O}}\underline{\mathcal{S}}_{\mathcal{T}}$ is $A\mathcal{O}$-semiadditive.

(b) The unique map $\mathcal{O}^\otimes \to \mathcal{N}^\otimes_{A\mathcal{O}\infty}$ is an equivalence.

Furthermore, for any almost-E-unital weak indexing system I and I-symmetric monoidal ∞-category $\mathcal{C}^\otimes$, the I-symmetric monoidal ∞-category $\underline{\mathrm{CAlg}}^\otimes_I\mathcal{C}$ is cocartesian.

Proof. By Lemma 2.3 and Theorem 2.2, Condition (a) is equivalent to the condition that $\underline{\mathrm{Alg}}^\otimes_{\mathcal{O}}(\mathcal{C})$ is $A\mathcal{O}$-cocartesian for all $\mathcal{C}$. In fact by Theorem 2.2, this is equivalent to existence of the first equivalence in

$$\mathrm{CAlg}^\otimes_{A\mathcal{O}}\underline{\mathrm{Alg}}^\otimes_{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{CAlg}}_{A\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}),$$

which by Yoneda's lemma is equivalent to the unique map $\mathcal{O}^\otimes \to \mathcal{N}^\otimes_{A\mathcal{O}\infty}$ being an equivalence, i.e. Condition (b). The remaining statement follows immediately from Theorem 2.2. $\square$

Corollary 2.4. *Let $\mathcal{O}^\otimes$ be a reduced I-operad. Then, the canonical map $F : \mathcal{N}^\otimes_{I\infty} \to \mathcal{N}^\otimes_{I\infty} \otimes \mathcal{O}^\otimes$ is an equivalence.*

Proof. By Theorem 2.2, the forgetful map

$$F^* : \mathrm{Alg}_{\mathcal{O}\otimes\mathcal{N}_{I\infty}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}^\otimes_{\mathcal{N}_{I\infty}}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{N}_{I\infty}}(\mathcal{C})$$

is an equivalence for all distributive G-symmetric monoidal categories $\mathcal{C}$; the statement follows by specializing to $\mathcal{C} := \underline{\mathcal{S}}_G$ and applying Proposition 1.11. $\square$

2.1.2. *Applications to the underlying $\mathcal{T}$-∞-category.* The following corollary follows immediately from Theorem A' and Theorem 2.2

Corollary 2.5. *Suppose I is almost-unital. Then, the restriction $U_{\mathrm{uni}} : \underline{\mathrm{Op}}_I^{\mathrm{uni}} \to \underline{\mathrm{Cat}}_{\mathcal{T}}$ is left $\mathcal{T}$-adjoint to $(-)^{I-\sqcup}$.*

Warning 2.6. Corollary 2.5 shows that no nontrivial $\mathcal{T}$-colimit of one-color $\mathcal{T}$-operads has one color; in particular, no one-color $\mathcal{T}$-operads are the result of a nontrivial induction. ◁

Furthermore, Theorem 2.2 yields equivalences

$$\begin{aligned}\mathrm{CAlg}_{\mathcal{T}} \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}^{I-\sqcup}) &\simeq \mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{CAlg}}_{\mathcal{T}}^{\otimes}(\mathcal{C}^{I-\sqcup}) \\ &\simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\sqcup}),\end{aligned}$$

for all $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}^{\mathrm{uni}}$, so another application of Theorem 2.2 implies the following.

Corollary 2.7. *Suppose $\mathcal{O}^{\otimes}$ is a unital I-operad and $\mathcal{C}$ admits I-indexed coproducts. Then, the I-symmetric monoidal category $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}\left(\mathcal{C}^{I-\sqcup}\right)$ is cocartesian.*

We use this to compute the $\mathcal{T}$-category underlying BV tensor products.

Proposition 2.8. *The underlying category $U|_{\mathrm{uni}} : \mathrm{Op}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Cat}_{\mathcal{T}}$ functor sends*

$$U\left(\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}\right) \simeq U(\mathcal{O}^{\otimes}) \times U(\mathcal{P}^{\otimes}).$$

Proof. Corollaries 2.5 and 2.7 together yield a string of equivalences

$$\begin{aligned}\mathrm{Fun}_{\mathcal{T}}\left(U\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right), \mathcal{C}\right) &\simeq \mathrm{Alg}_{\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{P}}\left(\mathcal{C}^{I-\sqcup}\right) \\ &\simeq \mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}\left(\mathcal{C}^{I-\sqcup}\right) \\ &\simeq \mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{Fun}}_{\mathcal{T}}\left(U(\mathcal{P}^{\otimes}), \mathcal{C}\right)^{I-\sqcup} \\ &\simeq \mathrm{Fun}_{\mathcal{T}}\left(U(\mathcal{O}^{\otimes}), \underline{\mathrm{Fun}}_{\mathcal{T}}\left(U(\mathcal{P}^{\otimes}), \mathcal{C}\right)\right) \\ &\simeq \mathrm{Fun}_{\mathcal{T}}\left((U(\mathcal{O}^{\otimes}) \times U(\mathcal{P}^{\otimes}), \mathcal{C}\right),\end{aligned}$$

so the result follows by Yoneda's lemma. □

Applying Observation 1.20 and Propositions 1.21 and 2.8, we acquire the following.

Corollary 2.9. *The full subcategories $\mathrm{Op}_{\mathcal{T}}^{\mathrm{red}} \subset \mathrm{Op}_{\mathcal{T}}^{a\mathrm{red}} \subset \mathrm{Op}_{\mathcal{T}}^{aE\,\mathrm{red}} \subset \mathrm{Op}_{\mathcal{T}}$ are closed under tensor products.*

2.2. **The smashing localization for $\mathcal{N}_{I\infty}^{\otimes}$ and the main theorem.** Need to say an I-operad case!!! In view of Corollary 2.9, when I is an almost-reduced weak indexing category the map $\mathbb{E}_0^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$ witnesses $\mathcal{N}_{I\infty}^{\otimes}$ as an idempotent object in $\mathrm{Op}_{\mathcal{T}}^{\otimes}$. To conclude Theorem E from this, we will characterize the associated smashing localization.

2.2.1. *The smashing localization classified by $\mathcal{N}_{I\infty}^{\otimes}$.* We will prove the following theorem.

Theorem 2.10. *Let I be an almost essentially unital weak indexing system. Then, an at-most one color $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ possesses an equivalence $\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{O}^{\otimes}$ if and only if the following conditions are satisfied:*

(a) $c(\mathcal{O}) = c(I)$.

(b) The canonical map $\mathrm{Bor}_I^{\mathcal{T}} \mathcal{O}^{\otimes} \to \mathcal{N}_I^{\otimes}$ is an equivalence.

Remark 2.11. Condition (b) of Theorem 2.10 is equivalent to the condition that, for all $\mathcal{P}^{\otimes} \in \mathrm{Op}_{I \cap c(\mathcal{O})}$ and $\mathcal{C} \in \mathrm{Cat}_{\mathcal{T}}^{\otimes}$, the forgetful map $\mathrm{Alg}_{\mathcal{P}} \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is an equivalence; by Theorem 2.2, this in turn is equivalent to the condition that, for all $\mathcal{C}$ (or just $\mathcal{C} = \underline{\mathcal{S}}_{\mathcal{T}}$) and all I-admissible $c(\mathcal{O})$-sets S, the S-indexed tensor products in $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ are indexed coproducts. ◁

In fact, by the arity support computation Proposition 1.21, Theorem 2.10 is equivalent to the following.

Proposition 2.12. *Let I be an almost-unital weak indexing system. Then, a one color $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ satisfies $\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \simeq \mathcal{O}^{\otimes}$ if and only if $\mathrm{Bor}_I^{\mathcal{T}} \mathcal{O}^{\otimes} \simeq \mathcal{N}^{\otimes}_{I\infty}$.*

Proof. First assume that $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \simeq \mathcal{O}^{\otimes}$. By Corollary 2.4, we have

$$\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathbb{E}^{\otimes}_{0,v(I)} \simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}^{\otimes}_{0,v(I)},$$

so Proposition 1.31 implies that $\mathcal{O}^{\otimes}$ is $v(I)$-unital, i.e. $\mathrm{Bor}_I^{\mathcal{T}} \mathcal{O}^{\otimes}$ is a unital I-operad. Thus, in light of Remark 2.11, it suffices to note that the equivalence $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \simeq \mathcal{O}^{\otimes}$ demonstrates that the canonical map

$$\begin{aligned} \mathrm{CAlg}_I(\underline{\mathcal{S}}_{\mathcal{T}}) &\xleftarrow{\sim} \mathrm{Alg}_{\mathrm{Bor}_I^{\mathcal{T}} \mathcal{O}} \underline{\mathrm{CAlg}}^{\otimes}_I(\underline{\mathcal{S}}_{\mathcal{T}}) \\ &\simeq \mathrm{CAlg}^{\otimes}_I \underline{\mathrm{Alg}}^{\otimes}_{\mathrm{Bor}_I^{\mathcal{T}} \mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}}) \\ &\to \mathrm{Alg}_{\mathrm{Bor}_I^{\mathcal{T}} \mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}}) \end{aligned}$$

is an equivalence, so Proposition 1.11 proves that $\mathrm{Bor}_I^{\mathcal{T}} \mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{I\infty}$ is an equivalence. In particular, this implies the converse to Proposition 1.31; the converse to the proposition then follows by noting that each of the above arguments works in reverse. □

2.2.2. *The proof of the main theorem.* We are finally ready for Theorem E. We start with the unital case.

Proposition 2.13. *When I and J are almost-unital, there is an equivalence $\mathcal{N}^{\otimes}_{I\infty} \overset{BV}{\otimes} \mathcal{N}^{\otimes}_{J\infty} \simeq \mathcal{N}^{\otimes}_{I \vee J\infty}$.*

Proof. By [CSY20, Prop 5.1.8], $\mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{J\infty} \in \mathrm{Op}^{\mathrm{red}}_{\mathcal{T}} \simeq \mathrm{Op}^{\mathrm{red}}_{\mathcal{T},\mathbb{E}_0/}$ is an idempotent object classifying the conjunction of the properties which are classified by $\mathcal{N}^{\otimes}_{I\infty}$ and $\mathcal{N}^{\otimes}_{\infty}$; that is, a unital $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ is fixed by $(-) \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{J\infty}$ if and only if $\underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive and J-semiadditive. By Proposition 1.3, this is equivalent to the property that $\underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is $I \vee J$-semiadditive , i.e. $\mathcal{O}^{\otimes}$ is fixed by $(-) \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I \vee J}$. Thus, we have

$$\mathcal{N}^{\otimes}_{I \vee J} \simeq \mathcal{N}^{\otimes}_{I \vee J} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\overset{\mathrm{BV}}{\otimes}}_{J\infty} \simeq \mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}.$$ □

We may now conclude the full theorem, which we restate in the orbital case.

Theorem E'. *The functor $\mathcal{N}^{\otimes}_{(-)\infty} : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T}}$ lifts to a fully faithful $\mathcal{T}$-right adjoint*

$$\begin{tikzcd} \underline{\mathrm{wIndex}}_{\mathcal{T}} \arrow[rr, bend right, "\mathcal{N}^{\otimes}_{(-)\infty}"'] \arrow[rr, phantom, "\perp"] & & \underline{\mathrm{Op}}_{\mathcal{T}} \arrow[ll, bend right, "A"'] \end{tikzcd}$$

whose restriction $\underline{\mathrm{wIndex}}^{aE\mathrm{uni}}_{\mathcal{T}} \subset \underline{\mathrm{Op}}_{\mathcal{T}}$ is symmetric monoidal. Furthermore, the resulting tensor product on $\mathrm{wIndex}^{aE\mathrm{uni},\otimes}_{\mathcal{T}}$ is computed by the Borelified join

$$I \otimes J = \mathrm{Bor}^{\mathcal{T}}_{\mathrm{cSupp}(I \cap J)}(I \vee J);$$

in particular, when I and J are almost-E-unital weak indexing systems, we have

$$\begin{aligned} \mathcal{N}^{\otimes}_{I\infty} \otimes \mathcal{N}^{\otimes}_{J\infty} &\simeq \mathcal{N}^{\otimes}_{(I \vee J)\infty} \otimes \mathrm{triv}^{\otimes}_{c(I \cap J)} \\ \mathcal{N}^{\otimes}_{I\infty} \times \mathcal{N}^{\otimes}_{J\infty} &\simeq \mathcal{N}^{\otimes}_{(I \cap J)\infty} \\ \mathrm{Res}^W_V \mathcal{N}^{\otimes}_{I\infty} &\simeq \mathcal{N}^{\otimes}_{\mathrm{Res}^W_V I\infty} \\ \mathrm{CoInd}^W_V \mathcal{N}^{\otimes}_{I\infty} &\simeq \mathcal{N}^{\otimes}_{\mathrm{CoInd}^W_V I\infty}. \end{aligned}$$

Hence W-norms of I-commutative algebras are $\mathrm{CoInd}^W_V I$-commutative algebras, and when I, J are almost-unital, we have

$$\underline{\mathrm{CAlg}}^{\otimes}_I \underline{\mathrm{CAlg}}^{\otimes}_J(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_{I \vee J}(\mathcal{C}). \tag{10}$$

Proof of Theorem E'. The $\mathcal{T}$-adjunction is precisely Proposition 1.23, the equations are immediate from the symmetric monoidal adjunction, the statement about norms of I-commutative algebras is Construction 1.27, and Eq. (10) follows immediately from symmetric monoidality of $\mathcal{N}_{(-)\infty}^{\otimes}$. We are left with proving that the adjunction is symmetric monoidal in the aE-unital case.

In view of Proposition 1.21, to prove that this is a $\mathcal{T}$-symmetric monoidal adjunction with the prescribed tensor product, it suffices to prove that the collection of aE-unital weak $\mathcal{N}_\infty$-operads is $\overset{\mathrm{BV}}{\otimes}$-closed, for which it suffices to prove that for all aE-unital weak indexing systems I and J, the unique map $\varphi : \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \to \mathcal{N}_{I\vee J}^{\otimes}$ is an equivalence. In fact, by **??**, it suffices to prove that $\mathrm{Bor}_{v(I\cap J)}^{\mathcal{T}}(\varphi)$ is an equivalence, i.e. we may assume that I and J are unital. Then, the statement is precisely Proposition 2.13. □

2.2.3. *The smashing localization classified by $\mathcal{N}_{I\infty}^{\otimes}$ in the many-colors case.* We now take a detour into a variant of Theorem 2.10 that the author has not found to be much more useful, but nevertheless conceptually illuminating. We begin with a contrived observation.

Observation 2.14. The computation of [HA, § 2.3.1] and resulting theory may be stated simply: the operad $\mathbb{E}_0^{\otimes}$ is an idempotent object in $\mathrm{Op}^{\otimes}$ under the unique map $\mathrm{triv}^{\otimes} \to \mathbb{E}_0^{\otimes}$, and the corresponding smashing localization classifies unital (colored) operads. In particular, a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is $(\mathbb{E}_0^{\otimes} \overset{\mathrm{BV}}{\otimes} -)$-local if and only if the unit object $1_{\mathcal{C}} \in \mathcal{C}^{\otimes}$ is initial, i.e. for all $X \in \mathcal{C}$, the unique map $X^{\sqcup\varnothing} \to X^{\otimes\varnothing}$ is an equivalence; that is, $\left(\mathbb{E}_0^{\otimes} \overset{\mathrm{BV}}{\otimes} -\right)$-locality classifies the I^0-cocartesian symmetric monoidal ∞-categories. ◁

This motivates the following propositions. We say $\mathcal{O}^{\otimes}$ is *I-cocartesian* if the identity on $\mathcal{O}$ is adjunct to an equivalence $\mathrm{Bor}_I^{\mathcal{T}} \mathcal{O}^{\otimes} \xrightarrow{\sim} \mathcal{O}^{I-\sqcup}$. We begin with a more general version of Theorem 2.2 and Lemma 2.3.

Proposition 2.15. *Given I an almost-unital weak indexing category, the following are equivalent for $\mathcal{Q}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$.*

(b) $\mathcal{Q}^{\otimes}$ is I-cocartesian.
(c) For all unital I-operads $\mathcal{O}^{\otimes}$, the forgetful functor $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{Q}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O}, \mathcal{Q})$ is an equivalence.
(d) The forgetful functor $\underline{\mathrm{CAlg}}_I(\mathcal{Q}) \to \mathcal{Q}$ is an equivalence.

Moreover, given $\mathcal{P}^{\otimes} \in \mathrm{Op}_I$, the conditions of Lemma 2.3 are equivalent to the following.

(j) For all $\mathcal{O}^{\otimes} \in \mathrm{Op}_I^{\mathrm{uni}}$ and $\mathcal{Q}^{\otimes} \in \mathrm{Op}_I$, the forgetful functor

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \mathcal{C})$$

is an equivalence

Proof. The implications (f) $\implies$ (j) $\implies$ (g) are obvious. The remaining arguments follow nearly verbatim, only needing $\mathcal{C}$ to be replaced with $\mathrm{Bor}_I^{\mathcal{T}} \mathcal{O}$ and the reference to Lemma 2.3 replaced with Proposition 2.15. □

We can now classify $\mathcal{N}_{I\infty}^{\otimes}$-modules in the general case.

Proposition 2.16. *Let I be an almost essentially unital weak indexing category and $\mathcal{O}^{\otimes}$ a $\mathcal{T}$-operad. Then, $\mathcal{O}^{\otimes}$ admits an (essentially unique) $\mathcal{N}_{I\infty}^{\otimes}$-module structure if and only if the following conditions hold:*

(a) $c(\mathcal{O}) = c(I)$, and
(b) $\mathcal{O}^{\otimes}$ is I-cocartesian.

Proof. By the same argument as Theorem 2.10, we reduce to the almost-unital case. In this case, by a standard two-out-of-three argument, $\mathcal{C}^{\otimes}$ is local for the smashing localization associated with $\mathcal{N}_{I\infty}^{\otimes}$ if and only if pullback along the localization map $\mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ induces an equivalence $\mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}\otimes\mathcal{N}_{I\infty}}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ for all $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$, which is equivalent to the condition that $\mathrm{CAlg}_I^{\otimes}(\mathcal{C}) \to \mathcal{C}^{\otimes}$ is an equivalence by Yoneda's lemma; this is equivalent to $\mathcal{C}^{\otimes}$ being I-cocartesian by Proposition 2.15. □

2.2.4. *More on unitality.* The specialization of Theorem 2.2 and Proposition 2.16 to $I \cap I_0$ is ubiquitously useful, so we state it explicitly here. We continue to fix I an almost-unital weak indexing system.

Corollary 2.17. *Given $\mathcal{O}^{\otimes}$ an I-operad, the following conditions are equivalent:*

(a) For all unital I-operads $\mathcal{P}^{\otimes}$, the forgetful I-operad map $\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{O}) \to \mathcal{O}^{\otimes}$ is an equivalence.

(b) *The forgetful I-operad map* $\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{0,v(I)}}(\mathcal{O}) \to \mathcal{O}^{\otimes}$ *is an equivalence.*
(c) $\mathcal{O}^{\otimes}$ *is unital.*
(d) *There exists an equivalence* $\mathcal{O}^{\otimes} \simeq \mathbb{E}^{\otimes}_{0,v(I)} \overset{BV}{\otimes} \mathcal{O}^{\otimes}$.
(e) *For all I-operads $\mathcal{C}^{\otimes}$, the forgetful I-operad map* $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{0,v(I)}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ *is an equivalence.*
(f) *For all I-operads $\mathcal{C}^{\otimes}$, the I-operad* $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ *is unital.*
(g) *The $\mathcal{T}$-∞-category* $\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{S})$ *is $v(I)$-pointed.*

In particular, $\mathrm{Op}_I^{\mathrm{uni}} \subset \mathrm{Op}_I$ *is a smashing localization and cosmashing colocalization:*

$$\begin{tikzcd}
\mathrm{Op}_I^{\mathrm{uni}} \arrow[r, hook] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathrm{Op}_I \arrow[l, bend right=50, "\mathbb{E}^{\otimes}_{0,v(I)} \overset{BV}{\otimes} (-)"'] \arrow[l, bend left=50, "\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_0}(-)"]
\end{tikzcd}$$

The following implications are labeled with references to statements that directly imply them.

$$\begin{tikzcd}
(c) \arrow[r, Leftrightarrow, "2.16"] \arrow[d, Rightarrow, "2.2"'] & (d) \arrow[r, Leftrightarrow, "\text{Yoneda}"] & (e) \arrow[r, Rightarrow, "2.2"] & (f) \arrow[d, Rightarrow, "1.29"] \\
(a) \arrow[r, Rightarrow, "2.2"'] & (b) \arrow[ul, Rightarrow, "2.2"'] & & (g) \arrow[ull, Rightarrow, "2.18"']
\end{tikzcd}$$

Lemma 2.18. *(g) $\implies$ (d)*

Proof. If (g), then by Observation 1.29 and (c) $\implies$ (b), (d) applied to $\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{S})$ we have

$$\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{S}) \simeq \underline{\mathrm{Mon}}_{\mathbb{E}_{0,v(I)}}(\mathcal{S}) \simeq \underline{\mathrm{Mon}}_{\mathcal{O}\otimes\mathbb{E}_{0,v(I)}}(\mathcal{S}),$$

so Proposition 1.11 yields (d). □

Proof of Corollary 2.17. We have shown that all of the above conditions are equivalent. It follows from Corollary 2.4 that $\mathbb{E}^{\otimes}_{0,v(I)}$ is an idempotent algebra, and (c) $\iff$ (d) shows that the corresponding smashing localization is $\mathrm{Op}_I^{\mathrm{uni}} \subset \mathrm{Op}_I$. Moreover, (c) $\implies$ (f) shows that $\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_0}(-)$ lands in unital I-operads and (c) $\implies$ (e) together with the usual equivalence

$$\left(\Gamma^{\mathcal{T}} U \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{Q})\right)^{\simeq} \simeq \mathrm{Map}_{\mathrm{Op}_I}(\mathcal{P}, \mathcal{Q})$$

shows that it is right adjoint to the inclusion $\mathrm{Op}_I^{\mathrm{uni}} \hookrightarrow \mathrm{Op}_I$. □

2.3. **Failure of the nonunital equivariant Eckmann-Hilton argument.** We will say that a $\mathcal{T}$-operad with at most one color $\mathcal{O}^{\otimes}$ is *n-connected* if the nonempty structure spaces $\mathcal{O}(S)$ are each n-connected. We denote write the full subcategory of n-connected $\mathcal{T}$-operads as

$$\mathrm{Op}^{\le \mathrm{oc}}_{\mathcal{T},\ge n} \subset \mathrm{Op}^{\le \mathrm{oc}}_{\mathcal{T}}.$$

By Proposition 1.11, this is equivalent to the condition that the forgetful functor $\mathrm{CAlg}_{A\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is an equivalence, which itself is equivalent to the same condition in the case $\mathcal{C} \simeq \mathcal{S}_{\le n+1}$.

Corollary 2.19. *When $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ are n-connected almost essentially reduced $\mathcal{T}$-operads, $\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}$ is an n-connected almost essentially reduced $\mathcal{T}$-operad.*

Proof. In view of Theorem E', we have a string of natural equivalences

$$\begin{aligned}
\mathrm{Mon}_{\mathcal{O}\otimes\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\le n+1}) &\simeq \mathrm{Mon}_{\mathcal{O}}\underline{\mathrm{Mon}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\le n+1}) \\
&\simeq \mathrm{CMon}_{A\mathcal{O}}\underline{\mathrm{CMon}}_{A\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\le n+1}) \\
&\simeq \mathrm{CMon}_{A\mathcal{O}\vee A\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T},\le n+1}) \\
&\simeq \mathrm{CMon}_{A(\mathcal{O}\otimes\mathcal{P})}(\underline{\mathcal{S}}_{\mathcal{T},\le n+1}),
\end{aligned}$$

induced by the unique map $\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes \to \mathcal{N}^\otimes_{A(\mathcal{O}\otimes\mathcal{P})}$. By Proposition 1.11, this implies that $\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes$ is n-connected. $\square$

Remark 2.20. The unit object $\mathrm{triv}^\otimes_{\mathcal{T}} \in \mathrm{Op}_{\mathcal{T}}$ is n-connected for all n, so n-connected $\mathcal{T}$-operads are closed under k-fold tensor products for all $k \in \mathbb{N}$. ◁

The example $\mathrm{triv}^\otimes_{\mathcal{T}} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^\otimes \simeq \mathcal{O}^\otimes$ demonstrates that this is the best we can say without further assumptions on the $\mathcal{T}$-operads in question; the author hopes to return to this question in forthcoming work, constructing analogues to [SY19] under the assumption that $A\mathcal{O} = A\mathcal{P}$. For the rest of this section, we will demonstrate that this fails without the *almost essentially unital* assumption, and connect it to failure of the nonunital Eckmann-Hilton argument.

Observation 2.21. Fix I a weak indexing system. By Proposition 1.21, there is a contractible space of diagrams of the following form:

$$\mathcal{N}^\otimes_{I\infty} \simeq \mathcal{N}^\otimes_{I\infty} \otimes^{\mathrm{BV}} \mathrm{triv}^\otimes_{\mathrm{cSupp}(I)} \xrightarrow{\mathrm{id}\otimes^{\mathrm{BV}}\mathrm{can}} \mathcal{N}^\otimes_{I\infty} \otimes^{\mathrm{BV}} \mathcal{N}^\otimes_{I\infty} \to \mathcal{N}^\otimes_{I\infty};$$

furthermore, the composite $\mathcal{N}^\otimes_{I\infty} \to \mathcal{N}^\otimes_{I\infty}$ is homotopic to the identity since $\mathcal{N}^\otimes_{I\infty}$ has contractible endomorphism space. In particular, this implies that there is a unique natural *split codiagonal* diagram

$$\begin{tikzcd} & \mathrm{CAlg}_I \underline{\mathrm{CAlg}}^\otimes_I(-) \arrow[dr, "U"] & \\ \mathrm{CAlg}_I(-) \arrow[ur, "\delta"] \arrow[rr, equal] & & \mathrm{CAlg}_I(-) \end{tikzcd}$$

δ takes a structure to two interchanging copies of itself, and U simply forgets one of the structures. ◁

A weak ∞-categorical form of the *Eckmann-Hilton argument* for I-commutative algebras would state that the functor U is an equivalence, or equivalently, δ is an equivalence; the weakening of this to algebras in $(n+1)$-categories is that δ is an equivalence on $(n+1)$-categories, i.e. $\mathcal{N}^\otimes_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^\otimes_{I\infty}$ is n-connected. Unfortunately, this does not hold for all weak indexing systems I. The following simple counterexample to nonunital Eckmann-Hilton was pointed out to the author by Piotr Pstrągowski.

Example 2.22. Let R be a nonzero commutative ring. Then, the Abelian group underlying R supports an $\mathrm{Comm}^\otimes_{nu} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^\otimes_{nu}$ structure given by the two multiplications

$$\mu(r,s) = rs, \qquad\qquad \mu_0(r,s) = 0,$$

which are easily seen to satisfy interchange but be distinct. In particular, the associated $\mathrm{Comm}^\otimes_{nu} \otimes \mathrm{Comm}^\otimes_{nu}$-algebra is not in the essential image of the codiagonal

$$\mathrm{Alg}_{\mathrm{Comm}_{nu}}(\mathbf{Ab}) \to \mathrm{Alg}_{\mathrm{Comm}_{nu}} \underline{\mathrm{Alg}}_{\mathrm{Comm}_{nu}}(\mathbf{Ab}),$$

so δ is not an equivalence; by Proposition 1.11, this implies that $\mathrm{Comm}^\otimes_{nu} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}^\otimes_{nu}$ is not connected. ◁

In the positive direction, [SY19] yields a classification of $\overset{\mathrm{BV}}{\otimes}$-idempotent algebras in *reduced* ∞-operads. In fact, Example 2.22 shows that the associated unitality assumption only misses one example among nonequivariant one-color weak $\mathcal{N}_\infty$-operads.

Corollary 2.23. *A weak $\mathcal{N}_\infty$-$*$-operad $\mathcal{O}^\otimes$ possesses a map $\mathrm{triv}^\otimes \to \mathcal{O}^\otimes$ inducing an equivalence*

$$\mathcal{O}^\otimes \xrightarrow{\sim} \mathcal{O}^\otimes \otimes^{\mathrm{BV}} \mathcal{O}^\otimes$$

if and only if $\mathcal{O}^\otimes$ is equivalent to $\mathrm{triv}^\otimes$, $\mathbb{E}^\otimes_0$, or $\mathbb{E}^\otimes_\infty$.

Proof. [SY19, Cor 5.3.4] covers the unital case, so it suffices to assume that $\mathcal{O}(\varnothing) = \varnothing$ and show that $\mathcal{O}^\otimes \simeq \mathrm{triv}^\otimes$. Note that Comm_{nu} is the terminal nonunital $\mathcal{N}_\infty$-$*$-operad, i.e. there exists a map $\mathcal{O}^\otimes \to \mathrm{Comm}_{nu}$, yielding a diagram

$$\begin{tikzcd} \mathcal{O}^\otimes \otimes \mathcal{O}^\otimes \arrow[r] & \mathrm{Comm}^\otimes_{nu} \otimes \mathrm{Comm}^\otimes_{nu} \\ \mathcal{O}^\otimes \arrow[u] \arrow[r] & \mathrm{Comm}^\otimes_{nu} \arrow[u] \end{tikzcd}$$

Pulling back Example 2.22, we find that if $\mathcal{O}(n) = *$ for any $n \neq 1$, then R has a $\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{O}^\otimes$-structure that is not in the image of the diagonal; hence $\mathcal{O}(n) = \varnothing$ when $n \neq 1$, i.e. it's equivalent to $\mathrm{triv}^\otimes$. □

By [Ste24], this is precisely the list of almost-unital weak indexing systems for $*$. In this section, we introduce an equivariant analogue to this argument in order to prove the following proposition.

Proposition 2.24. *Suppose $\mathcal{N}_{I\infty}^\otimes \overset{BV}{\otimes} \mathcal{N}_{I\infty}^\otimes$ is connected. Then, I almost essentially unital.*

By combining Proposition 2.24 and Corollary 2.4, we conclude the remaining part of Corollary C.

Corollary 2.25. *$\mathcal{N}_{I\infty}^\otimes \otimes \mathcal{N}_{I\infty}^\otimes$ is a weak $\mathcal{N}_\infty$-operad if and only if I is almost essentially unital; in particular, there exists a (necessarily unique) map $\mathrm{triv}^\otimes \to \mathcal{N}_{I\infty}^\otimes$ inducing an equivalence $\mathcal{N}_{I\infty}^\otimes \xrightarrow{\sim} \mathcal{N}_{I\infty}^\otimes \overset{BV}{\otimes} \mathcal{N}_{I\infty}^\otimes$ if and only if I is almost-unital.*

To show Proposition 2.24, we pass to a universal case. First, the weak indexing system.

Recollection 2.26. In [Ste24], we computed the terminal weak indexing system with unit family $\mathcal{F}$ to be

$$\mathbb{F}_{\mathcal{F}^\perp - nu, V} = \begin{cases} \mathbb{F}_V & V \in \mathcal{F}; \\ \mathbb{F}_V - \{S \mid \forall U \in \mathrm{Orb}(S),\, U \in \mathcal{F}\} & V \notin \mathcal{F}; \end{cases}$$

in particular, $\underline{\mathbb{F}}_I$ fails to be almost essentially unital if and only if there is some non-contractible W-set in $\mathbb{F}_{v(I)^\perp - nu, W} \cap \mathbb{F}_{I,W}$ for some $W \in v(I)^\perp$. We refer to the associated weak indexing category as $I_{\mathcal{F}^\perp - nu}$; note that $I_{\mathcal{F}^\perp - nu} \subset \mathbb{F}_{\mathcal{T}}$ is the wide subcategory of maps $T \to S$ such that either $S, T \in \mathbb{F}_{\mathcal{F}}$ or $S, T \in \mathbb{F}_{\mathcal{F}^\perp}$. ◁

Now, we construct a family of problematic $\mathcal{N}_{I_{\mathcal{F}^\perp - nu}}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I_{\mathcal{F}^\perp - nu}}^\otimes$-algebras.

Construction 2.27. Let M be a $\mathcal{T}$-commutative monoid in pointed sets. We define a new functor

$$M^0 \colon h_1 \operatorname{Span}_{I_{\mathcal{F}^\perp - nu}}(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Set}_*$$

which agrees with M on objects, backwards maps, forwards maps lying in $\mathbb{F}_{\mathcal{F}}$, but whose forward maps lying in $\mathbb{F}_{\mathcal{F}^\perp}$ are zero. This is evidently functorial on backwards and forward maps, and the restriction to backwards maps is product-preserving. We're left with verifying the double coset formula that, given a cartesian square as on the left such that $S, T, R \in \mathcal{T}$ and $f, f' \in I_{\mathcal{F}^\perp - nu}$, the right square commutes, where $(-)_*$ denotes covariant functoriality and $(-)^*$ contravariant.

$$\begin{tikzcd} & R \times_S T \arrow[dl, "g'"'] \arrow[dr, "f'"] \arrow[dd, phantom, "\lrcorner"] & \\ R \arrow[dr, "f"'] & & T \arrow[dl, "g"] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} & M^0(R \times_S T) \arrow[dr, "f'_*"] & \\ M^0(R) \arrow[ur, "g'^*"] \arrow[dr, "f_*"'] & & \widetilde{M}^0(T) \\ & M^0(R) \arrow[ur, "g^*"'] & \end{tikzcd}$$

The assertions that $f, f' \in I_{\mathcal{F}^\perp - nu}$ and that $\mathcal{F}$ is a family together imply that $T \in \mathcal{F}$ if and only if the entire diagram lives in $\mathbb{F}_{\mathcal{F}}$, and $T \in \mathcal{F}^\perp$ if and only if the entire diagram lives in $\mathbb{F}_{\mathcal{F}^\perp}$. In the former case, the right diagram commutes by the double coset formula for M, and in the latter case it commutes as each composite map is zero. ◁

Lemma 2.28. *M and M_0 interchange.*

Proof. First note that $\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} I_{\mathcal{F}^\perp - nu}$ is unital, so Corollary 2.4 implies that $\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} I_{\mathcal{F}^\perp - nu}$-algebras interchange with themselves; in particular, the interchange relation of Recollection 1.14 for M implies that M and M_0 interchange for all V-equivariant pairs of operations with $V \in \mathcal{F}$.

Moreover, note that a diagram of $\mathcal{T}$-coefficient systems in a 1-category commutes if and only if the V-fixed point diagram commutes for all $V \in \mathcal{T}$; the V-fixed points of the diagram in **??** in our case correspond with the diagram

$$\begin{tikzcd} \left(X^T\right)^S \arrow[d, "(\mathrm{tr}_T)_S"'] & \simeq & X^{S \times T} & \simeq & \left(X^S\right)^T \arrow[r, "(\mathrm{tr}_S^0)_T"] & X^T \arrow[d, "\mathrm{tr}_T"] \\ X^S \arrow[rrrrr, "\mathrm{tr}_S^0"] & & & & & X \end{tikzcd}$$

where tr_* is the indexed multiplication in M and tr_*^0 is the indexed multiplication in M^0; when $V \in \mathcal{F}$, this commutes by the above argument, and when $V \in \mathcal{F}^\perp$, this commutes as each of the composites factor through a zero map. ☐

Construction 2.29. Since the "isomorphism classes of objects" functor $\pi_0 \colon \mathrm{Cat} \to \mathrm{Set}$ preserves limits, pushforward along it lifts to a functor

$$\underline{\pi}_0 \colon \mathrm{Cat}_{\mathcal{T}}^{\otimes} \simeq \mathrm{CMon}_{\mathcal{T}}(\mathrm{Cat}) \to \mathrm{CMon}_{\mathcal{T}}(\mathrm{Set});$$

we define the *Burnside $\mathcal{T}$-commutative monoid* as $\underline{A}_{\mathcal{T}} \coloneqq \underline{\pi}_0 \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$; we denote its image under the maps

$$\mathrm{CMon}_{\mathcal{T}}(\mathrm{Set}) \simeq \mathrm{CMon}_{\mathcal{T}}(\mathrm{Set}_*) \to \mathrm{CMon}_{I_{\mathcal{F}^\perp-nu}}(\mathrm{Set}_*)$$

implied by Corollaries 1.39 and 2.4 by $\widetilde{\underline{A}}_{\mathcal{T}}$. ◁

Lemma 2.30. *The S-indexed multiplication in $\widetilde{\underline{A}}_{\mathcal{T}}$ and $\widetilde{\underline{A}}_{\mathcal{T}}^0$ are distinct for all $S \in \mathbb{F}_{\mathcal{F}^\perp-nu,V} - \{*_V\}$ and $V \in \mathcal{F}^\perp$.*

Proof. It suffices to prove that, for all $S \neq \varnothing_V \in \mathbb{F}_V$, the S-ary multiplication of $\underline{A}_{\mathcal{T}}$ takes some element to another element other than the unit; unraveling definitions, this is equivalent to the property that some nonempty V-set can be expressed as an S-indexed coproduct. S provides such an example. ☐

Proof of Proposition 2.24. Note that

$$\begin{aligned} \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes} \text{ is connected} &\iff h_1 \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{N}_{A\mathcal{N}_{I\infty} \otimes \mathcal{N}_{I\infty}} \simeq \mathcal{N}_{I\infty}^{\otimes} \\ &\implies \mathrm{CAlg}_I(\mathrm{Set}_*) \to \mathrm{Alg}_{\mathcal{N}_{I\infty} \otimes \mathcal{N}_{I\infty}}(\mathrm{Set}_*) \text{ is essentially surjective.} \end{aligned}$$

Furthermore, Lemmas 2.28 and 2.30 constructs an $\mathcal{N}_{v(I)^\perp-nu\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{v(I)^\perp-nu\infty}^{\otimes}$-algebra A satisfying the condition that its two individual structure maps $A(S) \to A(*_V)$ differ whenever $V \in v(I)^\perp$ and $S \neq *_V$. Since I is not almost essentially unital, it must admit some noncontractible $S \in \mathbb{F}_{I,V}$ for $V \in v(I)^\perp$, so the pullback $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ structure on A has two distinct underlying I-algebra structures, implying it is outside of this essential image. The contrapositive shows that $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ is not connected. ☐

Remark 2.31. Using the above argument, one can show that if $\mathcal{O}^{\otimes}$ is a idempotent object in $\mathcal{T}$-operads, then its nullary spaces $\mathcal{O}(\varnothing_V)$ are nonempty. If additionally $\mathcal{O}(\varnothing_V)$ are assumed to be contractible (i.e. $\mathcal{O}^{\otimes}$ is almost-unital), then Proposition 2.8 shows that the underlying fixed point catgeories $\mathcal{O}_V$ are all idempotent algebras, i.e. they are contractible. Hence $\mathcal{O}^{\otimes}$ will be shown to be almost-reduced. In forthcoming work, the author hopes to develop an equivariant lift of [SY19], which would imply that every idempotent almost-unital $\mathcal{T}$-operad is a weak $\mathcal{N}_\infty$-operad. ◁

3. Corollaries in higher algebra

3.1. Coherences and restrictions of equivariant Boardman-Vogt tensor products.

In [Ste25a] we proved the following.

Proposition 3.1. $\mathrm{Env}\left(\mathrm{triv}_{\mathcal{T}}^{\otimes}\right) \in \mathrm{Cat}_{\mathcal{T}}^{\otimes}$ *is the unit under the mode structure, and there is an equivalence*

$$\mathrm{Env}\left(\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}\right) \simeq \mathrm{Env}\left(\mathcal{O}^{\otimes}\right) \otimes^{\mathrm{Mode}} \mathrm{Env}\left(\mathcal{P}^{\otimes}\right).$$

We would like to use this to construct coherences for $\overset{\mathrm{BV}}{\otimes}$, but it is currently not know whether Env yields a monomorphism in Cat, so we can not use the exact same strategy as [BS24a]. Instead, we proved in Corollary 2.25 that $\mathrm{Comm}_{\mathcal{T}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}_{\mathcal{T}}^{\otimes}$ is terminal, so the unique map $\mathrm{triv}_{\mathcal{T}}^{\otimes} \to \mathrm{Comm}_{\mathcal{T}}^{\otimes}$ yields an equivalence

$$\mathrm{Comm}_{\mathcal{T}}^{\otimes} \simeq \mathrm{Comm}_{\mathcal{T}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{\mathcal{T}}^{\otimes} \xrightarrow{\sim} \mathrm{Comm}_{\mathcal{T}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}_{\mathcal{T}}^{\otimes};$$

that is, $\mathrm{Comm}_{\mathcal{T}}^{\otimes}$ bears a unique structure as an idempotent $\mathbb{E}_0$-algebra (or *idempotent object* in the sense of [HA, Rmk 4.8.2.1], noting that the definition only depends on an $\mathbb{E}_0$-structure on the ambient ∞-category). In particular, together imply that Env induces an idempotent $\mathbb{E}_0$-structure on $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \in \mathrm{Cat}_{\mathcal{T}}^{\otimes}$ under the mode structure. By [HA, Prop 4.8.2.9] this canonically lifts to an $\mathbb{E}_\infty$-structure, so [HA, Thm 2.2.2.4] constructs a symmetric monoidal structure on $\mathrm{Cat}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$ whose underlying tensor functor has value

$$\mathcal{C} \circledast \mathcal{D} \xrightarrow{\pi_{\mathcal{C}} \circledast \pi_{\mathcal{D}}} \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \circledast \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \xrightarrow{\sim} \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}.$$

and whose unit is

$$\mathrm{Env}\left(\mathrm{triv}_{\mathcal{T}}^{\otimes}\right) \xrightarrow{\eta} \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}.$$

Corollary D'. *$\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes} \subset \underline{\mathrm{Cat}}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$ is a symmetric monoidal subcategory under the mode structure, with unit corresponding with $\mathrm{triv}_{\mathcal{T}}^{\otimes}$ and tensor bifunctor corresponding with $\overset{BV}{\otimes}$. Hence there exists a unique symmetric monoidal $\mathcal{T}$-∞-category lifting $\overset{BV}{\otimes}$ such that the $\mathcal{T}$-functor*

$$\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes} \to \underline{\mathrm{Cat}}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$$

admits a symmetric monoidal structure.

Proof. We're tasked with proving that the image of $\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}(-)$ contains the unit and is closed under tensor products. The unit is Proposition 3.1, and for tensor products, the above argument constructs a diagram

$$\begin{tikzcd}
\mathrm{Env}\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right) \arrow[r, "\mathrm{Env}(\pi_{\mathcal{O}} \overset{\mathrm{BV}}{\otimes} \pi_{\mathcal{P}})"] \arrow[d, phantom, "\wr"] & \mathrm{Env}\left(\mathrm{Comm}_{\mathcal{T}}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Comm}_{\mathcal{T}}^{\otimes}\right) \arrow[d, phantom, "\wr"] & \mathrm{Env}\left(\mathrm{Comm}_{\mathcal{T}}^{\otimes}\right) \arrow[l, "\mathrm{Env}(\mathrm{id} \otimes \eta)", "\sim"'] \arrow[d, phantom, "\wr"] \\
\mathrm{Env}(\mathcal{O}^{\otimes}) \circledast \mathrm{Env}(\mathcal{P}^{\otimes}) \arrow[r, "\pi_{\mathrm{Env}(\mathcal{O}^{\otimes})} \circledast \pi_{\mathrm{Env}(\mathcal{P}^{\otimes})}"'] & \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \circledast \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} & \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup} \arrow[l, "\sim", "\mathrm{id} \circledast \eta"']
\end{tikzcd}$$

In particular, by inverting both $\mathrm{Env}(\mathrm{id} \otimes \eta)$ and $\mathrm{id} \circledast \eta$, we construct an equivalence

$$\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right) \simeq \mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}\left(\mathcal{O}^{\otimes}\right) \circledast \mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}\left(\mathcal{P}^{\otimes}\right)$$

over $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$, as desired. □

Corollary 3.2. *When $\mathcal{T} = *$, there is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Op}_*^{\otimes} \simeq \mathrm{Op}^{\otimes},$$

where the latter is the Boardman-Vogt symmetric monoidal ∞-category of [BS24a]. In particular, this takes $\overset{BV}{\otimes}$ to the Boardman-Vogt tensor product of [HM23; HA].

Proof. In [Ste25a] we supplied an equivalence $\mathrm{Op}_* \simeq \mathrm{Op}$, so it suffices to upgrade this to a symmetric monoidal equivalence. In fact, the forgetful functor $\mathrm{Cat}^{\otimes}_{\infty,\mathbb{F}^{\sqcup}} \to \mathrm{Cat}^{\otimes}_{\infty}$ is symmetric monoidal (as all "unslicing" forgetful functors are), so Corollary D' constructs a symmetric monoidal structure on the composite induced $\mathrm{Op}_*^{\otimes} \to \mathrm{Cat}_{\infty}^{\otimes}$, the latter having the mode symmetric monoidal structure. Thus [BS24a, Thm E] constructs a symmetric monoidal equivalence extending the equivalence $\mathrm{Op}_* \simeq \mathrm{Op}$ and shows that $\overset{\mathrm{BV}}{\otimes}$ is the tensor product of [HA]. □

Corollary 3.3. *Let I be a one color weak indexing system and $n \in \mathbb{N} \cup \{\infty\}$. Then, $\mathrm{Op}_I \subset \mathrm{Op}_{\mathcal{T}}$ is a symmetric monoidal subcategory, $\mathrm{Op}_I^{\mathrm{uni}} \subset \mathrm{Op}_I$ is a smashing localization, and the following are symmetric monoidal subcategory inclusions:*

$$\begin{array}{ccccccc}
\mathrm{Op}_{I,\geq n}^{aE\,\mathrm{red}} & \subset & \mathrm{Op}_{I}^{aE\,\mathrm{red}} & \subset & \mathrm{Op}_{I}^{aE\mathrm{uni}} & \subset & \mathrm{Op}_I \\
\mathrm{Op}_{I,\geq n}^{\mathrm{red}} & \subset & \mathrm{Op}_{I}^{\mathrm{red}} & \subset & \mathrm{Op}_{I}^{\mathrm{uni}} & &
\end{array}$$

Proof. The first statement follows from Proposition 1.21 and the second from Proposition 1.32. $\mathrm{triv}_{\mathcal{T}}^{\otimes}$ and $\mathbb{E}_0^{\otimes}$ are ∞-connected, so in particular, the symmetric monoidal units are compatible with each of the above subcategory inclusions. Thus we're left with verifying that each subcategory inclusion is closed under tensor products. The lefthand inclusions both follow from Corollary 2.19; the middle inclusions follow from Proposition 2.8; the righthand inclusion is Corollary 2.9. □

We acquire a convenient structural property.

Corollary 3.4. *$\underline{\mathrm{Op}}_I^{\otimes}$, $\underline{\mathrm{Op}}_I^{\mathrm{uni},\otimes}$, and $\underline{\mathrm{Op}}_I^{\mathrm{uni},J-\mathrm{cocart},\otimes}$ are presentably symmetric monoidal $\mathcal{T}$-∞-categories.*

To see this, $\underline{\mathrm{Op}}_I^{\otimes}$ is presentable by the localizing inclusion $\underline{\mathrm{Op}}_I \subset \underline{\mathrm{Cat}}_{\mathcal{T},/\mathbb{F}_I^{I-\sqcup}}$, and it is distributive by the tensor-hom $\mathcal{T}$-adjunction $(-) \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes} \dashv \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(-)$. The remaining cases follow from the following lemma.

Lemma 3.5. *If $L\colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ is a smashing $\mathcal{T}$-localization and $\mathcal{D}^{\otimes}$ is a presentably symmetric monoidal $\mathcal{T}$-∞-category, then $\mathcal{C}^{\otimes}$ is a presentably symmetric monoidal ∞-category.*

Proof. It's clear that $\mathcal{C}$ is a presentable $\mathcal{T}$-∞-category, so we're left with verifying that $-\otimes C\colon \mathcal{C} \to \mathcal{C}$ possesses a $\mathcal{T}$-adjoint right adjoint; by the usual argument, it suffices to show this on fixed points, so we may assume $\mathcal{T} = *$.

We claim that $-\otimes C \dashv \mathrm{hom}(C,-)$, for which it suffices to verify that $\mathrm{hom}(C,D)$ is L-local for all $D \in \mathcal{D}$. We apply the standard argument: if $f\colon X \to Y$ is an L-equivalence, then $C\otimes f \sim C\otimes Lf\colon C\otimes X \to C\otimes Y$ is an equivalence, so the horizontal arrows in the following are equivalences.

$$\begin{tikzcd} \mathrm{Map}(Y,\mathrm{hom}(C,D)) \arrow[r, "f^*"] \arrow[d, phantom, "\wr"] & \mathrm{Map}(X,\mathrm{hom}(C,D)) \arrow[d, phantom, "\wr"] \\ \mathrm{Map}(Y\otimes C,D) \arrow[r, "(C\otimes f)^*"] & \mathrm{Map}(X\otimes C,D) \end{tikzcd}$$

The fact that the top arrow is an equivalence is the desired locality. □

3.2. **Disintegration and equivariant Boardman-Vogt tensor products.** We show the following generalization of the main results of [HA, § 2.3.3-2.3.4] in Appendix B.3.

Theorem 3.6 (Disintegration and assembly)**.** *Let X be a $\mathcal{T}$-space. Taking fibers yields an equivalence*

$$\underline{\mathrm{Op}}_{I,/X^{I-\sqcup}} \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(X, \underline{\mathrm{Op}}_I).$$

The counit of this equivalence specifies a natural equivalence.

$$\underline{\mathrm{colim}}_{x\in X}\left(\underline{\mathcal{O}^{\otimes}_{\mathrm{stab}(x)}} \times_{\underline{X^{I-\sqcup}_{\mathrm{stab}(x)}}} \{x\}\right) \xrightarrow{\sim} \mathcal{O}^{\otimes}$$

Given $\mathcal{O}^{\otimes}$ an almost-unital $\mathcal{T}$-operad, I an almost-unital $\mathcal{T}$-weak indexing category, and $x \in \mathcal{O}_V$ a V object, we define the *reduced endomorphism I_V-operad of x* to be the pullback

$$\begin{tikzcd} \mathrm{End}_x^{I,\mathrm{red}}(\mathcal{O}) \arrow[r, "\iota_x"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Res}_V^{\mathcal{T}} \mathcal{O}^{\otimes} \arrow[d, "\eta"] \\ \mathcal{N}^{\otimes}_{I_V\infty} \arrow[r, "\{x\}"'] & \mathrm{Res}_V^{\mathcal{T}} U\mathcal{O}^{I-\sqcup} \end{tikzcd}$$

In the case $I = \mathcal{T}$, we simply write $\mathrm{End}_x^{\mathrm{red}}(\mathcal{O}) := \mathrm{End}_x^{\mathcal{T},\mathrm{red}}(\mathcal{O})$. We acquire the following.

Corollary 3.7. *Suppose $\mathcal{O}^{\otimes}$ is an almost-unital $\mathcal{T}$-operad whose underlying $\mathcal{T}$-∞-category $U\mathcal{O}$ is a $\mathcal{T}$-space and I is an almost-unital weak indexing system. Then, the inclusion maps ι_x assemble to a $\mathcal{T}$-colimit diagram in almost-unital I-operads:*

$$\underline{\mathrm{colim}}_{x\in U\mathcal{O}} \mathrm{End}_x^{I,\mathrm{red}}(\mathcal{O}) \xrightarrow{\sim} \mathrm{Bor}_I^{\mathcal{T}} \mathcal{O}.$$

In essence, this says that an almost-unital $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ whose underlying $\mathcal{T}$-∞-category is a $\mathcal{T}$-space disintegrates into a $U\mathcal{O}$-local system of almost-reduced $\mathcal{T}$-operads, and the $U\mathcal{O}$-indexed colimit $\mathcal{T}$-operad (i.e. the Grothendieck construction) assembles $\mathcal{O}^{\otimes}$ from this local system. In particular, $\mathcal{O}$-algebras are $U\mathcal{O}$-indexed systems of $\mathrm{End}_x^{\mathrm{red}}(\mathcal{O})$-algebras:

$$\begin{aligned} \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) &\simeq \underline{\mathrm{Alg}}_{\underline{\mathrm{colim}}_{x\in U\mathcal{O}} \mathrm{End}_x^{\mathrm{red}}(\mathcal{O})}(\mathcal{C}) \\ &\simeq \underline{\lim}_{x\in\mathcal{O}} \underline{\mathrm{Alg}}_{\mathrm{End}_x^{\mathrm{red}}(\mathcal{O})}(\mathcal{C}). \end{aligned}$$

The corresponding picture for $\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes$-algebras is $U\mathcal{O} \times U\mathcal{P}$-local systems of $\mathrm{End}^{\mathrm{red}}_x(\mathcal{O}) \overset{\mathrm{BV}}{\otimes} \mathrm{End}^{\mathrm{red}}_y(\mathcal{P})$-algebras: that is, we can compute tensor products of almost-unital $\mathcal{T}$-operads in terms of almost-reduced $\mathcal{T}$-operads, as long as they are $\mathcal{T}$-space colored.

Corollary 3.8 (Disintegration of tensor products)**.** *Suppose $\mathcal{O}^\otimes, \mathcal{P}^\otimes, \mathcal{Q}^\otimes$ are almost-unital $\mathcal{T}$-operads whose underlying $\mathcal{T}$-∞-categories are $\mathcal{T}$-spaces and $\varphi\colon \mathcal{O}^\otimes \overset{BV}{\otimes} \mathcal{P}^\otimes \to \mathcal{Q}^\otimes$ is a map such that*

(a) the underlying map of $\mathcal{T}$-spaces $U\varphi\colon U\mathcal{O} \times U\mathcal{P} \to U\mathcal{Q}$ is an equivalence, and
(b) for all pairs $(x,y) \in U\mathcal{O} \times U\mathcal{P}$, the associated map

$$\varphi_{(x,y)}\colon \mathrm{End}^{\mathrm{red}}_x(\mathcal{O}) \overset{BV}{\otimes} \mathrm{End}^{\mathrm{red}}_y(\mathcal{P}) \to \mathrm{End}^{\mathrm{red}}_{(x,y)}(\mathcal{Q})$$

is an equivalence.

Then φ is an equivalence.

Proof. Corollaries 3.4 and 3.7 construct equivalences of arrows

$$\begin{tikzcd}
\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes \arrow[d, "\varphi"] \arrow[r, phantom, "\simeq"] & \underline{\mathrm{colim}}_{x \in U\mathcal{O}} \mathrm{End}^{\mathrm{red}}_x(\mathcal{O}) \overset{\mathrm{BV}}{\otimes} \underline{\mathrm{colim}}_{y \in U\mathcal{P}} \mathrm{End}^{\mathrm{red}}_y(\mathcal{P}) \arrow[d] \arrow[r, phantom, "\simeq"] & \underline{\mathrm{colim}}_{(x,y) \in U\mathcal{O} \times U\mathcal{P}} \mathrm{End}^{\mathrm{red}}_x(\mathcal{O}) \overset{\mathrm{BV}}{\otimes} \mathrm{End}^{\mathrm{red}}_y(\mathcal{P}) \arrow[d, "\sim"] \\
\mathcal{Q}^\otimes \arrow[r, phantom, "\simeq"] & \underline{\mathrm{colim}}_{(x,y) \in U\mathcal{O} \times U\mathcal{P}} \mathrm{End}^{\mathrm{red}}_X(\mathcal{Q}) \arrow[r, equal] & \underline{\mathrm{colim}}_{(x,y) \in U\mathcal{O} \times U\mathcal{P}} \mathrm{End}^{\mathrm{red}}_X(\mathcal{Q})
\end{tikzcd}$$

The right vertical arrow is an equivalence by assumption, so φ is an equivalence by two out of three. ☐

We will make crucial use of this in forthcoming work concerning variants of $\mathbb{E}_V^\otimes$ with tangential structure.

3.3. **Norms of right-modules over I-commutative algebras.** Let I be an indexing category, $\mathcal{C}^\otimes$ an I-symmetric monoidal ∞-category, $t\colon W \to V$ an I-admissible transfer, A an I_V-commutative algebra, and M a right module over the associative algebra underlying $\mathrm{Res}^V_W A$. Then, we may define the *A-module norm of M* by the base assignment

$${}_A N^V_W M \coloneqq A \otimes_{N^V_W \mathrm{Res}^V_W A} N^V_W M;$$

that is, the normed multiplication recognizes A as an $N^V_W \mathrm{Res}^V_W A$-module, and the A-module norm of M is the free A-module on the normed $N^V_W \mathrm{Res}^V_W A$-algebra of M; see [Yan23] for a detailed account in the C_p-equivariant case.

In this subsection, we use the equivalence $\mathcal{N}^\otimes_{I\infty} \simeq \mathbb{E}_1^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{N}^\otimes_{I\infty}$ to lift this to an I-symmetric monoidal structure, yielding coherent functoriality and a coherent double coset formula for A-module norms. To do this, we begin by bootstrapping G-symmetric monoidality of the right module construction from the non-equivariant case.

Observation 3.9. By [HA, Rmk 4.8.3.8], functors $F\colon \mathrm{Tot}\mathcal{O}^\otimes \to \mathrm{Cat}^{\mathrm{Alg}}$ correspond with pairs

$$\begin{tikzcd}
 & \mathcal{C}^\otimes \arrow[d, "\pi_F"] \\
\mathbb{E}_1^\otimes \times \mathrm{Tot}\mathcal{O}^\otimes \arrow[ur, "A_F"] \arrow[r, equal] & \mathbb{E}_1^\otimes \times \mathrm{Tot}\mathcal{O}^\otimes
\end{tikzcd}$$

such that π is a cocartesian fibration whose fibers $\mathcal{C}_V^\otimes \to \mathbb{E}_1^\otimes$ are the unstraightenings of small monoidal ∞-categories and such that the composite arrows $\mathbb{E}_1^\otimes \times \{O\} \hookrightarrow \mathcal{C}^\otimes$ are associative algebras. Moreover, by unwinding definitions, the condition that F is an $\mathcal{O}$-monoid corresponds with the condition that π is an $\mathbb{E}_1 \overset{\mathrm{BV}}{\otimes} \mathcal{O}$-monoidal ∞-category and A is an $\mathbb{E}_1 \overset{\mathrm{BV}}{\otimes} \mathcal{O}$-algebra. ◁

We acquire functorial arrows as in the following diagram

$$\begin{tikzcd}
\mathrm{Tot}\mathcal{O}^{\otimes} \arrow[rrrrd, bend left=10, "\mathrm{RMod}_A(\mathcal{C})^{\otimes}"] \arrow[rrd, bend left=5, "(A{,}\mathcal{C}^{\otimes})"] \arrow[rd, dashed] \arrow[rddd, equal, bend right] & & & & \\
& \mathrm{Coeff}^{\mathcal{T}}\mathrm{Cat}^{\mathrm{Alg}} \times_{\mathrm{Coeff}^{\mathcal{T}}\mathrm{AlgCat}} \mathrm{Tot}\mathcal{O}^{\otimes} \arrow[r] \arrow[dd] \arrow[rdd, phantom, "\lrcorner", very near start] & \mathrm{Coeff}^{\mathcal{T}}\mathrm{Cat}^{\mathrm{Alg}} \arrow[r, "\Theta"'] \arrow[d, "U"] & \mathrm{Coeff}^{\mathcal{T}}\mathrm{Cat}^{\mathrm{Mod}} \arrow[r, "Y"'] & \mathrm{Coeff}^{\mathcal{T}}\mathrm{Cat} \\
& & \mathrm{Coeff}^{\mathcal{T}}\mathrm{Alg}(\mathrm{Cat}) \arrow[d, phantom, "\wr"] & & \\
& \mathrm{Tot}\mathcal{O}^{\otimes} \arrow[r, "\mathcal{C}^{\otimes}"] & \mathrm{Alg}(\mathrm{Coeff}^{\mathcal{T}}\mathrm{Cat}) & &
\end{tikzcd}$$

θ and Y are product preserving functors, so $\mathrm{RMod}_A(\mathcal{C})^{\otimes}$ is an $\mathcal{O}$-monoid in Cat, i.e. an $\mathcal{O}$-monoidal ∞-category. Unwinding definitions, this proves the following proposition.

Proposition 3.10. *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-operad, let $\mathcal{C}^{\otimes}$ be an $\mathbb{E}_1 \otimes \mathcal{O}$-monoidal ∞-category. Then, there is a factorization*

$$\begin{tikzcd}
& & \mathrm{Cat}_{\mathcal{O}}^{\otimes} \arrow[d, "\Gamma^{\mathcal{T}}"] \\
\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathcal{C}) \arrow[r] \arrow[rru, bend left=10, "\mathrm{RMod}^{\otimes}_{(-)}(\mathcal{C})"] & \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \arrow[r, "\mathrm{RMod}_{(-)}(\mathcal{C})"] & \mathrm{Cat},
\end{tikzcd}$$

natural separately in $\mathcal{O}^{\otimes}$ and $\mathcal{C}^{\otimes}$; that is, left modules over $\mathbb{E}_1 \otimes \mathcal{O}$-algebras bear a natural $\mathcal{O}$-algebra structure.

We immediately acquire the following corollary, confirming a hypothesis of [Hil17, Rmk 3.15]

Corollary 3.11. *If $\mathcal{O}^{\otimes}$ is a $\mathcal{T}$-operad whose underlying I^{∞}-operad is $\mathbb{E}_{\infty}$ and $\mathcal{C}^{\otimes}$ an $\mathcal{O}\otimes\mathbb{E}_1$-monoidal ∞-category, then there is a lift*

$$\begin{tikzcd}
& & \mathrm{Cat}_{\mathcal{O}}^{\otimes} \arrow[d] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r] \arrow[rru, bend left=10, "\mathrm{RMod}^{\otimes}_{(-)}(\mathcal{C})"] & \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \arrow[r, "\mathrm{RMod}_{(-)}(\mathcal{C})"] & \mathrm{Cat}
\end{tikzcd}$$

natural separately in $\mathcal{P}^{\otimes}$ and $\mathcal{C}^{\otimes}$. In particular, if I is an indexing category, the ∞-category of right-modules over an $\mathcal{N}_{I\infty}$-algebra admits a natural I-symmetric monoidal structure.

3.4. **Equivariant infinitary Dunn additivity.** In [Bon19], a *genuine operadic nerve* 1-categorical functor was constructed between a model of graph-G operads and a model for G-operads. In [Ste25a], we lifted this to a conservative functor of ∞-categories $N^{\otimes} \colon g\mathrm{Op}_G \to \mathrm{Op}_G$. We define the G-operad

$$\mathbb{E}_V := N^{\otimes} D_V,$$

where D_V is the *little V-disks graph G-operad* of [GM17], whose n-ary $G \times \Sigma_n$ space has

$$D_V(n) \simeq \mathrm{Conf}_n(V)$$

by [GM17, Lem 1.2]. The resulting unital G-operad $\mathbb{E}_V$ was studied in [Hor19], who showed for instance that

$$\mathbb{E}_V(S) \simeq \mathrm{Conf}_S^H(V),$$

where

$$\mathrm{Conf}_S^H(V) := \operatorname*{colim}_{\substack{W \subset V \\ \text{fin.dim}}} \mathrm{Conf}_S^H(W)$$

in view of the fact that the assignment $\mathcal{O} \mapsto \mathcal{O}(S)$ preserves sifted colimits [Ste25a].

A weak form of the following claim appears to be folklore.

Proposition 3.12. *Let G be a topological group, $H \subset G$ a closed subgroup, $S \in \mathbb{F}_H$ a finite H-set admitting an configuration $\iota : S \hookrightarrow W$, and V, W real orthogonal G-representations whose associated map*

$$\mathrm{Conf}_S^H(V) \hookrightarrow \mathrm{Conf}_S^H(V \oplus W)$$

is an equivalence. Then, $\mathrm{Conf}_S^H(V)$ is contractible.

Proof. Note that linear interpolation to ι yields a deformation of $\mathrm{Map}^H(S, V \oplus W)$ onto the subspace $\mathrm{Map}^H(S, W)$ consisting of maps whose image has zero projection to V. The path of a point beginning in the subspace $\mathrm{Conf}_S^H(V) \subset \mathrm{Conf}_S^H(V \oplus W)$ consisting of configurations with zero projection to W lands within $\mathrm{Conf}_S^H(V \oplus W)$ at all times; composing this deformation after the deformation retract $\mathrm{Conf}_S^H(V \oplus W) \xrightarrow{\sim} \mathrm{Conf}_H^S(V)$ thus yields a deformation retract of $\mathrm{Conf}_S^H(V \oplus W)$ onto $\{\iota\}$, so it is contractible.[5] By the equivalence $\mathrm{Conf}_S^H(V) \simeq \mathrm{Conf}_S^H(V \oplus W)$, the space $\mathrm{Conf}_S^H(V)$ is contractible as well. □

Remark 3.13. This argument only produces *contractibility*, whereas the nonequivariant argument using Fadell and Neuwirth's fibration [FN62] sharply characterizes *n-connectivity* of $\mathrm{Conf}_k(\mathbb{R}^n)$; we hope to equivariantize this in forthcoming work. ◁

We say that V is a *weak universe* if it is a direct sum of infinitely many copies of a collection of irreducible real orthogonal G-representations; equivalently, there is an equivalence $V \simeq V \oplus V$. Given V an orthogonal G-representation, we let $AV := A\mathbb{E}_V$, i.e. AV corresponds with the weak indexing system $\underline{\mathbb{F}}^V = \underline{\mathbb{F}}_{AV}$ of finite H-sets admitting an embedding into V. The following corollary follows immediately from Proposition 3.12.

Corollary 3.14. *If there exists an equivalence $\mathbb{E}_V^\otimes \simeq \mathbb{E}_{V \oplus W}^\otimes$, then the canonical map $\mathrm{Bor}_{AW}^{\mathcal{T}} \mathbb{E}_V^\otimes \to \mathcal{N}_{AW}^\otimes$ is an equivalence; in particular, if V is a weak universe, then the canonical map*

$$\mathbb{E}_V^\otimes \to \mathcal{N}_{AV}^\otimes$$

is an equivalence.

Observation 3.15. If V is a *universe* (i.e. it is a weak universe admitting a positive-dimensional fixed point locus), then it admits embeddings of all finite sets; hence it is not just a weak $\mathcal{N}_\infty$-operad, but an $\mathcal{N}_\infty$-operad. ◁

Because of the above observation, much study has been dedicated to the less general setting of *universes*; Rubin has given a complete and simple characterization of those indexing systems (equivalently, transfer systems) occurring as the arity-support of an $\mathbb{E}_V$-operad in [Rub19] for G abelian, where they are modelled via *Steiner operads.*

An inclusion $V \subset W$ yields a map of graph G-operads $D_V \subset D_W$, and hence a map of G-operads $\mathbb{E}_V^\otimes \to \mathbb{E}_W^\otimes$. This yields a map of weak indexing systems $\underline{\mathbb{F}}^V \to \underline{\mathbb{F}}^W$; in [Ste24] we showed that this is additive, i.e.

$$\underline{\mathbb{F}}^V \vee \underline{\mathbb{F}}^W = \underline{\mathbb{F}}^{V \oplus W}. \tag{11}$$

Corollary 3.16 (Equivariant infinitary Dunn additivity)**.** *Let G be a finite group and V, W real orthogonal G-representations satisfying at least one of the following conditions:*

(a) V, W are weak G-universes, or
(b) the canonical map $\mathbb{E}_V^\otimes \simeq \mathbb{E}_{V \oplus W}^\otimes$ is an equivalence.

Then the canonical map

$$\mathbb{E}_V^\otimes \overset{BV}{\otimes} \mathbb{E}_W^\otimes \to \mathbb{E}_{V \oplus W}^\otimes$$

is an equivalence; equivalently, for any G-symmetric monoidal category $\mathcal{C}$, the pullback functors

$$\mathrm{Alg}_{\mathbb{E}_V} \underline{\mathrm{Alg}}_{\mathbb{E}_W}^\otimes(\mathcal{C}) \leftarrow \mathrm{Alg}_{\mathbb{E}_{V \oplus W}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_W} \underline{\mathrm{Alg}}_{\mathbb{E}_V}^\otimes(\mathcal{C})$$

are equivalences.

Proof. Given Corollary 3.14, case (a) follows from Theorem E and Eq. (11) and case (b) follows from Corollary C. □

[5] Said explicitly, let $h : [0,1] \to \mathrm{Conf}_S^H(V \oplus W)$ be the deformation retract onto those configurations with zero projection to W. Then, our deformation retract h' onto $\iota(w)$ is computed by

$$h'(t) = \begin{cases} h(2t) & t \le \frac{1}{2}, \\ (2 - 2t) \cdot h(1) + (2t - 1)\iota & t \ge \frac{1}{2}. \end{cases}$$

Remark. In the thesis [Szc23], an ostensibly similar result to Corollary 3.16 is proved: given D_V the *little Disks graph G-operad*, Szczesny constructs a non-homotopical Boardman-Vogt tensor product $\otimes$ and a canonical map $D_V \otimes D_W \to D_{V\oplus W}$, which he shows to be a weak equivalence of graph G-operads in [Szc23, Thm 4.5.5]. Neither this result nor Corollary 3.16 imply each other.

On one hand, Szczesny's result concerns a tensor product with no known homotopical properties, so it is incomparable with results concerning ∞-categories of algebras defined by homotopy-coherent universal properties. On the other hand, while Corollary 3.16 is homotopical, it only concerns cases where at least one of the representations induces I-symmetric monoidal ∞-categories of algebras whose indexed tensor products are indexed coproducts; this property will not be satisfied for any nontrivial indexed tensor products in the finite-dimensional case, so the range of representations in Szczesny's result is significantly larger. The author hopes to address the general case in forthcoming work. ◁

3.5. **Iterated Real topological Hochschild homology.** In classical algebra, there are two well-known tensor products of functors $F, G : \mathcal{C} \to \mathcal{D}$: when $\mathcal{D}$ is monoidal, the *pointwise tensor product* sets $F \otimes G(C) := F(C) \otimes G(C)$, and when additionally $\mathcal{C}$ is monoidal, the *Day convolution product* sets $F \circledast G(-)$ to be the left Kan extension of the functor $F(-) \otimes G(-) : \mathcal{C}^2 \to \mathcal{D}$ along the tensor functor $\mathcal{C}^2 \to \mathcal{C}$. [NS22] constructed each, and we use the former.

Theorem 3.17 ([NS22, Thm 3.3.1, 3.3.3])**.** *Let $\mathcal{K}$ be a $\mathcal{T}$-∞-category, and $\mathcal{C}^\otimes$ a $\mathcal{T}$-operad. Then, there exists a unique (functorial) I-operad structure $\mathrm{Fun}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}$ on $\mathrm{Fun}_{\mathcal{T}}(\mathcal{K},\mathcal{C})$ satisfying the universal property*

$$\mathrm{Alg}_{\mathcal{O}}(\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}))$$

for $\mathcal{O} \in \mathrm{Op}_I$. Furthermore, when $\mathcal{C}^\otimes$ is I-symmetric monoidal, $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}$ is I-symmetric monoidal and satisfies the universal property

$$\mathrm{Fun}_{\mathcal{T}}^{I-\otimes}\left(\mathcal{D}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}\right) \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{K}, \underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})\right).$$

If S is I-admissible, then the S-indexed tensor product of $(F_U) \in \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})_S^{\otimes-\mathrm{ptws}}$ has values

$$\begin{tikzcd} \mathcal{D}_V \arrow[r, "\Delta^S"] \arrow[rrr, bend right, "\bigotimes_U^S F_U"'] & \mathcal{D}_S \arrow[r, "(F_U)"] & \mathcal{C}_S \arrow[r, "\bigotimes^S"] & \mathcal{C}_V \end{tikzcd}$$

Observation 3.18. Suppose $F : \mathcal{K}' \to \mathcal{K}$ is a functor. Then, the restriction and left Kan extension natural transformations

$$F_! : \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{K}', \underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})\right) \rightleftarrows \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{K}, \underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\otimes}(\mathcal{D},\mathcal{C})\right) : F^*$$

yield I-symmetric monoidal functors $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}',\mathcal{C})^{\otimes-\mathrm{ptws}} \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}}$ extending the left Kan extension and restriction functors between functor categories via Yoneda's lemma. In particular, give $X \in \Gamma^{\mathcal{T}}\mathcal{K}$ this yields an I-symmetric monoidal lift $\mathrm{ev}_X : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}} \to \mathcal{C}^\otimes$ of the ordinary evaluation $\mathcal{T}$-functor $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\{X\},\mathcal{C}) \simeq \mathcal{C}$. ◁

The following proposition is easy.

Proposition 3.19. *There exists an equivalence $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{\otimes-\mathrm{ptws}} \simeq \underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}(\mathcal{K})}(\mathcal{C})$.*

The structure functor $\mathrm{Env}(\mathcal{O}) \to \mathcal{O}$ is adjunct to a $\mathcal{T}$-operad map $\mathrm{triv}(\mathrm{Env}(\mathcal{O}))^\otimes \to \mathcal{O}^\otimes$, which yields a natural pullback $\mathcal{T}$-symmetric monoidal functor

$$U : \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathrm{Env}(\mathcal{O}),\mathcal{C})^{\otimes-\mathrm{ptws}}.$$

In particular, this constructs a G-symmetric monoidal lift for *genuine equivariant factorization homology.*

Corollary 3.20. *Given M a V-framed smooth G-manifold, M-factorization homology lifts to a G-symmetric monoidal functor*

$$\int_M : \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_V}(\mathcal{C}) \to \mathcal{C}^\otimes;$$

in particular, it further lifts to a G-symmetric monoidal functor

$$\int_M : \underline{\mathrm{CAlg}}^{\otimes}_{AV}(\mathcal{C}) \to \underline{\mathrm{CAlg}}^{\otimes}_{AV}(\mathcal{C}).$$

Proof. In the notation of [Hor19], let $\iota^\otimes : \underline{\mathrm{Disk}}^{G,V-fr,\sqcup} \to \underline{\mathrm{Mfld}}^{G,V-fr,\sqcup}$ be the symmetric monoidal inclusion of V-framed G-disks into V-framed G-manifolds. By [Hor19, Horev 4.1.4], $\int_M$ may be presented as the G-value of a composition

$$\int_M : \underline{\mathrm{Alg}}_{\mathbb{E}_V}(\mathcal{C}) \simeq \underline{\mathrm{Fun}}_G^{\otimes}\left(\underline{\mathrm{Disk}}^{G,V-fr}, \mathcal{C}\right) \xrightarrow{U} \underline{\mathrm{Fun}}_G\left(\underline{\mathrm{Disk}}^{G,V-fr}, \mathcal{C}\right) \xrightarrow{\iota_!} \underline{\mathrm{Fun}}_G\left(\underline{\mathrm{Mfld}}^{G,V-fr}, \mathcal{C}\right) \xrightarrow{\mathrm{ev}_M} \mathcal{C}.$$

To construct the lift of $\int_M$, we may compose G-symmetric monoidal lifts of U, $\iota_!$, and ev_M; these are given by the above work and Observation 3.18. ☐

Corollary 3.21. *Real topological Hochschild homology lifts to a C_2-symmetric monoidal functor*

$$\mathrm{THR} : \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_\sigma}(\mathrm{Sp}) \to \underline{\mathrm{Sp}}_{C_2};$$

in particular, THR *lifts to a C_2-symmetric monoidal endofunctor*

$$\mathrm{THR} : \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{V+\infty\sigma}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{V+\infty\sigma}}(\mathcal{C}).$$

Given $A \in \mathrm{Alg}_{\mathbb{E}_{V+\infty\sigma}}(\mathcal{C})$, there is an equivalence

$$\mathrm{THR}(A) \simeq \mathrm{colim}_{S^\sigma} A,$$

with colimit taken in $\mathrm{Alg}_{\mathbb{E}_{V+\infty\sigma}}(\mathcal{C})$, naturally in A.

Proof. The last sentence is the only part which does not follow immediately from combining Horev's facorization homology formula [Hor19, Rmk 7.1.2] with Corollary 3.20 in view of Corollary 3.16. It suffices to show the colimit property for $\mathcal{O} \simeq \mathcal{O} \otimes \mathbb{E}_\sigma$-algebras whenever $\mathrm{Bor}^{\mathcal{T}}_{A\sigma}\mathcal{O}^\otimes \simeq \mathcal{N}^{\otimes}_{A\sigma\infty}$, which holds for $\mathbb{E}_{V+\infty\sigma}$ by Proposition 3.12. In any case, applying operations yields a diagram

$$\begin{tikzcd}
\vdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A^{\otimes T_3} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & A^{\otimes T_2} \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d] & A \arrow[r] \arrow[d] & \mathrm{THR}(A) \arrow[d, "\varphi"] \\
\vdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A^{\sqcup T_3} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A^{\sqcup T_2} \arrow[r, shift left=1] \arrow[r, shift right=1] & A \arrow[r] & \mathrm{colim}_{S^\sigma} A;
\end{tikzcd}$$

where T_n is the n-element "dihedral" C_2-set, i.e. the unique σ-admissible C_2-set of size n, and each row is a geometric realization diagram. When the domain category is $A\sigma$-semiadditive, the vertical maps between the bar constructions are equivalences, so φ is an equivalence. The result then follows by $A\sigma$-semiadditivity of $\mathcal{O}$-algebras, as in Theorem E'. ☐

Remark 3.22. The computation $\mathrm{THR}(A) = \mathrm{colim}_{S^\sigma} A$ when A is pulled back from a C_2-commutative algebra is not new; indeed, it appears as [QS19, Rmk 5.4]. In fact, the ambiguity induced by the potential discrepancy between our construction $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ and that of [NS22, Thm 5.3.4] vanishes for the I-symmetric monoidal structure on $\mathrm{CAlg}_I(\mathcal{C})$ by applying Theorem A' in view of the fact that each are cocartesian [NS22, Thm 5.3.9]. The new element of this identification is that the operation on C_2-commutative algebras is induced canonically from the operation on $\mathbb{E}_\sigma$-algebras and that the colimit formulas need only an $\mathbb{E}_{\infty\sigma}$-algebra structure. ◁

Appendix A. Cartesian and cocartesian I-symmetric monoidal ∞-categories

Fix $\mathcal{P} \subset \mathcal{T}$ an atomic orbital subcategory and $I \subset \mathbb{F}^{\mathcal{P}}_{\mathcal{T}}$ an almost-unital weak indexing category. We define the *∞-category of Γ-I-preoperads* as

$$\mathrm{PreOp}^{\Gamma}_I := \mathrm{Cat}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathbb{E}_{I,*}},$$

so that the Segal envelope of [BHS22] recognizes $\mathrm{Op}_I \subset \mathrm{PreOp}^{\Gamma}_I$ as a localizing subcategory. Here, Γ is a reference to Segal's category Γ.

This appendix can be understood as a lift of [HA, § 2.4.1-2.4.3] to the setting of (co)cartesian I-symmetric monoidal ∞-categories, working in the specific model of Γ-I-preoperads; we proceed by an essentially similar strategy, complicated only by less convenient combinatorics.

First, define the $\mathcal{T}$-1-category $\underline{\Gamma}^*_I$ to have V-values

$$\Gamma^*_{I,V} := \left\{ U_+ \xrightarrow{s.i.} S_+ \;\middle|\; U \in \mathcal{T}_{/V} \right\} \subset \mathrm{Ar}(\underline{\mathbb{F}}_{I,*})_V;$$

that is, the objects of $\Gamma^*_{I,V}$ are pointed I-admissible V-sets with a distinguished orbit, and the morphisms of $\Gamma_{I,V}$ preserve distinguished orbits. This possesses a tautological forgetful functor $\underline{\Gamma}^*_I \to \underline{\mathbb{F}}_{I,*}$. We use this to construct an ∞-category $\mathcal{C}$ over $\underline{\mathbb{F}}_{I,*}$ in Appendix A.1 satisfying the following universal property.

Proposition A.1. *Given $\mathcal{C}$ a $\mathcal{T}$-∞-category, there exists an ∞-category $\mathcal{C}^{I-\sqcup}$ over $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ satisfying the universal property that there is a natural equivalence*

$$\mathrm{Fun}_{/\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}(\mathcal{D}, \mathrm{Tot}\mathcal{C}^{I-\sqcup}) \simeq \mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}(\mathcal{D} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I, \mathcal{C});$$

that is, the functor $(-) \times_{\mathrm{Tot}\underline{\mathbb{F}}_{I,*}} \mathrm{Tot}\underline{\Gamma}^*_I : \mathrm{Cat}_{\infty,/\underline{\mathbb{F}}_{I,*}} \to \mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}}$ possesses a right adjoint $(-)^{I-\sqcup}$.

Second, define the (non-full) $\mathcal{T}$-subcategory $\Gamma^\times_I \subset \mathrm{Ar}(\underline{\mathbb{F}}_{I,*})$ has V-objects given by summand inclusions of pointed V-sets $\overline{S} \hookrightarrow S$ and morphisms of V-objects given by maps $\alpha : S \to T$ with the property that $\alpha^{-1}(\overline{T}) \subset \overline{S}$. In Appendix A.1 we prove the following.

Proposition A.2. *Given $\mathcal{C}$ a $\mathcal{T}$-∞-category, there exists an ∞-category $\mathcal{C}^{I-\sqcup}$ over $\underline{\mathbb{F}}_{I,*}$ satisfying the universal property that there is a natural equivalence*

$$\mathrm{Fun}_{/\underline{\mathbb{F}}_{I,*}}(K, \mathrm{Tot}\widetilde{\mathcal{C}}^{I-\times}) \simeq \mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}(K \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^\times_I, \mathcal{C}).$$

that is, the functor $(-) \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^\times_I : \mathrm{Cat}_{\infty,/\underline{\mathbb{F}}_{I,*}} \to \mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}}$ possesses a right adjoint $\widetilde{(-)}^{I-\times}$.

Note that there is an equivalence

$$\{S_+\} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^\times_I \simeq \mathscr{P}_{\underline{V}}(S),$$

where $\mathscr{P}_{\underline{V}}(S)$ is the V-poset with U-value given by subsets of $\mathrm{Res}^V_U S$ ordered under inclusion. In particular, for $S_+ \in \underline{\mathbb{F}}_{I,*}$, we view objects in $\widetilde{\mathcal{C}}^{I-\times}_{S_+}$ as V-functors $\mathscr{P}_{\underline{V}}(S)^{\mathrm{op}} \to \mathcal{C}_V$. Let $\mathcal{C}^{I-\times} \subset \widetilde{\mathcal{C}}^{I-\times}$ be the full $\mathcal{T}$-subcategory spanned by those functors $\mathscr{P}_{\underline{V}}(S)^{\mathrm{op}} \to \mathcal{C}_{\underline{V}}$ satisfying the property that, for all $U \to V$ and $T \subset \mathrm{Res}^V_U S$, the maps

$$\mathrm{Res}^V_W F(T) \to F(W)$$

exhibit $F(T)$ as the T-indexed product $F(T) \simeq \prod^T_W F(U)$ in $\mathcal{C}$.

Following Appendix A.1, we characterize algebras and I-symmetric monoidal functors into $\mathcal{C}^{I-\sqcup}$ and $\mathcal{C}^{I-\times}$ in Appendices A.2 and A.3. We spell out a corollary in Appendix A.4 relating L_{Op_I}-equivalences to the Morita theory of algebraic patterns.

A.1. **Quasicategories modeling $\mathcal{C}^{I-\sqcup}$ and $\mathcal{C}^{I-\times}$.** Let $\mathcal{T}^{\mathrm{op}}$ be a quasicategory and $\mathcal{C} \in \mathrm{sSet}^{\mathrm{cocart}}_{/\mathcal{T}}$ a cocartesian fibration to $\mathcal{T}$. There exists a simplicial set $\mathcal{C}^{I-\sqcup}$ satisfying the universal property

$$\mathrm{Hom}_{/\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}(K, \mathcal{C}^{I-\sqcup}) \simeq \mathrm{Hom}_{/\mathcal{T}^{\mathrm{op}}}(K \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I, \mathcal{C}). \tag{12}$$

Lemma A.3. *The map of simplicial sets $\mathcal{C}^{I-\sqcup} \to \underline{\mathbb{F}}_{I,*}$ is an inner fibration; hence $\mathcal{C}^{I-\sqcup}$ is a quasicategory.*

Proof. The proof is exactly analogous to the analogous part of [HA, Prop 2.4.3.3]; that is, we may apply the universal property

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f_0"] \arrow[d] & \mathcal{C}^{I-\sqcup} \arrow[d] \\ \Delta^n_{(S_{0,+} \to \cdots \to S_{n,+})} \arrow[r] \arrow[ur, dashed] & \mathrm{Tot}\underline{\mathbb{F}}_{I,*} \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} \Lambda^n_i \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I \arrow[d] & \simeq & \coprod\limits_{\substack{U \in \mathrm{Orb}(S) \\ f(U) \in S^\circ_{n,+}}} \Lambda^n_i \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^n \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^*_I & \simeq & \coprod\limits_{\substack{U \in \mathrm{Orb}(S) \\ f(U) \in S^\circ_{n,+}}} \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{T}^{\mathrm{op}} \end{tikzcd}$$

after which the lifting problem on the RHS has solutions in bijection with the tuples of solutions to the lifting problems made up of the summands, which exist by assumption that the functor $\mathcal{C} \to \mathcal{T}$ is a cocartesian fibration (hence an inner fibration).

The remaining claim follows by noting that $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ is a quasicategory, so the composite map of simplicial sets $\mathcal{C}^{I-\sqcup} \to \underline{\mathbb{F}}_{I,*} \to *$ is an inner fibration. $\square$

Proof of Proposition A.2. Unwinding the above work, we've verified that $\mathcal{C}^{I-\sqcup}$ is a quasicategory over $\underline{\mathbb{F}}_{I,*}$. Fixing some quasicategory $\mathcal{D}$ over $\underline{\mathbb{F}}_{I,*}$ and applying Eq. (12) for $K := \mathcal{D} \times \Delta^n$, we find that $\mathrm{Fun}(K, \mathcal{C}^{I-\sqcup}) \simeq \mathrm{Fun}_{\mathcal{T}}(K \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_I^*, \mathcal{C})$. The result then follows by replacing "quasicategory" with "∞-category." □

Recollection A.4 ([NS22, Def 2.1.2])**.** A morphism f in $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ from $S_+ \in \mathbb{F}_{I,*,U}$ to $T_+ \in \mathbb{F}_{I,*,V}$ may be modelled as a morphism of spans

$$\begin{tikzcd} S \arrow[d] & & f^{-1}(T) \arrow[ll, hook'] \arrow[r, "f^\circ"] \arrow[d] & T \arrow[d] \\ & \mathrm{Res}_U^V S \arrow[ul] \arrow[dr] \arrow[ur, phantom, "\swarrow \iota_f"] & & \\ U & & V \arrow[ll] \arrow[r, equal] & V \end{tikzcd}$$

such that $f^\circ \in I$. Such a morphism is $\pi_{\underline{\mathbb{F}}_{I,*}}$-cocartesian if f° and ι_f are both equivalences, i.e. it witnesses an equivalence $\mathrm{Res}_U^V S_+ \xrightarrow{\sim} T_+$. ◁

Let $T_+ \to S_+$ be a map in $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ lying over an orbit map $U \to V$ and let $\overline{S} \subset S$ be an element of $\underline{\Gamma}_I^\times$ lying over S_+. We would like to construct a Cartesian edge landing on $\overline{S} \subset S$; we do so by setting $\overline{T} := f^{-1}(\mathrm{Res}_U^V \overline{S}) \subset f^{-1}(\mathrm{Res}_U^V S) \subset T$, and letting the associated map $t\colon \left(f^{-1}(\mathrm{Res}_U^V \overline{S}) \subset T\right) \to \left(\overline{S} \subset S\right)$ be the canonical one. The following lemma then follows by unwinding definitions, where $U\colon \underline{\Gamma}_I^\times \to \underline{\mathbb{F}}_{I,*}$ denotes the forgetful functor.

Lemma A.5. *t is a U-cartesian arrow; in particular, U is a cartesian fibration.*

The following lemma then follows from [HTT, Cor 3.2.2.12].

Lemma A.6. *Let $\widetilde{p}\colon \widetilde{\mathcal{C}}^{I-\times} \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ be the projection and let $\tilde{\alpha}\colon F \to G$ be a $\widetilde{\mathcal{C}}^{I-\times}$-morphism lying over a $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$-morphism $\alpha : T_+ \to S_+$ lying over an orbit map $U \to V$. Then, $\tilde{\alpha}$ is $\widetilde{p}$-cocartesian if and only if, for all $T' \subset T$, the induced map*

$$F(\alpha^{-1}(\mathrm{Res}_U^V T')) \to \mathrm{Res}_U^V G(T')$$

is an equivalence; in particular, $\tilde{p}$ is a cocartesian fibration of simplicial sets

A.2. **$\mathcal{O}$-comonoids and (co)cartesian rigidity.** An object of $\mathcal{C}^{I-\sqcup}$ may be viewed as S_+ a pointed V-set and $\mathbf{C} = (C_W) \in \mathcal{C}_S$ an S-tuple of elements of $\mathcal{C}$; a morphism $f : \mathbf{C} \to \mathbf{D}$ may be viewed as a $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$-map $(S_+ \to V_{S,+}) \xrightarrow{f} (T_+ \to V_{T,+})$ together with a collection of maps

$$\left\{f_W\colon \mathrm{Ind}_W^U C_W \to D_U \mid W \in f^{-1}(U)\right\}$$

for all $U \in \mathrm{Orb}(T)$. Unwinding definitions and applying [HTT, Cor 3.2.2.13], we find the following.[6]

Proposition A.7. *A morphism $f : (\mathbf{C}, S) \to (\mathbf{D}, T)$ is π-cocartesian if and only if $\{f_W\}$ witness D_U as the indexed coproduct*

$$\coprod_W^{f^{-1}(U)} C_W \xrightarrow{\sim} D_U$$

for all $U \in \mathrm{Orb}(T)$. In particular, f is inert if and only if the following conditions are satisfied:

(a) The projected morphism $\pi(f) : S \to T$ is inert.
(b) The associated map $C_{f^{-1}(U)} \to D_U$ is an equivalence for all $U \in \mathrm{Orb}(T)$.

Hence $\mathcal{C}^{I-\sqcup} \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ is an inert-cocartesian fibration.

Corollary A.8. *$\mathcal{C}^{I-\sqcup}$ is an I-operad which is an I-symmetric monoidal ∞-category if and only if $\mathcal{C}$ admits I-indexed coproducts.*

Proof. It follows from Proposition A.7 that $\mathcal{C}^{I-\sqcup} \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ is a cocartesian fibration if and only if $\mathcal{C}$ admits I-indexed coproducts, so it suffices to verify the following conditions:

[6] It is here that we use almost-unitality for the cocartesian setting; if I was not almost essentially unital, then there would exist some S whose I-admissible orbits do not together cover S, so $\mathcal{C}^{I-\sqcup} \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ would not be an inert-cocartesian fibration.

(b) Cocartesian transport yields an equivalence
$$\mathcal{C}_{S_+} \simeq \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_{U_+}$$
(c) Cocartesian transport yields an equivalence
$$\mathrm{Map}^{T \to S}_{\mathcal{O}^\otimes}(\mathbf{C}, \mathbf{D}) \simeq \prod_{U \in \mathrm{Orb}(S)} \mathrm{Map}^{T_U \to U}_{\mathcal{O}^\otimes}\left(\mathbf{C}_{\underline{U}}, D\right).$$

In fact, each condition follows from Proposition A.7. □

Observation A.9. It follows from Proposition A.7 that the indexed tensor product functor $\otimes^S : \mathcal{C}^{I-\sqcup}_S \to \mathcal{C}^{I-\sqcup}_V$ is left adjoint to Δ^S, i.e. indexed tensor products in $\mathcal{C}^{I-\sqcup}$ are indexed coproducts. ◁

Given $\mathcal{O}^\otimes$ a unital I-operad, define a diagram of Cartesian squares in $\mathrm{Cat}_{\mathcal{T}}$.

$$\begin{tikzcd} \mathcal{O} \arrow[r, hook, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^\otimes_\Gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^\otimes \arrow[d] \\ *_{\mathcal{T}} \arrow[r, hook] & \underline{\Gamma}^*_I \arrow[r] & \underline{\mathbb{F}}_{I,*} \end{tikzcd}$$

Note that the objects of $\mathcal{O}^\otimes_{\Gamma,V}$ consist of triples (S_+, U, X) where $U \in \mathrm{Orb}(S)$ and $X \in \mathcal{O}_S$, and the image of ι is equivalent to the triples where $S \in \mathcal{T}_{/V}$, hence $U = S$.

Further note that cocartesian transport along the inert morphism $U_+ \hookrightarrow S_+$ induces an equivalence
$$\mathrm{Map}_{\mathcal{O}^\otimes_{\Gamma,V}}(\iota Y, (S_+, U, X))) \simeq \mathrm{Map}_{\mathcal{O}^\otimes_{\Gamma,V}}(\iota Y, (U_+, U, X_U)))$$
for all $Y \in \mathcal{O}$.[7] In particular, ι witnesses $\mathcal{O}$ as a *colocalizing $\mathcal{T}$-subcategory*, with colocalization $\mathcal{T}$-functor
$$R(S_+, U, X) \simeq (U_+, U, X_U).$$

Lemma A.10. *Fix a $\mathcal{T}$-functor $A \colon \mathcal{O}^\otimes_\Gamma \to \mathcal{C}$. Then, the following are equivalent*

(a) The corresponding map $\mathcal{O}^\otimes \to \mathcal{C}^{I-\sqcup}$ is a functor of I-operads.
(b) For all morphisms α in $\mathcal{O}^\otimes_\Gamma$ whose image in $\mathcal{O}^\otimes$ is inert, $A(\alpha)$ is an equivalence in $\mathcal{C}$.
(c) If $f \colon (S_+, U, X) \to (U_+, U, X_U)$ is a cocartesian lift of the corresponding inert morphism, then $A(f)$ is an equivalence.
(d) A is $\mathcal{T}$-left Kan extended from $\mathcal{O}$.

Furthermore, every functor $F : \mathcal{O} \to \mathcal{C}$ admits a left Kan extension along $\mathcal{O} \hookrightarrow \mathcal{O}^\otimes_\Gamma$; in particular, the forgetful functor $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_G(\mathcal{O}, \mathcal{C})$ is an equivalence.

Proof. (a) $\iff$ (b) follows immediately from Proposition A.7. (b) $\iff$ (c) is immediate by definition. (c) $\iff$ (d) and the remaining statement both follow by the more general observation that the $\mathcal{T}$-left Kan extension of $F \colon \mathcal{C} \to \mathcal{D}$ along a $\mathcal{T}$-functor $L \colon \mathcal{C} \to \mathcal{E}$ with right adjoint R is given by the composite $FR \colon \mathcal{E} \to \mathcal{C} \to \mathcal{D}$. □

We would additionally like to characterize I-symmetric monoidal functors into $\mathcal{C}^{I-\sqcup}$. The following lemma follows immediately from Proposition A.7.

Lemma A.11. *Assume $\mathcal{C}$ has I-indexed coproducts and $\mathcal{D}^\otimes$ is an I-symmetric monoidal ∞-category. Then, TFAE for a lax I-symmetric monoidal functor $\varphi : \mathcal{D}^\otimes \to \mathcal{C}^{I-\sqcup}$:*

(1) φ is a map of I-symmetric monoidal categories.
(2) The corresponding $\mathcal{T}$-functor $F : \mathcal{D}^\otimes \to \mathcal{C}$ satisfies the property that, for all $(X_U) \in \mathcal{D}_S$, the canonical maps $\mathrm{Ind}_U^V F(X_U) \to F(X)$ exhibit $F(X)$ as the indexed coproduct
$$\coprod_U^S F(X_U) \simeq F(X).$$

[7] This assumes I-unitality of $\mathcal{O}^\otimes$, as we implicitly use that, for each orbit $U' \in \mathrm{Orb}(S)$ other than U, the space $\mathcal{O}(\varnothing_{U'}; X_{U'})$ is contractible.

We use this for the following fundamental proposition underlying (co)cartesian rigidity.

Proposition A.12. *Suppose $\mathcal{D}^{\otimes}$ is an I-symmetric monoidal category satisfying the condition that its action maps $f_{\otimes}: \mathcal{D}_S \to \mathcal{D}_V$ are left adjoint to the restriction map $f^*: \mathcal{D}_V \to \mathcal{D}_S$. Then, the forgetful functor*

$$U: \mathrm{Fun}_I^{\otimes}(\mathcal{D}^{\otimes}, \mathcal{C}^{I-\sqcup}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{D}, \mathcal{C})$$

is fully faithful with image spanned by the I-coproduct preserving functors; dually, if $\mathcal{E}^{\otimes}$ is an I-symmetric monoidal category satisfying the condition that its action maps $f_{\otimes}: \mathcal{E}_S \to \mathcal{E}_V$ are right *adjoint to the restriction map $f^*: \mathcal{E}_V \to \mathcal{E}_S$, then the forgetful functor*

$$U: \mathrm{Fun}_I^{\otimes}(\mathcal{E}^{\otimes}, (\mathcal{C}^{I-\times})^{v\,\mathrm{op}}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{E}, \mathcal{C})$$

is fully faithful with image spanned by the I-product preserving functors, $(-)^{v\,\mathrm{op}}$ denoting the fiberwise opposite over $\underline{\mathbb{F}}_{I,*}$.

Proof. The first statement follows by noting that those $\mathcal{T}$-functors $\mathcal{D}^{\otimes} \to \mathcal{C}$ satisfying the conditions of Lemma A.11 are precisely those which are left Kan extended along the (fully faithful) $\mathcal{T}$-functor $\mathcal{D} \hookrightarrow \mathcal{D}^{\otimes}$ from I-coproduct preserving functors. The second follows by taking fiberwise opposites. □

We are now ready to prove our main generalization for Theorem A' (see p. 19).

Proof of Theorem A'. The two cases are dual, so we prove it for $(-)^{I-\sqcup}$. To see that it's fully faithful, it suffices to note that the action maps in $\mathcal{C}^{I-\sqcup}$ are left adjoint to restriction and apply Proposition A.12. The compatibility with U is obvious, and the description of the image follows immediately from Proposition A.12. □

A.3. **$\mathcal{O}$-monoids.** Given a Γ-I-preoperad $\mathcal{O}^{\otimes}$, we say that an *$\mathcal{O}$-monoid in $\mathcal{C}$* is a $\mathcal{T}$-functor $\mathcal{O}^{\otimes} \to \mathcal{C}$ satisfying the condition that, for all $X \in \mathcal{C}_S$, the maps $\mathrm{Res}_U^V F(X) \to F(X_U)$ induced by cocartesian transport witness $F(X)$ as the indexed product

$$F(X) \simeq \prod_U^S F(X_U).$$

We are tasked with proving the following.

Proposition 1.36. *Fix $\mathcal{C}$ a $\mathcal{T}$-∞-category. Then, the postcomposition functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\times}) \to \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}^{\otimes}, \mathcal{C})$ is fully faithful with image spanned by the $\mathcal{O}$-monoids.*

Proposition A.13. *$\mathcal{C}^{I-\times} \to \mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ is a cocartesian fibration; moreover, its straightening is an I-symmetric monoidal ∞-category if and only if $\mathcal{C}$ admits I-indexed products.*

Observation A.14. $\mathcal{C}^{I-\times}$ is a cartesian I-symmetric monoidal ∞ category with underlying $\mathcal{T}$-∞-category $\mathcal{C}$, so we have not created a clash in notation. ◁

Observation A.15. The structure map $\mathcal{O}^{\otimes} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_I^{\times} \to \mathcal{O}^{\otimes}$ admits a left adjoint L sending $X \in \mathcal{O}^{\otimes}_{S_+}$ to $(X, S \subset S)$; the unit map of this adjunction is evidently an equivalence, so $L: \mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_I^{\times}$ is fully faithful. ◁

Fix a $\mathcal{T}$-functor $A: \mathcal{O}^{\otimes} \times_{\underline{\mathbb{F}}_{I,*}} \Gamma^{\times} \to \mathcal{C}$ with corresponding functor $\varphi: \mathcal{O}^{\otimes} \to \tilde{\mathcal{C}}^{I-\times}$ and restricted functor $A': \mathcal{O}^{\otimes} \to \mathcal{C}$. Lemma A.6 immediately implies the following.

Lemma A.16. *Suppose A' is a $\mathcal{T}$-functor. Then, the following conditions are equivalent:*

- *(a) The map φ is a functor of Γ-I-preoperads.*
- *(b) For all morphisms α in $\mathcal{O}^{\otimes} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_I^{\times}$ whose image in $\mathcal{O}^{\otimes}$ is inert $A(\alpha)$ is an equivalence in $\mathcal{C}$.*
- *(c) If $f: (\overline{S}_+ \to V_+, \overline{S}, F, X) \to (S_+ \to V_+, \overline{S}, F, X)$ is a cocartesian lift of the corresponding inert morphism, then $A(f)$ is an equivalence.*
- *(d) A is right Kan extended from A' along L.*

In this case, the composite map $\mathcal{O}^{\otimes} \to \tilde{\mathcal{C}}^{I-\times} \to \mathcal{C}$ is homotopic to A'.

We use this to finally identify Cartesian algebras in the following lemma, which also follows immediately from Lemma A.6.

Lemma A.17. *Suppose φ is a functor of I-operads. Then, the following conditions are equivalent:*

(a) φ *factors through the inclusion* $\mathcal{C}^{I-\times} \subset \tilde{\mathcal{C}}^{I-\times}$.
(b) A' *is an* $\mathcal{O}$*-monoid.*

Proof of Proposition 1.36. $\mathcal{C}^{I-\times} \hookrightarrow \tilde{\mathcal{C}}^{I-\times}$ is fully faithful, and hence it is a monomorphism in $\mathbf{Cat}$. This implies that the associated functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{I-\times}) \hookrightarrow \mathrm{Fun}^{\mathrm{int-cocart}}_{/\underline{\mathbb{F}}_{I,*}}\left(\mathcal{O}^{\otimes}, \tilde{\mathcal{C}}^{I-\times}\right) \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{O}^{\otimes}, \mathcal{C}\right)$$

is fully faithful. By Lemma A.17, its image is the $\mathcal{O}$-monoids. □

A.4. **A corollary concerning n-Morita equivalences.** A Segal morphism of algebraic patterns $\varphi\colon \mathfrak{O} \to \mathfrak{P}$ is called an *n-Morita equivalence* if, for all complete $(n+1)$-categories $\mathcal{C}$, the induced functor

$$f^*\colon \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$$

is an equivalence; in fact, it suffices to check this in the case $\mathcal{C} = \mathcal{S}_{\le n}$ [Bar23, Prop 2.1.9]. We have the following corollary.

Corollary A.18. *Suppose* $\mathfrak{P}, \mathfrak{O}$ *are inert-cocartesian fibrations over* $\mathsf{Span}(\mathbb{F}_{\mathcal{T}})$*. Suppose* $\varphi\colon \mathfrak{P} \to \mathfrak{O}$ *is an essentially surjective Segal morphism over* $\mathsf{Span}(\mathbb{F}_{\mathcal{T}})$ *under the induced algebraic pattern structure. Then,* φ *is an n-Morita equivalence if and only if the associated map of $\mathcal{T}$-operads* $L_{\mathrm{Op}_{\mathcal{T}}}\mathfrak{P} \to L_{\mathrm{Op}_{\mathcal{T}}}\mathfrak{O}$ *is an n-equivalence.*

Proof. There is an equivalence

$$\mathrm{Seg}_{\mathfrak{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathfrak{O}}(\mathcal{C}^{I-\times}) \simeq \mathrm{Alg}_{L_{\mathrm{Op}_{\mathcal{T}}}}(\mathcal{C}^{I-\times})$$

natural in Segal morphisms over $\mathsf{Span}(\mathbb{F}_{\mathcal{T}})$, so the result follows from the recognition result for n-equivalences of I-operads [Ste25a]. □

Appendix B. I-operadic disintegration and assembly

In this appendix, we assume familiarity with the minutiae of [BHS22; Ste25a; Ste25b] and prove Corollary 3.7

B.1. **The algebraic pattern for $\mathcal{C}$-families of I-operads.** Fix $\mathcal{C}$ a $\mathcal{T}$-∞-category.

Construction B.1. We give $\mathrm{Tot}\mathcal{C}$ two pattern structures. The *totally inert pattern structure on* $\mathcal{C}$ has

$$\left(\mathrm{Tot}^{\mathrm{int}}\mathcal{C}\right)^{\mathrm{el}} := \left(\mathrm{Tot}^{\mathrm{int}}\mathcal{C}\right)^{\mathrm{int}} := \mathrm{Tot}\mathcal{C}; \qquad \left(\mathrm{Tot}^{\mathrm{int}}\mathcal{C}\right)^{\mathrm{act}} := (\mathrm{Tot}\mathcal{C})^{\simeq}.$$

Now, $\mathrm{Tot}\mathcal{C}$ has a factorization system $\left(\mathrm{Tot}\mathcal{C}^{\mathrm{cocart}}, \mathrm{Tot}\mathcal{C}^{\mathrm{tdeg}}\right)$, the former denoting cocartesian arrows and the latter arrows which become identity in $\mathcal{T}^{\mathrm{op}}$. The *totally active pattern structure on* $\mathcal{C}$ has

$$\left(\mathrm{Tot}^{\mathrm{act}}\mathcal{C}\right)^{\mathrm{el}} := \left(\mathrm{Tot}^{\mathrm{act}}\mathcal{C}\right)^{\mathrm{int}} := \mathrm{Tot}\mathcal{C}^{\mathrm{cocart}}; \qquad \left(\mathrm{Tot}^{\mathrm{act}}\mathcal{C}\right)^{\mathrm{act}} := \mathrm{Tot}\mathcal{C}^{\mathrm{tdeg}}.$$ ◁

Construction B.2. More generally, we give $\mathrm{Tot}(\mathcal{C} \times \underline{\mathbb{F}}_{I,*})$ two pattern structures:

$$\mathrm{Tot}^{\mathrm{int}}\left(\mathcal{C} \times \underline{\mathbb{F}}_{I,*}\right) := \mathrm{Tot}\mathcal{C}^{\mathrm{int}} \times_{\mathcal{T}^{\mathrm{op,int}}} \mathrm{Tot}\underline{\mathbb{F}}_{I,*};$$
$$\mathrm{Tot}^{\mathrm{act}}\left(\mathcal{C} \times \underline{\mathbb{F}}_{I,*}\right) := \mathrm{Tot}\mathcal{C}^{\mathrm{act}} \times_{\mathcal{T}^{\mathrm{op,int}}} \mathrm{Tot}\underline{\mathbb{F}}_{I,*}.$$ ◁

The following observation is as crucial as it is immediate.

Observation B.3. If $\mathcal{C}$ is a $\mathcal{T}$-space, then $\mathrm{Tot}^{\mathrm{int}}\left(\mathcal{C} \times \underline{\mathbb{F}}_{I,*}\right)$ and $\mathrm{Tot}^{\mathrm{act}}\left(\mathcal{C} \times \underline{\mathbb{F}}_{I,*}\right)$ are synonymous. In any case, the functor $\mathrm{Tot}^{\mathrm{act}}(\mathcal{C} \times \underline{\mathbb{F}}_{I,*}) \to \underline{\mathbb{F}}_{I,*}$ is an inert-cocartesian fibration, and the domain has the induced pattern structure. ◁

We begin by identifying $\mathcal{C}$-indexed diagrams of I-operads.

Proposition B.4. *There exists an equivalence*

$$\mathrm{Fbrs}\left(\mathrm{Tot}^{\mathrm{int}}(\mathcal{C} \times \underline{\mathbb{F}}_{I,*})\right) \simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}, \underline{\mathrm{Op}}_I),$$

natural in $\mathcal{C}$ *and* I.

To prove this, we use the equifibered theory, focusing on the following lemmas.

Lemma B.5. *There exists a natural equivalence* $\mathrm{Seg}_{\mathrm{Tot}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*})}(\mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{CMon}}_I(\mathcal{D})\right)$.

Proof. First off, we get an embedding

$$\begin{aligned}\mathrm{Seg}_{\mathrm{Tot}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*})}(\mathcal{D}) &\subset \mathrm{Fun}(\mathrm{Tot}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*}), \mathcal{D});\\ &\simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*}, \underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{D}));\\ &\simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\underline{\mathbb{F}}_{I,*}, \underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}\right)\right)\end{aligned}$$

characterized by the Segal condition that the restricted functor $\mathrm{Tot}\left(\mathcal{C}\times\underline{\mathbb{F}}_{I,*}^{\mathrm{int}}\right) \to \mathcal{D}$ is right Kan extended from $\mathrm{Tot}\left(\mathcal{C}\times\underline{\mathbb{F}}_{I,*}^{\mathrm{el}}\right)$. Now, unwinding conditions, this corresponds with the condition that the value V-functors $\underline{\mathbb{F}}_{I_V,*}^{\mathrm{int}} \to \underline{\mathrm{Coeff}}^V\mathcal{D}$ are each right-Kan extended from $\underline{\mathbb{F}}_{I_V,*}^{\mathrm{int}}$, i.e. the corresponding functor factors through $\underline{\mathrm{CMon}}_I(\mathcal{D}) \subset \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\underline{\mathbb{F}}_{I,*}, \underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}\right)$. □

Lemma B.6. $\mathrm{EnvTot}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*})$ *corresponds naturally with the constant $\mathcal{T}$-functor over $\underline{\mathbb{F}}_{I,*}^{I-\sqcup}$; a natural transformation $F \to \mathrm{Env}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*})$ is equifibered if and only if it is pointwise-equifibered.*

Proof. This follows by explicitly identifying the active arrows in $\mathrm{Tot}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*})$ as products of equivalences and active arrows of $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$. □

Proof of Proposition B.4. The above work constructs a string of natural equivalences

$$\begin{aligned}\mathrm{Fbrs}(\mathrm{Tot}(\mathcal{C}\times\underline{\mathbb{F}}_{I,*})) &\simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{CMon}}_I(\mathcal{D})\right)^{\text{fiberwise-equifibered}}_{/\Delta\underline{\mathbb{F}}_{I,*}}\\ &\simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{CMon}}_I(\mathcal{D})^{\text{equifibered}}_{/\underline{\mathbb{F}}_{I,*}}\right)\\ &\simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{Op}}_I\right).\end{aligned}$$ □

B.2. **A morita equivalence to $\mathcal{C}^{I-\sqcup}$.** We now dedicate some study to the following functor.

Observation B.7. Fix $\mathcal{C}$ a $\mathcal{T}$-∞-category. Then, pullback along the projection $\mathcal{D}\times_{\underline{\mathbb{F}}_{I,*}}\underline{\Gamma}_I^* \to \mathcal{D}$ determines a natural transformation

$$\mathrm{Fun}_{/\mathrm{Tot}\underline{\mathbb{F}}_{I,*}}\left(-, \mathrm{Tot}\mathcal{C}^{I-\sqcup}\right) \simeq \mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}\left(-\times_{\underline{\mathbb{F}}_{I,*}}\underline{\Gamma}_I^*, \mathcal{C}\right) \leftarrow \mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}(-,\mathcal{C}) \simeq \mathrm{Fun}_{/\underline{\mathbb{F}}_{I,*}}\left(-, \mathcal{C}\times\underline{\mathbb{F}}_{I,*}\right),$$

which corresponds with a functor $\gamma\colon \mathcal{C}\times\underline{\mathbb{F}}_{I,*} \to \mathcal{C}^{I-\sqcup}$ under Yoneda's lemma. We may explicitly describe the values of γ as

$$\gamma(C,S) \simeq \Delta^S C.$$

Moreover, this is compatible with operadic structure in the case $\mathcal{C} = \mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes}$; we have a diagram

$$\begin{tikzcd}
\underline{\mathbb{F}}_{I,*}\times\mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes} \arrow[rr, "\mathrm{id}\times\pi_{\mathcal{P}}"] \arrow[dr, "\gamma"] \arrow[ddr, bend right, "\mathrm{pr}_1"'] & & \underline{\mathbb{F}}_{I,*}\times\underline{\mathbb{F}}_{I,*} \arrow[dr, "\wedge"] & \\
& (\mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes})^{I-\sqcup} \arrow[r, "\pi_{\mathcal{P}}^{I-\sqcup}"] \arrow[d, "\pi"] & \left(\underline{\mathbb{F}}_{I,*}\right)^{I-\sqcup} \arrow[r, "\vee"] & \underline{\mathbb{F}}_{I,*} \\
& \underline{\mathbb{F}}_{I,*} & &
\end{tikzcd}$$

Pulling back to $\mathrm{Span}_I(\mathbb{F}_V)$, we acquire a simpler diagram

$$\begin{tikzcd}
\mathrm{Span}_I(\mathbb{F}_V)\times\mathrm{Tot}\mathcal{P}^{\otimes} \arrow[r, "\mathrm{id}\times\pi_{\mathcal{P}}"] \arrow[dr, "\gamma"] \arrow[ddr, bend right, "\mathrm{pr}_1"'] & \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})\times\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \arrow[dr, "\wedge"] & \\
& (\mathrm{Tot}_V\mathcal{P}^{\otimes})^{I-\sqcup} \arrow[r, "\rho"] \arrow[d, "\pi"] & \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \\
& \mathrm{Span}_I(\mathbb{F}_V) &
\end{tikzcd}$$

so in particular, γ becomes a *bifunctor* under the alternative structure map ρ. ◁

In this paper, we mostly care about the case that $\mathcal{C}$ is a generic $\mathcal{T}$-space. We will use the following specialization case of Barkan's morita equivalence recognition result to Γ-I-preoperads (c.f. [HA, Thm 2.3.3.23, Thm 2.3.3.26]).[8]

Proposition B.8 ([BHS22, Prop 3.1.16, Thm 5.1.1])**.** *Suppose $f\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a strong Segal morphism of Γ-I-preoperads such that $\mathcal{P}^{\otimes}$ presents an I-operad and the following conditions hold:*

(1) the $\mathcal{T}$-functor $U\mathcal{O} \to U\mathcal{P}$ is an equivalence, and

(2) for every $O \in \mathcal{O}$, the map of spaces $\left(\mathcal{O}^{\mathrm{act}}_{/O}\right)^{\simeq} \to \left(\mathcal{P}^{\mathrm{act}}_{/f(O)}\right)^{\simeq}$ is an equivalence.

Then, the functors $f^*\colon \mathrm{Mon}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ and $f^*\colon \mathrm{Op}_{I,/\mathcal{P}^{\otimes}} \to \mathrm{Fbrs}(\mathcal{O})$ are equivalences.

Proposition B.9. *When X is a $\mathcal{T}$-space, the $\mathcal{T}$-functor $\gamma\colon X \times \underline{\mathbb{F}}_{I,*} \to \mathrm{Tot}_I X^{I-\sqcup}$ satisfies the conditions of Proposition B.8; in particular, γ is an L_{Op_I} localization map.*

Proof. By unwinding definitions we find that $\mathrm{Tot}\gamma$ is an iso-segal morphism, and in particular it is a strong Segal morphism. Moreover, condition (b) follows by unwinding definitions.

For claim (a), note that the elements of $\left(\mathrm{Tot}_I X^{I-\sqcup}\right)^{\mathrm{act}}_{/\gamma(x,S)}$ correspond with maps $f\colon T \to S$ in $\mathbb{F}_{\mathcal{T}}$ together with elements $(y_U)_T \in X^T$ with distinguished paths $y \sim x$; in particular, by contracting paths, we may construct a deformation retract of $\left(\left(\mathrm{Tot}_I X^{I-\sqcup}\right)^{\mathrm{act}}_{/\gamma(x,S)}\right)^{\simeq}$ onto the subspace $\mathbb{F}^{\simeq}_{I,/S} \subset \left(\left(\mathrm{Tot}_I X^{I-\sqcup}\right)^{\mathrm{act}}_{/\gamma(x,S)}\right)^{\simeq}$ of identity paths.

Similarly, we may perform a deformation retract of $\left(\left(X \times \underline{\mathbb{F}}_{i,*}\right)^{\mathrm{act}}_{/(x,S)}\right)^{\simeq}$ onto the summand $\mathbb{F}^{\simeq}_{I,/S} \subset \left(\left(X \times \underline{\mathbb{F}}_{i,*}\right)^{\mathrm{act}}_{/(x,S)}\right)^{\simeq}$ of identity paths. It follows by unwinding definitions that these are taken isomorphically onto each other– alternatively, one may note that the induced endomorphism of $\mathbb{F}^{\simeq}_{I,/S} \simeq \prod_{U \in \mathrm{Orb}(S)} \coprod_{T \in \mathbb{F}_{I,U}} B\mathrm{Aut}_U(T)$ is a product of coproducts of maps classified by torsor maps $\mathrm{Aut}_U(T) \to \mathrm{Aut}_U(T)$, which are automatically isomorphisms, implying that our map of 1-truncated spaces is an isomorphism on π_0 and on π_1 at all basepoints. □

Warning B.10. A closely related analog of Proposition B.9 is claimed in [HA, Rmk 2.4.3.6] in the case $\mathcal{T} = *$ *without the assumption that $\mathcal{C}$ is a space*; as pointed out in [KK24, Rmk 2.3] Lurie's claim (and hence proof) is incorrect in general, but the claim was verified in *loc. cit.* when $\mathcal{C}$ is a space. ◁

B.3. **Disintegration and assembly.** Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_{I,/X^{I-\sqcup}}$, define the pullback diagrams of Γ-I-preoperads

$$\begin{tikzcd} \mathrm{dis}^I(\mathcal{O}^{\otimes}) \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tot}_I \mathcal{O}^{\otimes} \arrow[d] \\ X \times \underline{\mathbb{F}}_{I,*} \arrow[r, "\gamma"] & \mathrm{Tot}_I X^{I-\sqcup} \end{tikzcd}$$

We refer to $\mathrm{dis}^I(-)$ as the *disintegration functor* and α as the *assembly map.*

Proposition B.11. *α is an L_{Op_I}-localization map.*

Proof. We verified in [Ste25a] that the conditions of Proposition B.8 are pullback-stable, so α is a Morita equivalence; by Corollary A.18 it is then an L_{Op_I}-equivalence. By assumption, $\mathrm{Tot}_I\mathcal{O}^{\otimes}$ is L_{Op_I}-local, proving the proposition. □

Proposition B.12. *$L_{\mathrm{Op}_I}\mathrm{dis}^I(\mathcal{O}^{\otimes})$ is the $\mathcal{T}$-colimit of the $\mathcal{T}$-functor $X \to \underline{\mathrm{Op}}_I$ associated with $\mathcal{O}^{\otimes}$.*

Proof. Note that $L_{\mathrm{Op}_I}\gamma_!\pi_{X^{I-\sqcup}!} \dashv \gamma^*\pi^*_{X^{I-\sqcup}}$, and the latter is equivalent to $\Delta\colon \mathrm{Op}_I \to \mathrm{Fun}_{\mathcal{T}}(X, \underline{\mathrm{Op}}_I)$; in particular, this left adjoint is the underlying I-operad functor, and indexed colimits are also left adjoint to Δ, so indexed colimits compute the underlying $\mathcal{T}$-operad of a fibrous $\mathrm{Tot}(X \times \underline{\mathbb{F}}_{I,*})$-pattern, and the claim follows. □

[8] To see this as a specialization of Barkan's result, note that by [Ste25a], $\mathrm{Tot}\underline{\mathbb{F}}_{I,*}$ is soundly extendable, so $\mathrm{Tot}_{\mathcal{T}}$ of an I-operad is soundly extendable by [BHS22, Lem 4.1.15]. The remaining modifications necessary are the observation that $\mathcal{O}^{\mathrm{el}} \simeq \mathrm{Tot}U\mathcal{O}$ (so condition (1) implies condition (1) of [BHS22, Thm 5.1.1]) and the identifications $\mathrm{Fbrs}(\mathcal{P}) \simeq \mathrm{Op}_{I,/\mathcal{P}^{\otimes}}$ of [Ste25a] and [BHS22, Cor 4.1.17] as well as $\mathrm{Mon}_{\mathcal{P}}(\mathcal{C}) \simeq \mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$ of cref .

We spell out the following corollary, which summarizes the full power of what we've proved.

Corollary B.13. *Let X be a $\mathcal{T}$-space. The assignment $x \mapsto \mathcal{O}_X \coloneqq \mathrm{Res}_V^{\mathcal{T}} \mathcal{O}^{\otimes} \times_{\mathrm{Res}_V^{\mathcal{T}} X^{I-\sqcup}} \mathcal{N}_{I\infty}^{I-\sqcup}$, yields an equivalence*

$$\underline{\mathrm{Op}}_{I,/X^{I-\sqcup}} \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(X, \underline{\mathrm{Op}}_I).$$

The unit of this equivalence specifies a natural equivalence.

$$\mathcal{O}^{\otimes} \simeq \underline{\mathrm{colim}}_{x\in X} \mathcal{O}_x^{\otimes},$$

Proof. The first claim follows by combining Observation B.3 and Propositions B.4, B.8 and B.9. The remaining claim is proved identically to Proposition B.12. □

References

[BH21] Tom Bachmann and Marc Hoyois. "Norms in motivic homotopy theory". In: *Astérisque* 425 (2021), pp. ix+207. ISSN: 0303-1179,2492-5926. DOI: 10.24033/ast. URL: https://arxiv.org/abs/1711.03061 (cit. on p. 4).

[Bar23] Shaul Barkan. *Arity Approximation of ∞-Operads*. 2023. arXiv: 2207.07200 [math.AT] (cit. on p. 41).

[BHS22] Shaul Barkan, Rune Haugseng, and Jan Steinebrunner. *Envelopes for Algebraic Patterns*. 2022. arXiv: 2208.07183 [math.CT] (cit. on pp. 2, 3, 8, 12, 14, 17, 36, 41, 43).

[BS24a] Shaul Barkan and Jan Steinebrunner. *Segalification and the Boardmann-Vogt tensor product*. 2024. arXiv: 2301.08650 [math.AT] (cit. on pp. 8, 29, 30).

[BS24b] Shaul Barkan and Jan Steinebrunner. *The equifibered approach to ∞-properads*. 2024. arXiv: 2211.02576 [math.AT]. URL: https://arxiv.org/abs/2211.02576 (cit. on p. 2).

[Bar14] C. Barwick. *Spectral Mackey functors and equivariant algebraic K-theory (I)*. 2014. arXiv: 1404.0108 [math.AT] (cit. on pp. 10, 11).

[BDGNS16] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah. *Parametrized higher category theory and higher algebra: A general introduction*. 2016. arXiv: 1608.03654 [math.AT] (cit. on p. 2).

[BGS20] Clark Barwick, Saul Glasman, and Jay Shah. "Spectral Mackey functors and equivariant algebraic K-theory, II". In: *Tunisian Journal of Mathematics* 2.1 (Jan. 2020), pp. 97–146. ISSN: 2576-7658. DOI: 10.2140/tunis.2020.2.97. URL: https://arxiv.org/abs/1505.03098 (cit. on p. 11).

[BH15] Andrew J. Blumberg and Michael A. Hill. "Operadic multiplications in equivariant spectra, norms, and transfers". In: *Adv. Math.* 285 (2015), pp. 658–708. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2015.07.013. URL: https://arxiv.org/abs/1309.1750 (cit. on pp. 1, 2, 6, 7).

[Bon19] Peter Bonventre. "The genuine operadic nerve". In: *Theory Appl. Categ.* 34 (2019), pp. 736–780. ISSN: 1201-561X. URL: https://arxiv.org/abs/1904.01465 (cit. on pp. 20, 33).

[CSY20] Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski. *Ambidexterity and Height*. 2020. arXiv: 2007.13089 [math.AT]. URL: https://arxiv.org/abs/2007.13089 (cit. on pp. 8, 24).

[CH21] Hongyi Chu and Rune Haugseng. "Homotopy-coherent algebra via Segal conditions". In: *Adv. Math.* 385 (2021), Paper No. 107733, 95. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2021.107733. URL: https://arxiv.org/abs/1907.03977 (cit. on p. 11).

[CHLL24a] Bastiaan Cnossen, Rune Haugseng, Tobias Lenz, and Sil Linskens. *Homotopical commutative rings and bispans*. 2024. arXiv: 2403.06911 [math.CT] (cit. on p. 3).

[CHLL24b] Bastiaan Cnossen, Rune Haugseng, Tobias Lenz, and Sil Linskens. *Normed equivariant ring spectra and higher Tambara functors*. 2024. arXiv: 2407.08399 [math.AT]. URL: https://arxiv.org/abs/2407.08399 (cit. on pp. 4, 6, 9).

[CLL24] Bastiaan Cnossen, Tobias Lenz, and Sil Linskens. *Parametrized higher semiadditivity and the universality of spans*. 2024. arXiv: 2403.07676 [math.AT] (cit. on pp. 5, 7, 9, 11, 20, 21).

[Cra11] James Cranch. *Algebraic theories, span diagrams and commutative monoids in homotopy theory*. 2011. arXiv: 1109.1598 [math.AT]. URL: https://arxiv.org/abs/1109.1598 (cit. on p. 3).

[Die09] Tammo tom Dieck. *Representation theory*. 2009. URL: https://ncatlab.org/nlab/files/tomDieckRepresentationTheory.pdf (cit. on p. 4).

[Dre71] Andreas W. M. Dress. *Notes on the theory of representations of finite groups. Part I: The Burnside ring of a finite group and some AGN-applications.* With the aid of lecture notes, taken by Manfred Küchler. Universität Bielefeld, Fakultät für Mathematik, Bielefeld, 1971, iv+158+A28+B31 pp. (loose errata) (cit. on p. 4).

[Dun88] Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258. ISSN: 0022-4049,1873-1376. DOI: 10.1016/0022-4049(88)90103-X. URL: https://people.math.rochester.edu/faculty/doug/otherpapers/Dunn.pdf (cit. on p. 9).

[Elm83] A. D. Elmendorf. "Systems of Fixed Point Sets". In: *Transactions of the American Mathematical Society* 277.1 (1983), pp. 275–284. ISSN: 00029947. URL: https://people.math.rochester.edu/faculty/doug/otherpapers/elmendorf-fixed.pdf (visited on 04/22/2023) (cit. on p. 4).

[FN62] Edward Fadell and Lee Neuwirth. "Configuration spaces". In: *Math. Scand.* 10 (1962), pp. 111–118. ISSN: 0025-5521,1903-1807. DOI: 10.7146/math.scand.a-10517. URL: https://www.mscand.dk/article/download/10517/8538 (cit. on p. 34).

[Gep19] David Gepner. *An Introduction to Higher Categorical Algebra.* 2019. arXiv: 1907.02904 [math.AT] (cit. on p. 9).

[GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus. "Universality of multiplicative infinite loop space machines". In: *Algebr. Geom. Topol.* 15.6 (2015), pp. 3107–3153. ISSN: 1472-2747,1472-2739. DOI: 10.2140/agt.2015.15.3107. URL: https://arxiv.org/pdf/1305.4550 (cit. on pp. 3, 20).

[GM11] Bertrand Guillou and J. P. May. *Models of G-spectra as presheaves of spectra.* 2011. arXiv: 1110.3571 [math.AT] (cit. on p. 6).

[GM17] Bertrand J. Guillou and J. Peter May. "Equivariant iterated loop space theory and permutative G-categories". In: *Algebr. Geom. Topol.* 17.6 (2017), pp. 3259–3339. ISSN: 1472-2747. DOI: 10.2140/agt.2017.17.3259. URL: https://arxiv.org/abs/1207.3459 (cit. on p. 33).

[Hau23] Rune Haugseng. *An allegendly somewhat friendly introduction to ∞-operads.* 2023. URL: https://folk.ntnu.no/runegha/iopd.pdf (cit. on p. 9).

[HHLN23] Rune Haugseng, Fabian Hebestreit, Sil Linskens, and Joost Nuiten. "Two-variable fibrations, factorisation systems and ∞-categories of spans". In: *Forum Math. Sigma* 11 (2023), Paper No. e111, 70. ISSN: 2050-5094. DOI: 10.1017/fms.2023.107. URL: https://arxiv.org/abs/2011.11042 (cit. on p. 11).

[HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. "On the nonexistence of elements of Kervaire invariant one". In: *Ann. of Math. (2)* 184.1 (2016), pp. 1–262. ISSN: 0003-486X. DOI: 10.4007/annals.2016.184.1.1. URL: https://people.math.rochester.edu/faculty/doug/mypapers/Hill_Hopkins_Ravenel.pdf (cit. on p. 4).

[Hil17] Michael A. Hill. *Equivariant chromatic localizations and commutativity.* 2017. arXiv: 1708.03017 [math.AT]. URL: https://arxiv.org/abs/1708.03017 (cit. on p. 33).

[HH16] Michael A. Hill and Michael J. Hopkins. *Equivariant symmetric monoidal structures.* 2016. arXiv: 1610.03114 [math.AT] (cit. on pp. 3, 6).

[Hil24] Kaif Hilman. *Parametrised Presentability over Orbital Categories.* 2024. arXiv: 2202.02594 [math.AT] (cit. on p. 19).

[HM23] Vladimir Hinich and Ieke Moerdijk. *On the equivalence of the Lurie's ∞-operads and dendroidal ∞-operads.* 2023. arXiv: 2206.14033 [math.CT] (cit. on pp. 3, 30).

[Hor19] Asaf Horev. *Genuine equivariant factorization homology.* 2019. arXiv: 1910.07226 [math.AT] (cit. on pp. 6, 33, 36).

[KK24] Manuel Krannich and Alexander Kupers. *∞-operadic foundations for embedding calculus.* 2024. arXiv: 2409.10991 [math.AT]. URL: https://arxiv.org/abs/2409.10991 (cit. on p. 43).

[Lin76] Harald Lindner. "A remark on Mackey-functors". In: *Manuscripta Math.* 18.3 (1976), pp. 273–278. ISSN: 0025-2611,1432-1785. DOI: 10.1007/BF01245921. URL: https://people.math.rochester.edu/faculty/doug/otherpapers/lindner.pdf (cit. on p. 4).

[HTT] Jacob Lurie. *Higher topos theory.* Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. DOI: 10.1515/9781400830558. URL: https://www.math.ias.edu/~lurie/papers/HTT.pdf (cit. on pp. 9, 38).

[HA] Jacob Lurie. *Higher Algebra*. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf (cit. on pp. 2, 3, 6, 8, 9, 17, 19, 25, 29–32, 36, 37, 43).

[Mar24] Gregoire Marc. *A higher Mackey functor description of algebras over an N_∞-operad*. 2024. arXiv: 2402.12447 [math.AT] (cit. on p. 20).

[Nar16] Denis Nardin. *Parametrized higher category theory and higher algebra: Exposé IV – Stability with respect to an orbital ∞-category*. 2016. arXiv: 1608.07704 [math.AT] (cit. on pp. 5, 9, 10, 20).

[Nar17] Denis Nardin. *Stability and distributivity over orbital ∞-categories (thesis)*. 2017. URL: https://dspace.mit.edu/handle/1721.1/112895?show=full (cit. on p. 20).

[NS22] Denis Nardin and Jay Shah. *Parametrized and equivariant higher algebra*. 2022. arXiv: 2203.00072 [math.AT] (cit. on pp. 2, 8, 9, 12, 13, 15, 18, 19, 35, 36, 38).

[NS18] Thomas Nikolaus and Peter Scholze. *On topological cyclic homology*. 2018. arXiv: 1707.01799 [math.AT]. URL: https://arxiv.org/abs/1707.01799 (cit. on p. 3).

[QS19] J. D. Quigley and Jay Shah. *On the parametrized Tate construction and two theories of real p-cyclotomic spectra*. 2019. arXiv: 1909.03920 [math.AT]. URL: https://arxiv.org/abs/1909.03920 (cit. on p. 36).

[Rub19] Jonathan Rubin. *Characterizations of equivariant Steiner and linear isometries operads*. 2019. arXiv: 1903.08723 [math.AT] (cit. on p. 34).

[Rub21] Jonathan Rubin. "Operadic lifts of the algebra of indexing systems". In: *J. Pure Appl. Algebra* 225.12 (2021), Paper No. 106756, 35. ISSN: 0022-4049,1873-1376. DOI: 10.1016/j.jpaa.2021.106756. URL: https://doi.org/10.1016/j.jpaa.2021.106756 (cit. on p. 7).

[SY19] Tomer M. Schlank and Lior Yanovski. "The ∞-categorical Eckmann-Hilton argument". In: *Algebr. Geom. Topol.* 19.6 (2019), pp. 3119–3170. ISSN: 1472-2747,1472-2739. DOI: 10.2140/agt.2019.19.3119. URL: https://arxiv.org/abs/1808.06006 (cit. on pp. 27, 29).

[Seg74] Graeme Segal. "Categories and cohomology theories". In: *Topology* 13 (1974), pp. 293–312. ISSN: 0040-9383. DOI: 10.1016/0040-9383(74)90022-6. URL: https://doi.org/10.1016/0040-9383(74)90022-6 (cit. on p. 3).

[Sha22] Jay Shah. *Parametrized higher category theory II: Universal constructions*. 2022. arXiv: 2109.11954 [math.CT] (cit. on p. 9).

[Sha23] Jay Shah. "Parametrized higher category theory". In: *Algebr. Geom. Topol.* 23.2 (2023), pp. 509–644. ISSN: 1472-2747,1472-2739. DOI: 10.2140/agt.2023.23.509. URL: https://arxiv.org/pdf/1809.05892.pdf (cit. on p. 9).

[Ste24] Natalie Stewart. *Orbital categories and weak indexing systems*. 2024. arXiv: 2409.01377 [math.CT]. URL: https://arxiv.org/abs/2409.01377 (cit. on pp. 1, 2, 5, 9, 10, 16, 19, 28, 34).

[Ste25a] Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT]. URL: https://arxiv.org/abs/2501.02129 (cit. on pp. 1, 2, 4–16, 20, 29, 30, 33, 41, 43).

[Ste25b] Natalie Stewart. *On tensor products of equivariant commutative operads (draft)*. 2025. URL: https://nataliesstewart.github.io/files/ninfty.html (cit. on p. 41).

[Szc23] Benjamin Szczesny. *On the Additivity of Equivariant Operads*. 2023. URL: https://escholarship.org/uc/item/4r40725w#main (cit. on p. 35).

[Yan23] Lucy Yang. *On normed $\mathbb{E}_\infty$-rings in genuine equivariant C_p-spectra*. 2023. arXiv: 2308.16107 [math.AT]. URL: https://arxiv.org/abs/2308.16107 (cit. on p. 32).