Problem Set 1

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Exercise 1.3

$\mathcal{G}_1$ is not an algebra.

- Take $B \in \mathcal{G}_1$. Its complement, B^c, is closed, and therefore not in $\mathcal{G}_1$. Hence, $\mathcal{G}_1$ is not closed under complements.

$\mathcal{G}_2$ is an algebra.

$\mathcal{G}_3$ is a σ-algebra.

Exercise 1.7

$\mathcal{A}$ is a σ-algebra of X. $\{\emptyset, X\} \subset \mathcal{A} \subset \mathcal{P}(X)$.

Part 1:

- By definition, $\emptyset \in \mathcal{A}$.
- $\emptyset \in \mathcal{A}$ implies $\emptyset^c = X \in \mathcal{A}$.
- So $\{\emptyset, X\}$ is the smallest σ-algebra of X.

Part 2:

- Suppose $\mathcal{P}(X)$ is not the largest σ-algebra on X.
- Then there is $B \in X$ such that $\mathcal{P}(X) \cup B$ is a σ-algebra of X, where $B \notin \mathcal{P}(X)$.
- But $B \in X$, so $B \in \mathcal{P}(X)$. This is a contradiction, so $\mathcal{P}(X)$ must be the largest σ-algebra.

Exercise 1.10

$\{S_\alpha\}$ is a family of σ-algebras on X.

- As $\emptyset \in \mathcal{S}_\alpha$ for all α, $\emptyset \in \cap_\alpha \mathcal{S}_\alpha$.
- Given $A \in \cap_\alpha \mathcal{S}_\alpha$, $A \in \mathcal{S}_\alpha$ for all α and $A^c \in \mathcal{S}_\alpha$ for all α. So $A^c \in \cap_\alpha \mathcal{S}_\alpha$.
- Given $A_1, A_2, \cdots \in \cap_\alpha \mathcal{S}_\alpha$, $A_1, A_2, \cdots \in \mathcal{S}_\alpha$ for all α. So $\cup_{n=1}^\infty A_n \in \mathcal{S}_\alpha$ for all α. Hence, $\cup_{n=1}^\infty A_n \in \cap_\alpha \mathcal{S}_\alpha$.

Therefore, $\cap_\alpha \mathcal{S}_\alpha$ is a σ-algebra.

Exercise 1.17

- $A, B \in \mathcal{S}$, $A \subset B$. $A \cup (B \cap A^c) = B$. As A and $B \cap A^c$ are disjoint, $\mu(A) + \mu(B \cap A^c) = \mu(B)$. This implies $\mu(A) \leq \mu(B)$.
- Suppose we form a sequence

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 \cap A_1^c \\ B_3 &= A_3 \cap (A_2^c \cup A_3^c) \\ &\vdots \\ B_n &= A_n \setminus \bigcup_{i=1}^{\infty} A_i \end{aligned}$$

  B_i is a sequence of disjoint sets, so $\mu(\cup_{i=1}^\infty B_i) = \sum_{i=1}^\infty \mu(B_i)$.
- As $B_i \subset A_i\ \forall i$, by monotonicity, $\sum_{i=1}^\infty \mu(B_i) \leq \sum_{i=1}^\infty \mu(A_i)$.
- Therefore $\mu(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu(A_i)$

Exercise 1.18

- $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$.
- $\lambda(\cup_{i=1}^\infty A_i) = \mu((\cup_i^\infty A_i \cap B) = \mu(\cup_{i=1}^\infty (A_i \cap B))$
- As $\{A_i\}$ is collection of disjoint sets,

$$\mu(\cup_{i=1}^\infty (A_i \cap B)) = \sum_{i=1}^{\infty} \mu(A_i \cap B) = \sum_{i=1}^{\infty} \lambda(A_i)$$

  so $\lambda(A)$ is a measure on $(X, \mathcal{S})$.

Exercise 1.20

- $\cap_{i=1}^n A_i = A_n$
- $\mu(\cap_{i=1}^\infty A_i) = \lim_{n\to\infty} \mu(\cap_{i=1}^n A_i) = \lim_{n\to\infty} \mu(A_n)$

Exercise 2.10

- μ^* is an outer measure, so it satisfies $\mu^*(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu^*(A_i)$

- Considering $B \cap E$ and $B \cup E^c$, by countably subadditivity, $\mu^*((B \cap E) \cup (B \cap E^C)) = \mu^*(B) \leq \mu(B \cap E) + \mu(B \cap E^c)$
- Combining this with (*), we see that $\mu^*(B) = \mu(B \cap E) + \mu(B \cap E^c)$

Exercise 2.14

- An open set $(a, b) \subset \mathbb{R}$ can be expressed as
$$\bigcup_{n=N}^{\infty} \left(a, b - \frac{1}{n}\right]$$
so (a, b) is in $\sigma(\mathcal{A})$, as $\sigma(\mathcal{A})$ is closed under countable unions
- An open interval can be expressed as a disjoint union of open intervals, so any arbitrary open set O is also in $\sigma(\mathcal{A})$
- So $\sigma(\mathcal{O}) \subset \sigma(\mathcal{A}) \subset \mathcal{M}$, where $\sigma(\mathcal{O}) = \mathcal{B}(\mathbb{R})$

Exercise 3.1

- Consider a countable set $A \subset \mathbb{R}$, with $\{a_i\}_{i=1}^n \in A$
- Given $\epsilon < 0$, we can construct an open interval around a_i as
$$A_i = (a_i - \frac{\epsilon}{2^i}, a_i + \frac{\epsilon}{2^i})$$
- So $\mu(A_i) = \frac{\epsilon}{2^i}$ and $\mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i) = \epsilon$
- As $A \subset \cup_{i=1}^n A_i$ and ϵ is arbitrarily chosen, $\mu(A) = 0$

Exercise 3.4

$\{x \in X : f(x) \leq a\}$

- Consider $f(x) \in (-\infty, a)$
- As $\mathcal{M}$ is closed under complements, $(-\infty, a)^c = [a, \infty)$
- (a, ∞) can be written as $\bigcup_{n=N}^{\infty} [a - \frac{1}{n}, \infty)$, which is in $\mathcal{M}$
- $(a, \infty)^c$ is $(-\infty, a]$ which is in $\mathcal{M}$ given closedness under complements
- In the other direction, $(-\infty, a)$ can be expressed as
$$\bigcup_{n=N}^{\infty} \left(-\infty, a - \frac{1}{n}\right]$$

- As $\mathcal{M}$ is closed under countable unions, $f(x) \in (-\infty, a)$ is equivalent to $f(x) \in (-\infty, a]$

$\{x \in X : f(x) > a\}$

- Consider $f(x) \in (a, \infty)$
- As $\mathcal{M}$ is closed under complements, $(a, \infty)^c = (-\infty, a]$ is contained in $\mathcal{M}$
- So $f(x) \in (a, \infty)$ is equivalent to $f(x) \in (-\infty, a]$. As shown above, this is equivalent to $f(x) \in (-\infty, a)$

$\{x \in X : f(x) \geq a\}$

- Consider $f(x) \in [a, \infty)$
- As $\mathcal{M}$ is closed under complements, $[a, \infty)^c = (-\infty, a)$ is contained in $\mathcal{M}$
- So $f(x) \in (a, \infty)$ is equivalent to $f(x) \in (-\infty, a)$

Exercise 4.13

As $||f|| = f^+ + f^- < M$, then $f^+ <$ and $f^- < M$. So both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite, so $f \in \mathscr{L}^1(\mu, E)$