The Geometry of Other Quadrilaterals, Triangle Similarity, Right Triangles

by your tired classmate <3

The Introduction of Geometry for Parallelograms, Rhombus, Trapezoid, Kite, and the Core of Triangle Similarity and Properties of Right Triangles

Made with love and LaTeX
Made in TeXMaker
Massive appreciation to StackExchange for LaTeX and Overleaf Documentation
LaTeX distribution used: MikTeX
Compiled with LuaLaTeX

`If you are planning to print as a book FOR PERSONAL USE, then you have the authorization.`
`Don't waste the back page!`

I Am Most Grateful for My Classmates ★

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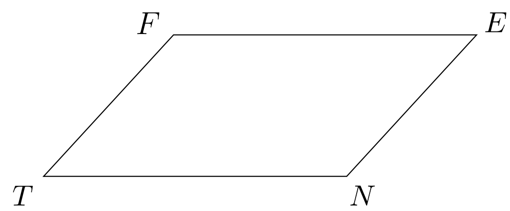
Before We Begin

Some terms or concepts mentioned can be seen in the Math Guide from 4Q, G8. Do not stress yourself to review two things. The terms or concepts mentioned will not be connected, as the main topic is primary.

All quadrilaterals and shapes are named from upper left, to upper right, to lower right, to lower left.

For example, when I say $\square FENT$, the parallelogram's labels are in the same order:



Any question marked with * have answers. Certain questions have hints alongside.

Preface

Over the course of the last two quarters, we covered the nature of quadratics and the concept of variation, laws of exponents, and radicals. However, the quarter now and the following quarter, we will deal with geometries. This is considered my weakness, and is something people get to be confused with. The quarter now covers geometries on quadrilaterals and other similiarities, and the quarter proceeding covers trigonometry; a topic in math that has not been seen before. My aim for this guide is to at least, try offer the best I can do to explain the spots that are not really understood mostly. The processes done in the theorems and definitions involving algebra uses the methods used formally in algebra. However, the way to deduct/induct the reasoning to start this process can be tricky or confusing to the inexperienced, like me. But of course, practice can be done to deduct and eventually a pattern can be seen but the steps are similar.

This little document has the aim to supplement the reader in their gripe to the form of geometry seen here. Unlike before, this is going to be a supplement and can help aid your learning. My original aims are still in play, but the main prior focus on this document is to supplement. If you still want more explanations, watching YouTube videos about these kinds of shapes can help improve your intuition than just learning by book or text.

I always have one wish when doing these guides: "to help you, and that you have at least read one thing from this, and I would appreciate it no matter what." (excerpt from the second guide.)

The following is an excerpt from the first guide to help show the extent of how you can expect and perform to the examinations upcoming:

> "It is no surprise that mathematics is a tough and desolate subject with rigor, abstractions, and paradoxical reasoning that the reader just does not want to embark. But, a step-by-step approach, to something so radical to the naked eye, can be turned simply *if you know what you are doing.*"

Changes from the Second Guide

I have come to realize that the more formal look on the document to make it look more appealing has no point, and the document class that I used and the margin size (memoir document class) makes it feel unnecessary. I do not like giving much effort, and I have realized that a more simpler approach to how the guide is presented can be more easier to make. To those that don't know by this point, all of the mathematics guides have been made through LaTeX because of how versatile and efficient it is compared to word processors like Microsoft Word. Because of its steep learning curve, it involves a lot of troubleshooting and heavy skimming on if the code would work, so the output would be more sensible. So I decided to tone down the formal-ness of the document, so that the document would not seem to be overwhelming. I do not recommend you learn LaTeX with no reason but to impress or to see it as an *alternative to Microsoft Word* because it is a steep learning curve and how its intention is to stylize research papers, books, guides, lecture notes, and more, dependent on STEM courses. I know you could also use it for this reason, but I prefer you not use it for that way.

The document-style from the previous reviewer will be reserved to when I publish my self-authored book about introductory algebra taken to an advanced level. This document style is going to be used for the next reviewer afterwards.

Another change is that I have moved the examples to its own sub-subsection so the topic proper can be focused even more. Included as a change is to separate exercises into their own sub-subsection like the examples sub-subsection, to further radicalize the focus on the explanations and the examples can be ratified to understand on a separate area to not influence or mix the definitions.

Recommendations

The following is a direct rip of the "Recommendations" section in the first ever booklet "An Advanced Synopsis of Polynomials In the Second Degree" but slightly rewritten:

> "I recommend to the reader to take a piece of paper, and something to write with. Take notes of what you understand, and understand the pieces DIRECTLY. If you need any superintendent aide, ask someone or a friend that discretely knows what you are tackling. This subject is not passable with no gaps, it is a thorn maze filled with tiny holes that leak information without grasp.
>
> I also want to note that: You won't understand everything at first glance and that is completely unavoidable. It is normal and not in an abstract way that people don't understand this. Topics can be understood after weeks or even years; something as basic as scientific notation can be understood after a long period. Interest in mathematics is not for the faint heart of those who see it deterring their future outcome. It is not an outcast. Having difficulties reading mathematics or such any topic is not a block. It is material to build with. Use any approach you would use to understand the reasoning, and you will get it more. Take your time; either long or short. Even if it won't be comprehension-certified within two hours or a day. It is something that you, and anyone, can do."

§1 Geometry on Quadrilaterals

We know what a *quadrilateral* is; a polygon with exactly four line segments connecting each other by different points with vertices. However, this section will include the geometry on the other quadrilaterals other than the square or rectangle.

§1.1 The Parallelogram

The figure below is a quadrilateral that has lines that are parallel if put on a plane. This is a kind of quadrilateral known as a parallelogram.

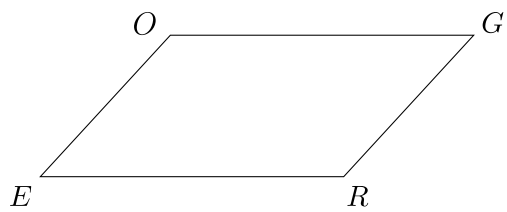


Figure 1.1: Parallelogam

Definition 1.1: The Parallelogram

A *parallelogram* is a quadrilateral wherein both pairs of sides opposing each other are *parallel*.

Seen on $\square OGRE$, lines parallel can be $\overline{OG}$ and $\overline{ER}$. They do not intersect if put on a plane.

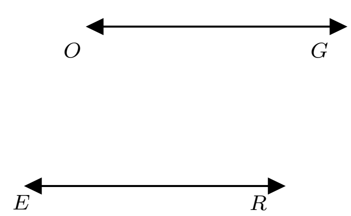


Figure 1.2: Parallel Lines $\overline{OG}$ and $\overline{ER}$

Note. We should begin with some terms deemed confusing:

1. A *consecutive side or angle* is a side/angle that is next to another. In $\square OGRE$, consecutive lines can be $\overline{OE}$ and $\overline{OG}$ because they are right next to each other. Consecutive angles can be $\angle O$ and $\angle E$ or $\angle E$ and $\angle R$.

2. An *opposing side or angle* is a side/angle that is directly opposite another. An example of opposite angles are $\angle O$ and $\angle R$ in Figure 1.1.

3. A *diagonal* is a line that connects any two opposing points. For example, between opposing points O and R, you can make a diagonal $\overline{OR}$ that connects them both.

We can begin our first theorem:

Theorem 1.1:

If a quadrilateral is a parallelogram, then the opposite sides are considered *congruent*.

Congruent means "equal." If we can see Figure 1.1, the opposing sides appear equal. Like in a rectangle, the longer sides are always the same with each other, and shorter sides are also the same with each other.

Infer in the figure below that if, example, $\overline{FA} \cong \overline{TE}$ are congruent, we conclude that it is the same as saying $\overline{FA} = \overline{TE}$.

Infer from Figure 1.1 that: opposite angles are equal. Examining the parallelogram shows that opposite angles appear to be the same measure. This is concluded in a theorem:

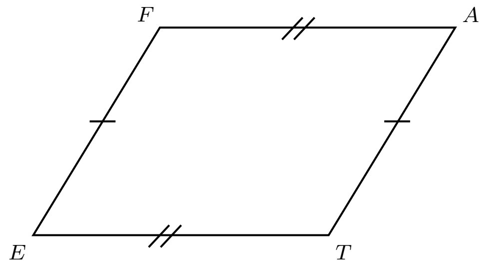


Figure 1.3: **Theorem 1.1** as a Parallelogram

Theorem 1.2:

If a quadrilateral is a parallelogram, then opposing angles are *congruent*.

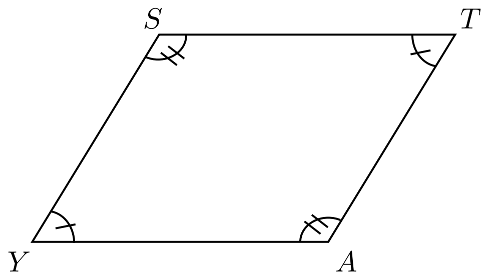


Figure 1.4: **Theorem 1.2** as a Parallelogram

Inference

We can also see that a diagonal $\overline{YT}$ creates an *isosceles triangle* $\triangle ATY$ showing $\angle T \cong \angle Y$.

Deducting from Figure 1.4, the diagonal $\overline{YT}$ forms a transversal line[1] and that it's same-side interior angles are *supplementary*. Same-side interior angles include $\angle S$ with $\angle Y$ or $\angle Y$ with $\angle A$ etc. We can infer to make a new theorem:

Theorem 1.3:

If a quadrilateral is a parallelogram, then the pairs of consecutive angles are *supplementary*.

Definition 1.2: Complementary and Supplementary Angles

Angles are considered either:

a. *Complementary* if their sum is 90°.
b. *Supplementary* if their sum is 180°.

[1]Do not worry about what this is if you don't know. Just refer to the reviewer from 4Q, G8 if you really need to.

If we were to draw the two diagonals, they intersect at a certain point or at a *bisection*. Something bisects another if it divides that other thing into two equal parts. This can be illustrated:

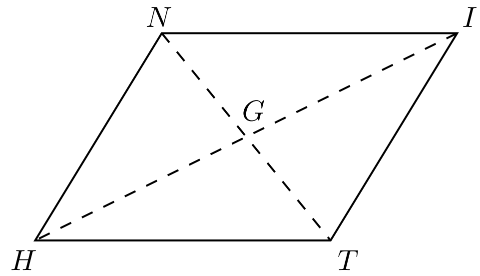


Figure 1.5: Parallelogram $\square THIN$ With Diagonals $\overline{HI}$ and $\overline{TN}$.

This concludes and creates the next theorem:

Theorem 1.4:

If a quadrilateral is a parallelogram, then the diagonals bisect each other.

The diagonals *bisect* in $\square THIN$. We can see $\overline{NG} \cong \overline{GT}$. Could you identify the other pair?

Infer from Definition 1.1. If we were to consider Theorem 1.1, and reverse it, the statement still holds true.

Theorem 1.5: Converse of Theorem 1.1

If opposing sides *(of a quadrilateral)* are congruent, then the quadrilateral is a *parallelogram*.

Infer from Figure 1.3. If both pairs of opposing sides are congruent and do not intersect by their respective pairs, they are parallel. But if a pair of opposing sides are congruent but intersect by another pair, it is not a parallelogram (trapezoids, kites, etc.)

We can also restate Theorems 1.2 and 1.4 by reversing them. It would still hold true, explained after the restatement:

Theorem 1.6: Converse of Theorem 1.2

If the opposing angles *(of a quadrilateral)* are congruent, then the quadrilateral is a parallelogram.

Theorem 1.7: Converse of Theorem 1.4

If the diagonals *(of a quadrilateral)* bisect each other, then the quadrilateral is a parallelogram.

We already know that a parallelogram consists of lines that are parallel when put on a plane. It has two pairs of it representing opposing sides. We can conclude that the opposing sides of a quadrilateral are congruent make a parallelogram.

Applying Theorem 1.5, and the idea about parallel lines, we can say:

Theorem 1.8:

If a pair of opposite sides of a quadrilateral is *both* parallel and congruent, then the quadrilateral is a parallelogram.

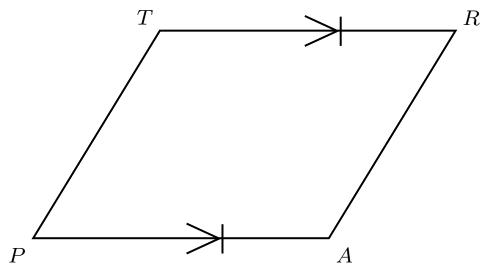


Figure 1.6: **Theorem 1.8** as a Parallelogram

The arrowheads mean they only signal going one direction. The lines mean that the pair is congruent.

§1.1.1 Examples

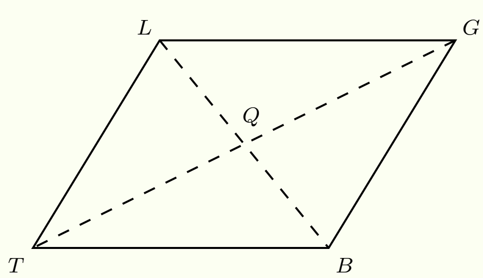
The frontmatter has been given and now we can use the theorems and definitions seen above to determine measurements or determine which is which.

Example 1.1: From *Do Basic*, p. 165; adjusted

Find the following:

a) All pairs of opposite angles and sides.
b) All pairs of consecutive angles.
c) All diagonals.
d) All the pairs of congruent angles and sides.
e) All the pairs of supplementary angles.
f) All pairs of congruent segments formed by the diagonals.
g) All pairs of congruent angles formed by the diagonals.

Use the parallelogram $\square LGBT$ below:



Solution. Chronological:

a) $\angle L, \angle B$ and $\angle G, \angle T$; $\overline{LT}, \overline{GB}$ and $\overline{LG}$ and $\overline{TB}$.

b) $\angle T, \angle B$; $\angle T, \angle L$; $\angle B, \angle G$; and $\angle L, \angle G$

c) $\overline{LB}, \overline{TG}$.

d) $\angle L \cong \angle B$ and $\angle T \cong \angle G$; $\overline{LG} \cong \overline{TB}$ and $\overline{LT} \cong \overline{GB}$.

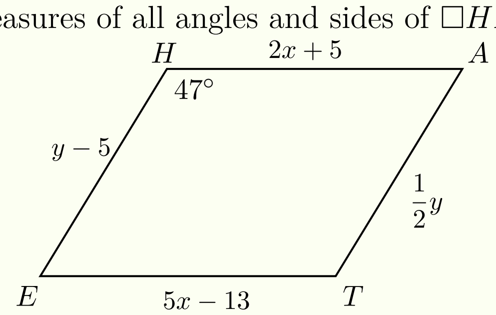
e) $\angle T + \angle B$, $\angle T + \angle L$, $\angle B + \angle G$, and $\angle L + \angle G$.

f) $\overline{LQ} \cong \overline{QB}$ and $\overline{TQ} \cong \overline{QG}$.

g) Recall from the 1st Year of HS, "opposite angles are congruent;" $\angle LQT \cong \angle GQB$ and $\angle LQG \cong \angle TQB$.

Example 1.2: Applying Algebra as Measures

Find the values of x, y and find the measures of the given seen in the parallelogram below. Complete the measures of all angles and sides of $\square HATE$.



Solution. Solve for the values of the each side and angle specified.

i: Solve for x. Use **Theorem 1.1** and equate the sides that are opposing. The example shows that the equations with variable x are opposite each other, therefore you should equate them as equations.

$$\begin{aligned} 2x + 5 &= 5x - 13 \\ 2x + 5 - 2x + 13 &= 5x - 13 - 2x + 13 \\ 3x &= 18 \\ \frac{3x}{3} &= \frac{18}{3} \\ x &= \boxed{6} \end{aligned}$$

We are *not done*. We must know that x is merely just a *value* that makes both equations equal. We still need to find what number the equation makes, so let's re-substitute the value of x back to the equality:

$$\begin{aligned} 2(6) + 5 &= 5(6) - 13 \\ 12 + 5 &= 30 - 13 \\ 17 &\stackrel{\checkmark}{=} 17 \end{aligned}$$

Therefore, $\overline{HA} = 17$ and $\overline{TE} = 17$, therefore verifying that $\overline{HA} \cong \overline{TE}$.

ii: Solve for y. Use **Theorem 1.1** again. Same idea as i.

$$\begin{aligned} \frac{1}{2}y &= y - 5 \\ \cancel{2}\left(\frac{1}{\cancel{2}}y\right) &= 2(y - 5) \\ y &= 2y - 10 \\ y - 2y &= 2y - 10 - 2y \\ -y &= -10 \\ \frac{-y}{-1} &= \frac{-10}{-1} \\ y &= \boxed{10} \end{aligned}$$

Again, we are *not done*. Re-substitute y back to the equality like what we did already:

$$\frac{1}{2}(10) = 10 - 5$$
$$5 \stackrel{\checkmark}{=} 5$$

We conclude that, $\overline{HE} = 5$ and $\overline{AT} = 5$, therefore $\overline{HE} \cong \overline{AT}$.

iii: Find the value of m$\angle A$. Use **Theorem 1.3**. Infer that m$\angle A$ is the missing angle, but m$\angle H = 47°$. We know that m$\angle H +$ m$\angle A = 180\,\text{deg}$, so we just need to add both angles and find m$\angle A$.

$$\begin{aligned} \text{m}\angle A + 47° &= 180° \\ \text{m}\angle A + 47° - 47° &= 180° - 47° \\ \text{m}\angle A &= \boxed{133°} \end{aligned}$$

iv: Find the measures of m$\angle E$ and m$\angle T$. Use **Theorem 1.2** and discover that opposite angles to the above are $\angle A$ and $\angle H$ respectively. Noting that, we can also use the theorem to declare m$\angle H \cong$ m$\angle T$ and m$\angle A \cong$ m$\angle E$. Therefore,

$$\begin{aligned} \text{m}\angle A &\cong \text{m}\angle E \\ 133° &\cong \text{m}\angle E \\ \text{m}\angle E &= \boxed{133°} \\ \text{m}\angle H &\cong \text{m}\angle T \\ 47° &\cong \text{m}\angle T \\ \text{m}\angle T &= \boxed{47°} \end{aligned}$$

Example 1.3: True or False; Modified *Go Beyond*, p.167 no. 7

If $\square TRIP$ is a quadrilateral, then the following must be true. Determine which is false.

1. $\angle T \cong \angle I$ and $\angle R \cong \angle P$ makes it a parallelogram.
2. The sides of a quadrilateral need to be congruent for it to be a parallelogram.
3. The angles of a quadrilateral need to be 90° for it to be a parallelogram.
4. $\overline{TR} + \overline{RI} = 180°$

Solution. Apply theorems and use counterexamples to help verify the claims' truth.

i: **TRUE**. Illustrating the parallelogram shows that they are congruent following **Theorem 1.2**.

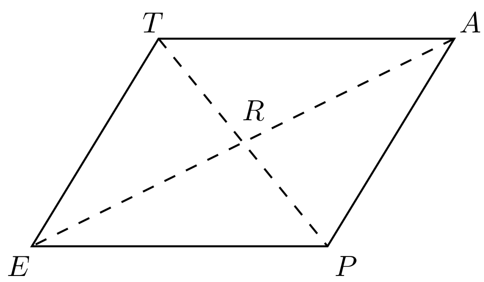
ii: **FALSE**. The *rectangle* has two pairs of sides that are longer than the other, and it is still a parallelogram. (Why?)

iii: **FALSE**. The *rhombus* has angles less than 90° yet it is considered a parallelogram. When we flip the rhombus to its sides, it appears like a regular parallelogram (or because it follows **Theorem 1.8**).

iv: **TRUE**. Any parallelogram (square, rectangle, rhombus) follows **Theorem 1.3**.

§1.1.2 Exercises

I: In $\square TAPE$ with point R below,



Find the following inferring from the parallelogram above:

a) All pairs of opposite angles.

b) All diagonals.

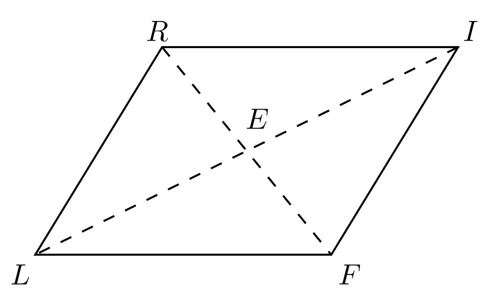
c) All the pairs of congruent angles and sides.

d) All pairs of consecutive angles, and supplementary angle pairs.

e) All pairs of congruent segments formed by the diagonals.

f) All pairs of congruent angles formed by the diagonals.

II: *Inspired from "Geometry" by Merle S. Alferez and Alvin E. Lambino.* Examine $\square RIFL$ with point E below,



Find all of the following requested.

1. If $\overline{RL} = 3x - 16$ and $\overline{IF} = x - 4$, find x, $m\overline{RL}$ and m$\overline{IF}$.
2. If m$\angle R = 3x - 13$ and m$\angle L = x + 7$, find the measure of both angles.
3. Using the answer from number two, find the measures of $\angle I$ and $\angle F$ if they equal $5x - 7$ and $3x - 1$ respectively. *Hint: Use Theorem 1.2*
4. "A quadrilateral has all angles sum to 360°." If $\angle R = x + 10, \angle I = 2x + 5$, find the value of the other angles and the measure of each angle.*
5. If $\overline{RE} = 0.7x + 11$ and $\overline{EF} = 0.3x + 17$, find the measures of the line segments.*
6. m$\angle I = x^2 - 6$ and m$\angle L = x$, find the measures of the angles mentioned.*

III: Write T if it is true, and F if not and write your reason.

1. A square is considered a parallelogram.
2. Any diagonal bisecting halves a parallelogram into two congruent triangles on certain shapes.*
3. If two opposite sides are congruent, then it is considered a parallelogram.
4. If only one pair of angles is 90°, it is considered a parallelogram.*

§1.2 The Trapezoid and Rhombus

The quadrilaterals are more *diverse* than what can be seen. Literally, *quad* means four, and *lateral* means side. If any shape can be conjoined by at least four sides, it is a quadrilateral. The following diagram shows the diagram of quadrilateral classification.

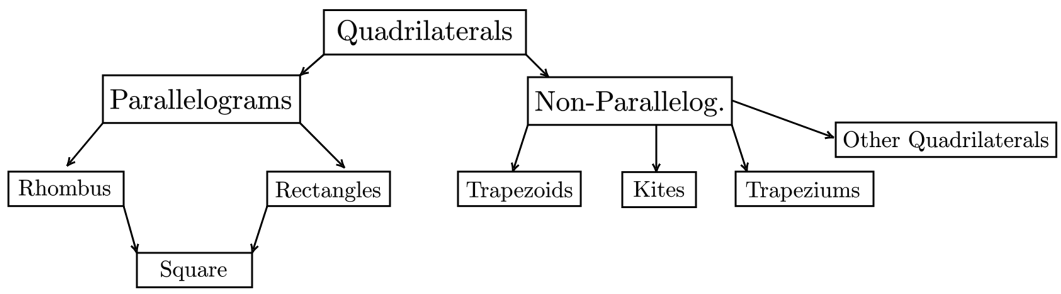


Figure 1.7: Quadrilateral Family Tree

EDITORIAL: Figure 1.7 might appear of slight inferior quality because it is not using TikZ, rather a PNG using `includegraphics` in the `figure` environment. It would straight up appear black on certain devices (typically older); I have no fix for that unfortunately as of now, so if you have the textbook refer to p. 171. If not, use the internet if you can. Don't go mega-detailed, just one that is simple.

A *trapezoid* is a shape with one pair of parallel lines. A *rhombus* is a shape with a set of congruent sides.

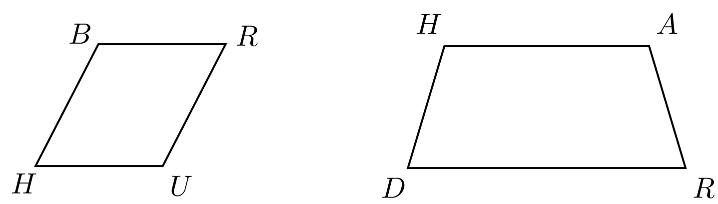


Figure 1.8: Rhombus[2] $\square BRUH$ and Trapezoid $\square HARD$.

As for a parallelogram as usual, we can draw a pair of diagonals (by Figure 1.7 rhombi are parallelograms.)

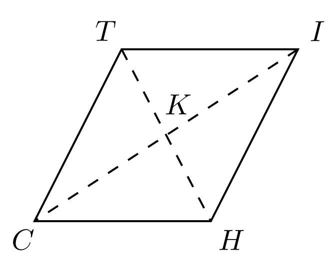


Figure 1.9: Rhombus $\square THIC$ with point K

Theoretically, we can *rotate* the rhombus to where it's furthest bottom end is standing upright. The diagonals hence become straight lines from inference.

[2]All rhombi being illustrated by me are not always *actual* rhombi as the GUI editor I'm using to make these shapes does need me to manually make one unfortunately.

After rotating the rhombus, we get a shape that looks like the diamond[3]. Illustrated:

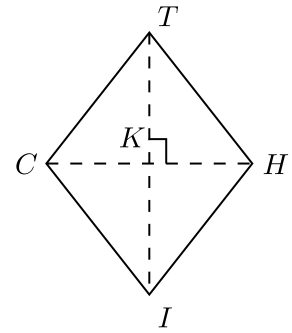


Figure 1.10: Figure 1.9, rotated upright

We can see that the diagonals form an intersection of perpendicular lines. Recall that *perpendicular lines* are lines that intersect at an angle of 90°. This is the exact description of our next theorem.

Theorem 1.9:

The diagonals of a rhombus bisect perpendicularly.

From $\square THIC$, we can see $\overline{TI} \perp \overline{CH}$. They are both a supplementary pair however only if both angles are equal (Why?).

Notice that in the diagonals formed, they *always* bisect with two of its angles that are opposing. This means that the rhombus gets divided to two equal triangles if one diagonal halves the rhombus, and the triangle is exactly the same if you picked another angle pair.

Compared to a parallelogram, one side is longer and the other is shorter, because of the sides. But, since the rhombus has all sides congruent, then no matter what angle pair you make a diagonal for, the rhombus is *always* halved to two equal triangles of same length to the other pair.

Theorem 1.10:

The diagonals of a rhombus bisects two of its angles.

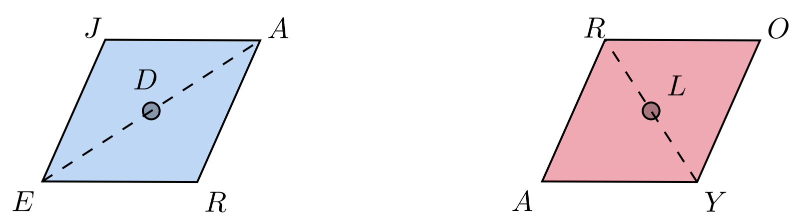


Figure 1.11: $\square JARE$ with point D and $\square ROYA$ with point L

The shaded colors represent what portion the rhombus is being halved. In the blue rhombus, it is being halved to two different parts that are still equal to the red rhombus' halved portions. Clearly you can see that the triangles formed are *not congruent to each other* but that is because the figure shows a parallelogram instead of a rhombus, I just can not make a good illustration.

What this defines is that: A pair of angles using the diagonals is always bisected or *split to two equal parts. Any angle pair that has the same vertex, and uses the diagonal as a leg, is congruent.* Seen in the blue rhombus, from what I described, we can see that two pairs of angles seem to be congruent using the diagonal as a part of its leg: one of them is $\angle RAD$ and $\angle JAD$. We write angles using its vertex as the center of the name. Could you identify the other pair? In the red rhombus, we see the angle pair $\angle OYL$ and $\angle AYL$.

[3] A diamond is a special case of a rhombus that has two pairs of opposite acute angles and obtuse angles respectively. Seen in the figure is *not* a diamond, as it is still a rhombus.

We shall continue to the *trapezoid*. Recall that this quadrilateral has only one pair of parallel lines. Seen in Figure 1.8, $\square HARD$ has only one pair of parallel lines $\overline{HA} \| \overline{RD}$.

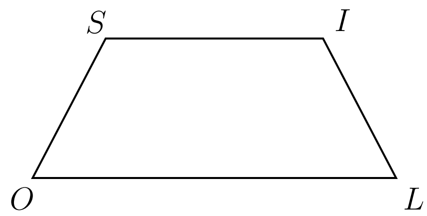


Figure 1.12: Trapezoid $\square SILO$

Look at the new trapezoid $\square SILO$. We shall consider the parts of the trapezoid illustrated.

Definition 1.3:

The following three are the parts of the trapezoid and will be used.

- The *bases* of the trapezoid are the pair of parallel lines $\overline{SI}$ and $\overline{LO}$.
- The *base angles* of the trapezoid are the angles that are from the parallel lines $\angle S, \angle I$ and $\angle L, \angle O$. Consider them the consecutive angles on the top and bottom sides.
- The *legs* of the trapezoid are the sides that intersect/are not parallel $\overline{SO}$ and $\overline{IL}$.

A trapezoid can either have congruent legs or a right angle for a leg. If a trapezoid has a right-angled leg, then it is a *right trapezoid*. However, this text will focus on a special case of trapezoid: an *isosceles trapezoid*.

Recall that an isosceles triangles has two angles and sides equal. Using that idea for a trapezoid, we can note that if the base angles are congruent, as well as the legs, we can safely say that the trapezoid is isosceles.

Definition 1.4:

A trapezoid is considered *isosceles* if the legs and base angles are congruent.

Let us consider a trapezoid $\square LMAO$.

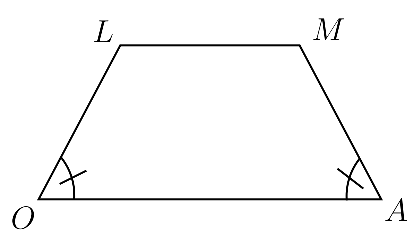


Figure 1.13: Trapezoid $\square LMAO$

See in the figure that the two angles are marked congruent. We infer from the 2nd Year reviewer (you don't need to review that at all, I will explain) that I mentioned two theorems that define an isosceles triangle:

Inference

- **Theorem 2.6**: If two sides of a triangle are congruent, then the angles opposite them are congruent.
- **Theorem 2.7** (Converse of Theorem 2.6): If two angles of a triangle are congruent, then the sides opposite them are congruent.

We can use this to give a bit of a better understanding of how an isosceles trapezoid is. Let us consider base angles and legs $\angle A, \angle O$ and $\overline{MA}, \overline{LO}$. We see that the opposite side of $\angle A$ is $\overline{LO}$. We can tell that we can make two sides from just an angle alone. We note Definition 1.4, we see that the angle congruent to $\angle A$ is $\angle O$ as the side formed by it is $\overline{LO}$. The same applies for the other angle. If that happens and there exists the bases $\overline{LM}$ and $\overline{AO}$, we can make an isosceles trapezoid[4].

[4] This is *not* a proof. I am giving a different perspective if you can consider it that. The theorems I am stating are for triangles, so this is improper but informal enough to be understood.

We can redefine some ideas into theorems, starting off with how the base angles are congruent.

Theorem 1.11:

If a trapezoid is identified isosceles, then its base angles form a congruence.

The converse is also true.

Theorem 1.12: Converse of Theorem 1.11

If a trapezoid's base angles form a congruence, then it is identified isosceles.

Forming diagonals on a new trapezoid $\square DRAK$ with point E leads us to our next theorem and its converse.

Theorem 1.13:

If a trapezoid is identified isosceles, then its diagonals form a congruence.

Theorem 1.14: Converse of Theorem 1.12

If a trapezoid's diagonals form a congruence, then it is identified isosceles.

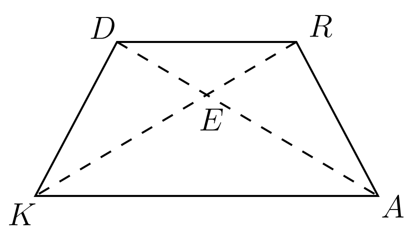


Figure 1.14: **Theorem 1.11** and **Theorem 1.13** as a Trapezoid

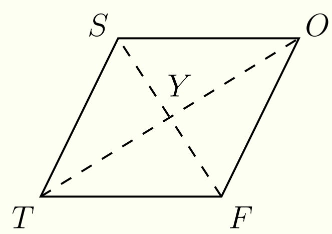
§1.2.1 Examples

We should know by now that the theorems applied can help solve problems with equations representing measures. I will split the rhombus and trapezoid to two parts so that it is clearer to see what can be seen. I will illustrate only some examples as including many would be painful to LaTeX manually.

PART I: The Rhombus

Example 1.4: Go Beyond, p. 183

For anyone reading outside of my school, first off hi. Secondly, this part really messed with my brain and with some others too. If I had to guess, it is because it sounded a little obscure. This is 11) in our textbook.
See the illustration below.



If $m\angle YSO = 5x - 12.5$ and $m\angle TSY = 3x + 8.5$, find the measures of $m\angle TSO$ and $m\angle FTO$.

Solution. We use **Theorem 1.10**. I described the theorem as *any angle pair that has the same vertex, and uses the diagonal as a leg, is congruent.* We note that $m\angle YSO$ and $m\angle TSY$ have the same vertex S and use one of the diagonals. Hence, we equate them since they are congruent.

$$\begin{aligned} 5x - 12.5 &= 3x + 8.5 \\ 5x - 12.5 - 3x + 12.5 &= 3x + 8.5 - 3x + 12.5 \\ 2x &= 21 \\ \frac{2x}{2} &= \frac{21}{2} \\ x &= 10.5 \end{aligned}$$

As typical, substitute this back to find the measurements of $m\angle YSO$ and $m\angle TSY$.

$$\begin{aligned} 5(10.5) - 12.5 &= 3(10.5) + 8.5 \\ 52.5 - 12.5 &= 31.5 + 8.5 \\ 40 &\stackrel{\checkmark}{=} 40 \end{aligned}$$

Therefore $m\angle YSO = 40$ and $m\angle TSY = 40$ since they are congruent. Next, we determine the other values of the other angles requested.

Finding $m\angle TSO$ involves the definition of **Theorem 1.10**. We note that the definition of bisection involves splitting into two equal parts. We *split* it. Meaning, we directly *halved* the object. Since the angles' measures is 40, we see that 40 is the value halved from the full angle $m\angle TSO$. So, we add 40 by itself to get the *full measurement* or multiply by two because we halved it. I hope you understand. I tried my best.

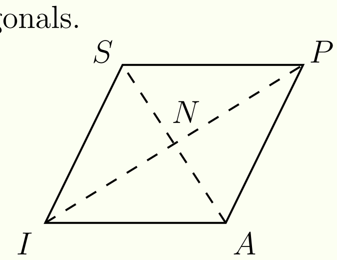
$$40 + 40 = 80$$

$\boxed{m\angle TSO = 80^\circ}$. Next we find $m\angle FTO$. We use a previous theorem, **Theorem 1.3**. We note that a rhombus is a parallelogram. And, we also know the value of $m\angle TSO = 80^\circ$ and it is a consecutive angle to $m\angle FTO$. Add them both to 180.

$$\begin{aligned} m\angle FTO + 80^\circ &= 180^\circ \\ m\angle FTO + 80^\circ - 80^\circ &= 180^\circ - 80^\circ \\ m\angle FTO &= \boxed{100^\circ} \end{aligned}$$

Example 1.5:

Consider a rhombus with diagonals.



Consider that $m\angle PNA = 3x - 15$ and $m\angle SNI = 10x - 30$. Find the value of x to complete the measurement of the angles.

Solution. We begin by using **Theorem 1.9**. We note that the diagonals bisect perpendicularly. We note that we are using an angle with the vertex of the intersection of the two diagonals N. Hence, we should equate the angles to 90° because it is perpendicular, meaning they are both 90. It won't matter if the value of x is different, because in the end they will both be 90°.

1. Solve $m\angle PNA$

$$\begin{aligned} m\angle PNA &= 90^\circ \\ 3x - 15 &= 90 \\ 3x - 15 + 15 &= 90 + 15 \\ 3x &= 105 \\ \frac{3x}{3} &= \frac{105}{3} \\ x &= \boxed{35} \end{aligned}$$

2. Solve m$\angle SNI$

$$\begin{aligned}
\text{m}\angle SNI &= 90^\circ \\
10x - 30 &= 90 \\
10x - 30 + 30 &= 90 + 30 \\
10x &= 120 \\
\frac{10x}{10} &= \frac{120}{10} \\
x &= \boxed{12}
\end{aligned}$$

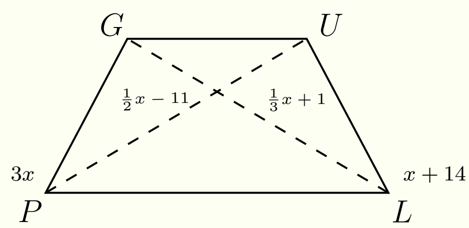
The reader is left to verify if these values of x make their respective angles perpendicular.

PART II: The Trapezoid

I will illustrate **Theorem 1.11** and **Theorem 1.13**.

Example 1.6:

Consider a trapezoid.



Find the measurements of the diagonals and base angles.

Solution. We start by using theorems.

1. **Theorem 1.11**. Equate the base angles together.

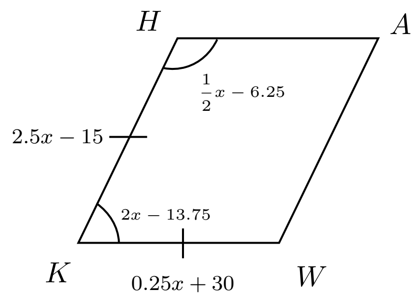
$$\begin{aligned}
3x &= x + 14 \\
3x - x &= x + 14 - x \\
2x &= 14 \\
\frac{2x}{2} &= \frac{14}{2} \\
x &= \boxed{7}
\end{aligned}$$

2. **Theorem 1.13**. Equate the diagonals together.

$$\begin{aligned}
\frac{1}{2}x - 11 &= \frac{1}{3}x + 1 \\
6\left(\frac{1}{2}x - 11\right) &= 6\left(\frac{1}{3}x + 1\right) \\
3x - 66 &= 2x + 6 \\
3x - 2x &= 66 + 6 \\
x &= \boxed{72}
\end{aligned}$$

§1.2.2 Exercises

I: Consider a rhombus $\square HAWK$.



Solve for the following. Afterwards, for number 3 and onwards, find the value of requested from the rhombus.

1. $\overline{AH}$ and $\overline{AW}$. *Hint: A rhombus has all sides congruent*

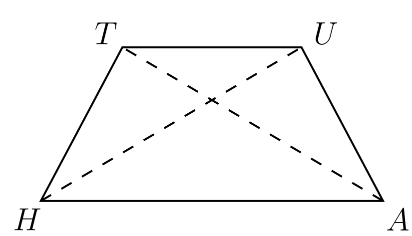
2. $\angle W$ and $\angle A$. *Hint: Congruent angles.*

3. If a diagonal $\overline{HW}$ with point S bisects the rhombus, and $\angle HAS$ is 90°, find $\angle HSK$.*

4. If another diagonal $\overline{AK}$ with the same point bisects the rhombus, and $\angle WSK = 2x - 14$, find the value of x that makes the diagonals' intersection perpendicular.

5. If diagonals $\overline{HW} = \dfrac{1}{4}x + 33$ and $\overline{AK} = \dfrac{3}{2}x - 14$, find the measures of both diagonals as decimal form. Use long division, not your calculator. *Hint: Multiply both sides by the common denomination of 2 and 4.*

II: Consider an isosceles trapezoid $\square TUAH$.



Solve for the following.

1. If $\angle A = x^2$ and $\angle H = 8x - 16$, find the value of both angles.*

2. If $\overline{TA} = 0.25x - 2.5$ and $\overline{UH} = 0.75x - 11.5$, find the value of the diagonals.

3. If the legs $\overline{TH} = 5x - \dfrac{3}{2}$ and $\overline{UA} = 3x + \dfrac{7}{4}$, find the values of the legs. *Hint: What makes a trapezoid isosceles? Use Definition 1.4.*

§1.3 The Rectangle

This subsection is brief. Consider a rectangle $\square ROFL$.

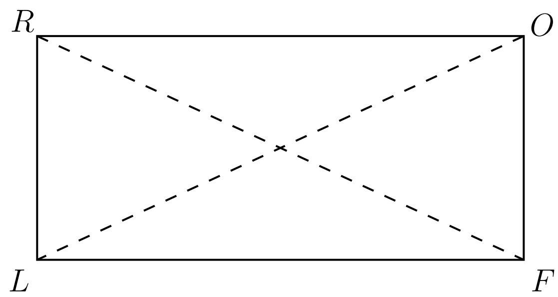


Figure 1.15: Rectangle $\square ROFL$

Consider the following theorem:

Theorem 1.15:

The diagonals of a rectangle are congruent.

§1.3.1 Examples

Let's apply this to an example. Use the same rectangle.

Example 1.7:

From the original rectangle $\square ROFL$, if $\overline{RF} = 4x - 5$ and $\overline{OL} = 2x + 17$, find the measurements of the diagonals.

Solution. Use **Theorem 1.15**. Equate both diagonals.

$$\begin{aligned} \overline{RF} &\cong \overline{OL} \\ 4x - 5 &= 2x + 17 \\ 4x - 2x &= 17 + 5 \\ 2x &= 22 \\ \frac{2x}{2} &= \frac{22}{2} \\ x &= 11 \end{aligned}$$

Find the values after.

$$\begin{aligned} 4(11) - 5 &= 2(11) + 17 \\ 44 - 5 &= 22 + 17 \\ 39 &\stackrel{\checkmark}{=} 39 \end{aligned}$$

Therefore, $\boxed{\overline{RF} = 39}$ and $\boxed{\overline{OL} = 39}$.

§1.4 The Midline of a Triangle and the Median of a Trapezoid

We can always draw lines in the direct middle of a shape. In a trapezoid, this is called the *median* of the trapezoid.

Definition 1.5: The Median of a Trapezoid

The *median* of a trapezoid is a segment that joins the midpoint of the legs.

The *midpoint* is the point directly on the middle of the line segment.

For a triangle, we call such a line segment the *midline*.

Definition 1.6: The Midline of a Triangle

The *midline* of a triangle is a segment that joins the midpoint of the two sides adjacent to the base.

I know this section is named "Geometry on Quadrilaterals" but has a subsection with a triangle as its basis. Anyway, look on the figure with a midline. We can deduct that the length of the midline appears to be quite smaller than the length of the base. The theorem after says why.

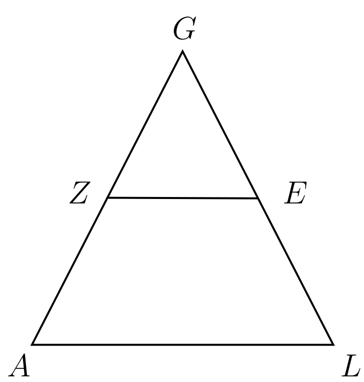


Figure 1.16: Triangle $\triangle GLA$ with midline $\overline{ZE}$

Theorem 1.16: Measure of the Midline

The midline of a triangle is *parallel* to its base. Its length is half as long as the base.

Remark. We know for a fact that since the midline is a half of the length of the base, the base is double the length of the midline.

Look on the following figure of a trapezoid with a median. This one is harder to deduct. The length of the median can be seen with our next theorem. Ignore the name of the quadrilateral, I thought of it a while back to name one like it.

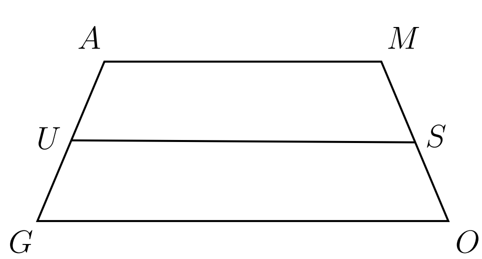


Figure 1.17: Trapezoid $\square AMOG$ with median $\overline{US}$

Theorem 1.17: Measure of the Median

The median of a trapezoid is *parallel* to its bases. Its length is half the sum of the bases of the trapezoid.

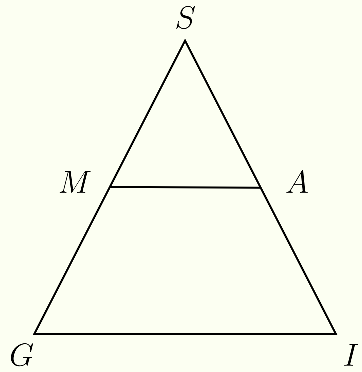
Remark. With the same idea as the prior remark, since the median is half the length of the sum of the bases, the sum of the bases is double the length of the median.

§1.4.1 Examples

Two examples, two theorems. Remarks included and not sold separately™.

Example 1.8:

Consider the following triangle $\triangle SIG$ with midline $\overline{MA}$.



Find:

- If $\overline{GI} = 4x + 20$ and $\overline{MA} = 4x$, find the length of the midline and the base.
- If $\overline{GI} = 3x + 15$ and $\overline{MA} = 2x - 13$, find the length of the midline using the base as reference.

Solution. Apply *Theorem 1.15.*

1. The midline is half the length of the base.

$$\begin{aligned}
\overline{MA} &= \frac{1}{2}\overline{GI} \\
4x &= \frac{1}{2}(4x + 20) \\
2(4x) &= 2\left(\frac{1}{2}(4x + 20)\right) \\
8x &= 4x + 20 \\
4x &= 20 \\
\frac{4x}{4} &= \frac{20}{4} \\
x &= 5
\end{aligned}$$

Find the lengths separately. Start with the base.

$$\begin{aligned}
4(5) + 20 &= \overline{GI} \\
\overline{GI} &= \boxed{40}
\end{aligned}$$

Then the midline. We know that the midline is half the length of the base, so we can just half the result of $\overline{GI}$ or the base to yield the midline.

$$\frac{1}{2}40 = \boxed{20}$$

2. Basing of the base means we are finding the base (kinda tongue twister.) Start by using the remark we made from **Theorem 1.16**. We double the length of the midline to find the original length of the base.

$$\begin{aligned}
\overline{GI} &= 2\overline{MA} \\
3x + 15 &= 2(2x - 13) \\
3x + 15 &= 4x - 26 \\
4x - 3x &= 15 + 26 \\
x &= 41
\end{aligned}$$

Next, find the length of the midline. We can input it to the length of the midline $(2x - 13)$ but we will do another way.

Again, half the base is the median's length. Express as formula,

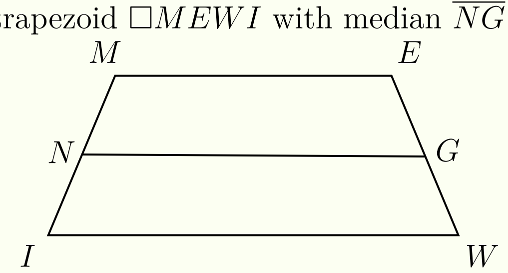
$$\frac{1}{2}\overline{GI}$$

We know what $\overline{GI}$ is, so the next step is to substitute x.

$$\begin{aligned}
\frac{1}{2}(3x+15) &= \overline{MA} \\
\frac{1}{2}(3(41)+15) &= \overline{MA} \\
\frac{1}{2}(138) &= \overline{MA} \\
\overline{MA} &= \boxed{69}
\end{aligned}$$

Example 1.9:

Consider the following trapezoid $\square MEWI$ with median $\overline{NG}$.



If $\overline{IW} = 3x+7$ and $\overline{ME} = 2x+3$ and $\overline{NG} = 45$, find the value of x and the measure of the bases.

Solution. By **Theorem 1.17**: We infer that $\overline{NG} = \frac{1}{2}(\overline{ME}+\overline{IW})$.

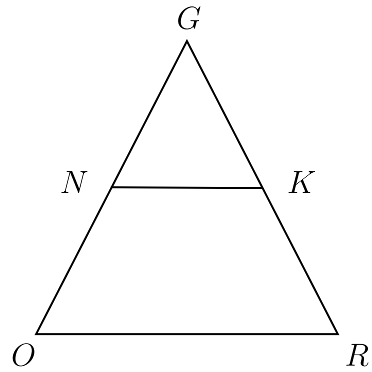
$$\begin{aligned}
45 &= \frac{1}{2}(2x+3+3x+7) \\
45 &= \frac{1}{2}(5x+10) \\
2 \cdot 45 &= \cancel{2}\left(\frac{1}{\cancel{2}}(5x+10)\right) \\
90 &= 5x+10 \\
90-10 &= 5x \\
5x &= 80 \\
\frac{5x}{5} &= \frac{80}{5} \\
x &= 16
\end{aligned}$$

Plug in x to the measure of the bases.

$$\begin{aligned}
\mathrm{m}\overline{ME} &= 2x+3 \\
2(16)+3 &= \boxed{35} \\
\mathrm{m}\overline{IW} &= 3x+7 \\
3(16)+7 &= \boxed{55}
\end{aligned}$$

§1.4.2 Exercises

I: Consider a triangle $\triangle GRO$ with a midline $\overline{NK}$.

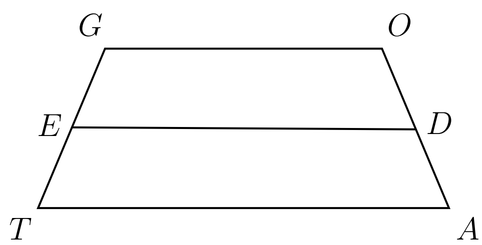


Find:

1. Find the value of the midline (using the base as reference) if $\overline{NK} = 8x + 17$ and $\overline{RO} = 82$. *Hint:* $2\overline{NK} = \overline{RO}$.

2. Find the length of the midline if we know the length of the base is 175.

II: Consider a trapezoid $\square GOAT$ with median $\overline{ED}$.



Find:

1. Find the length of the base if the two bases are $3x - 11$ and $2x + 33$.

2. Find the length of the median if the median is $4x - 5$ and the bases are 15 and 26.

If any of you get stuck, use ChatGPT. Just kidding, but you *can* try. 75% of the time it is correct or accurate. The 25% is really the kicker.

§1.5 The Kite

A *kite* is a shape that two pairs of congruent sides but opposing sides aren't.

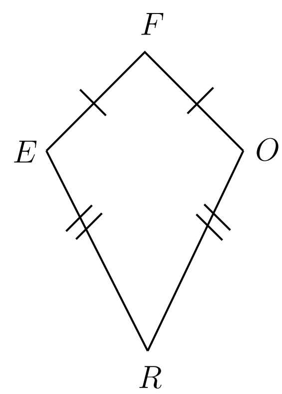


Figure 1.18: Kite $\square FORE$

EDITORIAL: This took me, WAY too long to make using just straight lines. Don't ask why. The image editor did not have a kite preset. Ugh.

Drawing diagonals on the kite shows that they are perpendicular, like on a rhombus. This can be seen as our next theorem:

Theorem 1.18:

The diagonals of a kite are perpendicular.

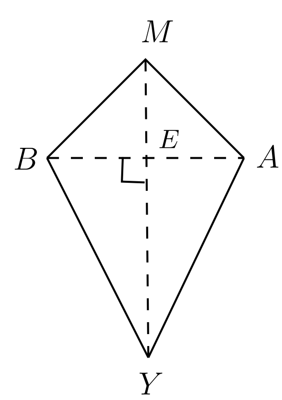


Figure 1.19: Kite $\square MAYB$ with point E

Notice that one some ends of a kite, there are two opposing angles that are equal. Segue:

Theorem 1.19:

A kite' has exactly one pair of opposite angles that are congruent.

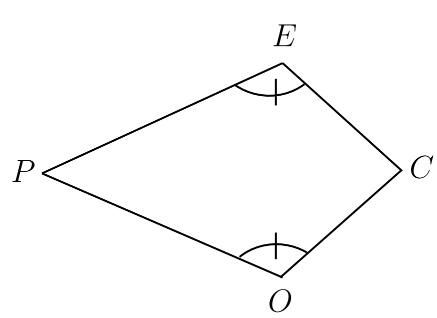


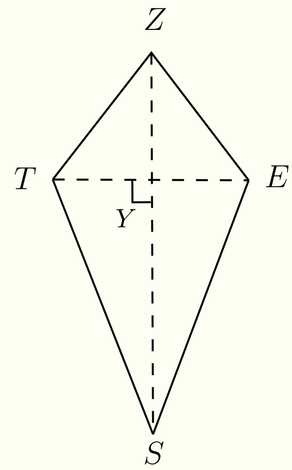
Figure 1.20: Kite $\square COPE$

§1.5.1 Examples

This is the final subsubsection for §1! Let's begin: again, two theorems, two examples. Remarks sold separately™.

Example 1.10:

Consider a kite $\square ZEST$ with point Y.



Find:

1. A value of t that makes $\angle S = 7t + 6$ perpendicular.
2. $\angle T$ and $\angle E$ if they equal $12y - 55$ and $7y + 70$ respectively.

Solution. Apply the theorems curated for the kite.

1. Use **Theorem 1.18**.

$$\begin{aligned} 7t + 6 &= 90 \\ 7t &= 90 - 6 \\ 7t &= 84 \\ \frac{7t}{7} &= \frac{84}{7} \\ t &= \boxed{12} \end{aligned}$$

2. Use **Theorem 1.19**.

$$\begin{aligned} 12y - 55 &= 7y + 70 \\ 12y - 7y &= 70 + 55 \\ 5y &= 125 \\ \frac{5y}{5} &= \frac{125}{5} \\ y &= 25 \end{aligned}$$

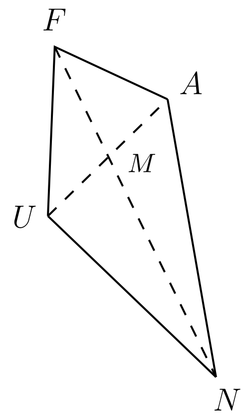
Re-substitute.

$$\begin{aligned} 12(25) - 55 &= 7(25) + 70 \\ 300 - 55 &= 175 + 70 \\ 245 &\stackrel{\checkmark}{=} 245 \end{aligned}$$

Therefore, $\angle T \cong \angle E$ hence $\boxed{\angle T, \angle E = 245^\circ}$.

§1.5.2 Exercises

I: Consider a kite $\square FANU$ with point M.



Find the following:

1. A value of k that makes $\angle M = 5k - 14$ perpendicular. Express your answer as a decimal.

2. $\angle U$ and $\angle A$ if they equal $x^2 - 45$ and 19.*

II: Construct a kite $\square SILO$ with point G. Show that:

1. The diagonals are perpendicular. *Hint: Measure*

2. The other pair of angles are not equal. *Hint: Visuals*

This is not a proof. I am requesting a reason why these work. It can be short or long, as long as it is a reason people can agree with generally and not rigorously.

§2 Proportion, Represented Geometrically

I remember the times of the pandemic. When we were all still Grade 6. 3rd year of middle school, fun times. Being locked in your house for two years does something to you. Anyway, a topic we discussed during that grade was *proportion*. Ideally, we already have seen it in Q2 or last quarter. But this time, it will be represented geometrically and re-defined with properties not seen as a sixth grader.

§2.1 Proportion, Represented Algebraically

Recall that a *ratio* is a comparison between two positive numbers a and b or quantities, expressed as either $a : b$ or $\frac{a}{b}$. It is read as "a is to b."

There can be more than one ratio, like $a : b : c$. Given a quantity to another, it is comparing them to it. For example, 25 quantities and 45 other quantities can be seen as "25 is to 45" or if we were reduced "5 is to 9" because as a fraction, you can reduce it lower. Remember, we can always represent a ratio as a fraction.

$$\frac{25}{45} = \frac{5}{9}$$

We will not cover *all* kinds of proportion, because this is not included in our textbooks. However, the definition of two ratios being equal is as follows:

Definition 2.1:

A *proportion* is a statement of equality between two ratios $a : b = c : d$ or $\frac{a}{b} = \frac{c}{d}$.

Remark. b and c are the *means* of a proportion. a and d are the *extremes* of a proportion.

There are four properties that fulfill the definition of a proportion. They help justify the definition of proportions to two ratios. Proportions can have three ratios, keep in mind.

Property 2.1 (Cross Product Property)**.** In a proportion, the product of the extremes is equal to the product of the means.

$$\frac{a}{b} = \frac{c}{d} := ad = bc$$

$:=$ means "defined to be equal to." You don't need to study it, it's just like equals.

If the means are the same, meaning they are *equal*, using **Property 2.1**:

$$\frac{a}{x} = \frac{x}{b} := x^2 = ab \text{ or } \boxed{x = \sqrt{ab}}$$

The result $x = \sqrt{ab}$ is considered the *geometric mean* in between of the two numbers a and b.

Property 2.2 (Reciprocation of Ratios)**.** In a proportion, the reciprocals of the two ratios are also equal.

$$\frac{a}{b} = \frac{c}{d} := \frac{b}{a} = \frac{d}{c}$$

Property 2.3 (Interchanging of Quantity)**.** In a proportion, means can be interchanged and extremes can be interchanged separately.

$$\frac{a}{b} = \frac{c}{d} := \frac{d}{b} = \frac{c}{a}$$
$$\frac{a}{b} = \frac{c}{d} := \frac{a}{c} = \frac{b}{d}$$

Property 2.4 (Denominator Property)**.** In a proportion, the value of each ratio's denominator can be added to or subtracted from the numerator and it would still stay equal.

$$\frac{a}{b} = \frac{c}{d} := \frac{a+b}{b} = \frac{c+d}{d}$$
$$\frac{a}{b} = \frac{c}{d} := \frac{a-b}{b} = \frac{c-d}{d}$$

Now we have represented the properties of proportions following the definition. Now, we can represent these geometrically. Examples will be done later.

§2.2 Proportions on Geometric Figures

The lengths of quadrilaterals can change but the angle themselves are the same. We declare that two figures are *similar* by this notation: $A \sim B$. The sizes of the sides have changed but the measure of the angles remains the same.

Remark (Polygon Similarity)*.* Two polygons are similar if and only if the corresponding angles are congruent and the corresponding sides are in a proportion.

If a polygon's sides are literally out of ratio, then it is not similar. It is like scaling up the size of a triangle. The shape is still the same but the sides are larger.

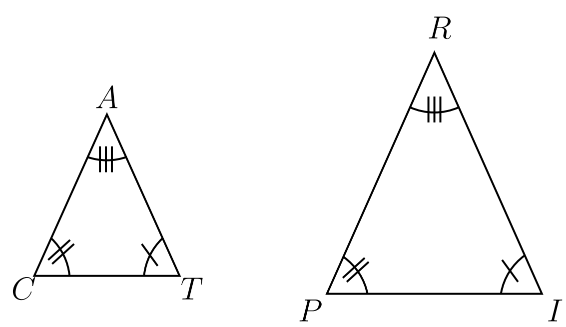


Figure 2.1: Triangles $\triangle ACT$ and $\triangle RIP$

We can clearly see that the triangles' sides are distinct. But, the angles are marked and clearly are congruent. We can infer that, by our Remark:

1. $\overline{AT} : \overline{RI}$
2. $\overline{AC} : \overline{RP}$
3. $\overline{CT} : \overline{PI}$

Any of these ratios are titled the *scale factors* of the two similar figures. Therefore, we can conclude that $\triangle ACT \sim \triangle RIP$.

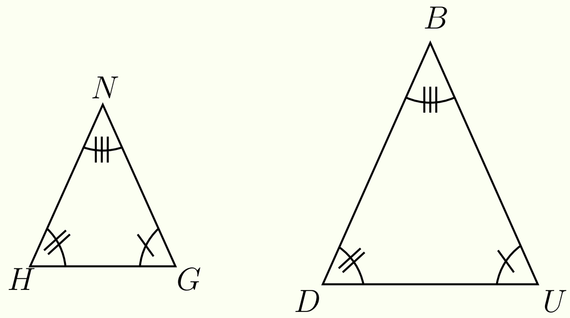
We can say the same for quadrilaterals. As long as we are *only* scaling it up and not stretching or squeezing any part of the shape, it *still* has both pairs of angles congruent.

We can apply these to isosceles triangles too, considering their congruence on one pair of sides. We will elaborate later in the example subsection.

§2.2.1 Examples

Example 2.1:

Consider $\triangle NGH \sim \triangle BUD$ and both are isosceles.



Find the following:

1. If $\overline{NH} : \overline{BD}$, and $\overline{NG} : \overline{BU}$ considering that $\overline{NH} = 2x+1, \overline{BD} = 3, \overline{NG} = x+4, \overline{BU} = 5$, find the value of x.
2. Find the similarity statement between the two triangles.
3. If the side lengths are $3, 9, 10$ going right, bottom, left, find the value of x to find the missing ratios of $12, x, 40$. I will reword this later, hard to explain.

Solution. We stated that both of these triangles are isosceles.

1. We know from an isosceles triangle that the legs are equal. With this in mind, when we equate the first ratio to the second, it becomes bigger. We see that $\overline{NH} : \overline{BD}$ has the same proportion as $\overline{NG} : \overline{BU}$. Compare the sizes of the legs. They look similar for both cases. Hence they are equal.

   Substitute with the values mentioned, and solve the equation:

$$\overline{NH} : \overline{BD} = \overline{NG} : \overline{BU}$$
$$\frac{2x+1}{3} = \frac{x+4}{5}$$

   By **Property 2.1**:

$$\begin{aligned} 5(2x+1) &= 3(x+4) \\ 10x+5 &= 3x+12 \\ 10x-3x &= 12-5 \\ 7x &= 7 \\ \frac{7x}{7} &= \frac{7}{7} \\ x &= \boxed{1} \end{aligned}$$

   Question. Since we know x, what are the values of the legs?

2. What I mean by *similarity statement* is enumerating all of it's proportional sides. **Statement.** $\overline{NH} : \overline{BD}, \overline{NG} : \overline{BU}, \overline{HG} : \overline{NU}$. Therefore, $\triangle NGH \sim \triangle BUN$.

3. Since we stated we label the sides' measures going right, bottom, to left, we enumerate as $\overline{NG} = 3, \overline{HG} = 9, \overline{NH} = 10$. We need to find the value of x that satisfies the statement for the larger triangle. Hence, $3 : 12, 9 : x, 10 : 40$.

   We need to find x that makes 9 a ratio to. Simply, we can always use induction (or pattern recognition). We see that $3 : 12$, twelve is four times as three. Similarly, $10 : 40$ is four times as ten. We expect that x is four times as nine, or $\boxed{36}$[5].

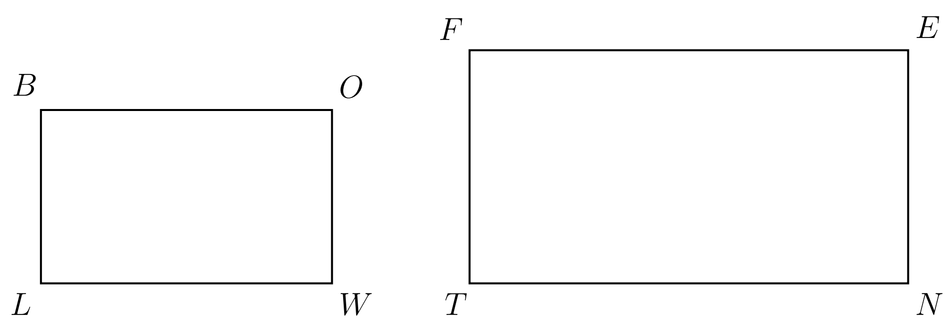
[5] Not ALL ratios like this are this easy to solve. I will omit this from the exercises, I do not expect something like this on the exam.

§2.2.2 Exercises

I: Determine what property was used to complete the proportion. Some have two properties applied.

1. $\frac{2}{7} = \frac{8}{28} := 42 = 42$
2. $\frac{x}{y} = \frac{2x}{3y} := \frac{y}{x} = \frac{3y}{2x} := \frac{2x}{x} = \frac{3y}{y}$
3. $\frac{3}{x} = \frac{9}{x+1} := \frac{3+x}{x} = \frac{9+x+1}{x+1} = \frac{x+3}{x} = \frac{x+10}{x+1}$
4. $\frac{x}{27} = \frac{3}{x} := x^2 = 81 = x = 9$*
5. $\frac{3x+1}{2} = \frac{2x+5}{10} := \frac{3x+1-2}{2} = \frac{2x+5-10}{10} := 10(3x-1) = 2(2x-5)$*

II: Consider two similar rectangles $\square BOWL \sim \square FENT$.



Find the proportions and solve for x:

1. If $\overline{BO} = x+1, \overline{FE} = 9$, and $\overline{LW} = 2, \overline{TN} = x+1$, find x as a positive integer.
2. If $\overline{BL} = 4, \overline{FT} = x$, and $\overline{OW} = x, \overline{EN} = 2x-4$, find the value of x and the measures of the sides.
3. Form the similarity statement of the two rectangles.

Question (HARD)**.** This question came up in my head while I was thinking for exercises. This might be a little hard, but *Hint: Involves linear systems.* Reminder this is not necessary, just try it out yourself if you still have mathematical maturity!

I assume you know the proportion statements of the rectangle already. If:

- $\overline{BL} = x-2$
- $\overline{FT} = y$
- $\overline{OW} = 2$
- $\overline{EN} = 3$
- $\overline{BO} = 1$
- $\overline{FE} = 2$
- $\overline{LW} = x+1$
- $\overline{TN} = y+3$

Find the values of x and y that satisfy BOTH proportions. Once done, are they negative? If so, can we consider them actual measures?*

§2.3 Triangle Similarity

Back in Year 2, we discovered the three postulates and one theorem of triangle congruence. They are the SSS postulate, SAS postulate, ASA postulate, and SAA postulate respectively. Recall that a *postulate* is a statement accepted without proof.

In this subsection, we discuss about *triangle similarity*. The direct comparison between two triangles that are similar has been talked about, but this time we will expand upon the knowledge preceded before us. We shall begin with our first postulate:

Postulate 1 (Angle-Angle Similarity (AA)). *If the two angles of a triangle are congruent to the two angles of another triangle, then the two triangles have similarity.*

See the figure below.

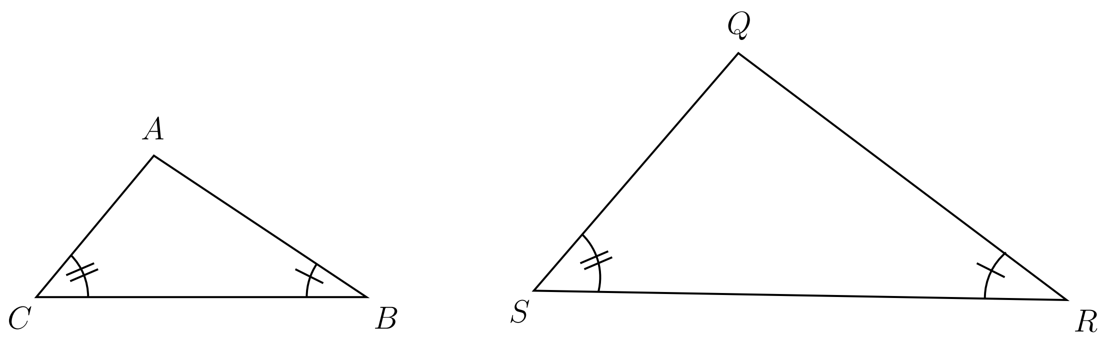


Figure 2.2: Similar Triangles $\triangle ABC \sim \triangle QRS$

We see angles $\angle C$ and $\angle S$ are congruent, as well as $\angle B$ and $\angle R$. By our postulate, they are equal.

We can visualize this by constructing a triangle with two angle measures. One is slightly larger, yet the angles stay the same. Same gist as similar polygons.

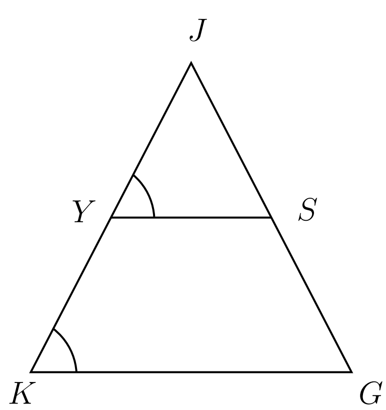


Figure 2.3: Similar Triangles $\triangle JGK \sim \triangle JSY$

We can also imagine inscribed measures, like a midline on a triangle. Another theorem that we can bring up related to triangle similarity is about proportion of the side lengths including the congruent angle.

Theorem 2.1: SAS Similarity

If an angle of a triangle is congruent to an angle of another triangle, and the lengths of the sides including these angles are proportional, then the triangles have similarity.

Illustrated next page.

We illustrate our theorem:

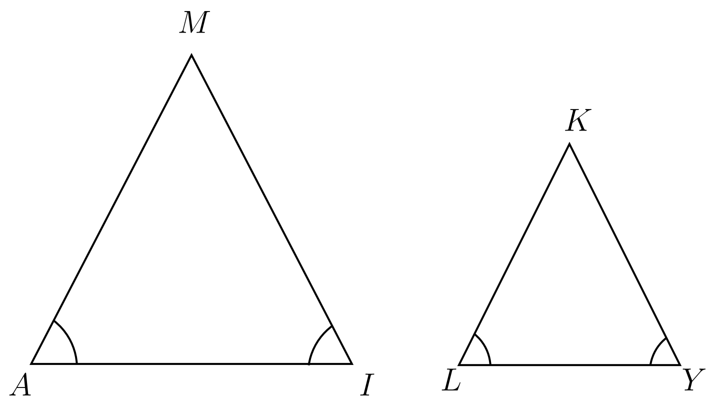


Figure 2.4: Similar Triangles $\triangle MIA \sim \triangle KYL$

We illustrate that: $\angle A \cong \angle L$. We can also use the other pair but I will use this one. We see that $\overline{AM}$: $\overline{KL}$ and $\overline{MI}$: $\overline{KY}$. This therefore forms the similarity statement $\triangle MIA \sim \triangle KYL$.

The next theorem has the same idea, but applied for all three sides (based of the SSS postulate.)

Theorem 2.2: SSS Similarity

If the lengths of the corresponding sides of two triangles are proportional, then the two triangles form similarity.

Illustrated on the next figure:

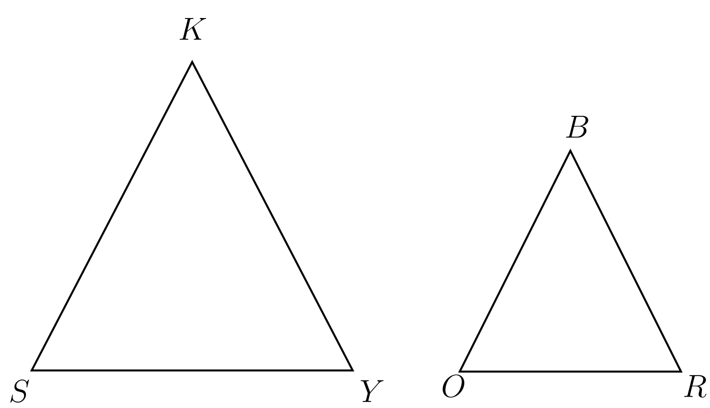


Figure 2.5: Similar Triangles $\triangle KYS \sim \triangle BRO$

We illustrate that: $\overline{KS} : \overline{BO} = \overline{KY} : \overline{BR} = \overline{SY} : \overline{OR}$. Therefore, the similarity statement is formed, $\triangle KYS \sim \triangle BRO$.

Our next theorem involves the midline. It involves how it halves the triangle into half. We will refer to the midline as the "parallel line" because of how it is labelled in the book.

Theorem 2.3:

If a line parallel to one side of a triangle intersects the remaining sides, then it divides the two sides as a proportion.

Next page.

Illustrated, using triangle $\triangle PUN$.

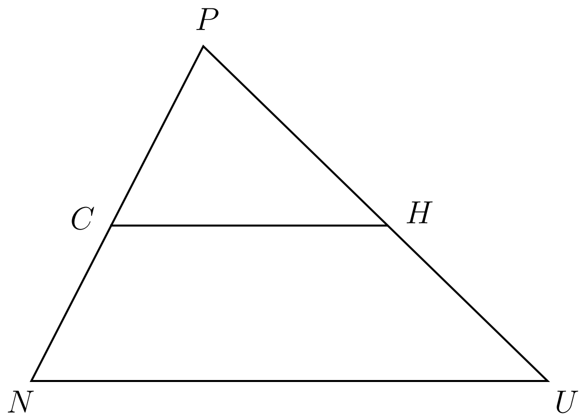


Figure 2.6: Triangle $\triangle PUN$ with parallel line $\overline{CH}$

From what was stated in **Theorem 2.3**, $\overline{PC} : \overline{CN} = \overline{PH} : \overline{HU}$. We also denote that the converse of this theorem is still true.

Theorem 2.4: Converse of Theorem 2.3

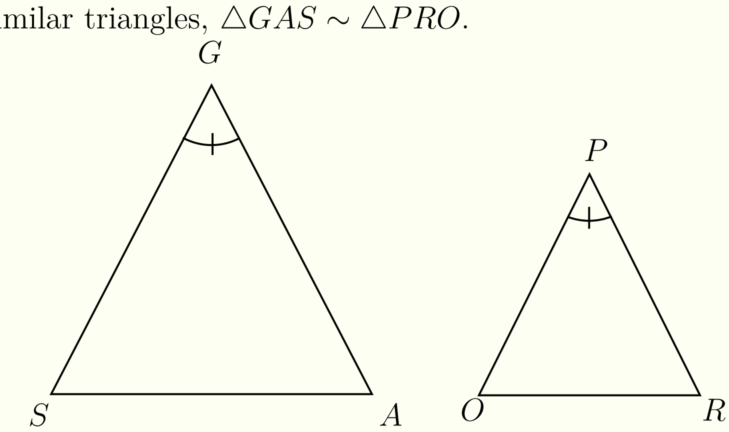
If a line divides the two sides of a triangle as a proportion, then it is parallel to the third side of the triangle.

§2.3.1 Examples

I had to grind this late at night. Haha. Anyways, examples.

Example 2.2: SAS Similarity

Consider two similar triangles, $\triangle GAS \sim \triangle PRO$.



Consider the following.

1. If $\angle G$ and $\angle P$ equate to $2y + 5$ and $3y + 1$ respectively, find y.
2. If $\overline{GS} : \overline{PO} = \overline{GA} : \overline{PR}$ are $x^2 : 4 = (0.25x + 3) : 1$, find the value of x such that $\overline{GS}$ is the longest side overall.

Solution. We begin by utilizing **Theorem 2.1**.

1. The angles are congruent.

$$\begin{aligned} 2y + 5 &= 3y + 1 \\ 5 - 1 &= 3y - 2y \\ y &= \boxed{4} \end{aligned}$$

The reader is tasked to find the value of the angles alone.

2. Form the proportion as fractions. Use **Property 2.1**.

$$\begin{aligned}\frac{x^2}{4} &= \frac{0.25x+3}{1}\\ x^2 &= 4(0.25x+3)\\ x^2 &= x+12\\ x^2-x-12 &= 0\\ (x-4)(x+3) &= 0\\ x &= 4, -3\end{aligned}$$

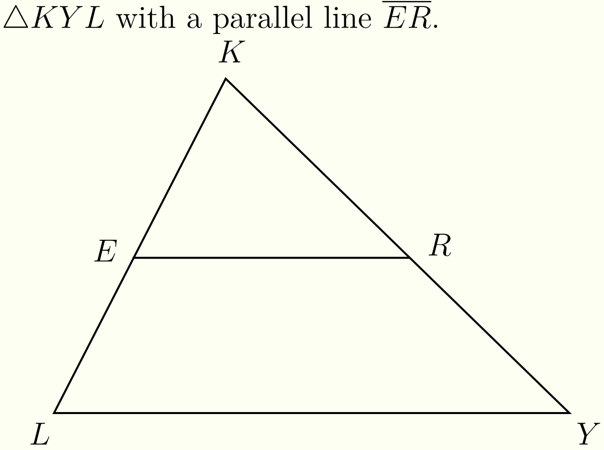
Note that the $\overline{GS} = x^2$. We can assume that when $x = 4$, the side length is longest. So we consider $\boxed{x = 4}$ as our answer, but we can also consider $x = -3$ as an answer but we have a conditional.

$$(4^2) = 16 \text{ is bigger than } (-3)^2 = 9$$

I will be leaving applications of **Theorem 2.2** and **Postulate 1** as exercises. You should probably know by now what parts should be congruent and not.

Example 2.3:

Consider a triangle $\triangle KYL$ with a parallel line $\overline{ER}$.



Consider that from **Theorem 2.3**:

$$\frac{\overline{KE}}{\overline{EL}} = \frac{\overline{KR}}{\overline{RY}}$$

Find the value of u when $\overline{KE} = u^2, \overline{EL} = 5, \overline{KR} = 2u-5, \overline{RY} = 1$.

Solution. By applying **Theorem 2.3**:

$$\begin{aligned}\frac{u^2}{5} &= \frac{2u-5}{1}\\ u^2 &= 5(2u-5)\\ u^2 &= 10u-25\\ u^2-10u+25 &= 0\\ (u-5)^2 &= 0\\ \sqrt{(u-5)^2} &= \sqrt{0}\\ u-5 &= 0\\ u &= \boxed{5}\end{aligned}$$

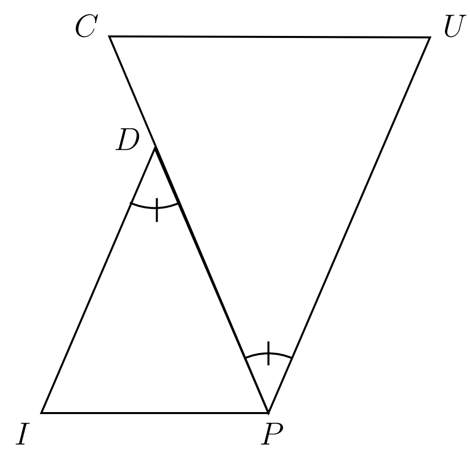
I humbly request you find the sides separately as an exercise.

§2.3.2 Exercises

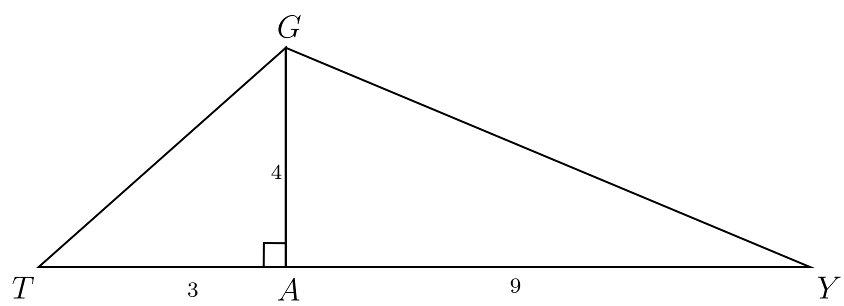
I: Identify out of the three what was used to form similarity.

- SAS for **SAS Similarity**.
- SSS for **SSS Similarity**.
- AAP for **AA Postulate**.

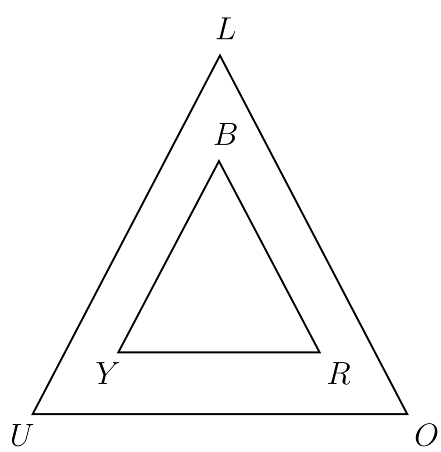
1. $\triangle CUP \sim \triangle DIP$.



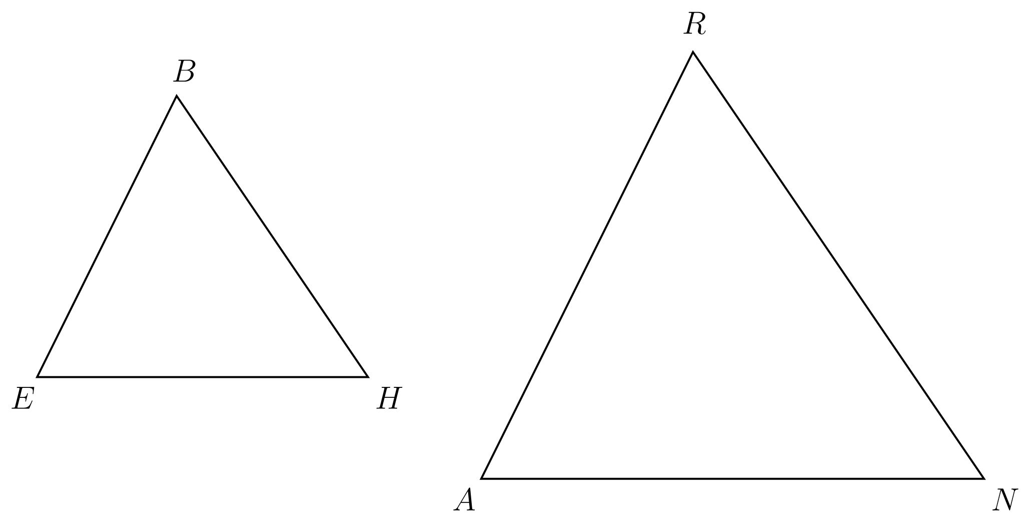
2. $\triangle GYA \sim \triangle GTA$.*



3. $\triangle LOU \sim \triangle BRY$. I know you're reading this you two. The triangles are NOT a double entendre.*



II: Consider two similar triangles $\triangle BHE \sim \triangle RNA$. This is the ultimate part that might take me forever to think about (just kidding. UNLESS???)



Find the following: (for 1-3, I added what was used to solve.)

1. **AA Postulate**. Find an angle pair that makes them similar.
2. **SAS Similarity**. On $\triangle BHE$, it has measures $\overline{BE} = 5$, $\overline{BH} = 8$, and $\angle A = 11$. If $\triangle RNA$ is quadrupled in size, what are the lengths now?
3. **SSS Similarity**. *Inspired from Example 4.1*; If:

$$\frac{2}{3} = \frac{\overline{BH}}{9} = \frac{10}{\overline{AN}}$$

   Find the values of the missing segments.*
4. A parallel line $\overline{XY}$ intersects the other two sides of $\triangle BHE$. If:

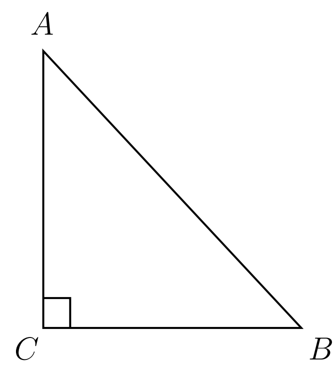
$$\frac{x+1}{3x} = \frac{2}{5}$$

   Find the value of x.

Class Editorial: I recommend you study on Example 4.1 from the book on page 203. This might appear on the test (for the class I am in, not to the outsider reader.) Some other examples might appear in the test that I am unaware of, so practice as much as you can.
This has also been brought up to me earlier, study quadrilaterals on extensive detail (not too extensive.)

§3 The Pythagorean Theorem

The *Pythagorean theorem* is one of many famous theorems in general mathematics, and it is one of the most influential. Alone, it has been proven in as many as 370 ways (from one source.) It is the root of trigonometry, some calculus, proofs in analytic geometry, and more. This pre-subsection is meant to give some historical background to the Pythagorean theorem.



The Pythagorean theorem stems from Greek mathematician *Pythagoras* of Samos. He was a Greek philosopher, and polymath. He created a philosophy known as *Pythagoreanism*. Albeit, this is a bit of a misnomer. Babylonians and Indians centuries before used this theorem, but it was more radicalized to himself of its introduction to Greece.

One famous legend that existed was of the constant $\sqrt{2}$. It was discovered from one of his students, thought to be Hippasus. When he discovered it, many Pythagoreans kept it secret. It was invalid in their philosophy, so (according to poor sources) he was murdered for it. But again, this is theorization and I am not a historian.

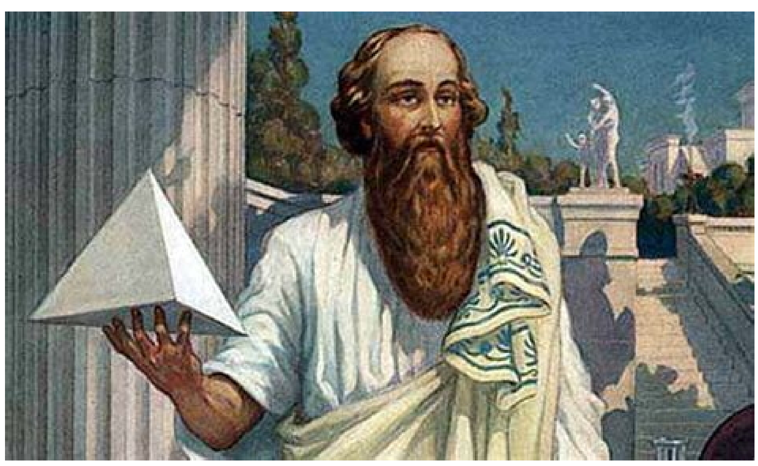


Figure 3.1: Pythagoras of Samos

We shall proceed to the iconic theorem and its elementary results on certain right triangles.

§3.1 The Groundwork

I knew this theorem when I was in Grade 4. I didn't know the nitty-gritty but I knew how it worked.

Theorem 3.1: The Pythagorean Theorem

The sums of the squares of the legs of a right triangle is the measure of the longest side.

As an equation, if we have a right triangle,

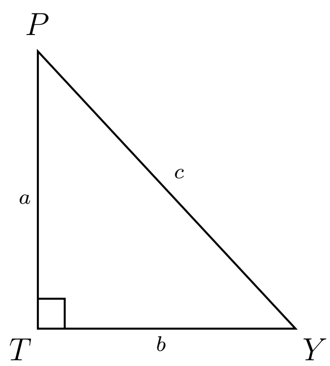


Figure 3.2: Right Triangle $\triangle PYT$

The side lengths a, b with longer side length c form the iconic equation[6]:

$$a^2 + b^2 = c^2$$

If we were to derive for c, another equation can be formed. It is somewhat familiar to more math-driven people.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ \sqrt{a^2 + b^2} &= \sqrt{c^2} \\ c &= \sqrt{a^2 + b^2} \end{aligned}$$

§3.1.1 Examples

This is just an examples section. It is not going to hurt, I am just going to list three basic examples from the book we're using.

Example 3.1:

In the original right triangle $\triangle PYT$, find the value of each of the missing in the following:

1. $a = 3, b = 8, c = ?$
2. $a = ?, b = 24, c = 25$
3. $a = 10, b = ?, c = 4\sqrt{13}$

Solution. Note that the general equation is $a^2 + b^2 = c^2$.

1. Solve for c.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (3)^2 + (8)^2 &= c^2 \\ c^2 &= 64 + 9 \\ c^2 &= 73 \\ \sqrt{c^2} &= \sqrt{73} \\ c &= \boxed{\sqrt{73}} \end{aligned}$$

[6] Am I glazing too much, no?

2. Solve for a.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + (24)^2 &= (25)^2 \\ a^2 + 576 &= 625 \\ a^2 &= 625 - 576 \\ a^2 &= 49 \\ \sqrt{a^2} &= \sqrt{49} \\ a &= \boxed{7} \end{aligned}$$

3. Solve for b.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (10)^2 + b^2 &= (4\sqrt{13})^2 \\ 100 + b^2 &= 4^2 \cdot (\sqrt{13})^2 \\ 100 + b^2 &= 208 \\ b^2 &= 208 - 100 \\ b^2 &= 108 \\ \sqrt{b^2} &= \sqrt{108} \\ b &= \sqrt{36}\sqrt{3} \\ b &= \boxed{6\sqrt{3}} \end{aligned}$$

If you have a keen eye, you noticed we did not add $\pm$ to any of the final results. That is because *negative measures are invalid.* In algebra, negative values exist. But measuring negative values is common sense, illogical. So we exclude negative results unless the results themselves turn to positive measures[7].

§3.2 Side Properties of the Pythagorean Theorem

The converse of the Pythagorean theorem is also true.

Theorem 3.2: Converse of the Pythagorean Theorem

If the square of the longest side of a triangle is the sum of the squares of all remaining sides, then it is a right triangle.

From now on, I will call the longest side the *hypotenuse.*

We shall also make another deduction. If c^2 is smaller or bigger than $a^2 + b^2$, the triangle is no longer at a $90°$ angle. Instead, it can either be an acute or obtuse triangle. This goes to the next two theorems.

Theorem 3.3:

If the square of the hypotenuse of a triangle is greater than the sum of the squares of the remaining sides, the triangle is *obtuse.*

Theorem 3.4:

If the square of the hypotenuse of a triangle is less than the sum of the squares of the remaining sides, the triangle is *acute.*

[7] See Appendix A.

See the illustration below:

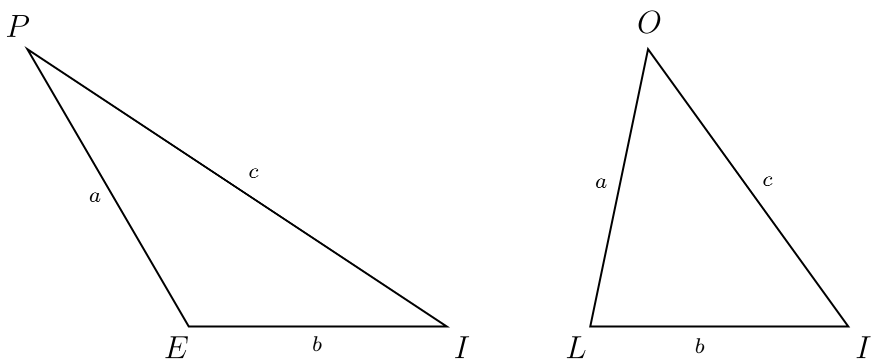


Figure 3.3: Obtuse Triangle $\triangle PIE$ and Acute Triangle $\triangle OIL$

We note in $\triangle PIE$, it is obtuse. **Theorem 3.3** in eqaution is:

$$c^2 > a^2 + b^2$$

In $\triangle OIL$, it is acute. **Theorem 3.4** in equation is:

$$c^2 < a^2 + b^2$$

§3.2.1 Examples

Example 3.2: Example 5.3, p. 224; adjusted

Use the following side-lengths and identify if: **R** if right triangle, **A** if acute, **O** for obtuse.

1. 8,10,12
2. $\sqrt{7}, 2\sqrt{2}, 4$
3. 5,12,13

Solution. Start from using the theorems mentioned. We can just stem from the original Pythagorean theorem and infer from there.

1. **A**.

$$\begin{aligned} 12^2 &= 8^2 + 10^2 \\ 144 &= 64 + 100 \\ 144 &\neq 164 \end{aligned}$$

We see 144 is less than 164. We determine it is an acute triangle.

2. **O**.

$$\begin{aligned} 4^2 &= (\sqrt{7})^2 + (2\sqrt{2})^2 \\ 16 &= 7 + 2^2 \cdot (\sqrt{2})^2 \\ 16 &= 7 + 8 \\ 16 &\neq 15 \end{aligned}$$

We see 16 is greater than 15. We determine it is an obtuse triangle.

3. **R**.

$$\begin{aligned} 13^2 &= 5^2 + 12^2 \\ 169 &= 25 + 144 \\ 169 &\stackrel{\checkmark}{=} 169 \end{aligned}$$

Since they are equal, we determine it is a right triangle.

§3.2.2 Exercises

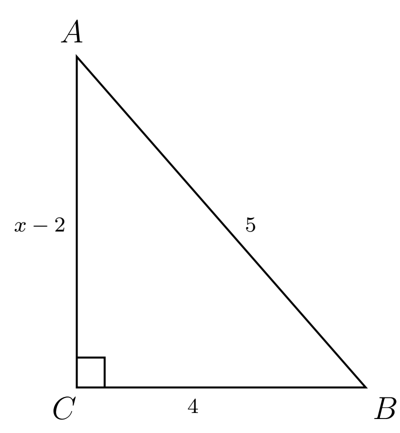
I: Find the missing values requested each number. Express in simplified form.

1. $a = 8, b = 15, c = ?$
2. $a = 3, b = ?, c = 5$
3. $a = 1, b = 1, c = ?$
4. $a = ?, b = 9, c = 14$
5. $a = ?, b = 24, c = 26$
6. $a = 7, b = ?, c = 25$
7. $a = 2, b = 4, c = ?$
8. $a = 10, b = 10, c = ?$
9. $a = 20, b = ?, c = 29$
10. $a = 8, b = ?, c = 17$
11. $a = 3, b = 6, c = ?$
12. $a = ?, b = 17, c = 20$
13. $a = ?, b = 2, c = 7$
14. $a = 5, b = ?, c = 8$

II: Identify whether the following is an: **A** for acute, **O** for obtuse, **R** for right triangle.

1. 12, 35, 37
2. 6, 8, 10
3. 10, 20, 30
4. 7, 9, 10
5. 2, 7, 8
6. 3, 9, 11
7. 7, 8, 12
8. 10, 11, 14
9. $\sqrt{3}, 3, 2\sqrt{3}$
10. $5\sqrt{3}, \sqrt{6}, 9$
11. $\sqrt{3}, 2\sqrt{3}, 3$
12. $\sqrt{11}, 1, 4$
13. $\sqrt{15}, \sqrt{11}, 6$
14. $\sqrt{2}, \sqrt{3}, 3$
15. $\sqrt{11}, 9, 10$

Question (Medium)**.** The 3, 4, 5 triplet of side-lengths is perhaps most famous when talking about right triangles. Let's spice it up. Find x to satisfy this right triangle.



§3.3 Special Right Triangles

A right triangle with 45°-45°-90° angles have a special property. This kind of right triangle with equal base-angle measure is called an *isosceles right triangle*.

Theorem 3.5: Sides of a 45°-45°-90° Triangle

In a 45°-45°-90° triangle, the length of the hypotenuse is $\sqrt{2}$ times the length of each of the legs.

Since it is an isosceles right triangle, we assume that the two sides are equal (not the hypotenuse.)

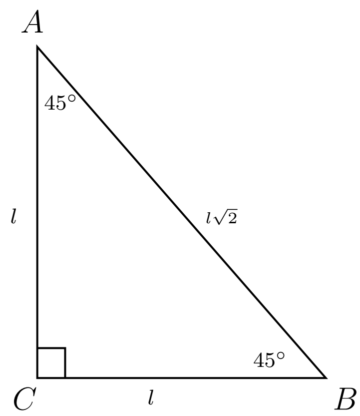


Figure 3.4: A 45°-45°-90° Triangle

We note, the length of the hypotenuse h is:

$$h = l\sqrt{2}$$

Another special right triangle is the 30°-60° -90°right triangle.

Theorem 3.6: Sides of a 30°-60°-90° Triangle

In a 30°-60°-90° triangle, the following are true:

- The length of the hypotenuse is twice the length of the shorter leg.
- The length of the longer leg is $\sqrt{3}$ times the length of the shorter leg.

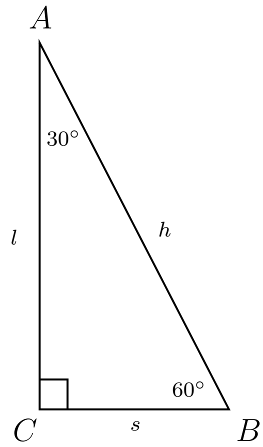


Figure 3.5: A 30°-60°-90° Triangle

We define the lengths of the longer leg and the hypotenuse as:

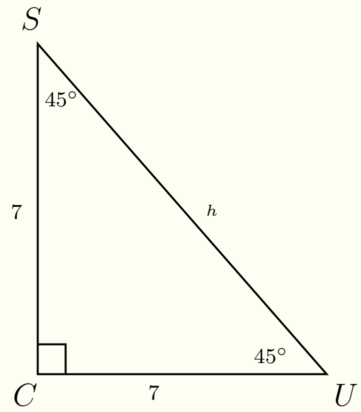
$$h = 2s$$
$$l = s\sqrt{3}$$

§3.3.1 Examples

MY ENDLESS TREMOR OF TYPING ALMOST ENDS!

Example 3.3:

Consider a 30°-60°-90° triangle $\triangle SUC$.



Find the hypotenuse.

Solution. Quite easy.

$$l = s\sqrt{2} = 7\sqrt{2}$$

Example 3.4:

I won't give an illustration for this example. If in a 30°-60°-90° triangle, the longer leg is $3\sqrt{3}$, what is the length of the shorter leg?

Solution. Recreating the figure will not be necessary. Just infer from the formula for the longer leg. Solve for s.

$$\begin{aligned} l &= s\sqrt{3} \\ \frac{l}{\sqrt{3}} &= \frac{s\sqrt{3}}{\sqrt{3}} \\ s &= \frac{l}{\sqrt{3}} \end{aligned}$$

We will rationalize the denominator[8]. We get $s = \dfrac{l\sqrt{3}}{3}$.

$$\begin{aligned} s &= \frac{3\sqrt{3} \cdot \sqrt{3}}{3} \\ s &= \frac{3 \cdot 3}{3} \\ s &= 9 \div 3 \\ s &= \boxed{3} \end{aligned}$$

The reader is tasked to double check this. What about the hypotenuse?

$$h = 2s = 6$$

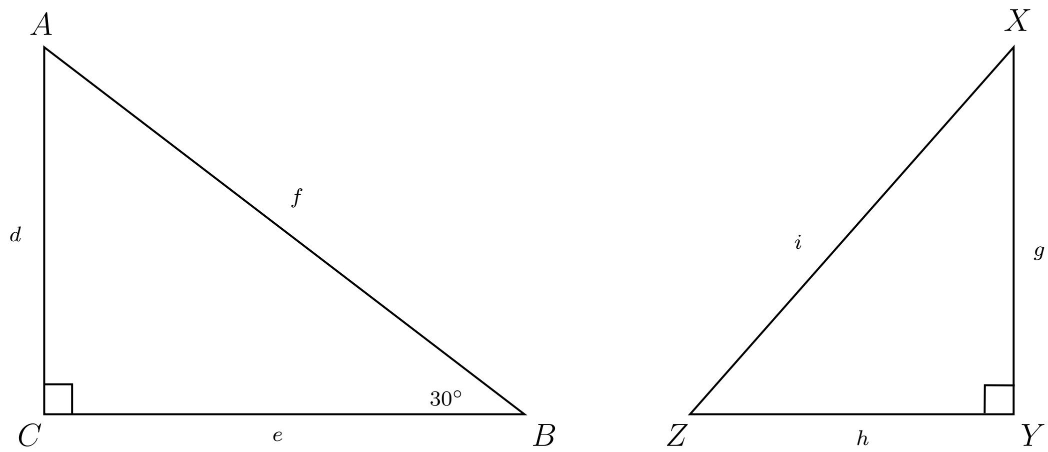
If you need to rationalize the denominator, go for it. But I would suggest keeping it as the normal form $\dfrac{l}{\sqrt{3}}$ for now, as you do not still know how to rationalize the denominator. Later on, I will update the second quarter reviewer to include the lesson.

[8] Was meant to be a second quarter topic but our instructor omitted the topic altogether due to time constraints.

§3.3.2 Exercises

Final sub-subsection of this guide...

I: *Go Beyond 9-15, adjusted.* Using the two triangles below,



Complete the following tables. Each row has a list of values that you need to fill in with the measures of each other side to infer from.

1. In $\triangle ABC$. *Hint: What special right triangle has* 30°*?*

Side	**Shorter**	**Longer**	**Hypotenuse**
d	9		
e		$5\sqrt{3}$	
f			20

2. In $\triangle XYZ$. *Hint: This triangle has two equal angles. What special triangle has two congruent angles?**

Side	**Leg #1**	**Leg #2**	**Hypotenuse**
g	1	1	
e	$\sqrt{2}$		
f			$5\sqrt{2}$

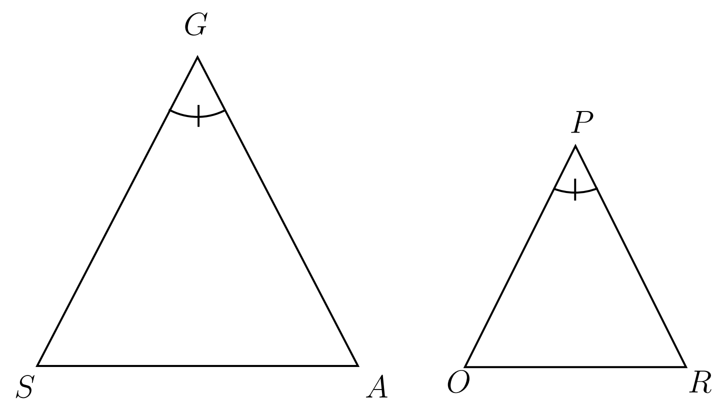
EDITORIAL: Use the book for more examples, and answer some questions there. Sam is not here, so he won't mess around saying "answering id advance is cheating!1!11!!!1"

Appendices

§A Negative Measure vs. Negative Values

We know that measure can NOT be negative. Logically, it can't happen. Only in Sci-Fi we would know what negative measure is.

However, in the guide, I have specified *negative values* at least once or twice. It is basically a value of x that gives a positive measure. For example, recall back to **Example 2.2**.



We see that the value of x can either be -3 or 4. Leaving the condition "... $\overline{GS}$ is the longest side." aside, when we square -3, we get $+9$. Positive nine is a literal measure. It exists. BUT, if we were to assume the equation makes a negative output, like $x^2 - 10$, we cannot consider the result valid.

$$\begin{aligned}(-3)^2 - 10 &= 9 - 10 \\ &= -1\end{aligned}$$

We can only consider negative values if the equation which we can substitute the value back to does not yield a negative output, or a negative measure. The value of x, y, or whatever only gives what number makes the measure. So it doesn't matter if the value is negative, unless it yields a negative output (like prior.)

§B Solutions to Selected Exercises

NOT ALL WILL BE FULL SOLUTIONS. Unless they are. I forgot.

1.1.2

Answers to **II**:

4. Use **Theorem 1.2**.

$$\begin{aligned} x + 10 + 2x + 5 + x + 10 + 2x + 5 &= 360 \\ 6x + 30 &= 360 \\ 6x &= 360 - 30 \\ 6x &= 330 \\ \frac{6x}{6} &= \frac{330}{6} \\ x &= \boxed{55} \end{aligned}$$

5. They are bisecting, hence congruent.

$$\begin{aligned} 0.7x + 11 &= 0.3x + 17 \\ 0.7x - 0.3x &= 17 - 11 \\ 0.4x &= 6 \\ \cancel{5}\left(\frac{2}{\cancel{5}}x\right) &= 5 \cdot 6 \\ 2x &= 30 \\ \frac{2x}{2} &= \frac{30}{2} \\ x &= 15 \end{aligned}$$

For the segments,

$$\begin{aligned} 0.7x + 11 &= 0.3x + 17 \\ 0.7(15) + 11 &= 0.3(15) + 17 \\ 10.5 + 11 &= 4.5 + 17 \\ 21.5 &\stackrel{\checkmark}{=} 21.5 \end{aligned}$$

Therefore both of the lengths of each segment is $\boxed{21.5}$

6. We will only consider one case. Pick the other and it will still hold true. Use **Theorem 1.2**.

$$\begin{aligned} x^2 - 6 &= x \\ x^2 - x - 6 &= 0 \\ (x - 3)(x + 2) &= 0 \\ x &= 3, -2 \end{aligned}$$

I will use 3.

$$\begin{aligned} (3)^2 - 6 &= 3 \\ 9 - 6 &= 3 \\ 3 &\stackrel{\checkmark}{=} 3 \end{aligned}$$

Therefore both of the angles' measure is 3.

Answers to **III**:

2. **T**. Any diagonal can bisect to create two congruent triangles for a parallelogram. For any two diagonals, all four triangles are not always congruent.

4. **F**. A trapezium can have a 90°angle.

1.2.2

Answers to **I**:

3. Double 90 to get $\boxed{180}$.

Answers to **II**:

1. They're congruent.

$$\begin{aligned} x^2 &= 8x - 16 \\ x^2 - 8x + 16 &= 0 \\ (x-4)^2 &= 0 \\ \sqrt{(x-4)^2} &= \sqrt{0} \\ x - 4 &= 0 \\ x &= 4 \\ (4)^2 &= 8(4) - 16 \\ 16 &= 32 - 16 \\ 16 &\overset{\checkmark}{=} 16 \end{aligned}$$

The angles' measure is 16.

1.5.2

2. Congruence.

$$\begin{aligned} x^2 - 45 &= 19 \\ x^2 &= 19 + 45 \\ x^2 &= 64 \\ \sqrt{x^2} &= \sqrt{64} \\ x &= \pm 8 \end{aligned}$$

We will accept +8.

$$\begin{aligned} (8)^2 - 45 &= 19 \\ 64 - 45 &= 19 \\ 19 &\overset{\checkmark}{=} 19 \end{aligned}$$

Hence both the angles' measure is 19.

2.2.2

Answers to **I**:

4. **Property 2.1**.

5. **Property 2.4** and **Property 2.1**.

Answer to the Hard Question:

$$\begin{gathered} \frac{\overline{BL}}{\overline{FT}} = \frac{\overline{OW}}{\overline{EN}} \text{ and } \frac{\overline{BO}}{\overline{FE}} = \frac{\overline{LW}}{\overline{TN}} \\ \frac{x-2}{y} = \frac{2}{3} \text{ and } \frac{1}{2} = \frac{x+1}{y+3} \\ 3(x-2) = 2y \text{ and } y + 3 = 2(x+1) \\ 3x - 2y = 6 \text{ and } 2x - y = 1 \end{gathered}$$

Solve the linear system.

$$\begin{cases} 3x - 2y = 6 \\ 2x - y = 1 \end{cases}$$

$$\begin{aligned} 2x - y &= 1 \\ y &= 2x - 1 \\ 3x - 2(2x - 1) &= 6 \\ 3x - 4x + 2 &= 6 \\ -x + 2 &= 6 \\ x &= -4 \\ 2(-4) - y &= 1 \\ -8 - y &= 1 \\ y &= -9 \end{aligned}$$

Therefore, $\boxed{x = -4, y = -9}$. The result is that the values are negative, they do not yield actual negative measures; we consider them legitimate.

2.3.2

Answers to **I**:

2. SAS

Answers to **II**:

3. $$\begin{aligned} \frac{2}{3} &= \frac{\overline{BH}}{9} \\ 18 &= 3\overline{BH} \\ \overline{BH} &= \boxed{6} \\ \frac{6}{9} &= \frac{10}{\overline{AN}} \\ 6\overline{AN} &= 90 \\ \overline{AN} &= \boxed{15} \end{aligned}$$

3.3.2

2. For $\triangle XYZ$:

Side	**Leg #1**	**Leg #2**	**Hypotenuse**
g	1	1	$\sqrt{2}$
e	$\sqrt{2}$	$\sqrt{2}$	2
f	5	5	$5\sqrt{2}$