Problem Set 1

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Exercise 1.3

$\mathcal{G}_1$ is not an algebra.

- Take $B \in \mathcal{G}_1$. Its complement, B^c, is closed, and therefore not in $\mathcal{G}_1$. Hence, $\mathcal{G}_1$ is not closed under complements.

$\mathcal{G}_2$ is an algebra.

$\mathcal{G}_3$ is a σ-algebra.

Exercise 1.7

$\mathcal{A}$ is a σ-algebra of X. $\{\emptyset, X\} \subset \mathcal{A} \subset \mathcal{P}(X)$.

Part 1:

- Take $A \in \mathcal{A}$. Then $A^c \in \mathcal{A}$. So $A \cup A^c = X \in \mathcal{A}$.
- $X \in \mathcal{A}$ implies $X^c = \emptyset \in \mathcal{A}$.
- So $\{\emptyset, X\}$ is the smallest σ-algebra of X.

Part 2:

Exercise 1.10

$\{S_\alpha\}$ is a family of σ-algebras on X.

- As $\emptyset \in \mathcal{S}_\alpha$ for all α, $\emptyset \in \cap_\alpha \mathcal{S}_\alpha$.
- Given $A \in \cap_\alpha \mathcal{S}_\alpha$, $A \in \mathcal{S}_\alpha$ for all α and $A^c \in \mathcal{S}_\alpha$ for all α. So $A^c \in \cap_\alpha \mathcal{S}_\alpha$.
- Given $A_1, A_2, \dots \in \cap_\alpha \mathcal{S}_\alpha$, $A_1, A_2, \dots \in \mathcal{S}_\alpha$ for all α. So $\cup_{n=1}^\infty A_n \in \mathcal{S}_\alpha$ for all α. Hence, $\cup_{n=1}^\infty A_n \in \cap_\alpha \mathcal{S}_\alpha$.

Therefore, $\cap_\alpha \mathcal{S}_\alpha$ is a σ-algebra.

Exercise 1.18

- $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$.