Problem Set 5

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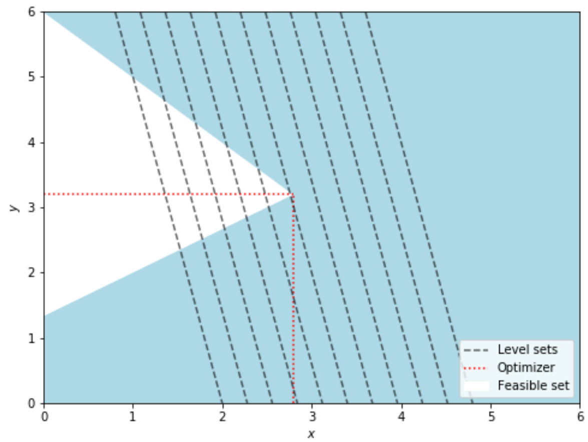
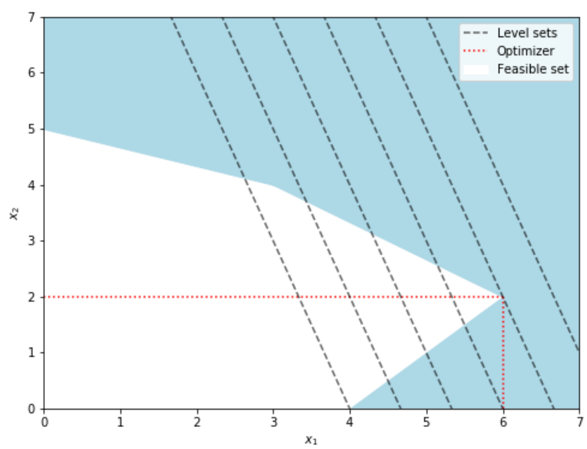
Exercise 8.1



The optimizer is (2.8, 3.2).

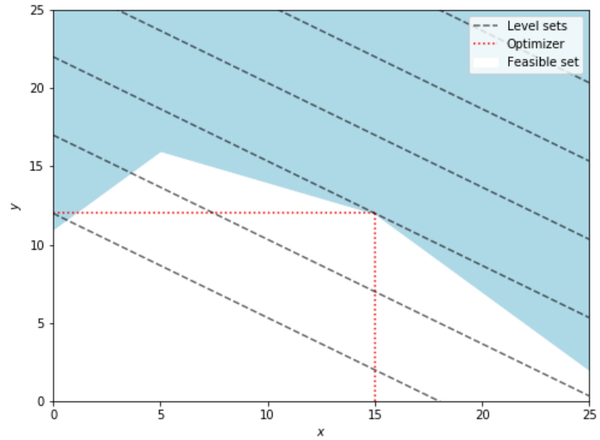
Exercise 8.2

Part i)



The optimizer is (6, 2) with an optimal value of 20.

Part ii)



The optimizer is (15, 12) with an optimal value of 132.

Exercise 8.3

x_1 is a GI Barb soldier, x_2 is a Joey doll. The optimisation problem is

$$\begin{aligned} \max_{x_1,x_2} \ & 4x_1 + 3x_2 \\ \text{s. t.} \ & x_2 \leq 200 \\ & 15x_1 + 10x_2 \leq 1800 \\ & 2x_1 + 2x_2 \leq 300 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Exercise 8.4

The optimisation problem is

$$
\begin{aligned}
\min_{x_{ij}} \quad & 2x_{AB} + 5x_{BC} + 2x_{CF} + 9x_{BF} + 3x_{EF} + 7x_{BE} + 2x_{BD} + 4x_{DE} + 5x_{AD} + 2x_{BD} \\
\text{s. t.} \quad & x_{AD} + x_{AB} = 10 \\
& -x_{AB} + x_{BC} + x_{BF} + x_{BE} + x_{BD} = 1 \\
& -x_{BC} + x_{CF} = -2 \\
& -x_{AD} - x_{BD} + x_{DE} = -3 \\
& -x_{DE} - x_{BE} + x_{EF} = 4 \\
& -x_{EF} - x_{BF} - x_{CF} = -10 \\
& 0 \leq x_{i,j} \leq 6
\end{aligned}
$$

Exercise 8.5

* Dictionaries adapted from Reiko's code.

Part i)

$$
\begin{array}{ccccccc}
\zeta & = & & & 3x_1 & + & x_2 \\
\hline
w_1 & = & 15 & - & x_1 & - & 3x_2 \\
w_2 & = & 18 & - & 2x_1 & - & 3x_2 \\
w_3 & = & 4 & - & x_1 & + & x_2
\end{array}
$$

w_3 is binding, so we increase x_1 by 4.

$$
\begin{array}{ccccccc}
\zeta & = & 12 & + & 4x_2 & - & 3w_3 \\
\hline
w_1 & = & 11 & - & 4x_2 & + & w_3 \\
w_2 & = & 10 & - & 5x_2 & + & 2w_3 \\
x_1 & = & 4 & + & x_2 & - & w_3
\end{array}
$$

w_2 is binding, so we increase x_2 by 2.

$$
\begin{array}{ccccccc}
\zeta & = & 20 & - & \frac{4}{5}w_2 & - & \frac{7}{5}w_3 \\
\hline
w_1 & = & 3 & + & \frac{4}{5}w_2 & - & \frac{3}{5}w_3 \\
x_2 & = & 2 & - & \frac{1}{5}w_2 & + & \frac{2}{5}w_3 \\
x_1 & = & 6 & - & \frac{1}{5}w_2 & - & \frac{3}{5}w_3
\end{array}
$$

We can no longer increase the variables in the objective function, so the optimum value of the objective function is 20 and the optimal point is (6, 2).

Part ii)

$$\begin{array}{rcrcrcr} \zeta & = & & & 4x & + & 6y \\ \hline w_1 & = & 11 & + & x & - & y \\ w_2 & = & 27 & - & x & - & y \\ w_3 & = & 90 & - & 2x & - & 5y \end{array}$$

w_2 is binding, so we increase x by 27.

$$\begin{array}{rcrcrcr} \zeta & = & 108 & - & 4w_2 & + & 2y \\ \hline w_1 & = & 38 & - & w_2 & - & 2y \\ x & = & 27 & - & w_2 & + & y \\ w_1 & = & 26 & - & 2w_2 & - & 3y \end{array}$$

w_3 is binding, so we increase y by 12.

$$\begin{array}{rcrcrcr} \zeta & = & 132 & - & \frac{16}{3}w_2 & - & \frac{2}{3}w_3 \\ \hline w_1 & = & 14 & + & \frac{7}{3}w_2 & - & \frac{2}{3}w_3 \\ x & = & 15 & - & \frac{5}{3}w_2 & - & \frac{1}{3}w_3 \\ y & = & 12 & - & \frac{2}{3}w_2 & - & \frac{1}{3}w_3 \end{array}$$

We can no longer increase the variables in the objective function, so the optimum value of the objective function is 132 and the optimal point is (15, 12).

Exercise 8.6

$$\begin{array}{rcrcrcr} \zeta & = & & & 4x_1 & + & 3x_2 \\ \hline w_1 & = & 200 & & & - & x_2 \\ w_2 & = & 360 & - & 3x_1 & - & 2x_2 \\ w_3 & = & 150 & - & x_1 & - & x_2 \end{array}$$

w_2 is binding, so we increase x_1 by 120.

$$\begin{array}{rcrcrcr} \zeta & = & 480 & - & \frac{4}{3}w_2 & + & \frac{1}{3}x_2 \\ \hline w_1 & = & 200 & & & - & x_2 \\ x_1 & = & 120 & - & \frac{1}{3}w_2 & - & \frac{2}{3}x_2 \\ w_3 & = & 30 & + & \frac{1}{3}w_2 & - & \frac{1}{3}x_2 \end{array}$$

w_3 is binding, so we increase x_2 by 90.

$$\begin{array}{rcrcrcr}
\zeta & = & 510 & - & w_2 & - & \frac{1}{3}w_3 \\
\hline
w_1 & = & 110 & - & w_2 & + & w_3 \\
x_1 & = & 60 & - & w_2 & - & \frac{2}{3}w_3 \\
x_2 & = & 90 & + & w_2 & - & w_3
\end{array}$$

We can no longer increase the variables in the objective function, so the maximum profit \$510 with supply of 60 GI Barb toys and 90 Joey dolls.

Exercise 8.7

Part i)

As one of the constraints has a negative constant, the origin is infeasible. We will first need to solve the auxiliary problem, given as

$$\begin{array}{rcrcrcrcr}
\zeta & = & & & & & & & -x_o \\
\hline
w_1 & = & -8 & + & 4x_1 & + & 2x_2 & + & x_0 \\
w_2 & = & 6 & + & 2x_1 & - & 3x_2 & + & x_0 \\
w_3 & = & 3 & - & x_1 & & & + & x_0
\end{array}$$

This problem is infeasible. We will pivot with w_1, the most negative constraint.

$$\begin{array}{rcrcrcrcr}
\zeta & = & -8 & + & 4x_1 & + & 2x_2 & - & w_1 \\
\hline
x_0 & = & 8 & - & 4x_1 & - & 2x_2 & + & w_1 \\
w_2 & = & 14 & - & 2x_1 & - & 5x_2 & + & w_1 \\
w_3 & = & 11 & - & 5x_1 & - & 2x_2 & + & w_1
\end{array}$$

We need the objective function of the auxiliary problem to be 0, so we will pivot x_1 with its binding constraint x_0.

$$\begin{array}{rcrcrcrcr}
\zeta & = & & & & & & & -x_o \\
\hline
x_1 & = & 2 & + & \frac{1}{2}w_1 & - & \frac{1}{2}x_2 & - & \frac{1}{4}x_0 \\
w_2 & = & 10 & + & \frac{1}{2}w_1 & - & 4x_2 & + & \frac{1}{2}x_0 \\
w_3 & = & 1 & - & \frac{1}{4}w_0 & + & \frac{1}{2}x_2 & + & \frac{5}{4}x_0
\end{array}$$

Part ii)

As one of the constraints has a negative constant, the origin is infeasible. We will first need to solve the auxiliary problem, given as

$$\begin{array}{rcrcrcrcr}
\zeta & = & & & & & & & -x_o \\
\hline
w_1 & = & 15 & - & 5x_1 & - & 3x_2 & + & x_0 \\
w_2 & = & 15 & - & 3x_1 & - & 5x_2 & + & x_0 \\
w_3 & = & -12 & - & 4x_1 & + & 3x_1 & + & x_0
\end{array}$$

This problem is infeasible. We will pivot with w_3, the most negative constraint.

$$\begin{array}{rcrcrcrcr}
\zeta & = & -12 & - & 4x_1 & + & 3x_2 & - & w_1 \\
\hline
w_1 & = & 27 & - & x_1 & - & 6x_2 & + & w_1 \\
w_2 & = & 12 & - & x_1 & - & 8x_2 & + & w_1 \\
x_0 & = & 12 & + & 4x_1 & - & 3x_2 & + & w_1
\end{array}$$

We need the objective function of the auxiliary problem to be 0, so we will pivot x_2 with its binding constraint w_2.

$$\begin{array}{rcrcrcrcr}
\zeta & = & -\frac{15}{8} & - & \frac{29}{8}x_1 & - & \frac{3}{8}w_1 & - & \frac{5}{8}w_2 \\
\hline
w_1 & = & \frac{27}{4} & - & \frac{7}{4}x_1 & + & \frac{3}{4}w_1 & + & \frac{1}{4}w_2 \\
x_2 & = & \frac{27}{8} & + & \frac{1}{8}x_1 & - & \frac{1}{8}w_1 & + & \frac{1}{8}w_2 \\
x_0 & = & \frac{15}{8} & + & \frac{29}{8}x_1 & + & \frac{3}{8}w_1 & + & \frac{5}{8}w_2
\end{array}$$

As the value of the objective function is negative, the original problem is infeasible.

Part iii)

$$\begin{array}{rcrcrcr}
\zeta & = & & & -3x_1 & + & x_2 \\
\hline
w_1 & = & 4 & & & - & x_2 \\
w_2 & = & 6 & + & 2x_1 & - & 3x_2
\end{array}$$

We can only increase x_2. The binding constraint is w_2, so we increase x_2 by 2.

$$\begin{array}{rcrcrcr}
\zeta & = & 2 & - & 2x_1 & - & \frac{1}{3}w_2 \\
\hline
w_1 & = & 2 & - & x_1 & + & \frac{1}{3}w_2 \\
x_2 & = & 2 & + & x_1 & - & \frac{1}{3}w_2
\end{array}$$

The optimal value is therefore 2 at (0, 2).

Exercise 8.12

$$\begin{array}{rcrcrcrcrcr}
\zeta & = & & & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4 \\
\hline
w_1 & = & & & -x_1 & + & 3x_2 & + & x_3 & - & 2x_4 \\
w_2 & = & & & -x_1 & + & 11x_2 & + & 5x_3 & - & 18x_4 \\
w_3 & = & 1 & - & x_1 & & & & & &
\end{array}$$

We will pivot x_1 with w_1.

$$\begin{array}{rcrcrcrcrcr}
\zeta & = & & & -10w_1 & - & 27x_2 & + & x_3 & - & 44x_4 \\
\hline
x_1 & = & & & -w_1 & + & 3x_2 & + & x_3 & - & 2x_4 \\
w_2 & = & & & w_1 & + & 8x_2 & + & 4x_3 & - & 20x_4 \\
w_3 & = & 1 & + & w_1 & - & 3x_2 & - & x_2 & + & 2x_4
\end{array}$$

We will pivot x_3 with w_3.

$$\begin{array}{rcrcrcrcrcr}
\zeta & = & 1 & - & -9w_1 & - & 30x_2 & - & w_3 & & \\
\hline
x_1 & = & 1 & & & & & - & w_3 & & \\
w_2 & = & 4 & + & 5w_1 & - & 4x_2 & - & 4w_3 & - & 12x_4 \\
x_3 & = & 1 & + & w_1 & - & 3x_2 & - & w_3 & + & 2x_4
\end{array}$$

The optimal value is 1 at the point (1, 0, 1, 0).

Exercise 8.15

We have $A\mathbf{x} \leq \mathbf{b}$ and $A^T\mathbf{y} \geq \mathbf{c}$.

$$\begin{aligned}
A\mathbf{x} &\leq \mathbf{b} \\
\mathbf{x}^T A^T &\leq \mathbf{b}^T \\
\mathbf{x}^T A^T \mathbf{y} &\leq \mathbf{b}^T \mathbf{y} \\
\mathbf{x}^T \mathbf{c} &\leq \mathbf{b}^T \mathbf{y} \\
\mathbf{c}^T \mathbf{x} &\leq \mathbf{b}^T \mathbf{y} \quad \text{as } \mathbf{c}^T\mathbf{x} \text{ is scalar}
\end{aligned}$$

Exercise 8.17

Consider the primal problem

$$\begin{aligned}
\max\ & \mathbf{c}^T\mathbf{x} \\
\text{s.t. } & A\mathbf{x} \preceq \mathbf{b} \\
& \mathbf{x} \preceq \mathbf{0}
\end{aligned}$$

With the dual problem

$$\begin{aligned} \min\ & \mathbf{b}^T\mathbf{y} \\ \text{s.t. } & A^T\mathbf{y} \succeq \mathbf{c} \\ & \mathbf{y} \succeq \mathbf{0} \end{aligned}$$

This is equivalent to

$$\begin{aligned} \max\ & -\mathbf{b}^T\mathbf{y} \\ \text{s.t. } & -A^T\mathbf{y} \preceq -\mathbf{c} \\ & \mathbf{y} \succeq \mathbf{0} \end{aligned}$$

Taking the dual of this, we have

$$\begin{aligned} \min\ & -\mathbf{c}^T\mathbf{x} \\ \text{s.t. } & -(A^T)^T\mathbf{x} \succeq -\mathbf{b} \\ & \mathbf{x} \succeq \mathbf{0} \end{aligned}$$

which is equivalent to the primal problem.

Exercise 8.18

Solving the primal problem...

$$\begin{array}{ccccccc} \zeta & = & & & x_1 & + & x_2 \\ \hline w_1 & = & 3 & - & 2x_1 & - & x_2 \\ w_2 & = & 5 & - & x_1 & - & 3x_2 \\ w_3 & = & 4 & - & 2x_1 & - & 3x_2 \end{array}$$

If we increase x_1, w_1 is binding.

$$\begin{array}{ccccccc} \zeta & = & \frac{3}{2} & - & \frac{1}{2}w_1 & + & \frac{1}{2}x_2 \\ \hline x_1 & = & \frac{3}{2} & - & \frac{1}{2}w_1 & - & \frac{1}{2}x_2 \\ w_2 & = & \frac{7}{2} & - & \frac{1}{2}w_1 & - & \frac{5}{2}x_2 \\ w_3 & = & 1 & + & w_1 & - & 2x_2 \end{array}$$

If we increase x_2, w_3 is binding.

$$
\begin{array}{rcrcrcr}
\zeta & = & \frac{7}{4} & - & \frac{1}{4}w_1 & - & \frac{1}{4}w_2 \\
\hline
x_1 & = & \frac{5}{4} & - & \frac{3}{2}w_1 & + & \frac{1}{4}w_3 \\
w_2 & = & \frac{9}{4} & - & \frac{3}{4}w_1 & + & \frac{5}{4}w_3 \\
x_2 & = & \frac{1}{2} & + & \frac{1}{2}w_1 & - & \frac{1}{2}w_3
\end{array}
$$

We can no longer increase the variables in the objective function, so the optimum value of the objective function is $\frac{7}{4}$ and the optimal point is $(\frac{5}{4}, \frac{1}{2})$.

The dual problem is

$$
\begin{aligned}
\min_{y_1,y_2,y_3} \; & 3y_1 + 5y_2 + 4y_3 \\
\text{s.t. } & 2y_1 + y_2 + 2y_3 \geq 1 \\
& y_1 + 3y_2 + 3y_3 \geq 1 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$