Problem Set 2

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Exercise 3.1

<u>Part i)</u>

We can write

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle + \|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\langle \mathbf{x},\mathbf{y}\rangle + \|\mathbf{y}\|^2 \tag{1}$$

as $\langle \mathbf{y},\mathbf{x}\rangle = \overline{\langle \mathbf{x},\mathbf{y}\rangle} = \langle \mathbf{x},\mathbf{y}\rangle$ in $\mathbb{R}^n$.

Using $\langle \mathbf{x},-\mathbf{y}\rangle = (-1)\langle \mathbf{x},\mathbf{y}\rangle$, we can write

$$\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - \langle \mathbf{x},\mathbf{y}\rangle - \langle \mathbf{y},\mathbf{x}\rangle + \|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\langle \mathbf{x},\mathbf{y}\rangle + \|\mathbf{y}\|^2 \tag{2}$$

Combining,

$$\frac{1}{4}(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2) = \frac{1}{4}(4\langle \mathbf{x},\mathbf{y}\rangle) = \langle \mathbf{x},\mathbf{y}\rangle$$

<u>Part ii)</u>

Combining equations 1 and 2, we find

$$\frac{1}{2}(\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2) = \frac{1}{2}(2\langle \mathbf{x},\mathbf{y}\rangle) = \langle \mathbf{x},\mathbf{y}\rangle$$

Exercise 3.3

<u>Part i)</u>

$\langle \mathbf{x},\mathbf{x}^5\rangle = \int_0^1 x^6 dx = \frac{x^7}{7}\Big|_0^1 = \frac{1}{7}$
$\langle \mathbf{x},\mathbf{x}\rangle = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} \implies \|\mathbf{x}\| = \sqrt{\frac{1}{3}}$
$\langle \mathbf{x}^5,\mathbf{x}^5\rangle = \int_0^1 x^{10} dx = \frac{x^{11}}{11}\Big|_0^1 = \frac{1}{11} \implies \|\mathbf{x}^5\| = \sqrt{\frac{1}{11}}$

$\cos(\theta) = \frac{\sqrt{3}\sqrt{11}}{7} = \frac{\sqrt{33}}{7} \implies \theta \approx 35°$

Part ii)

$\langle \mathbf{x}^2, \mathbf{x}^4 \rangle = \int_0^1 x^6 dx = \frac{x^7}{7}\Big|_0^1 = \frac{1}{7}$
$\langle \mathbf{x}^2, \mathbf{x}^2 \rangle = \int_0^1 x^4 dx = \frac{x^5}{5}\Big|_0^1 = \frac{1}{5} \implies \|\mathbf{x}^2\| = \sqrt{\frac{1}{5}}$
$\langle \mathbf{x}^4, \mathbf{x}^4 \rangle = \int_0^1 x^8 dx = \frac{x^9}{9}\Big|_0^1 = \frac{1}{9} \implies \|\mathbf{x}^4\| = \sqrt{\frac{1}{9}}$

$\cos(\theta) = \frac{\sqrt{9}\sqrt{5}}{7} = \frac{\sqrt{45}}{7} \implies \theta \approx 17°$

Exercise 3.8

i)

$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\sin(t)dt = 0$
$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\cos(2t)dt = 0$
$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\sin(2t)dt = 0$
$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2t)\sin(t)dt = 0$
$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\cos(t)dt = 1$
$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(t)\sin(t)dt = 1$
$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(2t)\sin(2t)dt = 1$
Therefore, S is an orthonormal set.

ii) $\|t\| = \sqrt{\langle t, t \rangle} = \int_{-\pi}^{\pi} t^2 dt = \frac{t^3}{3}\Big|_{-\pi}^{\pi} = \frac{2}{3}\pi^2$

iii) $\text{proj}_X(\cos(3t)) = \sum_{i=1}^{m} \langle \mathbf{x}_i, \cos(3t) \rangle \mathbf{x_i} = 0$

iv) $\text{proj}_X(t) = \sum_{i=1}^{m} \langle \mathbf{x}_i, t \rangle \mathbf{x_i} = 1$

Exercise 3.9

By Theorem 3.2.15, a matrix Q is orthonormal if and only if $Q^H Q = QQ^H = 1$.

The rotation matrix is given by

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Calculating $R_\theta R$, we find

$$\begin{aligned}
R_\theta R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}
\end{aligned}$$

So R_θ is an orthonormal transformation.

Exercise 3.10

<u>Part i)</u>

Assume Q is orthonormal, which implies $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle Q\mathbf{e}_i, Q\mathbf{e}_j \rangle$.

$$\begin{aligned}
\mathbf{e}_i^H \mathbf{e}_j &= \langle \mathbf{e}_i, \mathbf{e}_j \rangle \\
&= \langle Q\mathbf{e}_i, Q\mathbf{e}_j \rangle \\
&= (Q\mathbf{e}_i)^H (Q\mathbf{e}_j) \\
&= \mathbf{e}_i^H Q^H Q \mathbf{e}_j
\end{aligned}$$

$\mathbf{e}_i^H \mathbf{e}_j = \mathbf{e}_i^H Q^H Q \mathbf{e}_j$ only if $Q^H Q = I$.