CS3102 Theory of Computation

Warm up:

How do I know X isn't in the list?

$\mathbb{N}$	$\mathbb{R}$											
$f(1) =$	3	.	1	4	1	5	9	2	6	5	3	…
$f(2) =$	1	.	0	0	0	0	0	0	0	0	0	…
$f(3) =$	2	.	7	1	8	2	8	1	8	2	8	…
$f(4) =$	1	.	4	1	4	2	1	3	5	6	2	…
$f(5) =$	0	.	3	3	3	3	3	3	3	3	3	…
…	…											

$X = 0.\ 2\ \ 1\ \ 9\ \ 3\ \ 4\ \ldots \in \mathbb{R}$

Take-away Ideas

- All finite sets are countable
- Anything with a bijection to the naturals is countable
- A subset of a countable set is countable
- A union of countably many sets is countable
  - Formal proof of this is homework
- To be computable by Java, the set of possibilities must be countable!

Are the Real Numbers Countable?

- Things that don't work:
  - List out every real number that starts with 1, then 2, then 3, …
  - List out every real number that has one number after the decimal, then 2, then 3, …
- How would we prove it wasn't?

Diagonalization

- Used to prove that a set is not countable
  - Shows that there cannot be a bijection with the Natural Numbers

1. Assume toward a contradiction there is a bijection with the natural numbers
2. Treat this arbitrary bijection as an ordered list containing all items (item 0 is the thing which maps to 0, etc.)
3. Show that this list must always be missing something

Diagonalization

Assume toward a contradiction that $f: \mathbb{N} \leftrightarrow \mathbb{R}$, show that f cannot be onto (something from $\mathbb{R}$ is not mapped to)

$\mathbb{N}$	$\mathbb{R}$											
$f(1) =$	3	.	1	4	1	5	9	2	6	5	3	…
$f(2) =$	1	.	0	0	0	0	0	0	0	0	0	…
$f(3) =$	2	.	7	1	8	2	8	1	8	2	8	…
$f(4) =$	1	.	4	1	4	2	1	3	5	6	2	…
$f(5) =$	0	.	3	3	3	3	3	3	3	3	3	…
…	…											

$X = 0.2\ 1\ 9\ 3\ 4 \ldots \in \mathbb{R}$

This number X cannot appear anywhere in the list.

It's different from each $f(i)$ at digit i

Is the set of all Languages Countable?

$\mathbb{N}$	ε	a	b	aa	ab	ba	bb	aaa	…			
$f(1) =$	1	1	1	1	1	1	1	1	1	1	1	…
$f(2) =$	1	0	1	0	1	0	1	0	1	0	1	…
$f(3) =$	0	1	0	1	0	1	0	1	0	1	0	…
$f(4) =$	1	1	0	1	1	0	1	1	0	1	1	…
$f(5) =$	0	0	0	1	0	0	1	1	1	0	1	…
…	…											
L=	0	1	1	0	1	…						

Each row represents a language which includes string i provided column i has a 1

This Language L cannot appear anywhere in the list. It’s different from each $f(i)$ because its containment of string i is opposite

Correlary

- Some languages cannot be decided by Java

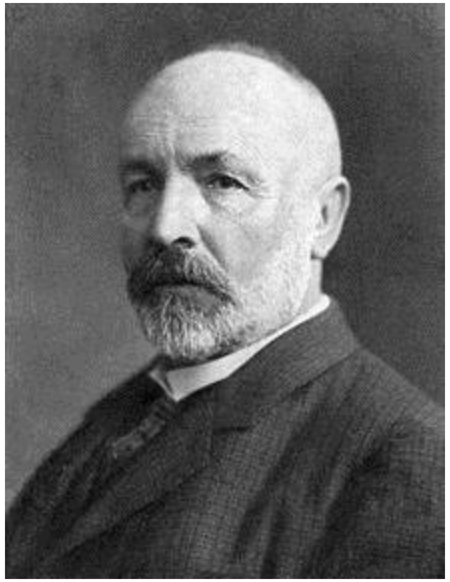
Another Correlary

- Some languages have no finite description!

Yet Another Correlary

- Some languages (problems) cannot be described!
- Can any of these be decided by Java?

Cantor's Theorem



- For any set S, $|2^S| < S$
  - Holds when S is finite (homework)
  - What about when S is infinite?
  - If S is countably infinite: diagonalization
- Assume toward contradiction we have $f: S \leftrightarrow 2^S$
  - Let $T = \{x \in S | x \in f(x)\}$
  - Note that $T \subseteq S$, so there must be some x_t s.t. $f(x_t) = T$
  - Is $x_t \in T$?

Continuum Hypothesis

- We know that $|\mathbb{N}| < |\mathbb{R}|$
- Is there a set S s.t. $|\mathbb{N}| < |S| < |\mathbb{R}|$?
- Answer:
  – Unanswerable

Godel’s Incompleteness Theorem

- Says any axiomatic system is at least one of:
  1. **Inconsistent**: There are false things that you can prove
  2. **Incomplete**: There are true things that you cannot prove
  3. **Weak**: You can’t talk about prime numbers
- Proof idea: Show that any system can construct the paradox “This statement cannot be proven”

Incompleteness in CS*

- Expectation Maximization Problem
  - You want to put ads on your website
  - You don't know yet who will visit your website
  - Select ads to maximize the maximum number of potential customers
- Answering this problem requires "tools" not yet addressed by set theory!

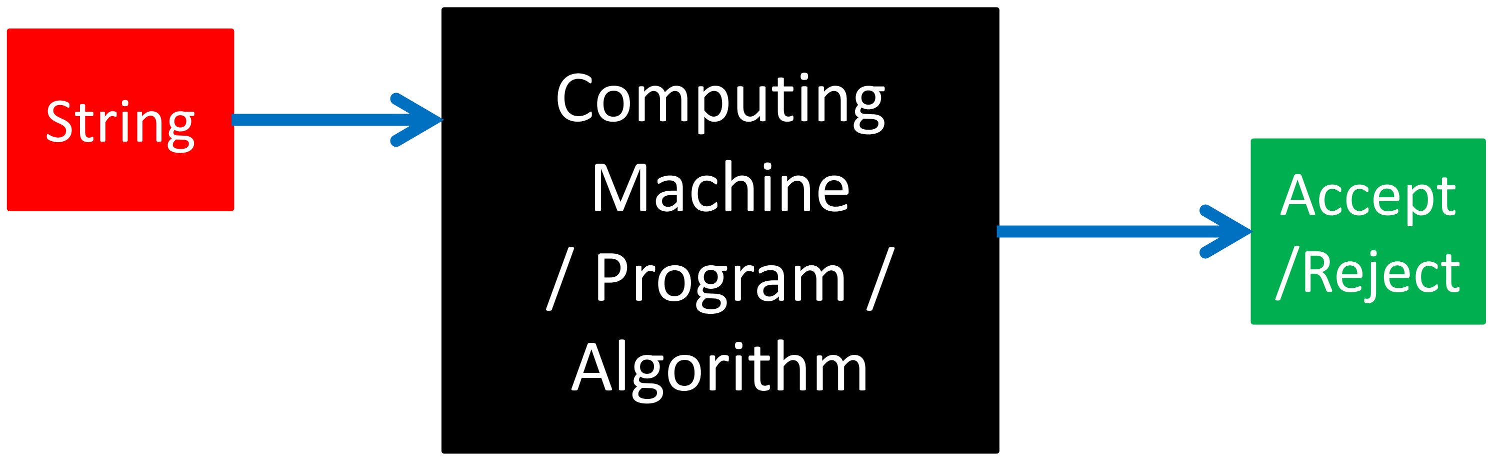
* https://www.nature.com/articles/s42256-018-0002-3

End of Phase 1

- Until now:
  - Mathematical foundations
  - Proof strategies
  - Key ideas/insights
  - Main takeaway: Some languages (and numbers) cannot be computed by Java (or anything else)
    - Why? There are more language (numbers) than there are Java programs (or even finite descriptions)

Phase 2

- Now we start filling in this box
  - First option: finite state machine



Operations on Strings

- Length
  - $|s|$= Number of characters in the string s
  - $|Ringo| = 5$
- Concatenation
  - $s \cdot t = st =$ string which has all of the characters from s followed by all of the characters from t
  - $John \cdot Paul = JohnPaul$
  - $|s \cdot t| = |s| + |t|$
- Exponentiation
  - s^k =The string created by concatenation s with itself k times
  - $(George)^5 = GeorgeGeorgeGeorgeGeorgeGeorge$
  - $\left|s^k\right| = |s| \cdot k$

Empty String (“”)

- Notation for this class: ε
  - \varepsilon in Latex
- $|\varepsilon| = 0$
- $s \cdot \varepsilon = s$
- $\varepsilon^k = \varepsilon$
- $s^0 = \varepsilon$

Operations on Languages

- Everything we can do on sets (∪,∩, −, …)
- Concatenation
- Exponentiation
- Kleene Closure

Language Concatenation

- $L_1 \cdot L_2$ or $L_1 L_2$
  - Notation is the same as string concatenation
  - Every possible way to concatenate a string from L_1 with a string from L_2 (in that order)
  - Idea: take $L_1 \times L_2$ and concatenate the strings that are paired
  - $\{john, paul\} \cdot \{george, ringo\} = \{johngeorge, jonringo, paulgeorge, paulringo\}$
  - $|L_1 L_2| \leq |L_1 \times L_2|$
  - $\{a, aa, aaa\} \cdot \{a, aa\} = \{aa, aaa, aaaa, aaaaa\}$

Language Exponentiation

- L^k
  - L concatenated with itself k times
  - $L^5 = L \cdot L \cdot L \cdot L \cdot L$
  - $\{a, b\}^3 = \{aaa, aab, aba, abb, baa, bab, bba, bbb\}$
  - $L^0 = \{\varepsilon\}$

Kleene Closure

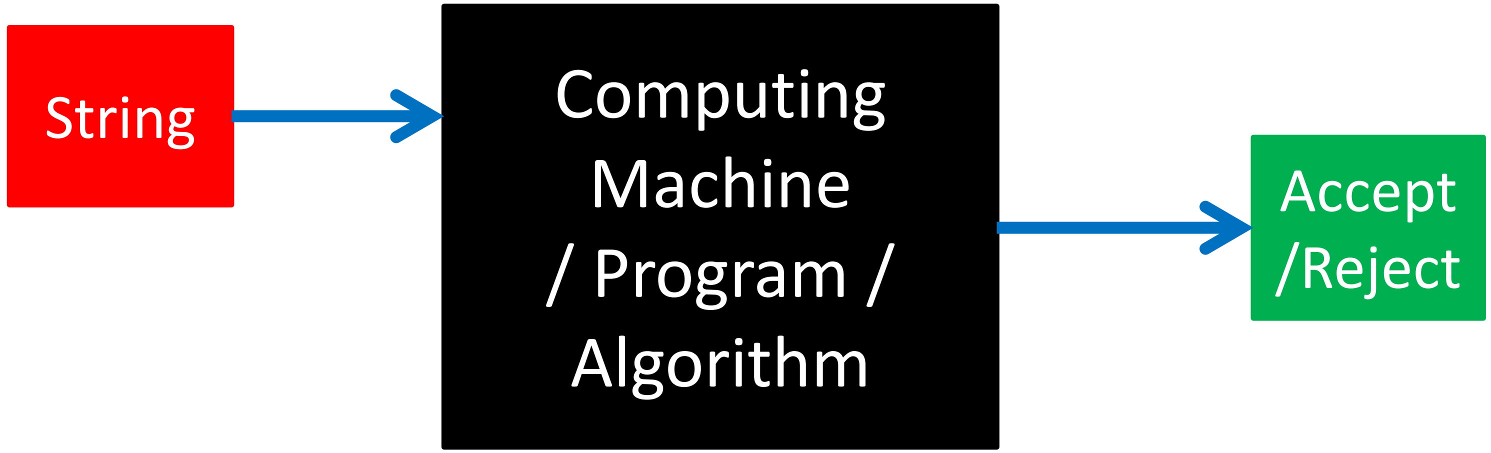
- L^*
  - L concatenated with itself 0 or more times
  - $L^* = L^0 \cup L^1 \cup L^2 \cup L^3 \ldots$
  - $\{a, bb\}^* = \{\varepsilon, a, bb, aa, abb, bba, bbbb, aaa, \ldots\}$
  - $\emptyset^* = \{\varepsilon\}$
  - $\{\varepsilon\}^* = \{\varepsilon\}$
  - For any other language L, L^* is infinite

Sigma Star

- We denote our alphabet as Σ
  - \Sigma in Latex
- A character is just a really short string, so an alphabet is a language
- Σ^* is the set of all strings using the alphabet Σ
- 2^{Σ^*} is the set of all languages using Σ

What Shall we put in the box?

- Goal: start with something easy to prove things about
- We’ve talked about Java, but that’s complex



Finite State Automaton

- Simple model of computation
- Represents computation without memory
- Kind of decider
- Our machine reads the input string only once, and one character at a time
- After reading each character, enters a new “state”
- State transition rules depend only on the current state and the current character (no looking back!)
- There are only finitely many states

Gumball Machine

- Our gumball machine takes only coins and does not give change
- Each gumball costs 7 cents
- $\Sigma = \{p, n\}$ (penny, nickel)
- We need to decide the language of sequences of coins adding up to at least 7 cents

Gumball Machine

- What are all the possible “states” the machine could be in?
- 0c, 1c, 2c, 3c, 4c, 5c, 6c, 7+c
- Which “state” should the machine start in?
- Which “state” means we’ve sold a gumball?
- 6c plus a penny is always 7c, no matter how I got to 6c ($pppppp$, or pn, or np)

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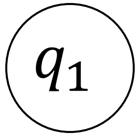
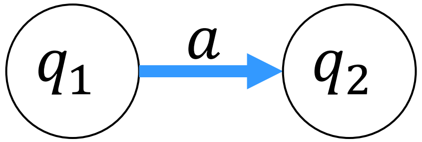
Finite State Automata

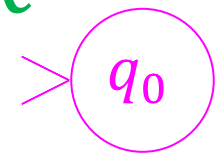
- Basic idea: a FA is a "machine" that changes states while processing symbols, one at a time.
- Finite set of states: $Q = \{q_0, q_1, \dots q_7\}$



- Transition function: $\delta: Q \times \Sigma \to Q$



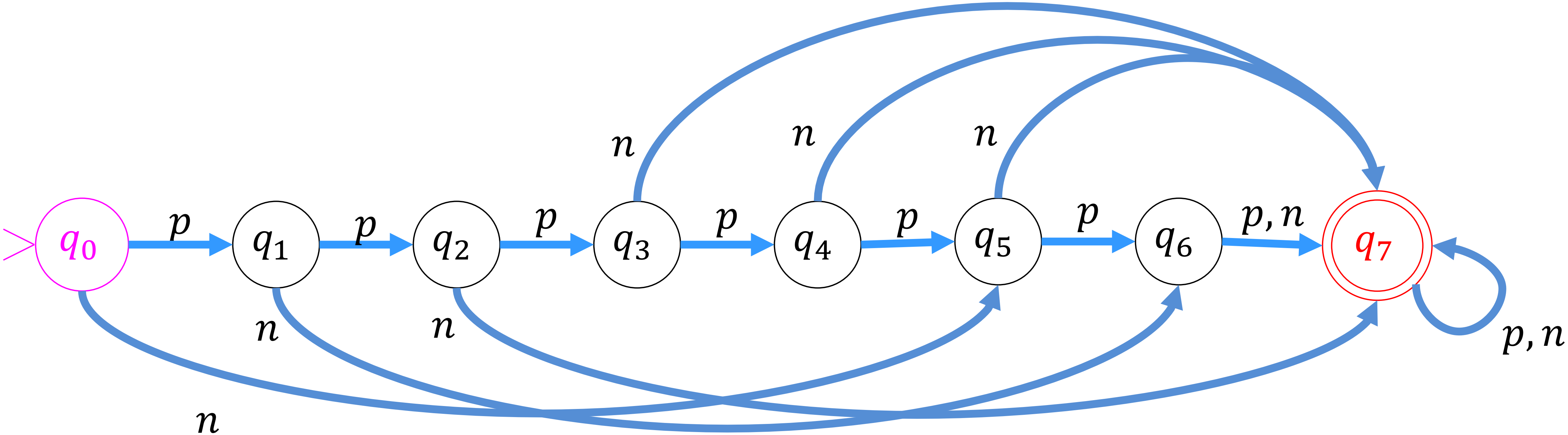
- Initial state: $q_0 \in Q$



- Final states: $F \subseteq Q$

- Finite state automaton is $M = (Q, \Sigma, \delta, q_0, F)$
- Accept if we end in a Final state, otherwise Reject

FSA for our Gumball Machine



Strings this accepts:

$pppppppp$

$nnnnnnn$

pnp

ppn

Strings this rejects:

ppp

n

np

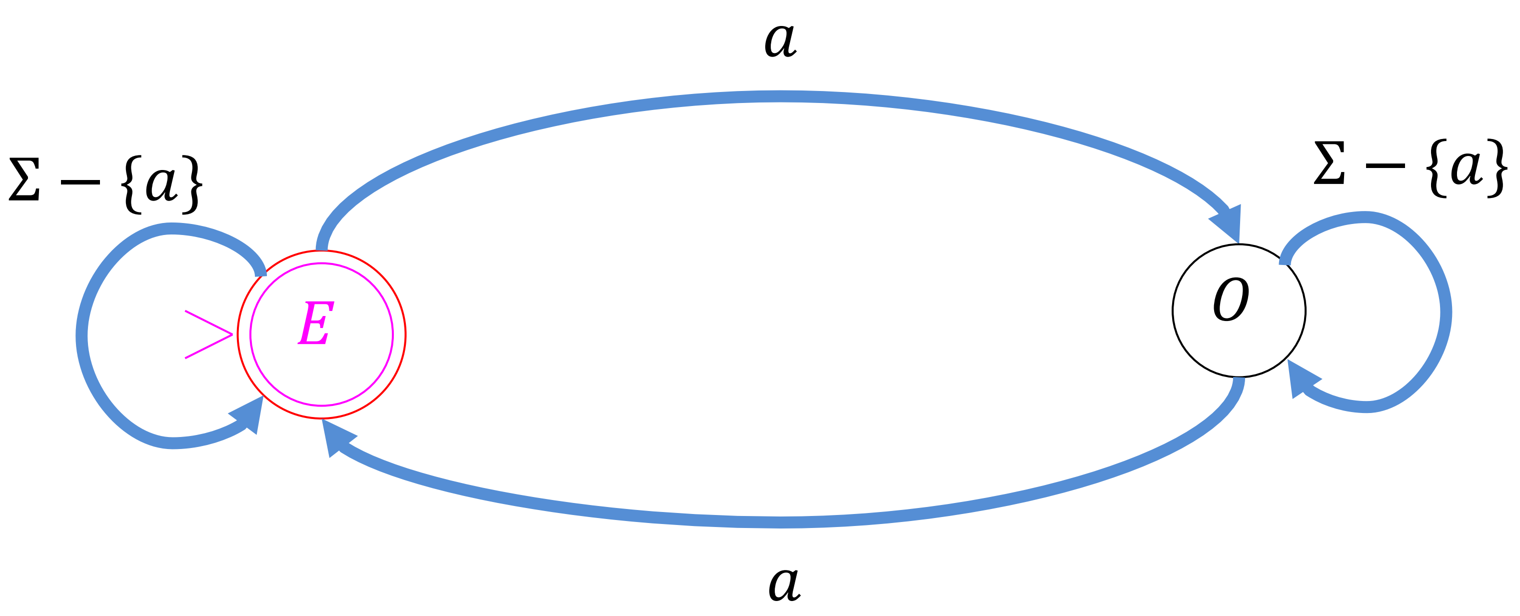
EvenA

- In HW1 you were asked to give a decider for EvenA (accepts all strings with an even number of A’s)
- How did you do it?

EvenA using FSA

1. What's our alphabet? (pick Σ)
2. What should our states be? (pick Q)
3. Which states are the accept states? (pick F)
4. Which state is the start state? (pick q_0)
5. How should we transition? (pick δ)

Let's Draw It!



EvenAOddB

- Let's make a finite state automaton which accepts strings that have an even number of a's followed by an odd number of b's (in that order)
- It should accept:
  - $b, bbb, aab, aaaabbbbb, \ldots$
- It should reject:
  - $bb, ab, baa, aba, aaabb$

EvenAOddB using FSA

1. What’s our alphabet? (pick Σ)
2. What should our states be? (pick Q)
3. Which states are the accept states? (pick $\mathbf{F}$)
4. Which state is the start state? (pick q_0)
5. How should we transition? (pick δ)