Problem Set 3

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Exercise 4.2

The derivative operator can be written in matrix form as

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Solving for λ such that $\det(D - \lambda I) = 0$ yields $\lambda = 0$. This eigenvalue has an associated eigenvector of $[x_1\ 0\ 0]^T$. Therefore, the algebraic multiplicity is 3 (λ appears 3 times) and the geometric multiplicity is 1 (there is 1 associated eigenvector).

Exercise 4.4

By 4.3, the characteristic polynomial of $A^H = A$ is

$$p(\lambda) = \lambda^2 - tr(A)\lambda + \det(A)$$

where A is a 2×2 matrix of the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The diagonal elements of a Hermitian matrix are real. So the diagonal elements of A are also real. We can check that the roots of the characteristic polynomial are real by finding the discriminant.

$$\begin{aligned} [tr(A)]^2 - 4\det(A) &= (a+d)^2 - 4(ad - bc) \\ &= a^2 + 2ad + d^2 - 4ad + 4bc \\ &= a^2 - 4ad + d^2 + 4bc \\ &= (a-d)^2 + 4bc \end{aligned}$$

<u>Part i)</u>

As A^H is Hermitian, $b = \bar{c}$, so $4bc \geq 0$. Therefore the discriminant is positive and the roots are real.

Part ii)

If $A^H = -A$, then $b = -\bar{c}$, so $4bc \leq 0$. Therefore, the discriminant is negative and the roots are imaginary.

Exercise 4.6

Consider a matrix A that is upper triangular. We know that the determinant of an upper triangular matrix is the product of its diagonal elements, ie. $\det(A) = \prod_{i=1}^{n} a_{ii}$.

The eigenvalues of a matrix are such that $\det(\lambda I - A) = 0$.

Therefore,

$$\begin{aligned}\det(\lambda I - A) &= \prod_{i=1}^{n}(\lambda_i - a_{ii}) = 0 \\ &= (\lambda_1 - a_{11}) \cdot (\lambda_2 - a_{22}) \cdots (\lambda_n - a_{nn}) = 0\end{aligned}$$

The roots of the characteristic polynomial (ie. the eigenvalues) are given by the diagonal elements of A.

Exercise 4.13

Solving for the spectrum of A gives $\sigma(A) = \{\frac{2}{5}, 1\}$, with corresponding eigenvectors of $[1, -1]^T$ and $[2, 1]^T$. Let P be the transition matrix, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} \frac{2}{5} & 0 \\ 0 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

Solving $P^{-1}AP$ gives

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0.8 & 0.4 \\ 0,2 & 0.6 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & 0 \\ 0 & 1 \end{bmatrix} = D$$

Exercise 4.18

We know that λ satisfies $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x}$. Taking the transpose, we see $\mathbf{x}^T A^T = \lambda\mathbf{x}^T$. This is equivalent to $\det(A^T - \lambda I) = 0$.

The determinant of a matrix A is equal to the determinant of its transpose, so

$$\det(A^T - \lambda I) = \det(A - \lambda I) = 0$$

As we know $\det(A - \lambda I) = 0$ holds for some $\mathbf{x}$, there exists a vector $\mathbf{x}^T$ such that $\det(A^T - \lambda I) = 0$ holds.

Exercise 4.20

If A^H and B are orthornormally similar, then there is U such that $B = U^H A^H U$.

$$B^H = (U^H A^H U)^H = U^H A U = U^H A^H U = B$$

Therefore, B is also Hermitian.

Exercise 4.24

With the standard inner product on $\mathbb{F}_n$

$$\langle \mathbf{x}, A\mathbf{x} \rangle = \langle \mathbf{x}, \lambda \mathbf{x} \rangle = \lambda \mathbf{x}^H \mathbf{x} = \lambda \|\mathbf{x}\|_2$$

Therefore, as Hermitian matrices only have real eigenvalues, the Rayleigh quotient only takes real values. As skew-Hermitian matrices only have imaginary eigenvalues, the Rayleigh quotient only takes imaginary values.

Exercise 4.25

Part i)

$$\begin{aligned}(\mathbf{x}_1\mathbf{x}_1^H + \cdots + \mathbf{x}_n\mathbf{x}_n^H)\mathbf{x}_j &= \mathbf{x}_1\mathbf{x}_1^H\mathbf{x}_j + \cdots + \mathbf{x}_n\mathbf{x}_n^H\mathbf{x}_j \\ &= \mathbf{x}_j\end{aligned}$$

As $\mathbf{x}_i^H\mathbf{x}_j = 0$, with $i \neq j$, and $\mathbf{x}_i^H\mathbf{x}_i = 1$, as the eigenvectors $[\mathbf{x}_1\ \mathbf{x}_2 \cdots \mathbf{x}_n]$ are orthonormal. Therefore, $(\mathbf{x}_1\mathbf{x}_1^H + \cdots + \mathbf{x}_n\mathbf{x}_n^H) = I$.

Part ii)

$$\begin{aligned}(\lambda_1\mathbf{x}_1\mathbf{x}_1^H + \cdots + \lambda_n\mathbf{x}_n\mathbf{x}_n^H)\mathbf{x}_j &= \lambda_1\mathbf{x}_1\mathbf{x}_1^H\mathbf{x}_j + \cdots + \lambda_n\mathbf{x}_n\mathbf{x}_n^H\mathbf{x}_j \\ &= \lambda_j\mathbf{x}_j \\ &= A\mathbf{x}_j \qquad \text{as } A\mathbf{x}_j = \lambda_j\mathbf{x}_j\end{aligned}$$

Similar to part i), we can therefore write, $(\lambda_1\mathbf{x}_1\mathbf{x}_1^H + \cdots + \lambda_n\mathbf{x}_n\mathbf{x}_n^H) = A$.

Exercise 4.27

A is a positive definite matrix, so it is Hermitian. The diagonal elements of a Hermitian matrix are real, as they are their own complex conjugate.

It also is true that $\langle\mathbf{x}, A\mathbf{x}\rangle = \mathbf{x}^H A\mathbf{x} > 0$ for any vector $\mathbf{x}$. Consider the vector $e_1 = [1\ 0\ 0\ \cdots\ 0]^T$. $e_1^H A e_1$ selects the first diagonal element of A. Likewise, the vector e_2 selects the second diagonal element of A. As $\langle\mathbf{x}, A\mathbf{x}\rangle = \mathbf{x}^H A\mathbf{x} > 0$ holds for all vectors, the diagonal elements are all positive.

Exercise 4.36

The following matrix has an eigenvalue of -1 and a singular value of one.

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Exercise 3.38

i) $AA^\dagger A = U_1\Sigma_1V_1^HV_1\Sigma_1^{-1}U_1^HU_1\Sigma_1V_1^H = U_1\Sigma_1V_1^H = A$

ii) $A^\dagger AA^\dagger = V_1\Sigma_1^{-1}U_1^HU_1\Sigma_1V_1^HV_1\Sigma_1^{-1}U_1^H = V_1\Sigma_1^{-1}U_1^H = A^\dagger$

iii) $(AA^\dagger)^H = ((V_1\Sigma_1U_1^H)(U_1\Sigma_1^{-1}V_1^H))^H = V_1\Sigma_1^{-1}U_1^HU_1\Sigma_1V_1^H = A^\dagger A$

iv) $(A^\dagger A)^H = ((V_1\Sigma_1^{-1}U_1^H)(U_1\Sigma_1V_1^H))^H = V_1\Sigma_1U_1^HU_1\Sigma_1^{-1}V_1^H = V_1V_1^H = A^\dagger A$