Problem 3.1

Hypothesis $\{s_n\}$ is a sequence in $\mathbb{R}$.

Claim $\{s_n\}$ converges $\implies \{|s_n|\}$ converges

Proof Assume $\{s_n\}$ converges to $p \in \mathbb{R}$. By the definition of convergence, we have

$$\forall \epsilon > 0 \ \exists N : n > N \implies |p - s_n| < \epsilon$$

For all $x, y \in \mathbb{R}$, we have $||x| - |y|| \leq |x - y|$. So for all n, we have $|p - s_n| < \epsilon \implies ||p| - |s_n|| < \epsilon$. Therefore,

$$\forall \epsilon > 0 \ \exists N : n > N \implies ||p| - |s_n|| < \epsilon$$

So $\{|s_n|\}$ converges to $|p|$. □

Claim $\{|s_n|\}$ converges $\not\Longrightarrow \{s_n\}$ converges

Proof Counterexample: $s_n = (-1)^n$. □

Problem 3.2

Hypothesis $\{p_n\}$ is a sequence in $\mathbb{R}$ such that $p_n = \sqrt{n^2 + n} - n$

Claim $\{p_n\}$ converges to $\frac{1}{2}$.

Proof First, I will show that $\{p_n\}$ is bounded. For all n, we have $\sqrt{n^2 + n} > \sqrt{n^2} = n$, so $\sqrt{n^2 + n} - n > 0$. Seeking a contradiction, assume that $\frac{1}{2}$ is not an upper bound for $\{p_n\}$. For some n, we have

$$\begin{aligned} \sqrt{n^2 + n} - n &> \frac{1}{2} \\ \sqrt{n^2 + n} &> \frac{1}{2} + n \\ n^2 + n &> \frac{1}{4} + n + n^2 \\ 0 &> \frac{1}{4} \end{aligned}$$

We have reached a contradiction, therefore $\frac{1}{2}$ must be an upper bound for $\{p_n\}$.

Todo Now I will show that $\{p_n\}$ is strictly increasing.

$\{p_n\}$ is monotonic and bounded, so by theorem 3.14 we can conclude that it converges. In the proof of this theorem, we see that $\{p_n\}$ must converge to its least upper bound.

Todo I will now show that $\{p_n\}$ has a least upper bound of $\frac{1}{2}$.

□

Problem 3.3

Hypothesis Define $\{s_n\}$ by $s_1 = \sqrt{2}$ and $s_n = \sqrt{2 + \sqrt{s_{n-1}}}$ for $n = 2, 3, \ldots$

Claim $(\forall n) \ s_n < 2$

Proof I will prove this by induction on n.

Base case: by definition, $s_1 < 2$.

Induction hypothesis: Assume $s_{n-1} < 2$.

We have $s_n = \sqrt{2 + \sqrt{s_{n-1}}} < \sqrt{2 + \sqrt{2}} < \sqrt{4} = 2$. $\square$

Claim $\{s_n\}$ converges.

Proof Clearly, $(\forall n)\ s_n \geq 0$. Combining this with the previous result, we see $\{s_n\}$ is bounded.

Todo Now I will show that $\{s_n\}$ is strictly increasing.

Invoking theorem 3.14, we can see that $\{s_n\}$ converges. $\square$

Problem 3.4

Hypothesis Let $\{s_n\}$ be defined by $s_1 = 0$, $s_{2m} = \frac{s_{2m-1}}{2}$, $s_{2m+1} = \frac{1}{2} + s_{2m}$ for $m = 1, 2, \ldots$

Claim $s_{2m+1} = 1 - \frac{1}{2^m}$ for $m = 0, 1, \ldots$

Proof I will prove this by induction on m.

Base case: For $m = 0$, we have $s_1 = 1 = 1 - \frac{1}{2^0}$.

Inductive hypothesis: Assume $s_{2m+1} = 1 - \frac{1}{2^m}$. Then we have $s_{2m+2} = \frac{s_{2m+1}}{2} = \frac{1}{2} - \frac{1}{2^{m+1}}$. And so $s_{2(m+1)+1} = s_{2m+3} = \frac{1}{2} + s_{2m+2} = 1 - \frac{1}{2^{m+1}}$.

Note that $\{s_{2m+1}\}$ is strictly increasing and bounded above by 1. $\square$

Claim $s_{2m} = \frac{1}{2} - \frac{1}{2^m}$ for $m = 1, 2, \ldots$

Proof By the definition of s_{2m+1} we have

$$\begin{aligned} s_{2m} &= s_{2m+1} - \frac{1}{2} \\ &= 1 - \frac{1}{2^m} - \frac{1}{2} \\ &= \frac{1}{2} - \frac{1}{2^m} \end{aligned}$$

$\square$

Claim $\limsup_{n\to\infty} s_n = 1$ and $\liminf_{n\to\infty} s_n = \frac{1}{2}$

Proof Let E consist of all the subsequential limits of $\{s_n\}$. I will show that $E = \{\frac{1}{2}, 1\}$.

Let $\{s_{n_k}\}$ be a subequence of $\{s_n\}$ indexed by k. Let $I = \{n_k : k \in \mathbb{N}\}$ be the set of indices. There are three cases:

Case 1 I contains infinitely many odd numbers and finitely many even numbers.

Note that there exists K such that $k \geq K \implies n_k$ is odd. Consider an arbitrary $\epsilon > 0$. Let z be the smallest integer that has these two properties:

1. $2z + 1 \geq K$

2. $1 - \epsilon < 1 - \frac{1}{2^z} = s_{2z+1}$

From the first property, we have $n_k \geq 2z + 1 \implies n_k$ is odd. Taking into account the second property and the fact that $\{s_{2m+1}\}$ is increasing, we have $n_k \geq 2z + 1 \implies 1 - \epsilon < s_{n_k}$. Finally, we note that $\{s_{2m+1}\}$ is bounded by 1, and conclude $n_k \geq 2z + 1 \implies 1 - \epsilon < s_{n_k} < 1$. Let K' be the smallest element of I that is greater than or equal to $2z+1$. We have shown:

$$\forall \epsilon > 0 \;\exists K' : k \geq K' \implies |1 - s_{n_k}| < \epsilon$$

Therefore, $\{s_{n_k}\}$ converges to 1. $*$

Case 2 I contains finitely many odd numbers and infinitely many even numbers.
Note that there exists K such that $k \geq K \implies n_k$ is even. Consider an arbitrary $\epsilon > 0$. Let z be the smallest integer that has these two properties:

1. $2z \geq K$

2. $\frac{1}{2} - \epsilon < \frac{1}{2} - \frac{1}{2^z} = s_{2z}$

From the first property, we have $n_k \geq 2z \implies n_k$ is even. Taking into account the second property and the fact that $\{s_{2m}\}$ is increasing, we have $n_k \geq 2z \implies \frac{1}{2} - \epsilon < s_{n_k}$. Finally, we note that $\{s_{2m}\}$ is bounded by $\frac{1}{2}$, and conclude $n_k \geq 2z+1 \implies \frac{1}{2} - \epsilon < s_{n_k} < \frac{1}{2}$. Let K' be the smallest element of I that is greater than or equal to $2z$. We have shown:

$$\forall \epsilon > 0 \;\exists K' : k \geq K' \implies \left|\frac{1}{2} - s_{n_k}\right| < \epsilon$$

Therefore, $\{s_{n_k}\}$ converges to $\frac{1}{2}$. $*$

Case 3 I contains infinitely many odd numbers and infinitely many even numbers.
As shown above, the subsequence of $\{s_{n_k}\}$ consisting of all odd indices converges to 1. The subsequence consisting of all even indices converges to $\frac{1}{2}$. A sequence converges to p if and only if every subsequence converges to p, so since $\{s_{n_k}\}$ has two subsequences that converge to different points, $\{s_{n_k}\}$ does not converge. $*$

I have proved $E = \{\frac{1}{2}, 1\}$. Therefore, the upper limit of $\{s_n\}$ is 1, and the lower limit is $\frac{1}{2}$. $\square$

Problem 4.1

Hypothesis Let $f : \mathbb{R} \to \mathbb{R}$. For $x \in \mathbb{R}$, let $g_x : \mathbb{R}_{>0} \to \mathbb{R} : h \to f(x+h) - f(x-h)$. For every x, g_x has the property that that $\lim_{h\to 0} g_x(h) = 0$.

Claim f is continuous.

Proof First, I'll expand the information above using the definition of limit:

$$\forall x \in \mathbb{R}, \epsilon > 0 \;\exists \delta > 0 : \forall h \in (0, \delta) \;\; |g_x(h)| < \epsilon$$

Now, substituting in the definition of g_x:

$$\forall x \in \mathbb{R}, \epsilon > 0 \;\exists \delta > 0 : \forall h \in (0, \delta) \;\; |f(x+h) - f(x-h)| < \epsilon \tag{1}$$

We want to show:

$$\forall x \in \mathbb{R}, \epsilon > 0 \;\exists \delta > 0 : \forall h \in \mathbb{R} \;\; |x - h| < \delta \implies |f(x) - f(h)| < \epsilon \tag{2}$$

Fix an $x \in \mathbb{R}$ and $\epsilon > 0$. Let $h \in \mathbb{R}$.

Let $x' = (x+h)/2$. Let $h' = |x' - x|$. So we have $\{x, h\} = \{x' + h', x' - h'\}$. By statement 1, for x' and ϵ, there exists a δ with the property that $\forall h'' \in (0, \delta)$, $|f(x' + h'') - f(x' - h'')| < \epsilon$. So $h' < \delta \implies |f(x' + h') - f(x' - h')| = |f(x) - f(h)| < \epsilon$. Note that $h' = |x - h|/2$, so $h' < \delta \iff |x - h| < 2\delta$. So we have $|x - h| < 2\delta \implies |f(x) - f(h)| < \epsilon$, which proves statement 2. Therefore, f is continuous.

Problem 4.12

Hypothesis $f : X \to Y$ and $g : f(X) \to Z$ are uniformly continuous.

Claim $g \circ f$ is uniformly continuous.

Proof Let $\epsilon_1 > 0$. We have

$$\exists \epsilon_2 > 0 : \forall y_1, y_2 \in f(X) \;\; d_Y(y_1, y_2) < \epsilon_2 \implies d_Z(g(y_1), g(y_2)) < \epsilon_1$$

Further, we have

$$\exists \epsilon_3 > 0 : \forall x_1, x_2 \in X \;\; d_X(x_1, x_2) < \epsilon_3 \implies d_Y(f(x_1), f(x_2)) < \epsilon_2$$

Putting these together, we get

$$\forall x_1, x_2 \in X \;\; d_X(x_1, x_2) < \epsilon_3 \implies d_Y(f(x_1), f(x_2)) < \epsilon_2 \implies d_Z((g \circ f)(x_1), (g \circ f)(x_2)) < \epsilon_1$$

This proves that $g \circ f$ is uniformly continuous.

Problem 4.13

Hypothesis Let E be a dense subset of metric space X, and let f be a uniformly continuous real function on E.

Claim f has a continuous extension from E to X.

Proof For every $p \in X$, either $p \in E$ or p is a limit point of E. In the latter case, there must exist a sequence $\{e_n\}$ such that each $e_n \in E$ and it converges to p. We now define the function $g : X \to \mathbb{R}$ as follows:

$$g(p) = \begin{cases} f(p) & p \in E \\ \lim_{n\to\infty} f(e_n) & p \notin E \end{cases}$$

where $\{e_n\}$ is a sequence with the above properties. To show g is well-defined, we will show that for any two sequences $\{p_n\}$ and $\{q_n\}$ that converge to p, $\{f(p_n)\}$ and $\{f(q_n)\}$ converge to the same point in $\mathbb{R}$.

From the definition of convergence, we have

$$\forall \delta > 0 \;\; \exists N : n > N \implies d(p, p_n) < \delta/2$$

$$\forall \delta > 0 \;\; \exists N : n > N \implies d(p, q_n) < \delta/2$$

Using the triangle inequality, we have

$$\forall \delta > 0 \ \exists N : n > N \implies d(p_n, q_n) \le d(p_n, p) + d(p, q_n) < \delta$$

From the definition of continuity, we have

$$\forall \epsilon > 0 \ \exists \delta > 0 : d(p_n, q_n) < \delta \implies |f(p_n) - f(q_n)| < \epsilon$$

Combining the previous two equations, we get

$$\forall \epsilon > 0 \ \exists N : n > N \implies |f(p_n) - f(q_n)| < \epsilon$$

Therefore $\{f(p_n)\}$ and $\{f(q_n)\}$ converge to the same point in $\mathbb{R}$, so g is well-defined.

Now we will show that g is continuous. We need to show the following:

$$\forall p \in X, \epsilon > 0 \ \exists \delta > 0 : \forall q \in X \ \ d(p, q) < \delta \implies |g(p) - g(q)| < \epsilon \tag{3}$$

Fix an arbitrary $p \in X$ and $\epsilon > 0$. Let $\{p_n\}$ be a sequence in E that converges to p (we know this exists because if $p \in E$, we can use $p_n = p$, and if $p \notin E$, then p is a limit point of E, so there must exist such a sequence). Because f is uniformly continuous and $\forall p \in E, f(p) = g(p)$ we can define $\delta > 0$ such that

$$\forall x, y \in E \ \ d(x, y) < \delta \implies |g(x) - g(y)| < \epsilon/2 \tag{4}$$

Let $q \in X$ such that $d(p, q) < \delta/2$. Let $\{p_n\}$ be a sequence in E that converges to q. We have

$$\exists N : n > N \implies d(p, p_n) < \delta/4 \wedge d(q, q_n) < \delta/4$$

Using the triangle inequality, we get

$$\exists N : n > N \implies d(p_n, q_n) \le d(p_n, p) + d(p, q) + d(q, q_n) < \delta/4 + \delta/2 + \delta/4 = \delta \tag{5}$$

Combining (5) with (4), we get

$$\exists N : n > N \implies |g(p_n) - g(q_n)| < \epsilon/2 \tag{6}$$

Note that by the definition of g, $\{g(p_n)\}$ converges to $g(p)$ and $\{g(q_n)\}$ converges to $g(q)$. So we have

$$\exists N : n > N \implies |g(p) - g(p_n)| < \epsilon/4 \wedge |g(q) - g(q_n)| < \epsilon/4 \tag{7}$$

Let N be an integer with the properties of both (6) and (7). Let $n > N$.

$$\begin{aligned} |g(p) - g(q)| &\le |g(p) - g(p_n)| + |g(p_n) - g(q_n)| + |g(q_n) - g(q)| \\ &< \epsilon/4 + \epsilon/2 + \epsilon/4 \\ &= \epsilon \end{aligned}$$

I have showed that $\forall q \in X, d(p, q) < \delta/2 \implies |g(p) - g(q)| < \epsilon$. This proves (3).

Problem 4.15

Definition A function $f : X \to Y$ is *open* if for every subset V of X, V is open $\implies f(V)$ is open.

Hypothesis $f : \mathbb{R} \to \mathbb{R}$ is continuous and open.

Claim f is monotonic.

Proof

Definition A function $f : \mathbb{R} \to \mathbb{R}$ has a *semi maximum* at $x \in \mathbb{R}$ if $\exists\, y < x : f(y) < f(x)$ and $\exists\, y > x : f(y) < f(x)$. *Semi minimum* is defined similarly.

Seeking a contradiction, assume f has a semi maximum at some point x. There exists $y_1 < x$ such that $f(y_1) < f(x)$ and $y_2 > x$ such that $f(y_2) < f(x)$. f is continuous on $[y_1, y_2]$, therefore it takes a maximum value at some $z \in [y_1, y_2]$. The max cannot be located at y_1 or y_2 by definition, therefore we have $z \in (y_1, y_2)$ such that $(\forall w \in (y_1, y_2))\ f(w) \leq f(z)$. So we have $f(z) \in f((y_1, y_2))$, but $f(z)$ is not an interior point of $f((y_1, y_2))$, which means that $f((y_1, y_2))$ is not open. This contradicts our assumption that f is open. Therefore, f does not have a semi maximum. By a similar argument, f does not have a semi minimum.

Todo Prove that $(\exists\, a, b \in \mathbb{R} : a < b \wedge f(a) < f(b)) \implies f$ is monotonically increasing. This will make the cases below much easier.

Now choose an $x \in \mathbb{R}$. We consider two cases:

Case 1 $\exists\, y < x : f(y) < f(x)$

Seeking a contradiction, assume $\exists\, z < x : f(z) > f(x)$. If $z < y$, then f would have a semi minimum at y. If $y < z$, then f would have a semi maximum at z. Therefore, no such z can exist. We conclude that $(\forall z < x)\ f(z) \leq f(x)$.

We have shown that $(\forall x, z \in \mathbb{R})\ z < x \implies f(z) \leq f(x)$. f is monotonically increasing. $*$

Case 2 $\forall y < x \ \ f(y) \geq f(x)$

If $\exists\, y < x : f(y) > f(x)$, we can use a similar argument as in the first case to show that $(\forall z < x)\ f(z) \geq f(x)$.

So we have $(\forall x, z \in \mathbb{R})\ z < x \implies f(z) \geq f(x)$, so f is monotonically decreasing.

On the other hand, if $\forall y < x \ \ f(y) = f(x)$, then we consider 3 cases.

We conclude that f is monotonically decreasing. $*$

$\square$

Problem 4.20

Hypothesis E is a nonempty subset of a metric space X. $\rho_E : X \to \mathbb{R} : x \to \inf_{z \in E} d(x, z)$.

Claim (a) $\rho_E(x) = 0 \iff x \in \overline{E}$

Proof Consider $x \in \overline{E}$. We have $\forall \epsilon > 0$, $\exists z \in E : d(x, z) < \epsilon$. Therefore, $\inf_{z \in E} d(x, z) = 0$.

Now consider $x \notin \overline{E}$. This means $x \in \overline{E}^C$, which is open. So we have $\exists \epsilon > 0 : d(x, z) < \epsilon \implies z \notin \overline{E}$. Using the contrapositive: $\exists \epsilon > 0 : z \in \overline{E} \implies d(x, z) \geq \epsilon$. So $\inf_{z \in E} d(x, z) \geq \epsilon > 0$.

Claim (b) ρ_E is uniformly continuous on X.

Proof First, I will show $\forall x, y \in X$, $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$.

Case 1: $x, y \in E$. By part (a), we have $\rho_E(x) = \rho_E(y) = 0$, so $|\rho_E(x) - \rho_E(y)| = 0 \leq d(x, y)$.

Case 2: One of x, y is in E, and the other one isn't. Without loss of generality, let's say $x \notin E$ and $y \in E$. We have $\rho_E(y) = 0$. We also know that $d(x, y) \in \{d(x, z) : z \in E\}$. Therefore,

$$|\rho_E(x) - \rho_E(y)| = \rho_E(x) = \inf_{z \in E} d(x, z) \leq d(x, y)$$

Case 3: $x, y \notin E$. By the definition of ρ_E, we have

$$\forall z \in E \quad \rho_E(x) \leq d(x, z)$$

Using the triangle inequality:

$$\forall z \in E \quad \rho_E(x) \leq d(x, y) + d(y, z) \tag{8}$$

Now I will show that

$$\forall \epsilon > 0 \ \exists z \in E : d(y, z) \leq \rho_E(y) + \epsilon \tag{9}$$

Seeking a contradiction, assume this is not true for some ϵ. Then $\rho_E(y) + \epsilon$ is a lower bound of $\{d(y, z) : z \in E\}$, so $\rho_E(y)$ is not the greatest lower bound, which contradicts the definition of ρ_E. Therefore, (9) is true.

Combining (8) and (9), we get

$$\forall \epsilon > 0 \quad \rho_E(x) \leq d(x, y) + \rho_E(y) + \epsilon$$

This is equivalent to

$$\rho_E(x) \leq d(x, y) + \rho_E(y)$$

Therefore, $\rho_E(x) - \rho_E(y) \leq d(x, y)$. Using the same argument with x and y swapped, we can conclude $\rho_E(y) - \rho_E(x) \leq d(x, y)$, so therefore $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$.

I've shown that $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$ in all cases. So for some $\epsilon > 0$, we have $d(x, y) < \epsilon \implies |\rho_E(x) - \rho_E(y)| < \epsilon$, which proves that ρ_E is uniformly continuous.

Problem 4.21

Hypothesis K and F are disjoint subsets of a metric space X. K is compact. F is closed.

Claim $\exists \delta > 0 : (\forall p \in K, q \in F) \ d(p, q) > \delta$

Proof

Todo Clean this up

Seeking a contradiction, assume not. So we have:

$$\forall \delta > 0 \ \exists p \in K, q \in F : d(p, q) \leq \delta$$

$$\forall \delta > 0 \ \exists p \in K : \rho_F(p) \leq \delta$$

This implies 0 is a limit point of $\rho_F(K)$. We know that $0 \notin \rho_F(K)$ because $\rho_F(K)$ is positive and continuous on K. So this contradicts the fact that $\rho_F(K)$ is closed □

Problem 5.1

Hypothesis $f : \mathbb{R} \to \mathbb{R}$, and $\forall x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq (x - y)^2$.

Claim f is constant.

Proof We have

$$\forall x, y \in \mathbb{R} \quad |f(x) - f(y)| \leq |x-y|^2$$

Divide both sides by $|x-y|$:

$$\forall x, y \in \mathbb{R} \quad \left|\frac{f(x)-f(y)}{x-y}\right| \leq |x-y|$$

Taking the limit of each side as $y \to x$ gives us

$$\forall x \in \mathbb{R} \quad |f'(x)| \leq 0$$

By theorem 5.11b, f is constant.

Problem 5.2

Hypothesis $\forall x \in (a,b),\ f'(x) > 0.$

Claim f is strictly increasing in (a,b). If g is the inverse function of f, g is differentiable, and $\forall x \in (a,b),\ g'(f(x)) = 1/f'(x)$.

Proof Let $x, y \in (a,b)$ with $x < y$. f is differentiable on (x,y), so by theorem 5.10 we have

$$\exists z \in (x,y) : f(y) - f(x) = (y-x)f'(z)$$

$y-x$ is positive, and $f'(z)$ is positive by our hypothesis, therefore $f(y)-f(x)$ is positive. This proves

$$\forall x, y \in (a,b),\ x < y \implies f(x) < f(y)$$

So f is strictly increasing on (a,b). This means f has an inverse function $g : f((a,b)) \to (a,b)$ with the property that $\forall x \in (a,b),\ g(f(x)) = x$.

Now I will show that $g'(x)$ exists for all $x \in (a,b)$:

$$\begin{aligned}
\lim_{s\to y} \frac{g(y)-g(s)}{y-s} &= \lim_{t\to x} \frac{g(f(x))-g(f(t))}{f(x)-f(t)} \\
&= \lim_{t\to x} \frac{x-t}{f(x)-f(t)} \\
&= \lim_{t\to x} \frac{1}{\frac{f(x)-f(t)}{x-t}} \\
&= \frac{1}{\lim_{t\to x} \frac{f(x)-f(t)}{x-t}} \\
&= \frac{1}{f'(x)}
\end{aligned}$$

$f'(x)$ exists and is nonzero for all $x \in (a,b)$, therefore $g'(x)$ also exists.

By the definition of g, we have $\forall x \in (a,b),\ (g \circ f)(x) = x$. Therefore, $\forall x \in (a,b),\ (g \circ f)'(x) = 1$. By theorem 5.5, we also have $\forall x \in (a,b),\ (g \circ f)'(x) = g'(f(x))f'(x)$. So $1 = g'(f(x))f'(x)$, and we can conclude $g'(f(x)) = 1/f'(x)$.

Problem 5.3

Hypothesis $g : \mathbb{R} \to \mathbb{R}$. g is differentiable on $\mathbb{R}$. $\exists M > 0 : \forall x \in \mathbb{R}$, $|g'(x)| < M$.

For each $\epsilon > 0$, define $f_\epsilon : \mathbb{R} \to \mathbb{R} : x \to x + \epsilon g(x)$.

Claim $\exists z > 0 : \epsilon < z \implies f_\epsilon$ is 1-1.

Proof Let $0 < \epsilon < 1/M$. f_ϵ is the sum of two differentiable functions, so by theorem 5.3 it is differentiable.

$$\begin{aligned} f'_\epsilon(x) &= \lim_{t \to x} \frac{x - t}{x - t} + \lim_{t \to x} \frac{\epsilon g(x) - \epsilon g(t)}{x - t} \\ &= \lim_{t \to x} 1 + \epsilon \lim_{t \to x} \frac{g(x) - g(t)}{x - t} \\ &= 1 + \epsilon g'(x) \end{aligned}$$

From the bound on g', we know that $1 - \epsilon M < f'_\epsilon(x) < 1 + \epsilon M$. We also know from the definition of ϵ that $0 < \epsilon M < 1$. Combining these, we get $0 < f'_\epsilon(x) < 2$. By the logic in the previous exercise, we see that f_ϵ is strictly increasing. This implies it is 1-1.

I have proved that $0 < \epsilon < 1/M \implies f_\epsilon$ is 1-1.

Problem 5.4

Claim $(\forall x, y \in \mathbb{R}, n \in \mathbb{N})\ x^n - y^n = (x - y) \sum_{j=1}^{n} x^{n-j} y^{j-1}$

Todo Prove this

Hypothesis $C_0, C_1, \ldots, C_n \in \mathbb{R}$ such that $\sum_{i=0}^{n} C_i/(i+1) = 0$. $f : \mathbb{R} \to \mathbb{R} : x \to \sum_{i=0}^{n} C_i x^i$.

Claim $\exists x \in (0, 1) : f(x) = 0$

Proof Let $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$g(x) = \sum_{i=0}^{n} \frac{C_i}{i+1} x^{i+1}$$

Clearly, $g(0) = 0$. By our hypothesis, we also have

$$g(1) = \sum_{i=0}^{n} \frac{C_i}{i+1} = 0$$

Now I will show that g is differentiable and that $g' = f$.

$$\begin{aligned}
g'(x) &= \lim_{t\to x} \frac{g(x)-g(t)}{x-t} \\
&= \lim_{t\to x} \frac{\sum_{i=0}^{n} \frac{C_i}{i+1} x^{i+1} - \sum_{i=0}^{n} \frac{C_i}{i+1} t^{i+1}}{x-t} \\
&= \lim_{t\to x} \frac{\sum_{i=0}^{n} \frac{C_i}{i+1} \left(x^{i+1} - t^{i+1}\right)}{x-t} \\
&= \lim_{t\to x} \frac{\sum_{i=0}^{n} \frac{C_i}{i+1} (x-t) \sum_{j=1}^{i+1} x^{i+1-j} t^{j-1}}{x-t} && \text{by our previous result} \\
&= \lim_{t\to x} \sum_{i=0}^{n} \frac{C_i}{i+1} \sum_{j=1}^{i+1} x^{i+1-j} t^{j-1} \\
&= \sum_{i=0}^{n} \frac{C_i}{i+1} \sum_{j=1}^{i+1} x^{i+1-j} x^{j-1} && \text{because we have a continuous function (theorem 4.6)} \\
&= \sum_{i=0}^{n} \frac{C_i}{i+1} \sum_{j=1}^{i+1} x^{i} \\
&= \sum_{i=0}^{n} \frac{C_i}{i+1} (i+1) x^{i} \\
&= \sum_{i=0}^{n} C_i x^{i} \\
&= f(x)
\end{aligned}$$

Now by theorem 5.10, we have

$$\exists\, x \in (0,1) : g(1) - g(0) = (1-0) f(x)$$

Which proves that $\exists\, x \in (0,1) : f(x) = 0$. $\square$

Problem 5.5

Hypothesis $f : (0,\infty) \to \mathbb{R}$. f is differentiable on $(0,\infty)$. $\lim_{x\to\infty} f'(x) = 0$.
$g : (0,\infty) \to \mathbb{R} : x \to f(x+1) - f(x)$.

Claim $\lim_{x\to\infty} g(x) = 0$

Proof We need to show:

$$\forall \epsilon > 0 \;\; \exists\, C \in \mathbb{R} : c > C \implies |g(c)| < \epsilon \tag{10}$$

Fix an $\epsilon > 0$. We have

$$\exists\, C \in \mathbb{R} : c > C \implies |f'(c)| < \epsilon$$

Fix a $c > C$. By theorem 5.10, we have

$$\begin{aligned}
\exists\, x \in (c, c+1) : f(c+1) - f(c) &= (c+1-c) f'(x) \\
g(c) &= f'(x)
\end{aligned}$$

$x > c > C$, so $|f'(x)| < \epsilon$, and therefore $|g(c)| < \epsilon$. This proves (10). $\square$