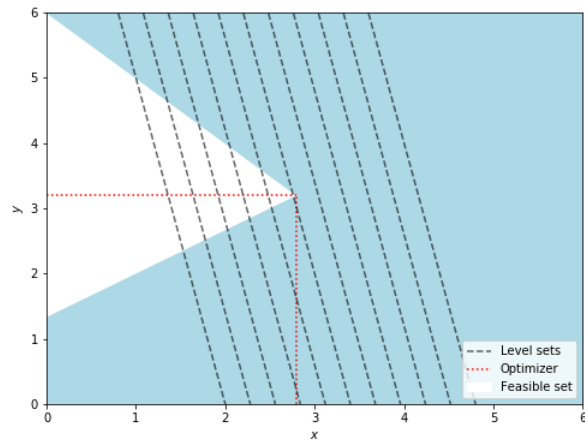


Problem Set 5

Natasha Watkins

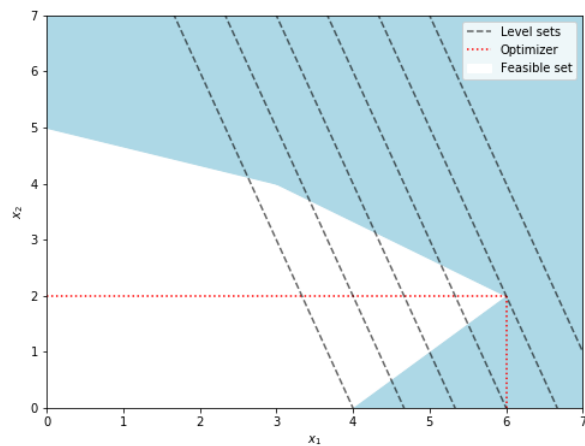
Exercise 8.1



The optimizer is $(2.8, 3.2)$.

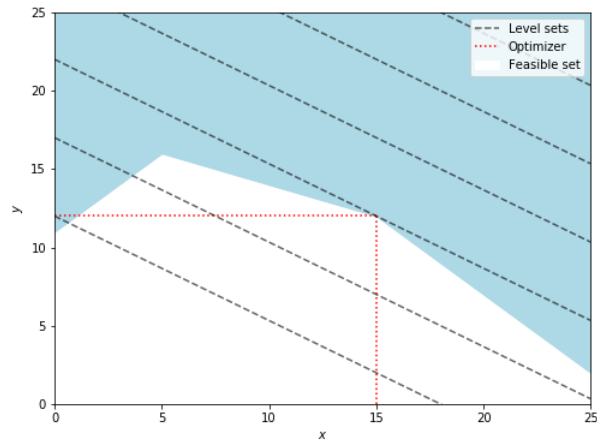
Exercise 8.2

Part i)



The optimizer is $(6, 2)$ with an optimal value of 20.

Part ii)



The optimizer is $(15, 12)$ with an optimal value of 132.

Exercise 8.3

x_1 is a GI Barb soldier, x_2 is a Joey doll. The optimisation problem is

$$\begin{aligned} \max_{x_1, x_2} \quad & 4x_1 + 3x_2 \\ \text{s. t.} \quad & x_2 \leq 200 \\ & 15x_1 + 10x_2 \leq 1800 \\ & 2x_1 + 2x_2 \leq 300 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Exercise 8.4

The optimisation problem is

$$\begin{aligned}
\min_{x_{ij}} \quad & 2x_{AB} + 5x_{BC} + 2x_{CF} + 9x_{BF} + 3x_{EF} + 7x_{BE} + 2x_{BD} + 4x_{DE} + 5x_{AD} + 2x_{BD} \\
\text{s. t.} \quad & x_{AD} + x_{AB} = 10 \\
& -x_{AB} + x_{BC} + x_{BF} + x_{BE} + x_{BD} = 1 \\
& -x_{BC} + x_{CF} = -2 \\
& -x_{AD} - x_{BD} + x_{DE} = -3 \\
& -x_{DE} - x_{BE} + x_{EF} = 4 \\
& -x_{EF} - x_{BF} - x_{CF} = -10 \\
& 0 \leq x_{i,j} \leq 6
\end{aligned}$$

Exercise 8.5

* Dictionaries adapted from Reiko's code.

Part i)

$$\begin{array}{rclclcl}
\zeta & = & & 3x_1 & + & x_2 \\
\hline
w_1 & = & 15 & - & x_1 & - & 3x_2 \\
w_2 & = & 18 & - & 2x_1 & - & 3x_2 \\
w_3 & = & 4 & - & x_1 & + & x_2
\end{array}$$

w_3 is binding, so we increase x_1 by 4.

$$\begin{array}{rclclcl}
\zeta & = & 12 & + & 4x_2 & - & 3w_3 \\
\hline
w_1 & = & 11 & - & 4x_2 & + & w_3 \\
w_2 & = & 10 & - & 5x_2 & + & 2w_3 \\
x_1 & = & 4 & + & x_2 & - & w_3
\end{array}$$

w_2 is binding, so we increase x_2 by 2.

$$\begin{array}{rclclcl}
\zeta & = & 20 & - & \frac{4}{5}w_2 & - & \frac{7}{5}w_3 \\
\hline
w_1 & = & 3 & + & \frac{4}{5}w_2 & - & \frac{3}{5}w_3 \\
x_2 & = & 2 & - & \frac{1}{5}w_2 & + & \frac{2}{5}w_3 \\
x_1 & = & 6 & - & \frac{1}{5}w_2 & - & \frac{3}{5}w_3
\end{array}$$

We can no longer increase the variables in the objective function, so the optimum value of

the objective function is 20 and the optimal point is (6, 2).

Part ii)

$$\begin{array}{rcl} \zeta & = & 4x + 6y \\ \hline w_1 & = & 11 + x - y \\ w_2 & = & 27 - x - y \\ w_3 & = & 90 - 2x - 5y \end{array}$$

w_2 is binding, so we increase x by 27.

$$\begin{array}{rcl} \zeta & = & 108 - 4w_2 + 2y \\ \hline w_1 & = & 38 - w_2 - 2y \\ x & = & 27 - w_2 + y \\ w_3 & = & 26 - 2w_2 - 3y \end{array}$$

w_3 is binding, so we increase y by 12.

$$\begin{array}{rcl} \zeta & = & 132 - \frac{16}{3}w_2 - \frac{2}{3}w_3 \\ \hline w_1 & = & 14 + \frac{7}{3}w_2 - \frac{2}{3}w_3 \\ x & = & 15 - \frac{5}{3}w_2 - \frac{1}{3}w_3 \\ y & = & 12 - \frac{2}{3}w_2 - \frac{1}{3}w_3 \end{array}$$

We can no longer increase the variables in the objective function, so the optimum value of the objective function is 132 and the optimal point is (15, 12).

Exercise 8.6

$$\begin{array}{rcl} \zeta & = & 4x_1 + 3x_2 \\ \hline w_1 & = & 200 - x_2 \\ w_2 & = & 360 - 3x_1 - 2x_2 \\ w_3 & = & 150 - x_1 - x_2 \end{array}$$

w_2 is binding, so we increase x_1 by 120.

$$\begin{array}{rcl} \zeta & = & 480 - \frac{4}{3}w_2 + \frac{1}{3}x_2 \\ \hline w_1 & = & 200 - x_2 \\ x_1 & = & 120 - \frac{1}{3}w_2 - \frac{2}{3}x_2 \\ w_3 & = & 30 + \frac{1}{3}w_2 - \frac{1}{3}x_2 \end{array}$$

w_3 is binding, so we increase x_2 by 90.

$$\begin{array}{rclclcl}
 \zeta & = & 510 & - & w_2 & - & \frac{1}{3}w_3 \\
 \hline
 w_1 & = & 110 & - & w_2 & + & w_3 \\
 x_1 & = & 60 & - & w_2 & - & \frac{2}{3}w_3 \\
 x_2 & = & 90 & + & w_2 & - & w_3
 \end{array}$$

We can no longer increase the variables in the objective function, so the maximum profit \$510 with supply of 60 GI Barb toys and 90 Joey dolls.

Exercise 8.7

Part i)

As one of the constraints has a negative constant, the origin is infeasible. We will first need to solve the auxiliary problem, given as

$$\begin{array}{rclclcl}
 \zeta & = & & & & & -x_o \\
 \hline
 w_1 & = & -8 & + & 4x_1 & + & 2x_2 & + & x_0 \\
 w_2 & = & 6 & + & 2x_1 & - & 3x_2 & + & x_0 \\
 w_3 & = & 3 & - & x_1 & & & + & x_0
 \end{array}$$

This problem is infeasible. We will pivot with w_1 , the most negative constraint.

$$\begin{array}{rclclcl}
 \zeta & = & -8 & + & 4x_1 & + & 2x_2 & - & w_1 \\
 \hline
 x_0 & = & 8 & - & 4x_1 & - & 2x_2 & + & w_1 \\
 w_2 & = & 14 & - & 2x_1 & - & 5x_2 & + & w_1 \\
 w_3 & = & 11 & - & 5x_1 & - & 2x_2 & + & w_1
 \end{array}$$

We need the objective function of the auxiliary problem to be 0, so we will pivot x_1 with its binding constraint x_0 .

$$\begin{array}{rclclcl}
 \zeta & = & & & & & -x_o \\
 \hline
 x_1 & = & 2 & + & \frac{1}{2}w_1 & - & \frac{1}{2}x_2 & - & \frac{1}{4}x_0 \\
 w_2 & = & 10 & + & \frac{1}{2}w_1 & - & 4x_2 & + & \frac{1}{2}x_0 \\
 w_3 & = & 1 & - & \frac{1}{4}w_0 & + & \frac{1}{2}x_2 & + & \frac{5}{4}x_0
 \end{array}$$

Part ii)

As one of the constraints has a negative constant, the origin is infeasible. We will first need to solve the auxiliary problem, given as

$$\begin{array}{rcl}
\zeta & = & -x_o \\
\hline
w_1 & = & 15 - 5x_1 - 3x_2 + x_o \\
w_2 & = & 15 - 3x_1 - 5x_2 + x_o \\
w_3 & = & -12 - 4x_1 + 3x_2 + x_o
\end{array}$$

This problem is infeasible. We will pivot with w_3 , the most negative constraint.

$$\begin{array}{rcl}
\zeta & = & -12 - 4x_1 + 3x_2 - w_1 \\
\hline
w_1 & = & 27 - x_1 - 6x_2 + w_1 \\
w_2 & = & 12 - x_1 - 8x_2 + w_1 \\
x_0 & = & 12 + 4x_1 - 3x_2 + w_1
\end{array}$$

We need the objective function of the auxiliary problem to be 0, so we will pivot x_2 with its binding constraint w_2 .

$$\begin{array}{rcl}
\zeta & = & -\frac{15}{8} - \frac{29}{8}x_1 - \frac{3}{8}w_1 - \frac{5}{8}w_2 \\
\hline
w_1 & = & \frac{27}{4} - \frac{7}{4}x_1 + \frac{3}{4}w_1 + \frac{1}{4}w_2 \\
x_2 & = & \frac{27}{8} + \frac{1}{8}x_1 - \frac{1}{8}w_1 + \frac{1}{8}w_2 \\
x_0 & = & \frac{15}{8} + \frac{29}{8}x_1 + \frac{3}{8}w_1 + \frac{5}{8}w_2
\end{array}$$

As the value of the objective function is negative, the original problem is infeasible.

Part iii)

$$\begin{array}{rcl}
\zeta & = & -3x_1 + x_2 \\
\hline
w_1 & = & 4 - x_2 \\
w_2 & = & 6 + 2x_1 - 3x_2
\end{array}$$

We can only increase x_2 . The binding constraint is w_2 , so we increase x_2 by 2.

$$\begin{array}{rcl}
\zeta & = & 2 - 2x_1 - \frac{1}{3}w_2 \\
\hline
w_1 & = & 2 - x_1 + \frac{1}{3}w_2 \\
x_2 & = & 2 + x_1 - \frac{1}{3}w_2
\end{array}$$

The optimal value is therefore 2 at $(0, 2)$.

Exercise 8.15

We have $A\mathbf{x} \leq \mathbf{b}$ and $A^T\mathbf{y} \geq \mathbf{c}$.

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x}^T A^T &\leq \mathbf{b}^T \\ \mathbf{x}^T A^T \mathbf{y} &\leq \mathbf{b}^T \mathbf{y} \\ \mathbf{x}^T \mathbf{c} &\leq \mathbf{b}^T \mathbf{y} \\ \mathbf{c}^T \mathbf{x} &\leq \mathbf{b}^T \mathbf{y} \quad \text{as } \mathbf{c}^T \mathbf{x} \text{ is scalar} \end{aligned}$$

Exercise 8.17

Consider the primal problem

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \preceq \mathbf{b} \\ & \mathbf{x} \preceq \mathbf{0} \end{aligned}$$

With the dual problem

$$\begin{aligned} \min \quad & \mathbf{b}^T \mathbf{y} \\ \text{s.t.} \quad & A^T \mathbf{y} \succeq \mathbf{c} \\ & \mathbf{y} \succeq \mathbf{0} \end{aligned}$$

This is equivalent to

$$\begin{aligned} \max \quad & -\mathbf{b}^T \mathbf{y} \\ \text{s.t.} \quad & -A^T \mathbf{y} \preceq -\mathbf{c} \\ & \mathbf{y} \succeq \mathbf{0} \end{aligned}$$

Taking the dual of this, we have

$$\begin{aligned} \min \quad & -\mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & -(A^T)^T \mathbf{x} \succeq -\mathbf{b} \\ & \mathbf{x} \succeq \mathbf{0} \end{aligned}$$

which is equivalent to the primal problem.

Exercise 8.18

Solving the primal problem...

$$\begin{array}{rclclcl}
 \zeta & = & & x_1 & + & x_2 \\
 \hline
 w_1 & = & 3 & - & 2x_1 & - & x_2 \\
 w_2 & = & 5 & - & x_1 & - & 3x_2 \\
 w_3 & = & 4 & - & 2x_1 & - & 3x_2
 \end{array}$$

If we increase x_1 , w_1 is binding.

$$\begin{array}{rclclcl}
 \zeta & = & \frac{3}{2} & - & \frac{1}{2}w_1 & + & \frac{1}{2}x_2 \\
 \hline
 x_1 & = & \frac{3}{2} & - & \frac{1}{2}w_1 & - & \frac{1}{2}x_2 \\
 w_2 & = & \frac{7}{2} & - & \frac{1}{2}w_1 & - & \frac{5}{2}x_2 \\
 w_3 & = & 1 & + & w_1 & - & 2x_2
 \end{array}$$

If we increase x_2 , w_3 is binding.

$$\begin{array}{rclclcl}
 \zeta & = & \frac{7}{4} & - & \frac{1}{4}w_1 & - & \frac{1}{4}w_2 \\
 \hline
 x_1 & = & \frac{5}{4} & - & \frac{3}{2}w_1 & + & \frac{1}{4}w_3 \\
 w_2 & = & \frac{9}{4} & - & \frac{3}{4}w_1 & + & \frac{5}{4}w_3 \\
 x_2 & = & \frac{1}{2} & + & \frac{1}{2}w_1 & - & \frac{1}{2}w_3
 \end{array}$$

We can no longer increase the variables in the objective function, so the optimum value of the objective function is $\frac{7}{4}$ and the optimal point is $(\frac{5}{4}, \frac{1}{2})$.

The dual problem is

$$\begin{array}{ll}
 \min_{y_1, y_2, y_3} & 3y_1 + 5y_2 + 4y_3 \\
 \text{s.t.} & 2y_1 + y_2 + 2y_3 \geq 1 \\
 & y_1 + 3y_2 + 3y_3 \geq 1 \\
 & y_1, y_2, y_3 \geq 0
 \end{array}$$