ON DOUBLE-NEGATION SHEAVES IN THE COPRESHEAF TOPOS ON A COFREE COMONOID

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This note fixes a polynomial functor $p \in \mathbf{Poly}$ and studies the copresheaf topos $\mathscr{C}_p := \mathscr{T}_P\text{-}\mathbf{Set}$; we fix $\Omega \in \mathscr{C}_p$ the subobject classifier (given by the copresheaf of sieves on $\mathscr{T}_p^{\mathrm{op}}$).

1. Presheaves

1.1. **$\mathscr{C}_p$ as a topos of dynamical systems.** A presheaf $F \in \mathscr{T}_p$ consists of the following data (subject to no restrictions):

(1) For each p-tree $i \in \mathscr{T}_p$, a set $F(i)$.
(2) For each morphism $f : i \to j$ in $\mathscr{T}_p$ corresponding with an inclusion of a subtree at height one, a function $F(f) : F(i) \to F(j)$.

This corresponds with a "dynamical system" with "positions with future knowledge" corresponding with p-trees, states in each position corresponding with each set $F(i)$, and "transition functions" between the states in these positions where one moves to a possible "next state" according to the futures predicted by the p-tree structure.[1] I will henceforth refer to a presheaf $F \in \mathscr{T}_p$ as a *p-dynamical system*.

1.2. **Subsystems.** Note that a morphism of p-dynamical systems $F \to G$ is precisely a map $F(i) \to G(i)$ from the i-states of F to the i-states of G for each $i \in \mathscr{T}_p$, compatible with the transition functions in F and G. A morphism is monic iff each of the constituent morphisms are monic; that is, a subobject corresponds with a subset of the states which is preserved under the transition functions.

As with any topos, the set of subobjects on a p-dynamical system $\mathrm{Sub}(F)$ comes equipped with a Heyting algebra stricture; the join and meet are given by union and intersection, and the implication $A \implies B$ is given by **FILL IN THE RELEVANT CHARACTERIZATION**. In particular, the *negation* $\neg A = (A \implies 0)$ is given by the largest dynamical system contained in the set-theoretic complement of A, and hence $\neg \neq A$ is given by the largest dynamical system whose complement is the same as A.

As with any topos, $\neg\neg : \Omega \to \Omega$ is a Lawvere-Tierney topology on $\mathscr{C}_p$. We will seek to characterize the sheaves with respect to this topology, for which it will be useful to name the following dynamical (sub)systems.

Example 1.1:
Let $s \in F(i)$ be a state in a dynamical system, and let $i = i_0 \to i_1 \to \cdots$ be an $\mathbb{N}$-indexed sequence of composable morphisms. Define $I := (i_n)_{n \in \mathbb{N}}$.[a] Then, we may define the dynamical system $U_{s,I}$, called the *future of s along I* by

$$U_{s,I}(j) = \begin{cases} 1 & j \in I, \\ 0 & \text{otherwise.} \end{cases}$$

The transition functions are canonically defined, and there is a unique monic $U_{s,I} \rightarrowtail F$ sending $U_{s,I}(i)$ to S. We may combine these to yield a subsystem of *futures of S*:

$$U_s(j) := \bigcup_{I \in (\mathscr{T}_p)^{\mathbb{N}} \text{ s.t. } I_0 = i} U_{s,I} = \begin{cases} 1 & \exists \text{ morphism } i \to j, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.2. *Any dynamical system F is generated by the associated subsystems U_s; that is,*

$$F = \bigcup_{s \in \coprod_i F(i)} U_s.$$

[1] This sentence makes no sense.

Similarly, for a $\mathbb{Z}$-indexed sequence I of composable morphisms with $I_0 = i$ and such that each morphism having codomain i has image containing s, there is a system $V_{s,I}$ called the *eternity of s along I*, defined by

$$V_s(j) = \begin{cases} 1 & j \in I \\ 0 & \text{otherwise.} \end{cases}$$

and we may combine these into the *eternities of s*:

$$V_s(j) := \bigcup_I V_{s,I} = \begin{cases} 1 & \exists \text{ morphism } i \to j \text{ or } \exists \text{ morphism } f : j \to i \quad \text{s.t.} \quad s \in \operatorname{im} f, \\ 0 & \text{otherwise.} \end{cases}$$

There

[a]The notation here is confusing, and should be fixed.

It is perhaps troublesome that U_s is defined in reference to a particular state of a dynamical system, as the systems themselves depend only on the position that the state resides in. We can simply define U_i for position i by the above description; these are contained in the following example.

Example 1.3:
The *all-ones system* 1_p has $1_p(i) = \{i\}$ for all $i \in \mathscr{F}_p$, with transition functions given by the unique endomorphism of 1. Note that there are unique monics $U_i \rightarrowtail V_i \rightarrowtail 1_p$.

Consider mentioning that they generate all subobjects under union–they always do, but this time it's particularly simple.

2. Sheaves

2.1. Density.

The following lemma is well known:

Lemma 2.1. *In a presheaf topos, a subobject $A \subset C$ is $\neg\neg$-dense iff all nonzero subobjects $0 \subsetneq B \subset C$ intersect A.*

This powers the following (easy) proposition, which shows that dense subsystems correspond intuitively with *attainable win conditions*:

Proposition 2.2. *A subsystem $F \subset G$ is dense iff there is no state $s \in F(i)$ and sequence of morphisms $i = i_0 \xrightarrow{f_0} i_1 \xrightarrow{f_1} i_2 \xrightarrow{f_2} \cdots$ such that $f_n(s) \notin G$ for all $n \in \mathbb{N}$.*

Proof. Suppose there is such a state s and sequence I; then, the subobject $U_{s,I} \subset G$ is nonzero and does not intersect F, so it is not dense.

Now suppose that no such sequence exists, and suppose $B \subset G$ is a nonzero subobject containing a state $s \in B(i)$. Then, picking some sequence $I = i_0 \to i_1 \to \cdots$, we have $B \cap F \supset U_{s,I} \cap F \neq 0$, so F is dense. □

The following dense monics will be useful.

Example 2.3:

Let F be a p-dynamical system. Let I be the sequence $i_0 \xrightarrow{f_1} i_1 \xrightarrow{f_2} \cdots$, and let $I_{\geq n}$ be the suffix $i_n \to i_{n+1} \to \cdots$. Then, there is a factorization

$$U_{f_n(s), I_{\geq n}} \rightarrowtail U_{s,I} \rightarrowtail F.$$

It quickly follows from Proposition 2.2 that the monic $U_{f_n(s), I_{\geq n}} \rightarrowtail U_{s,I}$ is dense.

Similarly, for I a $\mathbb{Z}$-diagram and $I_{\geq 0}$ its truncation to $\mathbb{N}$, the monic $U_{s,I_{\geq 0}} \rightarrowtail V_{s,I}$ is monic.[a]

[a]The latter subsumes the former.

One characterization of the closure of $F \subset G$ is the largest closed subobject of G containing F as a dense subobject.[2] Using this and Proposition 2.2, we have the following lemma.

[2]Is this true?

Lemma 2.4. *Let $F \subset G$ be a p-dynamical subsystem. Then, we may compute the closure of F as* ***CHARACTERIZATION GOES HERE.***

$$\neg\neg F$$

2.2. **Sheaves as ancient systems which remember history.** We say that a p-dynamical system F is *ancient* if all transition functions $F(i) \to F(j)$ are epic, and say that F *remembers history* if $F(i) \to F(j)$ are monic.

Proposition 2.5. *A p-dynamical system F is a $\neg\neg$-separated presheaf iff it remembers history. A p-dynamical system F is a $\neg\neg$-sheaf iff it is ancient and remembers history.*

Proof. We first verify the statement for separated presheaves. We have to verify that, F remembers history iff for all dense monics $A \rightarrowtail E$, the induced map

$$\mathrm{Hom}_{\mathscr{C}_p}(E, F) \to \mathrm{Hom}_{\mathscr{C}_p}(A, F)$$

is monic.

First suppose that F does not remember history, and there are two distinct states $s, t \in F(i)$ and a morphism $f : i \to j$ such that $f(s) = f(t)$. Then, since $U_s = U_t$ as (dense) subobjects of $U_{f(s)} = U_{f(t)}$, we have two distinct extensions

$$\begin{tikzcd}
U_s \arrow[dr, tail] \arrow[drr, tail, bend left] & & \\
& U_{f(s)=f(t)} \arrow[r, tail] & F \\
U_t \arrow[ur, tail] \arrow[urr, bend right] & &
\end{tikzcd}$$

so that F is not separated.

Conversely, suppose that F does remember history, and suppose we have two morphisms $\varphi, \varphi' : E \to F$ which each restrict to A identically. At position j possessing morphism $f : j \to i$, this is represented by commuting parallel morphisms

$$\begin{tikzcd}
E(i) \arrow[r] & F(i) \\
E(j) \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & F(j) \arrow[u, tail]
\end{tikzcd}$$

which must coincide since the morphism $F(j) \rightarrowtail F(i)$ is monic.[3]

For $\neg\neg$-sheaves, it suffices to prove that the morphism $\mathrm{Hom}_{\mathscr{C}_p}(E, F) \to \mathrm{Hom}_{\mathscr{C}_p}(A, F)$ is epic iff F is ancient. This will work by a similar argument; if F is not ancient, we may pick a state $s \in \coprod_i F(i)$ and morphism $f : j \to i$ such that $F(j)$ is nonempty (say, containing $t \in F(j)$) and such that $s \notin \operatorname{im} f$, and consider the embedding $U_s \rightarrowtail F$. Then, U_s is isomorphic as a p-dynamical system to $U_{f(t)} \subset U_t$, but there is no morphism $U_t \to F$ extending the given morphism.

Conversely, if F is ancient, we probably have to assume AoC, and then just choose preimages to extend a morphism $A \to F$ to E.

$\square$

[3] This needs to be corrected later; it isn't necessarily the case that any $E(i)$ is totally subsumed by A. Instead, we have to make some argument using colimits, which I don't particularly want to do..