University of Siena

Information Engineering and Mathematical Sciences Department

Automata and Queueing Systems

Project of Discrete Event Systems

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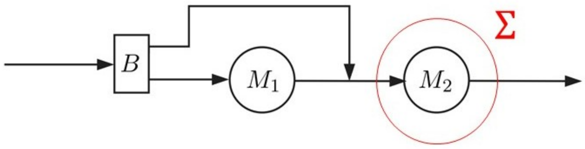
System description

Consider the production system shown in the figure, composed of a one-place buffer B followed by the series interconnection of two machines M_1 and M_2.



A fraction $q = 2/5$ of the arriving parts needs to be processed in M_2 only, while the other parts need to be processed in both machines. Parts arriving when B is full, are rejected. If M_1 terminates a job, and M_2 is working, M_1 keeps the part (and therefore it remains unavailable for another job) until M_2 terminates the ongoing job. Parts arriving from M_1 have priority to M_2 over those waiting in B.

Assume that raw parts arrive at the system as generated by a Poisson process, and processing times in M_1 and M_2 have exponential distributions. In the following, let t_δ denote the minimum time such that all the state probabilities settle around their limit values $\pm\delta$, with $\delta = 0.001$. We say that the system is practically at steady state for $t \geq t_\delta$.

Question 1

Model the logic of the system using a stochastic state automaton.

The stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, p_{x0}, F)$ is defined by:

- stochastic state automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, p_{x0})$;
- stochastic clock structure $F = \{F_a, F_{d1}, F_{d2}\}$ of the lifetimes of events where:
  - $F_a = 1 - e^{-\lambda t}$ with $t \geq 0$, rate $\lambda > 0$ [arrivals/min];
  - $F_{d1} = 1 - e^{-\mu_1 t}$ with $t \geq 0$, rate $\mu_1 > 0$ [services/min];
  - $F_{d2} = 1 - e^{-\mu_2 t}$ with $t \geq 0$, rate $\mu_2 > 0$ [services/min].

  Since all the distributions F_e are exponential we could say that F is a Poisson clock structure.

The stochastic state automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, p_{x0})$ is defined by:

- set of events $\mathcal{E} = \{a, d_1, d_2\}$ with events:
  - a - arrival of a part;
  - d_1 - termination of a job in M_1;
  - d_2 - termination of a job in M_2;
- set of states $\mathcal{X} = \{x_1, x_2, ..., x_{11}\}$; $x_i = \begin{bmatrix} x_B \\ x_{M1} \\ x_{M2} \end{bmatrix}$ where:
  - $x_B \in \{$0 (empty), 1 (one part waiting for M_1), 2 (one part waiting for M_2)$\}$;
  - $x_{M1} \in \{$0 (idle), 1 (busy), B (blocked)$\}$;
  - $x_{M2} \in \{$0 (idle), 1 (busy)$\}$.

Γ and p are described by the following graph with $q = 2/5$:

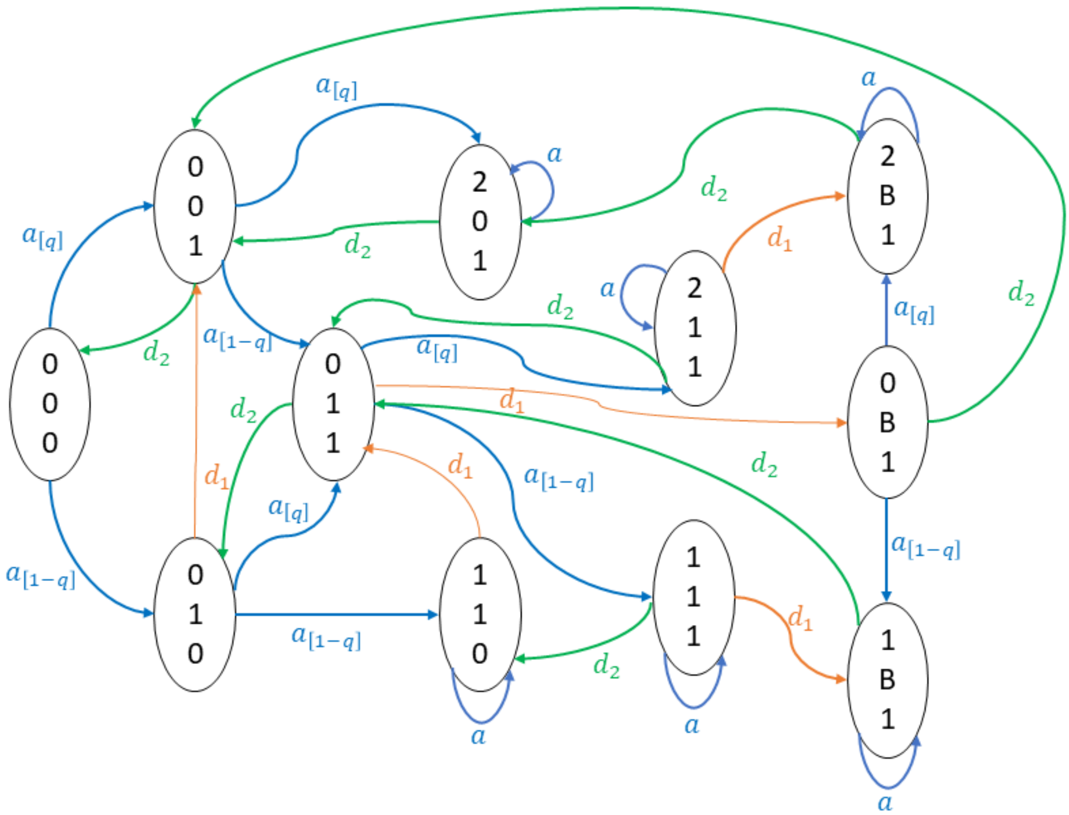


Figure 1: The model of stochastic state automaton.

Our production system didn't specify an initial state, so we didn't draw it in Fig. 1, but for simplicity we chose $p_{x0}([0\ 0\ 0]') = 1$ as initial condition for the π_0 vector in question 2.

Question 2

For a fixed $\varepsilon > 0$, determine the parameters of the event lifetime distributions so that all the limit state probabilities (computed analytically) belong to the interval $[(1-\varepsilon)\frac{1}{n}, (1+\varepsilon)\frac{1}{n}]$, where n is the number of states, and $t_\delta \in [10, 15]$ min. By trial and error, make ε as small as you can, and describe the difficulties encountered in making it smaller.

Firstly, we transformed the stochastic timed automaton into a continuous-time homogeneous Markov chain (Fig. 2) which is stochastically equivalent to it and is specified by:

- the state set $\mathcal{X}$;
- the transition rate matrix Q;
- the vector of initial state probabilities π_0 (chosen randomly).

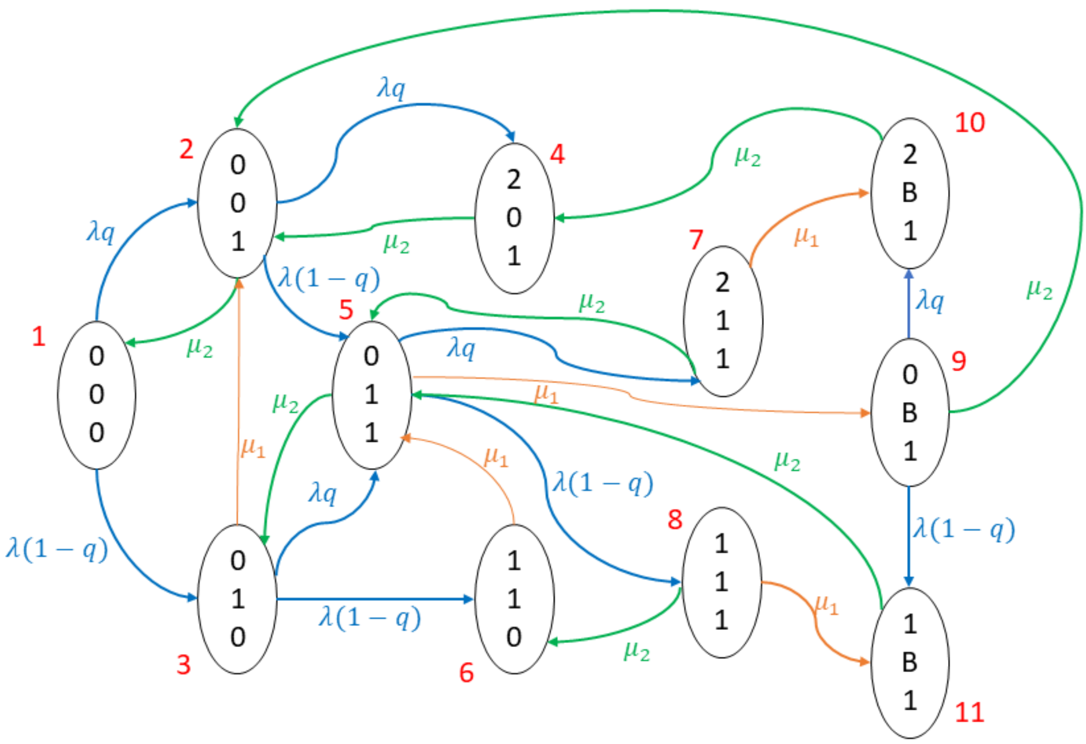


Figure 2: Equivalent CTHMC model.

We could compute the transition rate matrix Q:

$$Q = \begin{bmatrix} -\lambda & q\lambda & (1-q)\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_2 & -(\lambda+\mu_2) & 0 & q\lambda & (1-q)\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & -(\lambda+\mu_1) & 0 & q\lambda & (1-q)\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & -\mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & -(\lambda+\mu_1+\mu_2) & 0 & q\lambda & (1-q)\lambda & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_1 & -\mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & 0 & -(\mu_1+\mu_2) & 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & -(\mu_1+\mu_2) & 0 & 0 & \mu_1 \\ 0 & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_2) & q\lambda & (1-q)\lambda \\ 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 & 0 & 0 & -\mu_2 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 & 0 & 0 & -\mu_2 \end{bmatrix}$$

which depends on 3 distinct parameters:

- λ: the rate of arrivals per minute;
- μ_1: the rate of the termination of the machine M_1 per minute;
- μ_2: the rate of the termination of the machine M_2 per minute.

To determine these three unknown parameters we had to respect the constraints of time t_δ and the range for the limit state probabilities vector π depending on ε, which can be computed solving the following linear system:

$$\begin{cases} \pi Q = 0 \\ \sum \pi_i = 1 \end{cases}$$

To find the suitable values of the unknowns, we have followed two steps:

1. choosing fixed arbitrary values for λ, μ_1 and μ_2 so that ε is minimized to the lowest possible value;
2. for the minimized ε we continued to adjust the parameters λ, μ_1 and μ_2 in such a way, that the limit state probabilities π_i for $i \in 1, 2, ..., 11$ (where i represents the number of states) belonged to the interval $[(1-\varepsilon)\frac{1}{n}, (1+\varepsilon)\frac{1}{n}]$.

To tackle this problem we have also taken into account the topology of the graph (i.e. if state number 1 or 2 were having high limit state probabilities, one solution would have been to diminish λ and so to have less arrivals). The difficult part of this task was that in some cases while adjusting one of the three parameters in order to decrease the highest π_i the lowest one was decreasing too to such extend, that it did not fit in the interval boundaries anymore.

To finalize our decision for the parameters, we have also tested (Fig. 3) whether the system reached the steady state at $t_\delta \in [10, 15]$ min with the equation:

$$\pi(t) = \pi(0)e^{Qt}$$

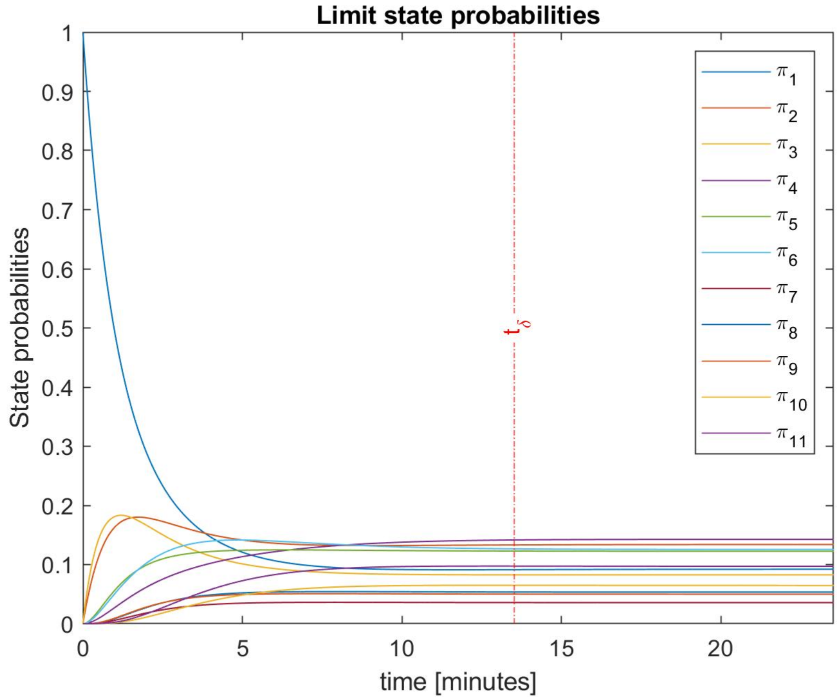


Figure 3: Limit state probabilities: the system reached the steady state at time $t_\delta = 13.53$ min, for $\varepsilon = 0.61$, $\lambda = 0.8$ [arrivals/min], $\mu_1 = 0.55$ [services/min] and $\mu_2 = 0.55$ [services/min].

Question 3

Estimate the limit state probabilities using simulations over a time horizon such that the steady state is reached. Show the trend of the variance of the estimates versus the number of samples using tables and/or figures, discussing them in the light of what you know from the theory.

To estimate $\hat{\pi}_i = \hat{P}(X = i)$ values for $\forall i \in X$, we can apply the law of large numbers by using the properties of the estimator:

$$\mathbf{1}^{(i)}_{\{X=i\}} = \begin{cases} 1 \text{ if } (X = i) \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

which is a random variable such that:

- $E[\mathbf{1}_{\{X=i\}}] = P(X = i)$
- $Var[\mathbf{1}_{\{X=i\}}] = P(X = i) * [1 - P(X = i)] \leq 0.25$

Since $E[\mathbf{1}_{\{X=i\}}] = P(X = i)$, an estimate of $\hat{E}[\mathbf{1}_{\{X=i\}}]$ is an estimate of $\hat{P}(X = i)$.

For the central limit theorem we know, that $\hat{P}(X = i) \sim N(\mu, \frac{\sigma^2}{N})$ where:

- $\mu = E[\mathbf{1}_{\{X=i\}}]$
- $\sigma^2 = Var[\mathbf{1}_{\{X=i\}}]$

for N sufficiently large.

Therefore, we can expect the mean value $E[\mathbf{1}_{\{X=i\}}]$ to be close to the real μ, and to be almost invariant to changes of N, while the variance we expect to decrease proportionally as the number N increases.

The experimental data shown below confirm the theory:

N	$E[\mathbf{1}^{(1)}]$	$E[\mathbf{1}^{(2)}]$	$E[\mathbf{1}^{(3)}]$	$E[\mathbf{1}^{(4)}]$	$E[\mathbf{1}^{(5)}]$	$E[\mathbf{1}^{(6)}]$	$E[\mathbf{1}^{(7)}]$	$E[\mathbf{1}^{(8)}]$	$E[\mathbf{1}^{(9)}]$	$E[\mathbf{1}^{(10)}]$	$E[\mathbf{1}^{(11)}]$
10	0.095	0.133	0.084	0.139	0.118	0.132	0.035	0.052	0.050	0.059	0.100
100	0.091	0.132	0.083	0.140	0.123	0.127	0.035	0.052	0.049	0.064	0.098
1000	0.091	0.133	0.082	0.141	0.123	0.126	0.035	0.053	0.050	0.065	0.097
10000	0.091	0.133	0.082	0.141	0.122	0.126	0.035	0.053	0.050	0.064	0.097

Table 1: Mean of the estimator $\mathbf{1}^{(i)}$ according to N

N	$V[\mathbf{1}^{(1)}]$	$V[\mathbf{1}^{(2)}]$	$V[\mathbf{1}^{(3)}]$	$V[\mathbf{1}^{(4)}]$	$V[\mathbf{1}^{(5)}]$	$V[\mathbf{1}^{(6)}]$	$V[\mathbf{1}^{(7)}]$	$V[\mathbf{1}^{(8)}]$	$V[\mathbf{1}^{(9)}]$	$V[\mathbf{1}^{(10)}]$	$V[\mathbf{1}^{(11)}]$
10	0.0088	0.0113	0.0074	0.0121	0.0101	0.0110	0.0035	0.0051	0.0047	0.0054	0.0090
100	0.0008	0.0012	0.0009	0.0012	0.0011	0.0011	0.0004	0.0005	0.0005	0.0006	0.0010
1000*	0.0821	0.1161	0.0781	0.1157	0.1105	0.1104	0.0359	0.0474	0.0533	0.0625	0.0897
10000*	0.0085	0.0116	0.0071	0.0114	0.0114	0.0109	0.0036	0.0046	0.0047	0.0060	0.0091

Table 2: Variance of the estimator $\mathbf{1}^{(i)}$ according to N; $* = 1.0e-03$

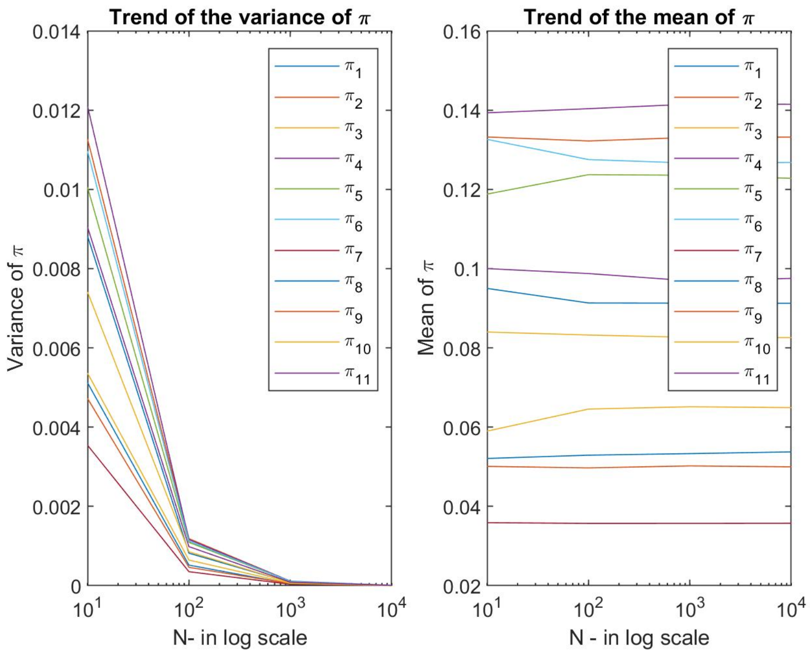


Figure 4: Trend of the variance (left) and the mean (right) of $\mathbf{1}^{(i)}_{\{X=i\}}$ for different N observations and $M = 1000$ estimations.

Question 4

Estimate the probability that at least five arriving parts are rejected over the interval $[t_\delta; 2.5t_\delta]$.

Let event $A = \{\text{having at least five rejected arrivals over the time } [t_\delta; 2.5t_\delta]\}$.

Then, an estimate of $\hat{P}(A)$ might be computed as $\frac{N_A}{N}$ where N_A is the number of observations of the event A over all the N observations.

The algorithm is as followed:

1. We assign to the time t_δ and $2.5t_\delta$ a corresponding index (a, b)
2. We run a for loop from a to $b - 1$ where:
   (a) if we have an event a (arrival) and the state is in blocking state we increase the number rejections (rej) by 1.
   (b) if the number of rejections is equal to 5 we break the loop and we increase the counter for rejections (counter$_{rej}$) by 1.
   (c) we repeat for each N observation and M estimation.
3. We calculate the $P(A)$ as the mean of the counter.

For $M = 1000$ estimations of $N = 100$ observations is the estimated $\hat{P}(A) = 0.81$.

Question 5

Estimate λ_{eff} and μ_{eff} at steady state using simulations, verifying the condition $\lambda_{eff} = \mu_{eff}$ with an error not exceeding 0.001.

The CTHMC is:

- irreducible - every state in the system is reachable from any other state from the system;
- with all positive recurrent states as $\pi_i > 0, \forall i \in X$.

Hence, it is ergodic and the limit state probabilities π_i can be represented as fractions of time spent in each state (as steady state). For that reason, we could compute λ_{eff} and μ_{eff}:

- analytically;
- using simulations.

For computing it **analytically** we defined:

- $\lambda_{eff} = \lambda(\pi_1 + \pi_2 + \pi_3 + \pi_5 + \pi_9) \simeq 0.3838$ [arrivals/minutes]
- $\mu_{eff} = \mu(\pi_2 + \pi_4 + \pi_5 + \pi_7 + \pi_8 + \pi_9 + \pi_{10} + \pi_{11}) \simeq 0.3846$ [services/minutes]

where $\lambda\pi_i$ represents the average number of accepted arrivals over the fraction of time π_i and $\mu\pi_i$ represents the average number of processed parts leaving the system over the fractions of time π_i. The condition $\lambda_{eff} = \mu_{eff}$ was verified with an error of 0.000808.

While **using simulations**, to get λ_{eff} we counted how many arrivals were accepted from t_δ to the end of each simulation, dividing it by the time length and then we took the mean, obtaining the result: $\lambda_{eff} \simeq 0.38413$ [arrivals/minutes]. The same was done for μ_{eff}, counting instead how many terminations of M_2 occurred, obtaining $\mu_{eff} \simeq 0.0.38414$. The condition $\lambda_{eff} = \mu_{eff}$ was therefore verified with an error of 0.000014.

In these computations we simply needed a large time T after the steady state, therefore we used $M = 1$ estimation and $N = 1$ observation with $k_{max} = 1000000$ events.

Question 6

Estimate $E[S_\Sigma], E[X_\Sigma]$ and λ_Σ at steady state using simulations for the subsystem Σ formed by M_2, verifying the Little's law with an error not exceeding 0.01.

To verify the Little's law we need to prove the following relationship:

$$E[X_\Sigma] = \lambda_\Sigma * E[S_\Sigma]$$

We have solved this problem, firstly, by comparing the quantities of interest:

- $E[S_\Sigma]$ - the average system time inside of M_2;
- $E[X_\Sigma]$ - the average number of parts inside the system;
- λ_Σ - the average rate of admitted parts in the system;

computed:

- analytically;
- using simulations.

We computed the 3 values **analytically** using the following formulas:

- $E[X_\Sigma] = 0 * (\pi_1 + \pi_3 + \pi_6) + 1 * (\pi_2 + \pi_4 + \pi_5 + \pi_7 + \pi_8 + \pi_9 + \pi_{10} + \pi_{11}) = 0.6997$
- $\lambda_\Sigma = q\lambda(\pi_1 + \pi_3) + \mu_1(\pi_3 + \pi_6) + \mu_2(\pi_4 + \pi_7 + \pi_9 + \pi_{10} + \pi_{11}) = 0.3848$, which could be also represented by λ_{eff}
- $E[S_\Sigma] = \frac{1}{\mu_2} = 1.8182$, where $\frac{1}{\mu_2}$ represents the service time of machine M_2

To prove the Little's Law, we compute the average system time as:

$$E[S_\Sigma] = \frac{E[X_\Sigma]_{est}}{\lambda_{\Sigma_{est}}} = 1.8182[min]$$

Then, **using simulations**:

- The average number of parts inside the system $E[X_\Sigma]_{est}$:
  - Starting from t=t_δ, we counted how many times a part was inside M_2, multiplying each part by the time it was in;

    - At the end of each observation we divided this 'weighted counter' by the total time T (from t_δ to the end of the observation time) and finally we computed the mean value $E[X_\Sigma]_{est} = 0.6971$.

- The average rate of admitted parts in the system $\lambda_{\Sigma_{est}}$:
    - Under the same previous condition, we checked how many arrivals occurred in M_2 (taking into account that they could come from B as well as from M_1), imposing in the code precise state's transitions for being sure that the arrival we were interested in was accepted.
    - Then we divided the counter of the arrivals of each observation by the total time and computed the mean value $\lambda_{\Sigma_{est}} = 0.3829$ [arrivals/min].

- the average system time $E[S_\Sigma]_{est}$ inside of M_2:
    - For each arrival in M_2, we took the time from when the arrival has occurred until the occurrence of a d_2 event and we stored this system time in a vector;
    - In the end we computed the mean value $E[S_\Sigma]_{est} = 1.8204$ [min].

To prove the Little's Law, we compute again the average system time as in the analytical part:

$$E[S_\Sigma] = \frac{E[X_\Sigma]_{est}}{\lambda_{\Sigma_{est}}} = 1.8204[min]$$

and then we compare it with the estimated one to check if the difference didn't exceed our limit condition, i.e. 0.01, indeed the difference is 0.0092.

For this result, actually, we didn't need a large number of simulations, but simply a long one - for example with $M = 1$ estimation of $N = 1$ observation and $k_{max} = 1000000$ events it is possible to get the suiting result. This also proves that our process is ergodic.

Question 7

Finally, consider the file dati-gruppo-07, containing measurements of the lifetimes of the events. Verify whether the system admits steady state, generating the lifetimes of the events according to the empirical distributions estimated with measured data.

From the above mentioned file we extracted the empirical cdf of the given clock structure $V = \{V_a, V_{d1}, V_{d2}\}$ of the lifetimes of events:

- a - arrival of a part;
- d_1 - termination of a job in M_1;
- d_2 - termination of a job in M_2.

Then, we used these empirical cdf to generate independent random variables of the lifetimes of the events needed to compute the limit state probabilities π_i assessing whether they admit the steady state or not (Fig. 5). This computation was made for each iteration over N by applying the inverse method.

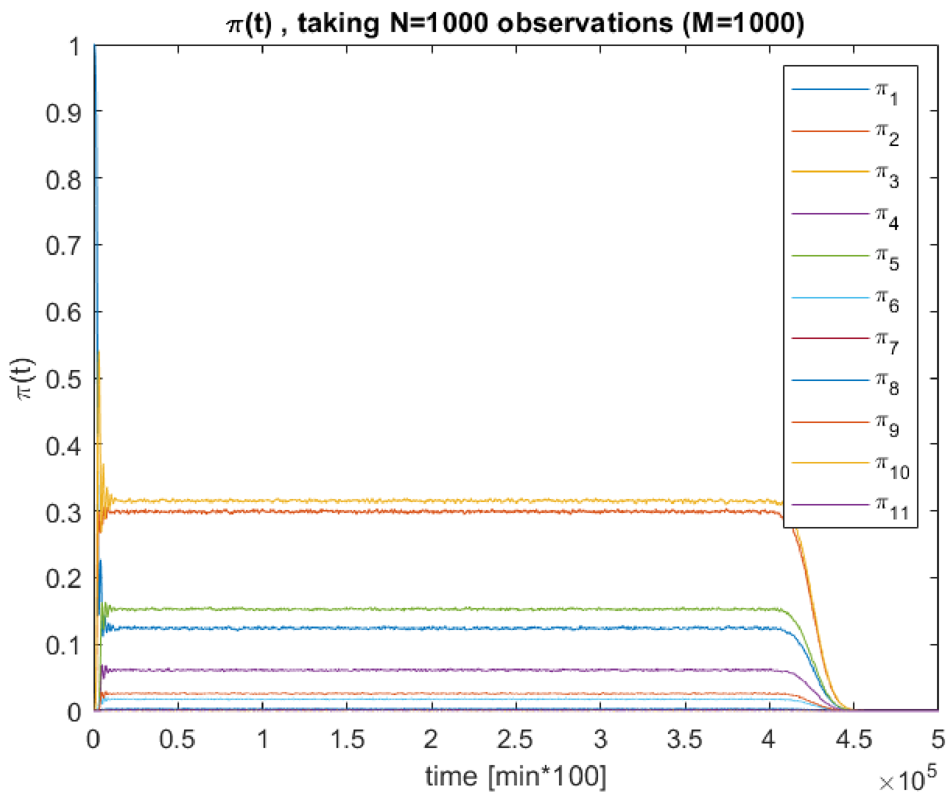


Figure 5: Limit state probabilities with generated lifetimes of events from the measured data: The steady state is reached.