Research statement: Relating Fukaya categories using combinatorics, operads, and nonlinear elliptic PDEs

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1 Introduction

I work in the field of symplectic geometry, which is concerned with $2n$-dimensional manifolds M equipped with an area measure ω, and with the n-dimensional submanifolds on which ω vanishes, called **Lagrangians**. Symplectic geometry originated with classical mechanics: the phase space of a Hamiltonian system is a symplectic manifold, and if there exist n independent commuting conserved quantities, then the level sets are Lagrangians.

Many symplectic concepts can be formulated as Lagrangians: for instance, a smooth map $\varphi\colon (M, \omega_M) \to (N, \omega_N)$ intertwines the symplectic structures if and only if its graph Γ_φ is Lagrangian. This led Alan Weinstein to declare [Wei] that *Everything is a Lagrangian submanifold!* In the same article, Weinstein defined the **symplectic "category"**, whose objects are symplectic manifolds, where the morphisms from M to N are the Lagrangians in $M^- \times N$. During the 30 years hence, it has transpired that Lagrangians connect symplectic geometry with algebraic geometry and low-dimensional topology.

On the other hand, symplectic geometry has been revolutionized by the use of **pseudoholomorphic curves**, beginning with Floer and Gromov's seminal work [Fl, Gr] leading to the **Floer cohomology** $HF^*(L_1, L_2)$, which is an enhanced intersection theory of the Lagrangians $L_1, L_2 \subset M$. One major success is the construction of the **Fukaya (A_∞)-category** $\text{Fuk}(M)$ [FuOhOhOn, Se2], which is an invariant of the symplectic manifold M that encodes the Lagrangians and the Floer cohomology groups. The Fukaya category has been a major focus of study over the past two decades, in particular because of the **homological mirror symmetry conjecture**, a deep connection between symplectic and algebraic geometry which was initiated by Kontsevich [Ko] and refined by many other since.

Connections between low-dimensional topology and symplectic geometry motivated Wehrheim and Woodward to develop **pseudoholomorphic quilts** [WeWo2], which they used to associate to a Lagrangian correspondence a functor between Fukaya categories, analogously to Fourier–Mukai transforms in algebraic geometry. However, they encountered a new and mysterious type of singularity formation, which produced an algebraic obstruction to their construction of functors and forced them to limit themselves to situations where such singularities can be excluded.

My work captures this obstruction in terms of a new type of bubbling. This is analogous to work of Fukaya's in the 1990s: in the cochain complex $CF(L_1, L_2)$, the differential d may not have $d^2 = 0$ as a result of singularity formation, which led Floer to only define $HF(L_1, L_2)$ under hypotheses where these singularities can be excluded. Fukaya recognized that this obstruction can be captured as disk bubbling, which led him to define a much richer structure, the Fukaya category. Completely analogously, I capture the obstruction noted by Wehrheim–Woodward in terms of a new type of bubbling, called **figure eight bubbling**.

The goal of my research program is to use and significantly extend this insight to construct a complete functoriality framework for the Fukaya category. I recognized that the analytic issues presented by figure eight bubbling can be tamed, and moreover that the figure eight bubble suggests a larger construction that significantly extends Wehrheim–Woodward's collection of functors. This construction is an enhancement of both Weinstein's symplectic category and the Fukaya category into a single object called Symp, the **symplectic** $(A_\infty, 2)$**-category**, which is the minimal object that includes Fukaya categories, functors between them, and coherences among these functors. Constructing Symp is a major undertaking with many components, many of which I have completed. Symp will streamline the manipulation of Fukaya categories, and provide a unified mechanism for transferring computations between Fukaya categories.

In the following list, I summarize the components of my research program that I will discuss in §§2–5. These mathematical contributions are motivated by the construction of Symp, but touch on several areas of mathematics. After this list, I note two concrete consequences of my work, despite Symp still being under construction.

- In [Bo1, Bo3, BoCa], I establish an algebraic framework for Symp. The key objects are a new family of singular quilts called **witch balls**, which are a natural generalization of figure eight bubbles. This framework is based on a new family of abstract polytopes called **2-associahedra**, which form the combinatorial backbone for the new notion of $(A_\infty, 2)$**-categories**. This extension of Wehrheim–Woodward's work is analogous to Fukaya's construction of the Fukaya category, which extended Floer cohomology by introducing operations built over associahedra. *See §2.*
- In work with Oblomkov [BoOb], I construct a complexification $\overline{2M}_{\mathbf{n}}^{\mathbb{C}}$ of the 2-associahedra closely related to the moduli space of curves $\overline{M}_{0,n}^{\mathbb{C}}$. $\overline{2M}_{\mathbf{n}}^{\mathbb{C}}$ can be thought of as a compactified moduli space of configurations of pointed vertical lines in $\mathbb{C}^2$. Our construction is necessary for the construction of Symp, as it equips the 2-associahedra with a suitable smooth structure. In addition, a remarkable feature of our complexified spaces is that even though they are moduli space of two-dimensional configurations, they have only very mild singularities. In ongoing work, we aim to cast this as an instance of a version of Fulton–MacPherson compactification for pairs of spaces, and to explore its implications for enumerative geometry. *See §3.*
- With Katrin Wehrheim, I developed analytic techniques [Bo4, BoWe] to deal with the new singularity phenomena that emerge in quilt theory (**"figure-eight bubbling"** and **"strip-shrinking"**), which are fundamentally different from the analysis in classical symplectic geometry. In ongoing work with Wehrheim, I am constructing Symp in the analytic framework of **polyfolds** [HoWyZe], which will enable a modular construction so that others can build off this work with maximum flexibility. *See §4.*
- While Symp in general will be a count of abstractly perturbed solutions, I have developed techniques to concretely compute the structure maps. As a proof of concept, I developed a technique for constructing quilts that relate the Fukaya categories of M and its symplectic quotient $M/\!\!/G$, which I used in [Bo2] to confirm in two examples predictions made in [Bo5, BoWe] about the algebraic effect of figure eight bubbling. These are the first constructions of nontrivial families of quilts. In ongoing work with Ritter, I am working to apply this construction technique to understand how the Fukaya category changes under symplectic reduction. *See §5.*

Finally, I note that the ideas I have developed for use in the construction and computation of Symp have already found application. One is my work with Oblomkov, which I described in the second bullet above: we construct a new collection of compactified moduli spaces, which are of interest both in algebraic geometry and in the study of the topology of configuration spaces. Another is a very recent paper [Se1], in which Paul Seidel uses ideas from several of my papers in his construction of a new invariant of a monotone symplectic manifold, which takes the form of a formal group and is related to quantum Steenrod operations.

1.1 Technical introduction: A blueprint for the symplectic $(A_\infty, 2)$-category Symp

In this technical introduction, I will give an overview of Symp, which, as mentioned earlier in §1, is a 2-category-like structure that is the minimal coherent framework for constructing functors between Fukaya categories. In the following sections, I will describe the progress I have made toward constructing Symp, which is the central objective of my research program.

For any symplectic manifold M, $\mathrm{Fuk}(M)$ is the A_∞-category whose objects are Lagrangians and where $\hom(L, L')$ consists of sums of points in $L \cap L'$, as long as L and L' intersect transversely. The structure maps $\mu^d : \hom(L^{d-1}, L^d) \otimes \cdots \otimes \hom(L^0, L^1) \to \hom(L^0, L^d)$ are defined by counting pseudoholomorphic polygons, i.e. maps from a disk with d input and 1 output boundary marked points to M and with boundary conditions in the L^i's, as indicated on the left of Fig. 1.

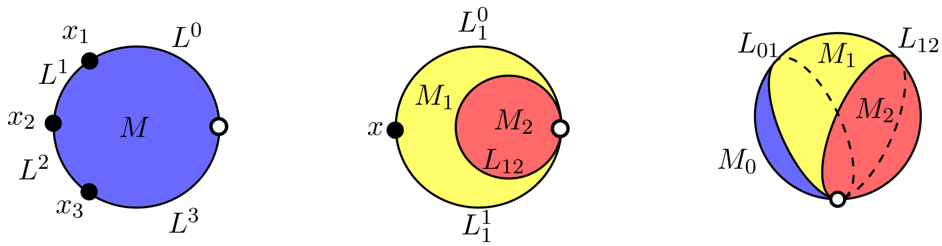


Figure 1: Domains for maps whose counts define, from left to right: the structure map μ^3 in $\mathrm{Fuk}(M)$; part of $\Phi(L_{12})$, a functor $\mathrm{Fuk}(M_1) \to \mathrm{Fuk}(M_2)$; the figure-eight count on L_{01} and L_{12}. The domains are labeled by the manifolds, submanifolds, and points to which the parts map.

Ma'u–Wehrheim–Woodward showed in [MaWeWo] that a Lagrangian $L_{12} \subset M_1^- \times M_2$ in a product of symplectic manifolds induces a functor $\Phi^\#(L_{12})\colon \mathrm{Fuk}^\#(M_1) \to \mathrm{Fuk}^\#(M_2)$. $\mathrm{Fuk}^\#$ is a less-geometric version of the Fukaya category, which [MaWeWo] worked with in order to avoid dealing with the singularity of the quilted disk whose domain is indicated in the middle of Fig. 1. They were only able to construct these functors for a restricted class of manifolds, due to new phenomena called **figure eight bubbling** and **strip-shrinking**. Moreover, they constructed a homotopy

$$\Phi^\#(L_{12}) \circ \Phi^\#(L_{01}) \simeq \Phi^\#(L_{01} \circ L_{12}), \tag{1}$$

but did not develop the larger structure hinted at by this collection of functors and homotopies.

I have proposed in [Bo1, Bo3, Bo4, Bo5, BoCa, BoWe] a significant enhancement of [MaWeWo]: by dealing head-on with the quilted disk's singularity, we will be able to upgrade $\Phi^\#(L_{12})$ to a functor $\Phi(L_{12})\colon \mathrm{Fuk}(M_1) \to \mathrm{Fuk}(M_2)$, where $\Phi(L_{12})$ is defined on morphisms by counting pseudoholomorphic quilted disks as in Fig. 1 (see §4). Furthermore, the figure eight bubbles that [MaWeWo] avoided, in addition to the polygons whose counts define the composition maps in the Fukaya category and the quilted disks used to define $\Phi(L_{12})$, are a particular instance of **witch balls**, and counting witch balls will give rise to a 2-category-like structure Symp whose objects are symplectic manifolds, and where $\hom(M_1, M_2)$ is $\mathrm{Fuk}(M_1^- \times M_2)$. (In particular, $\hom(\mathrm{pt}, M)$ recovers $\mathrm{Fuk}(M)$.) In particular, this will allow us to remove the restrictive geometric hypotheses that [MaWeWo] needed in order to exclude figure eight bubbling. Figure eight bubbles should therefore be thought of not as obstacles, but as part of a coherent naturality structure for Fukaya categories.

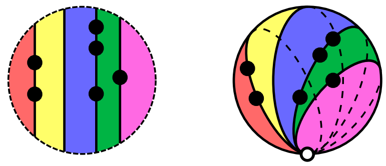


Figure 2: Two views of the domain of a witch ball: $\mathbb{R}^2$ (left) and $S^2 = \mathbb{R}^2 \cup \{\infty\}$ (right).

It is illuminating to describe the first few pieces of structure in Symp:

- For any objects X, Y in an $(A_\infty, 2)$-category, $\hom(X, Y)$ is an A_∞-category. In Symp, this says that $\mathrm{Fuk}(M_0^- \times M_1)$ should be an A_∞-category, which is true for any Fukaya category. The unusual thing here is that we view the structure maps in $\mathrm{Fuk}(M_0^- \times M_1)$ as being defined by witch balls with one seam, with patches mapping to M_0 and M_1; these can be identified with pseudoholomorphic polygons in $M_0^- \times M_1$.
- For any X, Y, Z in an $(A_\infty, 2)$-category, there is a **horizontal composition** operation $\big(\hom(X, Y), \hom(Y, Z)\big) \to \hom(X, Z)$ which is an A_∞-bifunctor. In Symp, this says that there should be a bifunctor $\big(\mathrm{Fuk}(M_0^- \times M_1), \mathrm{Fuk}(M_1^- \times M_2)\big) \to \mathrm{Fuk}(M_0^- \times M_2)$, which is defined by counting witch balls with two seams. We can interpret the algebraic role of figure eight bubbling in terms of this bifunctor: the count of figure eight bubbles is the curvature term of the composition bifunctor.

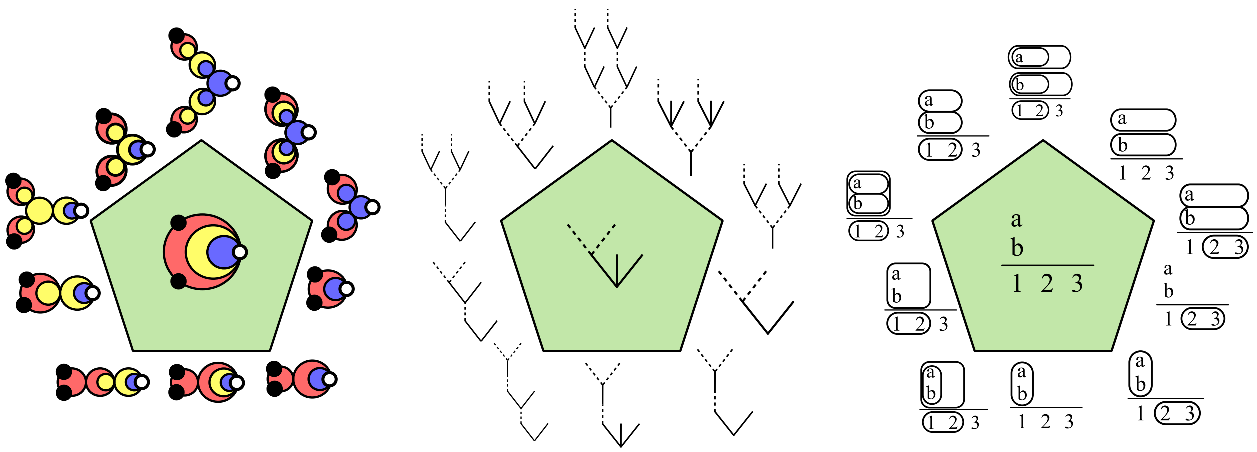
Witch balls neatly unify the maps pictured in Fig. 1, but present new challenges. First, counts of witch balls must be formulated as structure maps in some algebraic object. Second, we must reckon with the analytic issues presented by the witch ball's "singularity", where the seams intersect. In the following sections, we describe significant progress on these problems.

2 2-associahedra and the algebraic structure of an $(A_\infty, 2)$-category

Counting pseudoholomorphic polygons in M gives rise to an A_∞-category $\mathrm{Fuk}(M)$ [FuOhOhOn, Se2] because of the combinatorial structure of the moduli space of domains. Indeed, the configuration space of disks with r boundary points (one distinguished) up to Möbius transformations forms a CW complex $\overline{M}_r$. The underlying poset K_r indexing the cells of $\overline{M}_r$ is called an **associahedron** [Stash1]. These posets are equipped with maps $K_r \times K_s \to K_{r+s-1}$, and $(K_r)_{r \geq 1}$ together with these maps form an operad whose combinatorics gives rise to the notion of A_∞-category.

Similarly, to understand the structure that results from counting witch balls, we must understand for $r \geq 1$ and $\mathbf{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{\mathbf{0}\}$ the degenerations that can occur in the domain moduli space $2M_{\mathbf{n}}$ of **witch curves**, whose interior parametrizes configurations of r vertical lines in $\mathbb{R}^2$ with n_i marked points on the i-th line up to translations and positive dilations. (By identifying $\mathbb{R}^2 \cup \{\infty\} \simeq S^2$, we can also view elements of $2M_{\mathbf{n}}$ as configurations of marked circles on S^2.)

$2M_{\mathbf{n}}$ is not compact, because points on a single line can collide, or lines can collide. We compactify $2M_{\mathbf{n}}$ to the space $\overline{2M}_{\mathbf{n}}$ of **nodal witch curves** like so: when a collection of lines collide, then wherever the marked points on these lines are as this collision happens, we bubble off another configuration of lines and points. To define $\overline{2M}_{\mathbf{n}}$ precisely, we need to specify the allowed degenerations, and this is where the 2-associahedra [Bo1] come in: for $r \geq 1$ and $\mathbf{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{\mathbf{0}\}$ we define the **2-associahedron** $W_{\mathbf{n}}$ to be the poset of allowed degenerations in $\overline{2M}_{\mathbf{n}}$. We illustrate this in the following figure: on the left is the compactified moduli space $\overline{2M}_{200}$, and in the middle and on the right are two presentations of W_{200}.



In [Bo1] I also establish several fundamental properties of 2-associahedra, collected here:

Theorem 1 ([Bo1])**.** *For any $r \geq 1$ and $\mathbf{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{\mathbf{0}\}$, the 2-associahedron $W_{\mathbf{n}}$ is a poset, the collection of which satisfies the following properties:*

(ABSTRACT POLYTOPE) $\widehat{W_{\mathbf{n}}} \coloneqq W_{\mathbf{n}} \cup \{F_{-1}\}$ *is an abstract polytope.*

(FORGETFUL) *There are forgetful maps $\pi\colon W_{\mathbf{n}} \to K_r$ to the associahedra.*

(RECURSIVE) *The closure of any facet of $W_{\mathbf{n}}$ can be decomposed as a product of fiber products over the associahedra of lower-dimensional 2-associahedra.*

In [Bo3], I show that witch curves realize the 2-associahedra:

Theorem 2 ([Bo3])**.** *The moduli space $\overline{2M}_{\mathbf{n}}$ of witch curves can be given the structure of a compact metrizable stratified space, with the poset of strata equal to the 2-associahedron $W_{\mathbf{n}}$.*

This justifies my introduction of 2-associahedra: they really do govern the degenerations that can take place in $\overline{2M}_{\mathbf{n}}$.

In the next subsection, I will describe the algebraic structure that the 2-associahedra parametrize. It is called an $(A_\infty, 2)$-category, of which Symp will be an example. To conclude the current section, I note that the 2-associahedra are interesting combinatorial and topological objects in their own right. For instance, in a recent talk [Stash2] surveying the associahedra and the resulting objects and ideas, James Stasheff highlighted the 2-associahedra as the most recent development. Finally, in recent work with my undergraduate student Dylan Mavrides, we established an important combinatorial property of the 2-associahedra:

Theorem 3 ([BoMa])**.** *The completed 2-associahedron $W_{\mathbf{n}} \cup \{F_{min}\}$ is Eulerian, i.e. the alternating sum over each nontrivial interval is 0. In particular, the Euler characteristic of $W_{\mathbf{n}}$ is 1.*

This is additional evidence that the 2-associahedra can be realized as convex polytopes, and moreover allows one to associate an important combinatorial invariant called the ***cd*-index** to the 2-associahedra, as in [Stan].

2.1 $(A_\infty, 2)$-spaces and $(A_\infty, 2)$-categories

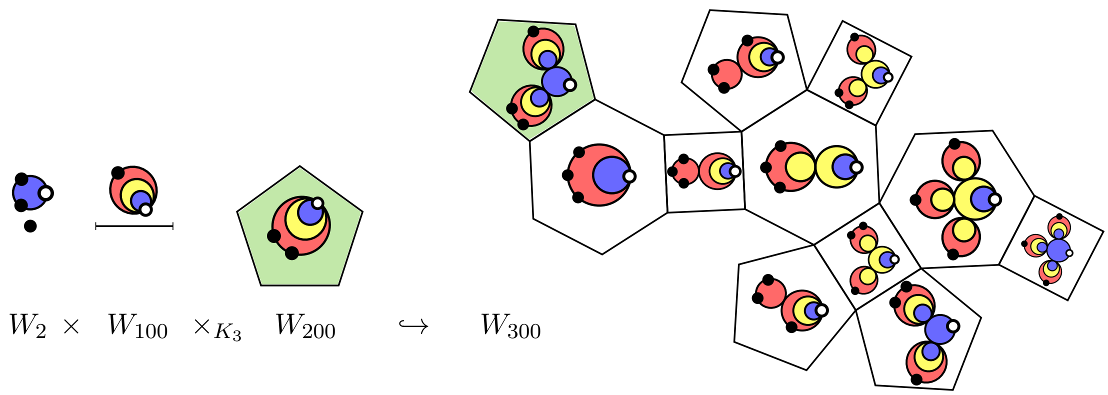
The 2-associahedra described in §2 have a rich algebraic structure: they form a **2-operad relative to the associahedra**. This structure gives rise to the new notion of an $(A_\infty, 2)$**-category**. This is essential for the ongoing construction of Symp, the structure which is the focus of my research, since Symp will be an $(A_\infty, 2)$-category. In this subsection we will explain the notion of a relative 2-operad, which I defines with Shachar Carmeli in [BoCa].

An **operad** is a collection $(\mathcal{O}_r)_{r\geq 1}$ of objects in a category $\mathcal{C}$ with a product operation, with maps $\mathcal{O}_r \times \mathcal{O}_s \to \mathcal{O}_{r+s-1}$ which satisfy a natural set of coherences. As described in §2, the associahedra form an operad whose structure underlies the notion of A_∞-category. Moreover, the associahedra can be realized as configuration spaces of disks with boundary marked points, which are the domain moduli spaces for the pseudoholomorphic polygons whose counts define the operations in the Fukaya category. The Fukaya category is therefore an A_∞-category.

My proposal for the symplectic $(A_\infty, 2)$-category Symp involves counting witch balls instead of pseudoholomorphic polygons, so it is natural to define the notion of $(A_\infty, 2)$-category by equipping the 2-associahedra $(W_\mathbf{n})$ with an operad-like structure. However, $(W_\mathbf{n})_{r\geq 1, \mathbf{n}\in\mathbb{Z}^r_{\geq 0}\setminus\{\mathbf{0}\}}$ does not form an operad: for one thing, the spaces are not indexed by the positive integers. In fact, 2-associahedra form a 2-categorical version of an operad:

Theorem 4 ([BoCa])**.** *The 2-associahedra $(W_\mathbf{n})$ form a 2-operad relative to the associahedra.*

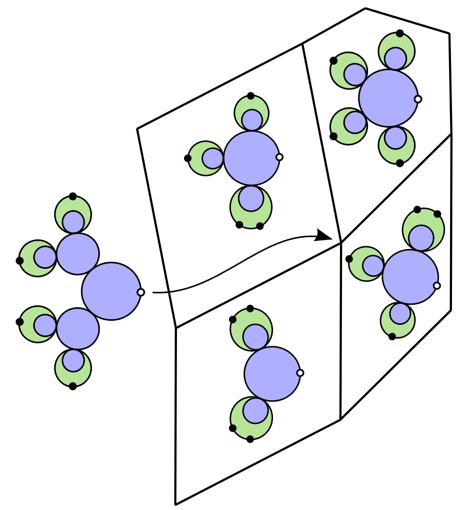
The notion of a relative 2-operad is new. Its complete definition would take us too far afield, but in the case of the 2-associahedra, the relative 2-operadic structure consists of the forgetful maps from the 2-associahedra to the associahedra noted in Thm. 1, and structure maps from products of fiber products of 2-associahedra to other 2-associahedra. In fact, these structure maps are inclusions in this case, and one of them is illustrated below. (In this figure, the maps from W_{100} and W_{200} to K_3 measure the width of the yellow strip. Also, W_{300} is a polyhedron; here we have depicted its net.)



The relative 2-operadic structure of the 2-associahedra enables me and Carmeli to define in [BoCa] the notion of an $(A_\infty, 2)$**-category**.

3 A complexification of the 2-associahedra and connections with $\overline{M}_{0,n}$

As I described in §2, my construction of the symplectic $(A_\infty, 2)$-category Symp depends crucially on the compactified moduli space $\overline{2M}_{\mathbf{n}}$ of witch curves, i.e. configurations of marked vertical lines in $\mathbb{R}^2$ up to translations and positive dilations. In Theorems 1 and 2 I equipped this space with the structure of a compact metrizable stratified space. In order to use $\overline{2M}_{\mathbf{n}}$ in the definition of Symp, the last remaining task is to equip it with a smooth structure.



$\overline{2M}_{\mathbf{n}}$ is not a manifold with corners: for example, on the right I show one of the corners in the 3-dimensional space $\overline{2M}_{40}$, at which four 2-faces meet. However, $\overline{2M}_{\mathbf{n}}$ fits naturally into Joyce's recently-introduced framework of **manifolds with g-corners** [Jo] — indeed, part of Joyce's motivation was to provide a smooth structure suitable for the domain moduli spaces used in [MaWeWo], which $\overline{2M}_{\mathbf{n}}$ generalizes.

In a very recent paper with Alexei Oblomkov, I have equipped $\overline{2M}_{\mathbf{n}}$ with this smooth structure. Specifically, we proved the following theorem about the complex analogue $\overline{2M}^{\mathbb{C}}_{\mathbf{n}}$, which is the compactified moduli space of marked vertical lines in $\mathbb{C}^2$:

Theorem 5 ([BoOb])**.** *For any $r \geq 1$ and $\mathbf{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{\mathbf{0}\}$, $\overline{2M}^{\mathbb{C}}_{\mathbf{n}}$ is a complex algebraic variety stratified by the 2-associahedron $W_{\mathbf{n}}$, and $\overline{2M}^{\mathbb{C}}_{\mathbf{n}}$ is compact and locally toric. These spaces are equipped with forgetful morphisms $\overline{2M}^{\mathbb{C}}_{\mathbf{n}} \to \overline{M}_{0,r+1}$.*

A consequence of this theorem is that $\overline{2M}^{\mathbb{C}}_{\mathbf{n}}$ is a manifold with g-corners.

While our original motivation was the smooth structure on $\overline{2M}_{\mathbf{n}}$, this theorem connects with several other topics. One connection is with the moduli space $\overline{M}^{\mathbb{C}}_{g,n}$ of stable n-pointed genus-g curves [DM, FM]. Specifically, $\overline{2M}^{\mathbb{C}}_{\mathbf{n}}$ fits into the following analogy:

$$(r-2)\text{-dim. associahedron} \;:\; \overline{M}_{0,r+1} \;::\; \overline{2M}^{\mathbb{R}}_{\mathbf{n}} \;:\; \overline{2M}^{\mathbb{C}}_{\mathbf{n}}. \tag{2}$$

That is, $\overline{2M}^{\mathbb{C}}_{\mathbf{n}}$ is a "2-dimensional" version of $\overline{M}_{0,r+1}$. $\overline{M}_{0,r+1}$ is of great importance in algebraic geometry — for instance, these spaces are essential to defining Gromov–Witten invariants, which are fundamental to modern enumerative geometry — and the above analogy suggests several questions about $\overline{2M}^{\mathbb{C}}_{\mathbf{n}}$. For instance, the Chow ring $A^*(\overline{M}^{\mathbb{C}}_{g,n})$ contains a subring $R^*(\overline{M}^{\mathbb{C}}_{g,n})$ called the **tautological ring**. The tautological ring is an extremely rich object, and it has been studied intensively by algebraic geometers since the 1970s, most recently by Pandharipande–Pixton [PanPi] and Pandharipande–Pixton–Zvonkine [PanPiZv]. In ongoing work, Oblomkov and I plan to study the cohomology $H^*(\overline{2M}^{\mathbb{C}}_{\mathbf{n}})$ and the tautological ring $R^*(\overline{2M}^{\mathbb{C}}_{\mathbf{n}})$, where the elements in the latter ring are defined via the maps that forget either a point or an unpointed line in a configuration in $\overline{2M}^{\mathbb{C}}_{\mathbf{n}}$.

Another connection is with the Fulton–MacPherson compactification of configurations of points on an algebraic variety [FM]. We can view our construction in Theorem 5 as part of a Fulton–MacPherson-like compactification for the pair $\mathbb{C}^2 \xrightarrow{\pi_1} \mathbb{C}$, where the interior parametrizes configurations of points $x_1, \ldots, x_r$ in $\mathbb{C}$ together with points $y_{i1}, \ldots, y_{in_i} \in \pi_1^{-1}\{x_i\}$ for $1 \leq i \leq r$. In our next paper, we plan to make this construction for a more general class of pairs $X \to Y$.

4 Analytic phenomena in Symp

Significant analytic issues arise from the witch ball's "singularity", the point where all the domain's seams intersect tangentially, and from the related phenomenon that in a moduli space of witch balls, the width of one of the strips in the domain can shrink to zero. In [Bo4] and [BoWe], I (jointly with Wehrheim in [BoWe]) show how these challenges can be overcome in the case of figure eight bubbles; the same analysis applies to general witch balls. The analytic core of these results, which we will describe below, is a substantial strengthening of the strip-shrinking estimates in [WeWo1] which I accomplished in [Bo4], using an unusual modification of the Sobolev space H^k.

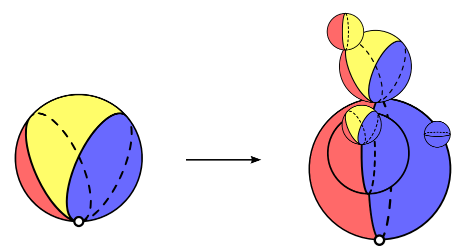
The first result is a "removal of singularity" for figure eight bubbles. Such a bubble can be viewed as a tuple of finite-energy pseudoholomorphic maps

$$w_0 : \mathbb{R} \times (-\infty, 0] \to M_0, \qquad w_1 : \mathbb{R} \times [0, 1] \to M_1, \qquad w_2 : \mathbb{R} \times [0, \infty) \to M_2$$

satisfying the seam conditions $(w_0(s, 0), w_1(s, 0)) \in L_{01}$ and $(w_1(s, 1), w_2(s, 0)) \in L_{12}$ for $s \in \mathbb{R}$. In [Bo4] I establish the following property of figure eights, as conjectured in [WeWo4].

Theorem 6 (Removal of singularity, [Bo4])**.** *If the composition $L_{01} \circ L_{12}$ is cleanly immersed, then w_0 resp. w_2 extend to continuous maps on $D^2 \cong (\mathbb{R}\times(-\infty, 0])\cup\{\infty\}$ resp. $D^2 \cong (\mathbb{R}\times[0, \infty))\cup\{\infty\}$, and $w_1(s, -)$ converges to constant paths as $s \to \pm\infty$.*

Without a removal of singularity, witch balls would have no chance at defining structures such as the $(A_\infty, 2)$-category Symp.



The second result concerns strip-shrinking, a phenomenon new to quilted Floer theory [WeWo1]: in a moduli space of quilted maps, the width of a strip or annulus in the domain of a pseudoholomorphic quilt may shrink to zero, as in the figure to the right. To understand the topology of moduli spaces of maps from such domains, we need a "Gromov Compactness Theorem": given a sequence of quilts in which strip-shrinking occurs and in which the energy is bounded, a subsequence of the maps must converge $\mathcal{C}^\infty_{\mathrm{loc}}$ away from finitely many points where the gradient blows up, and at each blowup point a tree of quilted spheres forms. In joint work with Wehrheim [BoWe], I establish full $\mathcal{C}^\infty_{\mathrm{loc}}$-convergence. This is stated in the following theorem, in which we describe the bubbling effects exhibited by strip-shrinking moduli spaces that [WeWo1] could not, and therefore excluded by monotonicity assumptions. In particular, we remove the restrictive hypothesis that $L_{01} \circ L_{12}$ be an embedded Lagrangian.

Theorem 7 (Gromov compactness, [BoWe])**.** *Say that $\underline{Q}^\nu$ is a sequence of pseudoholomorphic quilted maps, whose domains have a strip Q_1^ν of width $\delta^\nu \to 0$. Denote the target of Q_1^ν by M_1, and the targets of the neighboring patches M_0, M_2; call the Lagrangians defining the adjacent seam conditions L_{01}, L_{12}. Under the assumptions of Theorem 6, there is a subsequence that converges up to bubbling to a punctured quilt, and the energy that concentrates at each puncture is captured in a bubble tree consisting of disks, spheres, and figure eight quilts.*

4.1 Ongoing work: family polyfolds

The last serious hurdle in my construction of the symplectic $(A_\infty, 2)$-category is to regularize the moduli spaces of witch balls. There are a number of tools to accomplish this; I am to work in the framework of **polyfolds**. Polyfolds are a technology currently under development by Helmut Hofer and his collaborators [HoWyZe], which was introduced to provide a flexible framework for constructing moduli spaces of pseudoholomorphic objects. The essential difficulty here is the phenomenon of strip shrinking in moduli spaces of witch balls, as described earlier in §4.

This is an **adiabatic limit**, because energy is conserved in the limit. Motivated by the construction of Symp and the problem of strip-shrinking in particular, Wehrheim and I have developed an approach to set up adiabatic limits using polyfolds:

Project (with Wehrheim)**.** *Construct a theory of* **family polyfolds** *as a framework for moduli spaces with adiabatic limits, specifically strip-shrinking degenerations.*

The essential difficulty here is that to build a polyfold that includes strip-shrinking, we must locally phrase strip-shrinking as taking place on an unchanging domain, but strip-shrinking involves a strip in the domain of varying width $\epsilon \in [0,1)$. We deal with this by replacing the Cauchy–Riemann operator $\overline{\partial} := \partial_s + J\partial_t$ on a width-ϵ strip by $\overline{\partial}_\epsilon := \epsilon\partial_s + J\partial_t$ on a width-1 strip. In the $\epsilon \to 0$ limit, this operator becomes ∂_t, which is not Fredholm even when coupled with the operators on the neighboring strips. Wehrheim and I plan to overcome this by exploiting the fact that $\overline{\partial}_0$ becomes Fredholm when we restrict the Banach space of maps to those that are constant in the t-direction.

Once Wehrheim and I have completed this project, the preparation will finally be complete for the construction of Symp:

Project. *Construct an* $(A_\infty, 2)$*-category* Symp*, where the objects are symplectic manifolds and* $\hom(M,N) := \mathrm{Fuk}(M^- \times N)$*, and the structure maps are defined by counting witch balls.*

5 A method for constructing quilts mapping to M and $M/\!\!/G$

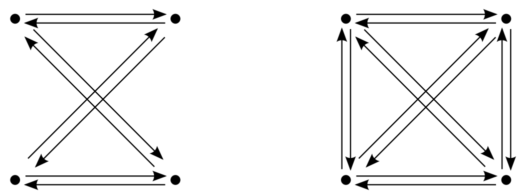
As I described in §1.1, I have built on the work of [MaWeWo] to propose that witch balls encode a complete naturality structure for the Fukaya category. In a recent paper [Bo2], I proposed the first general method for constructing moduli spaces of witch balls, in the context of **symplectic reduction**. Specifically, when M is a symplectic manifold with an action of a Lie group G by Hamiltonian diffeomorphisms, there is a **symplectic quotient** $M/\!\!/G$ and a Lagrangian correspondence Λ_G from $M/\!\!/G$ to M, and one can exploit Λ_G to compare pseudoholomorphic invariants of M against those of $M/\!\!/G$. This is an active topic of research, see e.g. [EvLe, Fuk].

As a proof of concept, I constructed moduli spaces of quilts in two examples, both in the case of S^1 acting on $\mathbb{CP}^2$ with quotient $\mathbb{CP}^2/\!\!/S^1 = \mathbb{CP}^1$:

- In the first, I produced the first example of figure eight bubbling, a phenomenon which has been discussed theoretically in a number of papers (e.g. [WeWo1, WeWo3, Bo4, BoWe]) but had never been seen in a nontrivial example. Specifically, I demonstrated the precise algebraic obstruction due to figure eight bubbles to the fundamental "composition commutes with categorification" isomorphism (1). This experimentally confirms the prediction of this obstruction that I made with Wehrheim in [BoWe].

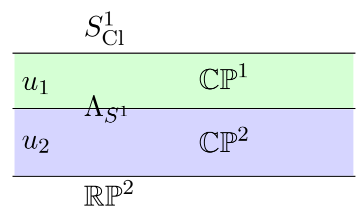
- In the second example, I answered a question posed by Akveld–Cannas da Silva–Wehrheim in 2015. They noticed that the Floer chain groups $CF(S^1_{\mathrm{Cl}}, \mathbb{RP}^1)$ and $CF(L_{\mathrm{AC}}, \mathbb{RP}^1 \circ \Lambda_{S^1})$ are different, where L_{AC} is the Lagrangian $\mathbb{RP}^2$ studied in [Ca] and $S^1_{\mathrm{Cl}} \subset \mathbb{CP}^1$ is the Clifford circle. This would contradict (1), hence it implies the existence of a rigid figure eight bubble; they asked whether such a bubble can be explicitly produced. My technique allowed me to explicitly produce such a bubble.

We will now explain the first example, which concerns the Floer chain groups $CF(\gamma, S^1_{\mathrm{Cl}})$ and $CF(\mathbb{RP}^2, T^2_{\mathrm{Cl}})$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$, where γ is the connected double-cover of $\mathbb{RP}^1 \subset \mathbb{CP}^1$. In the absence of figure eight bubbling, these chain groups would be isomorphic — but they are not, since the differential squares to zero in the former group but not the latter. We can see the difference in the following diagram:



$CF(\gamma, S^1_{\mathrm{Cl}})$ $\qquad$ $CF(\mathbb{RP}^2, T^2_{\mathrm{Cl}})$

The dots are generators, while the arrows are contributions to the differentials. Each arrow corresponds to a rigid pseudoholomorphic strip.



We can relate these collections of rigid strips by studying the 1-dimensional space of quilted strips as in the figure below and on the right. The height of the interior seam is allowed to vary, and the boundary points of this moduli space are exactly the limiting configurations as the seam hits the top or bottom boundary. Some of these limits are arrows in the figure above, and others are configurations where a figure eight bubbles off. In [Bo2], I explicitly produced this 1-dimensional moduli space and showed that there are four limits that exhibit figure eight bubbling, corresponding to the fact that four of the arrows on the right of the figure above do not appear on the left.

This confirms the prediction made in [BoWe] that figure eight bubbling is a codimension-1 phenomenon, and is the first instance where figure eight bubbling has been seen in an example.

5.1 Ongoing work: A functor $\mathrm{Fuk}(\mathbb{CP}^2) \to \mathrm{Fuk}(\mathcal{O}_{\mathbb{CP}^2}(-1))$, toward a symplectic analogue of the Bondal–Orlov theorem

This technique for constructing quilts in the context of symplectic reduction should be widely useful, and in this subsection I explain an application that I am currently exploring with Ritter.

In §10 of [RiSm], Ritter–Smith studied a Lagrangian correspondence $\Gamma \subset B^- \times E$ from the base of a negative complex line bundle $E \to B$ to the total space. In the case of the tautological line bundle $\mathcal{O}_{\mathbb{CP}^2}(-1) \to \mathbb{CP}^2$, this is the correspondence

$$\Gamma := \big\{\big(\ell, (\ell, z)\big) \in \mathbb{CP}^2 \times \mathcal{O}_{\mathbb{CP}^2}(-1) \mid z \in \ell,\ |z| = c\big\}, \qquad (3)$$

where c is chosen so that Γ has a technical property called monotonicity. One reason Γ is interesting is that $\mathbb{CP}^2 \subset \mathcal{O}_{\mathbb{CP}^2}(-1)$ is the local model for the exceptional divisor inside the blowup at a point of a complex 3-fold, so the induced functor $\Phi(\Gamma)\colon \mathrm{Fuk}(\mathbb{CP}^2) \to \mathrm{Fuk}(\mathcal{O}_{\mathbb{CP}^2}(-1))$ should be a useful tool for understanding how the Fukaya category changes under blowups. In particular, one hopes that there is an analogue for the Fukaya category of the following celebrated theorem of Orlov:

Theorem 8 (paraphrased from [Or]). *Let X be a smooth complex variety and $Y \subset X$ a smooth subvariety, and let $\widetilde{X}$ be the blowup of X along Y. Then the derived category $D^b_{\mathrm{coh}}(\widetilde{X})$ has a semiorthogonal decomposition in terms of $D^b_{\mathrm{coh}}(X)$ and* $\operatorname{codim} Y - 1$ *copies of $D^b_{\mathrm{coh}}(Y)$.*

Unfortunately, Ritter–Smith concluded that $\Phi(\Gamma)$ is the zero functor for technical reasons: while it does send the Clifford torus T^2_{Cl} to a monotone 3-torus T^3, which generate their respective Fukaya categories when equipped with the correct local systems, $\Phi(\Gamma)$ acts on local systems such that it is zero. (For this reason, $\Phi(\Gamma)$ is referred to as a "sobering example" in [RiSm].) In private communication, Ritter–Smith have told me that they hoped to salvage this example by equipping Γ with a bounding cochain b such that $\Phi(\Gamma, b)$ is nonzero. However, they gave up on this goal because it would require a good understanding of quilts with seam condition on Γ, of arbitrarily high Maslov index.

My construction technique applies here, and I can write down a formula for quilts of the relevant form. In joint work with Ritter, I therefore intend to pick up where Ritter–Smith left off:

Project (with Ritter). *Use my parametrization of the relevant moduli spaces of quilts to produce a bounding cochain b on Γ so that $\Phi(\Gamma, b)$ is nonzero. If this is successful, understand how the Fukaya category of a 3-fold changes under blowup at a point.*

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