ON DOUBLE-NEGATION SHEAVES IN THE COPRESHEAF TOPOS ON A COFREE COMONOID

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This note fixes a polynomial functor $p \in \mathbf{Poly}$ and studies the copresheaf topos $\mathscr{C}_p := \mathscr{F}_P\text{-}\mathbf{Set}$; we fix $\Omega \in \mathscr{C}_p$ the subobject classifier (given by the copresheaf of sieves on $\mathscr{F}_p^{\mathrm{op}}$).

1. Presheaves

1.1. $\mathscr{C}_p$ as a topos of dynamical systems. A presheaf $F \in \mathscr{F}_p$ consists of the following data (subject to no restrictions):

(1) For each p-tree $i \in \mathscr{F}_p$, a set $F(i)$.
(2) For each morphism $f : i \to j$ in $\mathscr{F}_p$ corresponding with an inclusion of a subtree at height one, a function $F(f) : F(i) \to F(j)$.

This corresponds with a "dynamical system" with "positions with future knowledge" corresponding with p-trees, states in each position corresponding with each set $F(i)$, and "transition functions" between the states in these positions where one moves to a possible "next state" according to the futures predicted by the p-tree structure.[1] I will henceforth refer to a presheaf $F \in \mathscr{F}_p$ as a *p-dynamical system*.

1.2. Subsystems. Note that a morphism of p-dynamical systems $F \to G$ is precisely a map $F(i) \to G(i)$ from the i-states of F to the i-states of G for each $i \in \mathscr{F}_p$, compatible with the transition functions in F and G. A morphism is monic iff each of the constituent morphisms are monic; that is, a subobject corresponds with a subset of the states which is preserved under the transition functions.

As with any topos, the set of subobjects on a p-dynamical system $\mathrm{Sub}(F)$ comes equipped with a Heyting algebra stricture; the join and meet are given by union and intersection, and the implication $A \implies B$ is given by **FILL IN THE RELEVANT CHARACTERIZATION**. In particular, the *negation* $\neg A = (A \implies 0)$ is given by the largest dynamical system contained in the set-theoretic complement of A, and hence $\neg \neq A$ is given by the largest dynamical system whose complement is the same as A.

As with any topos, $\neg\neg : \Omega \to \Omega$ is a Lawvere-Tierney topology on $\mathscr{C}_p$. We will seek to characterize the sheaves with respect to this topology, for which it will be useful to name the following dynamical (sub)systems.

> **Example 1.1:**
> Let $s \in F(i)$ be a state in a dynamical system. Define the system
> $$U_s(j) = \begin{cases} 1 & \exists f : i \to j, \\ 0 & \text{otherwise.} \end{cases}$$
> with transition functions are canonically defined. There is a unique monic $U_s \rightarrowtail F$ sending $U_s(i)$ to s.
> **Lemma 1.2.** *Any dynamical system F is generated by the associated subsystems U_s; that is,*
> $$F = \bigcup_{s \in \coprod_i F(i)} U_s.$$

It is perhaps troublesome that U_s is defined in reference to a particular state of a dynamical system, as the systems themselves depend only on the position that the state resides in. We can simply define U_i for position i by the above description; these are contained in the following example.

[1] This sentence makes no sense.

Example 1.3:
The *all-ones system* 1_p has $1_p(i) = \{i\}$ for all $i \in \mathscr{F}_p$, with transition functions given by the unique endomorphism of 1. Note that there are unique monics $U_i \rightarrowtail V_i \rightarrowtail 1_p$.

Consider mentioning that they generate all subobjects under union–they always do, but this time it's particularly simple.

2. Sheaves

2.1. **Density.** The following lemma is well known:

Lemma 2.1. *In a presheaf topos, a subobject $A \subset C$ is $\neg\neg$-dense iff all nonzero subobjects $0 \subsetneq B \subset C$ intersect A.*

This powers the following (easy) proposition, which shows that dense subsystems correspond intuitively with *attainable win conditions*:

Proposition 2.2. *A subsystem $F \subset G$ is dense iff there is no state $s \in F(i)$ and sequence of morphisms $i = i_0 \xrightarrow{f_0} i_1 \xrightarrow{f_1} i_2 \xrightarrow{f_2} \cdots$ such that $f_n(s) \notin G$ for all $n \in \mathbb{N}$.*

Proof. THIS NEEDS TO BE FIXED, AND IS ACTUALLY NOT TRUE. IT NEEDS TO INCORPORATE ALL FUTURES. SEE THE EXAMPLE OF THE $y^2 + 1$ TREE WHERE THE ROOT DIRECTIONS POINT TOWARDS THE CONSTANT TREE AND ITSELF; YOU CAN NEGLECT THE ENTIRE LEFT BRANCH IN A DENSE SUBSYSTEM. Suppose there is such a state s and sequence I; then, the subobject $U_{s,I} \subset G$ is nonzero and does not intersect F, so it is not dense.

Now suppose that no such sequence exists, and suppose $B \subset G$ is a nonzero subobject containing a state $s \in B(i)$. Then, picking some sequence $I = i_0 \to i_1 \to \cdots$, we have $B \cap F \supset U_{s,I} \cap F \neq 0$, so F is dense. □

The following dense monics will be useful.

Example 2.3:
Let F be a p-dynamical system. Suppose $s \in F(i)$ is a state such that there is no endomorphism $f : i \to i$ sending $F(f)(s) = s$. Then, we may define the *strict futures of s* by

$$\widetilde{U}_s(j) := \{t \in U_s(j) \mid t \neq s\}.$$

It quickly follows from Proposition 2.2 that the monic $U_{f_n(s), I_{\geq n}} \rightarrowtail U_{s,I}$ is dense.

One characterization of the closure of $F \subset G$ is the largest closed subobject of G containing F as a dense subobject.[2] Using this and Proposition 2.2, we have the following lemma.

Lemma 2.4. *Let $F \subset G$ be a p-dynamical subsystem. Then, we may compute the closure of F as* ***CHARACTERIZATION GOES HERE.***

$$\neg\neg F$$

2.2. **Characterizing sheaves when p is linear.** For now I'll cover a special case.

Proposition 2.5. *Suppose $p = Ay + B$ is a linear polynomial. Then, a p-dynamical system F is a $\neg\neq$-sheaf iff every morphism $f : i \to j$ in $\mathscr{F}_p$ induces a bijective transition function $F(f)$.*

We'll heavily use the following lemma.

Lemma 2.6. *Let $f : i \to j$ be a morphism in $\mathscr{F}_p$ for p a linear polynomial.*

- *(i) Suppose $F(f)$ fails to be injective for some $f : i \to j$. Then, there is some morphism $h : i \to k$ and two distinct elements distinct elements $s, t \in F(i)$ such that $F(g)(s) \neq s$ for all non-identity endomorphisms $g : i \to i$ and $F(f)(s) = F(f)(t)$.*
- *(ii) Suppose $F(f)$ fails to be surjective for some $f : j \to i$. Then, there is some $h : k \to i$ and element $s \in F(i) - \operatorname{im} h$ such that $F(g)(s) \neq s$ for all non-identity endomorphisms $g : i \to i$.*

[2] Is this true?

Proof. The lemma is immediate if there are no non-identity endomorphisms of i, so suppose that i has such an endomorphism g. Since p is linear, we have $\mathrm{End}(i) = \{g^n\}_{n \in \mathbb{N}}$.

Suppose contrapositively that all $s \in F(i)$ have some n_s where $F(g)^{n_s}(s) = s$. Then, $s \in \mathrm{im}\, F(g)^{n_s} \subset \mathrm{im}\, F(g)$, so $F(g)$ is surjective. Conversely, if s, t have $F(g)(s) = F(g)(t)$, then we have

$$s = F(g)^{n_s n_t}(s) = F(g)^{n_s n_t}(t) = t$$

so $F(g)$ is bijective, as desired. This implies that $F(g^n)$ is bijective for each n.

For condition (i), note that there is some composition $i \xrightarrow{f} j \to i$ expressing f as a prefix of a bijection; hence f is injective.

Now, consider some $F(f) : F(i) \to F(j)$ failing to be injective, so that $F(g^n)$ fails to be injective for each n, and let $S := \{s \in F(i) \mid g^n(s) \neq s \ \forall n \geq 1\}$. Note that $F(g)(F(i) - S) \subset F(i) - S$ and $F(g)|_{F(i)-S}$ is injective (by the above argument) Hence there must be some $s \in S$ and $t \in F(i)$ satisfying $F(g)(s) = F(g)(t)$, as desired.

Let $F(f) : F(j) \to F(i)$ fail to be surjective, so that $F(g^n)$ fails to be surjective for each n; a composition $i \to j \xrightarrow{f} i$ proves that the subset $S := \{s \in F(i) \mid g^n(s) \neq s \ \forall n \geq 1\}$ is nonempty. Recall that $F(g)(F(i) - S) \subset F(i) - S$, and note that $F(g)|_{F(i)-S}$ surjects onto $F(i) - S$; hence there must be some $s \in F(i) - S$ which is not in the image of g, as desired. $\square$

Proof of Proposition 2.5. First suppose that F has bijective transition functions, and suppose that $A \rightarrowtail E$ is a dense monic and $\tilde{\varphi} : A \to F$ is a morphism of p-dynamical systems. Then, for each $s \in E(i)$, there is a morphism $f : i \to j$ such that $f(s) \in A(j)$; we define an extension $\varphi : E \to F$ by $\varphi(s) := F(f)^{-1}(\tilde{\varphi}(f(s)))$. This is the unique choice of $\varphi(s)$ such that $F(f)(\varphi(s)) = \varphi(E(f)(s))$ for any f, so it is well defined as the unique extension of $\tilde{\varphi}$ to E.

Conversely, suppose first that $F(f)$ fails to be injective for some $f : i \to j$. Then, choose s, t, h as in Lemma 2.6, and note that $U_s = U_t$ as p-dynamical systems; however, they are distinct subsystems of F. Further, note that there are factorizations (where the leftmost monic is dense)

$$\begin{tikzcd}
 & U_s = U_t \arrow[dr, tail, "s"] & \\
\tilde{U}_s \arrow[ur, tail] \arrow[rr, tail] \arrow[dr, tail] & & F \\
 & U_s = U_t \arrow[ur, tail, "t"] &
\end{tikzcd}$$

which prove that the map $\mathrm{Hom}(E, F) \to \mathrm{Hom}(A, F)$ is not injective.

Next, suppose that $F(f)$ fails to be surjective for some $f : j \to i$ and choose some $s \in i$ as in Lemma 2.6. The system U_s occurs as a dense subobject of some system U who includes a state in position h; however, the morphism $U_s \rightarrowtail F$ doesn't extend to U, as desired. $\square$

This is left here for archival purposes; this argument works for linear systems with states in trees that are "non-referencial."

2.3. **Sheaves as ancient systems which remember history.** We say that a p-dynamical system F is *ancient* if all transition functions $F(i) \to F(j)$ are epic, and say that F *remembers history* if $F(i) \to F(j)$ are monic.

Proposition 2.7. *A p-dynamical system F is a $\neg\neg$-separated presheaf iff it remembers history. A p-dynamical system F is a $\neg\neg$-sheaf iff it is ancient and remembers history.*

Proof. We first verify the statement for separated presheaves. We have to verify that, F remembers history iff for all dense monics $A \rightarrowtail E$, the induced map

$$\mathrm{Hom}_{\mathscr{C}_p}(E, F) \to \mathrm{Hom}_{\mathscr{C}_p}(A, F)$$

is monic.

First suppose that F does not remember history, and there are two distinct states $s, t \in F(i)$ and a morphism $f : i \to j$ such that $f(s) = f(t)$. Then, since $U_s = U_t$ as (dense) subobjects of $U_{f(s)} = U_{f(t)}$, we have two distinct extensions

$$\begin{tikzcd}
U_s \arrow[dr, tail] \arrow[drr, tail, bend left] & & \\
& U_{f(s)=f(t)} \arrow[r, tail] & F \\
U_t \arrow[ur, tail] \arrow[urr, bend right] & &
\end{tikzcd}$$

so that F is not separated.

Conversely, suppose that F does remember history, and suppose we have two morphisms $\varphi, \varphi' : E \to F$ which each restrict to A identically. At position j possessing morphism $f : j \to i$, this is represented by commuting parallel morphisms

$$\begin{tikzcd}
E(i) \arrow[r] & F(i) \\
E(j) \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & F(j) \arrow[u, tail]
\end{tikzcd}$$

which must coincide since the morphism $F(j) \rightarrowtail F(i)$ is monic.[3]

For $\neg\neg$-sheaves, it suffices to prove that the morphism $\mathrm{Hom}_{\mathscr{C}_p}(E, F) \to \mathrm{Hom}_{\mathscr{C}_p}(A, F)$ is epic iff F is ancient. This will work by a similar argument; if F is not ancient, we may pick a state $s \in \coprod_i F(i)$ and morphism $f : j \to i$ such that $F(j)$ is nonempty (say, containing $t \in F(j)$) and such that $s \notin \operatorname{im} f$, and consider the embedding $U_s \rightarrowtail F$. Then, U_s is isomorphic as a p-dynamical system to $U_{f(t)} \subset U_t$, but there is no morphism $U_t \to F$ extending the given morphism.

Conversely, if F is ancient, we probably have to assume AoC, and then just choose preimages to extend a morphism $A \to F$ to E. I'll fill this in fully later.

$\square$

[3]This needs to be corrected later; it isn't necessarily the case that any $E(i)$ is totally subsumed by A. Instead, we have to make some argument using colimits, which I don't particularly want to do..