Robotic Arm Dynamics Simulation - A First Principles Analysis

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Contents

1 Abstract

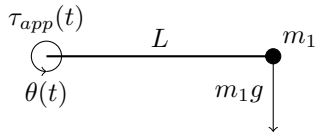
This paper provides an explanation of how to model robotic arm joints; it requires readers to possess knowledge of Newtonian mechanics and ordinary differential equations. It begins with the simple 1-joint problem, which models the behavior of one mass rotated by a motor. Next, the model is generalized for a system with n joints, the robot arm.

2 Introduction

Robotic arms are a useful part of modern industry. Most of the products we own have been touched at one time or another by a robotic arm during their manufacturing and shipping processes. Many people are captivated by the lifelike movements of these machines. Robotic arms have therefore become a common project for hobbyists and aspiring engineers. People in these circles enjoy the challenge of designing, building, and programming a robot arm themselves. While extensive modeling and simulation are done for robotic arms in industry, hobbyists tend to shy away from simulation. This makes the transition from the hobby world to the industrial one difficult. With a comprehensive and simple explanation of robotic arm modeling, starting from first principles, they will become accustomed to mathematical modeling and may come to find it enjoyable. For those taking their first differential equations course after completing Calculus 2, this paper demonstrates how a nonlinear differential equation has complex emergent properties, which have not yet been seen in our mathematical careers. Differential equations is a field where the dynamics of an equation can be explored, as opposed to being solved analytically or memorized.

3 The 1-joint problem

In this section, the equations of motion for a single motor, with a single mass, will be derived. The motor applies a torque $\tau_{app}(t)$ on a massless rod of length L. A point mass with a mass m_1 is connected to the end of the rod.



3.1 Deriving the equations of motion for the 1 joint problem

The angular acceleration of the system can be determined with Newton's second law of rotational motion: $\sum_i \tau_i = I\alpha$. We can determine the net torque and rotational inertia for our particular system:

$$\sum_i \tau_i = \tau_{app}(t) + \tau_{m_1 g}(t)$$

The torque applied by m_1 can be determined using the torque formula: $\tau_{m_1 g}(t) = L * F * cos(\theta)$. The net torque is therefore:

$$\sum_i \tau_i = \tau_{app}(t) - m_1 g L cos(\theta(t))L$$

The rotational inertia term, I, can be rewritten as $m_1 * L^2$, and the angular acceleration can be turned into the second derivative of the angular position. After reorganizing terms, we get:

$$\theta''(t) = \frac{\theta_{app}(t) - m_1 g L cos(\theta(t))}{m_1 L^2}$$

Adding a friction term, B, changes the equation slightly. This term is approximate, as friction cannot be modeled with a constant and the $\theta'(t)$ term should in reality be a $(\theta'(t))^2$ term. For low velocities, a $\theta'(t)$ term is sufficient.

$$\theta''(t) + B\theta'(t) = \frac{\theta_{app}(t) - m_1 g L cos(\theta(t))}{m_1 L^2}$$

We'd like to find an analytical solution to this equation, where $\theta(t) = ...$, but, it is impossible in this case. This is not an issue, however, as numerical methods can be used. To solve numerically, is helpful to disintegrate this equation into a system of first-order equations. This is done with a simple variable substitution:

$$\theta = y_1 \ \theta' = y_2$$

$$y_1' = y_2 \qquad y_2' = \frac{\tau_{app}(t) - m_1 g L cos(y_1)}{m_1 L^2} - B y_2$$

We will use this system of two first-order differential equations to obtain the numerical solution.

3.2 Programming the 1 joint problem

While the code can be accessed on GitHub here, explanations of the main parts of the code are in this document.

To begin, simulation parameters are defined.

```
L = 1 #in meters, also known as L
B = 0.2 #rotational friction
Mass_1 = 1 # in kg, also known as M
motor_torque = 10 # in Nm, also known as Tq(t)
grav = 9.81 # m/s^2
simulationDuration = 20 # seconds

#initial conditions
initial_angle = 0 #rad
initial_angular_velocity = 0 #rad/s
initial_values = [initial_angle, initial_angular_velocity]
```

Next, the differential equation is turned into a function that takes in the state of the system: $\vec{Y}$, and outputs its derivative: $\vec{Y}'$. The input vector has components $y_1 = Y[0]$ and $y_2 = Y[1]$. This defines the behavior of the system.

```
def slopes(t, Y):
    return([Y[1],
           (motor_torque - Mass_1*grav*L*np.cos(Y[0])) / (Mass_1*L**2) -
                B*Y[1]
           ])
```

The function is passed through solve_ivp, which solves the differential equation numerically. In this case, it uses a Runge Kutta 7-8 solver. The initial conditions are also plugged in, in vector form.

```
sol = solve_ivp(slopes, [0, simulationDuration], initial_values, t_eval
    =T, method = 'DOP853', rtol=1e-8, atol=1e-8)

for x in range(len(sol.y[0])):
    print("position: " + str(sol.y[0, x]) + "velocity: " + str(sol.y[1,
         x]))
```

Finally, the solution is animated. The angular position of the system over time is converted to an easy-to-plot vector of Cartesian coordinates. These are fed through an animation of the 1-joint problem.

```
x1 = np.cos(sol.y[0]) * L #convert angular solution coordinates to
    cartesian for animating
y1 = np.sin(sol.y[0]) * L

print(len(x1)) # prints how many frames the simulation has

def animate(i):
    x = [0, x1[i]]
    y = [0, y1[i]]
    line.set_data(x, y) # make a line between (0,0) and (x1[i], y1[i])
    dots.set_data(x, y) # make dots at (0,0) and (x1[i], y1[i])
    txt.set_text('Time = ' +
    '{:4.1f}'.format(i*(simulationDuration/len(x1))) +
    's' + "  Velocity: " + '{:4.1f}'.format(sol.y[1,i])
    + " rad/s") # display time and velocity
    return line, dots, txt

# call the animator.
anim = animation.FuncAnimation(fig, animate, init_func=init,
frames=len(x1), interval=10, blit=True)
# each frame iterates i by one. There is a 10ms delay between frames.
plt.show()
```

3.3 Some Examples...

Before moving on to multi-joint systems, let's investigate the properties of the one-joint system. The differential equation for this system provides valuable insight not understandable with a formulaic understanding of mechanics. Note that the arrows in the diagrams are normalized.

No torque When the motor has no torque, the system behaves as a simple pendulum. It creates the characteristic whirlpool vector field.

System property	
Length (L)	1 m
Mass1 (m_1)	1 kg
Torque	0 Nm
Friction	0.2

The whirlpools happen every 2π radians, and they are centered around -$\pi/2$, where the arm is vertically downwards. When the system is in the whirlpool, it is oscillating back and forth as a pendulum. If there is enough initial velocity, such as on the green Initial Value Problem, the pendulum makes a few full rotations before entering a whirlpool.

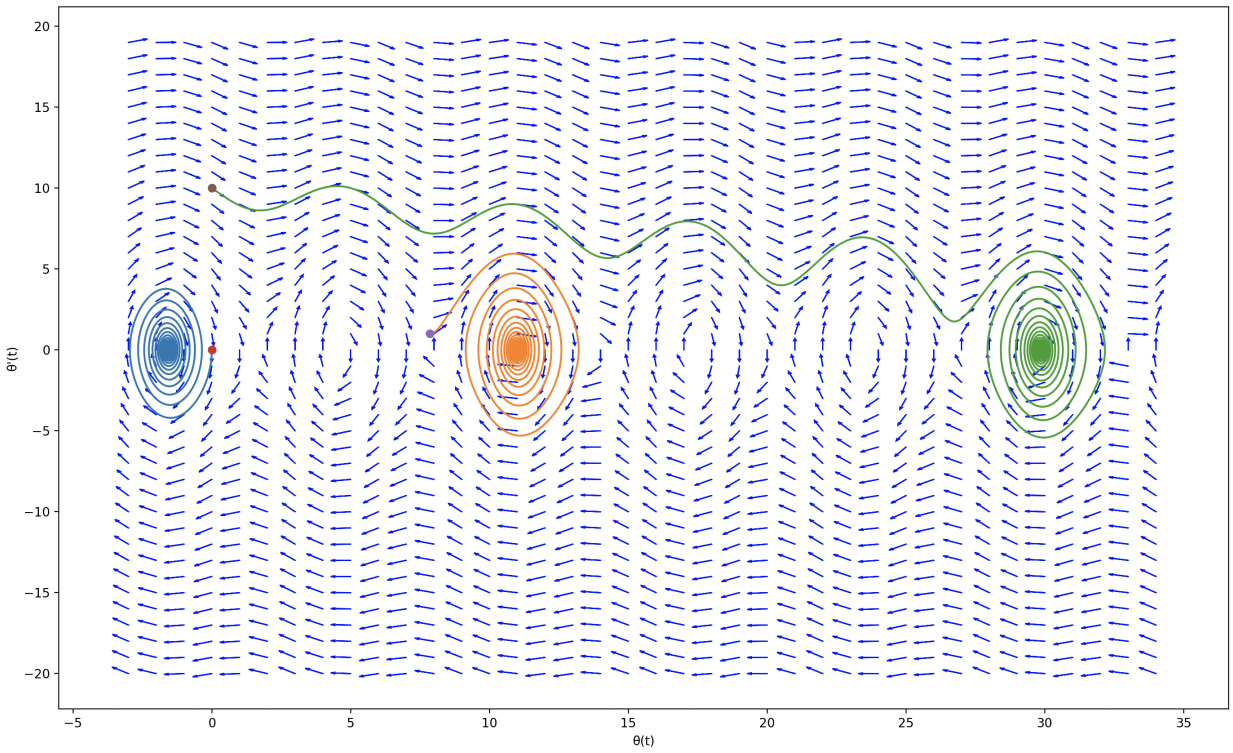


Figure 1: Vector field of a pendulum

Damped-driven 1-joint problem Here, the mass is acted on by a torque of 50Nm. This is just enough to make it overcome the force of gravity ($9.81N * 5m = 49.05Nm$ when the bar is horizontal). The mass accelerates until it reaches $4 \pm 0.5 \frac{rad}{s}$. This is the terminal velocity of the system, where the energy input by the motor is consumed entirely by friction. The steady-state velocity will vary between 4 and 5 radians per second, depending on whether the motor is fighting gravity or working with it.

System property	
Length (L)	5 m
Mass1 (m_1)	1 kg
Torque	50 Nm
Friction	0.5

3 cases are plotted on this diagram. The first (in blue) shows a bar with zero velocity starting in a horizontal position. It accelerates slowly until it reaches a quasi-sinusoidal steady state. The second (in green) shows a negative initial velocity, while the bar is still horizontal. The bar decelerates until it reaches zero velocity, then it climbs to the same steady state as the first case. The third case starts with a velocity slightly above the steady state. It decreases slowly and approaches the steady state.

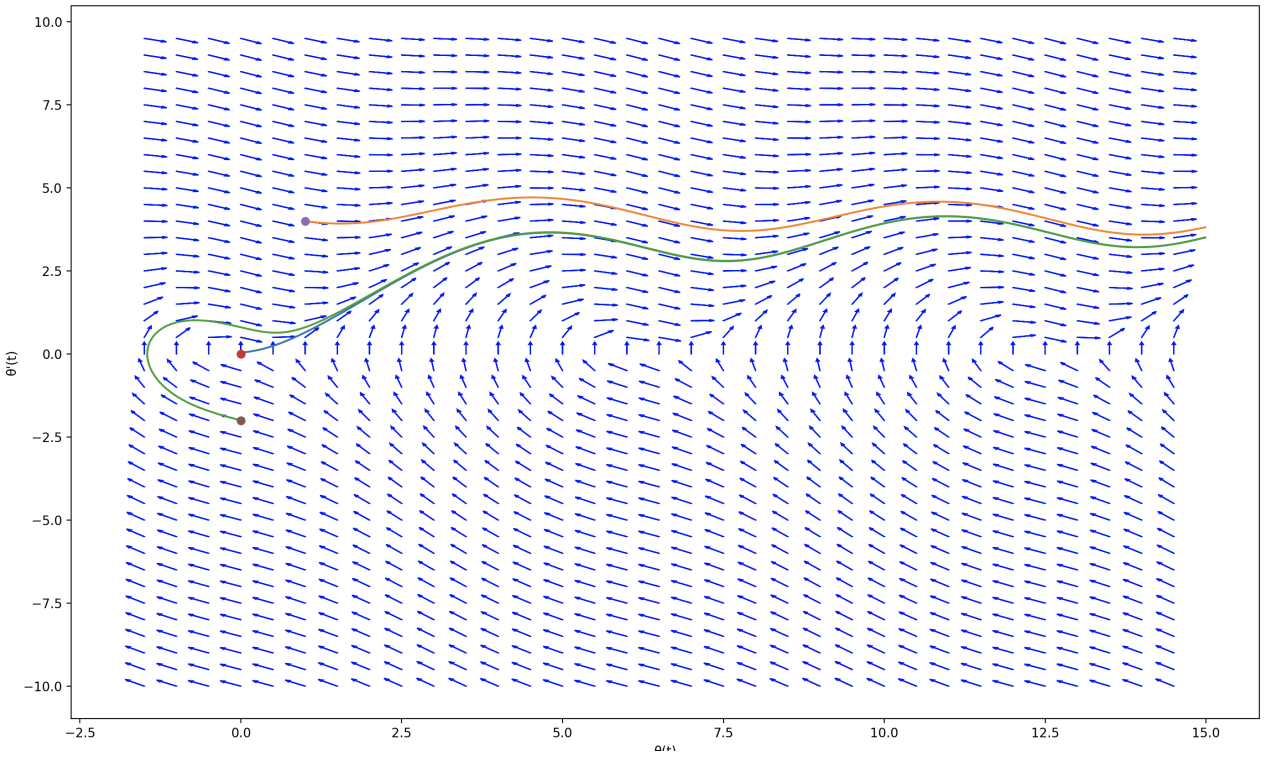


Figure 2: Vector field of a driven system

Under-Powered Motor When the motor does not have enough power to counter the force of gravity, it will not start spinning the mass. However, the wells of the vector field can be escaped with enough potential or kinetic energy. This will result in the mass rotating forever. Notice that the centers of the wells are slightly ahead of $-\pi/2 + 2\pi n$, at an angular velocity of zero. When the system is in the center of a well, the torque exerted by the motor and the torque exerted by gravity cancel out. The arm is left skewed from the vertical.

System property	
Length (L)	1 m
Mass1 (m_1)	1 kg
Torque	4 Nm
Friction	0.2

On this plot, three initial positions are set. The orange and blue cases do not have enough initial potential energy to escape the well. The orange case starts vertically downwards, while the blue one starts at $-3\pi/4$ radians. The green line starts at $-\pi$ radians and manages to escape the well. Once it does, the mass rotates forever.

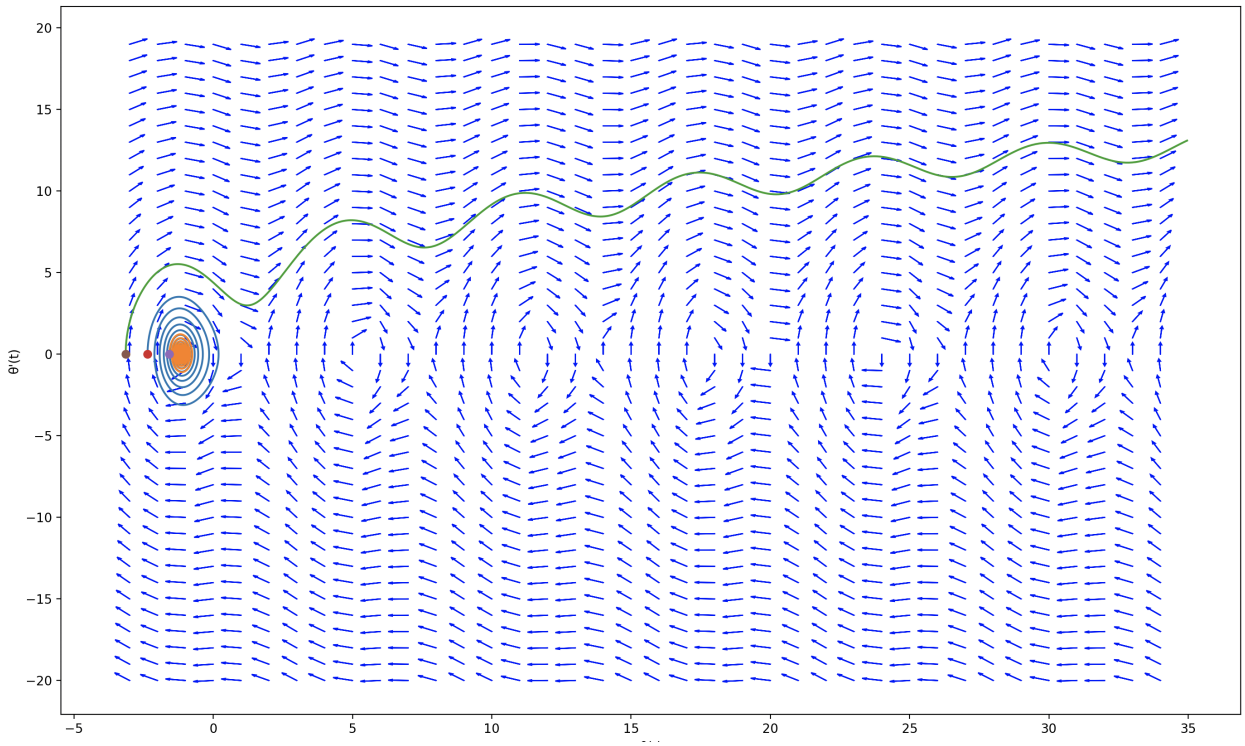


Figure 3: Vector field of an under-powered system, with differing initial positions

On this plot, three initial velocities are set, while the initial position is always horizontal. The orange and blue cases do not have enough initial kinetic energy to clear the vertical and escape the well. The orange spiral remains relatively round and the blue spiral displays an anomaly when it first falls into the well. It almost clears the vertical, but has slightly too little energy to do so, so it slowly falls back down into the well. With a slightly higher initial velocity, the green line clears the well and rotates perpetually.

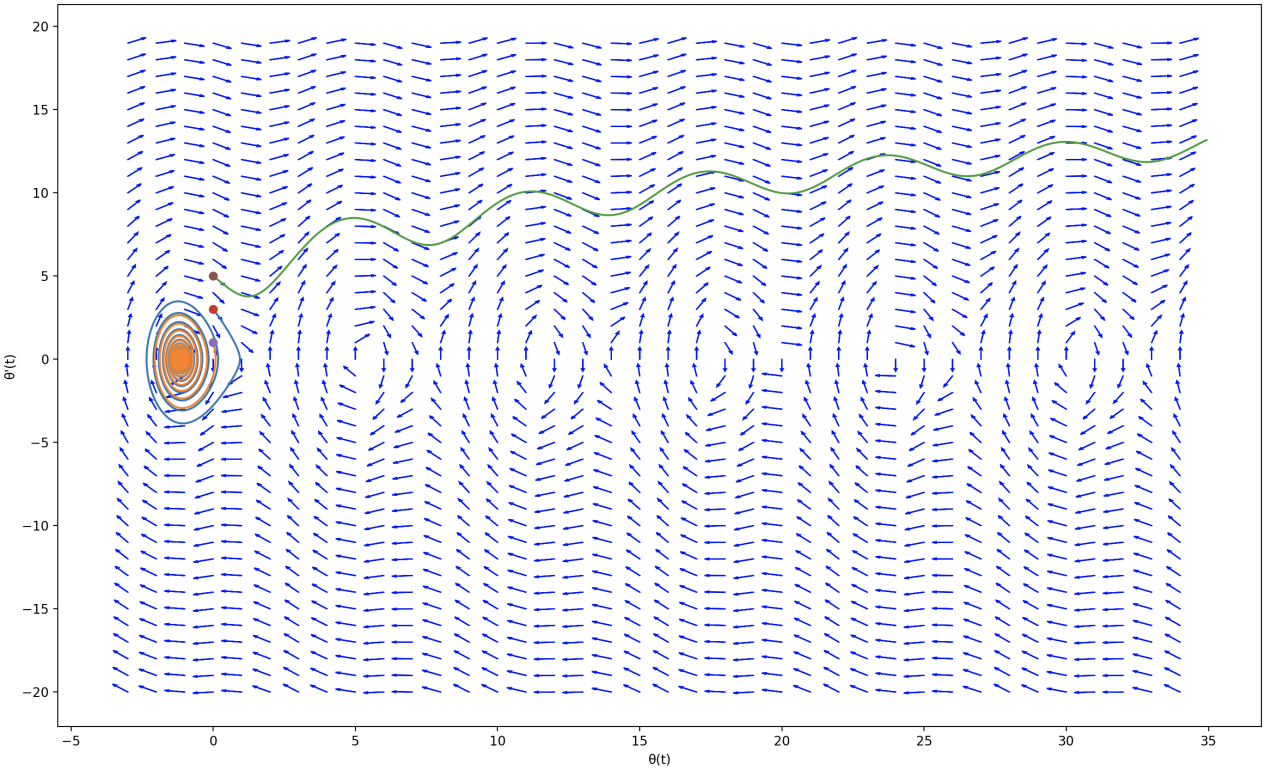


Figure 4: Vector field of an under-powered system, with differing positive initial velocities

When the initial velocity is negative, the torque of the motor opposes the direction of motion, and the system decelerates. Depending on its trajectory, the system will either fall into a well, or it will cross the x-intercept only once (come to a full stop). If it crosses the x-intercept once, it operates with the same dynamics as Figure 3. Notice that the orange and green trajectories avoid the well and cross the x-intercept. Their x-intercept positions are such that they have enough potential energy to begin spinning. The blue trajectory falls into a well.

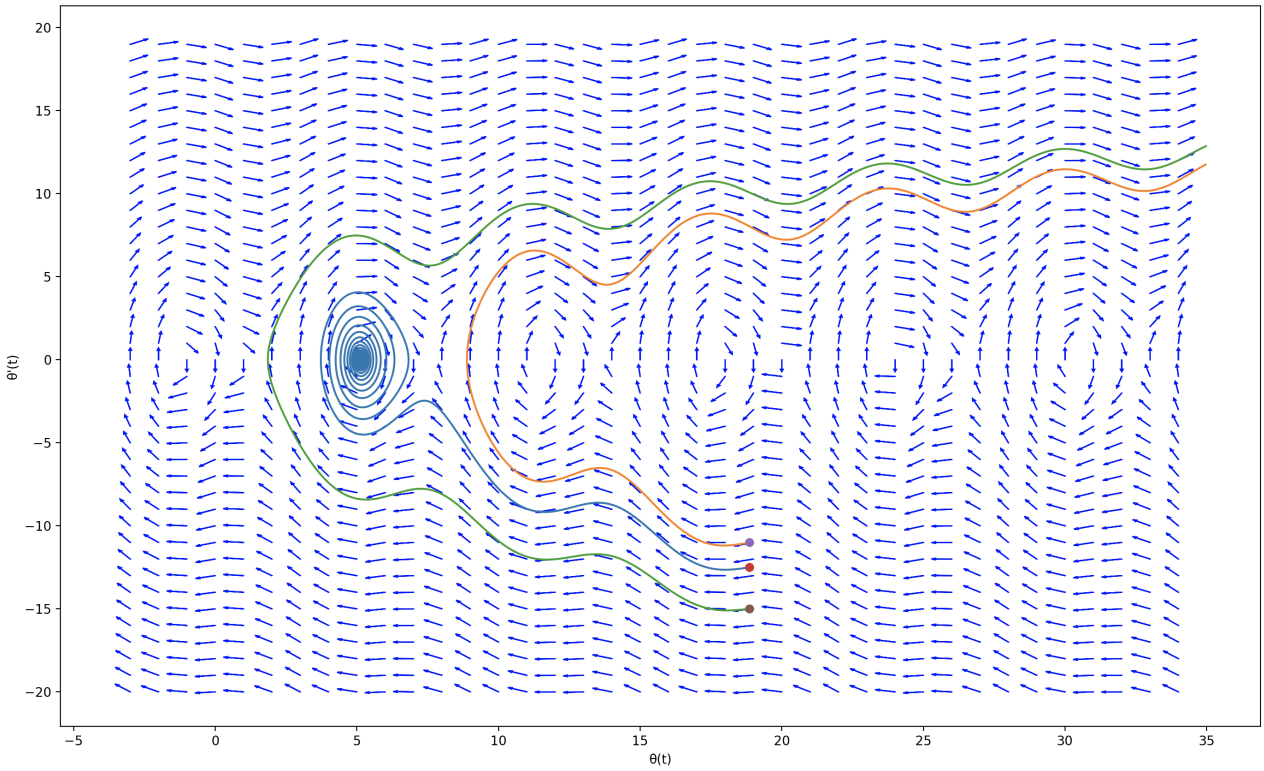
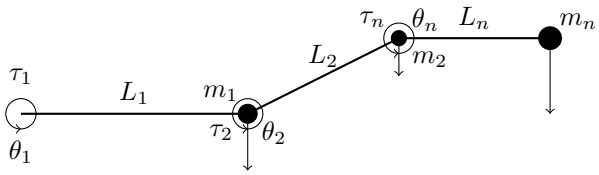


Figure 5: Vector field of an under-powered system, with differing negative initial velocities

4 The robot arm

This section covers the derivation and simulation of the equations of motion for an arm with an arbitrary number of joints. The robot arm is modeled as a collection of 1-joint problems organized sequentially:



Note that $\tau_1 = \tau_1(t)$. Time dependencies have been removed for simplicity.

4.1 Deriving the equations of motion for an arbitrary number of sequential 1-joint problems

Rotational Inertia Finding the rotational inertia and the torque exerted by gravity becomes complicated when more joints are involved. These quantities need to be computed with respect to each joint. For joint a in the system:

$$I_a = m_n L_{a,mn}^2 + m_{n-1} L_{a,mn-1}^2 ... + m_a L_a^2 \ ,$$

where I_a is the inertia calculated with respect to joint a, and $L_{a,mn}$, is the distance from joint a to the last mass. $L_{a,mn-1}$ is the distance from joint a to the second last mass. Only masses past the joint count toward its rotational inertia.
The lengths used in the inertia formula are computed by treating the 1 joint problems downstream of the joint as vectors of magnitude L_i and direction θ_i. These vectors can be summed to calculate the net vector. The vector $\vec{u}_{a,b}$ is computed from joint a (which we're calculating the inertia WRT to) to joint b, a joint we're trying to find the distance to. $\vec{u}_{a,b}$'s magnitude is $L_{a,mb}$.

$$\vec{u}_{a,b} = J_a + J_{a+1} + ... + J_b,$$

$$\text{where } J_a = \begin{bmatrix} L_a cos(\theta_a) \\ L_a sin(\theta_a) \end{bmatrix} \qquad |\vec{u}_{a,b}| = L_{a,b} \qquad \hat{u}_{a,b} = \theta_{a,b}$$

L_a denotes the length of joint a, and θ_a denotes the angle of joint a.
The rotational inertia with respect to a joint in the system is computed from this. It is computed starting at the last joint, and ending at joint a. The closest joint's inertia is not computed by vector sum, since it is just one segment.

$$I_a = m_n|\vec{u}_{a,n}|^2 + m_{n-1}|\vec{u}_{a,n-1}|^2 + ... + m_{a+1}|\vec{u}_{a,a+1}|^2 + m_A L_A^2$$

m_n is the last joint in the arm, m_{n-1} is the second last, and so on. $|\vec{u}_{a,a+1}|$ is the magnitude of a vector between joint a and joint a+1.

torque applied by gravity With the quantities just developed, the torque applied by gravity WRT to joint a can be computed:

$$\tau_a = m_a * L_a * cos(\theta_a) * g + m_{a+1} * L_{a,a+1} * cos(\theta_{a,a+1}) * g + ... + m_n * L_{a,n} * cos(\theta_{a,n}) * g$$

Here $\theta_{a,b}$ is the direction of $|\vec{u}_{a,b}|$. This tells us the angle at which the force of gravity is applied to the joint. This is used to compute the effect that gravity has on the torque of the joint.

Equations of motion The equations of motion of this system are a $2n$th order differential equation. Since this equation is nonlinear, its matrix form is quite messy. It will instead be written as a system of first-order equations:

$$\theta_1 = y_1 \quad \theta_1' = y_2 \quad \theta_2 = y_3 \quad \theta_2' = y_4 \quad .... \quad \theta_n = y_{2n-1} \quad \theta_n' = y_{2n}$$

$m_1 g L cos(y_1)$ and $m_1 L^2$ used in the 1-joint problem are now τ_1 and I_1, which are functions of $\theta_1, \theta_2, ..., \theta_n$

$$y_1' = y_2 \qquad y_2' = \frac{\tau_{app1}(t) - \tau_1}{I_1} - B y_2$$

$$y_3' = y_4 \qquad y_4' = \frac{\tau_{app2}(t) - \tau_2}{I_2} - B y_4$$

$$...$$

$$y_{2n-1}' = y_{2n} \qquad y_{2n}' = \frac{\tau_{appn}(t) - \tau_n}{I_n} - B y_{2n}$$

4.2 Programming and simulating the robot arm

A function that calculates the rotational inertia and torque applied by gravity with respect to each joint is defined:

```
def slopes(t, Input):     # Y is a vector of shape [x1, x2,..., xn]
    output = np.zeros(sn*2)

    #computes the distance between joints in an arm
    for joint_computed in range(sn):  #iterate joints
        rotational_inertia = 0 #rotational inertia WRT curr. joint
        net_torque_on_joint = 0 #net torque WRT curr joint
        for mass_computed in range((sn-1),(joint_computed-1),-1):
            lx, ly = 0, 0
            dist = 0
            #the vector sum
            for y in range(joint_computed, (mass_computed + 1)):
                lx = lx + Lengths[y]*np.cos(Input[y*2])
                ly = ly + Lengths[y]*np.sin(Input[y*2])
            dist = np.sqrt(lx**2 + ly**2) #The distance from Joint A to
                 B
            #The rotational inertia of the point mass computed is added
                 to the system:
            rotational_inertia += Masses[mass_computed] * dist**2
            net_torque_on_joint += Masses[mass_computed]*grav*dist*(lx/
                dist)
            # lx/dist = adj/hyp = cos
        #take the inertia and torques calculated, and use them to
            compute the ODE:
        output[joint_computed*2] = Input[2*joint_computed + 1]
        output[joint_computed*2 +1] = ((Torques[joint_computed] -
            net_torque_on_joint) / rotational_inertia)  - Bs[
            joint_computed]*Input[2*joint_computed + 1] #x2'
    return(output)
```

Note: sn is the number of segments in the arm

The simulation parameters are now vectorized, and they can be of an arbitrary length:

```
sn = 4

Lengths = [2, 1, 0.75, 2] # lenghths (m)
Bs = [0.5, 0.5, 0.5, 0.25] # friction terms
Masses = [3, 5, 1, 3] # mass terms (kg)
Torques = [300, 3, 50, 5] # torque terms (Nm)
Init_values = [0.1,0.3,0.1,0.3,0.1,0.3,0,0] # position1, velocity1,
    position2, velocity2, ....., positionn, velocityn

grav = 9.81
simulationDuration = 20 # seconds
```

The animation code is very similar to the 1-joint problem. The full code can be found here

4.3 Simulations

The n-joint arm exhibits interesting behavior. It can be played with extensively using the source code here

Inertia demo A 2-joint system can be used to model how an ice skater's angular velocity changes when she tucks her arms in. In this simulation, the friction terms have been exaggerated so that the effect of changing inertia on the system is more visible. The system has these parameters:

System property	Joint 1	Joint 2
Length	1.0 m	0.7 m
Mass	1 kg	3 kg
Torque	45 Nm	45 Nm
Friction	1.2	2

From this video, it is clear that the angular velocity of the central joint increases when the outer joint points inwards.

4.4 Torque as a function

Torque can easily be passed through the solver as a function of time. This is accomplished by turning the static torque array into a small function. This function then replaces the array in the differential equation input/output function.

```
def compute_torque(t):
    Torques = [100*t, 0, 0, 0]
    return(Torques)

# this function replaces the otherwise static Torques[x] wherever it is
     found in the I/O function.
```

5 Conclusion

The dynamics of a robot arm on a 2D plane have been explained and modeled in this paper. The code and theory developed here can be used in any robot arm project. The final version of the code, which models the dynamics of an arm with an arbitrary number of joints, can be found here. There are a few applications for this simulation code. It can be used to train machine learning algorithms to use robot arms. It can also be used, more practically, in programs that control robot arms. Most robot arms use servo motors to actuate their joints. A model like this one could give the computer a better idea of how the robot arm will move when the current supplied to the servo changes. There remains much work to be done to create a more useful model of the dynamics of an arm. First, a model that describes real motors, instead of the constant torque actuators used in the paper, can be used to model the arm dynamics. Next, a friction model that includes static friction instead of just dynamic friction can be included in the system. Overall, this paper demonstrates how the dynamics of a robot arm can be deconstructed into simple one-joint problems, which are 2nd order differential equations derived from basic laws of physics. The simulation of an arm is a demonstration of how a model based on basic laws, such as $F = ma$ and $\tau_g = mgrsin(\theta)$ can generate intricate emergent behavior. To a reader entering the field of engineering, this paper demonstrates a transition from rudimentary mathematical formulas, with no versatile use in the real world, to real-world models that deepen our understanding of the universe. This will hopefully reveal the beauty of mathematics to hobbyists and makers.

6 References

[1] Ling, S. J., Sanny, J., & Moebs, W. (2021). University physics. OpenStax, Rice University.

[2] Boyce, W. E., DiPrima, R. C. (1986). Elementary differential equations and boundary value problems (Vol. 4). John Wiley & Sons.