Problem Set 6

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Exercise 9.1

An unconstrained linear function can be written as $f(\mathbf{x}) = \mathbf{b}^T\mathbf{x}+c$. In the case where $\mathbf{b} = \mathbf{0}$, the function is constant $Df(\mathbf{x}) = 0 \ \forall \ \mathbf{x}$, and therefore the minimum is equal to c. In the case where $\mathbf{b} \neq \mathbf{0}$, $Df(\mathbf{x}) = \mathbf{b}^T \ \forall \ \mathbf{x}$, and therefore there is no minimum.

Exercise 9.2

$$\begin{aligned}
\|A\mathbf{x} - \mathbf{b}\|_2 &= (A\mathbf{x} - \mathbf{b})^T(A\mathbf{x} - \mathbf{b}) \\
&= \mathbf{x}^T A^T A\mathbf{x} - \mathbf{x}^T A^T \mathbf{b} - \mathbf{b}^T A\mathbf{x} + \mathbf{b}^T\mathbf{b} \\
&= \mathbf{x}^T A^T A\mathbf{x} - 2\mathbf{b}^T A\mathbf{x} + \mathbf{b}^T\mathbf{b} \qquad \text{as } \mathbf{b}^T A\mathbf{x} = \mathbf{x}^T A^T \mathbf{b} \text{ is scalar}
\end{aligned}$$

As $\mathbf{b}^T\mathbf{b}$ is a constant, this term can be dropped. As A is symmetric and positive definite, we can write

$$2A^T A\mathbf{x} - 2\mathbf{b}^T A\mathbf{x}$$

Setting this equal to 0 and transposing, we find $A^T A = A^T\mathbf{b}$.

Exercises 9.5 - 9.9

See Jupyter notebook.

Exercise 9.10

$$\begin{aligned}
f'(\mathbf{x}) &= \mathbf{x}^T Q - \mathbf{b}^T \\
f''(\mathbf{x}) &= Q^T
\end{aligned}$$

Setting $f'(\mathbf{x}) = \mathbf{0}$, we find $\mathbf{x}^* = Q^{-1}\mathbf{b}$.

Using Newton's method, given $\mathbf{x}_0$

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{x}_0 - (Q^T)^{-1}(\mathbf{x}_0^T Q - \mathbf{b}^T)^T \\
&= \mathbf{x}_0 - Q^{-1}Q\mathbf{x}_0 - Q^{-1}\mathbf{b} \qquad \text{as } Q^T = Q \\
&= Q^{-1}\mathbf{b}
\end{aligned}$$

Exercise 9.12

Given an eigenvector $\mathbf{x}_i$ of A, where $A\mathbf{x}_i = \lambda_i \mathbf{x}$, we have

$$\begin{aligned}
B\mathbf{x}_i &= (A + \mu I)\mathbf{x}_i \\
&= A\mathbf{x}_i + \mu\mathbf{x}_i \\
&= \lambda_i\mathbf{x}_i + \mu\mathbf{x}_i \\
&= (\lambda_i + \mu)\mathbf{x}_i
\end{aligned}$$

Therefore A and B have the same eigenvectors, and B has eigenvalues of the form $\lambda_i + \mu$.