

Relating Fukaya categories using combinatorics, operads, and nonlinear elliptic PDEs

1 Introduction

The PI works in the field of symplectic geometry, which is concerned with $2n$ -dimensional manifolds M equipped with an area measure ω , and with the n -dimensional submanifolds on which ω vanishes, called **Lagrangians**. Symplectic geometry originated with classical mechanics: the phase space of a Hamiltonian system is a symplectic manifold, and if there exist n independent commuting conserved quantities, then the level sets are Lagrangians.

Many symplectic concepts can be formulated as Lagrangians: for instance, a smooth map $\varphi: (M, \omega_M) \rightarrow (N, \omega_N)$ intertwines the symplectic structures if and only if its graph Γ_φ is Lagrangian. This led Alan Weinstein to declare [Wei] that *Everything is a Lagrangian submanifold!* In the same article, Weinstein defined the **symplectic “category”**, whose objects are symplectic manifolds, where the morphisms from M to N are the Lagrangians in $M^- \times N$, and where the composition of $L_{MN} \subset M^- \times N$ and $L_{NP} \subset N^- \times P$ is defined, under a generic transversality condition, to be $L_{MN} \circ L_{NP} := \pi_{MP}((L_{MN} \times L_{NP}) \cap (M \times \Delta_N \times P))$.

On the other hand, symplectic geometry has been revolutionized by the use of **pseudoholomorphic curves**, beginning with Floer and Gromov’s seminal work [Fl, Gr]. One major example is the **Fukaya A_∞ category** $\text{Fuk}(M)$ [FuOhOhOn, Se2], which encodes the Lagrangians in M and an intersection theory enhanced by counting pseudoholomorphic polygons in M .

The goal of the PI’s research program is to enhance both Weinstein’s symplectic category and the Fukaya category into a single object called **Symp**, the **symplectic $(A_\infty, 2)$ -category**. The objects of **Symp** are symplectic manifolds, and $\text{hom}(M, N)$ is $\text{Fuk}(M^- \times N)$. The PI uses **pseudoholomorphic quilts**, a new technology developed by Wehrheim and Woodward [WeWo3], in addition to tools from nonlinear PDEs, combinatorics, and algebraic topology. The PI’s construction of **Symp** will equip the Fukaya category with a complete naturality structure, which will be essential in turning Fuk into a tool that can be readily manipulated and computed.

The PI has made the following mathematical contributions, which are motivated by the construction of **Symp** but touch on several areas of mathematics:

- In [Bo1, Bo3, BoCa], the PI lays out an algebraic framework for the functoriality of the Fukaya category: a family of singular quilts called **witch balls** will enable the construction of **Symp**, which combines and relates the Fukaya categories of many different symplectic manifolds. This framework involves a new family of abstract polytopes called **2-associahedra**, which form the combinatorial backbone for the new notion of $(A_\infty, 2)$ -categories. See §2, §§2.1.1–2.1.2.
- The PI developed analytic techniques [Bo4, BoWe] to deal with new phenomena that emerge in quilt theory (“**figure-eight bubbling**” and “**strip-shrinking**”), which are fundamentally different from the analysis in classical Floer theory. See §2.1.3.
- The PI developed a technique for constructing quilts that relate the Fukaya categories of M and its symplectic quotient $M//G$, which he uses in [Bo2] to confirm in two examples predictions made in [Bo5, BoWe] about the algebraic effect of figure eight bubbling. These are the first constructions of nontrivial families of quilts. See §2.1.4.

The PI's project aims to complete the construction of **Symp** and to explore its applications:

- In ongoing work with Katrin Wehrheim described in §3.1.1, the PI is constructing **Symp** in the analytic framework of **polyfolds** [HoWyZe2], which will enable a modular construction so that others can build off this work with maximum flexibility.
- In §3.1.2 the PI describes two extensions of the construction technique described in §2.1.4. First, he applies it to the case of a correspondence $\mathbb{CP}^2 \rightarrow \mathcal{O}_{\mathbb{CP}^2}(-1)$. He aims to salvage Ritter–Smith's attempt to construct a nonzero functor $\text{Fuk}(\mathbb{CP}^2) \rightarrow \text{Fuk}(\mathcal{O}_{\mathbb{CP}^2}(-1))$ of geometric origin, which he views as the first step toward understanding how the Fukaya category of a 3-fold changes under blowup at a point. Second, he describes a strategy that aims at calculating how Floer cohomology changes under symplectic reduction.
- In §3.1.3–3.1.4 the PI describes two connections with other subjects which he is currently investigating: joint work-in-progress with Alexei Oblomkov to construct a complexification of the 2-associahedra closely related to the moduli space of curves $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$, and a continuation of the PI's project with Shachar Carmeli which aims to show that $(A_\infty, 2)$ -categories can be regarded as a model for $(\infty, 2)$ -categories. This will allow tools from modern homotopy theory to be brought to bear on the study of naturality properties for the Fukaya category.
- Finally, in §3.1.5 the PI explains a project to use the algebraic structure of **Symp** to refine the algebraic structure of an important invariant, the symplectic cohomology of a Liouville domain.

1.1 Quilted Floer theory and functors between Fukaya categories

We now give some context for the PI's proposal. For any symplectic manifold M , $\text{Fuk}(M)$ is the A_∞ -category whose objects are Lagrangians and where $\text{hom}(L, L')$ consists of sums of points in $L \cap L'$, as long as L and L' intersect transversely. The structure maps $\mu^d : \text{hom}(L^{d-1}, L^d) \otimes \cdots \otimes \text{hom}(L^0, L^1) \rightarrow \text{hom}(L^0, L^d)$ are defined by counting pseudoholomorphic polygons, i.e. maps from a disk with d input and 1 output boundary marked points to M and with boundary conditions in the L^i 's, as indicated on the left of Fig. 1.

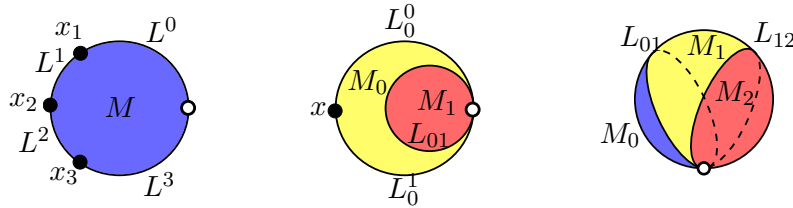


Figure 1: Domains for maps whose counts define, from left to right: the structure map μ^3 in $\text{Fuk}(M)$; part of $\Phi(L_{01})$, a functor $\text{Fuk}(M_0) \rightarrow \text{Fuk}(M_1)$; the figure-eight count on L_{01} and L_{12} . The domains are labeled by the manifolds, submanifolds, and points to which the parts map.

Ma'u–Wehrheim–Woodward showed in [MaWeWo] that a Lagrangian $L_{12} \subset M_1^- \times M_2$ in a product of symplectic manifolds induces a functor $\Phi^\#(L_{12}) : \text{Fuk}^\#(M_1) \rightarrow \text{Fuk}^\#(M_2)$. $\text{Fuk}^\#$ is a less-geometric version of the Fukaya category, which [MaWeWo] worked with in order to avoid dealing with the singularity of the quilted disk whose domain is indicated in the middle

of Fig. 1. They were only able to construct these functors for a restricted class of manifolds, due to new phenomena called **figure eight bubbling** and **strip-shrinking**. Moreover, they constructed a homotopy

$$\Phi^\#(L_{12}) \circ \Phi^\#(L_{01}) \simeq \Phi^\#(L_{01} \circ L_{12}), \quad (1)$$

but did not develop the larger structure hinted at by this collection of functors and homotopies.

The PI has proposed in [Bo1, Bo3, Bo4, Bo5, BoCa, BoWe] a significant enhancement of [MaWeWo]: by dealing head-on with the quilted disk’s singularity, we will be able to upgrade $\Phi^\#(L_{12})$ to a functor $\Phi(L_{12}): \text{Fuk}(M_1) \rightarrow \text{Fuk}(M_2)$, where $\Phi(L_{12})$ is defined on morphisms by counting pseudoholomorphic quilted disks as in Fig. 1 (see §2.1.3). Furthermore, the figure eight bubbles that [MaWeWo] avoided, in addition to the polygons whose counts define the composition maps in the Fukaya category and the quilted disks used to define $\Phi(L_{12})$, are a particular instance of **witch balls**, and counting witch balls will give rise to a 2-category-like structure **Symp** whose objects are symplectic manifolds, and where $\text{hom}(M_1, M_2)$ is $\text{Fuk}(M_1^- \times M_2)$. (The 1-morphisms in **Symp** are therefore Lagrangians in products $M_1 \times M_2$, which in particular includes ordinary Lagrangians $L \subset \text{pt} \times M$.) In particular, this will allow us to remove the restrictive geometric hypotheses that [MaWeWo] needed in order to exclude figure eight bubbling. Figure eight bubbles should therefore be thought of not as obstacles, but as part of a coherent naturality structure for Fukaya categories.

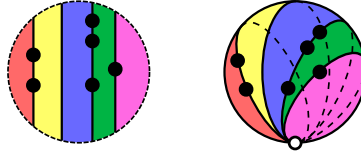


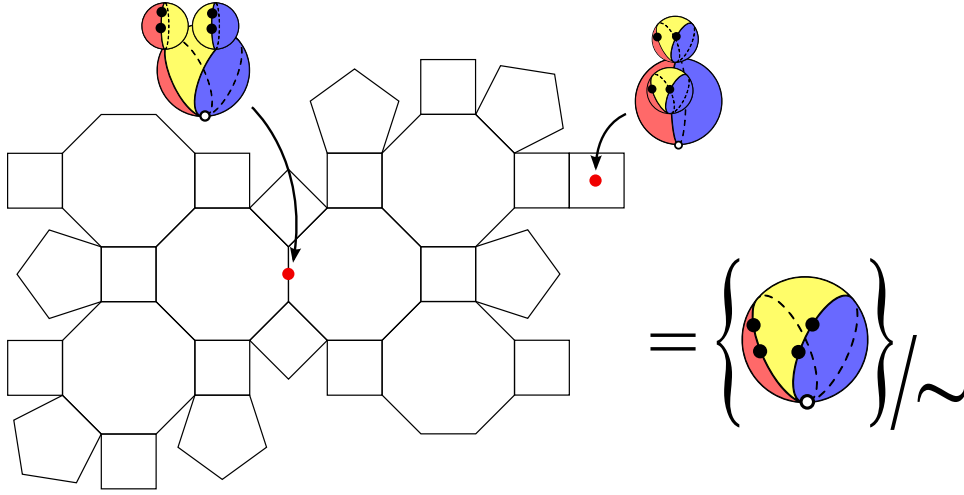
Figure 2: Two views of the domain of a witch ball: \mathbb{R}^2 (left) and $S^2 = \mathbb{R}^2 \cup \{\infty\}$ (right).

It is illuminating to describe the first few pieces of structure in **Symp**:

- For any objects X, Y in an $(A_\infty, 2)$ -category, $\text{hom}(X, Y)$ is an A_∞ -category. In **Symp**, this says that $\text{Fuk}(M_0^- \times M_1)$ should be an A_∞ -category, which is true for any Fukaya category. The unusual thing here is that we view the structure maps in $\text{Fuk}(M_0^- \times M_1)$ as being defined by witch balls with one seam, with patches mapping to M_0 and M_1 ; these can be identified with pseudoholomorphic polygons in $M_0^- \times M_1$.
- For any X, Y, Z in an $(A_\infty, 2)$ -category, there is a **horizontal composition** operation $(\text{hom}(X, Y), \text{hom}(Y, Z)) \rightarrow \text{hom}(X, Z)$ which is an A_∞ -bifunctor. In **Symp**, this says that there should be a bifunctor $(\text{Fuk}(M_0^- \times M_1), \text{Fuk}(M_1^- \times M_2)) \rightarrow \text{Fuk}(M_0^- \times M_2)$, which is defined by counting witch balls with two seams. We can interpret the algebraic role of figure eight bubbling in terms of this bifunctor: the count of figure eight bubbles is the curvature term of the composition bifunctor.

Witch balls neatly unify the maps pictured in Fig. 1, but present new challenges. First, counts of witch balls must be formulated as structure maps in some algebraic object. Second, we must reckon with the analytic issues presented by the witch ball’s “singularity”, where the seams intersect. In the following sections, we describe significant progress on these problems.

As we will describe in §2.1.1, the algebraic structure of \mathbf{Symp} is inherited from the moduli spaces of domains of witch balls. We give a preview of this structure now. Any $(A_\infty, 2)$ -category has an associated homology-level 2-category, so a consequence of this subproject will be a homology-level symplectic 2-category. In any 2-category, the interchange axiom intertwining horizontal and vertical composition must hold; in the case of \mathbf{Symp} , the interchange axiom is illustrated in the following figure:



This is an illustration of one of the domain moduli spaces, which in this case has dimension 3. (Here we have depicted the net, and left out the identifications of the outside edges for clarity.) The two red points are labeled by the nodal witch curves they correspond to, which in turn correspond to the two terms in the interchange axiom. There is of course a path between the red points through the interior, which is why the interchange axiom holds in the homology-level version of \mathbf{Symp} . A similar 2-categorical structure on homology-level information was constructed in [WeWo3]; as usual, Wehrheim–Woodward worked with the less-geometric extended Fukaya category, and only with symplectic manifolds and Lagrangian correspondences satisfying a somewhat restrictive technical condition.

2 Results from prior NSF support, including ongoing work

The PI is currently being supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship (DMS-1606435, for \$150,000, titled “Functoriality for Fukaya Categories of Symplectic Manifolds”, start date 06/01/2016), and was previously supported by an NSF Graduate Research Fellowship (2010 award year). Therefore the PI’s entire research program thus far is the result of NSF support. In particular, the papers [Bo4, BoWe] resulted from the PI’s Graduate Research Fellowship, and the papers [Bo1, Bo2, Bo3, BoCa] resulted from the PI’s Mathematical Sciences Postdoctoral Research Fellowship.

In this section we describe past and current results; these form the groundwork for the PI’s proposed project, which will be described in §3.

2.1 Intellectual Merit

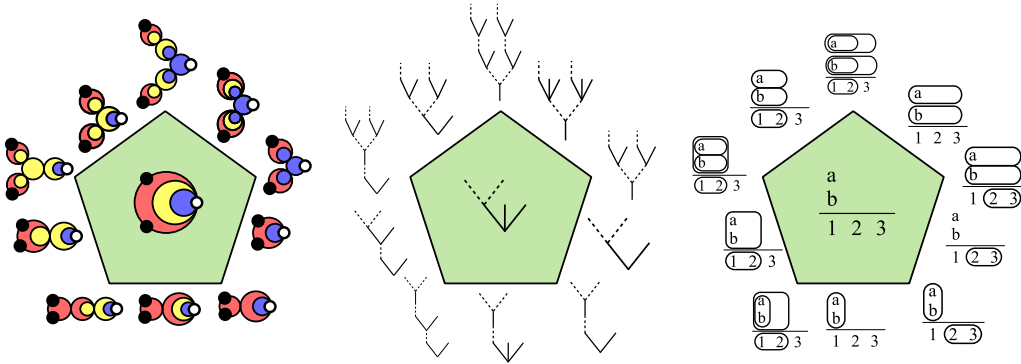
In this subsection we describe the intellectual merit of the work that resulted from the PI's prior NSF support. In particular, we will explain the results listed in §1.

2.1.1 2-associahedra

Counting pseudoholomorphic polygons in M gives rise to an A_∞ -category $\text{Fuk}(M)$ [FuOhOhOn, Se2] because of the combinatorial structure of the moduli space of domains. Indeed, the configuration space of disks with r boundary points (one distinguished) up to Möbius transformations forms a CW complex $\overline{\mathcal{M}}_r$. The underlying poset K_r indexing the cells of $\overline{\mathcal{M}}_r$ is called an **associahedron** [St]. These posets are equipped with maps $K_r \times K_s \rightarrow K_{r+s-1}$, and $(K_r)_{r \geq 1}$ together with these maps form an operad whose combinatorics gives rise to the notion of A_∞ -category.

Similarly, to understand the structure that results from counting witch balls, we must understand for $r \geq 1$ and $\mathbf{n} \in \mathbb{Z}_{\geq 0}^r \setminus \{\mathbf{0}\}$ the degenerations that can occur in the domain moduli space $2\mathcal{M}_{\mathbf{n}}$ of **witch curves**, whose interior parametrizes configurations of r vertical lines in \mathbb{R}^2 with n_i marked points on the i -th line up to translations and positive dilations. (By identifying $\mathbb{R}^2 \cup \{\infty\} \simeq S^2$, we can also view elements of $2\mathcal{M}_{\mathbf{n}}$ as configurations of marked circles on S^2 .)

$2\mathcal{M}_{\mathbf{n}}$ is not compact, because points on a single line can collide, or lines can collide. We compactify $2\mathcal{M}_{\mathbf{n}}$ to the space $\overline{2\mathcal{M}}_{\mathbf{n}}$ of **nodal witch curves** like so: when a collection of lines collide, then wherever the marked points on these lines are as this collision happens, we bubble off another configuration of lines and points. To define $\overline{2\mathcal{M}}_{\mathbf{n}}$ precisely, we need to specify the allowed degenerations, and this is where the 2-associahedra [Bo1] come in: for $r \geq 1$ and $\mathbf{n} \in \mathbb{Z}_{\geq 0}^r \setminus \{\mathbf{0}\}$ we define the **2-associahedron** $W_{\mathbf{n}}$ to be the poset of allowed degenerations in $\overline{2\mathcal{M}}_{\mathbf{n}}$. We illustrate this in the following figure: on the left is the compactified moduli space $\overline{2\mathcal{M}}_{200}$, and in the middle and on the right are two presentations of W_{200} .



In [Bo1] the PI also establishes several fundamental properties of 2-associahedra, collected here:

Theorem 1 ([Bo1]). *For any $r \geq 1$ and $\mathbf{n} \in \mathbb{Z}_{\geq 0}^r \setminus \{\mathbf{0}\}$, the 2-associahedron $W_{\mathbf{n}}$ is a poset, the collection of which satisfies the following properties:*

(ABSTRACT POLYTOPE) $\widehat{W}_{\mathbf{n}} := W_{\mathbf{n}} \cup \{F_{-1}\}$ is an abstract polytope.

(FORGETFUL) There are forgetful maps $\pi: W_{\mathbf{n}} \rightarrow K_r$ to the associahedra.

(RECURSIVE) The closure of any facet of $W_{\mathbf{n}}$ can be decomposed as a product of fiber products over the associahedra of lower-dimensional 2-associahedra.

In [Bo3], the PI shows that witch curves realize the 2-associahedra:

Theorem 2 ([Bo3]). *The moduli space $\overline{2\mathcal{M}}_{\mathbf{n}}$ of witch curves can be given the structure of a compact metrizable stratified space, with the poset of strata equal to the 2-associahedron $W_{\mathbf{n}}$.*

This justifies the PI's introduction of 2-associahedra: they really do govern the degenerations that can take place in $\overline{2\mathcal{M}}_{\mathbf{n}}$.

2.1.2 $(A_{\infty}, 2)$ -spaces and $(A_{\infty}, 2)$ -categories

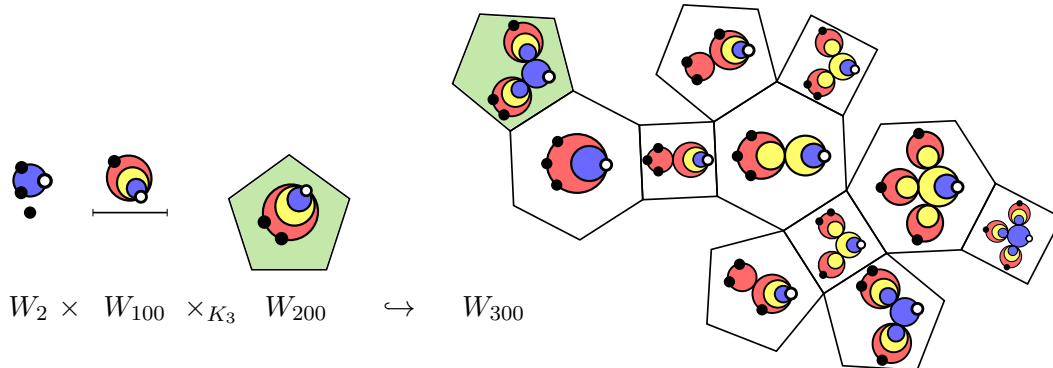
The 2-associahedra described in §2.1.1 have a rich algebraic structure: they form a **2-operad relative to the associahedra**. This structure gives rise to the new notion of an $(A_{\infty}, 2)$ -category. This is essential for the ongoing construction of **Symp**, the structure which is the focus of the PI's research, since **Symp** will be an $(A_{\infty}, 2)$ -category. In this subsection we will explain the notion of a relative 2-operad, which the PI defines with Shachar Carmeli in [BoCa].

An **operad** is a collection $(\mathcal{O}_r)_{r \geq 1}$ of objects in a category \mathcal{C} with a product operation, with maps $\mathcal{O}_r \times \mathcal{O}_s \rightarrow \mathcal{O}_{r+s-1}$ which satisfy a natural set of coherences. As described in §2.1.1, the associahedra form an operad whose structure underlies the notion of A_{∞} -category. Moreover, the associahedra can be realized as configuration spaces of disks with boundary marked points, which are the domain moduli spaces for the pseudoholomorphic polygons whose counts define the operations in the Fukaya category. The Fukaya category is therefore an A_{∞} -category.

The PI's proposal for the symplectic $(A_{\infty}, 2)$ -category **Symp** involves counting witch balls instead of pseudoholomorphic polygons, so it is natural to define the notion of $(A_{\infty}, 2)$ -category by equipping the 2-associahedra $(W_{\mathbf{n}})$ with an operad-like structure. However, $(W_{\mathbf{n}})_{r \geq 1, \mathbf{n} \in \mathbb{Z}_{\geq 0}^r \setminus \{\mathbf{0}\}}$ does not form an operad: for one thing, the spaces are not indexed by the positive integers. In fact, 2-associahedra form a 2-categorical version of an operad:

Theorem 3 ([BoCa]). *The 2-associahedra $(W_{\mathbf{n}})$ form a 2-operad relative to the associahedra.*

The notion of a relative 2-operad is new. Its complete definition would take us too far afield, but in the case of the 2-associahedra, the relative 2-operadic structure consists of the forgetful maps from the 2-associahedra to the associahedra noted in Thm. 1, and structure maps from products of fiber products of 2-associahedra to other 2-associahedra. In fact, these structure maps are inclusions in this case, and one of them is illustrated below. (In this figure, the maps from W_{100} and W_{200} to K_3 measure the width of the yellow strip. Also, W_{300} is a polyhedron; here we have depicted its net.)



The relative 2-operadic structure of the 2-associahedra enables the PI and his collaborator to define in [BoCa] the notion of an $(A_\infty, 2)$ -category.

2.1.3 Analysis of the witch ball’s singularity

Significant analytic issues arise from the witch ball’s “singularity”, the point where all the domain’s seams intersect tangentially, and from the related phenomenon that in a moduli space of witch balls, the width of one of the strips in the domain can shrink to zero. In [Bo4] and [BoWe], the PI (jointly with Katrin Wehrheim in [BoWe]) show how these challenges can be overcome in the case of figure eight bubbles; the same analysis applies to general witch balls. The analytic core of these results, which we will describe below, is a substantial strengthening of the strip-shrinking estimates in [WeWo1] which the PI accomplished in [Bo4], using an unusual modification of the Sobolev space H^k .

The first result is a “removal of singularity” for figure eight bubbles. Such a bubble can be viewed as a tuple of finite-energy pseudoholomorphic maps

$$w_0 : \mathbb{R} \times (-\infty, 0] \rightarrow M_0, \quad w_1 : \mathbb{R} \times [0, 1] \rightarrow M_1, \quad w_2 : \mathbb{R} \times [0, \infty) \rightarrow M_2$$

satisfying the seam conditions $(w_0(s, 0), w_1(s, 0)) \in L_{01}$ and $(w_1(s, 1), w_2(s, 0)) \in L_{12}$ for $s \in \mathbb{R}$. In [Bo4] the PI establishes the following property of figure eights, as conjectured in [WeWo4].

Theorem 4 (Removal of singularity, [Bo4]). *If the composition $L_{01} \circ L_{12}$ is cleanly immersed, then w_0 resp. w_2 extend to continuous maps on $D^2 \cong (\mathbb{R} \times (-\infty, 0]) \cup \{\infty\}$ resp. $D^2 \cong (\mathbb{R} \times [0, \infty)) \cup \{\infty\}$, and $w_1(s, -)$ converges to constant paths as $s \rightarrow \pm\infty$.*

The proof has two parts. First, the PI shows that in cylindrical coordinates, the gradients are uniformly bounded. This goes by contradiction: if not, we bubble off a nonconstant quilted sphere using the Gromov Compactness Theorem described below. Second, the PI establishes an isoperimetric inequality for the energy in the quilted setting; this crucially relies on the cleanness hypothesis (which holds for generic L_{01} and L_{12}). Without a removal of singularity, witch balls would have no chance at defining structures such as the $(A_\infty, 2)$ -category Symp.

The second result concerns strip-shrinking, a phenomenon new to quilted Floer theory [WeWo1]: in a moduli space of quilted maps, the width of a strip or annulus in the domain of a pseudoholomorphic quilt may shrink to zero. To understand the topology of moduli spaces of maps from such domains, we need a “Gromov Compactness Theorem”: given a sequence of quilts in which strip-shrinking occurs and in which the energy is bounded, a subsequence of the maps must converge $\mathcal{C}_{\text{loc}}^\infty$ away from finitely many points where the gradient blows up, and at each blowup point a tree of quilted spheres forms. In joint work with Katrin Wehrheim [BoWe], the PI establishes full $\mathcal{C}_{\text{loc}}^\infty$ -convergence. This is stated in the following theorem, which substantially extends the result in [WeWo1] by removing the restrictive hypothesis that $L_{01} \circ L_{12}$ be an embedded Lagrangian and by bounding the energy concentration by a geometric quantity:

Theorem 5 (Gromov compactness, [BoWe]). *Say that Q^ν is a sequence of pseudoholomorphic quilted maps, whose domains have a strip Q_1^ν of width $\delta^\nu \rightarrow 0$. Denote the target of Q_1^ν by M_1 , and the targets of the neighboring patches M_0, M_2 ; call the Lagrangians defining the adjacent seam conditions L_{01}, L_{12} . Under standard assumptions, there is a subsequence that converges up to bubbling to a punctured quilt, and at each puncture we can bubble off a nontrivial quilt.*

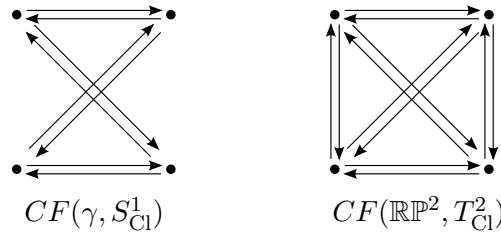
2.1.4 A method for constructing quilts mapping to M and $M//G$

As we described in the beginning of §2, the PI has built on the work of [MaWeWo] to propose that witch balls encode a complete naturality structure for the Fukaya category. In a very recent paper [Bo2], the PI proposed the first general method for constructing moduli spaces of witch balls, in the context of **symplectic reduction**. Specifically, when M is a symplectic manifold with an action of a Lie group G by Hamiltonian diffeomorphisms, there is a **symplectic quotient** $M//G$ and a Lagrangian correspondence Λ_G from $M//G$ to M , and one can exploit Λ_G to compare pseudoholomorphic invariants of M against those of $M//G$. This is an active topic of research, see e.g. [EvLe, Fuk].

As a proof of concept, the PI constructed moduli spaces of quilts in two examples, both in the case of S^1 acting on \mathbb{CP}^2 with quotient $\mathbb{CP}^2//S^1 = \mathbb{CP}^1$:

- In the first, the PI produced the first example of figure eight bubbling, a phenomenon which has been discussed theoretically in a number of papers (e.g. [WeWo1, WeWo3, Bo4, BoWe]) but had never been seen in a nontrivial example. Specifically, he demonstrated the precise algebraic obstruction due to figure eight bubbles to the fundamental “composition commutes with categorification” isomorphism (1). This experimentally confirms the prediction of this obstruction that the PI made with Katrin Wehrheim in [BoWe].
- In the second example, the PI answered a question posed by Akveld–Cannas da Silva–Wehrheim in 2015. They noticed that the Floer chain groups $CF(S_{\text{Cl}}^1, \mathbb{RP}^1)$ and $CF(L_{\text{AC}}, \mathbb{RP}^1 \circ \Lambda_{S^1})$ are different, where L_{AC} is the Lagrangian \mathbb{RP}^2 studied in [Ca] and $S_{\text{Cl}}^1 \subset \mathbb{CP}^1$ is the Clifford circle. This would contradict (1), hence it implies the existence of a rigid figure eight bubble; they asked whether such a bubble can be explicitly produced. The PI’s technique allowed him to explicitly produce such a bubble.

To close this subsection, we will explain the first example, which concerns the Floer chain groups $CF(\gamma, S_{\text{Cl}}^1)$ and $CF(\mathbb{RP}^2, T_{\text{Cl}}^2)$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$, where γ is the connected double-cover of $\mathbb{RP}^1 \subset \mathbb{CP}^1$. In the absence of figure eight bubbling, these chain groups would be isomorphic — but they are not, since the differential squares to zero in the former group but not the latter. We can see the difference in the following diagram:

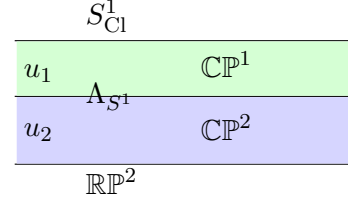


The dots are generators, while the arrows are contributions to the differentials. Each arrow corresponds to a rigid pseudoholomorphic strip.

We can relate these collections of rigid strips by studying the 1-dimensional space of quilted strips as in the figure below and on the right. The height of the interior seam is allowed to vary, and the boundary points of this moduli space are exactly the limiting configurations as the seam hits the top or bottom boundary. Some of these limits are arrows in the figure above, and others are configurations where a figure eight bubbles off. In [Bo2], the PI explicitly produced

this 1-dimensional moduli space and showed that there are four limits that exhibit figure eight bubbling, corresponding to the fact that four of the arrows on the right of the figure above do not appear on the left.

This confirms the prediction made in [BoWe] that figure eight bubbling is a codimension-1 phenomenon, and is the first instance where figure eight bubbling has been seen in an example.



2.2 Broader impact

The PI engaged regularly with undergraduates and junior researchers while being supported by the grants mentioned at the beginning of this section. Since he is continuing these broader impact projects, we describe all of them in detail in §3.2.

3 Proposed project

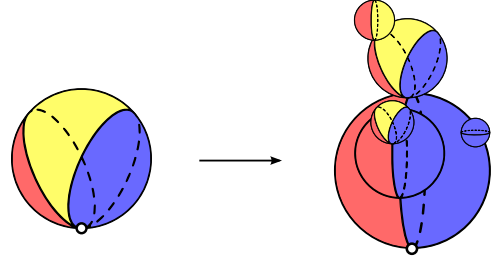
3.1 Intellectual merit

In this subsection we describe the five aspects of the intellectual merit portion of the PI's proposed project:

- In §3.1.1 we describe the continuation of the PI's main project, partly joint with Katrin Wehrheim: to implement functoriality for the Fukaya category by constructing an $(A_\infty, 2)$ -category in which the objects are symplectic manifolds and $\text{hom}(M, N) := \text{Fuk}(M^- \times N)$.
- Now that the PI has established the basic analysis and combinatorics of witch balls, the time has come to understand the moduli spaces of witch balls in some simple situations. §2.1.4 described some first steps toward this goal: the PI constructed moduli spaces of quilts which relate Floer groups which are already understood. The next task is to push these techniques to situations where the Floer theory is less well understood. In §3.1.2, we describe an attempt along these lines: we hope that our construction technique will allow us to produce a nontrivial functor $\text{Fuk}(\mathbb{CP}^2) \rightarrow \text{Fuk}(\mathcal{O}_{\mathbb{CP}^2}(-1))$ of geometric origin, which would salvage an attempt of Ritter–Smith and suggest a way to understand how the Fukaya category changes under blowups.
- In §3.1.3 we describe the PI's joint project with Alexei Oblomkov to study the complexification $\overline{2\mathcal{M}}_{\mathbf{n}}(\mathbb{C})$ of the realizations $\overline{2\mathcal{M}}_{\mathbf{n}}$ of the 2-associahedra, which we expect to have an intricate structure, similar to the moduli space $\overline{\mathcal{M}}_{0,n}$ of genus-0 curves.
- The 2-associahedra and the notions of $(A_\infty, 2)$ -categories and relative 2-operads are both new. In §3.1.4, we describe how the PI intends to further explore their connections with algebraic topology and higher category theory, in collaboration with Shachar Carmeli.
- In §3.1.5 we describe how the ideas from §3.1.1 should enable the PI to refine the algebraic structure of a symplectic invariant of central importance: the **symplectic cohomology** of a Liouville domain is a Batalin–Vilkovisky algebra, and understanding the relationship between the 2-associahedra and the Kimura–Stasheff–Voronov operad should allow one to lift this structure to the chain level, resolving a conjecture of Mohammed Abouzaid.

3.1.1 Family polyfolds and the $(A_\infty, 2)$ -category Symp

The last serious hurdle in the PI's construction of the symplectic $(A_\infty, 2)$ -category is constructing the moduli spaces of witch balls in the framework of **polyfolds**. Polyfolds is a technology currently under development by Helmut Hofer and his collaborators [HoWyZe2], which was introduced to provide a flexible framework for constructing moduli spaces of pseudoholomorphic objects. The essential difficulty here is the phenomenon of strip shrinking in moduli spaces of witch balls, as described in §2.1.3 and illustrated below and on the right. This is called “strip shrinking” because the yellow patch on the witch ball on the left is conformally a strip, and it shrinks to a line in this degeneration. Bubbling may occur on this line as the strip shrinks, as it does in this example.



This is an **adiabatic limit**, because energy is conserved in the limit. Motivated by the construction of Symp and the problem of strip-shrinking in particular, the PI and Katrin Wehrheim have developed an approach to set up adiabatic limits using polyfolds:

Subproject 1 (with Katrin Wehrheim). *Construct a theory of family polyfolds as a framework for moduli spaces with adiabatic limits, specifically strip-shrinking degenerations.*

The essential difficulty here is that to build a polyfold that includes strip-shrinking, we must locally phrase strip-shrinking as taking place on an unchanging domain, but strip-shrinking involves a strip in the domain of varying width $\epsilon \in [0, 1)$. We deal with this by replacing the Cauchy–Riemann operator $\bar{\partial} := \partial_s + J\partial_t$ on a width- ϵ strip by $\bar{\partial}_\epsilon := \epsilon\partial_s + J\partial_t$ on a width-1 strip. In the $\epsilon \rightarrow 0$ limit, this operator becomes ∂_t , which is not Fredholm even when coupled with the operators on the neighboring strips. The PI and Wehrheim plan to overcome this by exploiting the fact that $\bar{\partial}_0$ becomes Fredholm when we restrict the ambient Banach space of maps to those that are constant in the t -direction.

Once the PI and Wehrheim have completed Subproject 1, the preparation will finally be complete for the following subproject:

Subproject 2. *Construct an $(A_\infty, 2)$ -category Symp , in which the objects are symplectic manifolds and $\text{hom}(M_0, M_1) := \text{Fuk}(M_0^- \times M_1)$, and where the structure maps are defined by counting witch balls.*

3.1.2 Further computations in Symp

As we described in §2.1.4, the PI has developed a technique for constructing moduli spaces of quilts in the context of symplectic reduction. This technique should be widely useful, and in this subsection we explain two applications that the PI intends to explore.

A functor $\text{Fuk}(\mathbb{CP}^2) \rightarrow \text{Fuk}(\mathcal{O}_{\mathbb{CP}^2}(-1))$, toward a symplectic analogue of the Bondal–Orlov theorem

In §10 of [RiSm], Ritter–Smith studied a Lagrangian correspondence $\Gamma \subset B^- \times E$ from the base of a negative complex line bundle $E \rightarrow B$ to the total space. In the case of the tautological line

bundle $\mathcal{O}_{\mathbb{CP}^2}(-1) \rightarrow \mathbb{CP}^2$, this is the correspondence

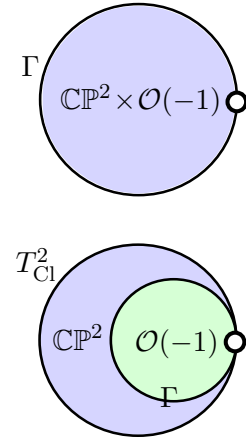
$$\Gamma := \{(\ell, (\ell, z)) \in \mathbb{CP}^2 \times \mathcal{O}_{\mathbb{CP}^2}(-1) \mid z \in \ell, |z| = c\}, \quad (2)$$

where c is chosen so that Γ has a technical property called monotonicity. One reason Γ is interesting is that $\mathbb{CP}^2 \subset \mathcal{O}_{\mathbb{CP}^2}(-1)$ is the local model for the exceptional divisor inside the blowup at a point of a complex 3-fold, so the induced functor $\Phi(\Gamma): \text{Fuk}(\mathbb{CP}^2) \rightarrow \text{Fuk}(\mathcal{O}_{\mathbb{CP}^2}(-1))$ should be a useful tool for understanding how the Fukaya category changes under blowups. In particular, one hopes that there is an analogue for the Fukaya category of the following celebrated theorem of Orlov:

Theorem 6 (paraphrased from [Or]). *Let X be a smooth complex variety and $Y \subset X$ a smooth subvariety, and let \tilde{X} be the blowup of X along Y . Then the derived category $D_{\text{coh}}^b(\tilde{X})$ has a semiorthogonal decomposition in terms of $D_{\text{coh}}^b(X)$ and $\text{codim } Y - 1$ copies of $D_{\text{coh}}^b(Y)$.*

Unfortunately, Ritter–Smith concluded that $\Phi(\Gamma)$ is the zero functor for technical reasons: while it does send the Clifford torus T_{Cl}^2 to a monotone 3-torus T^3 , which generate their respective Fukaya categories when equipped with the correct local systems, $\Phi(\Gamma)$ acts on local systems such that it is zero. (For this reason, $\Phi(\Gamma)$ is referred to as a “sobering example” in [RiSm].) In private communicate, Ritter–Smith have told the PI that they hoped to salvage this example by equipping Γ with a bounding cochain b such that $\Phi(\Gamma, b)$ is nonzero. However, they gave up on this goal because it would require a good understanding of quilts as on the right, of arbitrarily Maslov class.

The PI’s construction technique, as described in §2.1.4, applies equally well here, and he can write down a general formula for quilts as on the right. He therefore intends to pick up where Ritter–Smith left off:



Subproject 3. *Use the PI’s parametrization of the relevant moduli spaces of quilts to produce a bounding cochain b on Γ so that $\Phi(\Gamma, b)$ is nonzero. If this is successful, understand how the Fukaya category of a 3-fold changes under blowup at a point.*

Generalizing the example in §2.1.4 to understand how Fuk changes under symplectic reduction

In §2.1.4, we explained how the PI compared the Floer cochain groups $CF(\gamma, S_{\text{Cl}}^1)$ and $CF(\mathbb{RP}^2, T_{\text{Cl}}^2)$, where γ is the connected double-cover of $\mathbb{RP}^1 \subset \mathbb{CP}^1$ and S_{Cl}^1 and T_{Cl}^2 are the Clifford tori in \mathbb{CP}^1 resp. \mathbb{CP}^2 . The PI showed that the difference between these groups is due to figure-eight bubbling.

\mathbb{CP}^1 should be thought of here as the symplectic reduction of \mathbb{CP}^2 under the action of S^1 that rotates the last homogeneous coordinate. Equivalently, we can think of \mathbb{CP}^1 as the **GIT quotient** of \mathbb{CP}^2 under the analogous action of \mathbb{C}^* . The semistable locus of this action is $\mathbb{CP}_{\text{ss}}^2 := \mathbb{CP}^2 \setminus \{[0 : 0 : 1]\}$, and it is interesting to note that when we compute $CF(\mathbb{RP}^2, T_{\text{Cl}}^2)$ in $\mathbb{CP}_{\text{ss}}^2$, it is isomorphic to $CF(\gamma, S_{\text{Cl}}^1)$. In fact the moduli space of quilted strips discussed in §2.1.4 gives a cobordism between the collections of rigid pseudoholomorphic strips that define the differentials in these cochain groups. This leads us to formulate the following subproject:

Subproject 4. *Generalize the example discussed in §2.1.4 to understand how Floer groups in X and $X//G$ are related. Specifically, prove an isomorphism between certain Floer cochain groups in $X//G$ and X_{ss} , then understand how adding in the semistable locus affects the differential.*

3.1.3 A complexification of the 2-associahedra and connections with $\overline{\mathcal{M}}_{0,n}$

The moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{C})$ of stable n -pointed genus- g curves [DM, FM] is of great importance in algebraic geometry — for instance, these spaces are essential to defining Gromov–Witten invariants, which are fundamental to modern enumerative geometry. Its Chow ring $A^*(\overline{\mathcal{M}}_{g,n}(\mathbb{C}))$ contains a subring $R^*(\overline{\mathcal{M}}_{g,n}(\mathbb{C}))$ called the **tautological ring**, which is generated by pushforwards under the gluing maps of the “tautological classes” ψ_k and κ_i . The tautological ring is an extremely rich object, and it has been studied intensively by algebraic geometers since the 1970s. Most recently, Pandharipande–Pixton [PanPi] and Pandharipande–Pixton–Zvonkine [PanPiZv] proved a set of relations in the related tautological ring $R^*(\mathcal{M}_g(\mathbb{C}))$ that had been conjectured by Faber–Zagier decades earlier, and Pixton [Pi] conjectured an extension of these relations to $R^*(\overline{\mathcal{M}}_{g,n}(\mathbb{C}))$.

The genus-0 spaces $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$ can be viewed as complexifications of the (realizations of the) associahedra, as shown in [D], which led the PI to observe recently that the 2-associahedra can be complexified in the same fashion. We denote the \mathbf{n} -th complexification by $2\overline{\mathcal{M}}_{\mathbf{n}}(\mathbb{C})$. With Alexei Oblomkov, the PI is working to construct $2\overline{\mathcal{M}}_{\mathbf{n}}(\mathbb{C})$:

Subproject 5 (with Alexei Oblomkov). *For any $r \geq 1$ and $\mathbf{n} \in \mathbb{Z}_{\geq 0}^r \setminus \{\mathbf{0}\}$, construct $2\overline{\mathcal{M}}_{\mathbf{n}}(\mathbb{C})$ as a complex algebraic variety stratified by the 2-associahedron $W_{\mathbf{n}}$, and show that $2\overline{\mathcal{M}}_{\mathbf{n}}(\mathbb{C})$ is compact and has toric singularities. Equip these spaces with forgetful morphisms $2\overline{\mathcal{M}}_{\mathbf{n}}(\mathbb{C}) \rightarrow \overline{\mathcal{M}}_{0,r}(\mathbb{C})$. Study the cohomology $H^*(2\overline{\mathcal{M}}_{\mathbf{n}}(\mathbb{C}))$ and the tautological ring $R^*(2\overline{\mathcal{M}}_{\mathbf{n}}(\mathbb{C}))$.*

This project is interesting for two reasons: for one, it would produce a new family of compact complex varieties, which are well-behaved in the sense that their singularities are no worse than toric. For another thing, the PI and Oblomkov believe that it is possible to construct analogues of the “tautological” cohomology classes $\psi_i, \kappa_k \in H^*(\overline{\mathcal{M}}_{g,n}(\mathbb{C}))$, and hence to define a “tautological ring” $R^*(2\overline{\mathcal{M}}_{\mathbf{n}}(\mathbb{C}))$. The intricacy and importance of $R^*(\overline{\mathcal{M}}_{g,n}(\mathbb{C}))$ suggest that it would be very worthwhile to construct and study $R^*(2\overline{\mathcal{M}}_{\mathbf{n}}(\mathbb{C}))$.

3.1.4 Connections with homotopy theory

We will now describe a continuation of the project with Shachar Carmeli that we introduced in §2.1.2. As we described in that section, the PI and Carmeli showed that the 2-associahedra can be organized into a relative 2-operad, which allowed them to define the notion of an $(A_{\infty}, 2)$ -category.

The following fact establishes a connection between A_{∞} -categories and $(\infty, 1)$ -categories, which are important objects in modern algebraic topology:

Fact 7. *A_{∞} -categories form a model for stable $(\infty, 1)$ -categories.*

Symplectic geometers have begun to use this fact to study Fukaya categories, see e.g. [HLS].

The PI and Carmeli intend to prove a 2-categorical version of this fact:

Subproject 6 (with Shachar Carmeli). *Construct a natural equivalence from the model category of $(A_\infty, 2)$ -categories to stable $(\infty, 2)$ -categories.*

They have an idea of how to prove this, by exploiting a connection between algebras over 2-associahedra and Θ_2 -spaces, which form a model for certain $(\infty, 2)$ -categories.

A key to this strategy is to use another model for the 2-associahedra. Specifically, there is a relative 2-operad of spaces (\mathcal{C}_n) which is homotopy-equivalent to the relative 2-operad of 2-associahedra, where \mathcal{C}_n parametrizes configurations of strips and rectangles in the unit square. Rather than give the general definition, we illustrate one such configuration in \mathcal{C}_{203} in Fig. 3.

There are several things to gain from this equivalence: first, it will be evidence that the PI's definition of $(A_\infty, 2)$ -categories is the right one. Second, it will produce a convenient, workable model for stable $(\infty, 2)$ -categories, which so far are mysterious and unwieldy objects. Finally, it open the door to bringing modern homotopy theory to bear in the study of naturality for the Fukaya category. For instance, one version of the Barr-Beck theorem implies that the bounded derived category of coherent sheaves can sometimes be recovered from its restrictions to the elements in an open cover; see e.g. §3.4, [Lur2]. Key to this construction is working with the associated functors on the overlaps, on the chain level. It would represent a major advance to have a similar theorem for the Fukaya category.

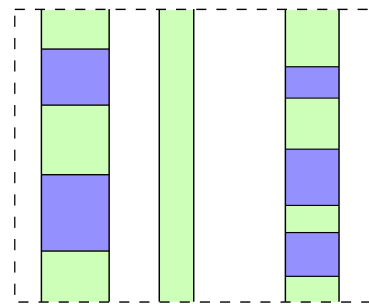


Figure 3: A configuration in \mathcal{C}_{203} .

3.1.5 Equipping symplectic cohomology with the structure of a homotopy BV algebra

A Liouville domain M is a compact symplectic manifold with boundary, equipped with a 1-form θ such that $d\theta =: \omega$ is symplectic and the vector field Z defined by $i_Z\omega = \theta$ points outward along ∂M . Symplectic cohomology is an invariant of Liouville domains; it measures the dynamics of a particular sort of Hamiltonian $H: S^1 \times M \rightarrow \mathbb{R}$. Among other applications, symplectic cohomology has found an important role in homological mirror symmetry: in favorable situations, the symplectic cohomology of a complex affine variety is the coordinate ring of its mirror.

Symplectic cohomology carries an important algebraic structure: it is a **BV algebra** [Ki], a notion which we explain now. As we explained in §2.1.2, an operad is a collection of objects $\mathcal{O}_1, \mathcal{O}_2, \dots$ in a category \mathcal{C} (e.g. spaces, posets, vector spaces, ...) together with structure maps $\mathcal{O}_r \times \mathcal{O}_s \rightarrow \mathcal{O}_{r+s-1}$. Moreover, there is a notion of an **algebra over an operad** [MaShSt]: this is an object $X \in \mathcal{C}$ together with maps $\mathcal{O}_r \times X^{\times r} \rightarrow X$ satisfying some coherences. We think of \mathcal{O}_r parametrizing operations on X taking r inputs. Let $(FD_d)_{d \geq 0}$ be the **framed little disks operad** [Ki], so that FD_d is the space of embeddings of from d copies of the unit disk in \mathbb{R}^2 to the disk, where each embedding is a combination of a translation, a dilation, and a rotation. The operad structure comes from the fact that if we are given configurations $C \in FD_d, C' \in FD_e$, and if we choose $i \in [1, d]$, then we can replace the i -th disk in C by a rescaled and rotated version of C' . A BV algebra is an algebra over the operad $(H_{-*}(FD_d))$ of vector spaces. The following result (see [Se1]) endows symplectic cohomology with a BV algebra structure.

Theorem (Thm. 12.4, [Se3]). *If M is a Liouville domain with $c_1(T^*M) = 0$, then its symplectic cohomology $SH^*(M)$ can be given the structure of a BV algebra.*

The BV structure of $SH^*(M)$ has found many applications, for instance [Pas] and [SeSo].

Just as the Fukaya A_∞ -category refines Floer homology, it is natural to ask for a chain-level version of this structure. Abouzaid formulates this as Conj. 2.6.1 in [Ab]: the chain complex $SC^*(M)$ for symplectic cohomology can be given the structure of a “homotopy BV algebra”, i.e. the structure of an algebra over the algebra of chain complexes $(C_{-*}(FD_d))_{d \geq 0}$, where $C_{-*}(FD_d)$ denotes the singular chains on FD_d with reversed grading. One way of resolving this conjecture would be to endow $(FD_d)_{d \geq 0}$ with a CW structure, and to define $C_{-*}(FD_d)$ to be the cellular chain complex. There is a connection between the 2-associahedra and the framed little disks operad, foreshadowed by an observation made in [Vo]; this connection suggests a method to prove Abouzaid’s conjecture:

Subproject 7. *Use the 2-associahedra to upgrade the BV algebra structure on symplectic cohomology to the chain level.*

Let us now indicate the first steps in this subproject. The **Kimura–Stasheff–Voronov operad** $(KSV_d)_{d \geq 1}$ [KiStVo] is an operad of CW complexes which is homotopy-equivalent to the framed little disks operad, in which KSV_d is the compactified moduli space of configurations of d disjoint framed points on S^2 . For $r \geq 1$, $\mathbf{n} \in \mathbb{Z}_{\geq 0}^r$, and $\sigma \in S_{\mathbf{n}}$ there is a map $f_{\mathbf{n}, \sigma}: 2\overline{\mathcal{M}}_{\mathbf{n}} \rightarrow KSV_{|\mathbf{n}|}$, which on the open stratum is defined by forgetting the seams and relabeling the points according to σ . This map is an embedding on the interior, and in low dimensions, the images of these maps produce CW decompositions of $KSV_{|\mathbf{n}|}$. The PI intends to construct in this fashion CW decompositions of all the spaces in the Kimura–Stasheff–Voronov operad, which would prove Abouzaid’s conjecture up to routine analysis.

An attractive feature of this proposed solution is that it would produce a *readily computable* chain-level version of the BV algebra structure on $SH^*(M)$, corresponding to the fact that the CW structure on $KSV_{|\mathbf{n}|}$ will involve only finitely many cells. This is analogous to the fact that the associahedra carry natural finite CW complex structures, which has the consequence that the structure equations for an A_∞ -category have only finitely many terms and are therefore amenable to algebraic manipulation.

3.2 Broader impacts

The PI views mentorship, dissemination, and engagement with the broader mathematical community as essential parts of mathematical research. The PI has demonstrated this commitment repeatedly in the past, as detailed in his biographical sketch. Over the next several years the PI plans to continue these efforts; in particular, the PI will focus on the following activities:

- Modern symplectic topology is a technical subject, and it can be difficult for undergraduates and beginning graduate students to gain entry to the field. The PI’s research is unusual in that it has a major combinatorial component, which is an ideal point of entry for undergraduates. For instance, the PI is currently advising the senior thesis of Dylan Mavrides, who is a student at Princeton University. Dylan is attempting to produce realizations of 2-associahedra as convex polytopes. Before that, the PI helped to advise another senior thesis [Ti], which was also concerned with the PI’s notion of 2-associahedra.

- The PI is a knowledgeable user of **polyfolds** [HoWyZe2], a new technology designed to systematize the construction of moduli spaces of pseudoholomorphic curves. (He is currently using polyfolds in his own research, as described in §3.1.1.) Polyfolds is one of the technologies that exist to regularize moduli spaces of pseudoholomorphic curves in situations where the “classical” techniques of e.g. [McDSa] are insufficient. Unfortunately, all of these technologies involve many hundreds (even thousands) of pages of intimidatingly technical documentation, which is an enormous barrier to entry for graduate students new to the field. The PI is currently working on a project, together with Joel Fish and Katrin Wehrheim, to ease and broaden access to polyfolds.

The centerpiece of the project is a wiki-style website which aims to present polyfolds knowledge in an accessible and publicly-editable fashion. We launched this project with a week-long workshop at UC Berkeley in June 2017, where the PI and the other two organizers gave lecture series in which we described the construction of the Fukaya algebra of a Lagrangian in a compact symplectic manifold using polyfolds. Approximately 25 graduate students attended, along with several postdocs and professors. In the next year, the PI plans to organize a workshop where the participants will add to the wiki, in particular by translating the technical material in the new book [HoWyZe1] into a form more accessible to junior researchers not yet familiar with this machinery.

This project builds on the success of two previous polyfolds summer schools and one previous polyfolds mini-conference:

- In January 2016, the PI co-organized a special session at the 2016 Joint Mathematical Meetings in Seattle, WA. The purpose of the special session was to exhibit several new applications-in-progress of polyfolds. After the formal session, the coorganizers invited all the participants to several informal sessions in which we workshopped the projects described in the talks. These informal sessions were an unusual way of including junior participants in the research process, and we particularly encouraged graduate students to attend. In addition, two of the five invited speakers were graduate students and one was a postdoc.
 - In July 2015, the PI conducted several problem sessions on polyfolds at the IHÉS summer school, and Joel Fish, Helmut Hofer, and Katrin Wehrheim gave lecture series about polyfolds.
 - In August 2012, the PI co-organized a workshop in Watsonville, CA, where 30 graduate students and postdocs learned about the construction of Gromov–Witten invariants using polyfolds. The participants took turns giving lectures, supervised by Helmut Hofer.
- The PI regularly travels to seminars and conferences to disseminate his research, including 24 invited talks. He is particularly interested to include junior participants: in addition to the 2012 workshop in Watsonville, CA mentioned above, the PI co-organized a similar workshop in 2013 on the topic of Gross–Siebert’s approach to Mirror Symmetry, supervised by Mark Gross. During Fall 2018, the PI will give a Princeton Undergraduate Math Club Colloquium, where he will describe the combinatorics underlying categorical symplectic geometry.