<u>Guide</u> for 4th Quarterly Examinations of *Mathematics* for 2^{nd} Year

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Hello dear reader. This is a guide for the upcoming 4^{th} Quarterly Periodical Examinations held during the final week of this school year in International Learning Academy. This 2^{nd} year has been absolutely wonderful, and some teachers or classmates or even friends may leave to depart on a new era of their lives. This document serves as a basic guide for the basis of the examination's content. This quarter tackled the basis of true plane geometry. This document serves as a review and guide on that, and a way to practice my LaTeX skills because they are lacking of some degree. With this prelude supposedly finished, let us introduce the concepts of geometry as a final send-off to this year of high school as a 2^{nd} year. Note that, this is NOT a reviewer. This is a guide on understanding more about the topics being discussed, and not a summary either as there may be more additional content to explain more thuroughly or to make a more bigger picture of the topic. If you want to practice these problems, use the book, or use online resources for practicing the topic thuroughly.

Created in LaTeX editor Overleaf.

Used packages: massymb, graphicx, tikz, microtype, csquotes, amsthm, lipsum, amsmath, tcolorbox (most), caption, hyperref subcaption, fontspec, and geometry.

<u>Took a minute to make a moment that you will cherish</u>. − app4r4tu5.

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1 The Basis of Geometry

Geometry is grounds for logic and patterns, defined for proofs for theorems. Proof in mathematics is a much broader sense than the typical high-school student can describe. For sakes of geometry, we should define *proof by induction*. This will be in more detail later on Section 2. We should start building on geometry by introducing the first steps for mathematical logic*.

*- *Mathematical logic in itself is a whole different topic, and is much tougher than the scope taught of a high school math class, typically meant for college students. For this reason, we will only introduce the core concepts of logic applied in geometry.*

1.1 Conditional Statements (If-Then)

A *conditional statement* is a statement that expresses conditions. These statements may be written structurally as "if ..., then ...". The *hypothesis* is the part of the statement that is after "if", and talks about the condition. The *conclusion* is the part of the statement that is after "then", and talks about the result or aftermath. The following are examples, and highlighted in italics is the conclusion:

1. If it is dark outside, *then it is night time.*
2. If there is no light, *then it is dark.*
3. If you have no paper, *then you can't write.*
4. If there was no screen, *then I couldn't see the content.*
5. If I have a good math textbook, *then I can learn math easily.*

If a statement looks like a conditional statement, you can describe them as conditional statements.

- The machine has to be on for the system to work.
    1. *Determine the subject and predicate.* The subject is "The machine" and the predicate is "has to be on for the system to work."
    2. *Create the conditional statement* The new sentence would be: "If the machine is on, then the system works."

The hypothesis is "If the machine is on" and the conclusion is "then the system works." Conditional statements have notation, but it is out of our bounds to discuss this.

Conditional statements may be true or false. They are always true if:

- The hypothesis is *true* and the conclusion is *true.*
- The hypothesis is *false* and the conclusion is *true.*
- The hypothesis is *false* and the conclusion is *false.*

A conditional statement is false if the hypothesis is *true* and the conclusion is *false.* Using this with normal language rather than sybolism:

1. *If a computer is fast, then it can run games.* The hypothesis is true, as a computer can be fast, and the conclusion is true, as it can run games in general. Therefore the statement is *true.*
2. *If a computer runs games, then it is fast.* The hypothesis is true, as a computer can run games (given those games can run on any computer), and the conclusion is false, as not all computers that can run games are fast. Therefore the statement is *false.*
3. *If division by zero is true, then it has an answer.* The hypothesis is false, as division by zero is impossible, and the conclusion is true, as every arithmetic problem has an answer in this statement. Therefore the statement is true.

A *counterexample* is a figure, explanation, or situation used to justify that a given conditional is false. An example:

- If you exercise regularly, then you will improve your physique.
  COUNTEREXAMPLE - If you have a health condition, exercising regularly may not be effective.

A *negation* is the opposite of a hypothesis or conclusion. Their notation varies by textbook. However, we will use $\sim$ or the tilde as the notation.

Definition of Negation

Let p be a statement. If $\sim p$, and p is true, then $\sim p$ is false. If $\sim p$, and p is false, then $\sim p$ is true.

There are various forms of conditional statements. Let p and q be the hypothesis and conclusion respectively. If we define the following:

Forms of Conditional Statements

1. If p then q, it is a *conditional statement* in general.
2. If q then p, it is a *converse* of a conditional statement.
3. If $\sim p$ then $\sim q$, it is an *inverse* of a conditional statement.
4. If $\sim q$ then $\sim p$, it is an *contrapositive* of a conditional statement.

A *biconditional* statement is when the conditional statement and its converse are combined. They are in the forms of ”... if and only if...” to denote that something MUST happen for another action to appear. Examples may be listed from using our previous examples:

1. I will sleep *if and only if* I am tired.
2. The system works *if and only if* the machine is on.
3. I can learn math easily *if and only if* i have a good math textbook.

Biconditional statements may be true when the converse and conditional is true.

1.2 Reasoning and Proof

In mathematics, logical expressions and patterns are used to make reasonable conjectures or conclusions. Let us try use reasoning for the following patterns:

1. *The Fibonacci Sequence*: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...
   ANSWER: This sequence has the next term be the sum of the previous term and the term after that. What would be the next number after 55? If we were to add 55 and 34, we would get 89.
2. *Arithmetic Sequence*: 4, 14, 24, 34, 44, 54, ...
   ANSWER: This sequence appears to have a pattern, that the next term is 10 more than the term before. What would be the next term after 54? Adding 10 to 54 would be 64.

The examples presented are logically shown with the use of reasoning. The first kind is *inductive reasoning*, which is when conclusion is obtained by observing a pattern or doing several observation. However, if there is no pattern, reasoning may be different. *Deductive reasoning* is when conclusions are based on facts, such as definitions of terms and properties, or on other related elements that seem to be applicable. The ways to write deductive reasoning:

1. Assume the hypothesis to be true. Identify the conclusion.
2. Enumerate the definition of terms or properties that can be used. Write an argument from the hypothesis to the desired conclusion. Statements in the argument should be in logical order.
3. Write the interpretation. Use ”given” and ”prove.”

Properties used to reason can include the Properties of Equality:

For all real number, a, b, c, and d, the following are true:

1. Addition Property of Equality (APE): If $a = b$, then $a + c = b + c$.
2. Subtraction Property of Equality (SPE): If $a = b$, then $a - c = b - c$.

3. Multiplication Property of Equality (MPE): If $a = b$, then $ac = bc$.
4. Division Property of Equality (DPE): If $a = b$, then $\frac{a}{c} = \frac{b}{c}$.
5. Distributive Property: $a(b + c) = ab + ac$
6. Substitution Property: If $a + b = c$ and $b = d$, then $a + d = c$.
7. Reflexive Property: $a = a$.
8. Symmetric Property: If $a = b$, then $b = a$.
9. Transitive Property: If $a = b$ and $b = c$, then $a = c$.

The following properties were introduced in the last quarter Grade 7, and are used for deductive reasoning.

1. *Property 1* - Angles in a linear pair are supplementary.
2. *Property 2* - Vertical angles are congruent.
3. *Property 3* - The sum of the measures of the interior angles in a triangle is equal to 180 degrees.
4. *Property 4* - The sum of the measures of the interior angles of a quadrilateral is 360 degrees.
5. *Property 5* - The sum of the measures of the interior angles of a polygon is equal to $180(n - 2)$ wherein n is the number of sides in the polygon.
6. *Property 6* - An interior angle of a polygon and its corresponding exterior angles are supplementary.
7. *Property 7* - The sum of the measures of the exterior angles at one at each vertex is equal to 360 degrees.
8. *Property 8* -
   - All radii of the same circle are congruent.
   - All diameters of the same circle are congruent.
   - The length of the diameter of a circle is twice the length of its radius.

Proving statements in geometry are most commonly represented as *two-column proofs*. One the left side is the statement, and the right is for reasoning. Here is an example:

If $2x - 3 = 15$, then $x = 9$. *Represented as a two-column proof*:

STATEMENT	**REASONING**
$2x - 3 = 15$	Given.
$2x - 3 + 3 = 15 + 3$	Subtraction Property of Equality.
$2x = 18$	Addition
$\frac{2x}{2} = \frac{18}{2}$	Division Property of Equality
$x = 9$	Division

A *conjecture* is a conclusion obtained from inductive reasoning that is not necessarily true. Conclusion from deductive reasoning is true when the hypothesis is true.

Proceed to the next page for continuation.

2 Geometry

In geometry, there exists *undefined terms.* They are defined, however, they are repeated definitions from other terms that may or may not be defined. We will not delve into these kinds of terms, rather we will just stick to an observed definition of the following structure of geometry.

2.1 Axiomatic Structure

The following are the basic structures of geometry, and it involves the three basic components: The point, the line, and the dot.

The *point* is a dot. It is an exact location. They are named with capital letters. See the figure below:

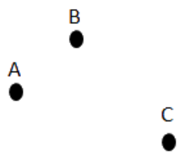

Figure 1: Points A, B, and C

A *line* is represented by a straight edge. It extends indefinitely, expressed with arrows on either sides. Therefore, a line has no length. Lines are named using any two points on the line, or a lowercase letter. In Figure 2, the line is named either Line l, or symbolically: $\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$, $\overleftrightarrow{AC}$ or $\overleftrightarrow{CA}$, and $\overleftrightarrow{BC}$ or $\overleftrightarrow{CB}$.

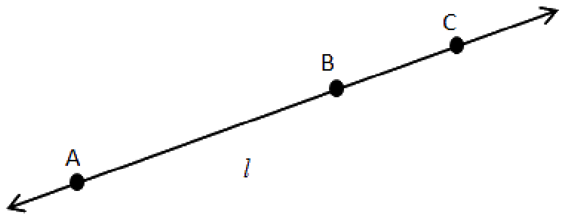

Figure 2: Line l

A *plane* is represented by a flat surface. It has no dimensions. A plane is named using three of its points that do not belong to the same line or a capital letter that does not represent any point. In Figure 3, The plane is named with either: plane P, ABD, ACD, or BCD. ABC is not represented because all the points are in the same line.
The following definitions are also important for determining characteristics for planes and lines:

Concepts in Geometry

A set of *collinear points* are points that lie in the same line.
A set of *coplanar points* are points that lie in the same plane.
A set of *coplanar lines* are lines that lie in the same plane.
An *intersection* is the set of all points that is common to both figures.
A *space* is the set of all possible points.

The logical study of geometry, statements are either expressed as definitions, postulates, or theorems. Definitions are already given previously.

Statements in Geometry

- A *postulate* is a statement that is accepted without proof.
- A *theorem* is a statement that needs proof before it is accepted as true.

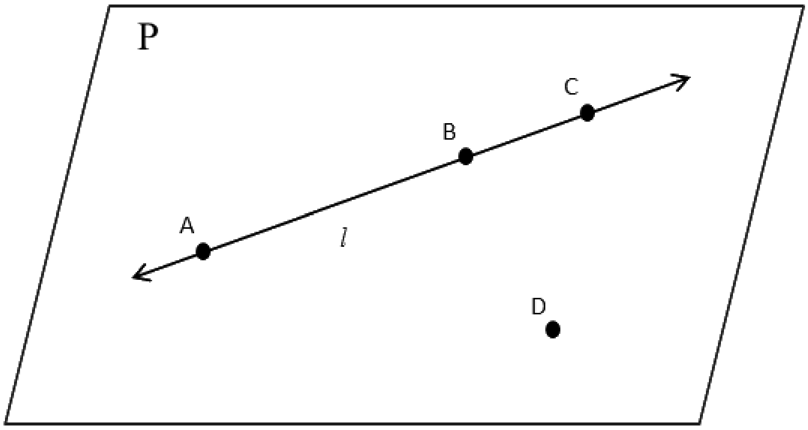

Figure 3: Plane P

The following are the six postulates related to lines, points, planes, and spaces, found in geometry.

1. *POSTULATE 1* - A line is formed by at least two different points. A plane is formed by at least three non-collinear points. A space is formed by at least four non-coplanar points.
2. *POSTULATE 2* - Two different points can create exactly ONE line. Any three non-collinear points can create exactly ONE plane.
3. *POSTULATE 3* - If two points lie in a plane, then if a line is formed by those two points, the line is also in the same plane.
4. *POSTULATE 4* - If two different planes intersect, then the intersection is a line.
5. *POSTULATE 5* - Every point on the real number line has a one-to-one correspondence with a real number.
6. *POSTULATE 6* - There is a unique distance between any two points on the number line, either obtained by counting or getting the absolute value of the difference between the coordinates of the two points.

Postulates 5 and 6 can be described with Figure 4, the number line:

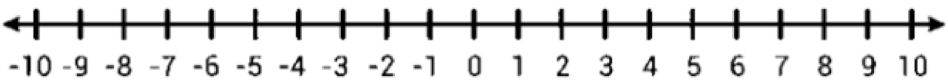

Figure 4: Number line

If we set coordinates $A(3)$ and $B(-2)$, they have points that go directly on the number line; point B is at -2, and point A is at 3 for reference. Their distance is the absolute value of their difference, $|-2-3| = |-5| = 5$.

There are various definitions that relate to the points on a number line, and can be described with the following:

Definitions

Definition 2.1 (Betweenness for Points)**.** If three points, A, B, and C are collinear, and parts of lines $\overline{AB} + \overline{BC} = \overline{AC}$, then $\underline{B \text{ is between } A \text{ and } C}$.

Definition 2.2 (Congruent Segments)**.** If two segments have equal lengths, they are congruent.

Definition 2.3 (Midpoint of a Segment)**.** If a point divides a segment into two congruent segments, then the point is called the midpoint of the segment.

Definition 2.4 (Bisector of a Segment)**.** If a line, segment, ray, or plane intersects a segment at its midpoint, then it is called a bisector of the segment.

In figure 4, if we assume the line is from -10 until 10, then their midpoint is 0. This midpoint divides the line in to two similar segments (assuming the line is a segment) and therefore, those two similar segments is congruent. If we scroll back up to Figure 2, between points A and C, there is a point B. $\overline{AB} + \overline{BC} = \overline{AC}$, and $\overline{AC}$ is the entire line. Therefore, in Figure 2, B is the midpoint. Looking at Figure 5:

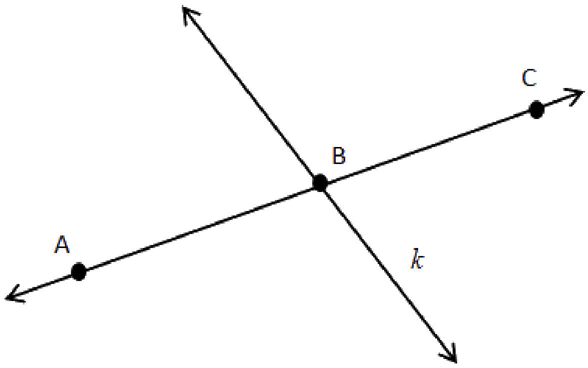

Figure 5: Bisector Example

Line k is a bisector of line $\overline{AC}$ at point B, as an example of **Definition 2.4**. Now after this, we will be defining four theorems, and we will first describe the first two. Figure 5 shows an intersection as a line passes through another line as a bisector, thus our first theorem:

Theorem 2.1: Intersection of Two Lines

If two different lines intersect, then their intersection is at ONLY one point.

Proof. Suppose two different lines exist, l and m, intersects at two points, A and B. Then, A and B lie at l, hence line $\overleftrightarrow{AB}$ is also in line l. A and B also lie in t, thus line $\overleftrightarrow{AB}$ also lies in m. Therefore, lines l and m are the same line. *This contradicts given that lines l and m are different lines.* Therefore, this proves **Theorem 2.1**.. $\square$

Our second theorem is nonetheless related to **Theorem 2.1**, and *Postulate 2*.

Theorem 2.2: Intersections in Planes

If two different lines intersect, then they are contained in ONLY one plane.

Proof. Assuming the same lines from the proof of **Theorem 2.1** exist, and three points exist, point A exists on line l, point B exists on line m, and point C exists as the point of intersection of both lines l and m (**Theorem 2.1**). Since A, B, and C are non-collinear points, meaning they do not lie on the same line, they form a plane (*Postulate 2*). Therefore, this proves **Theorem 2.2**. $\square$

The former can be illustrated in Figure 6.

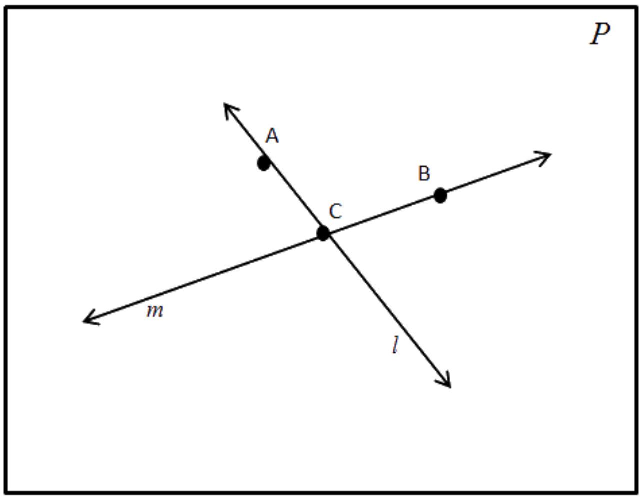

Figure 6: Intersection of two lines form a plane

As you can tell, Plane P is formed from the intersection of these two lines, therefore concluding that **Theorem 2.2** is true once more.

We shall now discuss our prelude to the following theorems, regarding midpoints and bisectors. Since we know for a fact that *a midpoint divides a line segment in to two equal/congruent parts*, and how *a bisector intersects a segment at its midpoint*, we can use these definitions for our next two theorems, and also define what are rays, and line segments, since they were never properly introduced, and including angles. We will call these definitions firsthand before we proceed.

2.1.1 Rays, Line Segments and Angles

A *ray* is a part of a line that only extends in one direction indefinitely. Look at Figure 7a. A *line segment* is a part of a line only, and it is just a portion of the line or the ray. Look at Figure 7b.

(a) A Ray (b) A Line Segment

Figure 7: Ray and Line Segment

As you can tell, on Figure 7a, there exists a ray $\overrightarrow{AB}$. On Figure 7b, there exists a line segment $\overline{AB}$.

An *angle* is a figure that is created with two rays having a common startpoint, called the *vertex.* Note that the startpoint of a ray is the point which does not extend indefinitely, like on Figure 7a, point A is the startpoint. All textbooks do not describe these as startpoints.

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For our case, we will not define the logistics on an angle (degrees, kinds of angles etc.) as we will presume the reader knows all five. The following figure will showcase an angle.

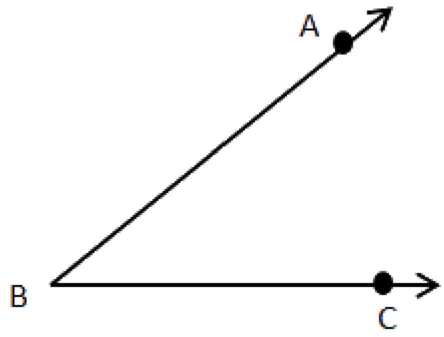

Figure 8: Angle $\angle ABC$

Naming an angle involves the vertex being in the middle of the name. $\angle CBA$ or $\angle ABC$ is acceptable, but not $\angle CAB$ as the vertex is not A.

Now, we can construct the theorems and definitions that involves rays and angles. We shall start with a theorem:

Theorem 2.3: Theorem of Midpoints

If M is the midpoint of line segment $\overline{PQ}$, then the following is true:

- $2\overline{MP} = \overline{PQ}$, and therefore $\overline{MP} = \frac{1}{2}\overline{PQ}$.
- $2\overline{MQ} = \overline{PQ}$, and therefore $\overline{MQ} = \frac{1}{2}\overline{PQ}$.

If $m\overline{MQ}$ and $m\overline{MP}$ denote the measures of the lengths in $\overline{MQ}$ and $\overline{MP}$ respectively, then they are congruent, meaning they are equal, therefore $m\overline{MQ} = m\overline{MP}$ no matter what. See in Figure 9:

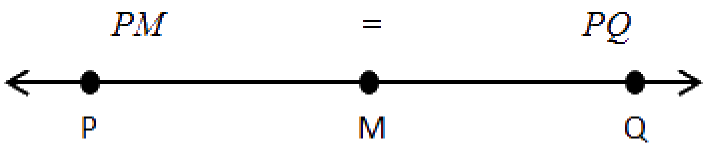

Figure 9: **Theorem 2.3** shown visually

Note that, *betweenness* is applied in rays. This is a direct transition to our next definition:

Definition of Betweenness for Rays

If three rays, $\overrightarrow{OP}$, $\overrightarrow{OM}$, and $\overrightarrow{OQ}$ exist, are coplanar, and $m\angle MOP + m\angle MOQ = m\angle POQ$, then $\overrightarrow{OM}$ is in between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

Look at the figure below. This figure describes that, a ray $\overrightarrow{OM}$ bisects the angle $m\angle POQ$. This also shows

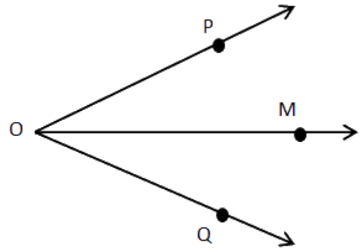

Figure 10: Angle Bisector

that, the same ray $\overrightarrow{OM}$ bisects angle $m\angle POQ$. This can lead to our next theorem.

Theorem 2.4: Theorem on Angle Bisectors

If $\overrightarrow{OM}$ is a bisector of $\angle POQ$, then the following are defined:

- $2m\angle MOP = m\angle POQ$, and therefore $m\angle MOP = \frac{1}{2}m\angle POQ$.
- $2m\angle MOQ = m\angle POQ$, and therefore $m\angle MOQ = \frac{1}{2}m\angle POQ$.

Let us solve an example of this using **Theorem 4.4**.

I. Within Figure 11, $\overrightarrow{BD}$ bisects $\angle ABC$. Find the measure of $\angle ABC$.

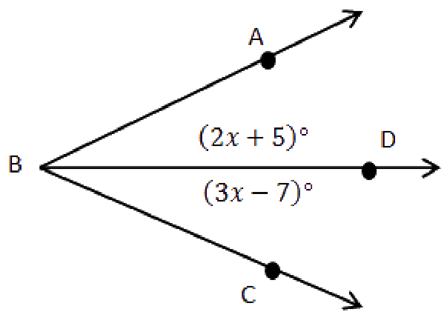

Figure 11: Example I

Let us use the two-column proof for us to find the measurement given.

STATEMENT	REASONING
$m\angle ABC$	Given.
$2x+5=3x-7$	**Theorem 2.4**.
$2x+5-5=3x-7-5$	Addition Property of Equality.
$2x=3x-12$	Subtraction.
$2x-3x=3x-3x-12$	Addition property of Equality.
$-x=-12$	Subtraction.
$\frac{-x}{-1}=\frac{-12}{-1}$	Division Property of Equality (To cancel the negative signs.)
$x=12$	Division.

After we obtain x, we must substitute it back to any equation in order to get the final measures of each angle being bisected. The reason why we only need one equation is that we assume the angle is being bisected. Any two figures being divided into two equal parts from a bisector *are equal*. Therefore, we just need one equation to find the measurement of the other equation, since we assumed both measurements are equal. We will use

$m\angle ABD$ for our sakes.

$$m\angle ABD = 2x + 5 \quad (1)$$
$$m\angle ABD = 2(12) + 5 \quad (2)$$
$$m\angle ABD = 24 + 5 \quad (3)$$
$$m\angle ABD = 29 \quad (4)$$

We can also observe Therefore, $m\angle ABD = 29°$. After that, multiply this number by 2. In order to get our FULL angle measurement, we must multiply this value by 2, according to **Theorem 2.4** once again. Therefore, $2(29) = 58$. Therefore, $m\angle ABC = 58°$ is the measurement of that entire angle.

2.2 Congruence of Triangles

The definition of *congruence* is the following:

Definition of Congruence

If $A = B$, then $A \cong B$.

In two triangles that are equal, *they are only congruent if and only if the corresponding angles and sides are congruent as well.* Consider the figure below:

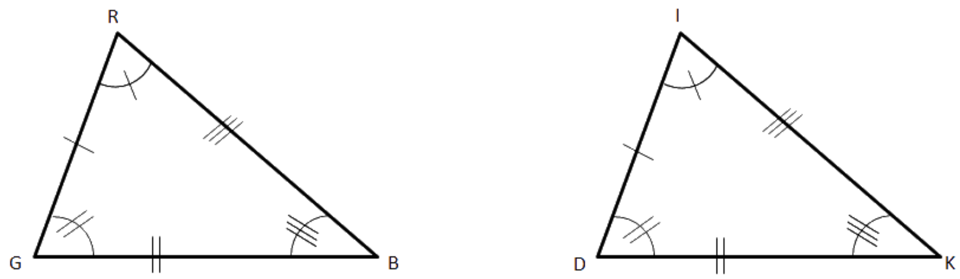

Figure 12: Example of Triangle Congruence

As you can tell from this Figure, we can say the following as to determine if the triangles are congruent:

Congruent Sides and Angles for Congruent Triangles

In $\triangle RGB$ and $\triangle IDK$, the following is considered:

$\overline{RG} \cong \overline{ID}$

$\overline{RB} \cong \overline{IK}$

$\overline{GB} \cong \overline{DK}$

And that,

$\angle R \cong \angle I$

$\angle G \cong \angle D$

$\angle B \cong \angle K$

Therefore, these lines on the angles and the sides determine that they are congruent by their respective sides *(if one side has one marked line, the other side that is congruent to it also has one marked line, so if two marked lines on one side/angle, that other side/angle marked with two lines is congruent with that other line, etc.)* If ever there is a question that asks "what are the potential sides or angles to make these triangles congruent," just remember what kinds of congruent sides/angles there are by the definition keyed above.

The following are three postulates and one theorem, to determine the congruence between two triangles. We will describe each of them after enumeration:

1. *POSTULATE 7* - Also known as *Side-Side-Side* or *SSS*, says that if three sides of a triangle are congruent to the corresponding sides of another triangle, then the two triangles are congruent.
2. *POSTULATE 8* - Also known as *Side-Angle-Side* or *SAS*, says that if two sides and an included angle (the angle that is made from the sides) are congruent to the corresponding parts of another triangle, then the two triangles are congruent.
3. *POSTULATE 9* - Also known as *Angle-Side-Angle*, or *ASA*, says that if two angles and an included side (the side made from the angles) are congruent to the corresponding parts of another triangle, then the two triangles are congruent.

Theorem 2.5: Side-Angle-Angle (SAA) Theorem

If two angles and a not-included side (a side that is adjacent or opposite either angle) between the angles of a triangle are congruent to the corresponding parts of another triangle, then the two triangles are congruent.

We can describe *Postulate 7* as the following figure:

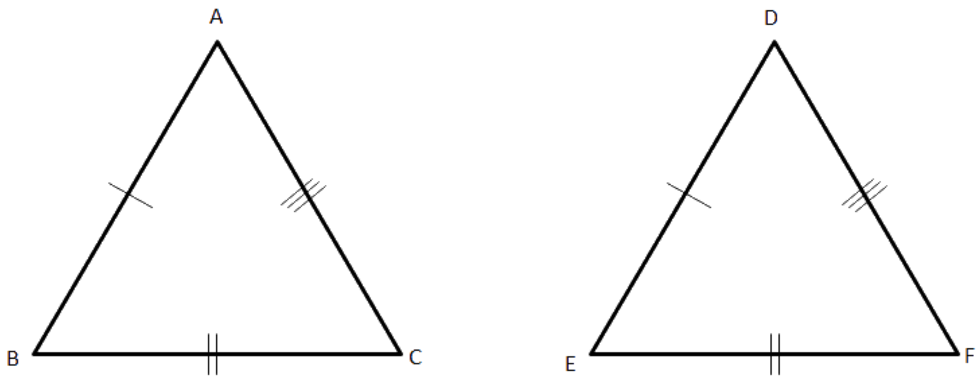

Figure 13: *SSS Postulate* Showcase

As you can tell, since $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\overline{BC} \cong \overline{EF}$, we can conclude that $\triangle ABC \cong \triangle DEF$ if they only listed the three sides. We can also describe *Postulate 8* as the following:

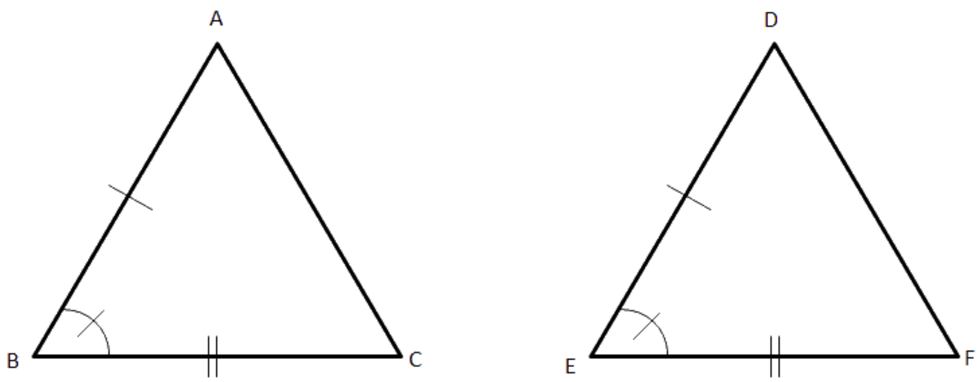

Figure 14: *SAS Postulate* Showcase

As shown in the figure, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and that the included angle $\angle B \cong \angle E$, we can therefore say that $\triangle ABC \cong \triangle DEF$ if they only have two sides and their included angle. To get a better understanding of what an included angle is, again, it is an angle formed by two sides of a triangle that have a common vertex, and for included angle $\angle B$, the common vertex of $\overline{AB}$ and $\overline{BC}$ is point B.

We can also describe *Postulate 9* as the following:

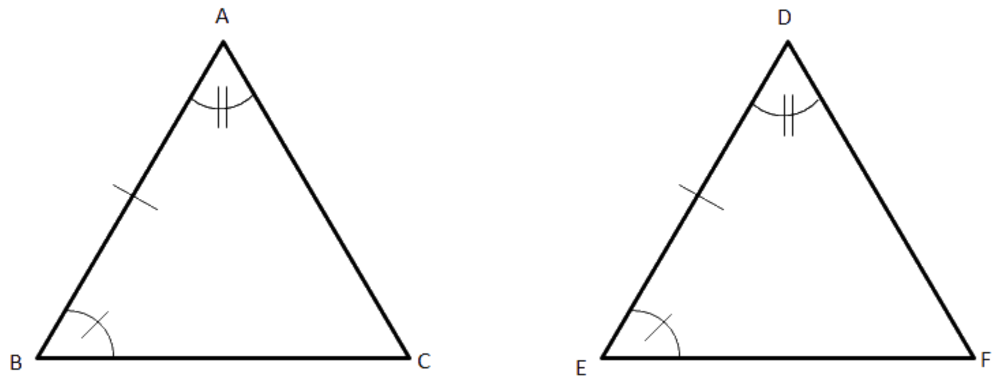

Figure 15: *ASA Postulate* Showcase

As shown in the figure, $\angle A \cong \angle D$, $\angle B \cong \angle E$, and that the included side $\overline{AB} \cong \overline{DE}$, we can therefore say that $\triangle ABC \cong \triangle DEF$. To get a better understanding what an included side is, again, it is a side that is formed by angles of a triangle that have a common ray, and for included side $\overline{AB}$, the common rays of $\angle A$ and $\angle B$ is ray $\overrightarrow{AB}$ (theoretically it is a line segment, but we will assume that it is a ray.) We can also describe **Theorem 2.5** as the following:

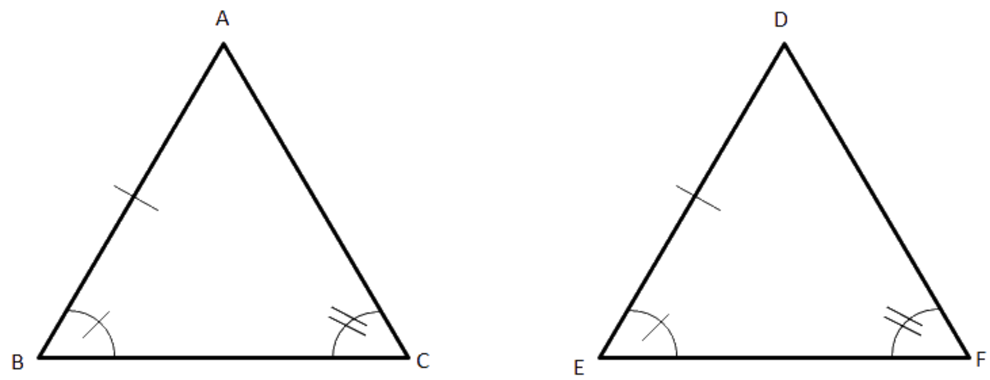

Figure 16: **SAA Theorem** Showcase

As shown in the figure, $\angle B \cong \angle E$ and $\angle C \cong \angle F$, and that the non-included side $\overline{AB} \cong \overline{DE}$. Therefore $\triangle ABC \cong \triangle DEF$. As you can tell, a non-included side is the opposite of the included side (duh.)

Proceed to the next page for continuation.

2.3 Triangle Inequality

Inequality talks about comparison between two numbers, or figures. If there are two figures, the bigger figure is greater, and the other is less, same for numbers. This section discusses on applying geometry to such inequality using triangles.

2.3.1 Defining Inequality & Isosceles Triangles

An *isosceles triangle* is a triangle with two equal sides. The side that are equal, or congruent, are the *legs* of the triangle, and the side not congruent is the *base*. The angles opposing the legs are the *base angles*, and the angle opposing the base is the *vertex angle*. Seen in the following figure:

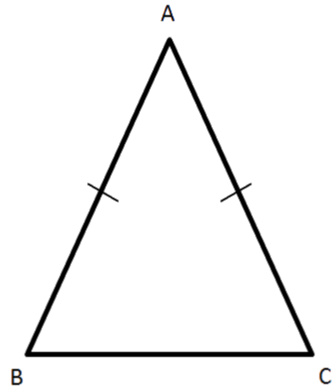

Figure 17: Isosceles Triangle

Seen in Figure 17, there are two congruent sides $\overline{AB} \cong \overline{AC}$. The base, $\overline{BC}$ is not congruent to any side at all. The base angles, $\angle B \cong \angle C$. The vertex angle, $\angle A$ is opposite the base angle seen in the figure. A theorem can be described by the congruence of two sides:

Theorem 2.6: Theorem of Isosceles Triangles

If two sides of a triangle are congruent, then the angles opposite them are congruent.

The converse of this theorem is also true:

Theorem 2.7: Converse of Theorem 2.6

If two angles of a triangle are congruent, then the sides opposite them are congruent.

Proof for **Theorem 2.6** is in the book (p. 242), and proof for **Theorem 2.7** can be found through this link. To show how both theorems work, look at Figure 17. $\angle B$, and $\angle C$. Their sides $\overline{AB} \cong \overline{AC}$. Infer that, if we were to construct a side with the degree of the angle, we would get the same congruent side (**Theorem 2.7**) and measuring the sides' angle will get the same degrees we constructed from (**Theorem 2.6**).

Proceed to the next page for continuation.

All sides and angles of a triangle may not be congruent, therefore inequality has to be performed; seen as the following definitions:

Definitions of Inequalities

Greater and Less Than

Definition 2.5. If $a = b + c$ and a, b, and c are all greater than 0, then $a > b$, and $a > c$.

Definition 2.6. If $a = b + c$ and a, b, are c all less than 0, then $a < b$, and $a < c$.

Definition 2.7 (Addition Property of Inequality)**.** a, b, and c, and d are real numbers. If $a > b$, then $a + c > b + d$.

Definition 2.8 (Subtraction Property of Inequality)**.** a, b, and c, and d are real numbers. If $a > b$, then $a - c > b - d$.

Definition 2.9 (Multiplication Property of Inequality)**.** a, b, and c, and d are real numbers.

- If $a > b$ and $c > 0$, then $ac > bc$.
- If $a > b$ and $c < 0$, then $ac < bc$.

Definition 2.10 (Division property of Equality)**.** a, b, and c, and d are real numbers.

- If $a > b$ and $c > 0$, then $a \div c > b \div c$.
- If $a > b$ and $c < 0$, then $a \div c < b \div c$

Definition 2.11 (Transitive Property)**.** For all real numbers a, b, and c, if $a > b$ and $b > c$, then $a > c$

Definition 2.12 (Trichotomy Property)**.** For all real numbers a and b, a must either be only:

- $a = b$
- $a > b$
- $a < b$

They can not be true equally at the same time.

An *interior angle* are angles that exist within a polygon. An *exterior angle* are angles that exist outside a polygon, typicalaly associated with a ray connecting a side. We will focus on exterior angles of triangles, and more specifically connected to our next theorem:

Theorem 2.8: Exterior Angle Theorem (EAT)

The measure of the exterior angle of a triangle is greater than the measure of any of its remote interior angles, as the measure of the exterior angle is equal to the sum of the measures of the opposing interior angles in the triangle (or polygon).

Let us try with a simple example:

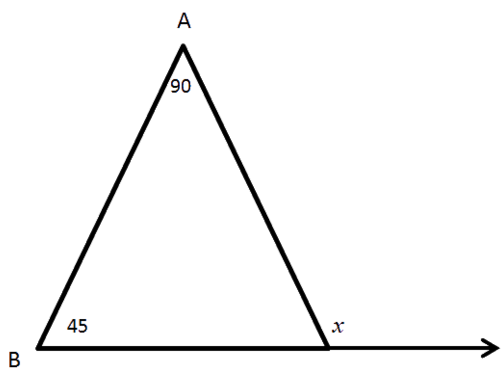

Figure 18: Finding x in an exterior angle

Using **Theorem 2.8**, we can determine that x is found by adding the opposing interior angles. Since $\angle A = 90$, and $\angle B = 45$, then $\angle A + \angle B = \angle C$ supposing $\angle C$ is the other angle. With this, $\angle C = x$, therefore $\angle C = 90+45$. Hence, $\angle C = 135$ or $x = 135$. Since $135 > 90$ or $135 > 45$ is true, then this example is true to the theorem.

2.3.2 Scalene Triangles

A *scalene triangle* is a triangle with no equal sides or angles. A figure can be drawn:

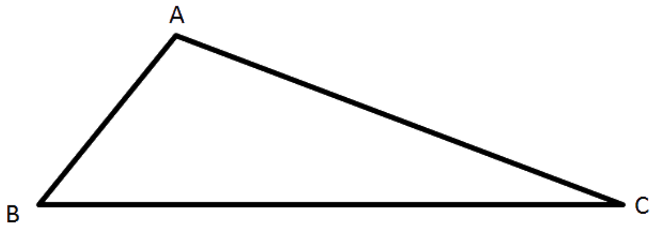

Figure 19: Scalene Triangle

Seen in the figure, if you were to measure any side or angle, it won't be equal. $\overline{AB} \ncong \overline{AC}$ or $\overline{BC} \ncong \overline{AC}$, and they hold true. Even if you were to take a ruler and protractor, they still won't be equal. Our next theorem involves scalene triangles:

Theorem 2.9: Theorem of Scalene Triangle Inequality

If one side of a triangle is longer than the other side, the the measure of the angle opposing the longer side is greater than the measure opposing the shorter side.

To visualize, look back at Figure 19. Compare $\overline{AC}$ to $\overline{AB}$. Angle $\angle ABC$ projects opposing to $\overline{AC}$. This measure is bigger than the measure of the angle projecting opposite to $\overline{AB}$, which is $\angle ACB$. Therefore, this example is true to the theorem. The converse is also true:

Theorem 2.10: Converse of Theorem 2.9

If the measure of one angle of a triangle is greater than the measure of another angle, then the side opposing the larger angle is longer than the side opposing the smaller angle.

Proving this involves the work of *indirect proving. Indirect proof* is when you start with an assumption of the given (or conclusion) being negated or of opposite truth. If the proof ends in a contradiction, then it is true. Let us use the following figure, and construct the proof using indirect proof.

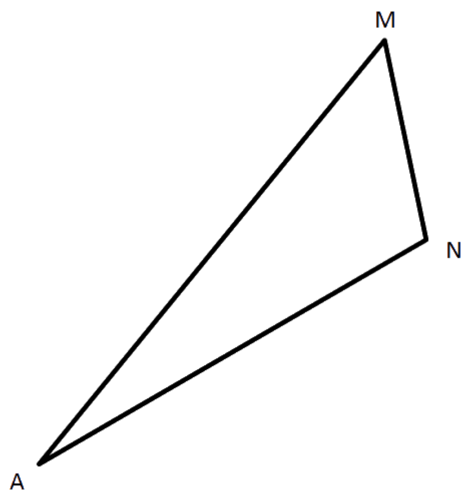

Figure 20: Scalene triangle $\triangle MAN$

Given $m\angle MAN > m\angle NMA$, prove $MN > AN$.

STATEMENT	REASONING
$MN = AN$	**Assumption.**
$\overline{MN} \cong \overline{AN}$	Definition of Congruence.
$\angle MAN \cong \angle NMA$	**Iscoceles Triangle Theorem**.
$m\angle MAN = m\angle NMA$	Definition of Congruent Angles.
However $m\angle MAN > m\angle MNA$	Given
Hence, $MN = AN$ cannot be true.	Contradiction of the Given.
New assumption: $MN < AN$ or $AN > MN$	Assumption.
$m\angle MAN < m\angle NMA$ or $m\angle NMA > m\angle MAN$	**Theorem of Scalene Triangle Inequality**.
However $m\angle MAN > m\angle NMA$	Given.
Hence, $MN < AN$ cannot be true.	Contradiction for statements 8 and 9.
$MN > AN$	Trichotomy Property

2.3.3 Triangle Inequality Theorems (in general)

The following theorem describes on why certain sets of three segments do not create a triangle.

Theorem 2.11: Theorem of Triangle Inequality

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Proof for this kind of theorem needs an *auxiliary figure*, which is an additional point, segment, ray, or line that may be drawn AS LONG as this figure does not contradict any definition, property, theorem, or postulate. Proof is given in the book (page 249.) As an example:

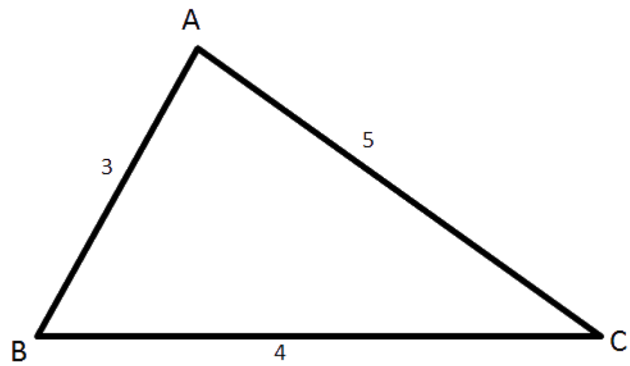

Figure 21: **Theorem of Triangle Inequality** Example

If, for example, $\overline{AB} + \overline{BC}$, or $3 + 4 = 7$. 7 is greater than 5, which means the sum of $\overline{AB}$ and $overline BC$ is greater than $\overline{AC}$, or the third side. Another example, if $\overline{AC} + \overline{AB} = 5 + 3 = 8$, $8 > 4$. So the sum of these sides is greater than $\overline{BC}$. Our next theorem describes the relations two triangles which have certain bigger lengths:

Theorem 2.12: The Hinge Theorem

If two triangles have two congruent sides, and one triangle has a side that is longer than the corresponding, then the side is opposite the larger included angle.

Theorem 2.13: Converse of Theorem 2.12

If two triangles have two congruent sides, and one triangle has an angle that is larger than the corresponding, then the angle is opposite the longer side.

Showcased as a figure:

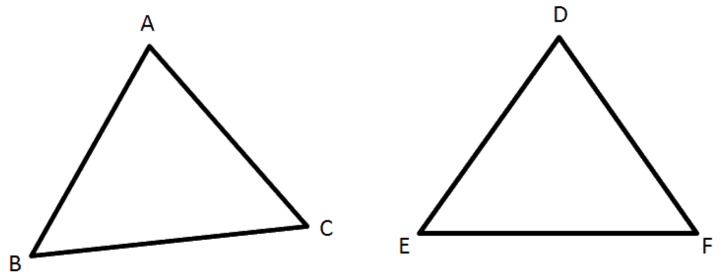

Figure 22: **Hinge Theorem**

The longer side is $\overline{AB}$ on $\triangle ABC$, compared to the corresponding shorter side $\overline{DE}$ on $\triangle DEF$. The angle opposite $\overline{AB}$, $\angle C$ is larger compared to the corresponding angle opposite $\overline{DE}$, $\angle F$ (**Theorem 2.12**.) Conversely, $\angle C$ is opposite to $\overline{AB}$, which means that the side is longer compared to the opposite side of $\angle F$, $\overline{DE}$ (**Theorem 2.13**.) Use the book for more resources (page 250-251), and visualize it by drawing triangles based off the definitions.

2.4 Parallel and Perpendicular Lines

We already know what a line is. Two sets of lines are *parallel* if they never meet at all. Two sets of lines are *perpendicular* if they intersect to create a 90° angle. If two parallel lines pass through a line, that line is a *transversal*. Look at the following:

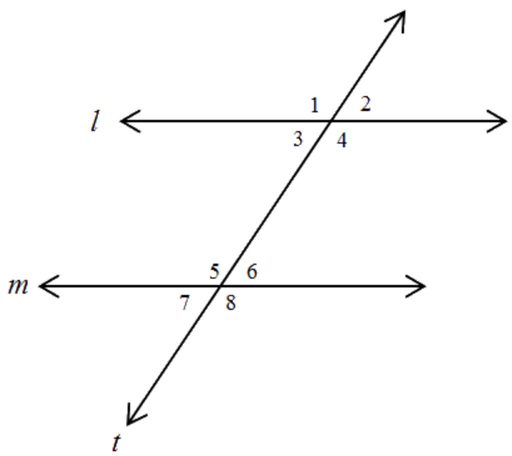

Figure 23: Transversal Line

Corresponding angles are angles that occupy the same position on the other line. A list of all corresponding angles are:

1. $\angle 1$ and $\angle 5$
2. $\angle 3$ and $\angle 7$
3. $\angle 2$ and $\angle 6$
4. $\angle 4$ and $\angle 8$

Exterior angles are angles that exist outside their respective parallel lines. A list of all exterior angles:

1. $\angle 1$
2. $\angle 2$
3. $\angle 7$
4. $\angle 8$

Interior angles are angles that exist inside their respective parallel lines. A list of all interior angles:

1. $\angle 3$
2. $\angle 4$
3. $\angle 5$
4. $\angle 6$

A set of *alternate angles* are angles that are on alternating sides of a transversal. All alternate exterior angles are:

1. $\angle 1$ and $\angle 8$
2. $\angle 2$ and $\angle 7$

All alternate interior angles are:

1. $\angle 3$ and $\angle 6$
2. $\angle 4$ and $\angle 5$

A set of *same-side angles* are angles that are on the same side of a transversal. All same-side exterior angles are:

1. $\angle 2$ and $\angle 8$
2. $\angle 1$ and $\angle 7$

All same-side interior angles are:

1. $\angle 3$ and $\angle 5$
2. $\angle 4$ and $\angle 6$

There are properties that should be known for such angles that exist in a transversal line:

I *Property 1* − Corresponding angles are congruent.

II *Property 2* − Alternate interior and exterior angles are congruent.

III *Property 3* − Same-side interior and exterior angles are supplementary

We shall describe these properties as theorems respectively, rather than labelling them as properties:

Theorems 2.14 & 2.15

If two parallel lines are cut by a transversal, then the pairs of alternate interior or exterior angles formed are congruent.

The theorem above is how we describe Property 2. Note that, the term *supplemetary* means that two angles' degrees have the sum of 180. For example, all angles in a triangle are supplementary triplets, since they add up to 180 degrees.

We can also describe Property 3:

Theorem 2.16

If two parallel lines are cut by a transversal, then the pairs of same-side interior angles or exterior angles formed are supplementary.

We shall also describe the converse of **Theorem 2.14 & 2.15**:

Theorem 2.17 & 2.18: Converse of Theoremd 2.14 & 2.15

If two lines are cut by a transversal, and a pair of alternate interior or exterior angles are congruent, then the two lines are parallel.

The converse of **Theorem 2.16** is also true:

Theorem 2.19: Converse of Theorem 2.16

If two lines are cut by a transversal and a pair of same-side interior or exterior angles are supplementary, then the two lines are parallel.

For Property 1, we can describe the following:

1. *POSTULATE 10* - If two parallel lines are cut by a transversal, then the pairs of corresponding angles formed are congruent.
2. *POSTULATE 11* - The *Converse of Postulate 10*, if two lines are cut by a transversal and a pair of corresponding angles are congruent, then the lins are parallel.

The lists above provide pairs of all kinds of angles formed with a transversal. Thsoe pairs that apply to the theorems stand true. Using the book can provide examples, more specifically Page 261. Another theorem is also true, and can be explained with the figure preceding it:

Theorem 2.20

If two lines are perpendicular to the same line, then they are parallel.

Shown in the following figure:

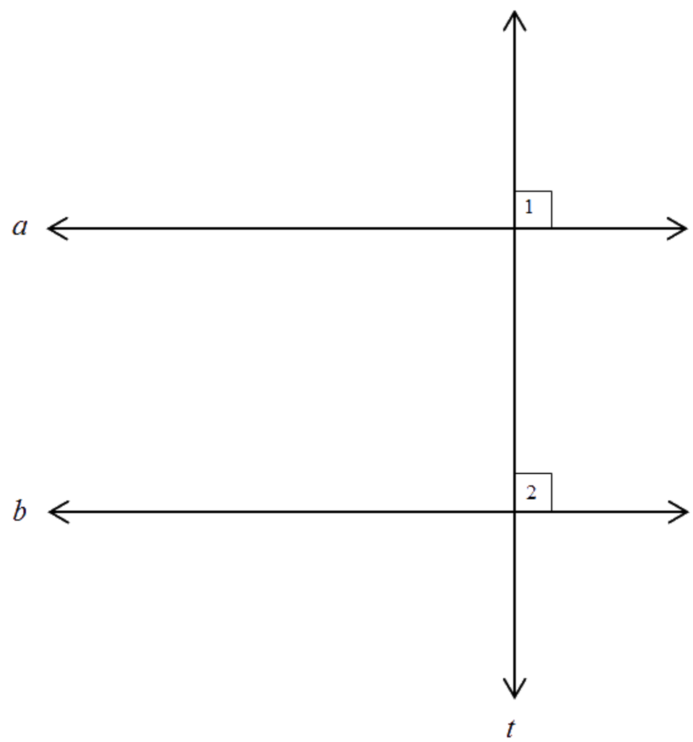

Figure 24: **Theorem 2.20**

Seen in the figure, $\angle 1$ and $\angle 2$ are perpendicular to lines a and b, whilst being cut by a transversal t. If those parallel lines form a right angle when a transversal line passes through one point of those parallel lines, then they are perpendicular. And, if we assumed that if those two lines just looked ordinary, if both of them are perpendicular to the same transversal, they would be parallel, as if you were to tilt a line, it would cause the line to hit a point on the other line (contradicting how parallel lines are defined).

Seems like you have finished the guide. I hope you thuroughly understood it. If not, then I will see on how to improve it. This has been in the making for one and a half weeks, so it would really be appreciated if you took the time to read at least one page or one paragraph. I appreciate it.

I want to thank everyone whoever was involved in this school year; teachers, classmates, friends, and the like. It has been a fun journey for second year of high school, so hope that it would be fun next year (even if some of us are leaving for better opportunities). Remember, it took a minute to make a moment that you will cherish. - by app4r4tu5.