The Nature of Quadratic Equations

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An Advanced Synopsis of Polynomials In the Second Degree

Use your bookmarks, and tap on any chapter/section and it will take you to your page.
This is not a reviewer! This is an outline, a guide, and as a reference.

Made with love and LaTeX
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Contents

0.1 Preface

Quadratic equations are the pinnacle of every high school student. It has its infamy for producing the *quadratic formula*, in popular culture. This kind of information has stuck, despite the fact that it really is not that hard to grasp with the right knowledge. My gripe is to aide the reader in successive terms such that they can understand quadratics in full detail. This task is however so inevitable to have issues. The reader can't know everything in possible succession if they don't know the prerequisites. The reader can be more susceptible to confusion and lack of generality for the subject. Most people don't even care about the mathematics they learn in school; they simply just want to learn it so they can have a passing grade in school. This is honestly the saddening part. I self-teach myself mathematics or anything that needs an exam to partake in. It is quite saddening that students cannot endure such coerce understandings without feeling inadequacy to the entire goal. Mathematics is a subject, not an object.

The entire premise of this little book is to ensure the reader's full comprehension of the topic. The rigor and the in-depth associations with the topic is a gripe that students do not love. This booklet is designed without the rigor, without the gripe of abstracts. While I understand students need to learn mathematics just to get a passing score, it is undesirable for it to be a reason anyway. It is no surprise that mathematics is a tough and desolate subject with rigor, abstractions, and paradoxical reasoning that the reader just does not want to embark. But, a step-by-step approach, to something so radical to the naked eye, can be turned simple *if you know what you are doing.*

0.2 Requirements

The prerequisites for the typical quadratic studies is nothing moreso than maturity in manipulating algebraic equations to solve for a given numerical value. The reader can enter without this requirement but it is not a foreseen success of full commitment to fully comprehend. I recommend a brief recourse of basic algebra that you have tackled in the last two grades. Nothing advanced or over-the-top, and nothing too rigorous, namely you just need to learn the core of: Integer arithmetic, the idea of exponents, properties of arithmetic and exponents, PEMDAS, factoring, and square roots. The nature of unheard topics will be explained in an appendix in addition at the very end of this scope. I adhere this as many people do not have the amends of understanding to know this, so the appendix can be read at all times to anyone if you need to.

0.2.1 Recommendations

I recommend the reader to take a piece of paper, and something to write with. Take notes of what you understand, and understand the pieces DIRECTLY. If you need any superintendent aide, ask someone or a friend that discretely knows what you are tackling. This subject is not passable with no gaps, it is a thorn maze filled with tiny holes that leak information without grasp.

I also want to note that: You won't understand everything at first gripe and that is completely unavoidable. It is normal and not in an abstract way that people don't understand this. I've seen people understand a topic after weeks or even years, even something basic as scientific notation can be understood after years. Interest in mathematics is not for the faint heart of those who see it deterring their future outcome. It is not an outcast. Having difficulties reading mathematics or such any topic is not a block. It is material to build off of. Find methodical ways to understand the reasoning, and you will understand it better. Take your time, even if it won't be comprehension certified in a day. It is something that you, and anyone, can do, and can willfully understand with the hope of common good between.

0.3 Practice Problems

Seeing these "Practice Problems" might make you feel like this is an artificial way to boost the formality of this document. But I want to emphasize: *these are for the reader*. Practice problems are INCREDIBLY beneficial especially for mathematics when you want to take an exam. They are just for practice and maintaining skill development and deduction. Do at least three or five, and try inferring a lot from the discussion seen. These are not required, but are INCREDIBLY beneficial if you perform them as they increase your familiarity with what you have read.

The only way to learn mathematics is to do mathematics.
– Paul Halmos, PhD

1 Introduction to Quadratic Equations

In a notion, quadratic equations are depicted as polynomials. A *polynomial* is an equation with terms that have the following constraints:

1. They do not divide by an unknown. $\left(\frac{1}{x}\right)$
2. They do not have a radical of an unknown. $\sqrt{x-11}$
3. They do not have fractional or negative exponents.[1]

Despite that, numerically, the goal of quadratic equations is *to find values of an unknown such that they make the quadratic lead to zero as a result.* This is depicted as *solving a quadratic equation.*

1.1 What It Depicts

A quadratic equation's values that equal zero are called *roots* or *zeroes* of a quadratic. Solving quadratic equations have four forms:

- By Extracting Roots
- By Completing the Square
- By Factoring
- By the Quadratic Formula

Ideally, these four methods are all famed to be ones that help complete a quadratic equation to zero.

1.2 Forms of a Quadratic

A quadratic equation has five forms. They typically correspond with what method of solving quadratics is required in each form.

Definition 1.1 (Forms of a Quadratic)**.** The following:

1. $ax^2 = c$
2. $ax^2 + c = 0$
3. $ax^2 + bx = 0$
4. $ax^2 + bx = c$ − Standard Form
5. $ax^2 + bx + c = 0$ − General Form

Note. I want to process some common notions in a polynomial. The *coefficients* are the numbers that are multiplying an unknown. The unknowns in a polynomial are the *variables*. The numbers that are lone, meaning they are just subtracting or adding something are called *constants*. As an example,

$$3x^2 - 12x + 16 = 0$$

[1]This is appended by the laws of exponents where a fractional exponent leads to a radical, and a negative leads to a fraction. More can be said in the next quarter of our school year.

Coefficients are 3 and -12, variable is x, and constant is 16. If some terms of a polynomial are on the right-hand side of the equation (opposite the equal sign) then move them to the other side by transposition.[2] If a term does not exist, then the coefficient or constant term (depends on what kind of term is there) is automatically zero.

The bx term in the general form is the *linear* term. The ax^2 in the general form is the *quadratic* term.

1.3 A Formal Definition

A quadratic equation is described as:

Definition 1.2 (The Quadratic)**.** If a, b, c are coefficients and $a \neq 0$, then a *quadratic equation* is considered as $ax^2 + bx + c = 0$ in terms of x as the general form.

Quadratic equations might seem unobvious if you consider this.

Example. Is $(x-3)(x+6) = 0$ a quadratic equation?

Yes. We must note the common notion of *expanding terms.* Using the FOIL method,

$$\begin{aligned}(x-3)(x+6) = 0 &= x^2 + 6x - 3x - 18 = 0 \\ &= x^2 + 3x - 18\end{aligned}$$

The common merit is that our leading term (first term) is a quadratic term. So, we can say that *it is indeed a quadratic.*

One thing I would want to consider is the order of polynomials. Quadratic equations, are more formally expressed with the quadratic term as the first term in the equation, then the linear, then the constant.

If it is expressed as $-7x + 49 + x^2$, move x^2 in the front as $x^2 - 7x + 49$. If the x^2 term is negative, like $4x - 4 - x^2$, switch the quadratic term to the front and multiply everything by -1 to switch all signs.

$$\begin{aligned}4x - 4 - x^2 &= -x^2 + 4x - 4 \\ -1(-x^2 + 4x - 4) &= (-1)(-x^2) + (-1)(4x) + (-1)(-4) \\ x^2 + (-4x) + 4 &= x^2 - 4x + 4\end{aligned}$$

If the linear term is not in the right place then switch it, goes for any term not expressed as the order in $ax^2 + bx + c = 0$.

If I were to ask what a, b, and c are in $x^2 - 5x + 6 = 0$, then $a = 1, b = -5, c = 6$. Account for the sign. If there is a term missing, then the missing term(s) equals 0, however if ANY TERM is on the other side ($x^2 - 5x = -6$), move it to the other side to which the quadratic term does not become negative ($x^2 - 5x = -6 \Rightarrow -x^2 + 5x - 6$). If so, multiply all by -1. Proper way would be $x^2 - 5x = -6 \Rightarrow x^2 - 5x + 6$. Ignore those arrow symbols. They just mean "implies." I won't go in depth with this jargon.

[2] More extended in the appendix.

2 Solving the Quadratic Equation

2.1 by Extracting Roots

Extracting roots involve tinkering with the equation. Extracting roots only work if $ax^2 = c$ or $ax^2 + c = 0$ or if we are completing the square.[3]

Definition 2.1. If there exists a quadratic equation $ax^2 = c$, then it can be solved as the process:

$$\begin{aligned} ax^2 &= c \\ \frac{ax^2}{a} &= \frac{c}{a} \\ x^2 &= \frac{c}{a} \\ \sqrt{x^2} &= \sqrt{\frac{c}{a}} \\ x &= \pm\sqrt{\frac{c}{a}} \end{aligned}$$

Divide by a if $a \neq 1$.

The symbol $\pm$ is called the *plus-or-minus*, *plus-negative*, or *positive-or-negative*, chose whatever expression suitable.

Remark. We know that a positive number times a positive number yields a positive number. It should also be said that a negative number times a negative number still yields a positive number. So, that is why we included a $\pm$ in our root, because we still account if the result is negative in order to get the same answer.

If we consult that: any equation that only has the quadratic term and a constant equaling it, dividing by the term's coefficient if it is not equal to one, taking the roots on both sides (with regards to plus-minus), leads to our simplified answer. A simple showcase can be shown by the following examples.

Example. Find the roots of each of the following:

(a) $3x^2 = 48$

(b) $b^2 - 4 = 12$

(c) $(p + 7)^2 + 12 = 11$

(d) $5z^2 - 90 = 0$

(e) $m^2 = \frac{54}{9}$

(a): Examine the following.

$$\begin{aligned} 3x^2 &= 18 \\ \frac{3x^2}{3} &= \frac{18}{3} \\ x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} \\ x &= \boxed{\pm 3} \end{aligned}$$

[3]A part of completing the square is expressing both sides as roots. Discussion will entail more on later.

Since we know that 9 is a perfect square, it means thrice of three, or three times itself, or negative three times itself. So our result is what is shown.

(b): Examine the following.

$$\begin{aligned} b^2 - 4 &= 12 \\ b^2 &= 12 + 4 \\ b^2 &= 16 \\ \sqrt{b^2} &= \sqrt{16} \\ x &= \boxed{\pm 4} \end{aligned}$$

We considered the fact that -4 is a constant on the left-hand side. We remove the -4 by *transposing* (or adding both sides by 4) to remove it, to then proceed to our step. It is ESSENTIAL to remember: Remove any constant from the side that has the quadratic term IF there is no linear term.

(c): Examine the following.

$$\begin{aligned} (p+7)^2 + 12 &= 11 \\ (p+7)^2 &= 11 - 12 \\ (p+7)^2 &= -1 \\ \sqrt{(p+7)^2} &= \sqrt{-1} \\ p + 7 &= \pm i \\ p &= \boxed{7 \pm i} \end{aligned}$$

Questions may head over. What is i? In mathematics, historically, there was no real expression of negative numbers under radicals. Why? Figure this: Negative times a negative is positive. Negative times a positive is negative (vice versa.) We can't really express a negative radical without having opposing signs. So, Leonard Euler declared the imaginary unit i as the universal symbol for negative numbers under radicals.[4]

The reason why we took the root of $(p+7)^2$ is because it *acts like a quadratic term*, just with extra terms underneath. Replace $p+7$ with any arbitrary symbol and we treat it normally; as a regular term.

(d): Examine the following.

$$\begin{aligned} 5z^2 - 90 &= 0 \\ 5z^2 &= 90 \\ \frac{5z^2}{5} &= \frac{90}{5} \\ z^2 &= 18 \\ \sqrt{z^2} &= \sqrt{18} \\ z &= \pm\sqrt{18} \\ z &= \boxed{\pm 3\sqrt{2}} \end{aligned}$$

[4]The scope of imaginary units *is* under the ninth-grade curricula, but we do not consider it in our discussions as we only follow from the textbook that the school uses. For more research is optional but not needful in the least.

New questions can come in; why is it $3\sqrt{2}$? Is this a new perfect square type? *No.* All we did was *reduce/simplify the radical.* Reducing a radical involves finding the biggest factor of the number in it that is a perfect square, breaking it up as a product, and simplifying the perfect square.[5] For now, let us just assume this axiomatically is correct.

(e): Examine the following.

$$\begin{aligned} m^2 &= \frac{54}{9} \\ \sqrt{m^2} &= \sqrt{\frac{54}{9}} \\ m &= \pm\sqrt{\frac{54}{9}} \\ m &= \pm\frac{\sqrt{54}}{\sqrt{9}} \\ m &= \pm\frac{3\sqrt{6}}{3} \\ m &= \boxed{\pm\sqrt{6}} \end{aligned}$$

I included this example as you can just divide 54 to 9 and get 6, and it will lead to the same result. But I wanted to show this solution as it is really needed. The idea of distributing radicals is simple: If a radical is expressed as $\sqrt{\frac{a}{b}}$, it is the same as $\frac{\sqrt{a}}{\sqrt{b}}$.

– PRACTICE PROBLEMS (A) –

Compute the following roots. Questions with * are with FULL solutions next page.

1. $x^2 = 289$
2. * $r^2 + 25 = 0$
3. $q^2 - 15 = 34$
4. $a^2 + 1 = 2$
5. * $t^2 - \frac{5}{4} = \frac{3}{4}$
6. $(y + 2)^2 = 9$
7. * $(h - 8)^2 - \frac{4}{25} = \frac{9}{25}$
8. $(2x - 9)^2 + 9 = 0$
9. $(3k + 5)^2 = 32$
10. * $(n - 1)(n + 2) = n$

[5] Adhered in the appendix.

ANSWERS TO SELECTED QUESTIONS

2.

$$\begin{aligned} r^2 + 25 &= 0 \\ r^2 &= -25 \\ \sqrt{r^2} &= \sqrt{-25} \\ r &= \pm 5i \end{aligned}$$

5.

$$\begin{aligned} t^2 - \frac{5}{4} &= \frac{3}{4} \\ t^2 &= \frac{5}{4} + \frac{3}{4} \\ t^2 &= \frac{8}{4} \\ t^2 &= 2 \\ \sqrt{t^2} &= \sqrt{2} \\ t &= \pm\sqrt{2} \end{aligned}$$

7.

$$\begin{aligned} (h-8)^2 - \frac{4}{25} &= \frac{9}{25} \\ (h-8)^2 &= \frac{9}{25} - \frac{4}{25} \\ (h-8)^2 &= \frac{5}{25} \\ \sqrt{(h-8)^2} &= \sqrt{\frac{5}{25}} \\ h - 8 &= \pm\frac{\sqrt{5}}{\sqrt{25}} \\ h - 8 &= \pm\frac{\sqrt{5}}{5} \\ h &= 8 \pm \frac{\sqrt{5}}{5} \end{aligned}$$

10.

$$\begin{aligned} (n-1)(n+2) &= n \\ n^2 + 2n - n - 2 &= n \\ n^2 + n - 2 &= n \\ n^2 + n - 2 - n &= 0 \\ n^2 - 2 &= 0 \\ n^2 &= 2 \\ \sqrt{n^2} &= \sqrt{2} \\ n &= \pm\sqrt{2} \end{aligned}$$

2.2 by Completing the Square

The sole idea of *completing the square* is to *force* a quadratic equation to factor as a perfect square trinomial such that you can simply extract the roots. We have to consider that: The definition of this is a little wordy, so I will explain it here:

Definition 2.2 (Completing the Square)**.** In order to complete the square of a general quadratic equation $ax^2 + bx + c = 0$:

1. Divide by a if $a \neq 1$.
2. Transpose (or move) c to both sides.
3. Take your b coefficient; divide it by two, and square that result $(\frac{b}{2})^2$
4. Add this result on both sides of your quadratic (considering you moved c on both sides already.)
5. Simplify the other side.
6. Factor the new quadratic to a perfect square trinomial.
7. Extract roots on both sides.
8. Move the constant term (from the once quadratic) to the other side.
9. You have your roots.

We can try the following examples:

Example. Solve the following:

(a) $x^2 - 4x - 12 = 0$

(b) $x^2 - x - 3 = 0$

(c) $4x^2 + 8x + 1 = 0$

(d) $2x^2 - x - 5 = 0$

(e) $x^2 + 4x = 7$

(a): Examine the following. Follow the steps.

$$\begin{aligned}
x^2 - 4x - 12 &= 0 \\
x^2 - 4x &= 12 \\
\left(\frac{4}{2}\right)^2 &= 4 \\
x^2 - 4x + 4 &= 12 + 4 \\
(x-2)^2 &= 16 \\
\sqrt{(x-2)^2} &= \sqrt{16} \\
x - 2 &= \pm 4 \\
x &= 2 \pm 4 \\
&= 2 + 4 = 6 \\
&= 2 - 4 = -2 \\
x &= \boxed{-2, 4}
\end{aligned}$$

At the end, if you have don't have any radicals on the other side, or if they all add up perfectly to a rational number, divide split the plus-minus to a plus and negative to get both answers. This is a mistake often students make, so beware.

(b): Examine the following.

$$\begin{aligned}
x^2 - x - 3 &= 0 \\
x^2 - x &= 3 \\
\left(\frac{1}{2}\right)^2 &= \frac{1}{4} \\
x^2 - x + \frac{1}{4} &= 3 + \frac{1}{4} \\
&= \frac{12}{4} + \frac{1}{4} \\
&= \frac{13}{4} \\
\left(x + \frac{1}{2}\right)^2 &= \frac{13}{4} \\
\sqrt{\left(x + \frac{1}{2}\right)^2} &= \sqrt{\frac{13}{4}} \\
x + \frac{1}{2} &= \pm\frac{\sqrt{13}}{\sqrt{4}} \\
&= \pm\frac{\sqrt{13}}{2} \\
x &= -\frac{1}{2} \pm \frac{\sqrt{13}}{2} \\
x &= \boxed{\frac{-1 \pm \sqrt{13}}{2}}
\end{aligned}$$

Notice I combined the fractions at the end? It is because they have the same denominator. If you fear adding fractions, just do the butterfly method. You know that already, but if you don't, you can always use the internet.

(c): Examine the following.

$$
\begin{aligned}
4x^2 + 8x + 1 &= 0 \\
\frac{4x^2}{4} + \frac{8x}{4} + \frac{1}{4} &= 0 \\
x^2 + 2x + \frac{1}{4} &= 0 \\
x^2 + 2x &= -\frac{1}{4} \\
\left(\frac{2}{2}\right)^2 &= 1 \\
x^2 + 2x + 1 &= -\frac{1}{4} + 1 \\
&= -\frac{1}{4} + \frac{4}{4} \\
&= \frac{-1+4}{4} \\
x^2 + 2x + 1 &= \frac{3}{4} \\
(x+1)^2 &= \frac{3}{4} \\
\sqrt{(x+1)^2} &= \sqrt{\frac{3}{4}} \\
x + 1 &= \pm\frac{\sqrt{3}}{\sqrt{4}} \\
&= \pm\frac{\sqrt{3}}{2} \\
x &= -1 \pm \frac{\sqrt{3}}{2} \\
x &= \frac{-2}{2} \pm \frac{\sqrt{3}}{2} \\
x &= \boxed{\frac{-2 \pm \sqrt{3}}{2}}
\end{aligned}
$$

(d): Examine the following:

$$\begin{aligned}
2x^2 - x - 5 &= 0 \\
\frac{2x^2}{2} - \frac{x}{2} - \frac{5}{2} &= 0 \\
x^2 - \frac{1}{2}x - \frac{5}{2} &= 0 \\
x^2 - \frac{1}{2}x &= \frac{5}{2} \\
\left(\frac{\frac{1}{2}}{2}\right)^2 &= \frac{1}{2} \cdot \frac{1}{2} \\
&= \left(\frac{1}{4}\right)^2 \\
x^2 - \frac{1}{2}x + \frac{1}{16} &= \frac{5}{2} + \frac{1}{16} \\
x^2 - \frac{1}{2}x + \frac{1}{16} &= \frac{80+2}{32} \\
x^2 - \frac{1}{2}x + \frac{1}{16} &= \frac{82}{32} = \frac{41}{16} \\
\left(x - \frac{1}{4}\right) &= \frac{41}{16} \\
\sqrt{\left(x - \frac{1}{4}\right)} &= \sqrt{\frac{41}{16}} \\
x - \frac{1}{4} &= \pm\frac{\sqrt{41}}{\sqrt{16}} \\
x - \frac{1}{4} &= \pm\frac{\sqrt{41}}{4} \\
x &= \boxed{\frac{1 \pm \sqrt{41}}{4}}
\end{aligned}$$

The reason why $\frac{x}{2}$ turned to $\frac{1}{2}x$ because $\frac{a}{b}x = \frac{a}{b} \cdot \frac{x}{1} = \frac{ax}{b}$.

(e) Examine the following.

$$\begin{aligned}
x^2 + 4x &= 7 \\
\left(\frac{4}{2}\right)^2 &= 4 \\
x^2 + 4x + 4 &= 7 + 4 \\
(x+2)^2 &= 11 \\
\sqrt{(x+2)^2} &= \sqrt{11} \\
x + 2 &= \pm\sqrt{11} \\
x &= \boxed{-2 \pm \sqrt{11}}
\end{aligned}$$

– PRACTICE PROBLEMS (B) –

Complete the squares of each quadratic equation and find their roots. Questions with * are with FULL solutions. Questions with (H) have hints.

1. $x^2 - 6x + 4 = 0$
2. $m^2 + 10m + 21 = 0$
3. * $t^2 + 3t - 88 = 0$
4. $k^2 + 14k + 46 = 0$
5. * $3d^2 + 17d + 10 = 0$
6. * $8u^2 + 12u + 3 = 0$
7. $6r^2 - 2r - 5 = -4$
8. * $2j^2 = 6j - 5$
9. $4g^2 = -12g - 11$
10. (H) $8y = -x^2 + 20$

ANSWERS/HINTS TO SELECTED QUESTIONS

10. *Hint*. Transpose one side to the other to make sure the quadratic term is not negative.

3.

$$\begin{aligned}
t^2 + 3t - 88 &= 0 \\
t^2 + 3t &= 88 \\
\left(\frac{3}{2}\right)^2 &= \frac{9}{4} \\
t^2 + 3t + \frac{9}{4} &= 88 + \frac{9}{4} \\
&= \frac{352 + 9}{4} \\
\left(t + \frac{3}{2}\right)^2 &= \frac{361}{4} \\
\sqrt{\left(t + \frac{3}{2}\right)^2} &= \sqrt{\frac{361}{4}} \\
t + \frac{3}{2} &= \pm\frac{\sqrt{361}}{\sqrt{4}} \\
&= \pm\frac{19}{2} \\
t &= \frac{-3 \pm 19}{2} \\
\frac{-3 - 19}{2} &= -11 \\
\frac{-3 + 19}{2} &= 8 \\
t &= -11, 8
\end{aligned}$$

5.

$$\begin{aligned}
3d^2 + 17d + 10 &= 0\\
\frac{3d^2}{3} + \frac{17d}{3} + \frac{10}{3} &= 0\\
d^2 + \frac{17}{3}d &= -\frac{10}{3}\\
\left(\frac{\frac{17}{3}}{2}\right)^2 &= \left(\frac{17}{3} \cdot \frac{1}{6}\right)^2\\
\left(\frac{17}{6}\right)^2 &= \frac{289}{36}\\
d^2 + \frac{17}{3}d + \frac{289}{36} &= -\frac{10}{3} + \frac{289}{36}\\
&= \frac{289}{36} - \frac{12}{12} \cdot \frac{10}{3}\\
\left(d + \frac{17}{6}\right)^2 &= \frac{169}{36}\\
\sqrt{\left(d + \frac{17}{6}\right)^2} &= \sqrt{\frac{169}{36}}\\
d + \frac{17}{6} &= \pm\frac{\sqrt{169}}{\sqrt{36}}\\
d + \frac{17}{6} &= \pm\frac{13}{6}\\
d &= \frac{-17 \pm 13}{6}\\
\frac{-17 + 13}{6} &= \frac{-4}{6} = -\frac{2}{3}\\
\frac{-17 - 13}{6} &= \frac{-30}{6} = -5\\
d &= -5, -\frac{2}{3}
\end{aligned}$$

6.

$$
\begin{aligned}
8u^2 + 12u + 3 &= 0 \\
\frac{8u^2}{8} + \frac{12u}{8} + \frac{3}{8} &= 0 \\
u^2 + \frac{3}{2}u &= -\frac{3}{8} \\
\left(\frac{\frac{3}{2}}{2}\right)^2 &= \left(\frac{3}{2} \cdot \frac{1}{2}\right)^2 \\
\left(\frac{3}{4}\right)^2 &= \frac{9}{16} \\
u^2 + \frac{3}{2}u + \frac{9}{16} &= -\frac{3}{8} + \frac{9}{16} \\
&= \frac{9}{16} - \frac{2}{2} \cdot \frac{3}{8} \\
\left(u + \frac{3}{4}\right)^2 &= \frac{3}{16} \\
\sqrt{\left(u + \frac{3}{4}\right)^2} &= \sqrt{\frac{3}{16}} \\
u + \frac{3}{4} &= \pm\frac{\sqrt{3}}{\sqrt{16}} \\
u + \frac{3}{4} &= \pm\frac{\sqrt{3}}{4} \\
u &= \frac{-3 \pm \sqrt{3}}{4}
\end{aligned}
$$

8.

$$
\begin{aligned}
2j^2 &= 6j - 5 \\
2j^2 - 6j + 5 &= 0 \\
\frac{2j^2}{2} - \frac{6j}{2} + \frac{5}{2} &= 0 \\
j^2 - 3j &= -\frac{5}{2} \\
\left(\frac{3}{2}\right)^2 &= \frac{9}{4} \\
j^2 - 3j + \frac{9}{4} &= -\frac{5}{2} + \frac{9}{4} \\
j^2 + 3j + \frac{9}{4} &= \frac{9}{4} - \frac{2}{2} \cdot \frac{5}{2} \\
\left(j + \frac{3}{2}\right)^2 &= \frac{-1}{4} \\
\sqrt{\left(j + \frac{3}{2}\right)^2} &= \sqrt{\frac{-1}{4}} \\
j + \frac{3}{2} &= \pm\frac{\sqrt{-1}}{\sqrt{4}} \\
j + \frac{3}{2} &= \pm\frac{i}{2} \\
j &= \frac{-3 \pm i}{2}
\end{aligned}
$$

2.3 by The Quadratic Formula

The *quadratic formula* is the formulaic representation of completing the square, essentially the shortcut method to it. The actual methodical reasoning of *completing the square* is the process to *derive* the quadratic formula. We can derive it here, however the discussion of deriving the quadratic formula can be excluded:

Definition 2.3 (Deriving the Quadratic Formula)**.** Begin with the general form of a quadratic $ax^2 + bx + c = 0$.

$$
\begin{aligned}
\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} &= 0 \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
\left(\frac{\frac{b}{a}}{2}\right)^2 &= \frac{b}{a} \cdot \frac{1}{2} \\
\left(\frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} \\
x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} &= \frac{b^2}{4a^2} - \frac{c}{a} \\
&= \frac{a}{a} \cdot \frac{b^2}{4a^2} - \frac{4a^2}{4a^2} \cdot \frac{c}{a} \\
\frac{ab^2 - 4a^2c}{4a^3} &= \frac{a(b^2 - 4ac)}{4a^3} \\
&= \frac{b^2 - 4ac}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
\sqrt{\left(x + \frac{b}{2a}\right)^2} &= \sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \boxed{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}
\end{aligned}
$$

Definition 2.4 (The Quadratic Formula)**.** The *quadratic formula* is expressed as: If any quadratic as $ax^2 + bx + c = 0$ exists, then

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is considered the formula for the polynomial.

All we have to do is solve for x by substituting all coefficients a, b, and c and solve from there. Since completing the square applies to ALL quadratics, then we can deduct that the quadratic formula applies to ALL quadratics alongside.

Example. Solve all of the following with the quadratic formula:

(a) $x^2 - 4x + 4 = 0$

(b) $3x^2 + 8x + 5 = 0$

(c) $2x^2 = 6x - 11$

(d) $x^2 - 2x + 6 = 0$

(e) $x^2 - 5x = 0$

(a): Examine the following. Follow the steps.

We know that our quadratic formula is

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so we substitute all coefficients that exist in this formula.

$$\begin{aligned}
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)} \\
\frac{4 \pm \sqrt{16 - 4(4)}}{2} &= \frac{4 \pm \sqrt{16 - 16}}{2} \\
\frac{4}{2} &= x = \boxed{2}
\end{aligned}$$

(b): Examine the following.

$$\begin{aligned}
3x^2 + 8x + 5 &= 0 \\
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-(8) \pm \sqrt{(8)^2 - 4(3)(5)}}{2(3)} \\
\frac{-8 \pm \sqrt{64 - 60}}{6} &= \frac{-8 \pm 2}{6} \\
\frac{-8 + 2}{6} &= -1 \\
\frac{-8 - 2}{6} &= -\frac{5}{6} \\
x &= \boxed{-1, -\frac{5}{6}}
\end{aligned}$$

(c): Examine the following.

$$2x^2 = 6x - 11 = 2x^2 - 6x + 11 = 0$$
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(11)}}{2(2)}$$
$$\frac{6 \pm \sqrt{36 - 88}}{4} = \frac{6 \pm \sqrt{-52}}{4}$$
$$\sqrt{-52} = \sqrt{-4}\sqrt{13} = 2i\sqrt{13}$$
$$\frac{6 \pm 2i\sqrt{13}}{4} = \frac{6}{4} \pm \frac{2i\sqrt{13}}{4}$$
$$= \frac{3}{2} \pm \frac{i\sqrt{13}}{2}$$
$$x = \boxed{\frac{3 \pm i\sqrt{13}}{2}}$$

Notice at the end I split up the fractions? I reduced the radical and I reduced the fractions. Since all of the coefficients in the numerator and the term in the denominator can be simplified, I just reduced the fraction in order to make it in its simplified form.

(d): Examine the following.

$$x^2 - 2x + 6 = 0$$
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(6)}}{2(1)}$$
$$\frac{2 \pm \sqrt{4 - 24}}{2} = \frac{2 \pm \sqrt{-20}}{2}$$
$$\sqrt{-20} = \sqrt{-4}\sqrt{5} = 2i\sqrt{5}$$
$$\frac{2 \pm 2i\sqrt{5}}{2} = \frac{2}{2} \pm \frac{2i\sqrt{5}}{2}$$
$$x = \boxed{1 \pm i\sqrt{5}}$$

(e): Examine the following.

$$x^2 - 5x = 0$$
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(0)}}{2(1)}$$
$$\frac{5 \pm \sqrt{25}}{2} = \frac{5 \pm 5}{2}$$
$$\frac{5 + 5}{2} = 5$$
$$\frac{5 - 5}{2} = 0$$
$$x = \boxed{0, 5}$$

Since there is no c or constant term, it means that $c = 0$ since it is not in the quadratic. So, if there is a term missing (THAT IS NOT THE QUADRATIC TERM[6]), substitute that term with 0.

[6] Removing the quadratic term would make the equation a linear equation $ax + b = 0$.

– PRACTICE PROBLEMS (C) –

Solve each quadratic equation by using the quadratic formula and find their roots.

1. $x^2 - 5x + 6 = 0$
2. $x^2 - 7x + 10 = 0$
3. $4x^2 - x - 3 = 0$
4. $2x^2 = -x - 3$
5. $12x - 36 = -x^2$
6. $x^2 - 7 = 0$
7. $x^2 + 2x + 9 = 0$
8. $x^2 - 10x + 30 = 0$
9. $2x^2 + 8x - 3 = 0$
10. $x^2 + 11 = 0$
11. $x^2 + 4x + 13 = 0$
12. $x^2 + 2x = -5$
13. $x^2 = x - 10$
14. (H) $18 = -x^2 - 3x + 36$
15. $3x^2 - 5x - 2 = -7$
16. (H) $2x^2 + 2x - 19 = -6x - 13$

HINTS TO SELECTED QUESTIONS

14. *Hint*. Transpose one side that does not make the quadratic term negative, and combine like terms.

16. *Hint*. Transpose one side that does not make the quadratic term negative, and combine like terms.

2.4 by Factoring

The idea of *factoring* a quadratic is similar to how factoring works ignoring the third degree polynomial. More about factoring is deferred to the appendix. We shall start with Common Monomial Factoring (CMF.) We must also adhere the *zero property of binomial products.*

Definition 2.5. If a product $(a)(b) = 0$, then $a = 0$ or $b = 0$.

This means that if there is a product of two a and b equate to zero, then either a or b is zero or BOTH are zero.

Example. Consider the following:

(a) $3x^2 - 18 = 0$

(b) $2x^2 + 8x = 0$

(c) $2x^2 + 4x - 12 = 0$

(a): Examine the following. The GCF of this equation is 3.

$$3x^2 - 18 = 0 = 3(x^2 - 9) = 0$$
$$x^2 - 9 = 0 = x^2 = 9$$
$$x = \boxed{\pm 3}$$

(b): Examine the following. The GCF of this equation is $2x$.

$$2x^2 + 8x = 0 = 2x(x + 4) = 0$$
$$x + 4 = 0$$
$$x = -4$$
$$2x = 0$$
$$x = 0$$
$$x = \boxed{-4, 0}$$

(c): Examine the following. The GCF of this equation is 2.

$$2x^2 + 2x - 12 = 0 = 2(x^2 + x - 6) = 0$$
$$x^2 + x - 6 = 0$$
$$\frac{-1 \pm \sqrt{(1)^2 - 4(1)(-6)}}{2(1)} = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{4}$$
$$\frac{1+5}{4} = \frac{3}{2}$$
$$\frac{1-5}{4} = -1$$
$$x = \boxed{-1, \frac{3}{2}}$$

We can also begin by using the difference of two squares as a factoring method. The perfect-square trinomial is self explanatory, as it is referenced in completing the square. I will append here with one example.

Example. Consider the following:

(a) $x^2 - 441 = 0$

(b) $x^2 = 100$

(c) $x^2 + 20x + 100 = 0$

(a): Examine the following.

$$x^2 - 441 = 0$$
$$(x+21)(x-21) = 0$$
$$x + 21 = 0 = x = -21$$
$$x - 21 = 0 = x = 21$$
$$x = \boxed{\pm 21}$$

I merged both 21's because they are differing signs, and it is the same as $\pm$.

(b): Examine the following.

$$x^2 = 100$$
$$x^2 - 100 = 0$$
$$(x+10)(x-10) = 0$$
$$x + 10 = 0 = x = -10$$
$$x - 10 = 0 = x = 10$$
$$x = \boxed{\pm 10}$$

We could have also extracted roots as well giving the same answer.

(c): Examine the following.

$$x^2 + 20x + 100 = 0$$
$$(x+10)^2 = 0$$
$$\sqrt{(x+10)^2} = \sqrt{0}$$
$$x + 10 = 0$$
$$x = \boxed{-10}$$

I would also bring awareness the idea of *equal roots*; where a quadratic equation has only one solution. These only occur in quadratics as forms of a perfect-square trinomial.

The similar idea can be said for factoring a general trinomial. I want to assume the reader knows the pre-requisite of this otherwise they are free to relearn it[7].

Example. Consider the following:

(a) $x^2 - 13x + 22 = 0$

(b) $x^2 - x - 20 = 0$

(c) $3x^2 + 7x + 4 = 0$

(a): Examine the following. Two numbers that add to -13 and multiply to 22 are $-2, -11$.

$$\begin{aligned}
x^2 - 13x + 22 &= 0 \\
(x-2)(x-11) &= 0 \\
x - 2 = 0 &= x = 2 \\
x - 11 = 0 &= x = 11 \\
x &= \boxed{2, 11}
\end{aligned}$$

(b): Examine the following. Two numbers that add up to -1 and multiply to -20 are $-5, 4$.

$$\begin{aligned}
x^2 - x - 20 &= 0 \\
(x+4)(x-5) &= 0 \\
x + 4 = 0 &= x = -4 \\
x - 5 = 0 &= x = 5 \\
x &= \boxed{-4, 5}
\end{aligned}$$

(c): Examine the following. Since $a \neq 1$, or the coefficient of the quadratic term, move then multiply a to the constant term c.

$$3x^2 + 7x + 4 = x^2 + 7x + (3)(4) = x^2 + 7x + 12$$

Then, factor. Two numbers that add up to 7 and multiply to 12 are $3, 4$.

$$\begin{aligned}
x^2 + 7x + 12 &= 0 \\
(x+3)(x+4) &= 0
\end{aligned}$$

Now, divide the constant term of each binomial with the a term. If it is not divisible, or considering that the term has been reduced and yet still isn't divisible, move the term to the beginning of the binomial it is situated.

$$\begin{aligned}
(x+3)(x+4) = 0 &= \left(x + \frac{3}{2}\right)\left(x + \frac{4}{2}\right) \\
(2x+3)(x+2) &= 0 \\
2x + 3 = 0 &= 2x = -3 \\
&= x = -\frac{3}{2} \\
x + 2 = 0 &= x = -2 \\
x &= \boxed{-2, -\frac{3}{2}}
\end{aligned}$$

[7] Adhered in the appendix.

– PRACTICE PROBLEMS (D) –

Solve the following by all processes of factoring, and apply them. Then, find their roots.

1. $4x^2 + 12 = 0$
2. $x^2 + 2x = x$
3. $3x^2 - 24 = 3$
4. (H) $x^2 - 2x = 3x^2$
5. $16x^2 - 32 = 48$
6. $-6 + 6x^2 = 0$
7. $15x^2 - 5 = 0$
8. * $4x^2 - 32x + 64 = 0$
9. $9x^2 - 16 = 0$
10. $x^2 - 64 = 0$
11. $144 - x^2 = 0$
12. $x^2 - 1 = 0$
13. $x^2 + 4 = 29$
14. $x^2 - 150 = -19$
15. $16x^2 - 81 = -56$
16. $x^2 + 17 = 21$
17. $x^2 - x - 42 = 0$
18. $x^2 + 13x + 36 = 0$
19. $x^2 + 4x - 12 = 0$
20. (H) $3x^2 - 12x + 9 = 0$
21. $2x^2 + 11x + 9 = 0$
22. $3x^2 - 14x + 8 = 0$
23. $4x^2 + 3x - 10 = 0$
24. $6x^2 + 17x - 3 = 0$

ANSWERS/HINTS TO SELECTED QUESTIONS

4. *Hint*. Transpose one side such that the quadratic term is not negative.

20. *Hint*. Use CMF first.

8.

$$
\begin{aligned}
4x^2 - 32x + 64 &= 0 \\
4(x^2 - 8x + 16) &= 0 \\
x^2 - 8x + 16 = 0 &= (x-4)^2 = 0 \\
\sqrt{(x-4)^2} &= \sqrt{0} \\
x - 4 &= 0 \\
x &= 4
\end{aligned}
$$

3 Transformations to Quadratics

The idea of *transforming an equation* is solving any equation that does not express as a polynomial, or it is being expressed with fractions, and making it to its polynomial's general form.

Recall that a polynomial have the following restraints:

1. They do not divide by an unknown $\frac{1}{x}$
2. They do not have a radical of an unknown $\sqrt{x - 11}$
3. They do not have fractional or negative exponents.

We can transform any equation that ignores these restraints and make them a polynomial.

3.1 Transforming Algebraic Expressions

Algebraic expressions are just fractions but with expressions. Typically, they do not abide by one of the constraints of a polynomial (They do not divide by an unknown) and in order to convert the algebraic expressions to a polynomial, we must do the following:

1. Multiply both sides by the LCD of the expressions.
2. Cancel out similar terms if they exist.
3. Solve.

Example. Consider the following:

(a) $\dfrac{x^2-2}{3} = \dfrac{x+6}{6}$

(b) $\dfrac{x+1}{x} = \dfrac{5x+12}{2}$

(c) $\dfrac{3x}{x+2} - \dfrac{x+3}{x-1} = \dfrac{x-1}{x+2}$

(a): Examine the following. The LCM of 3 and 6 is 6.

$$\frac{x^2-2}{3} = \frac{x+6}{3} \to 6\left(\frac{x^2-2}{3}\right) = 6\left(\frac{x+6}{6}\right)$$

$$\begin{aligned}
2(x^2-2) = x+6 &= 2x^2 - 4 = x + 6 \\
2x^2 - 4 - x - 6 &= 2x^2 - x - 10 \\
x^2 - x - (10)(2) &= x^2 - x - 20 \\
(x-5)(x+4) &= \left(x - \frac{5}{2}\right)\left(x + \frac{4}{2}\right) \\
(2x-5)(x+2) &= 0 \\
2x - 5 = 0 &= 2x = 5 \\
&= \frac{5}{2} \\
x + 2 = 0 &= x = -2 \\
x &= \boxed{-2, \frac{5}{2}}
\end{aligned}$$

(b): Examine the following. The LCM of $2x$ and 2 is $2x$.

$$\frac{x+1}{x} = \frac{5x+12}{2} \rightarrow 2x\left(\frac{x+1}{x}\right) = 2x\left(\frac{5x+12}{2}\right)$$

$$2(x+1) = x(5x+12) = 2x+2 = 5x^2+12x$$

$$5x^2+10x-2=0$$

$$\frac{-(10) \pm \sqrt{(10)^2-4(5)(-2)}}{2(5)} = \frac{-10 \pm \sqrt{100+40}}{10}$$

$$\frac{-10 \pm \sqrt{140}}{10} = \frac{-10 \pm 2\sqrt{35}}{10}$$

$$\frac{-10}{10} \pm \frac{2\sqrt{35}}{10} = x = \boxed{-1 \pm \frac{\sqrt{35}}{5}}$$

(c): Examine the following. The LCD of $x+2$, $x+2$, and $x-1$ is $(x+2)(x-1)$.

$$(x+2)(x-1)\left(\frac{3x}{x+2} - \frac{x+3}{x-1}\right) = (x+2)(x-1)\left(\frac{x-1}{x+2}\right)$$

$$3x(x-1) - (x+2)(x+3) = (x-1)(x-1)$$

$$3x^2-3x-(x^2+5x+6) = x^2-2x+1$$

$$3x^2-3x-x^2-5x-6 = x^2-2x+1$$

$$3x^2-3x-x^2-5x-6-x^2+2x-1=0$$

$$x^2-6x-7=0$$

$$(x-7)(x+1)=0$$

$$x-7=0=x=7$$

$$x+1=0=x=-1$$

$$x = \boxed{-1, 7}$$

3.2 Transforming Radical Equations

Recall that when we solve quadratic equations, at certain times we have to extract roots on both sides. The *reverse* of this is to *square both sides*. During situations, around when solving radical equations, we must square both sides as to *cancel* the radical. See the example:

Example. Solve $\sqrt{x+5} = 2x$. Examine the following.

$$\begin{aligned}
\sqrt{x+5} &= 2x \\
(\sqrt{x+5})^2 &= (2x)^2 \\
x+5 &= 4x^2 \\
4x^2 - x - 5 &= 0 \\
x^2 - x - (4)(5) &= 0 \\
x^2 - x - 20 &= 0 \\
(x-5)(x+4) &= 0 \\
\left(x - \frac{5}{4}\right)\left(x + \frac{4}{4}\right) &= (4x-5)(x+1) \\
4x - 5 = 0 &= 4x = 5 \\
&= \frac{5}{4} \\
x + 1 = 0 &= x = -1 \\
x &= \boxed{-2, \frac{5}{2}}
\end{aligned}$$

Before we proceed, doing radical equations might give out a result that is not applicable to the equation. To show:

$$\begin{aligned}
\sqrt{\frac{5}{4} + 5} &= 2\left(\frac{5}{4}\right) \\
\sqrt{\frac{5}{4} + \frac{20}{4}} &= \frac{10}{4} \\
\sqrt{\frac{25}{4}} &= \frac{5}{2} \\
\frac{5}{2} &= \frac{5}{2}
\end{aligned}$$

Of course, this is axiomatically true. However, consider the solution $x = -1$.

$$\begin{aligned}
\sqrt{-1+5} &= 2(-1) \\
\sqrt{4} &= -2 \\
2 &\neq -2
\end{aligned}$$

When there is a result or solution that is not applicable or results in a contradiction when checking if it is correct, then it is considered an *extraneous root*. This is a root that does not satisfy the equation but is "still" considered a root.

Let us try one more example. The gist of this part is easy to follow.

Example. Consider the following.

(a) $\sqrt{x+6}+x=0$

(b) $\sqrt{3x-1}=x-1$

(a) Examine the following.

$$\begin{aligned}
\sqrt{x+6}+x&=0\\
\sqrt{x+6}&=-x\\
(\sqrt{x+6})^2&=(-x)^2\\
x+6&=x^2\\
x^2-x-6&=0\\
(x-3)(x+2)&=0\\
x-3=0&=x=3\\
x+2=0&=x=-2\\
x&=-2,3
\end{aligned}$$

Find the extraneous root.

$$\begin{aligned}
\sqrt{3+6}+3&=0\\
\sqrt{9}+3&=0\\
3+3&=0\\
9&\neq 0
\end{aligned}$$

Our extraneous root is $x=3$. So, our solution is $\boxed{x=-2}$.

(b): Examine the following.

$$\begin{aligned}
\sqrt{3x-1}&=3x-1\\
(\sqrt{3x-1})^2&=(3x-1)^2\\
3x-1&=9x^2-6x+1\\
9x^2-6x+1-3x+1&=0\\
9x^2-9x+2&=0\\
\text{skipping some steps. }(3x-2)(3x-1)&=0\\
3x-2=0&=3x=2\\
&=\frac{2}{3}\\
3x-1=0&=3x=1\\
&=\frac{1}{3}\\
x&=\boxed{\frac{1}{3},\frac{2}{3}}
\end{aligned}$$

If you were to substitute x in the equation, both solutions qualify. So this means, this equation has *no extraneous roots.*

– PRACTICE PROBLEMS (E) –

Transform each quadratic equation. Then, solve for their roots. Find their extraneous roots if they have.

1. $\dfrac{x}{x-3} + \dfrac{2x}{3} = 1$
2. $\dfrac{x+5}{x^2-2x} = \dfrac{1}{x^2-2x} + 1$
3. $\dfrac{1}{u} + u = \dfrac{13}{6}$
4. $\dfrac{4}{m} + m - 5 = 0$
5. $\dfrac{t-1}{t+2} = \dfrac{3}{2t}$
6. $\sqrt{y^2-20} = 4$
7. (H) $\sqrt{v-10} - v = 10$
8. (H) $\sqrt{n^2-n+18} = \sqrt{2n^2-3n-6}$
9. $\sqrt{-3s+18} + s = 2s$
10. $\sqrt{2r-3} = \sqrt{r^2-5r+7}$

HINTS TO SELECTED QUESTIONS

7. *Hint*. Make one side of the equation JUST the radical only.
8. *Hint*. Square both sides.

4 The Properties of the Quadratic

The quadratic equation has properties that are unique to this degree of polynomial. Since we know that the values that make the polynomial equate to zero (roots), they have various intrinsic properties that make them valuable to the quadratic equation. This section will be very brief; practice problems will be simpler.

4.1 The Discriminant

The sole idea of the *discriminant* is to identify the nature of the roots of the quadratic. The following must be discerned.

Definition 4.1 (The Discriminant)**.** The discriminant, denoted as $D = b^2 - 4ac$ (or whatever is under the radical sign in the quadratic formula) defines what kind of roots the equation will hold. Consider if:

1. $D > 0$ and a perfect square: *Rational and inequal roots.*
2. $D > 0$ and irrational: *Irrational and inequal roots.*
3. $D = 0$: *Rational and equal.*
4. $D < 0$: *Imaginary*

These basically act as a way to predict what kind of root it will be. Consider the following:

Example. Examine the following's roots.

(a) $x^2 + x + 1 = 0$

(b) $4x^2 + 4x + 1 = 0$

(c) $x^2 + 5x - 14 = 0$

(d) $x^2 - 2x - 4 = 0$

(a): Examine the following.

$$\begin{aligned} D = b^2 - 4ac = D &= (1)^2 - 4(1)(1) \\ D &= -3 \end{aligned}$$

Since $D < 0$, then the roots are *imaginary.*

(b): Examine the following.

$$\begin{aligned} D = b^2 - 4ac = D &= (4)^2 - 4(4)(1) \\ D &= 0 \end{aligned}$$

Since $D = 0$, then the roots are *rational and equal.* They are rational and equal only IF they are perfect-square trinomials.

(c): Examine the following.

$$\begin{aligned} D = b^2 - 4ac = D &= (5)^2 - 4(1)(-14) \\ D &= 25 + 56 \\ D &= 81 \end{aligned}$$

Since $D > 0$ and is a perfect square ($9^2 = 81$), then the roots are *rational and inequal.*

(d): Examine the following:

$$\begin{aligned} D = b^2 - 4ac &= (-2)^2 - 4(1)(-4) \\ D &= 4 + 16 \\ D &= 20 \end{aligned}$$

Since $D > 0$ and is irrational (20 is not a perfect square), then the roots are *irrational and inequal.*

In the book, they refer to them as "Rational and not equal" and "Irrational and not equal." BE WEARY!

– PRACTICE PROBLEMS (F) –

Examine all of the following quadratics and find their discriminant. Denote "RE" if rational and equal, "RN" if rational and inequal, "IN" if irrational and inequal, and "IM" if imaginary.

1. $x^2 - 1 = 0$
2. $x^2 + 4x + 4 = 0$
3. $(x+3)(x-2) = -5$
4. $x^2 = -4x + 9$
5. $(x+1)(x-7) = 0$
6. $4x = -x^2 + 32$
7. $x^2 - 2x = 0$
8. $x^2 + 7x - 7 = 0$
9. $x^2 + 1 = 0$
10. $x^2 + 2x + 4 = 0$
11. $3x^2 + x + 9 = 0$
12. $2x^2 + x - 4 = 0$
13. $5x^2 + 5x + 5 = 0$
14. $4x^2 + 7x + 2 = -4x$
15. $7x^2 + x + 1 = -2$

4.2 Sum and Product of Roots

We will not go over how to derive them as it seems unnecessary for the reader. I highly encourage the reader to read the textbook at page 26 to further emphasize their knowledge but it is indeed optional and will not hurt your skill, but it is knowing and not useless to understand how this set is derived. Only six problems will be given to the reader.

The goal of the sum and product of roots is quite simple, yet have a unique approach.

Definition 4.2. The sum of the roots of a quadratic r_1 and r_2 as $r_1 + r_2$ is the same as the negative ratio of the b coefficient and a coefficient expressed as $r_1 + r_2 = -\dfrac{b}{a}$

This means that we can obtain the same result of adding the two roots if we have the negative ratio of the coefficients.

Example. Find the sum of the roots of $x^2 + 6x + 9 = 0$ and $3x^2 - x - 4$.

$$\begin{aligned} x^2 + 6x + 9 = 0 &\to -\frac{6}{1} \\ &= -3 \end{aligned}$$

CHECK: The roots of the quadratic is $x = 3$. If it is one root, assume that there are two of them.

$$3 + 3 = 6$$

Trivial; this is true.

$$\begin{aligned} 3x^2 - x - 4 = 0 &\to -\frac{-1}{3} \\ &= \frac{1}{3} \end{aligned}$$

CHECK: The roots of the quadratic is $x = \frac{4}{3}, -1$.

$$\frac{4}{3} + (-1) = \frac{4}{3} - \frac{3}{3} = \frac{1}{3}$$

Trivial; this is true.

Definition 4.3. The product of the roots r_1 and r_2 as $r_1 r_2$ is the same as the ratio of the c coefficient and a coefficient expressed as $r_1 r_2 = \dfrac{c}{a}$.

Same idea as the last part.

Example. Find the product of the roots of the same quadratic equations expressed in the last example.

$$\begin{aligned} x^2+6x+9=0 &\to \frac{9}{1} \\ &= 9 \end{aligned}$$

CHECK:

$$(3)(3) = 9$$

True.

$$\begin{aligned} 3x^2-x-4=0 &\to \frac{-4}{3} \\ &= -\frac{4}{3} \end{aligned}$$

CHECK:

$$\left(\frac{4}{3}\right)(-1) = \frac{-4}{3} = -\frac{4}{3}$$

Trivial.

– PRACTICE PROBLEMS (G) –

For 1-3, find the sum of the roots. For 4-6, find the product of the roots.

1. $x^2 - 2x - 24 = 0$
2. $x^2 - 3x = 10$
3. $8x^2 + 4x = 0$
4. $x^2 - 9 = 0$
5. $x^2 + 4x + 4 = 0$
6. $4x^2 - 16x + 16 = 0$

4.3 Creating a Quadratic Equation

Creating a quadratic equation at random is easy. But having a quadratic equation with specific roots is tedious, unless methodically approached. NOTE: We will only accept the use of *rational* numbers. The use of irrational and imaginary numbers will not be approached, yet are encouraged to be researched but optional.

Definition 4.4. Roots r_1 and r_2 in $(x-r_1)(x-r_2) = x^2-(r_1+r_2)x+(r_1r_2)$ or a quadratic equation.

If it seems complicated that is because it is a definition.

Example. Create a quadratic equation with these roots:

(a) $3, -6$

(b) $4, -2$

(c) $-\frac{1}{3}, \frac{2}{3}$

(a): Examine the following.

$$\begin{gathered} (x-(3))(x-(-6)) = (x-3)(x+6) \\ x^2+6x-3x-18 = x^2+3x-18 \end{gathered}$$

(b): Examine the following.

$$(x-(4))(x-(-2)=(x-4)(x+2)$$
$$x^2+2x-4x+8=x^2-2x+8$$

(c): Examine the following.

$$\left(x-\left(-\frac{1}{3}\right)\right)\left(x-\left(\frac{2}{3}\right)\right)$$

Since these are fractions; we move the denominator and multiply it to the x term of each. So, moving it becomes:

$$(3x-(-1))(3x-2)=(3x+1)(3x-2)$$
$$9x^2-6x+3x-3=9x^2-3x-3$$

– PRACTICE PROBLEMS (H) –

Only six items. Form the quadratic equation with the following roots:

1. $3, 7$
2. $-2, -8$
3. $9, -1$
4. $-9, 2$
5. $\frac{3}{2}, 5$
6. $\frac{1}{2}, -\frac{2}{5}$

5 Quadratic Inequalities

The term *inequality* means that the equation satisfies ONLY when a certain condition is met. In terms of this, we shall define:

Definition 5.1. Consider:

- $>$ and $<$ mean "greater than" and "less than" respectively.
- $\geq$ and $\leq$ mean "greater than or equal" and "less than or equal" respectively.

We all know how these symbols work by now, and it is a prerequisite. It is true that $3 > 2$ and $2 < 3$. We can also show that $4 \geq 2$ and $2 \leq 2$. On a number line,

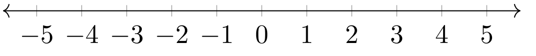

Figure 1: The Number Line From -5 to 5

We can plot what values can apply to certain equations. For example, $x > 3$ is TRUE for all values bigger than 3. And, $x \leq -1$ is TRUE for all values that is -1 or less than -1. On the number line, we draw an arrow to whatever direction it is stating. If x is greater than, then it is going tothe right. If x is less than, then it goes to the left.

We have to define that points on the line must be unanimous to what we are referring; if it is any valuethat is EXCLUDED (using $<$ or $>$), then we write a OPEN circle (blank). However, if it is any value that is INCLUDED (using $\leq$ or $\geq$), then we write an CLOSED circle (filled in.) Showcased:

Example. Plot $x > 3$ and $x \leq -1$.
Since we know that all values BIGGER than 3 satisfy $x > 3$, and all values SMALLER

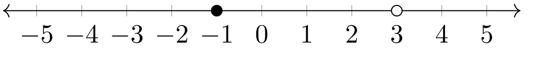

Figure 2: Plots of the Points Mentioned

OR EQUAL than -1 satisfy $x \leq -1$, we need to draw an arrow to wherever the equation is true.

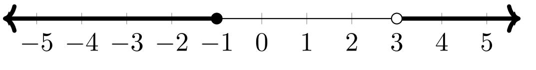

Figure 3: Plots of All Values Satisfactory to Each Equation

All values within the arrows satisfy both equations. These arrows on the line are called *intervals*. They withold all values that satisfy them. The next chapter will introduce the reader to intervals and set notation. Note there are no prerequisites. Just try understanding.

5.1 Intervals (Brief Re-introduction)

The idea of *intervals* is the plot on the number line that houses all potential values that satisfy an equation. *Solving an inequality* has the same process of solving a regular equation, however, the way we find the answers that satisfy the values are different. Let us solve a simple linear inequality first:

Example. Solve for x: $x - 3 \leq 2$

$$\begin{aligned} x - 3 &\leq 2 \\ x &\leq 5 \end{aligned}$$

Since we know that all values $x \leq 5$ satisfy the equation, the interval must be all values less or equal to five. Since it includes five (or uses any sign that has "or equal to"), we use a closed circle on the number line.

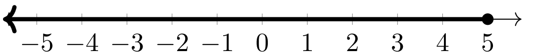

Figure 4: Plot of $x - 3 \leq 2$

This line houses all potential values that satisfy the inequality, including 5.

Example. Solve for x: $-3x + 3 < 0$

$$\begin{aligned} -3x + 3 &< 0 \\ -3x &< -3 \\ \frac{-3x}{-3} &< \frac{-3}{-3} \\ x &> 1 \end{aligned}$$

Since we are dividing by a negative number, we must flip the inequality sign. Remember this, this is a common mistake to not flip the sign when multiplying or dividing by a negative number. Since all values less than 1 satisfy the equation, but excluded 1, we use an open circle on the number line.

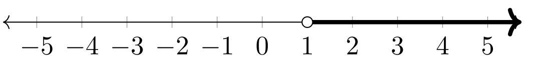

Figure 5: Plot of $-3x + 3 < 0$

The notion of using special notation for intervals is key. There are two types, but we will focus on *set builder notation.* The set builder notation of both examples are $\{x|x \leq 5\}$ and $\{x|x > 1\}$ respectively. It is read as "x such that x is..." and the statement.

These are examples of *infinite intervals*. They have values that satisfy the inequality from a value to negative or positive infinity. In quadratic inequalities, there exists intervals between numbers. Look at the following:

Example. Plot the inequality $-2 \leq x < 1$.

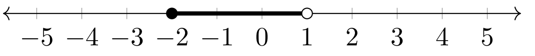

Figure 6: Plot of $-2 < x < 1$

This means that all values including -2 satisfy the inequality. It is read as "There is a set of numbers smaller than 1 but bigger than -2." In set builder notation, it is as $\{x| -2 \leq x < 1\}$ or $\{x|x \geq 2$ and $x < 1\}$. It is up to the instructor on what can be picked but pick yours to use.

5.2 Quadratic Inequalities

The gist of quadratic inequalities stems from finding test values that work on the inquality. To explain, let me elaborate on an example.

Example. Solve and find values of x: $x^2 - 2x \geq 3$.

$$\begin{aligned} x^2 - 2x &\geq 3 \\ x^2 - 2x - 3 &\geq 0 \\ (x+1)(x-3) &\geq 0 \\ x &\geq -1, 3 \end{aligned}$$

Are we done? *No.* To ensure that there are values that satisfy this equation, find test values that are bigger or smaller than the said numbers. The plot:

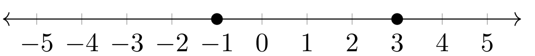

Figure 7: Plot of Roots of $x^2 - 2x \geq 3$

Test values between those intervals. Test out values less than -1 (more preferred the closest integer to -1), values between -1 and 3 (more preferred zero if zero is within the interval) and bigger than 3 (more preferred the closest integer to 3). Label each interval $A, B,$ and C respectively to prevent confusion. Table is not needed, just pick values.

Interval	Test Value	Inequality	Result
A	-2	$(-2)^2 - 2(-2) \geq 3$	TRUE
B	0	$(0)^2 - 2(0) \geq 3$	FALSE
C	4	$(4)^2 - 2(4) \geq 3$	TRUE

In whatever places it is true, graph the interval by drawing arrows.

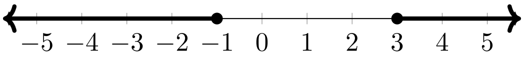

Figure 8: Solutions of $x^2 - 2x \geq 3$

Whatever is under the arrow satisfies the quadratic. The solution set in set builder is $\{x|x \leq -1 \text{ or } x \geq 3\}$. Label the intervals the same way as before to prevent confusion.

Let us try one with a finite interval between values.

Example. Solve and find all true values: $-x^2 + 4 > 0$.

Since our quadratic term is negative, multiply all by negative one. This must in turn flip the inequality sign (since it only happens when we are multiplying/dividing by a negative number) to a less than sign.

$$\begin{aligned} -x^2 + 4 &> 0 \\ -1(-x^2 + 4) &> 0 \cdot -1 \\ x^2 - 4 &< 0 \\ (x+2)(x-2) &< 0 \\ x &< -2, 2 \end{aligned}$$

Test out values smaller than -2, between -2 and 2, and bigger than 2.

Interval	Test Value	Inequality	Result
A	-3	$(-3)^2 - 4 < 0$	FALSE
B	0	$(0)^2 - 4 < 0$	TRUE
C	3	$(3)^2 - 4 < 0$	FALSE

Graph this on the number line, we find that only interval B, or values between -2 and 2 satisfy the inequality. The solution set is $\{x|x > -3 \text{ and } x < 3\}$ or $\{x| -3 < x < 3\}$.

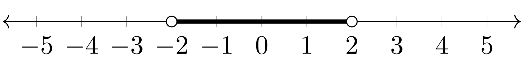

Figure 9: Solutions of $-x^2 + 4 > 0$

– PRACTICE PROBLEMS (I) –

Find all values that satisfy each quadratic inequality. Write it as set builder form, and graph it. Remember to use what appropriate circle to use (closed or open circle.) Always make one side equal to zero.

1. $x^2 - 4x + 3 \leq 0$
2. $-x^2 + 3x < -10$
3. (H) $7x - 10 > x^2$
4. $-x^2 \geq x - 6$
5. $2x^2 - 5 < 3$
6. $(x+2)(x-4) \geq 0$
7. $x^2 - 16 \leq -7$
8. $x^2 - 16 > 0$
9. $3x^2 - x < 0$
10. $x^2 > -x$
11. (H) $x^2 - 4x + 4 \leq 0$
12. $-6x < -x^2 - 9$

HINTS TO SELECTED QUESTIONS

3. Transpose one side that does not make the quadratic term negative.
10. (Graphing) Has an infinite interval.

6 The Quadratic Function

The idea of the *quadratic function* is that they are represented as $y = ax^2 + bx + c$. Typically, graphing a function involves finding points and connecting them. However, quadratics have more than that. *But, we will only tackle what is specified in the pages that were mentioned.* We will also not involve any graphing, according to the pointers.

6.1 The General Form of the Quadratic Function

The quadratic function, expressed as $y = ax^2 + bx + c$ or in its other forms act like graphing any other equation. This subsection is dedicated to making a list or table of values of a function.

Example. Find the table of values for $y = x^2$, $y = x^2 - 1$, and $y = x^2 - x - 2$.

(a) $y = x^2$

x	y
-2	4
-1	1
0	0
1	1
2	4

(b) $y = x^2 - 1$

x	y
-2	3
-1	0
0	-1
1	0
2	3

(c) $y = x^2 - x - 2$

x	y
-2	4
-1	0
0	-2
1	-2
2	0

These values can be anything; I chose unanimously -2 to 2 on these tables, but they can change depending on the context.

The reader will not be doing any exercises here. It is simply that easy.

Appendices

A Simplifying Radicals

Simplifying radicals involve taking a radical and simplifying it to a product of an irreducible radical and a whole number.

Example. Simplify $\sqrt{40}$, $\sqrt{160}$, $\sqrt{-144}$, $\sqrt{-125}$.

Firstly, find the LARGEST square number that is a factor of the radical you want to simplify. Then, express the number under the radical as a product with the square number. Afterwards, split the radical, then simplify the square number. This only works if the number under the radical can be expressed as a product of a square number and another number.

1. $\sqrt{40}$

$$\begin{aligned} \sqrt{40} &= \sqrt{4 \cdot 10} \\ \sqrt{4} \cdot \sqrt{10} &= \boxed{2\sqrt{10}} \end{aligned}$$

2. $\sqrt{160}$

$$\begin{aligned} \sqrt{160} &= \sqrt{16 \cdot 10} \\ \sqrt{16} \cdot \sqrt{10} &= \boxed{4\sqrt{10}} \end{aligned}$$

3. $\sqrt{-144}$. This is a special case. Since the base number is already a perfect square but is negative, extract -1.

$$\begin{aligned} \sqrt{-144} &= \sqrt{-1 \cdot 144} \\ \sqrt{-1} \cdot \sqrt{144} &= \boxed{12i} \end{aligned}$$

4. $\sqrt{-125}$. Exctract the perfect square, as a negative number to ensure that the simplified form does not have a negative under the radical.

$$\begin{aligned} \sqrt{-125} &= \sqrt{-25 \cdot 5} \\ \sqrt{-25} \cdot \sqrt{5} &= \boxed{5i\sqrt{5}} \end{aligned}$$

B The Imaginary Unit

This section is not necessary but can be used. The imaginary unit defined as $\sqrt{-1}$ is the solution to $x^2 + 1 = 0$, meaning if you square it i^2 you get -1. No real number gives you a negative number if you square it. Here is a pattern that you should know, without the proof and rigor:

1. $i^0 = 1$
2. $i^1 = i$
3. $i^2 = -1$
4. $i^3 = -i$
5. $i^4 = 1$

$$\vdots$$

Can you tell me what is i^{16}?

C LCM and Transposition

The term *LCM* means "Least Common Multiple." It is defined as a number which is the smallest multiple that exists on both.

The LCM of 12 and 15 is 60. The LCM of 2 and 5 is 10. The LCM of 3 and 6 is 6. The LCM of 10 and 20 is 20, and so on. The issue is when it comes to variables.

A neat trick is to consider the biggest exponent on one variable. That is your LCM. The LCM of $3x^2$ and $4x^7$ is $12x^7$. This is true, always. Split the coefficients and find the coefficients' LCM separately, and find the biggest exponent on each term.

Tranposition is the term of shortening the process of adding/subtracting both sides of an equation. What it does is that you switch a term to the other side and change its sign.

Example. Show transposing $x + 2 = -9$ and $x - 3 = 15$, and show how to do it normally.

$$\begin{aligned} x + 2 = -9 &= x = -2 - 9 \\ &= x = -11 \\ x - 3 = 15 &= x = 3 + 15 \\ &= x = 18 \end{aligned}$$

Normally, you add/subtract both sides if the operation is subtraction and addition respectively.

$$\begin{aligned} x + 2 = -9 &= x + 2 - 2 = -9 - 2 \\ &= x = -11 \\ x - 3 = 15 &= x - 3 + 3 = 15 + 3 \\ &= x = 18 \end{aligned}$$

Truthfully, transposing can get confusing sometimes, but really a stupid and foolish analogy I describe it is: *If you are a transgender person, "MTF" is male to female and "FTM" is female to male. Likewise, "PTN" means "Positive to Negative" and "NTF"* *means "Negative to Positive."*[8] It is shorter but personally, it's more better to use normal methods.

D Three Forms of Basic Factoring

(a) Common Monomial Factoring: $ax + ay = a(x + y)$

(b) Difference of Two Squares: $x^2 - y^2 = (x + y)(x - y)$

(c) Perfect-Square Trinomial:

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

The *FOIL* method is *First, Outer, Inner, Last.* It is a simpler process of multiplying two binomials.

Example. Use FOIL to expand $(2x^2 - 3)(x + 1)$.

$$\text{First: } 2x^2 \cdot x = 2x^3$$
$$\text{Outer: } 2x^2 \cdot 1 = 2x^2$$
$$\text{Inner: } -3 \cdot x = -3x$$
$$\text{Last: } -3 \cdot 1 = -3$$

Add them up:

$$2x^3 + 2x^2 - 3x - 3$$

Always go in this order, and combine like terms if there are any. Check the graphic below:

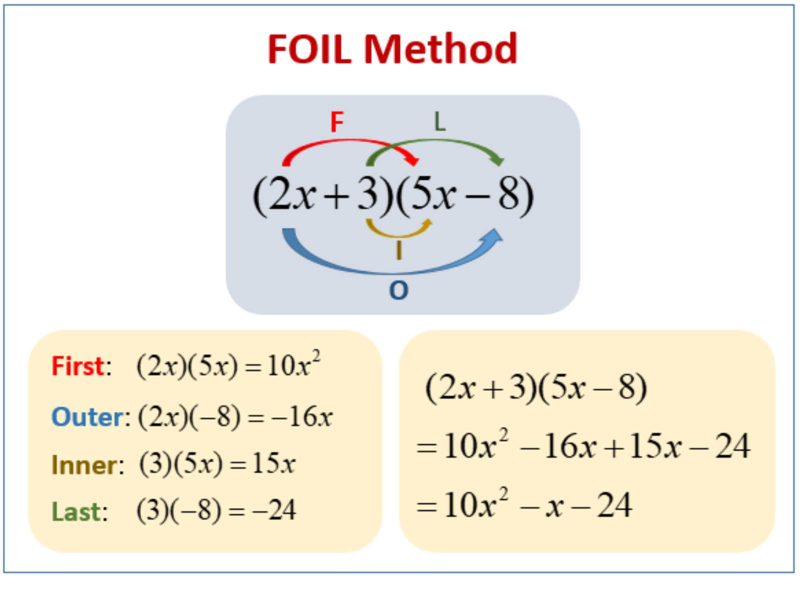

Figure 10: FOIL Method Visualized

[8] I'm not offending anyone here. It's just an analogy.