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Business Analytics & Machine Learning
Tutorial sheet 3: Logistic Regression

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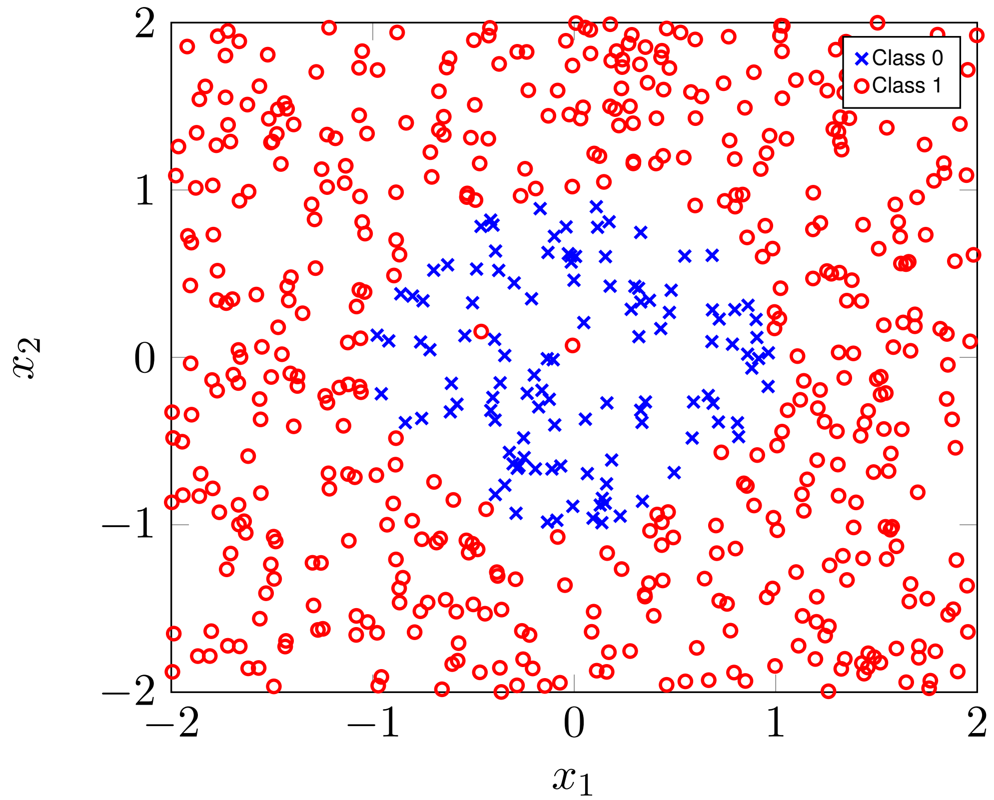
Exercise T3.1 *Logistic regression for a 2D classification problem*

You are given the data set in *2d-classification-data.csv* which is also visualized below. The data consists of two-dimensional points specified by coordinates x_1 and x_2, belonging to one of two classes: 0 or 1. For convenience, the data set also includes square coordinates, $z_1 = x_1^2$ and $z_2 = x_2^2$.



In this exercise, we are going to find a predicting model which can classify new data points.
Consider the logistic regression model:

$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

$$\iff$$

$$p(y=1|x_1,x_2) = \sigma(\beta_0 + \beta_1 x_1 + \beta_2 x_2) = \frac{e^{\beta_0+\beta_1 x_1+\beta_2 x_2}}{1+e^{\beta_0+\beta_1 x_1+\beta_2 x_2}}$$

a) For which values $x = (x_1, x_2)$ does the logistic regression model output $p(y=1|x) = p(y=0|x)$? Derive a functional description $x_2 = f(x_1)$ which describes the corresponding decision boundary.

b) The model in a) is obviously not appropriate for the data at hand. Looking at the plot, convince yourself that aside from some very random noise, class 0 data points are those that lie within some elliptic disc[1]. Come up with an appropriate logistic regression model, using the fact that the set of points of a standard ellipse satisfies $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq r^2$.
Hint: The simplest such model will have three coefficients.

c) Using Python and scikit-learn (or statsmodels), derive the optimal parameters for the logistic regression model. If you like, you may use the provided template notebook for this purpose.

d) You are now told that the actual data is class 1 with certainty outside the circular disc of radius 1, and class 1 with probability 2% if it is inside it. Draw the decision boundary of the optimal model and observe the classification of points about it. Can you explain why the model might perform worse (with respect to the number of misclassified samples) than the actual model you could have even come up with by yourself in b)?

Exercise T3.2 *Maximum likelihood estimation*

You are given the following dataset with the dependent binary variable y and the independent variable x.

x	y
1	0
2	0
2.5	1
4	1

Based on these data points we want to create a logistic regression model with the logistic function σ (or more broadly a sigmoid function):

$$\text{Pr}[Y|X] = p(x) = \sigma(\beta_0 + \beta_1 x) = \frac{e^{\beta_0+\beta_1 x}}{1+e^{\beta_0+\beta_1 x}}.$$

To estimate the logistic regression coefficients, we will use *maximum likelihood estimation*.
To simplify notation, let $p_i = p(x_i) = \sigma(z_i)$ and $z_i = \beta_0 + \beta_1 x_i$.

a) Determine the likelihood function $L(\beta)$.
Hint: To keep everything simple, it is sufficient to formulate L in terms of p_i (which includes the dependency on β).

b) Find the gradient for the log of the likelihood function $LL(\beta)$. The gradient is defined as:

$$\nabla LL(\beta) = \begin{pmatrix} \frac{\partial LL(\beta)}{\partial \beta_0} \\ \frac{\partial LL(\beta)}{\partial \beta_1} \end{pmatrix}.$$

Hint: Use the chain rule: $\frac{\partial LL}{\partial \beta_j} = \sum_{i=1}^{n} \frac{\partial LL}{\partial p_i} \frac{\partial p_i}{\partial z_i} \frac{\partial z_i}{\partial \beta_j}$ with $z_i = \beta_0 + \beta_1 x_i$. You may use the following derivative of the logistic function σ without proof: $\sigma'(z_i) = \sigma(z_i)(1 - \sigma(z_i))$.

c) Given the initial values $\beta^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and a learning rate $\alpha = 0.2$, calculate the coefficients after the first iteration of gradient ascent.

[1] You might even deduce that it's a circle!

d) If a linear regression model was fitted to a logistic regression dataset, what could be the problems w.r.t. the Gauss Markov properties?

Exercise T3.3 *Poisson regression*

You are provided the following numbers from the result of a *Poisson regression model*.

Variable	Estimate	Std. Error
Intercept	1.5499	0.0503
Age	-0.0047	0.0009

a) According to the model above, what *qualitative* effect does a change in the independent variable age (+1) have on the dependent variable dv.

b) According to the model above, what *quantitative* effect (on the incidence rate and log-incidence rate) does a change in the independent variable age (+1) have on the dependent variable dv.