Problem Set 1

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Exercise 1.3

$\mathcal{G}_1$ is not an algebra.

- Take $B \in \mathcal{G}_1$. Its complement, B^c, is closed, and therefore not in $\mathcal{G}_1$. Hence, $\mathcal{G}_1$ is not closed under complements.

$\mathcal{G}_2$ is an algebra.

$\mathcal{G}_3$ is a σ-algebra.

Exercise 1.7

$\mathcal{A}$ is a σ-algebra of X. $\{\emptyset, X\} \subset \mathcal{A} \subset \mathcal{P}(X)$.

Part 1:

- By definition, $\emptyset \in \mathcal{A}$.
- $\emptyset \in \mathcal{A}$ implies $\emptyset^c = X \in \mathcal{A}$.
- So $\{\emptyset, X\}$ is the smallest σ-algebra of X.

Part 2:

- Suppose $\mathcal{P}(X)$ is not the largest σ-algebra on X.
- Then there is $B \in X$ such that $\mathcal{P}(X) \cup B$ is a σ-algebra of X, where $B \notin \mathcal{P}(X)$.
- But $B \in X$, so $B \in \mathcal{P}(X)$. This is a contradiction, so $\mathcal{P}(X)$ must be the largest σ-algebra.

Exercise 1.10

$\{S_\alpha\}$ is a family of σ-algebras on X.

- As $\emptyset \in \mathcal{S}_\alpha$ for all α, $\emptyset \in \cap_\alpha \mathcal{S}_\alpha$.
- Given $A \in \cap_\alpha \mathcal{S}_\alpha$, $A \in \mathcal{S}_\alpha$ for all α and $A^c \in \mathcal{S}_\alpha$ for all α. So $A^c \in \cap_\alpha \mathcal{S}_\alpha$.
- Given $A_1, A_2, \cdots \in \cap_\alpha \mathcal{S}_\alpha$, $A_1, A_2, \cdots \in \mathcal{S}_\alpha$ for all α. So $\cup_{n=1}^\infty A_n \in \mathcal{S}_\alpha$ for all α. Hence, $\cup_{n=1}^\infty A_n \in \cap_\alpha \mathcal{S}_\alpha$.

Therefore, $\cap_\alpha \mathcal{S}_\alpha$ is a σ-algebra.

Exercise 1.17

- $A, B \in \mathcal{S}$, $A \subset B$. $A \cup (B \cap A^c) = B$. As A and $B \cap A^c$ are disjoint, $\mu(A) + \mu(B \cap A^c) = \mu(B)$. This implies $\mu(A) \leq \mu(B)$.

Exercise 1.18

- $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$.

Exercise 1.20

- $\cap_{i=1}^n A_i = A_n$
- $\mu(\cap_{i=1}^\infty A_i) = \lim_{n\to\infty} \mu(\cap_{i=1}^n A_i) = \lim_{n\to\infty} \mu(A_n)$