Problem Set 5

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Exercise 8.3

x_1 is a GI Barb soldier, x_2 is a Joey doll. The optimisation problem is

$$\begin{aligned}
&\max_{x_1,x_2} \; 4x_1 + 3x_2 \\
\text{s. t.} \quad & x_2 \leq 200 \\
& 15x_1 + 10x_2 \leq 1800 \\
& 2x_1 + 2x_2 \leq 300 \\
& x_1, x_2 \geq 0
\end{aligned}$$

Exercise 8.4

The optimisation problem is

$$\begin{aligned}
\min_{x_{ij}} \quad & 2x_{AB} + 5x_{BC} + 2x_{CF} + 9x_{BF} + 3x_{EF} + 7x_{BE} + 2x_{BD} + 4x_{DE} + 5x_{AD} + 2x_{BD} \\
\text{s. t.} \quad & x_{AD} + x_{AB} = 10 \\
& -x_{AB} + x_{BC} + x_{BF} + x_{BE} + x_{BD} = 1 \\
& -x_{BC} + x_{CF} = -2 \\
& -x_{AD} - x_{BD} + x_{DE} = -3 \\
& -x_{DE} - x_{BE} + x_{EF} = 4 \\
& -x_{EF} - x_{BF} - x_{CF} = -10 \\
& 0 \leq x_{i,j} \leq 6
\end{aligned}$$

Exercise 8.5

* Dictionaries adapted from Reiko's code.

Part i)

$$
\begin{array}{rcrcrcr}
\zeta & = & & & 3x_1 & + & x_2 \\
\hline
w_1 & = & 15 & - & x_1 & - & 3x_2 \\
w_2 & = & 18 & - & 2x_1 & - & 3x_2 \\
w_3 & = & 4 & - & x_1 & + & x_2
\end{array}
$$

w_3 is binding, so we increase x_1 by 4.

$$
\begin{array}{rcrcrcr}
\zeta & = & 12 & + & 4x_2 & - & 3w_3 \\
\hline
w_1 & = & 11 & - & 4x_2 & + & w_3 \\
w_2 & = & 10 & - & 5x_2 & + & 2w_3 \\
x_1 & = & 4 & + & x_2 & - & w_3
\end{array}
$$

w_2 is binding, so we increase x_2 by 2.

$$
\begin{array}{rcrcrcr}
\zeta & = & 20 & - & \frac{4}{5}w_2 & - & \frac{7}{5}w_3 \\
\hline
w_1 & = & 3 & + & \frac{4}{5}w_2 & - & \frac{3}{5}w_3 \\
x_2 & = & 2 & - & \frac{1}{5}w_2 & + & \frac{2}{5}w_3 \\
x_1 & = & 6 & - & \frac{1}{5}w_2 & - & \frac{3}{5}w_3
\end{array}
$$

We can no longer increase the variables in the objective function, so the optimum value of the objective function is 20 and the optimal point is (6, 2).

<u>Part ii)</u>

$$
\begin{array}{rcrcrcr}
\zeta & = & & & 4x & + & 6y \\
\hline
w_1 & = & 11 & + & x & - & y \\
w_2 & = & 27 & - & x & - & y \\
w_3 & = & 90 & - & 2x & - & 5y
\end{array}
$$

w_2 is binding, so we increase x by 27.

$$
\begin{array}{rcrcrcr}
\zeta & = & 108 & - & 4w_2 & + & 2y \\
\hline
w_1 & = & 38 & - & w_2 & - & 2y \\
x & = & 27 & - & w_2 & + & y \\
w_1 & = & 26 & - & 2w_2 & - & 3y
\end{array}
$$

w_3 is binding, so we increase y by 12.

$$
\begin{array}{rcrcrcr}
\zeta & = & 132 & - & \frac{16}{3}w_2 & - & \frac{2}{3}w_3 \\
\hline
w_1 & = & 14 & + & \frac{7}{3}w_2 & - & \frac{2}{3}w_3 \\
x & = & 15 & - & \frac{5}{3}w_2 & - & \frac{1}{3}w_3 \\
y & = & 12 & - & \frac{2}{3}w_2 & - & \frac{1}{3}w_3
\end{array}
$$

We can no longer increase the variables in the objective function, so the optimum value of the objective function is 132 and the optimal point is (15, 12).

Exercise 8.6

$$
\begin{array}{rcrcrcr}
\zeta & = & & & 4x_1 & + & 3x_2 \\
\hline
w_1 & = & 200 & & & - & x_2 \\
w_2 & = & 360 & - & 3x_1 & - & 2x_2 \\
w_3 & = & 150 & - & x_1 & - & x_2
\end{array}
$$

w_2 is binding, so we increase x_1 by 120.

$$
\begin{array}{rcrcrcr}
\zeta & = & 480 & - & \frac{4}{3}w_2 & + & \frac{1}{3}x_2 \\
\hline
w_1 & = & 200 & & & - & x_2 \\
x_1 & = & 120 & - & \frac{1}{3}w_2 & - & \frac{2}{3}x_2 \\
w_3 & = & 30 & + & \frac{1}{3}w_2 & - & \frac{1}{3}x_2
\end{array}
$$

w_3 is binding, so we increase x_2 by 90.

$$
\begin{array}{rcrcrcr}
\zeta & = & 510 & - & w_2 & - & \frac{1}{3}w_3 \\
\hline
w_1 & = & 110 & - & w_2 & + & w_3 \\
x_1 & = & 60 & - & w_2 & - & \frac{2}{3}w_3 \\
x_2 & = & 90 & + & w_2 & - & w_3
\end{array}
$$

We can no longer increase the variables in the objective function, so the maximum profit \$510 with supply of 60 GI Barb toys and 90 Joey dolls.

Exercise 8.7

Part i)

As one of the constraints has a negative constant, the origin is infeasible. We will first need to solvethe auxiliary problem, given as

$$\begin{array}{rcrcrcrcr}
\zeta & = & & & & & & & -x_o \\
\hline
w_1 & = & -8 & + & 4x_1 & + & 2x_2 & + & x_0 \\
w_2 & = & 6 & + & 2x_1 & - & 3x_2 & + & x_0 \\
w_3 & = & 3 & - & x_1 & & & + & x_0
\end{array}$$

This problem is infeasible. We will pivot with w_1, the most negative constraint.

$$\begin{array}{rcrcrcrcr}
\zeta & = & -8 & + & 4x_1 & + & 2x_2 & - & w_1 \\
\hline
x_0 & = & 8 & - & 4x_1 & - & 2x_2 & + & w_1 \\
w_2 & = & 14 & - & 2x_1 & - & 5x_2 & + & w_1 \\
w_3 & = & 11 & - & 5x_1 & - & 2x_2 & + & w_1
\end{array}$$

We need the objective function of the auxiliary problem to be 0, so we will pivot x_1 with its binding constraint x_0.

$$\begin{array}{rcrcrcrcr}
\zeta & = & & & & & & & -x_o \\
\hline
x_1 & = & 2 & + & \frac{1}{2}w_1 & - & \frac{1}{2}x_2 & - & \frac{1}{4}x_0 \\
w_2 & = & 10 & + & \frac{1}{2}w_1 & - & 4x_2 & + & \frac{1}{2}x_0 \\
w_3 & = & 1 & - & \frac{1}{4}w_0 & + & \frac{1}{2}x_2 & + & \frac{5}{4}x_0
\end{array}$$

Part ii)

As one of the constraints has a negative constant, the origin is infeasible. We will first need to solvethe auxiliary problem, given as

$$\begin{array}{rcrcrcrcr}
\zeta & = & & & & & & & -x_o \\
\hline
w_1 & = & 15 & - & 5x_1 & - & 3x_2 & + & x_0 \\
w_2 & = & 15 & - & 3x_1 & - & 5x_2 & + & x_0 \\
w_3 & = & -12 & - & 4x_1 & + & 3x_1 & + & x_0
\end{array}$$

This problem is infeasible. We will pivot with w_3, the most negative constraint.

$$\begin{array}{rcrcrcrcr}
\zeta & = & -12 & - & 4x_1 & + & 3x_2 & - & w_1 \\
\hline
w_1 & = & 27 & - & x_1 & - & 6x_2 & + & w_1 \\
w_2 & = & 12 & - & x_1 & - & 8x_2 & + & w_1 \\
x_0 & = & 12 & + & 4x_1 & - & 3x_2 & + & w_1
\end{array}$$

We need the objective function of the auxiliary problem to be 0, so we will pivot x_2 with its binding constraint w_2.

$$
\begin{array}{rcrcrcrcr}
\zeta & = & -\frac{15}{8} & - & \frac{29}{8}x_1 & - & \frac{3}{8}w_1 & - & \frac{5}{8}w_2 \\
\hline
w_1 & = & \frac{27}{4} & - & \frac{7}{4}x_1 & + & \frac{3}{4}w_1 & + & \frac{1}{4}w_2 \\
x_2 & = & \frac{27}{8} & + & \frac{1}{8}x_1 & - & \frac{1}{8}w_1 & + & \frac{1}{8}w_2 \\
x_0 & = & \frac{15}{8} & + & \frac{29}{8}x_1 & + & \frac{3}{8}w_1 & + & \frac{5}{8}w_2
\end{array}
$$

As the value of the objective function is negative, the original problem is infeasible.

Part iii)

$$
\begin{array}{rcrcrcr}
\zeta & = & & & -3x_1 & + & x_2 \\
\hline
w_1 & = & 4 & & & - & x_2 \\
w_2 & = & 6 & + & 2x_1 & - & 3x_2
\end{array}
$$

We can only increase x_2. The binding constraint is w_2, so we increase x_2 by 2.

$$
\begin{array}{rcrcrcr}
\zeta & = & 2 & - & 2x_1 & - & \frac{1}{3}w_2 \\
\hline
w_1 & = & 2 & - & x_1 & + & \frac{1}{3}w_2 \\
x_2 & = & 2 & + & x_1 & - & \frac{1}{3}w_2
\end{array}
$$

The optimal value is therefore 2 at (0, 2).

Exercise 8.15

We have $A\mathbf{x} \leq \mathbf{b}$ and $A^T\mathbf{y} \geq \mathbf{c}$.

$$
\begin{aligned}
A\mathbf{x} &\leq \mathbf{b} \\
\mathbf{x}^T A^T &\leq \mathbf{b}^T \\
\mathbf{x}^T A^T \mathbf{y} &\leq \mathbf{b}^T \mathbf{y} \\
\mathbf{x}^T \mathbf{c} &\leq \mathbf{b}^T \mathbf{y} \\
\mathbf{c}^T \mathbf{x} &\leq \mathbf{b}^T \mathbf{y} \quad \text{as } \mathbf{c}^T\mathbf{x} \text{ is scalar}
\end{aligned}
$$

Exercise 8.17

Consider the primal problem

$$
\begin{aligned}
\max\ & \mathbf{c}^T\mathbf{x} \\
\text{s.t. } & A\mathbf{x} \preceq \mathbf{b} \\
& \mathbf{x} \preceq \mathbf{0}
\end{aligned}
$$

With the dual problem

$$\begin{aligned} &\min\ \mathbf{b}^T\mathbf{y} \\ &\text{s.t. } A^T\mathbf{y} \succeq \mathbf{c} \\ &\mathbf{y} \succeq \mathbf{0} \end{aligned}$$

This is equivalent to

$$\begin{aligned} &\max\ -\mathbf{b}^T\mathbf{y} \\ &\text{s.t. } -A^T\mathbf{y} \preceq -\mathbf{c} \\ &\mathbf{y} \succeq \mathbf{0} \end{aligned}$$

Taking the dual of this, we have

$$\begin{aligned} &\min\ -\mathbf{c}^T\mathbf{x} \\ &\text{s.t. } -(A^T)^T\mathbf{x} \succeq -\mathbf{b} \\ &\mathbf{x} \succeq \mathbf{0} \end{aligned}$$

which is equivalent to the primal problem.

Exercise 8.18

Solving the primal problem...

$$\begin{array}{ccccccc} \zeta & = & & & x_1 & + & x_2 \\ \hline w_1 & = & 3 & - & 2x_1 & - & x_2 \\ w_2 & = & 5 & - & x_1 & - & 3x_2 \\ w_3 & = & 4 & - & 2x_1 & - & 3x_2 \end{array}$$

If we increase x_1, w_1 is binding.

$$\begin{array}{ccccccc} \zeta & = & \frac{3}{2} & - & \frac{1}{2}w_1 & + & \frac{1}{2}x_2 \\ \hline x_1 & = & \frac{3}{2} & - & \frac{1}{2}w_1 & - & \frac{1}{2}x_2 \\ w_2 & = & \frac{7}{2} & - & \frac{1}{2}w_1 & - & \frac{5}{2}x_2 \\ w_3 & = & 1 & + & w_1 & - & 2x_2 \end{array}$$

If we increase x_2, w_3 is binding.

$$
\begin{array}{rcrcrcr}
\zeta & = & \frac{7}{4} & - & \frac{1}{4}w_1 & - & \frac{1}{4}w_2 \\
\hline
x_1 & = & \frac{5}{4} & - & \frac{3}{2}w_1 & + & \frac{1}{4}w_3 \\
w_2 & = & \frac{9}{4} & - & \frac{3}{4}w_1 & + & \frac{5}{4}w_3 \\
x_2 & = & \frac{1}{2} & + & \frac{1}{2}w_1 & - & \frac{1}{2}w_3
\end{array}
$$

We can no longer increase the variables in the objective function, so the optimum value of the objective function is $\frac{7}{4}$ and the optimal point is $(\frac{5}{4}, \frac{1}{2})$.

The dual problem is

$$
\begin{aligned}
\min_{y_1, y_2, y_3} \; & 3y_1 + 5y_2 + 4y_3 \\
\text{s.t. } & 2y_1 + y_2 + 2y_3 \geq 1 \\
& y_1 + 3y_2 + 3y_3 \geq 1 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$