Variation, Laws of Exponentiation and Radicals, and Radical Equations

An Insight to Exponentiation, the Radical, and The Idea of Variation

Use your bookmarks, and tap on any chapter/section and it will take you to your page. *This is not a reviewer!* This is a guide, and as a reference, or to review with clearer explanations if possible.

Made with love and LaTeX
Made in TeXMaker
LaTeX distribution used: MikTeX
Compiled with XeLaTeX

If you are planning to print as a book FOR PERSONAL USE, then you have the authorization.
I highly recommend double-sided to save paper.

Worked on for 2 weeks. This is a mere 60 pages. This is again not a reviewer!

Contents

Preface

In the third year of high school, the first quarter covers the material involving quadratic equations. Rather easy, they are hard to grasp at times. However, the second quarter of our year (in terms of the ending of the first semester) is more calmer. The topics include The Forms of Variation, Laws of Exponents and Radicals, and Radical Equations. Despite that, some topics were omitted due to constraints in schedule and constant class suspensions from blue weather. The constant lack of discussions have rather limited the lessons done. Whilst my aim for this reviewer is to educate the reader on how to aid their acts on their gripe with mathematics to pass the quarterly exams, I also want to put aware on how this can affect the learning done in the fourth and final year of high school as certain pre-requisites are not met. However, this is not the main constant for this document.

This large little book has the aim to aid the reader in their gripe against the topics mentioned. Despite their simplicity, the amount of deduction needed can be a little abstract and rigor-ful. It is with heavy pleasure to educate and teach the Advanced Synopsis of Polynomials in Second Degree, and using various supplements (such as YouTube videos). Learning these topics can be time consuming but using enough supplements is vital to fully understanding the topic at hand, like this little pamphlet. I hope that the reader gets a full grasp of what they will embark on despite the calming of the topics aforementioned. The following is a small excerpt from my original reviewer, to show the extent that you can do to perform well for the examinations:

> ”It is no surprise that mathematics is a tough and desolate subject with rigor, abstractions, and paradoxical reasoning that the reader just does not want to embark. But, a step-by-step approach, to something so radical to the naked eye, can be turned simply *if you know what you’re doing.*

My goal for this document is to do everything with lessening the abstraction, filler, and rigor (of course rigor is sometimes needful). My only wish for this document is to help you, and that you have at least read one thing from this, and I would appreciate it no matter what.

Ambitious Goal

As an addendum chapter to only this document, I am currently planning on embarking a motive. I am going to write a book. It will cover basic mathematics to the elementary level but with an advanced approach. The overall plan is to release it the next summer, but I have no promises. However, I do plan to ask for your feedback when it comes with the content and the exercises done in that book. I want to go with an unconventional approach to introducing algebra whilst making it beginner-friendly. Thank you for acknowledging.

Recommendations

The following is a direct rip of the ”Recommendations” section in the first ever booklet ”An Advanced Synopsis of Polynomials In the Second Degree” but slightly rewritten:

”I recommend to the reader to take a piece of paper, and something to write with. Take notes of what you understand, and understand the pieces DIRECTLY. If you need any superintendent aide, ask someone or a friend that discretely knows what you are tackling. This subject is not passable with no gaps, it is a thorn maze filled with tiny holes that leak information without grasp.

I also want to note that: You won't understand everything at first glance and that is completely unavoidable. It is normal and not in an abstract way that people don't understand this. Topics can be understood after weeks or even years; something as basic as scientific notation can be understood after a long period. Interest in mathematics is not for the faint heart of those who see it deterring their future outcome. It is not an outcast. Having difficulties reading mathematics or such any topic is not a block. It is material to build with. Use any approach you would use to understand the reasoning, and you will get it more. Take your time; either long or short. Even if it won't be comprehension-certified within two hours or a day. It is something that you, and anyone, can do.”

Practice Problems

These are for the reader. I firmly believe doing exercises is INCREDIBLY beneficial especially for mathematics. They are just for practice. That's the reason teachers ask students to do exercises: to help you understand better. These are not required, but help A LOT.

A quote that I always follow is, by Paul Halmos, PhD:

> ”The only way to learn mathematics is to do mathematics”

Chapter 1

The Use of Variation

"Be careful of careless mistakes."

— Rodel Calsas; Mathematics Teacher

The idea of *variation* involves the influx or reduction of an amount in contrast to another amount in particular, e.g one amount increases twice, another thrice.

There are kinds of variations that have several applications. These applications range from rate, projections of sales, estimated time of completion, or speed. Variations are used in the daily life for several reasons. Considering the fact variations are used in these situations, some scenarios involving them can bring up to a math text. However, I will try my best to omit word problems, however I will include only five word problems to improve the reader's intuition. However, the reader can choose to omit as they are most likely to not exist in the exams. They are however encouraged, to improve deduction and mobility. You are not required; reiterating but in different words to also affect your attention span.

Proceed to the next page. The formatting on this area is kinda "gulo."

1.1 Direct Variation

TO BE COVERED:

a) The idea of direct variation.

b) Solving problems involving direct variation.

c) Extend knowledge of solving such problems with applications*

* - can be omitted.

When people have a proportional amount, one value can influence the other.

Example 1. Consider:

1. Mia and Kyler both have an increasing amount of paper proportional to each other. When Kyler gets 20 pieces of paper, Mia gets double; 40.

2. Andee and Caleb had a dare; pledging if: One of them drinks a litre of water, another would drink a half more. Andee drank 1.5 litres; Caleb drank 2.

3. Bherna and Anthony run at a distance as a fun activity. When one of them reaches the finish line, the loser's distance gets doubled. Bherna made it to the finish line first at 250 meters. Hence, Anthony's distance doubles to 500 meters (half a kilometer.)

These are considered *proportions*. They are an alternate term to variation and are used interchangeably by different authors and mathematicians.

The example with Mia and Kyler is an example of a *direct proportion*.

Note. k is considered the *constant of variation*.

Direct Variation

When one amount varies directly to another, it is *directly proportional*. It is when the quotient of two variables is a constant.

Definition 1.1.1. Let x and y be two quantities. Let k be the constant of variation, or the coefficient that modifies a quantity. If,

$$y = kx \text{ or } \frac{y}{x} = k \tag{1.1}$$

wherein $k \neq 0$, then y *is directly proportional to* x.

They can be read as the former or "*y varies directly to x*" or "*y is proportional to x*."

We can show some examples as well as an application involving prediction of sales. Keep in mind the variables can interchange when dealing with more concealed problems. However, the variable for the constant of variation is staying as is EXCEPT when it is necessary (formulas.)

Example 2. If $y = 160$ is directly proportional to $x = 8$, find the value of y when $x = 20$.

1 − Form the equation.

$$y = kx$$

2 − Substitute all the values. Afterwards, solve for k.

$$\begin{aligned} 160 &= 8k \\ \frac{160}{8} &= \frac{8k}{8} \\ k &= 20 \end{aligned}$$

3 − Substitute this back to $y = kx$ and find y by substituting the value of x by the case that $x = 20$.

$$\begin{aligned} y &= 20 \cdot 20 \\ y &= \boxed{400} \end{aligned}$$

How does this work?

Explanation. Once we solved for the constant of variation (k), we substitute it back to $y = kx$ now knowing what our value of k is. Now, we consider the case if $x = 20$. We already know that if $x = 8$, $y = 160$. If the case that $x = 20$, we should put it back in our new equation $y = 20k$ to find the value of y when we consider $x = 20$.

If we are solving for something other than k we do the same steps but in respect of the other variable.

Example 3. If $x = 14$ varies directly to $y = \frac{1}{3}$, find values of y when:

a) $x = 6$

b) $x = -210$

c) $x = \frac{21}{5}$

1 − Form the equation.

$$x = ky$$

Since it says "x varies directly to y", it should adhere to this form (recall it as a pattern.)

2 – Substitute all the values. Afterwards, solve for k.

$$\begin{aligned} 14 &= \frac{1}{3}k \\ 3 \cdot 14 &= 3\left(\frac{1}{3}k\right) \\ k &= 42 \end{aligned}$$

3 – Substitute this back to $x = ky$ and find the values of y we were tasked to solve.

a) $x = 6$:

$$\begin{aligned} 6 &= 42y \\ \frac{6}{42} &= \frac{42y}{42} \\ y &= \boxed{\frac{1}{7}} \end{aligned}$$

b) $x = -210$:

$$\begin{aligned} -210 &= 42y \\ \frac{-210}{42} &= \frac{42y}{42} \\ y &= \boxed{-5} \end{aligned}$$

c) $\frac{21}{5}$:

$$\begin{aligned} \frac{21}{5} &= 42y \\ 5\left(\frac{21}{5}\right) &= 5 \cdot 42y \\ 21 &= 210y \\ \frac{21}{210} &= \frac{210y}{210} \\ y &= \boxed{10} \end{aligned}$$

How does this work?

Explanation. Infer from the first explanation for better understanding. Since we know that when $y = \frac{1}{3}$, $x = 14$. We can do the converse; if $x = 14$ then $y = \frac{1}{3}$ in terms of direct variation. We are finding a value of y that if we use can give the value of x we are finding. For example: We are trying to find a value of y that can create $x = -210$. This value, when multiplied by our k which is 42, is $y = -5$. If we were to multiply k and y we would get x in that case. Same applies to each case of x.

This is the reference to the remark I made at the end of the first explanation.

Example 4. If $q = 4$ is being varied directly by $p = 20$, find p when $q + 5 = 7$.

Do not be scared or fooled! Break down the words. If being "varied directly by" then the one directly varying is the result.

Solve for q to get our value of q. $q + 5 = 7$; $q = 2$. Reword the question:

If $p = 20$ varies directly to $q = 4$, find p when $q = 2$.

1 – Form the equation.

$$p = kq$$

2 – Substitute the values. Afterwards, solve for k.

$$\begin{aligned} 20 &= 4k \\ \frac{20}{4} &= \frac{4k}{4} \\ k &= 5 \end{aligned}$$

3 – Substitute this back to $p = kq$ and find the value of p by substituting the value of q by the case that $q = 2$

$$\begin{aligned} p &= 5q \\ p &= 5 \cdot 2 \\ p &= \boxed{10} \end{aligned}$$

Same logic as the first example.

Example 5. Two vendors sell apples and bananas separately. One doubles their inventory when one sells a bigger stock.

This week, the apple vendor sold 200 apples and the banana vendor sold 100 bananas. *If we predict next week; the banana vendor sold 250 bananas, how many apples did the apple vendor sell?*

1 – Create variables to deduce who-is-who.

a) a be the apple vendor's sales.

b) b be the banana vendor's sales.

2 – Deduct that: Since we are asking how many apples did the apple vendor sell when the banana vendor sold 250 bananas, we are finding for a when $b = 250$.

When the apple vendor sold 200 apples, and the banana vendor sold 100, we can deduct that a is directly varying b.

As a word problem: *If $a = 200$ varies directly to $b = 100$, find a when* $b = 250$.

3 – Form the equation.

$$a = bk$$

4 – Substitute the values. Afterwards, solve for k.

$$200 = 100k$$
$$\frac{200}{100} = \frac{100k}{100}$$
$$k = 2$$

5 – Substitute this back to $a = bk$ and find the value of a by substituting the value of b by the case that $b = 250$.

$$a = 2 \cdot 250$$
$$a = \boxed{500}$$

Hence, the apple vendor $\boxed{\text{sold 500 apples}}$.

– PRACTICE PROBLEMS (A) –

On a piece of paper, try solving some of these problems. *Problems with a star have answers at the end of this document. Problems with (H) have a hint.*

1. If $y = 111$ is directly proportional to $x = 3$, find y when $x = 17$.
2. If $y = \frac{1}{3}$ is directly proportional to $x = 2$, find x when $y = 96$.*
3. If $g = 5$ varies directly to $h = -\frac{1}{5}$, find g when $h = 10$, $h = \frac{1}{2}$ and $h = -1$.*
4. If $m = -6$ is being directly varied by $n = \frac{1}{2}$, find n when $m = 12$ and $n = 2$.
5. If $r = 15$ is directly varied to $s + 3 = 6$, find r when $s - 4 = 1$.
6. If $l+1 = -2$ varies directly to $t-12 = 10$, find l when $2t-15 = 21$.*
7. If $h - 35 = 5$ varies directly to $2j = 20$, find j when $h = 200$ and $h - 6 = 54$.
8. A machine can execute 170 commands when given 5 instructions. How many commands can be executed when the machine was given 20 instructions?
9. Francis can solve 16 Rubik's cubes in 2 hours. If he continues at the same rate, how many Rubik's cubes can he solve in 5 hours?*
10. A company's stock price in 5 years would be 200$ per share. In 15 years, how much would the share be (considering the price increases at the same rate and is always trending up.)*

1.2 Inverse and Power Variation

TO BE COVERED:

a) The ideas of inverse and power variation.

b) Solving problems involving inverse and power variation as well as applying inferrence from the last section.

c) Extend knowledge of solving such problems with applications to inverse variations*

* - can be omitted.

We have skimmed over the parts involving direct variation. Now, we need to adapt the usage of inverse and power variation.

The steps from direct variation can be useful to infer to the steps when doing certain variations that we are using like typical. We shall start with *inverse variations.*

Inverse Variation

When one amount varies directly to the product of two variables is a constant, the variation being told is considered the *inverse variation.*

Definition 1.2.1. Let x and y be two quantities. Let k be the constant of variation. If,

$$y = \frac{k}{x} \text{ or } k = xy$$

wherein $k, x \neq 0$, then y *is indirectly proportional to* x.

Some authors call it *inverse variation* (for lack of confusion, we will use inverse variation.) It can be read as the former or "*y varies inversely as x*" or "*y is inversely proportional to x.*"

Example 6. If $y = 12$ varies indirectly to $x = 3$, find y when $x = 12$.

1 − Form the equation.

$$y = \frac{k}{x}$$

2 − Substitute all the values. Afterwards, solve for k.

$$12 = \frac{k}{3}$$
$$3 \cdot 12 = 3\left(\frac{k}{3}\right)$$
$$k = 36$$

3 – Substitute this back to $y = \dfrac{k}{x}$ and find y by substituting the value of x by the case that $x = 12$.

$$y = \frac{36}{12}$$
$$y = \boxed{3}$$

Example 7. An AI scouts through 1200 web-pages in 540 seconds. If the AI could only read 25 pages in 540 seconds, how long would it take for it to reach 1200 pages (the same amount) expressed in seconds?

1 – Create variables to deduce who-is-who.

a) p be the amount of web-pages.

b) t be the time.

2 – Deduct that: Since we are finding the *time* it takes for the AI to read 1200 pages knowing it takes 540 seconds to read 25, we are finding for t when $p = 1200$ as it is the *total amount of pages to read.*

Since we know previously the AI can read 1200 pages in 540 seconds, and now the AI has decrease in power to 25 pages in the same amount of time, we'd know the amount of pages has been *divided* in some way. So, we can tell that t varies indirectly to p.

As a word problem: If $t = 540$ inversely varies to $p = 1200$, find t when $p = 25$.

3 – Form the equation.

$$t = \frac{k}{p}$$

4 – Substitute all values. Afterwards, solve for k.

$$540 = \frac{k}{1200}$$
$$1200 \cdot 540 = 1200\left(\frac{k}{1200}\right)$$
$$k = 648000$$

5 – Substitute this back to $t = \dfrac{k}{p}$ and find t by substituting the value of p by the case that $p = 25$.

$$p = \frac{648000}{25}$$
$$p = \boxed{25920}$$

Hence, the AI would $\boxed{\text{take 25920 seconds}}$ to read 1200 pages at the rate of 25 pages per 540 seconds. It would take 7.2 hours to finish reading 1200 pages. Now imagine compiling it all as a document. Painful.

We have concluded the definition of *inverse variation.* We can now proceed to power variation.

Power Variation

When one amount varies directly to the product of the constant of variation and another quantity as the power of n, it is denoted as *power variation.*

Definition 1.2.2. Let x and y be two quantities. Let k be the constant of variation. If,

$$y = kx^n \text{ or } k = \frac{y}{x^n}$$

wherein $k \neq 0$, then y *varies directly as the nth power of* x.

It can also be read as y *varies as the nth power of* x" and "y *is directly proportional to the nth power of* x."

Keep in mind that *any* power can be used. For now, I will only use examples involving powers two and three.

Example 8. If $y = 180$ varies directly to the square of $x = 6$, find x when $y = 125$.

1 – Form the equation.

$$y = kx^2$$

2 – Substitute all values. Afterwards, solve for k.

$$\begin{aligned}180 &= (6)^2 \cdot k\\ 180 &= 36k\\ \frac{180}{36} &= \frac{36k}{36k}\\ k &= 5\end{aligned}$$

3 – Substitute this back to $y = kx^2$ and find x by substituting the value of y by the case that $y = 125$.

$$\begin{aligned}125 &= 5x\\ \frac{125}{5} &= \frac{5x}{125}\\ x &= \boxed{25}\end{aligned}$$

Example 9. If $y = 64$ varies as the cube of $x = 4$, find y when $x = 6$.

1 – Form the equation.

$$y = kx^3$$

2 – Substitute all values. Afterwards, solve for k.

$$\begin{aligned} 64 &= (4)^3 \cdot k \\ 64 &= 64k \\ \frac{64}{64} &= \frac{64k}{64} \\ k &= 1 \end{aligned}$$

3 – Substitute this back to $y = kx^3$ and find y by substituting the value of x by the case that $x = 6$.

$$y = 1 \cdot 6$$

$$y = \boxed{6}$$

– PRACTICE PROBLEMS (B) –

On a piece of paper, try solving some of these problems.

1. When $u = 22$ varies indirectly to $v = 3$, find u when $v = 2$.
2. $h - 2 = 5$ varies indirectly to $j = 7$. What is the value of h when $j + 2 = 3$ and $j = -3$?
3. If $v = 12$ inversely varies to $w - 1 = 3$, find v when $w = 6$, $w = \frac{6}{5}$, and $w = -1$.
4. If $r = 5$ is being indirectly varied by $s - 24 = -12$, find s when $r = 5$, $r = 25$ and $r = \frac{1}{25}$.
5. If $p = 64$ varies as the cube of $x = -4$, find p when $x = 6$.
6. If $x = 200$ directly varies to the square of $y = 10$, find x when $y = 50$ and $y = 1500$.*
7. When $f = 80$ varies as the square of $j - 4 = -2$, find f when $j - 2 = 5$ and $j + 3 = 7$.*
8. 8 people are grouped together to research. If they can do it in 12 hours, how long would it take for 16 people to research the same thing?
9. An unordinary ball runs down a hill. That ball can run down at 5mph in 2 minutes. How long would it take to run at 12mph?
10. A fantasy game shows a creature that can run at 5mph (average human speed) in just 30 seconds. How long would it take if it ran at 15mph?*

1.3 Joint Variation

We have covered the three basic and essential kinds of variation. However, we should revisit direct variation. Consider that, we have only talked about two variables, like for instance:

$$y = kx$$

y and x were the only variables (or quantities) being talked about. Sometimes, they can involve three quantities. This is the sole reason of our next topic is: the idea of *joint variation*.

Joint Variation

When one amount varies to the product of two or more other quantities and the constant of variation, it is considered a *joint variation*.

Definition 1.3.1. Let x, y and z be quantities. Let k be the constant of variation. If,

$$z = kxy \text{ or } k = \frac{z}{xy}$$

wherein $k \neq 0$, then *z varies directly to x and y*.

If there were more quantities, we would append them to the sentence e.g "*v varies directly to w, x, y and z*" or as $v = kwxyz$. We would only talk about in two and only few examples with three variables.

They are also read as either "*z varies directly as the product of x and y*", "*z varies jointly as x and y*" and a lot more forms. The following are examples:

Example 10. If $z = 96$ varies jointly to $x = 4$ and $y = 3$, find x when $y = 6$ and $z = 4800$.

1 − Form the equation.

$$z = kxy$$

2 − Substitute all values. Afterwards, solve for k.

$$\begin{aligned} 96 &= (4)(3)k \\ 96 &= 12k \\ \frac{96}{12} &= \frac{12k}{12} \\ k &= 8 \end{aligned}$$

3 – Substitute this back to $z = kxy$ and find x by substituting the values of y and z by the case that $y = 6$ and $z = 4800$.

$$\begin{aligned} 4800 &= (8)(6)k \\ 4800 &= 48k \\ \frac{4800}{48} &= \frac{48k}{48} \\ k &= \boxed{100} \end{aligned}$$

Example 11. When $w = 480$ varies directly as the product of $x = 2, y = 3$ and $z = 4$, what is w when $x = 5$, and $y, z = 2$?

1 – Form the equation.

$$w = kxyz$$

2 – Substitute all values. Afterwards, solve for k.

$$\begin{aligned} 480 &= (2)(3)(4)k \\ 480 &= 24k \\ \frac{480}{24} &= \frac{24k}{24} \\ k &= 20 \end{aligned}$$

3 – Substitute this back to $w = kxyz$ and find w.

$$\begin{aligned} w &= (20)(5)(2)(2) \\ w &= \boxed{400} \end{aligned}$$

Example 12 (Volume of a Rectangular Prism)**.** If a rectangular prism is 16 cm wide and 10cm tall with a total volume of 800 cm^3, what would be the volume of the prism if it became 20 cm wide and 13 cm tall?

1 – Determine the equation.

The formula for the volume of a rectangular prism is:

$$V = lwh$$

wherein l is length, w is width, and h is height.

2 – Substitute all values. Treat l as our constant of proportionality.

$$\begin{aligned} 800 &= (16)(10)l \\ 800 &= 160l \\ \frac{800}{160} &= \frac{160l}{160} \\ l &= 5 \end{aligned}$$

3 – Substitute this back to $V = lwh$ and find V.

$$\begin{aligned} V &= (5)(20)(13) \\ V &= \boxed{1300 \text{ cm}^3} \end{aligned}$$

1.4 Combined Variation

We have dealt over joint variation and the three kinds of variation that has been tackled. The sole idea of *joint variation* is that there are three or more variables. Now, we will mix up the kinds of variations with this being the definition of *combined variation.* It involves a mix of regular, inverse, or power variation with joint variation. This means that one variable can be varied inversely, another to the power, another directly, or a combination of all three.

Example 13. If $z = 100$ varies directly to $x = 90$ and inversely to $y = 18$, what is z when $x = 2$ and $y = 4$?

1 – Form the equations. A hint is to break the parts and combine them:

a) If z is directly varied to x, then $z = kx$

b) If z is inversely varied to y, then $z = \frac{k}{y}$

Combine them both and get the equation we're looking for:

$$z = \frac{kx}{y}$$

Don't combine the k. Just put it on the numerator always unless equations that flip it.[1]

2 – Substitute all values. Afterwards, solve for k.

$$\begin{aligned} 100 &= \frac{90k}{18} \\ 100 &= 5k \\ \frac{100}{5} &= \frac{5k}{5} \\ k &= 20 \end{aligned}$$

3 – Substitute this back to $\frac{kx}{y}$ and find z.

$$\begin{aligned} z &= \frac{20 \cdot 2}{4} \\ z &= \boxed{40} \end{aligned}$$

Example 14. When $y = 4$ jointly varies, indirectly, to $x = 3$ and $z = 6$, what is x when $z = 2$ and $y = 12$?

1 – Form the equation.

$$y = \frac{k}{xz}$$

[1]Reciprocals.

2 – Substitute the values. Afterwards, solve for k.

$$\begin{aligned}
4 &= \frac{k}{3 \cdot 6} \\
4 &= \frac{k}{18} \\
18 \cdot 4 &= 18\left(\frac{k}{18}\right) \\
k &= 72
\end{aligned}$$

3 – Substitute this back to $\frac{k}{xz}$ and find x.

$$\begin{aligned}
12 &= \frac{72}{2x} \\
2x \cdot 12 &= 2x\left(\frac{72}{2x}\right) \\
24x &= 72 \\
\frac{24x}{24} &= \frac{72}{24} \\
x &= \boxed{3}
\end{aligned}$$

– PRACTICE PROBLEMS (C) –

On a piece of paper, try solving some of these problems.

1. If $y = 24$ varies jointly by $x = 4$ and $z = 3$, find y when $x = 6$ and $z = 5$.

2. What is c when $b = 2$ and $y = 48$ considering $y = 80$ is being varied by the product of $x = 5$ and $x = 4$.

3. Consider $p = 189$ varies to three quantities: $q = 3$, $r = 4$, $s = 5$. Consider p at these values: $q = 6$, $r = 2$, $s = 5$.

4. If w varies directly to all three: $x = 1$, $y = 3$, $z = 4$, find w when x, y, z have been tripled.

5. If $a = 168$ varies directly to $b = 3$, $c = 4$, and $d = 7$. Determine the value of d that makes $a = 1080$ from $b = 4$ and $c = 27$.*

6. A rectangular prism has the volume of 400 cubic centimeters wherein its 5 cm wide and 10 cm long. If I shrunk it to 5 cm long with a volume of 80 cubic centimeters, find its width.

7. By some magical property, a metal bar with the volume of 720 cubic centimeters, 12 cm long, and 6 cm tall got shrunk down to 360 cubic meters with the same height. Find the new width.

8. Assume a comically big, perfectly sharp cornered plastic rectangular prism has a volume of 585 cubic meters. It has the width and length of 9 cubic meters and 5 cubic meters respectively. An extra

rectangular prism was attached that doubled the width and length whilst retaining the same height. Find the new volume.*

9. y is jointly varied by the square of x and the cube of z. If $y = 72$ when $x = 3$ and $z = 2$, find the value of y when $x = 4$ and $z = 1$.

10. When $t = 3$ varies to the inverse of $2u = 8$ and $v = 2$, what is u when $t = \frac{4}{3}$ and $v = 3$.*

11. $f = 18$ varies to the square of $g = 3$ and the inverse of $h = 2$, find f when $g = 5$ and $h = 4$.

12. $j = 12$ varies to the inverse of the square of $n = 3$. Find j when $n = 2$.

13. Consider $z = 2$ varies to the cube of $x = 2$ and the inverse of $y = 12$. Evaluate $x = 5$ and $y = 10$.*

14. For if $t = 16$ varies jointly to the square of $y = 4$ and the inverse of the cube of $u = 2$, determine y when $t + 15 = 19$ and $u + 6 = 10$.*

15. If $j = 96$ varies directly to $m = 3$ and the square of $n = 4$. Find j when m, n are doubled.

Chapter 2

The Laws of Exponents

"Beginners, amateurs, experienced, and professionals have one in common: they make mistakes. Sometimes the smartest of people can be as knowing like a beginner."

— Anonymous

Exponents are incredibly familiar to many people, some even during elementary. It's core concept is really simple.

Definition 2.0.1. a^n is considered:

$$\underbrace{a \cdot a \cdot a \cdot a \cdot \dots \cdot a}_{n \text{ amount of times.}}$$

wherein n is the *power*, a is the *base*.

There are many ways to read it. A notion to conceive is that if the power is 2, it is "a squared". If it is 3, "a cubed." This terminology will be needed.

How in this chapter we will introduce the laws of exponents? There are many; but I will simplify it to a place where it is understandable. Admittedly, this part is easy *if you have memory.* At least decent memory. All you know to do this is to identify *patterns.*

2.1 Integer (Integral) Exponents

TO BE COVERED:

a) The zero exponent and negative exponents.

b) Quotient and product rule.

c) Power rule.

d) Distribution of powers.

e) Solving equations within powers.

Integral exponents are not exponents with integrals[1]. We will refer them as *integer exponents* if we need to, but the exam might reference as integral.

Integer exponents are exponents with integers as their powers. We shall start with the most basic.

The Zero Exponent

When any amount, that is not zero (appended in Appendix A) is raised to the power of zero, it is ALWAYS 1.

Definition 2.1.1.

$$a^0 = 1, a \neq 0$$

Some people concern over if 0^0 is 1, or 0, or undefined. It is not really clarified so we will avoid such cases.

Since I have clarified earlier that the exponents we are discussing have *integer powers*, expect negative numbers to apply. But what sense is that? Here is the following.

Negative Exponents

When any amount that is not zero raised to a negative power, the result is its reciprocal.

Definition 2.1.2.

$$x^{-n} = \frac{1}{x^n}, x \neq 0$$

If we allow $x = 0$, then we would divide by zero.

[1]Calculus idea. Very hard.

Example 15. Simplify the following:

a) 4^{-3}

b) $\left(\frac{1}{2}\right)^{-5}$

SOLUTIONS:

a)

$$\begin{aligned} 4^{-3} &= \frac{1}{4^3} \\ &= \boxed{\frac{1}{64}} \end{aligned}$$

b)

$$\begin{aligned} \left(\frac{1}{2}\right)^{-5} &= \frac{1}{\left(\frac{1}{2}\right)^5} \\ \frac{1}{\frac{1}{32}} &= \frac{1}{1} \cdot 32 \\ &= \boxed{32} \end{aligned}$$

Let us consider how powers work when distributing them over a fraction or a product of numbers as a review:

Distribution of Powers to Products and Fractions

Consider all of the following.

Definition 2.1.3.

$$(xyz)^n = x^n y^n z^n$$

Definition 2.1.4.

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

These are essential and are a core.

We can now proceed with some notions to consider. They are just special cases that occur when the negative powers are under or distributed to a quotient of two numbers. The definitions can help ease with what I have just said on the next page.

Quotient of Powers

$$\frac{x^{-m}}{y^{-n}} = \frac{y^n}{x^m}$$
$$\left(\frac{x}{y}\right)^{-m} = \left(\frac{y}{x}\right)^m$$
$$\frac{x^m}{y^{-n}} = \frac{y^n}{x^{-m}}$$
$$\frac{x^{-m}}{y^n} = \frac{y^{-n}}{x^m}$$

So far I have bluntly explained how these works, but now I will give examples on how these work. Look on the following:

Example 16. Simplify all of the following:

a) $(2x^0yz)^3$

b) $(y^{-2})\left(\frac{x^2z^3}{y^{-2}}\right)$

c) $\frac{x^{-2}y}{z^{-3}w^2}$

d) $\left(\frac{x^{-1}}{y^{-2}}\right)^{-2}$

SOLUTIONS:

a)

$$\begin{aligned} (2x^0yz)^3 &= (1 \cdot 2yz)^3 \\ 2^3y^3z^3 &= \boxed{8y^3z^3} \end{aligned}$$

b) I chose the path that involves exponents rather than cancelling y^{-2}.

$$\begin{aligned} (y^-2)\left(\frac{x^2z^3}{y^{-2}}\right) &= (y^{-2})\left(\frac{x^2z^3}{\frac{1}{y^2}}\right) \\ y^{-2} \cdot \left(x^2z^3 \cdot y^2\right) &= y^{-2} \cdot x^2y^2z^3 \\ \frac{1}{y^2} \cdot x^2y^2z^3 &= \frac{1}{\cancel{y^2}} \cdot x^2\cancel{y^2}z^3 \\ &= \boxed{x^2y^3} \end{aligned}$$

c)

$$\frac{x^{-2}y}{z^{-3}w^2} = \frac{x^{-2}}{z^{-3}} \cdot \frac{y}{w^2}$$
$$\frac{z^3}{x^2} \cdot \frac{y}{w^2} = \boxed{\frac{yz^3}{w^2x^2}}$$

d)

$$\left(\frac{x^{-1}}{y^{-2}}\right)^{-2} = \left(\frac{y^{-2}}{x^{-1}}\right)^2$$
$$\left(\frac{x}{y^2}\right)^2 = \boxed{\frac{x^2}{y^4}}$$

On the last example, how did $(y^2)^2 = y^4$? This property is called the *power rule*. Enumerated:

The Power Rule of Exponents

An amount to the power of something, and it is raised to another power is that amount to the power of the product of the two exponents.

Definition 2.1.5. $(x^n)^m = x^{mn}$

This is why the last example has the denominator changed to the fourth power.

Accept as granted for now. The proof of this is not in our books at the moment. In the future I can reference it but for now it is not needed.

Another rule that we should be familiarized is the product and quotient rule. The *product rule* is the rule that applies to when two exponents with similar bases and differing powers interact as a product.

The Product Rule of Exponents

An amount multiplied by the same amount but both with differing powers is the same as the amount to the power of the sum of the powers used.

Definition 2.1.6.

$$x^n \cdot x^m = x^{m+n}$$

If there are three powers, then add all three, AS LONG as the bases Bare the same.

The *quotient rule* is the rule that applies to when two exponents with similar bases and differing powers interact as a quotient. There are three cases.

The Quotient Rule of Exponents

An amount divided by the same amount but both with differing powers is the same as the amount to the power of the difference of the numerator's power to the denominator's. This process is also related very closely with *cancelling exponents.*

Definition 2.1.7.

$$\frac{x^m}{x^n} = x^{m-n}$$

There are three cases:

1. If $m > n$, then it remains as is.
2. If $m = n$,
$$\frac{x^m}{x^n} = x^{m-n}$$
$$\text{but, } m = n$$
$$x^{m-m} = x^0 = 1$$
therefore, if $m = n$, it is 1.
3. If $m < n$, then $x^{m-n} = \dfrac{1}{x^{n-m}}$. The proof is kind of confusing.

An example for the third case is this:

Example 17 (Third Case Example)**.** Compute $\dfrac{2^3}{2^5}$.

$$\frac{2^3}{2^5} = 2^{3-5}$$
$$2^{-2} = \frac{1}{2^2}$$
$$= \boxed{\frac{1}{4}}$$

Now, let us proceed with the following examples.

Example 18. Simplify the following:

a) $\left(\dfrac{12x^7}{90}x^{-9}y^{-2}\right)^0$

b) $\left(\dfrac{6u^{-2}m^3}{k^{-2}}\right)^{-2}$

c) $\dfrac{3(r+s)^3}{(r+s)^{-2}}$

d) $\dfrac{7v^3w^{-2}}{49v^{-2}w^3}$

e) $\dfrac{3x^{-2}y}{9x^3y^{-3}}$

With these solutions, we will use the properties of exponents listed and only using cancellation when necessary. Whenever I am splitting the fractions, that just means I am breaking it apart to two products.[2]

a)

$$\left(\frac{12x^7}{90}x^{-9}y^{-2}\right)^0 = \boxed{1}$$

b)

$$\begin{aligned}
\left(\frac{6u^{-2}m^3}{k^{-2}}\right)^{-2} &= \left(\frac{k^{-2}}{6u^{-2}m^3}\right)^2 \\
\left(\frac{1}{6}\cdot\frac{k^{-2}}{u^{-2}}\cdot\frac{1}{m^3}\right)^2 &= \left(\frac{u^2}{k^2}\cdot\frac{1}{6}\cdot\frac{1}{m^3}\right)^2 \\
\left(\frac{u^2}{k^2}\right)^2\cdot\left(\frac{1}{6}\right)^2\cdot\left(\frac{1}{m^3}\right)^2 &= \frac{u^{2\cdot 2}}{k^{2\cdot 2}}\cdot\frac{1}{36}\cdot\frac{1}{m^{3\cdot 2}} \\
\frac{u^4}{k^4}\cdot\frac{1}{36}\cdot\frac{1}{m^6} &= \boxed{\frac{u^4}{36k^4m^6}}
\end{aligned}$$

c)

$$\begin{aligned}
\frac{3(r+s)^3}{(r+s)^{-2}} &= 3\cdot\frac{(r+s)^3}{(r+s)^{-2}} \\
3\cdot\frac{(r+s)^2}{(r+s)^{-3}} &= 3\cdot(r+s)^{2-(-3)} \\
3\cdot(r+s)^5 &= \boxed{3(r+s)^5}
\end{aligned}$$

d)

$$\begin{aligned}
\frac{7v^3w^{-2}}{49v^-2w^3} &= \frac{7}{49}\cdot\frac{w^{-2}}{v^{-2}}\cdot\frac{v^3}{w^3} \\
\frac{1}{7}\cdot\frac{v^2}{w^2}\cdot\frac{v^3}{w^3} &= \frac{v^2\cdot v^3}{7\cdot w^2\cdot w^3} \\
\frac{v^{2+3}}{7\cdot w^{2+3}} &= \boxed{\frac{v^5}{7w^5}}
\end{aligned}$$

e)

$$\begin{aligned}
\frac{3x^{-2}y}{9x^3y^{-3}} &= \frac{3}{9}\cdot\frac{x^{-2}}{y^{-3}}\cdot\frac{y}{x^3} \\
\frac{1}{3}\cdot\frac{y^3}{x^2}\cdot\frac{y}{x^3} &= \frac{y^3\cdot y}{3\cdot x^2\cdot x^3} \\
\frac{y^{3+1}}{3\cdot x^{2+3}} &= \boxed{\frac{y^4}{3x^5}}
\end{aligned}$$

[2]See Appendix B.

Note that: In EVERY variable, there is always an invisible power of 1. Why? Since a^n means we multiply a, n times, if a^1 then it's just a because there is no second a to multiply to. $a^1 = a$, and hence why there is an invisible 1. *This is INCREDIBLY important!*

2.2 Solving Basic Exponential Equations

TO BE COVERED:

a) Solving exponential equations.

b) Using laws of exponents to solve them.

An *exponential equation* is just an equation involving exponents; the equation is typically the power e.g u^{2x+7}. We need to consider the following statement firsthand:

Property of Equal Powers

Two powers are equal if and only if their bases are also equal and their exponents are equal.

Definition 2.2.1.

$$x^m = y^n \text{ if and only if } x = y, m = n$$

This can help solve basic exponential equations.

Example 19. Solve for x:

a) $x^3 = 216$

b) $5^x = 625$

c) $4^x = 512$

SOLUTIONS:

a)

$$\begin{aligned} x^3 &= 216 \\ x^3 &= 6^3 \\ x &= \boxed{6} \end{aligned}$$

b)

$$\begin{aligned} 5^x &= 625 \\ 5^x &= 5^4 \\ x &= \boxed{4} \end{aligned}$$

c)

$$\begin{aligned} 4^x &= 512 \\ (2^2)^x &= 512 \\ 2^{2x} &= 2^9 \\ 2x &= 9 \\ \frac{2x}{2} &= \frac{9}{2} \\ x &= \boxed{\frac{9}{2}} \end{aligned}$$

How does this work?

Explanation. Whenever we find x regularly, we try isolating the entire equation until we get an x equalling something. This case, we just equate it to the corresponding spot. If the x is in the power, equate it to the power on the other side of the equation. If x is in the base, same idea. If x is an equation $2x - 7$ equates to a different power, say 5, then equate it $2x - 7 = 5$. We get the power or the base by converting the other side to an exponent. This does need some memorization.

Example 20. Compute the following exponential equations:

a) $4^{3x-5} = 4^{2x+1}$

b) $\dfrac{3^{-3x+12}}{3^{-x-6}} = 9$

c) $\dfrac{8}{2^{x+2}} = 2^{3x-11}$

d) $3^{x+5} \cdot 3^{2x-3} = 3^5$

SOLUTIONS:

a)

$$\begin{aligned} 4^{3x-5} &= 4^{2x+1} \\ 3x - 5 &= 2x + 1 \\ 3x - 2x - 5 &= 2x + 1 - 2x \\ x - 5 &= 1 \\ x - 5 + 5 &= 1 + 5 \\ x &= \boxed{6} \end{aligned}$$

b) Since it is a fraction with denominator and numerator same base, we subtract the powers from the numerator to the denominator. Do not forget to distribute the negative sign when subtracting polynomials!

$$\begin{aligned}
\frac{3^{-3x+12}}{3^{-x-6}} = 9 = 3^{-3x+12-(-x-6)} &= 3^2 \\
3^{-3x+12+x+6} &= 3^2 \\
-3x + 12 + x + 6 &= 2 \\
-2x + 18 &= 2 \\
-2x + 18 - 18 &= 2 - 18 \\
\frac{-2x}{-2} &= \frac{-16}{-2} \\
x &= \boxed{8}
\end{aligned}$$

c) Convert the numerator to an exponent 2^3 and do the same process as b).

$$\begin{aligned}
\frac{8}{2^{x+2}} = 2^{3x-11} = \frac{2^3}{2^{x+2}} &= 2^{3x-11} \\
2^{3-(x+2)} &= 2^{3x-11} \\
2^{3-x-2} &= 2^{3x-11} \\
3 - x - 2 &= 3x - 11 \\
-x + 1 &= 3x - 11 \\
-x + 1 + 11 &= 3x - 11 + 11 \\
-x + 12 + x &= 3x + x \\
\frac{4x}{4} &= \frac{12}{4} \\
x &= \boxed{3}
\end{aligned}$$

d) Since it is a product of two exponents with same base, we add the powers.

$$\begin{aligned}
3^{x+5} \cdot 3^{2x-3} = 3^5 = 3^{x+5+2x-3} &= 3^5 \\
3^{x+5+2x-3} &= 3^5 \\
x + 5 + 2x - 3 &= 5 \\
3x + 2 &= 5 \\
3x + 2 - 2 &= 5 - 2 \\
\frac{3x}{3} &= \frac{3}{3} \\
x &= \boxed{1}
\end{aligned}$$

Just know: I want to provide something that needs addendum.

Remark 2.2.1. This is just to clarify something: If $-a^n$ and n is any number, the result is negative. But if $(-a)^n$, the result is *negative* if n is odd and *positive* if n is even.

– PRACTICE PROBLEMS (D) –

I : Simplify all of the following by the laws of exponents.

1. $a \cdot a^3 \cdot a^7 \cdot a^{14}$
2. $\dfrac{a^8}{a^3}$
3. $(-2x^2)^{-3}$
4. $6x^2 \cdot (-3x)^4$
5. $(x^2)^7 \cdot x^{-12}$
6. $(x+2) \cdot (x+2)^2$*
7. $7x^0 \cdot 5x^3 \cdot y^7 \cdot -3y^2 \cdot (x^2y^3)^2$*
8. $\dfrac{3x^{-2}}{9x^{-3}} \cdot 27x^3$
9. $\dfrac{7xyz \cdot 2x^2y \cdot 3y^9z^6}{x^{-2}z^{-1}}$*
10. $\dfrac{x^3y^9}{xy^3 \cdot xy^4}$
11. $((2x^2)^2)^3$ (H) Perform the power rule on the innermost parenthesis.
12. $((x^2y)^5)^3 \cdot ((xy^3)^7)^{-2}$*
13. $2^0x^3y^0 \cdot xy^3 \cdot \dfrac{x^{-2}y^3}{x^2y^5}$*
14. $a^{107} \cdot a^{-54} \cdot a^{-53}$ (H) Add the negative exponents.
15. $\dfrac{a^{-2} \cdot a^{-3} \cdot a^{-1}}{b^3 \cdot b^1 \cdot b^6}$ (H) Product rule.
16. $\dfrac{a^2 \cdot a^3 \cdot a^1}{b^{-3} \cdot b^{-1} \cdot b^{-6}}$

II : Solve for x in all of the following.

1. $x^5 = 32$
2. $9^x = 81$
3. $x^4 = 256$
4. $4^{2x} \cdot 4^{-x} = 1024$*
5. $\dfrac{9^{-x}}{9^{-2x}} = 27$
   (H) Subtract the exponents (distribute the subtraction sign); make 9 into a power of 3.
6. $2^x = 32768$

7. $5^{3x-2} = 5^{5x-12}$

8. $\dfrac{3^{5x-15}}{3^{2x+3}} = 27$

9. $\dfrac{2^{10x+15}}{32} = 1$ (H) Convert 32 and 1 as powers of 2.

10. $4^{x-10} \cdot 4^{x-6} = 4^{-2x}$ (H) Product rule.

11. $2^{-4x-15} \cdot 2^{7x-16} = 8$

12. $\dfrac{2^{-3x-2}}{2^{2x+1}} \cdot 2^{x+3} = 2^{12}*$

13. $5^x \cdot 5^{x+1} \cdot 5^3 = 1$ (H) Product rule; convert 1 to a power of 5.

2.3 The Idea of Rational Exponents

So far we have deducted the idea of exponents with integer powers (or integral powers.) We are now going to embark on the idea of a *rational* power. A number is *rational* if it can be expressed as a ratio of two numbers. $1, -9.5, -10223, \frac{2}{5}$ are rational because they are integers over 1 ($\frac{1}{1}, \frac{-10223}{1}$) or can be expressed as a fraction ($-9.5 = -\frac{19}{2}$) but $\sqrt{3}$ is not a rational number because it can not be expressed as a fraction or as a integer with a denominator of 1.

We expressed the idea a^n with integers, but with rational powers, we would need to turn to the radical. Simply put, a *radical* is the reverse of an exponent. It is finding the base. The square root of 25 is 5. The cube root of 27 is 3. Why? What number that we can square gives 25, and what number can we cube gives 27? 5 and 3. It is expressed as the $\sqrt[n]{a}$ wherein n is the nth root, or the "power."

The Rational Exponent (A)

Any number raised to a rational exponent $\frac{1}{n}$ is the nth root of that number.

Definition 2.3.1.

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

In this section, we will not consider turning to the radical in this sense, as this will be introduced later on in "Laws of Radicals." Before we accept this, we should note this:

Remember back when we were solving quadratics, some methods involve us *taking the square root on both sides*? It cancels the square because $\sqrt{a^2} = a$.

Try doing $\sqrt{4^2}$. It is also 4. Same goes if $\sqrt[3]{a^3} = a$. Essentially, $\sqrt[n]{x^n} = x$. We can convert this to a power:

$$\sqrt[n]{x^n} = (x^n)^{\frac{1}{n}}$$

If we used the power rule,

$$(x^n)^{\frac{1}{n}} = x^{n\times\frac{1}{n}}$$
$$x^{\cancel{n}\times\frac{1}{\cancel{n}}} = x^1 = \boxed{x}$$

This only works if $n > 0$. If it were, it is undefined.

We can try some examples:

Example 21. Solve all of the following:

a) $25^{\frac{1}{2}}$

b) $64^{\frac{1}{3}}$

c) $(x^4y^2)^{\frac{1}{2}}$

SOLUTIONS:

a)

$$25^{\frac{1}{2}} = \boxed{5}$$

b)

$$64^{\frac{1}{3}} = \boxed{4}$$

c)

$$(x^4y^2)^{\frac{1}{2}} = x^{4\times\frac{1}{2}} \cdot y^{2\times\frac{1}{2}}$$
$$x^{\frac{4}{2}} \cdot y^{\cancel{2}\times\frac{1}{\cancel{2}}} = \boxed{x^2y}$$

Now, what if $x^{\frac{m}{n}}$? We ask this rhetorical question as we have never asked this yet. We can show it here[3].

Proof. We let $x^{\frac{m}{n}}$ expressed as a product of two powers:

$$x^{\frac{m}{n}} = x^{m\times\frac{1}{n}}$$

We know that, since the power is a product, we reverse the power rule to make it raised to another power:

$$x^{m\times\frac{1}{n}} = (x^m)^{\frac{1}{n}}$$

This yield is how we express this power. □

[3] A proof is a convincing argument to show a mathematical statement is true and is agreeable in the language of math. The "proof" shown is a pseudo-proof and not a legitimate one.

This result can be expressed as a continuation of our definition:

The Rational Exponent (B)

Any number raised to a rational exponent $\frac{m}{n}$ is the nth root of the number which has been raised to a power m.

Definition 2.3.2.

$$x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}$$

All laws of exponents also apply to rational ones.

This can be shown in some examples:

Example 22. Solve all of the following:

a) $25^{\frac{3}{2}}$

b) $\left(\frac{9}{64}\right)^{-\frac{1}{2}}$

c) $\dfrac{x^{\frac{1}{2}}y^{-\frac{2}{3}}}{x^{-\frac{2}{5}}y^{\frac{1}{3}}}$

SOLUTIONS:

a)

$$25^{\frac{3}{2}} = (25^{\frac{1}{2}})^3 = 5^3 = \boxed{125}$$

If you noticed the switch of the exponents whilst doing the reversal of the power rule: if you multiply it it still gives $\frac{3}{2}$. It's multiplication; it is commutative.

b)

$$\left(\frac{9}{64}\right)^{-\frac{1}{2}} = \left(\frac{64}{9}\right)^{\frac{1}{2}}$$
$$\frac{64^{\frac{1}{2}}}{9^{\frac{1}{2}}} = \boxed{\frac{8}{3}}$$

c)

$$\begin{aligned}
\frac{x^{\frac{1}{2}}y^{-\frac{2}{3}}}{x^{-\frac{2}{5}}y^{\frac{1}{3}}} &= \frac{y^{-\frac{2}{3}}}{x^{-\frac{2}{5}}} \cdot \frac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}} \\
\frac{x^{\frac{2}{5}}}{y^{\frac{2}{3}}} \cdot \frac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}} &= \frac{x^{\frac{2}{5}}x^{\frac{1}{2}}}{y^{\frac{2}{3}}y^{\frac{1}{3}}} \\
\frac{x^{\frac{2}{5}+\frac{1}{2}}}{y^{\frac{2}{3}+\frac{1}{3}}} &= \frac{x^{\frac{2}{5}+\frac{1}{2}}}{y^{\frac{3}{3}}} \\
\frac{2}{5}+\frac{1}{2} &= \frac{2}{2}\cdot\frac{2}{5}+\frac{5}{5}\cdot\frac{1}{2} \\
\frac{4+5}{10} &= \frac{9}{10} \\
\frac{3}{3} &= 1 \\
&= \boxed{\frac{x^{\frac{9}{10}}}{y}}
\end{aligned}$$

We will try something from the book on Example 4.3, d., p. 132.

Example 23. Find $\frac{x^{-\frac{1}{4}}y^{\frac{3}{2}}}{x^{\frac{5}{4}}y^{-\frac{1}{2}}}$ when $x = 16$ and $y = 4$.

We can go two ways: The first is to substitute the values directly and the second is to substitute the values after we simplify the fraction itself. I will pick the second route as it is more *recommended* since doing the first route can lead to confusion and can lead to big numbers on other examples.

1. Simplify the exponents in the fraction.

$$\begin{aligned}
\frac{x^{-\frac{1}{4}}y^{\frac{3}{2}}}{x^{\frac{5}{4}}y^{-\frac{1}{2}}} &= \frac{x^{-\frac{1}{4}}}{y^{-\frac{1}{2}}} \cdot \frac{y^{\frac{3}{2}}}{x^{\frac{5}{4}}} \\
\frac{y^{\frac{1}{2}}}{x^{\frac{1}{4}}} \cdot \frac{y^{\frac{3}{2}}}{x^{\frac{5}{4}}} &= \frac{y^{\frac{1}{2}}y^{\frac{3}{2}}}{x^{\frac{1}{4}}x^{\frac{5}{4}}} \\
\frac{y^{\frac{1}{2}+\frac{3}{2}}}{x^{\frac{1}{4}+\frac{5}{4}}} &= \frac{y^{\frac{4}{2}}}{x^{\frac{6}{4}}} \\
&= \frac{y^2}{x^{\frac{3}{2}}}
\end{aligned}$$

2. Substitute the values and simplify.

$$\begin{aligned}
\frac{y^2}{x^{\frac{3}{2}}} &= \frac{4^2}{16^{\frac{3}{2}}} \\
\frac{4^2}{(16^{\frac{1}{2}})^3} &= \frac{16}{4^3} \\
\frac{16}{64} &= \boxed{\frac{1}{4}}
\end{aligned}$$

For the sake of simplicity, only three of these kinds of problems will be given on the practice problems part of this section, which is the next page.

– PRACTICE PROBLEMS (E) –

Some of these might be too hard! I went a little overkill. Skip some if needed.

I : Evaluate all of the following:

1. $144^{\frac{1}{2}}$
2. $(4^{\frac{1}{2}})^3$
3. $(16^{\frac{1}{4}})^2$
4. $(25^{\frac{1}{2}})^2$
5. $4^{\frac{3}{2}}$
6. $(25x^2)^{\frac{1}{2}}$ (H) Power rule.
7. $(225x^4)^{\frac{1}{2}}$*
8. $\dfrac{64^{\frac{2}{3}}}{16^{\frac{1}{2}}}$
9. $\dfrac{9^{\frac{1}{2}} \cdot 27^{\frac{2}{3}}}{81^{\frac{1}{2}}}$*
10. $\dfrac{10000^{\frac{1}{2}}}{100000000^{\frac{1}{8}}}$*

II : Simplify all of the following:

1. $\left(\dfrac{x^2y^4}{x^8y^2}\right)^{\frac{1}{2}}$ (H) Distribution property and power rule.
2. $\left(\dfrac{x^{-2}y^4}{x^6y^{-2}}\right)^{\frac{1}{2}}$ (H) Quotient of powers first then same hint as 1.
3. $\left(\dfrac{x^3y^6}{x^9y^{15}}\right)^{-\frac{1}{3}}$*
4. $\dfrac{x^{-\frac{3}{4}}y^{\frac{1}{2}}}{x^{\frac{1}{4}}y^{-\frac{1}{2}}}$*
5. $\dfrac{x^3y^{-\frac{1}{2}}}{(64x^3)^{\frac{1}{3}}y^{\frac{1}{3}}}$*
6. $\left(\dfrac{x^{-\frac{1}{2}}y^{-\frac{1}{4}}}{x^{-\frac{2}{3}}y^{-\frac{1}{2}}}\right)^{-1}$*
7. $\dfrac{x^3y^{-\frac{1}{2}}z^{-1}}{x^{-\frac{2}{3}}y^{\frac{1}{2}}z^{-\frac{1}{3}}}$*

8. $\dfrac{x^{-1}y^{-\frac{2}{3}}z}{x^3y^{-\frac{1}{3}}z^{-\frac{1}{2}}}$

III : Solve the following:

1. Find $\dfrac{x^{\frac{3}{2}}x^{-\frac{5}{4}}}{y \cdot y^{-\frac{1}{2}}}$ when $x = 81$ and $y = 81$.
   (H) Power rule on the numerator and denominator.

2. $\dfrac{x^{-\frac{1}{4}}y^{\frac{1}{3}}}{x^{\frac{1}{4}}y^{-\frac{8}{3}}}$ when $x = 4$ and $y = 2$.
   (H) Use Example 23 as a hint.

3. The answer for Part II, 4. when $x = 100$ and $y = 500$.

Chapter 3

The Essence of Radicals

"I know numbers are beautiful. If they aren't beautiful, nothing is."

— Paul Erdos

The core concept of a *radical* was explained in the previous chapter. However, a radical has two parts to it that make up its structure.

Parts of the Radical

If you take a look at this radical,

$$\sqrt[n]{x}$$

n is the *index*, x is the *radicand*, and the radical itself is the *radical sign*. The radical is read as the nth root.

Laws of the radical only apply *if and only if* the index is the same as the radical you are applying the law to. This means that square roots stick to square roots, cube roots stick to cube roots, etc. This also applies to operations involving radicals. Square roots operate under square roots, cube roots under cube roots, etc.

Just know, this section *will only lightly tackle roots of negative numbers.* Negative roots are negative if the index is odd, and negative roots are not real numbers if the index is even. The reason of this infers from Remark 2.2.1[1]. We can now proceed.

[1]Tap on this number to go back to what this Remark was. This applies to any reference to a Definition/Remark; tap/click on the number

3.1 A Skim Over Laws of Radicals

TO BE COVERED:

a) A slight summary of the laws of radicals.

b) How to perform the laws of radicals.

The laws of radicals are influenced from the laws of exponents since we know for a fact that radicals can be expressed as exponents (look at rational exponents) and if such, we can infer that all the rules follow the same way. Remember: These only work if the radicals being used are of the SAME index.

All Laws of Radicals

Definition 3.1.1. The following agree with the laws of exponents plus additional:

1. $\sqrt[n]{a^n} = a$
   a) n is odd: $\sqrt[n]{a^n} = a$
   b) n is even: $\sqrt[n]{a^n} = |a|$ or the nonnegative version of a.
2. If $a \geq 0$ and $b \geq 0$, then $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
3. If $a \geq 0$ and $b > 0$, then $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$.
4. $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$

These are also defined by the laws of exponents. Since the radical $\sqrt[n]{a^m}$ is the same as $a^{\frac{m}{n}}$ using Definition 2.3.2, we can also define it by the rational exponent.

All Laws of Radicals Expressed Through Exponents

Remark 3.1.1. The following agree with the laws of exponents plus additional:

1. $(a^n)^{\frac{1}{n}} = a$
   a) n is odd: $(a^n)^{\frac{1}{n}} = a$
   b) n is even: $(a^n)^{\frac{1}{n}} = |a|$ or the nonnegative version of a.
2. If $a \geq 0$ and $b \geq 0$, then $a^{\frac{m}{n}} \cdot b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$
3. If $a \geq 0$ and $b > 0$, then $\dfrac{a^{\frac{m}{n}}}{b^{\frac{m}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{m}{n}}$.

4. $(a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m = a^{\frac{m}{n}}$ referencing Proof 2.3.

This is just a remark. It does not affect anything proceeding it; it is just a remark, or something to be noted.

Example 24. Compute the following:

a) $\sqrt{75} \cdot \sqrt{3}$

b) $\sqrt[3]{-64}$

c) $\sqrt[6]{-64}$

d) $\sqrt[6]{(-2)^6}$

e) $\dfrac{\sqrt{270}}{\sqrt{3}}$

f) $\dfrac{\sqrt{60}}{\sqrt{15}}$

g) $(\sqrt{20})^2$

h) $(\sqrt[4]{2})^4$

SOLUTIONS:

a)

$$\begin{aligned} \sqrt{75} \cdot \sqrt{3} &= \sqrt{75 \cdot 3} \\ \sqrt{225} &= \boxed{15} \end{aligned}$$

b)

$$\begin{aligned} \sqrt[3]{-64} &= \sqrt[3]{(-4)^3} \\ &= \boxed{-4} \end{aligned}$$

c) $\sqrt[6]{-64}$ is not a real number because the index is even.

d)

$$\begin{aligned} \sqrt[6]{(-2)^6} &= \sqrt[\cancel{6}]{(-2)^{\cancel{6}}} \\ &= \boxed{-2} \end{aligned}$$

e)

$$\begin{aligned} \frac{\sqrt{270}}{\sqrt{3}} &= \sqrt{\frac{270}{3}} \\ \sqrt{90} &= \sqrt{9 \cdot 10} \\ \sqrt{9} \cdot \sqrt{10} &= \boxed{3\sqrt{10}} \end{aligned}$$

f)

$$\begin{aligned} \frac{\sqrt{60}}{\sqrt{15}} &= \sqrt{\frac{60}{15}} \\ \sqrt{4} &= \boxed{2} \end{aligned}$$

g) If we recall on the process to cancelling square roots from squares in solving quadratic equations, this can be familiar.

$$\sqrt{20})^2 = \boxed{20}$$

h)

$$\begin{aligned}(\sqrt[4]{2})^4 &= (\sqrt[\cancel{4}]{2})^{\cancel{4}} \\ &= \boxed{2}\end{aligned}$$

3.2 Adding and Subtracting Radicals

TO BE COVERED:

a) How to add and subtract radicals.

b) How to identify different indices to add/subtract to.

We all know how we add and subtract like terms.

$$x + y + 2x + 3y - x - 2y = x + 2x - x + 3y - 2y = 2x + y$$

This also applies to radicals. However, if two radicals are not like terms, if all can be reduced to one radical that is like to each term, then we can add all.

Note. Like terms also mean that they should be THE SAME index. Same idea from before. If there is a cube root and a square root, they can't add together unless one of them are the same.

Example 25. Solve the following:

a) $\sqrt{3} + 6\sqrt{3} - 5\sqrt{3}$

b) $\sqrt{360} + 2\sqrt{10} - \sqrt{1000}$

c) $\sqrt{3x^2} - \sqrt{27x^2}$

d) $\sqrt[3]{64x^3y} - \sqrt[3]{125y} + \sqrt[3]{8y}$

SOLUTIONS:

a)

$$\begin{aligned}\sqrt{3} + 6\sqrt{3} - 5\sqrt{3} &= 7\sqrt{3} - 5\sqrt{3} \\ &= \boxed{2\sqrt{3}}\end{aligned}$$

b)

$$\begin{aligned}\sqrt{360}+2\sqrt{10}-\sqrt{1000} &= \sqrt{36\cdot 10}+2\sqrt{10}-\sqrt{100\cdot 10}\\ \sqrt{36}\cdot\sqrt{10}+2\sqrt{10}-\sqrt{100}\cdot\sqrt{10} &= 6\sqrt{10}+2\sqrt{10}-10\sqrt{10}\\ 8\sqrt{10}-\sqrt{10} &= \boxed{-2\sqrt{10}}\end{aligned}$$

c)

$$\begin{aligned}\sqrt{3x^2}-\sqrt{27x^2} &= \sqrt{x^2}\cdot\sqrt{3}-\sqrt{9x^2\cdot 3}\\ x\sqrt{3}-3x\sqrt{3} &= \boxed{-2x\sqrt{3}}\end{aligned}$$

d)

$$\begin{aligned}\sqrt[3]{64x^3y}-\sqrt[3]{125y}+\sqrt[3]{8y} &= \sqrt[3]{64x^3\cdot y}-\sqrt[3]{125\cdot y}+\sqrt[3]{8\cdot y}\\ \sqrt[3]{64x^3}\cdot\sqrt[3]{y}-\sqrt[3]{125}\cdot\sqrt[3]{y}+\sqrt[3]{8}\cdot\sqrt[3]{y} &= 4x\sqrt[3]{y}-5\sqrt[3]{y}+2\sqrt[3]{y}\end{aligned}$$

Since $-5, 2$, and $4x$ are not like terms, we factor out $\sqrt[3]{y}$ and add like terms inside the parenthesis.

$$\begin{aligned}4x\sqrt[3]{y}-5\sqrt[3]{y}+2\sqrt[3]{y} &= \sqrt[3]{y}(4x-5+2)\\ &= \boxed{\sqrt[3]{y}(4x-3)}\end{aligned}$$

We now conceive an equation with roots of differing index, and adding the like terms of their respective indices.

Example 26. Solve the following:

a) $\sqrt{7}+\sqrt{63}-\sqrt[3]{128}-\sqrt[3]{16}$

b) $2\sqrt[3]{81}-3\sqrt[3]{3}+12\sqrt[4]{3}-2\sqrt[4]{48}$

SOLUTIONS:

a)

$$\begin{aligned}\sqrt{7}+\sqrt{63}-\sqrt[3]{128}-\sqrt[3]{16} &= \sqrt{7}+\sqrt{9}\cdot\sqrt{7}-\sqrt[3]{64}\cdot\sqrt[3]{2}-\sqrt[3]{8}-\sqrt[3]{2}\\ \sqrt{7}+3\sqrt{7}-4\sqrt[3]{2}-2\sqrt[3]{2} &= \boxed{4\sqrt{7}-6\sqrt[3]{2}}\end{aligned}$$

b)

$$2\sqrt[3]{81}-3\sqrt[3]{3}+12\sqrt[4]{3}-2\sqrt[4]{48} = 2\cdot\sqrt[3]{27}\cdot\sqrt[3]{3}-3\sqrt[3]{3}+12\sqrt[4]{3}-2\cdot\sqrt[4]{16}\cdot\sqrt[4]{3}$$

$$\begin{aligned}&= 2\cdot 3\cdot\sqrt[3]{3}-3\sqrt[3]{3}+12\sqrt[4]{3}-2\cdot 2\cdot\sqrt[4]{3}6\sqrt[3]{3}-3\sqrt[3]{3}+12\sqrt[4]{3}-4\sqrt[4]{3}\\ &= \boxed{3\sqrt[3]{3}+8\sqrt[4]{3}}\end{aligned}$$

3.3 Multiplication of Radicals

TO BE COVERED:

a) How to multiply radicals.

b) Using the laws of radicals to multiply radicals.

c) The idea of the conjugate.

The idea of multiplying radicals is already emphasized in Definition 3.1.1, number 2. *The reason why we are not discussing division of radicals is due to the list of items to review only goes up to this point.* In Example 24, on number 1., showcased an example of multiplying two radicals together.

This, again, only applies if the radicals you want to multiply are of same index unless you want to multiply radicals directly[2]. This is simply just either: a) distributing the radical, b) multiplying two radicals only.

For the sake of simplicity, I (the author) will not mix different roots together. We can infer from the laws of radicals and the properties of multiplication (distribution property) to go further.

We can start by this set of examples:

Example 27. Solve all of the following:

a) $\sqrt{27} \cdot \sqrt{3}$

b) $\sqrt{3}(\sqrt{2} - 2\sqrt{12})$

c) $5\sqrt[3]{2} \cdot \sqrt[3]{4}$

d) $2\sqrt[3]{2}(3\sqrt[3]{4} + 2\sqrt[3]{32})$

e) $\sqrt{8x} \cdot \sqrt{18x}$

f) $\sqrt[3]{xy^2}(\sqrt[3]{x^2} - \sqrt[3]{2y})$

g) $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$

h) $(\sqrt{3x} + \sqrt{x})(\sqrt{3x} - \sqrt{x})$

SOLUTIONS:

a)

$$\begin{aligned} \sqrt{27} \cdot \sqrt{3} &= \sqrt{27 \cdot 3} \\ \sqrt{81} &= \boxed{9} \end{aligned}$$

b)

$$\begin{aligned} \sqrt{3}(\sqrt{2} - 2\sqrt{12}) &= \sqrt{3} \cdot \sqrt{2} - 2 \cdot \sqrt{3} \cdot \sqrt{12} \\ \sqrt{3 \cdot 2} - 2\sqrt{3 \cdot 12} &= \sqrt{6} - 2\sqrt{36} \\ \sqrt{6} - 2 \cdot 6 &= \boxed{\sqrt{6} - 12} \end{aligned}$$

[2]This is beyond the scope of our topic listed in the pointers and is not in the textbook we are using.

c)

$$\begin{aligned} 5\sqrt[3]{2}\cdot\sqrt[3]{4} &= 5\cdot\sqrt[3]{2\cdot 4} \\ 5\cdot\sqrt[3]{8} = 5\cdot 2 &= \boxed{10} \end{aligned}$$

d) We separate all the coefficients (2, 3, and 2) after distributing to ease the flow.

$$\begin{aligned} 2\sqrt[3]{2}(3\sqrt[3]{4}+2\sqrt[3]{32}) &= (2\cdot 3)\cdot\sqrt[3]{2}\cdot\sqrt{4}+(2\cdot 2)\cdot\sqrt[3]{2}\cdot\sqrt[3]{32} \\ 6\sqrt[3]{2\cdot 4}+4\sqrt[3]{2\cdot 32} &= 6\sqrt[3]{8}+4\sqrt[3]{64} \\ 6\cdot 2+4\cdot 4 &= \boxed{28} \end{aligned}$$

e)

$$\begin{aligned} \sqrt{8x}\cdot\sqrt{18x} &= \sqrt{8x\cdot 18x} \\ \sqrt{144x^2} &= \boxed{12x} \end{aligned}$$

f)

$$\begin{aligned} \sqrt[3]{xy^2}(\sqrt[3]{x^2}-\sqrt[3]{2y}) &= \sqrt[3]{xy^2}\cdot\sqrt[3]{x^2}-\sqrt[3]{xy^2}\cdot\sqrt[3]{2y} \\ \sqrt[3]{xy^2\cdot x^2}-\sqrt[3]{xy^2\cdot 2y} &= \sqrt[3]{x^3y}-\sqrt[3]{2xy^3} \\ \sqrt[3]{x^3}\cdot\sqrt[3]{y}-\sqrt[3]{y^3}\cdot\sqrt[3]{2x} &= \boxed{x\sqrt[3]{y}-y\sqrt[3]{2x}} \end{aligned}$$

g) We use the difference of two squares because the form of the equation is familiar to that. We substitute $x^2 - y^2$ where $x = \sqrt{4}$ and $y = \sqrt{2}$.

$$\begin{aligned} (\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2}) &= (\sqrt[3]{\ })^2-(\sqrt{2})^2 \\ 3-2 &= \boxed{1} \end{aligned}$$

h) Same idea.

$$\begin{aligned} (\sqrt{3x}+\sqrt{x})(\sqrt{3x}-\sqrt{x}) &= (\sqrt{3x})^2-(\sqrt{x})^2 \\ 3x-x &= \boxed{2x} \end{aligned}$$

g) and h) evoke the idea of the conjugate, because they simplify down to a reduced term (rational number).

The Conjugate of a Radical

(for this case we will assume it is an index of 2 since higher indices are more tougher.)

Definition 3.3.1. The *conjugate* of a square root is something that you multiply to the radical to make it rational, or to cancel the roots.

The conjugate of $\sqrt{2}$ is itself because multiplying $\sqrt{2}$ by $\sqrt{2}$ makes 2. The conjugate of $\sqrt{x}+\sqrt{2}$ is $\sqrt{x}-\sqrt{2}$ because multiplying them together makes the difference of two squares which as we saw earlier makes a rational number in return.

– PRACTICE PROBLEMS (F) –

I : Simplify:

1. $\sqrt[5]{-32}$
2. $\dfrac{\sqrt[3]{32}}{\sqrt[3]{-4}}$
3. $\sqrt{-72} \cdot \sqrt{2}$ (H) Is the radicand negative?
4. $\dfrac{\sqrt{1732}}{\sqrt{7}}$
5. $(\sqrt[5]{-5})^5$

II : Add/subtract the following:

1. $13\sqrt{3} - \sqrt{12} - \sqrt{147}$
2. $3\sqrt{6} + 2\sqrt{150} + \sqrt{96}$
3. $3\sqrt{2} + \sqrt{8} - \sqrt{50}$
4. $4\sqrt{12x^4} - x\sqrt{27x^2}$*
5. $\sqrt[3]{24x} + \sqrt[3]{-81x} - \sqrt[3]{3x}$*
6. $\sqrt{125x^2} + 2\sqrt{80x^2} - 4\sqrt{20x^2}$
7. $5\sqrt[3]{32x^3} - 2\sqrt[3]{108x^3} + \sqrt[3]{3x^3}$

III : Multiply the following:

1. $\sqrt{3} \cdot \sqrt{5} \cdot \sqrt{15}$
2. $(2\sqrt{21} + \sqrt{15})(2\sqrt{21} - \sqrt{15})$
3. $\sqrt{3}(\sqrt{2} + 2\sqrt{3} - \sqrt{5})$*
4. $\sqrt{xy}(\sqrt{xy} + \sqrt{2xy})(\sqrt{xy} - \sqrt{2xy})$
   (H) Do the difference of two squares first.
5. $2 \cdot 3\sqrt{50x^2} \cdot \sqrt{8x^2}$
6. $(x + \sqrt{y})(x - \sqrt{y})$ (H) Difference of two squares still.
7. $\sqrt[3]{2x} \cdot \sqrt[3]{32x^2}$
8. $\sqrt[3]{2}(\sqrt[3]{4} + 4\sqrt[3]{36})$

IV : Find the conjugates.

1. $\sqrt{10}$
2. $\sqrt{2xy}$*
3. $\sqrt{3} - \sqrt{2}$
4. $\sqrt{5x} + \sqrt{5}$

Appendix **A**

Why is 0^0 invalid?

This short appendix will describe this situation.

This controversial statement that 0^0 is invalid has conflicting arguments on both sides.

RULE 1 – Any number raised to 0 is always 1. This is true due to the laws of exponents. $2^0, 4^0, 124923^0, (x+y)^0$ are all 1.
RULE 2 – Any power of 0 is always 0. This is true because 0 times itself is still 0. $0^2, 0^{19}, 0^{1291206324}$ are all 0.

How about 0^0?

Case of Argument 1 – 0^0 should be 1. They follow Rule 1.
Case of Argument 2 – 0^0 should be 0. They follow Rule 2.

There is no definitive answer because it leads to contradiction. If 0^0 then any power of 0 is always 0. But it is raised to 0, so it should equal to 1. But it violates the first rule so it should be 0. But it violates the second rule, and it always repeats in contradiction.

People just accept one truth to another; invalidating the contradiction and picking a side. On some textbooks, authors list 0^0 undefined. Others give an answer. There is no definitive result.

It is best recommended to say "0^0 is invalid" because it contradicts both statements back and forth.

Appendix **B**

Fraction Operations

Even if most of the content in this book only involved multiplication or division of fractions, let me clarify, this is important.

Remark B.0.1. We define the operations of fractions under the following:

1. Addition/Subtraction:
$$\frac{a}{b} \pm \frac{c}{d} = \frac{d}{d} \cdot \frac{a}{b} \pm \frac{b}{b} \cdot \frac{c}{d} = \frac{ad \pm bc}{bd}$$
2. Multiplication:
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$
The best way to interpret is to just multiply across. No fancy methods.
3. Division:
$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$
This is called "Keep, Change, Flip." It is an analogy on how we perform this operation on fraction.

STEP 1: *Keep* the first fraction as is.

STEP 2: *Change* the division sign to a multiplication sign.

STEP 3: *Flip* the second fraction's denominator and numerator.

If you have more than two fractions, work each of them one by one and follow the order of operations.

Remember, if it is a whole number, it is always a fraction with a denominator of 1 e.g $9 = \frac{9}{1}$.

Appendix C

List of Perfect Squares, Cubes, and Powers of 2

You do not need to memorize all of this! Just memorize all what is important.

PERFECT SQUARES. Until 20^2.

1^2: 1	6^2: 36	11^2: 121	16^2: 256
2^2: 4	7^2: 49	12^2: 144	17^2: 289
3^2: 9	8^2: 64	13^2: 169	18^2: 324
4^2: 16	9^2: 81	14^2: 196	19^2: 361
5^2: 25	10^2: 100	15^2: 225	20^2: 400

PERFECT CUBES. Until 10^3.

1^3: 1	6^3: 216
2^3: 8	7^3: 343
3^3: 27	8^3: 512
4^3: 64	9^3: 729
5^3: 125	10^3: 1000

POWERS OF 2. Until 2^{15}. Their number on the list corresponds to their power e.g 1. 2 means 2^1.

1. 2	6. 64	11. 2048
2. 4	7. 128	12. 4096
3. 8	8. 256	13. 8192
4. 16	9. 512	14. 16384
5. 32	10. 1024	15. 32768

Appendix D

Full Solutions to Selected Exercises

Exercises marked with an asterisk (*) are given full solutions here in this part of the document.

– (A) –

2.

$$\begin{aligned}
y &= kx \\
\frac{1}{3} &= 2k \\
\frac{1}{3} \cdot 3 &= 3 \cdot 2k \\
1 &= 6k \\
k &= \boxed{\frac{1}{6}} \\
y &= \frac{1}{6}x \\
96 &= \frac{1}{6}x \\
6 \cdot 96 &= 6 \cdot \frac{1}{6}x \\
x &= \boxed{576}
\end{aligned}$$

3.

$$\begin{aligned}
g &= hk \\
5 &= -\frac{1}{5}k \\
5 \cdot -5 &= -5 \cdot (-\frac{1}{5})k \\
k &= \boxed{-25} \\
g &= -25h
\end{aligned}$$

- $$\begin{aligned}
g &= -25 \cdot \frac{1}{2} \\
g &= \boxed{-\frac{25}{2}}
\end{aligned}$$

- $$\begin{aligned}
g &= -25 \cdot 10 \\
g &= \boxed{-250}
\end{aligned}$$

- $$\begin{aligned}
g &= -25 \cdot -1 \\
g &= \boxed{25}
\end{aligned}$$

6. Solve the equations mentioned then substitute when appropriate.

$$\begin{aligned}
l + 1 = -2 &= l = -2 \\
t - 12 = 10 &= t = 22 \\
2t - 15 = 21 &= 2t = 36 = t = 18
\end{aligned}$$

Solve.

$$\begin{aligned}
t &= lk \\
-2 &= 22k \\
\frac{-2}{22} &= \frac{22k}{22} \\
k &= \boxed{-\frac{1}{11}} \\
l &= -\frac{1}{11} \cdot 18 \\
\\
l &= \boxed{-\frac{18}{11}}
\end{aligned}$$

9. Let r = Rubik's cubes and t = time in hours.

$$
\begin{aligned}
r &= tk \\
16 &= 2k \\
\frac{16}{2} &= \frac{2k}{2} \\
k &= \boxed{8} \\
r &= 8t \\
r &= 8 \cdot 5 \\
r &= \boxed{40}
\end{aligned}
$$

10. Let s = share price and y = years.

$$
\begin{aligned}
s &= yk \\
200 &= 5k \\
\frac{200}{5} &= \frac{5k}{5} \\
k &= \boxed{40} \\
s &= 40y \\
s &= 40 \cdot 15 \\
s &= \boxed{600}
\end{aligned}
$$

– (B) –

6.

$$
\begin{aligned}
x &= ky^2 \\
200 &= k \cdot (10)^2 \\
200 &= 100k \\
\frac{200}{100} &= \frac{100k}{100} \\
k &= \boxed{2} \\
x &= 2y^2 \\
x &= 2(50)^2 \\
x &= 2 \cdot 2500 \\
x &= \boxed{5000} \\
x &= 2(1500)^2 \\
x &= 2 \cdot 2250000 \\
x &= \boxed{4500000}
\end{aligned}
$$

7.

$$
\begin{aligned}
j-4=-2=j&=2\\
f&=kj^2\\
80&=(2^2)k\\
80&=4k\\
\frac{80}{4}&=\frac{4k}{4}\\
k&=\boxed{20}\\
f&=20j^2
\end{aligned}
$$

Solve for the equations mentioned then substitute.

$$
\begin{aligned}
j-2=5=j&=7\\
j+3=7=j&=4\\
f&=20\cdot 7^2\\
f&=20\cdot 49\\
f&=\boxed{980}\\
f&=20\cdot 4^2\\
f&=20\cdot 16\\
f&=\boxed{320}
\end{aligned}
$$

10. Let $s=$ speed in mph and $t=$ time in seconds.

$$
\begin{aligned}
t&=\frac{k}{s}\\
30&=\frac{k}{5}\\
30\cdot 5&=5\cdot\frac{k}{5}\\
k&=\boxed{150}\\
t&=\frac{150}{k}\\
t&=\frac{150}{15}\\
t&=\boxed{10}
\end{aligned}
$$

– (C) –

5.

$$\begin{aligned} a &= kbcd \\ 168 &= k \cdot 3 \cdot 4 \cdot 7 \\ 168 &= 84k \\ \frac{168}{84} &= \frac{84k}{84} \\ k &= \boxed{2} \\ a &= 2bcd \\ 1080 &= 2 \cdot 4 \cdot 27 \cdot d \\ 1080 &= 216d \\ \frac{1080}{216} &= \frac{216d}{216} \\ d &= \boxed{5} \end{aligned}$$

8.

$$\begin{aligned} V &= lwh \\ 585 &= 9 \cdot 5 \cdot h \\ 585 &= \frac{45h}{45} \\ h &= \boxed{13} \\ V &= 13lw \\ V &= 13 \cdot 10 \cdot 18 \\ V &= \boxed{2340} \end{aligned}$$

10. Solve for the equation mentioned then substitute.

$$\begin{aligned} 2u = 8 &= u = 4 \\ t &= \frac{k}{uv} \\ 3 &= \frac{k}{4 \cdot 2} \\ 8 \cdot 3 &= 8 \cdot \frac{k}{8} \\ k &= \boxed{24} \\ t &= \frac{24}{uv} \\ \frac{4}{3} &= \frac{24}{3u} \\ \cancel{3}u \cdot \frac{4}{\cancel{3}} &= 3u \cdot \frac{24}{3u} \\ 4u &= 24 \\ \frac{4u}{4} &= \frac{24}{4} \\ u &= \boxed{6} \end{aligned}$$

13.

$$\begin{aligned}
z &= \frac{kx^3}{y} \\
2 &= \frac{k \cdot 2^3}{12} \\
2 &= \frac{8k}{12} \\
12 \cdot 2 &= 12 \cdot \frac{8k}{12} \\
8k &= 24 \\
\frac{8k}{8} &= \frac{24}{8} \\
k &= \boxed{3} \\
z &= \frac{3x^3}{y} \\
z &= \frac{3 \cdot 5^3}{10} \\
z &= \frac{375}{10} \\
\text{Simplify: } z &= \boxed{\frac{75}{2}}
\end{aligned}$$

14.

$$\begin{aligned}
t &= \frac{ky^2}{u^3} \\
16 &= \frac{k \cdot 4^2}{2^3} \\
16 &= \frac{16k}{8} \\
\frac{16}{2} &= \frac{2k}{2} \\
k &= \boxed{8} \\
t &= \frac{8y^2}{u^3} \\
2 &= \frac{8y^2}{4^3} \\
64 \cdot 2 &= 64 \cdot \frac{8y^2}{64} \\
8y^2 &= 128 \\
\frac{8y^2}{8} &= \frac{128}{8} \\
y^2 &= 16 \\
\sqrt{y^2} &= \sqrt{16} \\
x &= \boxed{4}
\end{aligned}$$

ONLY the positive result works. The negative result is a contradiction (Why?).

– (D) –

I :

6.

$$\begin{aligned} (x+2)\cdot(x+2)^2 &= \boxed{(x+3)^3} \\ \text{OR } (x+2)(x+2)(x+2) &= (x^2+4x+4)(x+2) \\ x^3+2x^2+4x^2+8x+4x+8 &= \boxed{x^3+6x^2+12x+8} \end{aligned}$$

7.

$$\begin{aligned} 7x^0\cdot 5x^3\cdot y^7\cdot -3y^2\cdot (x^2y^3)^2 &= 7\cdot 5x^3\cdot -3y^9\cdot x^4y^6 \\ -105x^3y^9\cdot x^4y^6 &= \boxed{-105x^7y^15} \end{aligned}$$

9.

$$\begin{aligned} \frac{7xyz\cdot 2x^2y\cdot 3x^9z^6}{x^{-2}z^{-1}} &= \frac{42x^{1+2}y^{1+1+9}z^{1+6}}{x^{-2}z^{-1}} \\ \frac{42x^3y^{11}z^7}{x^{-2}z^{-1}} &= 42y^{11}\cdot\frac{x^3z^7}{x^{-2}z^{-1}} \\ 42y^{11}\cdot x^{3-(-2)}y^{7-(-1)} &= \boxed{7x^5y^{11}z^8} \end{aligned}$$

12.

$$\begin{aligned} ((x^2y)^5)^3\cdot((xy)^3)^7)^{-2} &= (x^{10}y^5)^3\cdot(x^7y^{21})^{-2} \\ x^{30}y^{15}\cdot x^{-14}y^{-42} &= x^{30+(-14)}y^{15+(-42)} \end{aligned}$$

$$\boxed{x^{16}y^{-27}} \text{ or } \boxed{\frac{x^{16}}{y^{27}}}$$

15.

$$\begin{aligned} 2^0x^3y^0\cdot xy^3\cdot\frac{x^{-2}y^3}{x^2y^5} &= x^3\cdot xy^3\cdot\frac{x^{-2}y^3}{x^2y^5} \\ x^4y^3\cdot x^{-2-2}y^{3-5} &= x^4y^3\cdot x^{-4}y^{-2} \\ x^{4+(-4)}y^{3+(-2)} &= x^0y^1 = \boxed{y} \end{aligned}$$

II :

4.

$$\begin{aligned} 4^{2x}\cdot 4^{-x} = 1024 &= 4^{2x+(-x)} = 1024 \\ 4^x &= 1024 \\ 4^x &= 4^5 \\ x &= \boxed{5} \end{aligned}$$

12.

$$\begin{aligned}
\frac{2^{-3x-2}}{2^{2x+1}} \cdot 2^{x+3} = 2^{12} &= 2^{-3x-2-(2x+1)} \cdot 2^{x+3} = 2^{12} \\
2^{-3x-2-2x-1} \cdot 2^{x+3} &= 2^{12} \\
2^{-5x-3+x+3} &= 2^{12} \\
2^{-4x} &= 2^{12} \\
-4x &= 12 \\
\frac{-4x}{-4} &= \frac{12}{-4} \\
x &= \boxed{-3}
\end{aligned}$$

– (E) –

I :

7.

$$\begin{aligned}
(225x^4)^{\frac{1}{2}} &= 225^{\frac{1}{2}} x^{4\cdot\frac{1}{2}} \\
&= \boxed{15x^2}
\end{aligned}$$

9.

$$\begin{aligned}
\frac{9^{\frac{1}{2}} 27^{\frac{2}{3}}}{81^{\frac{1}{2}}} &= \frac{3 \cdot (27^{\frac{1}{3}})^2}{9} \\
\frac{3 \cdot 3^2}{9} &= \frac{3 \cdot 9}{9} \\
\frac{27}{9} &= \boxed{3}
\end{aligned}$$

10.

$$\begin{aligned}
\frac{10000^{\frac{1}{2}}}{100000000^{\frac{1}{8}}} &= \frac{100}{10} \\
&= \boxed{10}
\end{aligned}$$

II :

3.

$$\begin{aligned}
\left(\frac{x^3y^6}{x^9y^{15}}\right)^{-\frac{1}{3}} &= \frac{x^{3\cdot-\frac{1}{3}} y^{6\cdot-\frac{1}{3}}}{x^{9\cdot-\frac{1}{3}} y^{15\cdot-\frac{1}{3}}} \\
\frac{x^{-1}y^{-2}}{x^{-3}y^{-5}} &= \frac{x^3y^5}{xy^2} \\
x^{3-1}y^{5-2} &= \boxed{x^2y^3}
\end{aligned}$$

4.

$$\begin{aligned}
\frac{x^{-\frac{3}{4}}y^{\frac{1}{2}}}{x^{\frac{1}{4}}y^{-\frac{1}{2}}} &= \frac{y^{\frac{1}{2}}}{x^{\frac{1}{4}}} \cdot \frac{x^{-\frac{3}{4}}}{y^{-\frac{1}{2}}} \\
\frac{y^{\frac{1}{2}}}{x^{\frac{1}{4}}} &= \frac{y^{\frac{1}{2}}}{x^{\frac{3}{4}}} \\
\frac{y^{\frac{1}{2}+\frac{1}{2}}}{x^{\frac{1}{4}+\frac{3}{4}}} &= \boxed{\frac{y}{x}}
\end{aligned}$$

5.

$$\begin{aligned}
\frac{x^3y^{-\frac{1}{2}}}{(64x^3)^{\frac{1}{3}}y^{\frac{1}{3}}} &= \frac{x^3y^{-\frac{1}{2}}}{4xy^{\frac{1}{3}}} \\
\frac{1}{4} \cdot \frac{x^3}{x} \cdot \frac{y^{-\frac{1}{2}}}{y^{\frac{1}{3}}} &= \frac{1}{4} \cdot x^2 \cdot y^{-\frac{1}{2}-\frac{1}{3}} \\
\frac{x^2}{4} \cdot y^{-\frac{5}{6}} &= \frac{x^2}{4} \cdot \frac{1}{y^{\frac{5}{6}}} \\
&= \boxed{\frac{x^2}{4y^{\frac{5}{6}}}}
\end{aligned}$$

6.

$$\begin{aligned}
\left(\frac{x^{-\frac{1}{2}}y^{-\frac{1}{4}}}{x^{-\frac{2}{3}}y^{-\frac{1}{2}}}\right)^{-1} &= \frac{x^{-\frac{2}{3}}y^{-\frac{1}{2}}}{x^{-\frac{1}{2}}y^{-\frac{1}{4}}} \\
\frac{x^{\frac{1}{2}}y^{\frac{1}{4}}}{x^{\frac{2}{3}}y^{\frac{1}{2}}} &= x^{\frac{1}{2}-\frac{2}{3}}y^{\frac{1}{4}-\frac{1}{2}} \\
x^{-\frac{1}{6}}y^{-\frac{1}{4}} &= \boxed{\frac{1}{x^{\frac{1}{6}}y^{\frac{1}{4}}}}
\end{aligned}$$

7.

$$\begin{aligned}
\frac{x^3y^{-\frac{1}{2}}z^{-1}}{x^{-\frac{2}{3}}y^{\frac{1}{2}}z^{-\frac{1}{3}}} &= \frac{x^3}{y^{\frac{1}{2}}} \cdot \frac{y^{-\frac{1}{2}}z^{-1}}{x^{-\frac{2}{3}}z^{-\frac{1}{3}}} \\
\frac{x^3}{y^{\frac{1}{2}}} \cdot \frac{x^{\frac{2}{3}}z^{\frac{1}{3}}}{y^{\frac{1}{2}}z} &= \frac{x^{3+\frac{2}{3}}z^{\frac{1}{3}}}{y^{\frac{1}{2}+\frac{1}{2}}z} \\
\frac{x^{\frac{11}{3}}z^{\frac{1}{3}}}{yz} &= \frac{x^{\frac{11}{3}}}{y} \cdot \frac{z^{\frac{1}{3}}}{z} \\
\frac{x^{\frac{11}{3}}}{y} \cdot z^{\frac{1}{3}} - 1 &= \frac{x^{\frac{11}{3}}}{y} \cdot z^{-\frac{2}{3}} \\
\frac{x^{\frac{11}{3}}}{y} \cdot \frac{1}{z^{\frac{2}{3}}} &= \boxed{\frac{x^{\frac{11}{3}}}{yz^{\frac{2}{3}}}}
\end{aligned}$$

– (F) –

II :

4.

$$4\sqrt{12x^4} - x\sqrt{27x^2} = 4 \cdot \sqrt{4(x^2)^2} \cdot \sqrt{3} - x \cdot \sqrt{9x^2} \cdot \sqrt{3}$$
$$4 \cdot 2x^2 \cdot \sqrt{3} - x \cdot 3x \cdot \sqrt{3} = 8x^2\sqrt{3} - 3x^2\sqrt{3}$$
$$= \boxed{5x^2\sqrt{3}}$$

5.

$$\sqrt[3]{24x} + \sqrt[3]{-81x} - \sqrt[3]{3x} = \sqrt[3]{8} \cdot \sqrt[3]{3x} + \sqrt[3]{-27} \cdot \sqrt[3]{3x} - \sqrt[3]{3x}$$
$$2\sqrt[3]{3x} - 3\sqrt[3]{3x} - \sqrt[3]{3x} = \boxed{-2\sqrt[3]{3x}}$$

III :

3.

$$\sqrt{3}(\sqrt{2} + 2\sqrt{3} - \sqrt{5}) = \sqrt{3 \cdot 2} + 2 \cdot \sqrt{3 \cdot 3} - \sqrt{3 \cdot 5}$$
$$\sqrt{6} + 2\sqrt{9} - \sqrt{15} = \boxed{6 + \sqrt{6} - \sqrt{15}}$$

IV :

2. $\boxed{\sqrt{2xy}}$ because:

$$\sqrt{2xy} \cdot \sqrt{2xy} = (\sqrt{2xy})^2 = 2xy$$