Finite-time Reachability of a Class of PWA Systems over a Class of Polyhedra

Dieky Adzkiya Alessandro Abate

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1 Max-Plus-Linear Systems

As a case study, we would like to leverage the max-plus model developed in [1, Fig. 2.6]. By renumbering the states, we obtain a system given by:

$$\begin{array}{rcl} x_1(k) & = & 38 + x_8(k-1) \\ x_2(k) & = & 36 + x_8(k-1) \\ x_3(k) & = & \max\{55 + x_1(k-1), 54 + x_2(k-1)\} \\ x_4(k) & = & 35 + x_3(k-1) \\ x_5(k) & = & 54 + x_4(k-1) \\ x_6(k) & = & 58 + x_4(k-1) \\ x_7(k) & = & \max\{90 + x_5(k-1), 93 + x_6(k-1)\} \\ x_8(k) & = & 16 + x_7(k-1) \end{array} \tag{1}$$

where $x_i(k)$, $i = 1, \ldots, 8$, represent the k-th departure time of trains at the stations Den Haag CS to Amersfoort (via Utrecht), Rotterdam CS to Amersfoort (via Utrecht), Amersfoort to Zwolle, Zwolle to Leeuwarden and to Groningen, Leeuwarden to Amersfoort (via Zwolle), Groningen to Amersfoort (via Zwolle), Amersfoort to Utrecht, Utrecht to Den Haag CS and to Rotterdam CS, respectively.

2 Piecewise Affine Systems

The max-plus-linear system (1) can be transformed to a Piecewise Affine (PWA) System. The PWA system has 4 regions:

- Region 1. The region is given by $\{x : 55+x_1 \geq 54+x_2, 90+x_5 \geq 93+x_6\}$.

The dynamics is given by:

$$
\begin{array}{rcl}
x_1(k) & = & 38 + x_8(k-1) \\
x_2(k) & = & 36 + x_8(k-1) \\
x_3(k) & = & 55 + x_1(k-1) \\
x_4(k) & = & 35 + x_3(k-1) \\
x_5(k) & = & 54 + x_4(k-1) \\
x_6(k) & = & 58 + x_4(k-1) \\
x_7(k) & = & 90 + x_5(k-1) \\
x_8(k) & = & 16 + x_7(k-1)
\end{array}
\tag{2}
$$

- Region 2. The region is given by $\{x : 55+x_1 \leq 54+x_2, 90+x_5 \geq 93+x_6\}$. The dynamics is given by:

$$
\begin{array}{rcl}
x_1(k) & = & 38 + x_8(k-1) \\
x_2(k) & = & 36 + x_8(k-1) \\
x_3(k) & = & 54 + x_2(k-1) \\
x_4(k) & = & 35 + x_3(k-1) \\
x_5(k) & = & 54 + x_4(k-1) \\
x_6(k) & = & 58 + x_4(k-1) \\
x_7(k) & = & 90 + x_5(k-1) \\
x_8(k) & = & 16 + x_7(k-1)
\end{array}
\tag{3}
$$

- Region 3. The region is given by $\{x : 55+x_1 \geq 54+x_2, 90+x_5 \leq 93+x_6\}$. The dynamics is given by:

$$
\begin{array}{rcl}
x_1(k) & = & 38 + x_8(k-1) \\
x_2(k) & = & 36 + x_8(k-1) \\
x_3(k) & = & 55 + x_1(k-1) \\
x_4(k) & = & 35 + x_3(k-1) \\
x_5(k) & = & 54 + x_4(k-1) \\
x_6(k) & = & 58 + x_4(k-1) \\
x_7(k) & = & 93 + x_6(k-1) \\
x_8(k) & = & 16 + x_7(k-1)
\end{array}
\tag{4}
$$

- Region 4. The region is given by $\{x : 55+x_1 \leq 54+x_2, 90+x_5 \leq 93+x_6\}$. The dynamics is given by:

$$
\begin{array}{rcl}
x_1(k) & = & 38 + x_8(k-1) \\
x_2(k) & = & 36 + x_8(k-1) \\
x_3(k) & = & 54 + x_2(k-1) \\
x_4(k) & = & 35 + x_3(k-1) \\
x_5(k) & = & 54 + x_4(k-1) \\
x_6(k) & = & 58 + x_4(k-1) \\
x_7(k) & = & 93 + x_6(k-1) \\
x_8(k) & = & 16 + x_7(k-1)
\end{array}
\tag{5}
$$

We would like to consider the following problem. Given a PWA system, a time horizon N, a set of initial conditions X_0, a set S, we wanted to know whether the system can reach S within the given time horizon.

Let us give an example. The initial condition is $X_0 = \{x : 0 \leq x_i \leq 1, \text{ for all } i = 1, \dots, 8\}$. This means the time of first departure in all stations is between current time and 1 time unit later. Suppose that $N = 10$ and the unsafe set is $S = \{x : x_1 > 100\}$. This means that we do not want the time of the next 10 departures to be more than 100 time units from now.

References

[1] Subiono. *On classes of min-max-plus systems and their applications.* PhD thesis, Delft University of Technology, 2000.