Problem Set 2

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Exercise 3.1

Part i)

We can write

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle + \|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2 \tag{1}$$

as $\langle \mathbf{y}, \mathbf{x}\rangle = \overline{\langle \mathbf{x}, \mathbf{y}\rangle} = \langle \mathbf{x}, \mathbf{y}\rangle$ in $\mathbb{R}^n$.

Using $\langle \mathbf{x}, -\mathbf{y}\rangle = (-1)\langle \mathbf{x}, \mathbf{y}\rangle$, we can write

$$\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - \langle \mathbf{x}, \mathbf{y}\rangle - \langle \mathbf{y}, \mathbf{x}\rangle + \|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2 \tag{2}$$

Combining,

$$\frac{1}{4}(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2) = \frac{1}{4}(4\langle \mathbf{x}, \mathbf{y}\rangle) = \langle \mathbf{x}, \mathbf{y}\rangle$$

Part ii)

Combining equations 1 and 2, we find

$$\frac{1}{2}(\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2) = \frac{1}{2}(2\langle \mathbf{x}, \mathbf{y}\rangle) = \langle \mathbf{x}, \mathbf{y}\rangle$$

Exercise 3.2

$$
\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &= \frac{1}{4}(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2 + i\|\mathbf{x}-i\mathbf{y}\|^2 - i\|\mathbf{x}+i\mathbf{y}\|^2) \\
&= \frac{1}{4}(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2) + \frac{1}{4}(i\|\mathbf{x}-i\mathbf{y}\|^2 - i\|\mathbf{x}+i\mathbf{y}\|^2) \\
&= \langle \mathbf{x}, \mathbf{y} \rangle + \frac{1}{4}(i\|\mathbf{x}-i\mathbf{y}\|^2 - i\|\mathbf{x}+i\mathbf{y}\|^2) \\
&= \langle \mathbf{x}, \mathbf{y} \rangle + \frac{1}{4}(i\langle \mathbf{x}-i\mathbf{y}, \mathbf{x}-i\mathbf{y} \rangle - i\langle \mathbf{x}+i\mathbf{y}, \mathbf{x}+i\mathbf{y} \rangle) \\
&= \langle \mathbf{x}, \mathbf{y} \rangle + \frac{1}{4}(i\langle \mathbf{x}, \mathbf{x}-i\mathbf{y} \rangle + i^2\langle \mathbf{y}, \mathbf{x}-i\mathbf{y} \rangle - i\langle \mathbf{x}, \mathbf{x}+i\mathbf{y} \rangle + (-i)^2\langle \mathbf{y}, \mathbf{x}+i\mathbf{y} \rangle) \\
&= \langle \mathbf{x}, \mathbf{y} \rangle + \frac{1}{4}(i\langle \mathbf{x}, \mathbf{x} \rangle + i^2\langle \mathbf{x}, \mathbf{y} \rangle + i^2\langle \mathbf{y}, \mathbf{x} \rangle + i^3\langle \mathbf{y}, \mathbf{y} \rangle \\
&\qquad - i\langle \mathbf{x}, \mathbf{x} \rangle + (-i)^2\langle \mathbf{x}, \mathbf{y} \rangle + (-i)^2\langle \mathbf{y}, \mathbf{x} \rangle + (-i^3)\langle \mathbf{y}, \mathbf{y} \rangle) \\
&= \langle \mathbf{x}, \mathbf{y} \rangle
\end{aligned}
$$

Exercise 3.3

Part i)

$\langle \mathbf{x}, \mathbf{x}^5 \rangle = \int_0^1 x^6 dx = \frac{x^7}{7}\Big|_0^1 = \frac{1}{7}$
$\langle \mathbf{x}, \mathbf{x} \rangle = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} \implies \|\mathbf{x}\| = \sqrt{\frac{1}{3}}$
$\langle \mathbf{x}^5, \mathbf{x}^5 \rangle = \int_0^1 x^{10} dx = \frac{x^{11}}{11}\Big|_0^1 = \frac{1}{11} \implies \|\mathbf{x}^5\| = \sqrt{\frac{1}{11}}$

$\cos(\theta) = \frac{\sqrt{3}\sqrt{11}}{7} = \frac{\sqrt{33}}{7} \implies \theta \approx 35°$

Part ii)

$\langle \mathbf{x}^2, \mathbf{x}^4 \rangle = \int_0^1 x^6 dx = \frac{x^7}{7}\Big|_0^1 = \frac{1}{7}$
$\langle \mathbf{x}^2, \mathbf{x}^2 \rangle = \int_0^1 x^4 dx = \frac{x^5}{5}\Big|_0^1 = \frac{1}{5} \implies \|\mathbf{x}^2\| = \sqrt{\frac{1}{5}}$
$\langle \mathbf{x}^4, \mathbf{x}^4 \rangle = \int_0^1 x^8 dx = \frac{x^9}{9}\Big|_0^1 = \frac{1}{9} \implies \|\mathbf{x}^4\| = \sqrt{\frac{1}{9}}$

$\cos(\theta) = \frac{\sqrt{9}\sqrt{5}}{7} = \frac{\sqrt{45}}{7} \implies \theta \approx 17°$

Exercise 3.8

i)

$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\sin(t)dt = 0$
$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\cos(2t)dt = 0$
$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\sin(2t)dt = 0$
$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(2t)\sin(t)dt = 0$
$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\cos(t)dt = 1$
$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(t)\sin(t)dt = 1$
$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(2t)\sin(2t)dt = 1$
Therefore, S is an orthonormal set.

ii) $\|t\| = \sqrt{\langle t, t\rangle} = \int_{-\pi}^{\pi} t^2 dt = \frac{t^3}{3}\Big|_{-\pi}^{\pi} = \frac{2}{3}\pi^2$

iii) $\text{proj}_X(\cos(3t)) = \sum_{i=1}^{m}\langle \mathbf{x}_i, \cos(3t)\rangle \mathbf{x_i} = 0$

iv) $\text{proj}_X(t) = \sum_{i=1}^{m}\langle \mathbf{x}_i, t\rangle \mathbf{x_i} = 1$

Exercise 3.9

By Theorem 3.2.15, a matrix Q is orthonormal if and only if $Q^H Q = QQ^H = 1$.

The rotation matrix is given by

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Calculating $R_\theta R$, we find

$$\begin{aligned} R_\theta R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I} \end{aligned}$$

So R_θ is an orthonormal transformation.

Exercise 3.10

Part i)

Assume Q is orthonormal, which implies $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle Q\mathbf{e}_i, Q\mathbf{e}_j \rangle$.

$$\begin{aligned}
\mathbf{e}_i^H \mathbf{e}_j &= \langle \mathbf{e}_i, \mathbf{e}_j \rangle \\
&= \langle Q\mathbf{e}_i, Q\mathbf{e}_j \rangle \\
&= (Q\mathbf{e}_i)^H (Q\mathbf{e}_j) \\
&= \mathbf{e}_i^H Q^H Q \mathbf{e}_j
\end{aligned}$$

$\mathbf{e}_i^H \mathbf{e}_j = \mathbf{e}_i^H Q^H Q \mathbf{e}_j$ only if $Q^H Q = I$.

Part ii)

$$\begin{aligned}
\|Q\mathbf{x}\| &= \sqrt{\langle Q\mathbf{x}, Q\mathbf{x} \rangle} \\
&= \sqrt{\mathbf{x}^H Q^H Q \mathbf{x}} \\
&= \sqrt{\mathbf{x}^H \mathbf{x}} \qquad \text{as } Q^H Q = 1 \\
&= \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \\
&= \|\mathbf{x}\|
\end{aligned}$$

Part iii)

$Q^H Q = I$ so $Q^H = Q^{-1}$.

$$\begin{aligned}
\langle Q^{-1}\mathbf{x}, Q^{-1}\mathbf{x} \rangle &= \mathbf{x}^H (Q^{-1})^H Q^{-1} \mathbf{x} \\
&= \mathbf{x}^H (Q^H)^H Q^H \mathbf{x} \qquad \text{as } Q^H = Q^{-1} \\
&= \mathbf{x}^H Q Q^H \mathbf{x} \\
&= \mathbf{x}^H \mathbf{x} \qquad \text{as } Q Q^H = 1 \\
&= \langle \mathbf{x}, \mathbf{x} \rangle
\end{aligned}$$

So Q^{-1} is orthonormal.

Part iv)

$Q_{n,m}$ is an orthonormal matrix. We can write Q as

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_m \end{bmatrix}$$

Where $\mathbf{q}_i$ is a $(n \times 1)$ column of Q, and $\{\mathbf{q}_i\}_{i=1}^m$ is a set of columns vectors.

As Q is orthonormal, we know $Q^H Q = I_m$. So we can write

$$\begin{aligned}
Q^H Q &= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_m \end{bmatrix}^H \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_m \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{q}_1^H \\ \mathbf{q}_2^H \\ \vdots \\ \mathbf{q}_m^H \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_m \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{q}_1^H \mathbf{q}_1 & \mathbf{q}_2^H \mathbf{q}_1 & \cdots & \mathbf{q}_m^H \mathbf{q}_1 \\ \mathbf{q}_1^H \mathbf{q}_2 & \mathbf{q}_2^H \mathbf{q}_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \\ \mathbf{q}_1^H \mathbf{q}_2 & \cdots & & \mathbf{q}_m^H \mathbf{q}_m \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix}
\end{aligned}$$

which satisfies the definition of an orthonormal set, ie. $\langle \mathbf{q}_i, \mathbf{q}_j \rangle = \delta_{i,j}$ where

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Part v)

It is not true that $|\det(Q)| = 1$ implies Q is orthonormal. For example

$$Q = \begin{bmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

has $|\det(Q)| = 1$ but

$$Q^H Q = \begin{bmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix} \begin{bmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2\sqrt{3} - 3\sqrt{2} \\ 3\sqrt{2} - 2\sqrt{3} & -1 \end{bmatrix} \neq I$$

So Q is not orthonormal.

Part vi)

Q_1 and Q_2 are orthonormal, so $Q_1^H Q_1 = I$ and $Q_2^H Q_2 = I$ Given $Q = Q_1 Q_2$

$$\begin{aligned} Q^H Q &= (Q_1 Q_2)^H Q_1 Q_2 \\ &= Q_2^H Q_1^H Q_1 Q_2 \\ &= Q_2^H Q_2 \\ &= I \end{aligned}$$

Q is therefore orthonormal.

Exercise 3.23

By the triangle inequality,

$$\|\mathbf{x}\| = \|(\mathbf{x} - \mathbf{y}) + \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\| \Leftrightarrow \|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

By scale preservation,

$$\|\mathbf{y}\| = \|-1\| \cdot \|\mathbf{y}\| = \|-\mathbf{y}\| = \|\mathbf{x} - \mathbf{y} + \mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\| \Leftrightarrow \|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

Exercise 3.47

$P = A(A^H A)^{-1} A^H$.

i. $P^2 = A(A^H A)^{-1} A^H A (A^H A)^{-1} A^H$

ii. $P^H = (A(A^H A)^{-1} A^H)^H = A((A^H A)^{-1})^H A^H = A(A^H A)^H)^{-1} A^H = A(A^H A)^{-1} A^H = P$

iii. P is an idempotent matrix by i), which means $rank(P) = tr(P)$.

$$tr(P) = tr(A(A^H A)^{-1} A^H) = tr(A^H A (A^H A)^{-1}) = tr(I_n) = n = rank(P)$$