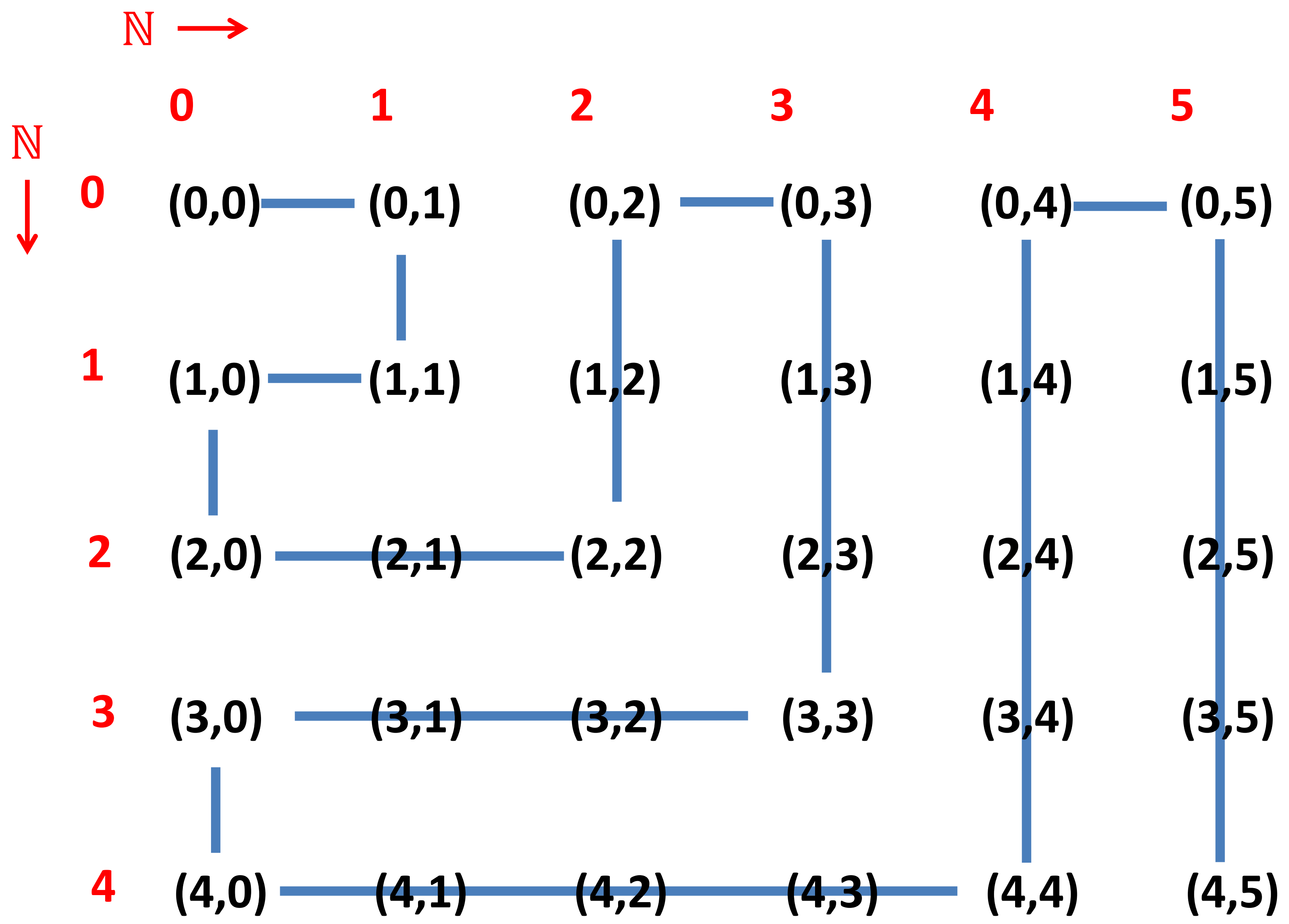
# CS3102 Theory of Computation

$$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$$

Show:
$|\mathbb{N}| = |\mathbb{Q}|$



# $|\mathbb{Q}| = |\mathbb{N}|$

Remove Duplicates

| $\mathbb{N} \downarrow$ \ $\mathbb{Z}_+ \rightarrow$ | 1 | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|---|
| 0 | $\frac{0}{1}$ | $\frac{0}{2}$ | $\cancel{\frac{0}{3}}$ | $\cancel{\frac{0}{4}}$ | $\cancel{\frac{0}{5}}$ | $\cancel{\frac{0}{6}}$ | $\cdots$ |
| 1 | $\frac{1}{1}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{6}$ | $\cdots$ |
| 2 | $\frac{2}{1}$ | $\cancel{\frac{2}{2}}$ | $\frac{2}{3}$ | $\cancel{\frac{2}{4}}$ | $\frac{2}{5}$ | $\cancel{\frac{2}{6}}$ | $\cdots$ |
| 3 | $\frac{3}{1}$ | $\frac{3}{2}$ | $\cancel{\frac{3}{3}}$ | $\frac{3}{4}$ | $\frac{3}{5}$ | $\cancel{\frac{3}{6}}$ | $\cdots$ |
| 4 | $\frac{4}{1}$ | $\cancel{\frac{4}{2}}$ | $\frac{4}{3}$ | $\cancel{\frac{4}{4}}$ | $\frac{4}{5}$ | $\cancel{\frac{4}{6}}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

# Countability

- As set is countable if:
  - It is finite
  - It has a bijection with the natural numbers
    - (countably infinite)
- Notation
  - $|\mathbb{N}| = \aleph_0$
    - Aleph-naught
- Is the set of all strings over alphabet $\{a, b\}$ countable?
  - What about other alphabets?

# The set of all strings is countable

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Length 0** | $\varepsilon^{0}$ | | | **1 string** | | | | | |
| **Length 1** | $a^{1}$ | $b^{2}$ | | | **2 strings** | | | | |
| **Length 2** | $aa^{3}$ | $ab^{4}$ | $ba^{5}$ | $bb^{6}$ | | | **4 strings** | | |
| **Length 3** | $aaa^{7}$ | $aab^{8}$ | $aba^{9}$ | $abb^{10}$ | $baa^{11}$ | $bab^{12}$ | $bba^{13}$ | $bbb^{14}$ | **8 strings** |
| **Length 4** | $aaaa$ | $aaab$ | $aaba$ | $aabb$ | $abaa$ | $abab$ | $abba$ | $abbb$ … | **16 strings** |
| **Length 5** | $aaaaa$ | $aaaab$ | $aaaba$ | $aaabb$ … | | | | | **32 strings** |
| ⋮ | | | | | | | | | ⋮ |

Important: A countable union of countable sets is countable

# Some more countable things

- Any language ever!
- Number of possible Java Programs
- The empty set
- The number of words in the English language
- Number of possible 410-page novels

Let’s prove them!

# All Languages are Countable

- Proof:

# The set of all possible Java programs is countable

- Proof:

# The Empty set is Countable

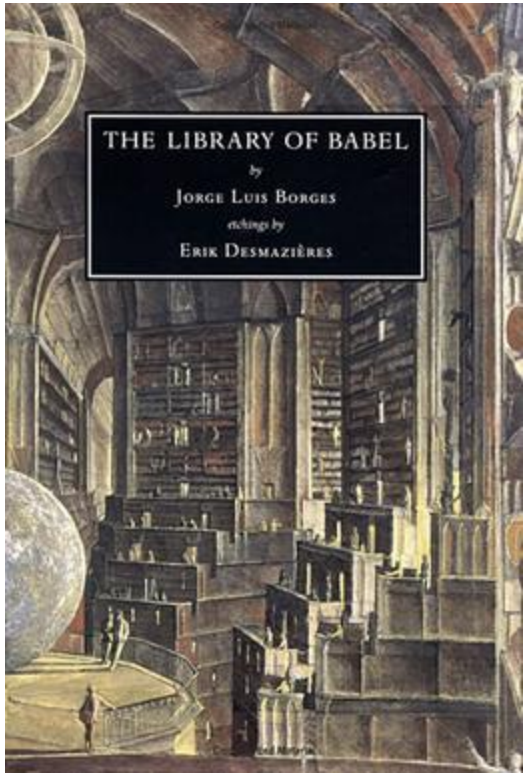
- Proof:

# The set of all English words is Countable

- Proof:

# The number of possible 410-page novels is countable

- Proof:



# Is $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ countable?

- Proof:

# Dovetailing

$\mathbb{N} \times \mathbb{N} \longrightarrow$

$f : \mathbb{N} \leftrightarrow \mathbb{N}$

| $\mathbb{N} \downarrow$ | $f(0)$ | $f(1)$ | $f(2)$ | $f(3)$ | $f(4)$ | $f(5)$ | |
|---|---|---|---|---|---|---|---|
| 0 | (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (0,5) | … |
| 1 | (1,0) | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | … |
| 2 | (2,0) | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | … |
| 3 | (3,0) | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | … |
| 4 | (4,0) | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) | … |
| | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

# Take-away Ideas

- All finite sets are countable
- Anything with a bijection to the naturals is countable
- A subset of a countable set is countable
- A union of countably many sets is countable
  - Formal proof of this is homework
- To be computable by Java, the set of possibilities must be countable!

# Are the Real Numbers Countable?

- Things that don't work:
  - List out every real number that starts with 1, then 2, then 3, …
  - List out every real number that has one number after the decimal, then 2, then 3, …
- How would we prove it wasn't?

# Diagonalization

- Used to prove that a set is not countable
  - Shows that there cannot be a bijection with the Natural Numbers

1. Assume toward a contradiction there is a bijection with the natural numbers
2. Treat this arbitrary bijection as an ordered list containing all items (item 0 is the thing which maps to 0, etc.)
3. Show that this list must always be missing something

# Diagonalization

Assume toward a contradiction that $f: \mathbb{N} \leftrightarrow \mathbb{R}$, show that $f$ cannot be onto (something from $\mathbb{R}$ is not mapped to)

| $\mathbb{N}$ | $\mathbb{R}$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(1) =$ | 3 . | 1 | 4 | 1 | 5 | 9 | 2 | 6 | 5 | 3 | … |
| $f(2) =$ | 1 . | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | … |
| $f(3) =$ | 2 . | 7 | 1 | 8 | 2 | 8 | 1 | 8 | 2 | 8 | … |
| $f(4) =$ | 1 . | 4 | 1 | 4 | 2 | 1 | 3 | 5 | 6 | 2 | … |
| $f(5) =$ | 0 . | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | … |
| … | … | | | | | | | | | | |

$X = 0.\ 2\ \ 1\ \ 9\ \ 3\ \ 4\ \ \ldots \in \mathbb{R}$

This number $X$ cannot appear anywhere in the list.

It's different from each $f(i)$ at digit $i$

# Is the set of all Languages Countable?

| $\mathbb{N}$ | $\varepsilon$ | $a$ | $b$ | $aa$ | $ab$ | $ba$ | $bb$ | $aaa$ | … | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(1) =$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | … |
| $f(2) =$ | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | … |
| $f(3) =$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | … |
| $f(4) =$ | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | … |
| $f(5) =$ | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | … |
| … | … | | | | | | | | | | | |
| L= | 0 | 1 | 1 | 0 | 1 | … | | | | | | |

Each row represents a language which includes string $i$ provided column $i$ has a 1

This Language $L$ cannot appear anywhere in the list. It's different from each $f(i)$ because its containment of string $i$ is opposite

# Correlary

- Some languages cannot be decided by Java

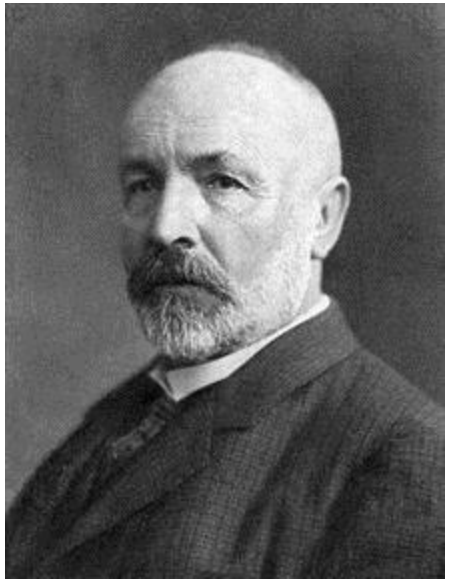
# Another Correlary

- Some languages have no finite description!

# Yet Another Correlary

- Some languages (problems) cannot be described!
- Can any of these be decided by Java?

# Cantor's Theorem



- For any set $S$, $|2^S| < S$
  - Holds when $S$ is finite (homework)
  - What about when $S$ is infinite?
  - If $S$ is countably infinite: diagonalization
- Assume toward contradiction we have $f: S \leftrightarrow 2^S$
  - Let $T = \{x \in S | x \in f(x)\}$
  - Note that $T \subseteq S$, so there must be some $x_t$ s.t. $f(x_t) = T$
  - Is $x_t \in T$?

# Continuum Hypothesis

- We know that $|\mathbb{N}| < |\mathbb{R}|$
- Is there a set $S$ s.t. $|\mathbb{N}| < |S| < |\mathbb{R}|$?
- Answer:
  - Unanswerable

# Godel’s Incompleteness Theorem

- Says any axiomatic system is at least one of:
  1. **Inconsistent**: There are false things that you can prove
  2. **Incomplete**: There are true things that you cannot prove
  3. **Weak**: You can’t talk about prime numbers
- Proof idea: Show that any system can construct the paradox “This statement cannot be proven”

# Incompleteness in CS*

- Expectation Maximization Problem
  - You want to put ads on your website
  - You don't know yet who will visit your website
  - Select ads to maximize the maximum number of potential customers
- Answering this problem requires "tools" not yet addressed by set theory!

* https://www.nature.com/articles/s42256-018-0002-3