Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science

# Assignment 3 for Block III
# of the course “Algorithms for NP-Hard Problems.”

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# Contents

**!** Difficulty of the problem is indicated by ☕. Each ☕ equals 1 point. You gain a point for each sub-task of the problem. Collect at least 10 points. Where specified, provide analytical and numerical solution, and illustration.

# 1 Bernoulli generator ☕

- Based on the Bernoulli generator scheme, construct a generator for binomial distribution and negative binomial distribution. Define them analytically and illustrate the distributions for different values of parameters (e.g., probability, number of trials).
- On a sample of 5000 check empirically the law of large numbers for frequencies. Provide illustration.
- Consider the game of flipping a coin, an endless sequence of independent trials with the flip of the fair coin. Let us define by $X_1, X_2, \ldots$ a sequence of independent identically distributed random variables, each of which takes on values 1 if in the corresponding trial the tail is drawn, and $-1$ otherwise (with probability $= 1/2$). Denote the total gain by $S_n = X_1 + X_2 + \ldots + X_n$. Perform N = 1000 trials and plot the trajectory of the process $X(t)$, $t \in [0, 1]$, which equals $X(n/N) = \frac{1}{\sqrt{N}} S_n$ at points $n/N$, where $n = 0, 1, \ldots, N$, and in the other cases is determined by piecewise linear interpolation (i.e., in the form of a broken line).

## 1.1 Analytical solution

Provide an analytical solution and theoretical background.

## 1.2 Illustration

Implement the solution and provide plots illustrating implementation for different values of parameters.

# 2 Symmetry property ☕☕☕☕

- Implement a Bernoulli scheme generator with a given probability of of success $p$.
- Construct a generator for a singular distribution that corresponds to the Cantor distribution function ("staircase" function).
- Verify the symmetry property with respect to 1/2 ($X$ is equally distributed with $1 - X$), and the property of self-similarity (the conditional distribution of $Y$ under the condition that $Y$ belongs to the interval $[0, 1/3]$ coincides with the distribution $Y/3$).
- Using the Cantor generator, check empirically the distribution function (on a sample volume of 1000).

*See Appendix for hints.*

### 2.1 Analytical solution

Provide an analytical solution and theoretical background.

### 2.2 Illustration

Implement the solution and provide plots illustrating implementation for different values of parameters.

## 3 From one distribution to another

- Construct an exponential distribution generator. Use it to verify the Poisson distribution generator.
- Build a standard normal distribution generator by simulating random variables in pairs with transition to polar coordinates.
- Construct a $\chi^2$ distribution generator with $n$ degrees of freedom.

*See Appendix for hints.*

### 3.1 Analytical solution

Provide an analytical solution and theoretical background.

### 3.2 Illustration

Implement the solution and provide plots illustrating implementation for different values of parameters.

## 4 LLN and CLT

Let $X(i) \sim N(a, b^2)$. Verify empirically the law of large numbers and the central limit theorem, i.e., investigate the behavior of the sum $S(n)/n$ and the empirical distributions of $\sqrt{n}(S(n)/n - a)$.

### 4.1 Analytical solution

Provide an analytical solution and theoretical background.

### 4.2 Illustration

Implement the solution and provide plots illustrating implementation for different values of parameters.

## 5 Integral

Calculate the integral

$$I = \int_0^{\infty} \dots \int_0^{\infty} \exp\left\{ -\left( x_1 + \dots + x_{10} + \frac{1}{2^{20} x_1 \dots x_{10}} \right) \right\} x_1^{\frac{1}{11}-1} x_2^{\frac{2}{11}-1} \dots x_{10}^{\frac{10}{11}-1} dx_1 \dots dx_{10}$$

a) by Monte Carlo method; b) by the quadrature method (reducing the problem to the calculation of the Riemann integral) evaluating the accuracy of calculations.

*Hint:* try representing the integrand as a sum of random variables following the same well-known distribution. Make use of LLN and CLT.

### 5.1 Analytical and numerical solution

Provide an analytical solution and theoretical background. Implement the solution and present numerical results.

## 6 Unit Simplex

Sample from $X = \{(x_1, x_2, x_3) | 0 \leq x_i \leq 1, x_1 + x_2 + x_3 = 1\}$ a) in a way that most of the samples are concentrated in the center and b) uniformly. Visualize in two dimensions.

*Hint:* for uniform sampling, use approach from the lectures or your own.

### 6.1 Analytical solution

Provide an analytical solution and theoretical background.

### 6.2 Illustration

Implement the solution and provide plots illustrating implementation for different values of parameters.

## References

[Buslenko et al.(1966)] Nikolai Panteleimonovich Buslenko et al. *The Monte Carlo method: the method of statistical trials*, volume 87. Pergamon Press, 1966.

[Shiryaev(2016)] Albert N Shiryaev. *Probability-1*, volume 95. Springer, 2016.

[Shiryaev(2019)] Albert N Shiryaev. *Probability-2*, volume 95. Springer, 2019.

[Shreider(2014)] Yu A Shreider. *The Monte Carlo method: the method of statistical trials*, volume 87. Elsevier, 2014.

[Sobol(2018)] Ilya M Sobol. *A primer for the Monte Carlo method.* CRC press, 2018.

[Sobol(1974)] IM Sobol. The monte carlo method. popular lectures in mathematics. 1974.

## Appendix

**Problem 2.** Page 190 in Shiryaev ”Probability-1”.

**Problem 3.** Page 189-190 in Shiryaev ”Probability-1”. Page 315 in Buslenko ”Monte-Carlo Method”. For standard normal distribution, proof the following: Let $r, \varphi$ be such that : $r^2 \sim Exp(1/2)$ and $\varphi \sim U[0, 2\pi)$. Then $X = r \sin \phi$ and $Y = r \cos \phi$ are normally distributed. Chi-distribution can be expressed as a sum of normally distributed variables. Use Box-Muller method to generate them.

**Problem 6.** Distribution on a unit simplex is also known as Dirichlet distribution.