An Introduction to Plane Trigonometry

by your... dumb classmate <3

The Basis of the Six Trigonometric Ratios, Their Properties and Identities, and Application to Elevation and Depression

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thank you ajr, boywithuke, mozart, and kessoku band for the bg music <3

Download this PDF again at *my website!*

Contents

Before We Begin

Things mentioned from the third quarter reviewer are seen here, especially ones related to triangles. You can download it here on my website, and the source code (if I do not forget), linked `nate0-mc.github.io/main_page/pdfs.html` alongside other guides and notes I have made. I already linked the section of the website with the document.

Many things in this reviewer may require memory, but the main thing to memorize are the six primary trigonometry ratios. Other tricks shall be mentioned in the document proper.

This unit is shorter than usual because of the actual length in the textbook. Elaborated this is not an issue, and I am not complaining. Just telling beforehand to seek if the topic is extensive and needs explanations detailed for understanding.

Our naming convention for right triangles denotes: If I say $\triangle GAY$, A is the angle with $90°$.

Preface

The fourth quarter of the third year has arrived for our class. It is incredibly concerning how time withers faster than a flower on a deathbed. All of the preceding quarters mentioned the nature of quadratics, variation, laws of exponents, and radicals, and the prior quarter defines the geometries of quadrilaterals, proportion and similarity, and the iconic Pythagorean theorem. This quarter covers a field in mathematics essential to most college mathematics like calculus: *trigonometry*.

This field of mathematics is split to three kinds: *Plane trigonometry* tackles trigonometry on geometric spaces with shapes, like triangles. *Analytic trigonometry* tackles trigonometry on the Cartesian plane, especially their graphs. This trigonometry might use the unit circle[1] to base the common ratios. *Hyperbolic trigonometry* tackles trigonometry on a bent surface, like a flapped piece of paper or a globe. Our curriculum tackles *plane trigonometry*. STEM courses in senior high-school might offer analytic trigonometry as a part of pre-calculus.

My aim for this guide is again, to at least, try to offer the best I can do to explain parts that are slightly blind to the reader. Our book does not cover many proofs for this topic, so it is not really needed to think really hard on theorems that may require it.

This document has the aim to supplement the reader in the discovery of trigonometry. A lot of us have never touched this topic literally; times in science where one of the ratios are used are only given a glance. This is a *supplementary form of material* to your studying. My original aims are always formed into play. More detailed explanations can't always be done by one person since they do have their hard limits, so watching YouTube videos about trigonometry (plane) can help improve deduction than just

[1] The *unit circle* is a circle with exactly one radius length. In the fourth year of high school, I believe third quarter, this might be a topic.

learning by the source material provided (book or text.)

One wish I put in these documents before sending off is "to help you, and that you have at least read one thing from this, and I would appreciate it no matter what." (excerpt from the second guide.) I dedicate a lot of time to these documents, even *sleepless nights* are countless making all three of these guides, so I would really appreciate some sort of gratitude in any way (even the most minimal.)

The following is an excerpt from the very first guide of this school year to help show the extent of how you can expect and perform to the examinations upcoming.

> "It is no surprise that mathematics is a tough and desolate subject with rigor, abstractions, and paradoxical reasoning that the reader just does not want to embark. But, a step-by-step approach, to something so radical to the naked eye, can be turned simply if you know what you are doing."

Changes from the Third Guide

I will keep this brief. A lot of the changes can be seen in the source code of the document. I have finally reorganized the massive mess that can be seen in the preamble, due to the immense wall of packages and the random macros and commands that are scattered around the document. I have now organized them to `.sty` file for the packages related to typesetting and graphics, and to a `.cls` file for the packages related to the document formatting (paper size, sectioning, etc.).

As you can tell from the document formatting, more has changed. The section symbol § and the section numbering now have coloring, in order to make it more slightly aesthetic. The `xcolor` color of the section color is `purple`. The `tcolorbox`es that I have used have now been revised, with a more simpler approach and newer ones that can be seen in the document this time. Alongside, Some colors have been changed.

If you have noticed, now, some of the important keyterms have been put under **boldface**. This is so the reader knows what is important and contrast which is meant to declare *emphasis*.

Recommendations

The following is a direct rip of the "Recommendations" section in the first ever booklet "An Advanced Synopsis of Polynomials In the Second Degree" but with all errors and mispells rewritten:

> "I recommend to the reader to take a piece of paper, and something to write with. Take notes of what you understand, and understand the pieces DIRECTLY. If you need any superintendent aide, ask someone or a friend that discretely knows what you are tackling. This subject is not passable with no gaps, it is a thorn maze filled with tiny holes that leak information without grasp.
>
> I also want to note that: You won't understand everything at first glance and that is completely unavoidable. It is normal and not in an abstract way that people don't understand this. Topics can be understood after weeks or even years; something as basic as scientific notation can be understood after a long period. Interest in mathematics is not for the faint heart of those who see it deterring their future outcome. It is not an outcast. Having difficulties reading mathematics or such any topic is not a block. It is material to build with. Use any approach you would use to understand the reasoning, and you will get it more. Take your time; either long or short. Even if it won't be comprehension-certified within two hours or a day. It is something that you, and anyone, can do."

§1 The Core of Trigonometry

The book describes the trigonometry being used as *right-angled trigonometry* and it is defined as "the basic kind of trigonometry". Take note, this might pop up on the exam. Reality, this is called plane trigonometry but we will follow our textbook.

§1.1 The Six Trigonometric Ratios

Let us consider a triangle $\triangle ABC$ as a current reference. The triangle is also right-angled, keep in mind.

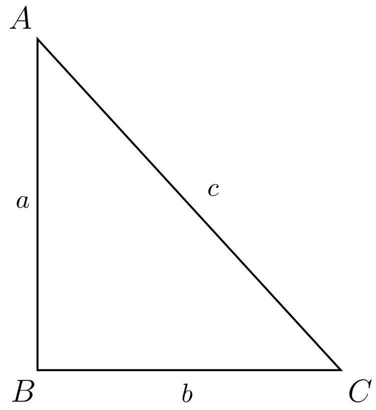


Figure 1.1: Triangle $\triangle ABC$

We declare the following: The **opposite side** of an angle is the side the angle is facing directly. The **adjacent side** of an angle is the side next to it that is not the **hypotenuse**, or the longest side of a right triangle (generally it is the part which is on a slant.)

For triangle $\triangle ABC$, we can see that the opposite side of $\angle C$ is a, and the adjacent side of $\angle A$ is also a. For now, we will ignore $\angle B$ as it is not included in our discussions[2].

[2]However, you can see such a thing in lessons on trigonometry utilizing the Cartesian plane.

Now, trigonometry's basis tackles with six ratios. The first three shall be enumerated.

Definition 1.1: The Three Primary Ratios

The **sine** of any angle θ (Greek letter theta) is the ratio of the opposite side to the hypotenuse.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

The **cosine** of any angle θ is the ratio of the adjacent side to the hypotenuse.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

The **tangent** of any angle θ is the ratio of the opposite side to its adjacent one.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Definition 1.1 describes the sine, cosine, and tangent ratios. These ratios are dependent by which angle is being used, like in $\triangle ABC$, the trigonometric ratios of A and C are not the same.

Remark (Acronym).

A neat little mnemonic is the acronym "SOH-CAH-TOA." This acronym is the abbreviation of the ratios and its values.

1. SOH means:
$$\textbf{s}\text{in}\theta = \frac{\textbf{o}\text{pposite}}{\textbf{h}\text{ypotenuse}}$$
2. CAH means:
$$\textbf{c}\text{os}\theta = \frac{\textbf{a}\text{djacent}}{\textbf{h}\text{ypotenuse}}$$
3. TOA means:
$$\textbf{t}\text{an}\theta = \frac{\textbf{o}\text{pposite}}{\textbf{a}\text{djacent}}$$

Next, we shall describe the other three ratios; the reciprocals of the primary ratios.

Definition 1.2

The reciprocal of sine is **cosecant**. The cosecant of any angle θ is its ratio of the hypotenuse to the opposite side.

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

The reciprocal of cosine is **secant**. The secant of any angle θ is its ratio of the hypotenuse to the adjacent side.

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

The reciprocal of tangent is **cotangent**. The cotangent of any angle θ is its ratio of the adjacent side to its opposite one.

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

The six trigonometric ratios really are important but sometimes their reciprocals are a little confusing to memorize. There is a pattern where if the initial letter of the function is an "s" or a "c", it switches to a "c" or an "s." I am not a writer, I don't know how to explain.

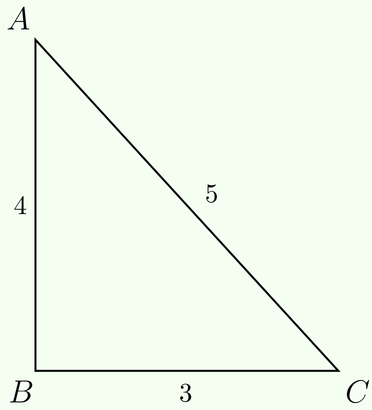
> *Remark* (Historical).
>
> Trigonometry was said to be first seen by Hipparchus of Nicaea (190 BCE to 120 BCE), the founder of geometry. He was the first to construct a table of trigonometric values.

§1.1.1 Examples

To clear up more confusion that I potentially just made, let us find the trigonometric ratios of a triangle $\triangle ABC$.

Example 1.1

Find the six trigonometric ratios of angles A and C in the triangle $\triangle ABC$.



solutions

Start by finding the ratios for, say, A; we can make a table for it rather than listing it down.

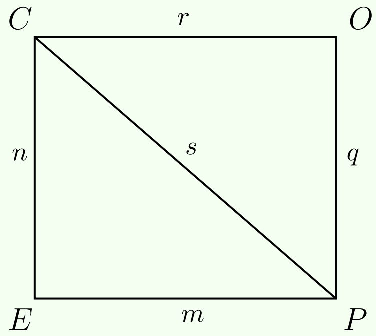
$\sin A$	$\frac{3}{5}$
$\cos A$	$\frac{4}{5}$
$\tan A$	$\frac{3}{4}$

For the other three ratios, just reciprocate the results of the first three and correspond them to their respective ratios.

$\sin A$	$\frac{3}{5}$	$\csc A$	$\frac{5}{3}$
$\cos A$	$\frac{4}{5}$	$\sec A$	$\frac{5}{4}$
$\tan A$	$\frac{3}{4}$	$\tan A$	$\frac{4}{3}$

Example 1.2: A More Complicated Triangle

In square $\square COPE$, find the ratios of C and P.



solutions

Noting that, since a square is a rhombus, it has every side equal. We will apply this later. Now, for the list of ratios of:

1. C:

$\sin C$	$\frac{m}{s}$	$\csc C$	$\frac{s}{m}$
$\cos C$	$\frac{n}{s}$	$\sec C$	$\frac{s}{n}$
$\tan C$	$\frac{m}{n}$	$\cot C$	$\frac{n}{m}$

2. P:

$\sin P$	$\frac{r}{s}$	$\csc P$	$\frac{s}{r}$
$\cos P$	$\frac{q}{s}$	$\sec P$	$\frac{s}{q}$
$\tan P$	$\frac{r}{q}$	$\cot P$	$\frac{q}{r}$

Recall to earlier; a square is a rhombus, hence all sides are equal; $m = r$, $n = q$. Using this, we can also say (alternatively):

$$\begin{aligned}
\sin C &= \sin P \\
\cos C &= \cos P \\
\tan C &= \tan P \\
\csc C &= \csc P \\
\sec C &= \sec P \\
\cot C &= \cot P
\end{aligned}$$

We will just accept the table from previously and link it to this.

Example 1.3: Finding a Missing Side

Go back to **Example 1.1**'s triangle. If we don't know the opposite side of C, and $\cos C = \frac{2}{3}$, find the missing side and the six trigonometric ratios of A.

solutions

Considering we know that,

$$\cos C = \frac{\text{adjacent}}{\text{hypotenuse}}$$

We know that our adjacent side of C is 2, and the hypotenuse is 3. We use the Pythagorean theorem to find the opposite side:

$$\begin{aligned}
a^2 + b^2 &= c^2 \\
a^2 + (2)^2 &= 3^2 \\
a^2 + 4 &= 9 \\
a^2 &= 9 - 4 \\
\sqrt{a^2} &= \sqrt{5} \\
a &= \boxed{\sqrt{5}}
\end{aligned}$$

Knowing $a = \sqrt{5}$, we can find the ratios by also knowing that $b = 2$ and $c = 3$.

$\sin C$	$\frac{\sqrt{5}}{3}$	$\csc C$	$\frac{3}{\sqrt{5}}$ or $\frac{3\sqrt{5}}{5}$
$\cos C$	$\frac{2}{3}$	$\sec C$	$\frac{3}{2}$
$\tan C$	$\frac{\sqrt{5}}{2}$	$\cot C$	$\frac{2}{\sqrt{5}}$ or $\frac{2\sqrt{5}}{5}$

What I did on the last column is I *rationalized the denominator*[3]. Soon I will update the second guide to include rationalizing the denominator, so this text might be outdated but whatever.

The next topic will involve a lot of this. So beware.

Ooh blank space........

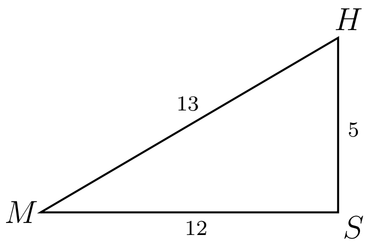
[3] I am currently in progress of updating the second guide to include rationalizing the denominator; a topic involving radicals would not feel complete without it.

§1.1.2 Exercises

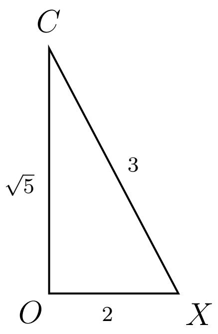
Exercise A

I: Find the six trigonometric ratios of the following triangles.

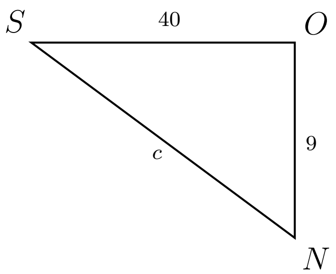
1. $\triangle HSM$ for M



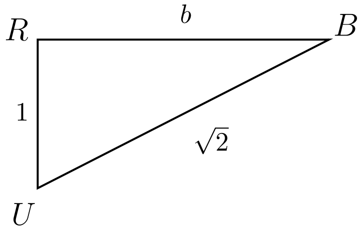
2. $\triangle COX$ for both C and X.*



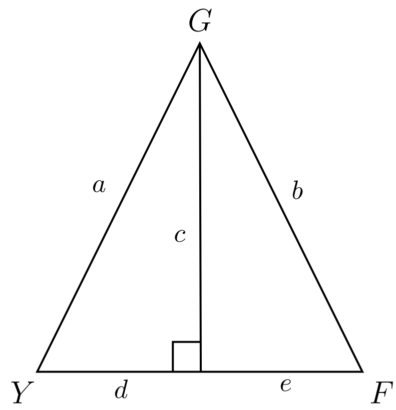
3. $\triangle SON$; find c then find the ratios for S and N.



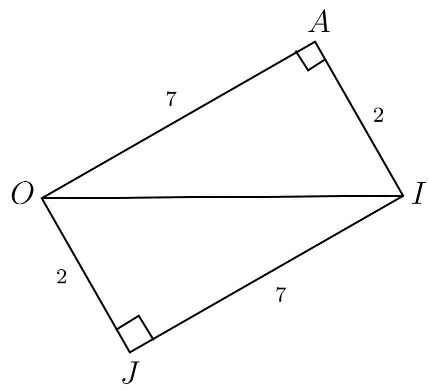
4. $\triangle RUB$; find b then find the ratios for B and R.



5. Find the ratios of Y, and F.



6. **(Medium)** Find the ratios in I and O.

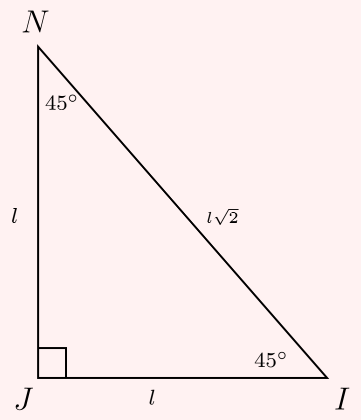


§1.2 Ratios on Special Right Triangles

Recall in the third quarter, we tackled the idea of the 45-45-90 and 30-60-90 triangle. If you do not have the PDF;

Theorem 1.1: 45°-45°-90° Triangle

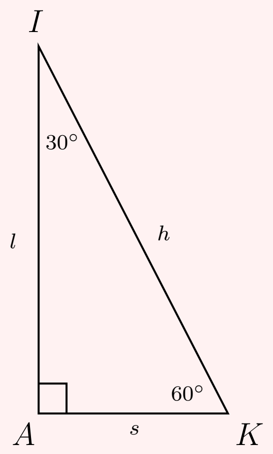
In a 45°-45°-90° triangle, the length of the hypotenuse is $\sqrt{2}$ times the length of each of the legs.



Theorem 1.2: 30°-60°-90° Triangle

In a 30°-60°-90° triangle, the following are true:

- The length of the hypotenuse is twice the length of the shorter leg.
- The length of the longer leg is $\sqrt{3}$ times the length of the shorter leg.



Remark (About the 45-45-90 Triangle).

In a unit 45-45-90 triangle, both legs are 1 unit, and the hypotenuse is $\sqrt{2}$ units. This is by the Pythagorean theorem:

$$1^2 + 1^2 = \sqrt{2}$$

Remark (About the 30-60-90 Triangle).

In a unit 30-60-90 triangle, the shorter leg is 1 unit, the longer leg is $\sqrt{3}$ units, and the the hypotenuse is 2 units. This is also by the Pythagorean theorem:

$$1^2 + (\sqrt{3})^2 = 2^2$$

Now, we shall proceed with finding the ratios of all three angles mentioned: 30°, 45°, and 60°.

For 45° –

Note.

The non-90° angles in a 45-45-90 triangle are equal, therefore this triangle is an **isosceles right triangle**[a].

[a]Recall that an isosceles triangle has two sides and base angles equal.

We note in $\triangle NIJ$, from **Theorem 1.1** $\angle N = \angle J$; $45° = 45°$.

$\sin 45$	$\frac{1}{\sqrt{2}}$	$\csc 45$	$\sqrt{2}$
$\cos 45$	$\frac{1}{\sqrt{2}}$	$\sec 45$	$\sqrt{2}$
$\tan 45$	1	$\cot 45$	1

Rationalize the denominator,

$\sin 45$	$\frac{\sqrt{2}}{2}$	$\csc 45$	$\sqrt{2}$
$\cos 45$	$\frac{\sqrt{2}}{2}$	$\sec 45$	$\sqrt{2}$
$\tan 45$	1	$\cot 45$	1

Note. $\sin 60 = \cos 60$, and $\csc 60 = \sec 60$, because the lengths of the legs are equal.

For 30° – Recall to $\triangle IKA$ from **Theorem 1.2**, on $\angle I$:

$\sin 30$	$\frac{1}{2}$	$\csc 30$	2
$\cos 30$	$\frac{\sqrt{3}}{2}$	$\sec 30$	$\frac{2}{\sqrt{3}}$
$\tan 30$	$\frac{1}{\sqrt{3}}$	$\cot 30$	$\sqrt{3}$

Rationalize the denominators,

$\sin 30$	$\frac{1}{2}$	$\csc 30$	2
$\cos 30$	$\frac{\sqrt{3}}{2}$	$\sec 30$	$\frac{2\sqrt{3}}{3}$
$\tan 30$	$\frac{\sqrt{3}}{3}$	$\cot 30$	$\sqrt{3}$

For 60° – Recall to the same triangle from earlier, this time on $\angle A$:

$\sin 60$	$\frac{\sqrt{3}}{2}$	$\csc 60$	$\frac{2}{\sqrt{3}}$
$\cos 60$	$\frac{1}{2}$	$\sec 60$	2
$\tan 60$	$\sqrt{3}$	$\cot 60$	$\frac{1}{\sqrt{3}}$

Rationalize the denominators,

$\sin 60$	$\frac{\sqrt{3}}{2}$	$\csc 60$	$\frac{2\sqrt{3}}{3}$
$\cos 60$	$\frac{1}{2}$	$\sec 60$	2
$\tan 60$	$\sqrt{3}$	$\cot 60$	$\frac{\sqrt{3}}{3}$

Remark.

It is kind of hard and unnecessary to represent the trigonometric ratio for 90° in a right triangle. For trivial reasons,

$\sin 90$	1	$\csc 90$	1
$\cos 90$	0	$\sec 90$	undefined
$\tan 90$	undefined	$\cot 90$	undefined

All the values will be summarized as a table.

EDITORIAL: For my classmates, PLEASE memorize this table. It will appear in our tests, so I would highly advise memorizing everything except for 90° since our textbook does not cover that.

Table 1.1: The Trigonometric Ratios for Special Angles

Degree	sin	cos	tan	csc	sec	cot
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	1	0	und.	1	und.	und.

If ever there is a time whenever you need to solve something like this,

$$\tan 45 - \frac{\sec 60}{2\csc 30}$$

Remember their values:

$$1 - \frac{2}{2 \cdot 2} = 1 - \frac{2}{4}$$
$$1 - \frac{1}{2} = \frac{1}{2}$$

Example 1.4: A More Complicated Version

Consider this equation:

$$x^2 - x \csc 45 \sec 45 + 2 \cos 60 = 8 \csc 30$$

Looks so scary on the outside. But convert your expressions using Table 1.1.

$$\begin{aligned} x^2 - x(\sqrt{2})(\sqrt{2}) + 2 \cdot \frac{1}{2} &= 8 \cdot 2 \\ x^2 - 2x + 1 &= 16 \\ (x-1)^2 &= 16 \\ x - 1 &= \pm 4 \\ x &= 5, -3 \end{aligned}$$

§1.3 The Complementary Angle Theorem

The textbook mentions this theorem VERY briefly, in fact it is just a margin note. I oppose this, so I will riot by dedicating an entire subsection for this theorem because it is more useful than what it said in the book.

Theorem 1.3: The Complementary Angle Theorem (in trigonometry)

For complementary angles A and B, the following hold true:

$$\begin{aligned} \sin A &= \cos(90 - A) \\ \tan A &= \cot(90 - A) \\ \sec A &= \csc(90 - A) \end{aligned}$$

and vice versa, for $B = 90 - A$. If so, then:

$$\begin{aligned} A + (90 - A) &= 90 \\ 90 &\overset{\checkmark}{=} 90 \end{aligned}$$

Explanation.

In a right triangle $\triangle ACB$, C is the right angle. We note if a triangle has the sum of all its angles to be $180°$, $A + B + C = 180$, then if $C = 90$, then $A + B = 90$.

$$\begin{aligned} A + B + 90 &= 180 \\ A + B &= 180 - 90 \\ A + B &= 90 \end{aligned}$$

Therefore A and B are complementary angles in a right triangle.

Note on Table 1.1, certain entries are equal to other angles. By following the complementary angle theorem:

$$\sin 30 = \cos 60$$
$$\tan 45 = \cot 45$$
$$\sec 60 = \csc 30$$

Or,

$$\sin 30 = \cos(90 - 30)$$
$$\tan 45 = \cot(90 - 45)$$
$$\sec 60 = \csc(90 - 60)$$

This was the same question that our teacher asked for plus points, worth it.

§1.3.1 Examples

Only one example.

Example 1.5

Using your scientific calculator after finding the missing value that makes it equal, verify its authenticity.

1. $\cos 33 = \sin x$
2. $\sec x = \csc 89$
3. $\tan x = \csc 64$

solutions

Begin with the first one. A trick we can use is by finding x using $90 - x = 33$.

$$90 - x = 33$$
$$90 - 33 = x$$
$$x = 57$$

We can tell that $\cos 33 = \sin 57$. Our scientific calculator says that both of them are approximately 0.8387.

For the second one, same thing.

$$90 - x = 89$$
$$90 - 89 = x$$
$$x = 1$$

If your scientific calculator does not have a function for cosecant, secant, and cotangent, just use $\frac{1}{\sin\theta}$, $\frac{1}{\cos\theta}$, and $\frac{1}{\tan\theta}$[4] respectively, for any angle θ.

Once calculated, we note that both of them are also approximately 1.0001.

For the third one, same thing (I can't keep on repeating this.)

$$\begin{aligned} 90 - x &= 64 \\ 90 - 64 &= x \\ 26 &= x \end{aligned}$$

Calculate the ratios; we see both of them are approximately 0.4877.

[4]Refer to **Definition 1.2**

§1.3.2 Exercises

Exercise B

I: Calculate the following, using Table 1.1.

1. $(\cot 45 - 2\sin 60 + \cot 30)^2 - \tan 45$
2. $(\csc 45)(\sec 45)(\cos 30)(\tan 60)$
3. $\sec 60 \left((\cos 60)^2 + \dfrac{\cot 45}{(\csc 45)^2 \csc 30} \right)$
4. For 4-6, solve for x: $\dfrac{\tan 45}{\cot 60} 2x \tan 60 - \csc 30 = 3 \sec 60$
5. $\sqrt{x \csc 30 \sin 45 \csc 45} = x + \csc 90^*$
6. $x + \sqrt{\dfrac{1}{\sin 45} x \sec 45 \csc 30 - x \tan 45} - \sec 60 (\tan 60)^2 = \cos 90^*$ (i am so sorry)

II: Perform the same test from **Example 1.5**.

1. $\tan 35 = \cot x$
2. $\cot x = \tan 70$
3. $\sin x = \sin 33$
4. $\csc 22 = \sec x$
5. $\sec 10 = \csc x$
6. $\tan x = \cot 75$
7. $\sin 50 = \cos x$
8. $\cos x = \sin 51$
9. $\csc 18 = \sec x$
10. $\sin 72 = \cos x$
11. $\cot x = \tan 27$
12. $\cos x = \sin 47$
13. $\tan x = \cot 26$
14. $\sec 65 = \csc x$
15. $\cot x = \tan 41$

§2 Applications and Solutions of Triangles

Beginning this section, we talk about the angles of elevation and depression. Then, we will bring the idea of solutions to all parts of a right triangle.

§2.1 The Angle of Elevation and Depression

Imagine yourself idle on Point A. There are two objects, object B above you and object C below you. The distance to see those objects is called the **horizontal**. The actual path light traces to your eyes (in our terms) is called the **line of sight**. Illustrated,

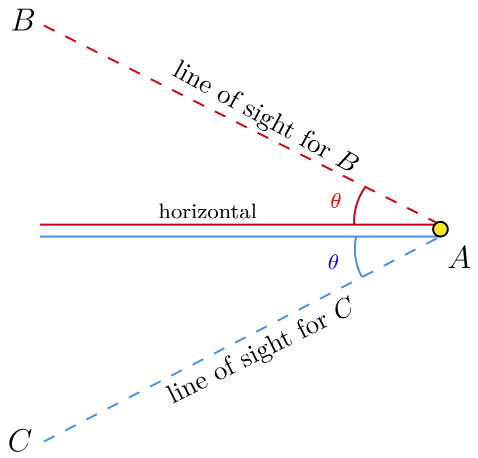


Figure 2.1: Line of Sight for Object A and Object B

The **angle of elevation** (in red) is the angle formed between the horizontal and the line of sight, looking upwards. The **angle of depression** (in blue) is the same idea but looking downwards.

(a) Angle of Elevation

(b) Angle of Depression

Figure 2.2: Angles of Elevation and Depression

§2.1.1 Examples

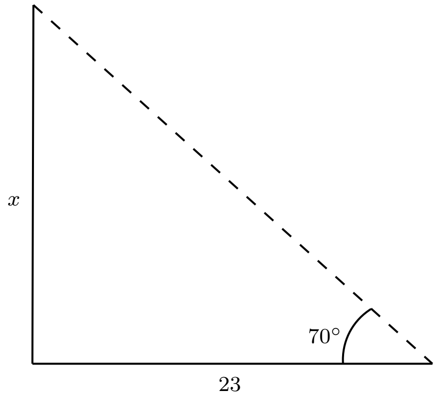
Right now, I have not explained how to actually solve for the missing lengths or angle of depression/elevation, but I will illustrate by showing examples.

Example 2.1: Angle of Elevation

A crane raises up its head with an angle of 70°. It was getting an object that was 23 m away from it. How high was the crane's head?

solutions

Illustrate:



I am not decorating it. It is stupid hard.

Explanation.

We are finding the missing side, the *opposite* of the angle we referenced. Our goal is to find this side's length.

What we will do is utilize a ratio that involves the opposite side, and our other side that we know of (the adjacent side; the horizontal). A ratio that involves the opposite and adjacent side is the tangent. We need to identify a ratio that involves the side we are looking for so that we can solve the equation for that side specifically.

$$\begin{aligned} \tan 70^\circ &= \frac{x}{23} \\ x &= 23 \tan 70^\circ \\ x &\approx \boxed{68.19 \text{ m}} \end{aligned}$$

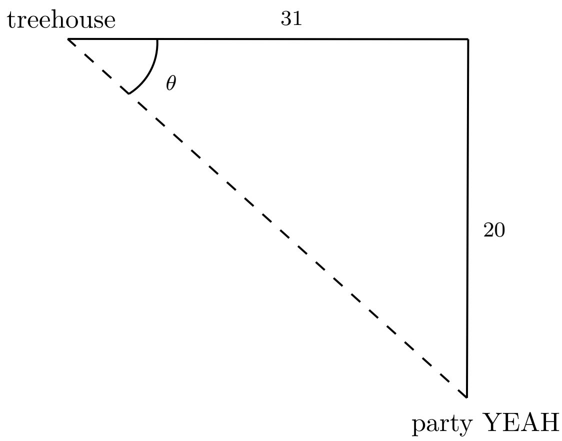
EDITORIAL: Our instructor said to round the angle to the nearest whole number, and the sides to two decimal places. Keep in mind for everything proceeding.

Example 2.2: Angle of Depression

In High School Musical 3: Senior Year, at the beginning, Troy and Gabriella are sitting atop in Troy's treehouse. Let's say, they both observe a party 20 meters below them with a distance of 31 meters. What was their angle of depression?

solutions

Illustrate, always:



Explanation.

Notice this time, we do not have our angle of depression. This is our missing part. We will label the missing angle θ. What we will do is use a ratio with the missing angle that involves the sides that we know of already. Tangent (again) works well, because we only have our opposite and adjacent sides. We write,

$$\tan\theta = \frac{20}{31}$$

We find the missing angle by using, on our calculator, the *inverse tangent* function. Press SHIFT, then TAN on your scientific calculator. Then, input $\frac{20}{31}$.

$$\begin{aligned}\tan\theta &= \frac{20}{31}\\ \theta &= \tan^{-1}\left(\frac{20}{31}\right)\\ \theta &\approx \boxed{33^\circ}\end{aligned}$$

Remark.

$\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ are **inverse functions**; they return the inputs of the sine, cosine, and tangent function respectively. Press SHIFT then either SIN, COS, TAN.

§2.1.2 Exercises

Exercise C

I: The following within this part involve angles of elevation.

1. In Bocchi The Rock!, Kessoku Band auditioned in a venue underground (I need verification, I can't tell.) with a descending staircase.

   We will deal with mock measures. Nijika, Kita, and Ryo go up the staircase that is 2.7 meters tall. Bocchi, staying at the bottom, is looking upwards to them at an angle of 34°. How far away is she from the other three?*

2. Caleb is about to pull a kamikaze belly flop on Kyler. He jumps around 3.4 meters, and Kyler is looking at Caleb's menacing aura 1.3 meters away. Before Kyler and Caleb got killed together (as lovers should), what was Kyler's angle of elevation?

3. Returning to the Kessoku Band, consider me being in the front row, right in front of Kita Ikuyo (who is 1.7 meters away me.) If my angle of elevation is 25°, how high is the stage?

4. In Fate/stay night: Unlimited Blade Works, a scene depicts Rin and Shirou looking upwards to see Lancer ontop of a structure willing to help the duo. Let's say, Lancer is standing 5 meters above, and they are gazing away at 3.2 meters. What was Rin and Shirou's angle of elevation?

5. Imagine Spiderman, atop a skyscraper, about to jump to swing. JJJ, about to write an article on him, looks up a 34 meter wide building with Spiderman above a whopping 161 meter tall skyscraper. What is the angle of elevation of JJJ?

6. In 2022, Bherna fell down the stairs with the description from one of her friends as "a flying angel." We consider her descent (the height) as 3.3 meters. One of her friends was at the start of the staircase witnessing disaster at 41.6 degrees. How far away was he before her doom manifested?*

7. Similarly, in 2024, Claire fell down the slides and fractured her ankle (or angle :P hehe). The slide was 5 meters long, and her friend was looking up at her with tremor with an angle of 30 degrees. How high was her fall? If it is a weird number, this is just an estimate.

8. Time for KonoSuba. Someone, specifically one of the generals of the Devil King, Verdia, was witnessing Megumin's absolutely destructive Explosion spell. He was gazing afar, actually very near to it, at 10.1 meters from his castle. The explosion's cloud reached 140 meters high. In his annoyance gaze, what was the angle of elevation?

II: The following within this part involves angle of depression.

1. Kyler is hopelessly daydreaming about Mia, and he dreams about her below him as if she was kneeling. He gazes down at an angle of 26 degrees and, in total, 1.2 meters tall. How far away was Mia from Kyler? They were hiking elevated terrain, by the way.

2. In the manga version of 100 Girlfriends (insert long name), Rentarou pushes off a personification of the production industry down the building with him. Three of his girlfriends gaze down 39 meters (by horizontal means) and 16 meters down to Rentarou. What was their angle of depression?

3. Let's start with an average anime battle scene. We will use TenSura as an example. If Rimuru was about the decimate the hell out of Gobta using Beelzebub at an angle of depression of 57°, and Gobta is about to be returned to sender 7 meters apart, how high was Rimuru's incredible wrath?

4. AJR performs a concert in Madison Square Garden. Jack, the vocalist, looks down to the front row with an angle of 20° and the front row is 25 centimeters away from him. In centimeters, how tall was the stage?

5. In the same concert, on the stands, a fan watches the performance of Karma which is 10 meters away from him, with him gazing at the large transparent screens of 6 meters. If he was filming, what was the angle of his hand position?

6. Maria looks down from a balcony with an angle of depression 60° down to Don. Ibarra. The vertical distance downward is 41 decimeters. Find the horizontal distance.

7. This will be a strange example. Imagine I am almost looking down at a 90° angle, say I am looking down at an angle of depression with 87°. A small corn that is on the floor is 5 decimeters below me. How far away is that potentially beyond 5-second-rule corn bud?*

8. Sir Roi, trying to find his laptop bag that Louise willingly took, looked at the bottom of the table at an angle of 30°. The table from his view is 17 inches long. What was the vertical distance to where his eyes were observing?*

§2.2 Solutions of Right Triangles

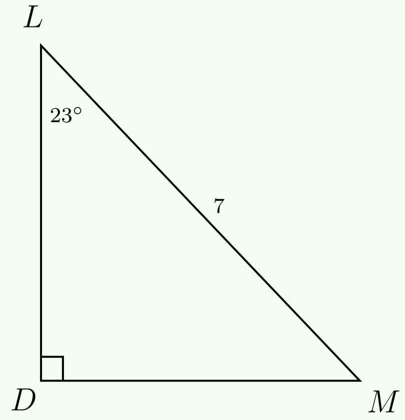
Last subsection, we tackled solving for a certain side or angle of depression/elevation using trigonometric ratios. Now, we will be using these ratios to try find all parts of a right triangle.

However, Certain sides of a right triangle can be found by using the Pythagorean theorem. This can be elaborated later on when we get to the examples, like right now. Segue.

§2.2.1 Examples

Example 2.3

Find $\overline{LD}$, $\overline{DM}$, and $\angle M$.



solutions

1. For $\angle M$, we note again the sum of all angles is 180°, and that $\angle D$ is already 90°:

$$\begin{aligned} 90 + 23 + \angle M &= 180 \\ \angle M &= 180 - 90 - 23 \\ \angle M &= \boxed{67^\circ} \end{aligned}$$

2. Finding $\overline{LD}$ involves the usage of a ratio that involves that side. Pick an angle to work with (I will pick $\angle L$.) Find a ratio that involves $\overline{LD}$, or the adjacent side and the hypotenuse.

$$\begin{aligned} \cos 23 &= \frac{\overline{LD}}{7} \\ \overline{LD} &= 7\cos 23 \\ \overline{LD} &\approx \boxed{6.44} \end{aligned}$$

Note (If we used $\angle M$).

We could also use $\angle M$. Since $\overline{LD}$ is opposite that angle, we use sine. Note that $\angle M$ is 67°.

$$\begin{aligned} \sin 67 &= \frac{\overline{LD}}{7} \\ \overline{LD} &= 7\sin 67 \\ \overline{LD} &\approx 6.44 \end{aligned}$$

Not convinced? Calculate $7\sin 67$.

3. For $\overline{DM}$, we can use either angles. I will pick $\angle M$. Note our missing side is $\overline{DM}$, or the adjacent side of $\angle M$. We can use either cosine or tangent, since they both involve the ratio of a known side to our unknown. Let's use both.

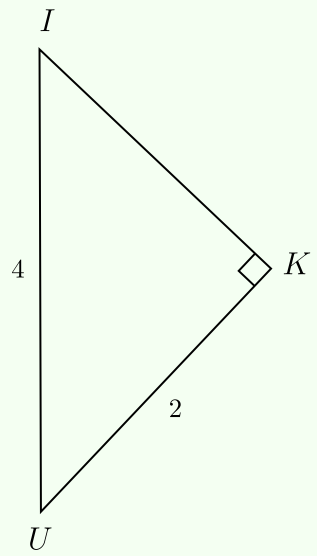
$$\begin{aligned} \tan 67 &= \frac{6.44}{\overline{DM}} \\ \overline{DM}\tan 67 &= 6.44 \\ \overline{DM} &= \frac{6.44}{\tan 67} \\ \overline{DM} &\approx \boxed{2.73} \\ \text{or}\ \ \cos 67 &= \frac{\overline{DM}}{7} \\ \overline{DM} &= 7\cos 67 \\ \overline{DM} &\approx \boxed{2.74} \end{aligned}$$

Both of them are different by a tenth, but really it is my fault because I rounded $\tan 67$. If you used your scientific calculator and typed $\frac{6.44}{\tan 67}$ it would be most accurate (I used the Windows Calculator.)

Try finding $\overline{DM}$ but this time using $\angle L$ as reference.

Example 2.4

Find $\angle I$, $\angle U$, $\overline{IK}$.



solutions

1. Let us find $\overline{IK}$ first. Note that there are no angles available. We can start by either finding an angle or a side, which is what we are doing.

$$\begin{aligned} 2^2 + b^2 &= 4^2 \\ b^2 &= 16 - 4 \\ b^2 &= 12 \\ b &= \sqrt{12} \\ b &= 2\sqrt{3} \\ b &\approx \boxed{3.46} \end{aligned}$$

2. We shall find $\angle U$. We can use either sine, cosine, or tangent, since we do not know what $\angle U$ is, and that we have all three sides unlocked. i will use sine.

$$\begin{aligned} \sin U &= \frac{3.46}{5} \\ U &= \sin^{-1}\frac{3.46}{5} \\ \angle U &\approx \boxed{43^\circ} \end{aligned}$$

Rounded to the nearest whole number.

3. We now find $\angle U$. We don't need trigonometry, since we already know $\angle I$ is $43°$.

$$\begin{aligned} \angle U &= 180 - \angle K - \angle I \\ \angle U &= 180 - 90 - 43 \\ \angle U &= \boxed{47°} \end{aligned}$$

Remark (An Alternative Method).

In **Example 2.3**, note that once we have two sides, we can find the third by trigonometric means (the third side we found was 2.74 units.) However, could've we just used the Pythagorean theorem? Turns out we can. Recall we had 6.44 and 7 as our side lengths:

$$\begin{aligned} a^2 + 6.44^2 &= 7^2 \\ a^2 &= 49 - 41.4736 \\ a^2 &= 7.5264 \\ \sqrt{a^2} &= \sqrt{7.5264} \\ a &\approx 2.74 \end{aligned}$$

It still leads the same yield. I would prefer you use trigonometry to find missing sides as it is our focus, but if you need to double check, use the Pythagorean theorem to find it to check for any mistakes.

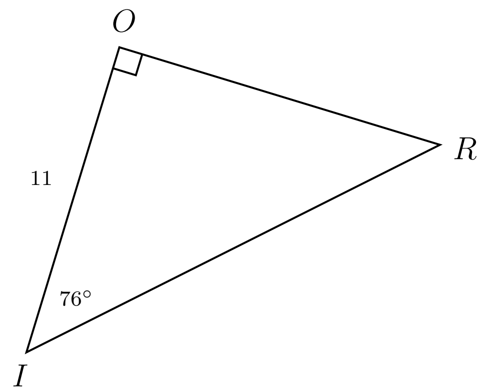
Always identify what is given. Then, think what ratio fits best to an angle you will use. For example, on an angle θ, you find the hypotenuse is a and the adjacent of that angle is b. If you need to find θ, use cosine; $\cos\theta = \frac{a}{b}$.

§2.2.2 Exercises

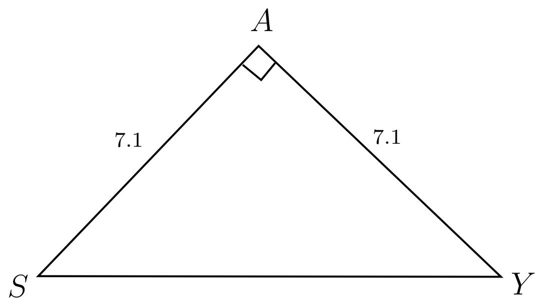
Exercise D

I: Find the requested parts of the right triangle. Use trigonometry to find the missing side, and not the Pythagorean Theorem.

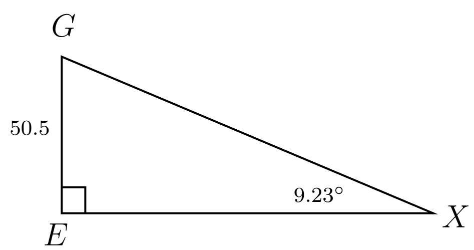
1. Find $\angle R$, $\overline{OR}$ and $\overline{IR}$.



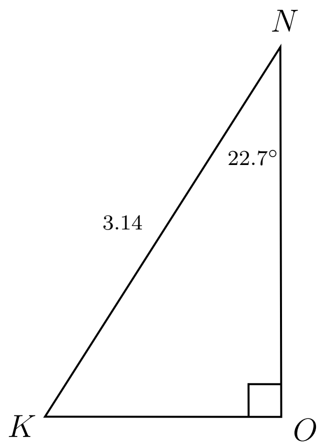
2. Find $\angle S$, $\angle Y$, and $\overline{YS}$.*



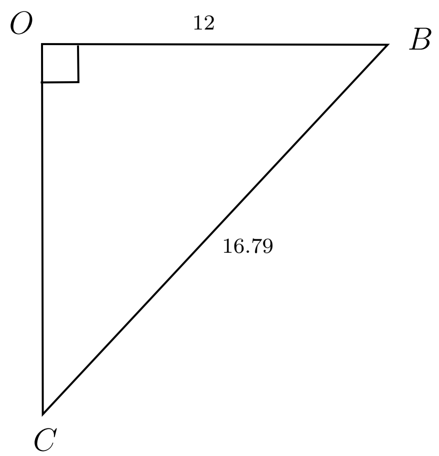
3. Find $\angle G$, $\overline{GX}$, and $\overline{EX}$.



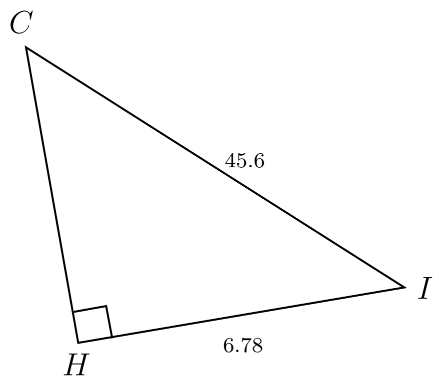
4. Find $\angle K$, $\overline{OK}$, and $\overline{NO}$.



5. Find $\angle B$, $\angle C$, and $\overline{OC}$.



6. Find $\angle C$, $\angle I$, and $\overline{CH}$.



§3 Trigonometry on Oblique Triangles

The oblique triangle is almost any triangle you draw. This section will tackle the *law of sines* and *law of cosines*. We also will tackle an extra topic, regarding the area of an oblique triangle[5].

§3.1 The Law of Sines

Before we begin, the elephant in the room is what is an oblique triangle. Following the definition:

Definition 3.1

An **oblique triangle** is a triangle that lacks a right angle.

If you draw any triangle that is not a right triangle, it is almost guaranteed to be an oblique triangle. Under the oblique triangle class, there exists:

Definition 3.2

A triangle is considered **acute** if all three of its angles are acute (less than ninety degrees.) A triangle is considered **obtuse** if *only* one angle is obtuse (greater than ninety degrees.). Under all categories, there exists a triangle with no equal sides (hence also no equal interior angles), or a **scalene** triangle.

Oblique triangles follow a theorem (or a law/property) that can enable us to solve for the sides and the angles.

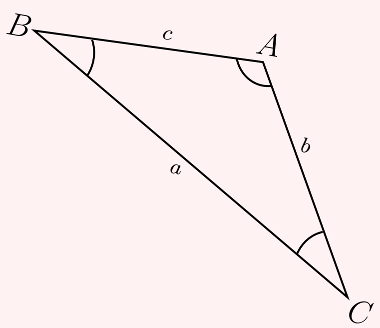
Theorem 3.1: The Law of Sines

Consider a triangle $\triangle ABC$. If there exists angles A, B, C, and opposing sides a, b, and c to their respective angles, then the following is true:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

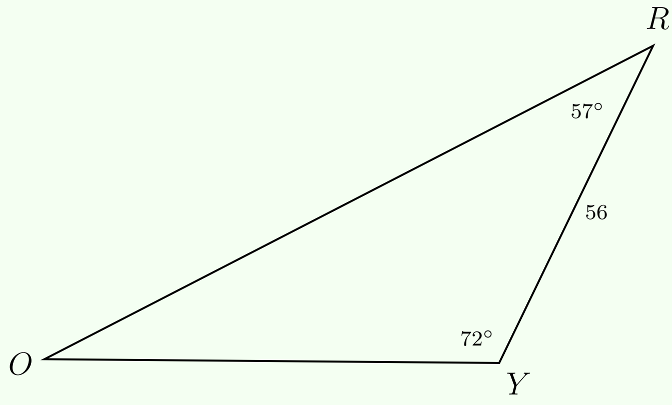


The proof is quite the length, check the book if you are interested/look it up on Google. If you're lazy, then I already did it for you.

[5] Actually not in this guide. I plan to add it once I update the second quarter guide.

§3.1.1 Examples

Example 3.1: Example 4.1 from our textbook



(Triangle name modified) Solve for all parts of the triangle $\triangle RYO$ noting that $\angle R = 57°$, $\angle Y = 72°$, and $\overline{RY} = 56$ units. *Round the sides to the nearest whole number.*

solutions

The book *doesn't illustrate* the triangle. If ever possible, illustrate the triangle for easier observation. Begin by finding the remaining angle. Using the 180 degree yada yada property,

$$\angle O = 180 - \angle R - \angle Y$$
$$\angle O = 180 - 57 - 72$$
$$\angle O = \boxed{51°}$$

Afterwards, finding the missing sides require **Theorem 3.1** or the law of sines. Find the opposite side (that has not been given) of a certain angle, I picked $\angle Y$. Then, use another one that already has an opposite side given, say $\angle O$. We can't use $\angle R$ because its opposite side does not have a given value, so we resort to the other available.

Using law of sines (note the small letters represent the opposite side of the angles):

$$\frac{\sin Y}{y} = \frac{\sin O}{o}$$
$$\frac{\sin 72}{y} = \frac{\sin 51}{56}$$
$$\sin 72 \approx 0.01y$$
$$y = \frac{\sin 72}{0.01}$$
$$y = \boxed{69}$$

Note that $y = \overline{RO}$.

Explanation.

Whenever we have to use the law of sines, find an angle and its opposite side that is already given. Then, use another pair that has either the angle missing or the side missing. Equate them both by the form in the law of sines, then solve for the missing part.

Once done, use the remaining pair that has a missing part with another pair that is completed (either the one you just solved or the other one that you used earlier.) Equate them like in the law of sines, and solve.

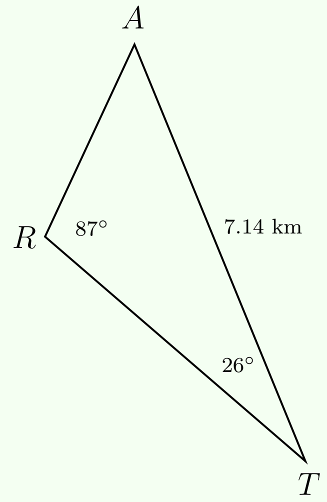
We then find the final side, $\overline{OY}$. Same process. Use $\angle R$ because it has the missing side, and $\angle O$. You could also do the same for $\angle Y$ (we already know the side for the angle.). I will use the reciprocal version.

$$\begin{aligned}\frac{r}{\sin R} &= \frac{o}{\sin O}\\ \frac{r}{\sin 57} &= \frac{56}{\sin 51}\\ r &= \frac{56 \sin 57}{\sin 51}\\ r &\approx \boxed{60}\end{aligned}$$

If we were to use $\angle Y$, there would be a rounding error (it would be 61.). For now, we will accept whatever was given. If you were going to construct a triangle, well just deal with it. (no seriously, I also don't know why there is a rounding error.)

Example 3.2: An SAA (side-angle-angle) Example

Find $\overline{AR}$, $\overline{RT}$, and $\angle A$.



Round the missing sides to two decimal places.

solutions

I will be a little abnormal and find one missing side. Find two pairs; one with a missing side and an angle given $\angle T$), and one that has both given ($\angle R$). Perform law of sines (small letters are the opposite sides of the angles.); using the reciprocal version would be easier.

$$\begin{aligned}\frac{t}{\sin T} &= \frac{r}{\sin R}\\ \frac{t}{\sin 26} &= \frac{7.14}{\sin 87}\\ t &= \frac{7.14 \sin 26}{\sin 87}\\ t &\approx \boxed{3.13 \text{ km}}\end{aligned}$$

To solve for the opposite side, we start with finding $\angle A$:

$$\begin{aligned}\angle A &= 180 - \angle R - \angle T\\ \angle A &= 180 - 87 - 26\\ \angle A &= \boxed{67^\circ}\end{aligned}$$

Now, we can find the final missing side. Same as earlier, pick two pairs and you get the point. I picked $\angle R$ again, and $\angle A$ obviously. The reciprocal version again.

$$\begin{aligned}\frac{a}{\sin A} &= \frac{r}{\sin R}\\ \frac{a}{\sin 67} &= \frac{7.14}{\sin 87}\\ a &= \frac{7.14 \sin 67}{\sin 87}\\ a &\approx \boxed{6.58 \text{ km}}\end{aligned}$$

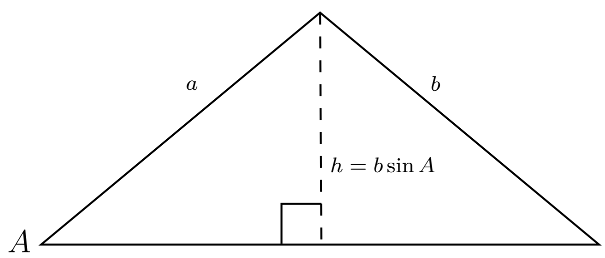
§3.2 The Ambiguous Case

Note that we only know how to solve for a triangle with only AAS (angle-angle-side) and ASA (angle-side-angle) given. Solving for a triangle with only SSS (side-side-side) or AAA (angle-angle-angle) is impossible, because neither angle or side exists in either respectively, and SAS (side-angle-side) is not possible because there is not enough data provided.

However, for an SSA case (side-side-angle), there can exist many possibilities. There can be two possible triangles because an angle exists the original's boundary, one triangle (the original), or no triangle. When there exists two or no possible triangle(s) in a solution, we call this an **ambiguous case**.

Remark (Ambiguous Cases).

If there exists an angle A and two sides a and b in the form of SSA, then the following must be considered (keep in mind $b \sin A$ is the altitude of the triangle):



1. If your angle A is acute and if $a < b$:
    - If $a < b \sin A$, then there is no solution.
    - If $a = b \sin A$, then the only solution is a right triangle.
    - If $a > b \sin A$, then there are two solutions.
2. If A is acute and $a \geq b$, then there is only one solution (this matters.) Refer to page 269 if you have the book.
3. If A is an obtuse or right angle:
    - If A is an obtuse or right angle, then if $a \leq b$, then there is no solution.
    - If $a > b$, there is only one solution (given we are using an obtuse triangle for reference.)

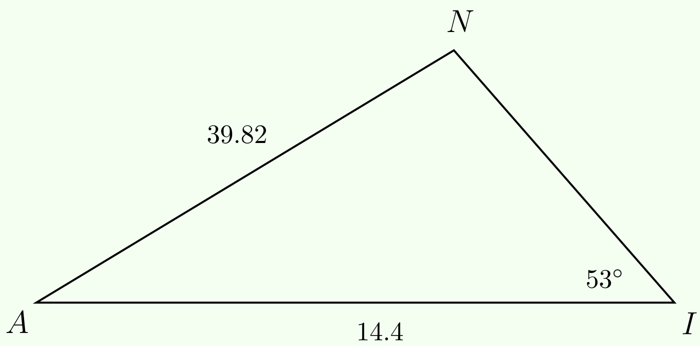
Note that: in order to solve problems of this form, refer to that we need to know our data first. If the angle given is acute, then refer to the cases for acute angles. If otherwise, then refer to the cases for obtuse/right angles. But do not forget, this is only for *SSA Cases*, when there is two sides and one angle adjacent to a side.

§3.2.1 Examples

We begin by defining examples. If you are dealing with acute angles, of all steps, first COMPUTE the product of the (given) non-opposite side and the sine of the reference angle, or $b \sin A$ in the preceding page. If you are dealing with an obtuse angle, just inspect. Just be warned the explanations are lengthy, so if you are still lost, there will be more detail when I solve any triangle following.

Example 3.3: An Acute Angle for Reference

Find the solution of the oblique triangle $\triangle ANI$. Round all angles to the nearest whole number, and the sides to two decimal places.



solutions

Pick the pairs with both the opposite side and its angle given, and the one with the missing angle. Unlike usual, we cannot solve this by using the 180 trick since it requires only one side missing for it to work. We need to solve by using law of sines. Remember, lowercase letters mean the opposite side of their respective angles.

$$\frac{\sin N}{n} = \frac{\sin I}{i}$$
$$\frac{\sin N}{14.4} = \frac{\sin 53}{39.82}$$
$$\sin N = \frac{14.4 \sin 53}{39.82}$$
$$\angle N = \sin^{-1} \frac{14.4 \sin 53}{39.82}$$
$$\angle N \approx \boxed{17^\circ}$$

Before we proceed, note we need to know if there is only one possible solution or two, since we are dealing with an SSA triangle. Noting that originally only $\angle I$ and its opposite side i had given values, 53° and 39.82 respectively. Compute the product of the given non-opposite side, and the sine of the (acute) reference angle; 14.4 is our non-opposite (or should I say adjacent) side.

$$39.82 \stackrel{?}{=} 14.4 \sin 53$$
$$39.82 \neq 11.5$$
$$39.82 > 11.5$$

A shortcut for *acute angles* is if the opposite side of it is greater than the given non-opposite side[6]. But, even so, doing the other way still helps to check if it does actually have two solutions. If so, then we must know that there are two potential triangles. To actually unlock the second solution, we subtract our angle (17°) from 180.

$$180 - 17 = 163$$

So we now know the second solution is based on our second angle, 163°. *However*, whilst we actually have the solutions, do they work? The verification is done by finding the final angle in the triangle that is missing ($\angle A$.)

Start by subtracting $\angle I$ and the original value of $\angle N$, 17°.

$$\angle A = 180 - 53 - 17 \rightarrow 110°$$

We now know our first solution has the third angle at 110°. What about our second angle?

$$\angle A = 180 - 53 - 163 \rightarrow -36°$$

Negative angle measures *do not exist in closed shapes*[7], so hence we declare this solution impossible. Therefore we only have one solution.

> *Explanation* (Two Solutions Down to One???).
>
> I do hope you are following, because even if we verified that there can be two triangles, we have to at least check if that is possible first. In a sense, this is the trigonometry equivalent of an *extraneous root*.

So, we proceed with the first solution since the second one is false ($\angle A = \boxed{110°}$ is our only solution.) We shall now find the missing side $\overline{NI}$, or small letter a. I pick angle pair $\angle N$ and $\overline{AI}$.

$$\begin{aligned} \frac{a}{\sin A} &= \frac{n}{\sin N} \\ \frac{a}{\sin 110} &= \frac{14.4}{\sin 17} \\ a &= \frac{14.4 \sin 110}{\sin 17} \\ a &\approx \boxed{46.28} \end{aligned}$$

[6]Recall back to the Ambiguous Case Remark a while back.
[7]Negative angles do exist if they are open, but it is unrelated.

Example 3.4: Acute Reference with Two Angles

Given $\triangle SEN$ has $S = 34°$, $s = 7.09$, and $e = 10.1$, solve the triangle (if there are two solutions, list them too.)

Same rounding/approximation requirements.

I will try doing this without illustrating to try save time.

Follow like usual. Solve for $\angle E$.

$$\frac{\sin E}{e} = \frac{\sin S}{s}$$
$$\frac{\sin E}{10.1} = \frac{\sin 34}{7.09}$$
$$\sin E = \frac{10.1 \sin 34}{7.09}$$
$$\angle E = \sin^{-1} \frac{10.1 \sin 34}{7.09}$$
$$\angle E \approx \boxed{53°}$$

Now, check if there are two solutions, or only one. Note that $s < e$, or the opposite side is less than the non-opposite (adjacent) side. Meaning, to prove this, we have to check if s is greater than $e \sin S$ (recall to the Remark from prior.)

$$7.09 \stackrel{?}{=} 10.1 \sin 34$$
$$7.09 \neq 5.65$$
$$7.09 > 5.65$$

Begin by finding the other solution's second angle.

$$180 - 53 = 127°$$

Double check if the triangle truly has two solutions by finding the third angle:

- $\angle N = 180 - 34 - 53 = \boxed{93°}$
- $\angle N = 180 - 34 - 127 = \boxed{19°}$

Therefore we have two solutions. Find the solutions to each one. Infer from what I will do next because it is important you do not mix it.

For the first solution:

$$\frac{n}{\sin N} = \frac{e}{\sin E}$$
$$\frac{n}{\sin 93} = \frac{10.1}{\sin 53}$$
$$n = \frac{10.1 \sin 93}{\sin 53}$$
$$n \approx \boxed{12.63}$$

For the second solution:

$$\begin{aligned}\frac{n}{\sin N} &= \frac{e}{\sin E}\\ \frac{n}{\sin 19} &= \frac{10.1}{\sin 127}\\ n &= \frac{10.1 \sin 19}{\sin 127}\\ n &\approx \boxed{4.11}\end{aligned}$$

Therefore:

1. **First Solution**
   - $\angle E = 53°$
   - $\angle N = 93°$
   - $n = 12.63$
2. **Second Solution**
   - $\angle E = 127°$
   - $\angle N = 19°$
   - $n = 4.11$

Explanation.

Imagine each solution is in a separate box. What I did was, that, for the first box, there were different items; our original value for $\angle E$ was $53°$ whilst with that value I got $\angle N = 93°$. Whilst so, our $n = 12.63$. However, on the second box, the values have changed. Reason is, we had to obtain the second solution by subtracting the first solution's value from 180.

Once we got $\angle E = 127°$ from subtracting $53°$ from 180, that is our $\angle E$ for our second box (solution). Using this, I solve for $\angle N$ but this time with a different value of $\angle E$, therefore yielding $\angle N = 19°$. After that, I used both angles from the second box (solution) to find that $n = 4.11$.

Example 3.5: An Oblique Triangle with No Solution

Imagine a given obtuse triangle $\triangle PAI$. We consider that $A = 120°$. Consider also that $a = 5$ and $p = 9.8$. Determine the solution of this obtuse triangle, if it exists (foreshadowing.)

solutions

Quickly we note that our given angle is obtuse. We note that, if there is no solution, the opposite side is less than the adjacent side given.

$$5 < 9.8$$

This is true. Therefore, our triangle has no solution. How can we say so? Take note of this remark.

> *Remark* (Range (potential output) for Sine).
>
> The output of $\sin x$ is only within the interval $-1 < \sin x < 1$ or $\sin x$ is always bigger than -1 but less than 1. This means that $\sin x$ can not be bigger than 1 or less than -1.

Try it on your calculator. Input $\sin^{-1}$ and then a value like 1.0001 or anything bigger than 1. It should receive an error. If not, take your calculator to the clinic.

How does this relate with our law of sines method?

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin P}{p} \\ \frac{\sin 120}{5} &= \frac{\sin P}{9.8} \\ \sin P &= \frac{9.8 \sin 120}{5} \\ P &= \sin^{-1} \frac{9.8 \sin 120}{5} \end{aligned}$$

Try to input this to your calculator. It will not work, because if you calculated $\frac{9.8 \sin 120}{5}$ alone you would get $1.6974\ldots$ and hence, $1 < 1.6974\ldots$, therefore leading the inverse sine function to not work. I do not have a reason why, its more advanced as to why sine can only have outputs within that interval, so I do bear apology.

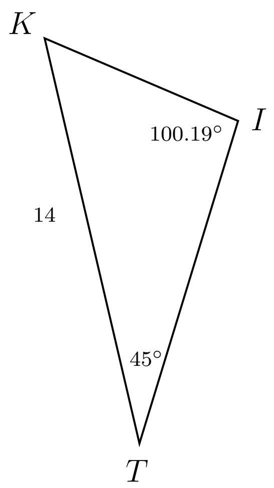
§3.2.2 Exercises

Exercise E

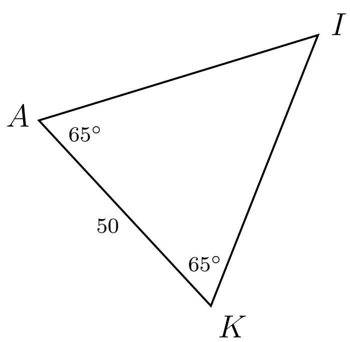
I: This part will involve the law of sines on four illustrated triangles, and six non-illustrated triangles (just the given, no drawing). For every odd number, solve for the indicated side or angle.

If any given angles do not have decimals, solve with angles rounded to the nearest whole number and the sides to two decimal places.

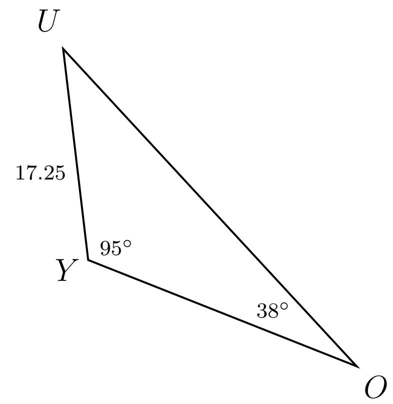
1. $\triangle KIT$. Solve for $\overline{KI}$ or t.



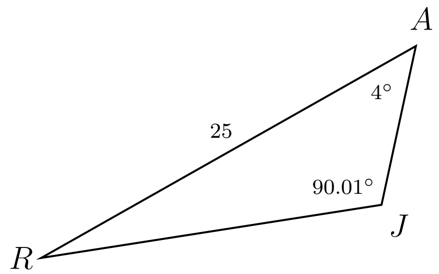
2. Solve for acute triangle $\triangle AIK$.*



3. In scalene triangle $\triangle UYO$, solve for $\overline{YO}$.*



4. Solve for obtuse triangle $\triangle AJR$.



5. Given we have a triangle $\triangle BWU$, solve for $\overline{BU}$ or w wherein $\angle W = 70°$, $\angle U = 100°$, and $u = 510.32$.*

6. Solve for triangle $\triangle JER$, given $\angle J = 54°$, $r = 46$, and $\angle R = 45°$.

7. Solve for $\overline{AK}$ (or b) in $\triangle BAK$ for if $\angle A = 10°$, $K = 150°$, and $k = 9$.*

8. Solve for $\triangle APT$ for if $\angle A = 60°$, $a = 27°$, and $\angle P = 33°$.

9. For $\triangle QED$, solve for $\overline{QE}$ or d if $\angle Q = 57°$, $q = 150$, and $\angle E = 41°$.

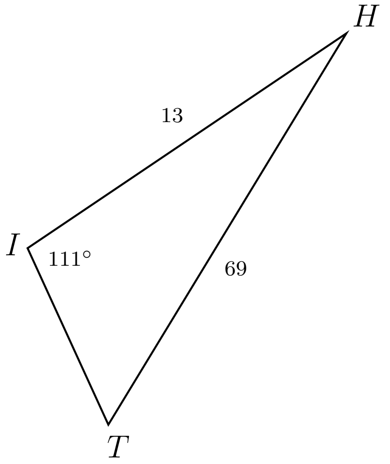
10. Solve for $\triangle AND$ if $\angle N = 139°$, $\angle D = 17°$, and $n = 22$.

II: This part also includes the law of sines, but this time involving the ambiguous case. If the amount of solutions is not given, fill in the blanks. NOTE: For number six, the given already says "**no** solution,=." Show your reason why instead.

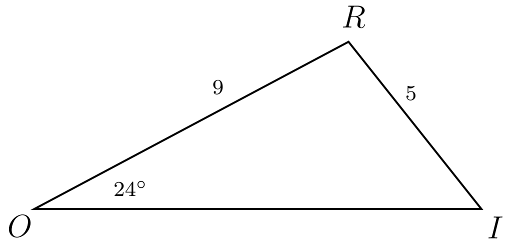
There will be four illustrated triangles, and six non-illustrated triangles. If any given angles do not have decimals, solve the angles rounded to the nearest whole number and the sides to two decimal places.

Solve the following oblique triangles.

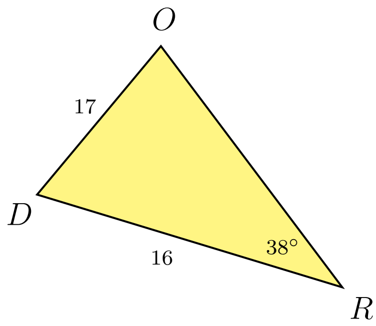
1. There is exactly ______ solution(s).*



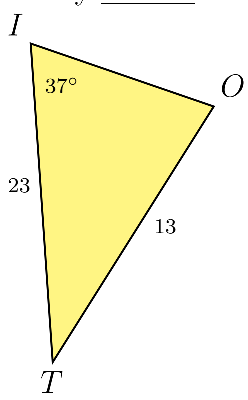
2. There is exactly **two** solutions.



3. There is exactly **one** solution.[a] *Hint: Is the opposite side bigger than the adjacent side?*



4. There is exactly ______ solution(s).



[a] If you know, you know where this dorito comes from.

5. For $\triangle TRY$, $\angle Y = 25°$, $y = 7.65$, and $r = 20$, there is exactly ______ solution(s).
6. For $\triangle FUM$, $\angle F = 129°$, $f = 78$, and $u = 81$, there is exactly **no** solution. Why?
7. For $\triangle BLE$, $\angle E = 60°$, $e = 88$, and $b = 55$, there is exactly ______ solution(s).
8. For $\triangle KYS$, $\angle S = 100°$, $e = 43$, and $s = 12$, there is exactly ______ solution(s).
9. For $\triangle ZER$, $\angle E = 64°$, $e = 33.$ and $z = 98$, there is exactly **two** solutions.
10. For $\triangle QWX$, $\angle Q = 86°$, $q = 74$, and $x = 50$, there is **one** solution.

§3.3 The Law of Cosines

The law of sines only works under ASA, SAA, and SSA (ambiguous case) conditions. For SSS (side-side-side) and SAS (side-angle-side) triangles, we resort to using the **law of cosines**.

Theorem 3.2: The Law of Cosines

Consider a triangle $\triangle ABC$. If there exists angles A, B, C, and opposing sides a, b, and c to their respective angles, then the following is true:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

or, solving for the angle's cosine:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Memorizing the patterns in between the three formulas is helpful to solve a triangle with SSS or SAS cases. There are tricks, but I confess, explaining them is not my forte.

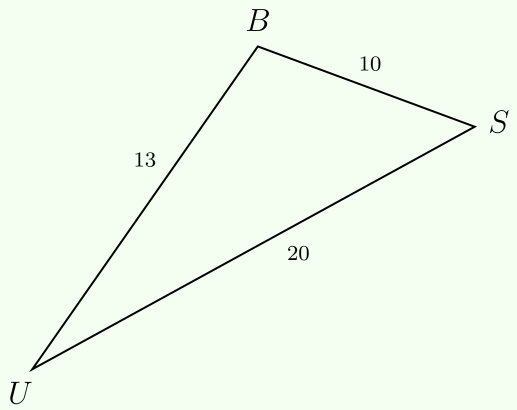
Use the *original* form if you are solving for an indicated side. Use the *derived* form if you are solving for an indicated angle.

§3.3.1 Examples

I know that explanation was brief, but really, it is just it. No word problems unfortunately, I am not skilled to do such a thing.

Example 3.6: An SSS Example

Solve for $\angle B$, $\angle U$, and $\angle S$.



solutions

If you are finding all three angles, you just need to find two first, then subtract them from 180 to get your final angle. Start by solving for B, my pick.

$$\begin{aligned}
\cos B &= \frac{s^2+u^2-b^2}{2su} \\
\cos B &= \frac{13^2+10^2-20^2}{2(13)(10)} \\
\cos B &= \frac{-131}{260} \\
\angle B &= \cos^{-1}\frac{-131}{260} \\
\angle B &\approx \boxed{120^\circ}
\end{aligned}$$

Now try to find $\angle U$. Same thing, but different values.

$$\begin{aligned}
\cos U &= \frac{b^2+s^2-u^2}{2bs} \\
\cos U &= \frac{20^2+13^2-10^2}{2(20)(13)} \\
\cos U &= \frac{469}{520} \\
\angle U &= \cos^{-1}\frac{469}{520} \\
\angle U &\approx \boxed{26^\circ}
\end{aligned}$$

Then, subtract the angles for $\angle S$.

$$\angle S = 180 - 26 - 120 = \boxed{34^\circ}$$

Example 3.7: An SAS Example w/ No Illustration

Given a triangle $\triangle XYZ$, $\angle Y = 131^\circ$, $x = 76$, and $z = 90$, find the missing parts.

solutions

Start by solving for the missing side. Use the original form, not the derived form for cosine of that angle.

$$\begin{aligned}
y^2 &= x^2 + z^2 - 2xz\cos Y \\
y^2 &= 76^2 + 90^2 - 2(76)(90)\cos 131 \\
y^2 &\approx 13876.66 \\
\sqrt{y^2} &= \sqrt{13876.66} \\
y &\approx \boxed{117.8}
\end{aligned}$$

Then solve for one of the missing angles using the derived form:

$$\begin{aligned}
\cos X &= \frac{y^2 + z^2 - x^2}{2yz} \\
\cos X &= \frac{117.8^2 + 90^2 - 76^2}{2(117.8)(90)} \\
\cos X &\approx 0.76 \\
\angle X &= \cos^{-1} 0.76 \\
\angle X &\approx \boxed{41^\circ}
\end{aligned}$$

Find the third angle by subtracting both of them from 180.

$$\angle Z = 180 - 131 - 41 = \boxed{9^\circ}$$

Outrageous numbers. I had to approximate 0.76 because my calculator (Casio 570ES) would not show the fraction form. I do not know why.

§3.3.2 Exercises

Exercise F

I will not provide any illustrations. I will assume that you already know how to visualize the triangle given, or how to start solving without an illustration. Of course, illustration is not banned. Again, round the angles to the nearest whole number and the sides to two decimal places.

1. For a triangle $\triangle UPD$, solve for the triangle if $\angle U = 66°$, $p = 19$, and $d = 20$.
2. For a triangle $\triangle PUT$, solve for the triangle if $p = 13.22$, $u = 66.6$, and $t = 6.66$.
3. For a triangle $\triangle DOG$, solve for the triangle if $d = 6.9$, $p = 69$, and $g = 69.9$.
4. For a triangle $\triangle BAD$, solve for the triangle if $\angle B = 42°$, $a = 420$, and $d = 4.22$.
5. For a triangle $\triangle ING$, solve for the triangle if $\angle I = 97°$, $n = 21$, and $g = 15$.
6. For a triangle $\triangle OSU$, solve for the triangle if $o = 60$, $s = 50$, and $u = 40.16$.
7. For a triangle $\triangle KHR$, solve for the triangle if $\angle K = 51.9°$, $h = 109$, and $r = 14$.
8. For a triangle $\triangle SUP$, solve for the triangle if $s = 32$, $u = 42$, and $p = 29$.
9. For a triangle $\triangle TUX$, solve for the triangle if $t = 18$, $u = 40$, and $x = 5$.
10. For a triangle $\triangle GRQ$, solve for the triangle if $\angle G = 167°$, $r = 56$, and $q = 70$.

Appendices

§A Rounding Off Decimals

Rounding off a decimal involves two criteria:

1. If the number after is less than five, do not add one and keep it as is.
2. If the number after is greater than five, add one.

For example:

- To three decimal places, 19.45719 is 19.457.
- To the first decimal place, 0.691 is 0.70, or 0.7.
- To the nearest whole number, 1.956761123 is 2.

Numbers use the ones, tens, hundreds, thousands place for numbers. However, decimals start at the tenths, hundredths, thousandths, etc.

1. The tenths place in a decimal is the first number in the decimal part
2. The hundredths place in a decimal is the second number in the decimal part
3. The thousandths place in a decimal is the third number in the decimal part

and so on. For example, in 3.1415,

- 1 is the tenths place.
- 4 is the hundredths place.
- 1 is the thousandths place.
- 5 is the ten thousandths place.

§B Solutions to Selected Exercises

Exercise A

2.

$\sin X$	$\frac{\sqrt{5}}{3}$	$\csc X$	$\frac{3\sqrt{5}}{5}$
$\cos X$	$\frac{2}{3}$	$\sec X$	$\frac{3}{2}$
$\tan X$	$\frac{\sqrt{5}}{3}$	$\cot X$	$\frac{3\sqrt{5}}{5}$

$\sin C$	$\frac{2}{3}$	$\csc C$	$\frac{3}{2}$
$\cos C$	$\frac{\sqrt{5}}{3}$	$\sec C$	$\frac{3\sqrt{5}}{5}$
$\tan C$	$\frac{2\sqrt{5}}{5}$	$\cot C$	$\frac{\sqrt{5}}{2}$

Exercise B

I:

5.

$$\begin{aligned}
\sqrt{(2)\left(\frac{\sqrt{2}}{2}\right)(\sqrt{2})x} &= x+1 \\
\sqrt{2\cdot\frac{2}{2}\cdot x} &= x+1 \\
\sqrt{2x} &= x+1 \\
(\sqrt{2x})^2 &= (x+1)^2 \\
2x &= x^2+2x+1 \\
x^2+1 &= 0 \\
x^2 &= -1 \\
\sqrt{x^2} &= \sqrt{1} \\
x &= \pm i
\end{aligned}$$

$\pm i$ is not a real number, therefore no real solutions exist.

6.

$$
\begin{aligned}
x + \sqrt{\frac{1}{\frac{\sqrt{2}}{2}} \cdot 2\sqrt{2} \cdot x - x} - 2(\sqrt{3})^2 &= 0 \\
x + \sqrt{\frac{\sqrt{2}}{2} \cdot 2\sqrt{2} \cdot x - x} - 2 \cdot 3 &= 0 \\
x + \sqrt{2x - x} - 6 &= 0 \\
x + \sqrt{x} - 6 &= 0 \\
\sqrt{x} &= -x + 6 \\
(\sqrt{x})^2 &= (-x + 6)^2 \\
x &= x^2 - 12x + 36 \\
x^2 - 13x + 36 &= 0 \\
(x - 9)(x - 4) &= 0 \\
x &= 9, 4
\end{aligned}
$$

9 is an extraneous root. Therefore the answer is 4.

Exercise C

No illustrations nor any unit of measurement.

I:

1.

$$
\begin{aligned}
\tan 34 &= \frac{2.7}{x} \\
x \tan 34 &= 2.7 \\
x &= \frac{2.7}{\tan 34} \\
x &\approx 4
\end{aligned}
$$

6.

$$
\begin{aligned}
\tan 41.6 &= \frac{3.3}{x} \\
x \tan 41.6 &= 3.3 \\
x &= \frac{3.3}{\tan 41.6} \\
x &\approx 3.72
\end{aligned}
$$

II:

7.

$$\begin{aligned}\tan 87 &= \frac{5}{x} \\ x\tan 87 &= 5 \\ x &= \frac{5}{\tan 87} \\ x &\approx 0.26\end{aligned}$$

8.

$$\begin{aligned}\tan 30 &= \frac{x}{17} \\ x &= 17\tan 30 \\ x &\approx 9.81\end{aligned}$$

Exercise D

2. Recall the 45-45-90 triangle. If so, then automatically $\angle S = 45°$ and $\angle Y = 45°$. Just find the hypotenuse.

$$\begin{aligned}\cos S &= \frac{7.1}{h} \\ \cos 45 &= \frac{7.1}{h} \\ h\cos 45 &= 7.1 \\ h &= \frac{7.1}{\cos 45} \\ h &\approx 10.04\end{aligned}$$

Exercise E

I:

2. Find the third angle first.

$$180 - 65 - 65 = 50$$

Solve.

$$\begin{aligned}\frac{k}{\sin K} &= \frac{i}{\sin I} \\ \frac{k}{\sin 65} &= \frac{50}{\sin 50} \\ k &= \frac{50\sin 65}{\sin 50} \\ k &\approx 60\end{aligned}$$

For the other side:

$$\begin{aligned}
\frac{k}{\sin K} &= \frac{a}{\sin A} \\
\frac{60}{\sin 65} &= \frac{a}{\sin 65} \\
a &= \frac{60\cancel{\sin 65}}{\cancel{\sin 65}} \\
a &= 60
\end{aligned}$$

Using another pair would lead to rounding errors as this is an isosceles triangle[8]

3. Start by finding the third angle.

$$180 - 95 - 38 = 47$$

Solve.

$$\begin{aligned}
\frac{u}{\sin U} &= \frac{o}{\sin O} \\
\frac{u}{\sin 47} &= \frac{17.25}{\sin 38} \\
u &= \frac{17.25 \sin 47}{\sin 38} \\
u &\approx 20.49
\end{aligned}$$

5. Solve.

$$\begin{aligned}
\frac{w}{\sin W} &= \frac{u}{\sin U} \\
\frac{w}{\sin 70} &= \frac{510.32}{\sin 100} \\
w &= \frac{510.32 \sin 70}{\sin 100} \\
w &\approx 486.94
\end{aligned}$$

7. Find the third angle.

$$180 - 150 - 10 = 20^\circ$$

Solve.

$$\begin{aligned}
\frac{k}{\sin K} &= \frac{b}{\sin B} \\
\frac{9}{\sin 150} &= \frac{b}{\sin 20} \\
b &= \frac{9 \sin 20}{\sin 150} \\
b &\approx 6.16
\end{aligned}$$

[8] A property an isosceles triangle has is that it has equal base angles.

Exercise E

II:

2. There is exactly **one** solution. The opposite side of $\angle I$ is greater than its adjacent, $69 > 13$.

$$\begin{aligned}
\frac{\sin T}{t} &= \frac{\sin I}{i} \\
\frac{\sin T}{13} &= \frac{\sin 111}{69} \\
\sin T &= \frac{13 \sin 111}{69} \\
\angle T &= \sin^{-1} \frac{13 \sin 111}{69} \\
\angle T &\approx 10
\end{aligned}$$

Find the third angle.

$$180 - 111 - 10 = 59$$

Find the missing side.

$$\begin{aligned}
\frac{h}{\sin H} &= \frac{t}{\sin T} \\
\frac{h}{\sin 59} &= \frac{13}{\sin 10} \\
h &= \frac{13 \sin 59}{\sin 10} \\
h &\approx 64.17
\end{aligned}$$

A Letter

To the dearest reader, you have come across the final guide for this school year. I hope you enjoyed your journey to get here. The last minute you spent reading at least one sentence or paragraph of my work always brings me joy, and I hope all that I have done helped some way. I dearly wish for you to bring onward the upcoming to be the brightest and most vivid era of your life. Especially for me, many teachers will soon end contracts and my class now has to adjust. Despite hurdles, there is always a fix or at least a fine-tuned solution. I am glad to be here, and I am glad that you are too.