CS4102 Algorithms

Fall 2018

Warm up

Dynamic Programming in General

Generic Top-Down Dynamic Programming Soln

Apply to Matrix Chain Multiplication

```
mem = {}
def myDPalgo(problem):
        if mem[problem] not blank:
                return mem[problem]
        if baseCase(problem):
                solution = solve(problem)
                mem[problem] = solution
                return solution
        for subproblem of problem:
                subsolutions.append(myDPalgo(subproblem))
        solution = OptimalSubstructure(subsolutions)
        mem[problem] = solution
        return solution
```

Matrix Chaining

$$\underset{30 \times 35}{M_1} \times \underset{35 \times 15}{M_2} \times \underset{15 \times 5}{M_3} \times \underset{5 \times 10}{M_4} \times \underset{10 \times 20}{M_5} \times \underset{20 \times 25}{M_6}$$

$$Best(i,j) = \min_{k=1}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$j =$	1	2	3	4	5	6	$i =$
	0	15750	7875	9375	11875	**15125**	1
		0	2625	4375	7125	10500	2
			0	750	2500	5375	3
				0	1000	3500	4
					0	5000	5
						0	6

$$Best(1,6) = \min \begin{cases} Best(1,1) + Best(2,6) + r_1 r_2 c_6 \\ Best(1,2) + Best(3,6) + r_1 r_3 c_6 \\ Best(1,3) + Best(4,6) + r_1 r_4 c_6 \\ Best(1,4) + Best(5,6) + r_1 r_5 c_6 \\ Best(1,5) + Best(6,6) + r_1 r_6 c_6 \end{cases}$$

Today’s Keywords

- Dynamic Programming
- Gerrymandering

CLRS Readings

- Chapter 15
- Chapter 16

Homeworks

- Homework 5 due Friday @ 11pm
  - Dynamic Programming
  - Python or Java

Dynamic Programming

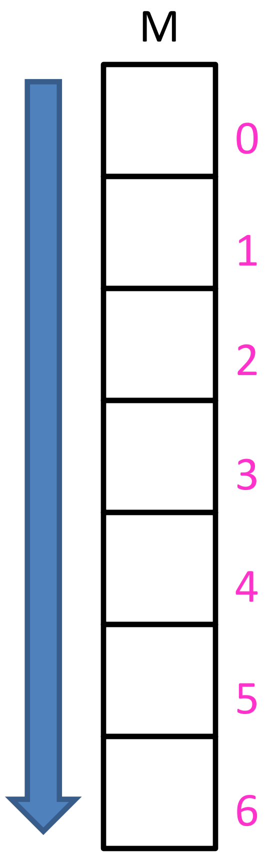
- Requires Optimal Substructure
  - Solution to larger problem contains the solutions to smaller ones
- Idea:
  1. Identify recursive structure of the problem
     - What is the “last thing” done?
  2. Select a good order for solving subproblems
     - “Top Down”: Solve each recursively
     - “Bottom Up”: Iteratively solve smallest to largest
  3. Save solution to each subproblem in memory

DP Algorithms so far

- $2 \times n$ domino tiling (Fibonacci)
- Log cutting
- Matrix Chaining
- Longest Common Subsequence
- Seam Carving

Domino Tiling

```
Tile(n):
    Initialize Memory M
    M[0] = 0
    M[1] = 0
    for i = 0 to n:
        M[i] = M[i-1] + M[i-2]
    return M[n]
```



Log Cutting

Solve Smallest subproblem first

$$Cut(4) = \max \begin{cases} Cut(3) + P[1] \\ Cut(2) + P[2] \\ Cut(1) + P[3] \\ Cut(0) + P[4] \end{cases}$$

$Cut(i)$:	0										
Length:	0	1	2	3	4	5	6	7	8	9	10

4

Matrix Chaining

$$\underset{30\times 35}{M_1} \times \underset{35\times 15}{M_2} \times \underset{15\times 5}{M_3} \times \underset{5\times 10}{M_4} \times \underset{10\times 20}{M_5} \times \underset{20\times 25}{M_6}$$

$$Best(i,j) = \min_{k=1}^{j-1}\Big(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\Big)$$

$$Best(i,i) = 0$$

$j =$ 1	2	3	4	5	6	$= i$
0	15750	7875	9375	11875	15125	1
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$$Best(1,6) = \min \begin{cases} Best(1,1) + Best(2,6) + r_1 r_2 c_6 \\ Best(1,2) + Best(3,6) + r_1 r_3 c_6 \\ Best(1,3) + Best(4,6) + r_1 r_4 c_6 \\ Best(1,4) + Best(5,6) + r_1 r_5 c_6 \\ Best(1,5) + Best(6,6) + r_1 r_6 c_6 \end{cases}$$

Longest Common Subsequence

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

$Y =$ \ $X =$		0	A 1	T 2	C 3	T 4	G 5	A 6	T 7
	0	0	0	0	0	0	0	0	0
T	1	0	0	1	1	1	1	1	1
G	2	0	0	1	1	1	2	2	2
C	3	0	0	1	2	2	2	2	2
A	4	0	1	1	2	2	2	3	3
T	5	0	1	2	2	3	3	3	4
A	6	0	1	2	2	3	3	4	4

To fill in cell (i,j) we need cells $(i-1,j-1), (i-1,j), (i,j-1)$
Fill from Top->Bottom, Left->Right (with any preference)

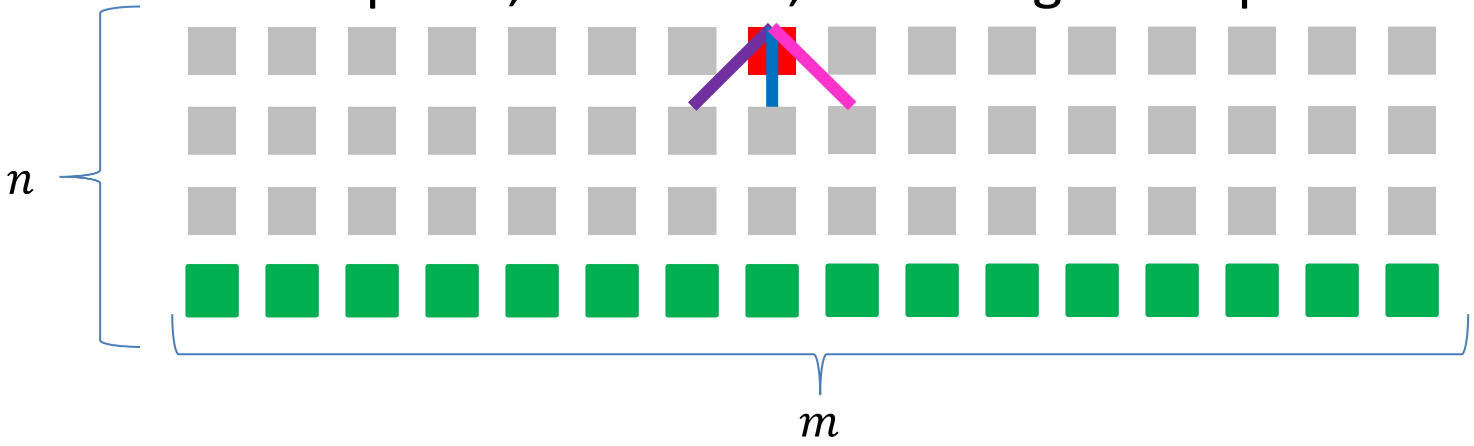
Seam Carving

Start from bottom of image (row 1), solve up to top

Initialize $S(1,k) = e(p_{1,k})$ for each pixel $p_{1,k}$

For $i > 2$ find $S(i,k) = \begin{cases} S(n-1,k-1) + e(p_{n,k}) \\ S(n-1,k) + e(p_{n,k}) \\ S(n-1,k+1) + e(p_{n,k}) \end{cases}$

Pick smallest from top row, backtrack, removing those pixels



Energy of the seam initialized to the energy of that pixel

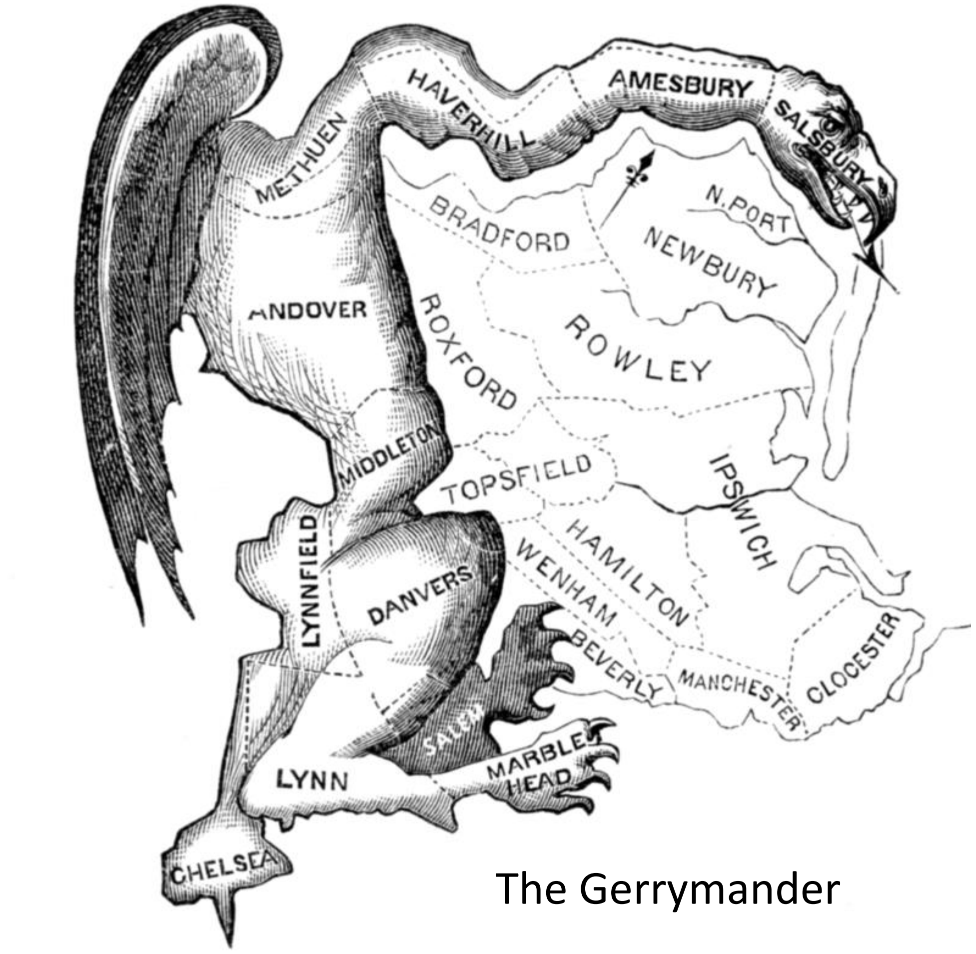
Supreme Court Associate Justice Anthony Kennedy gave no sign that he has abandoned his view that extreme partisan gerrymandering might violate the Constitution. I Eric Thayer/Getty Images

Supreme Court eyes partisan gerrymandering

Anthony Kennedy is seen as the swing vote that could blunt GOP's map-drawing successes.

Gerrymandering

- Manipulating electoral district boundaries to favor one political party over others
- Coined in an 1812 Political cartoon
- Governor Gerry signed a bill that redistricted Massachusetts to benefit his Democratic-Republican Party

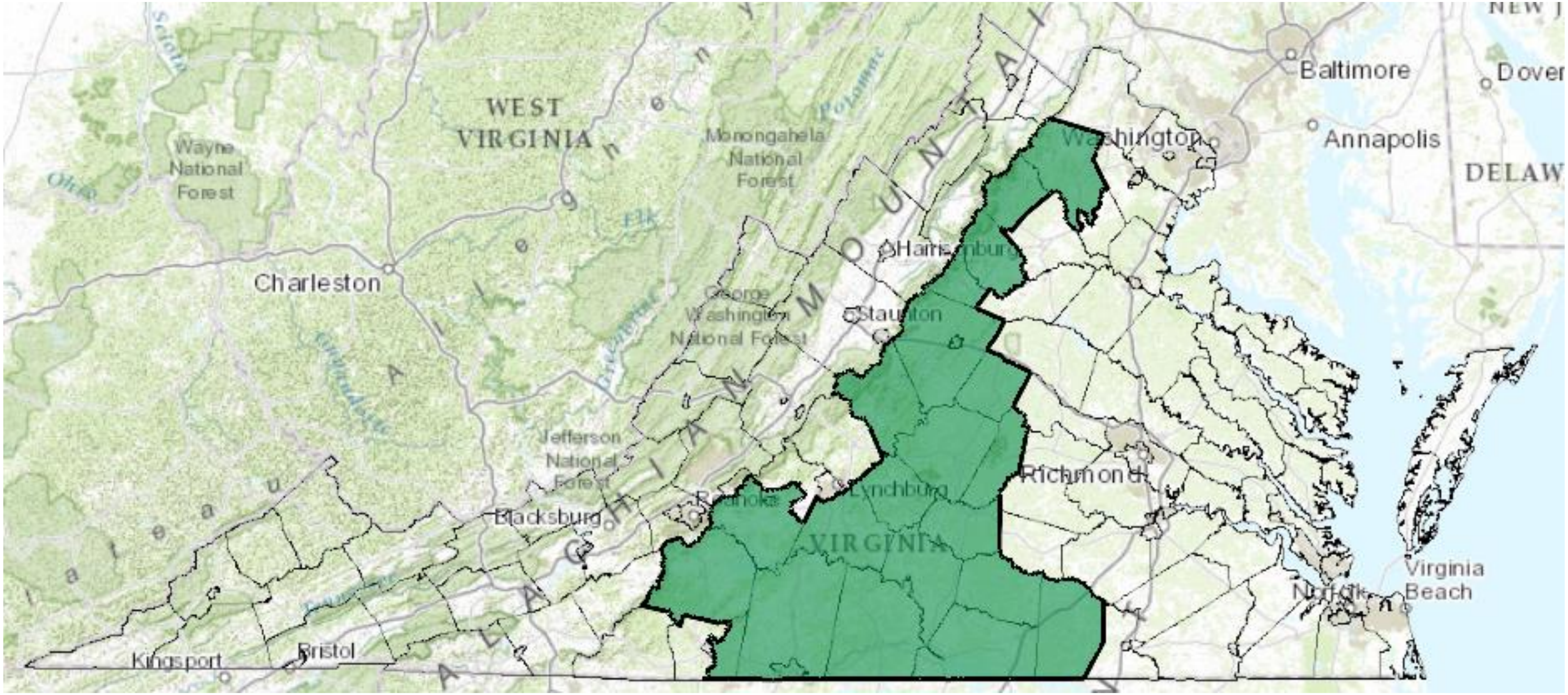


The Gerrymander

According to the Supreme Court

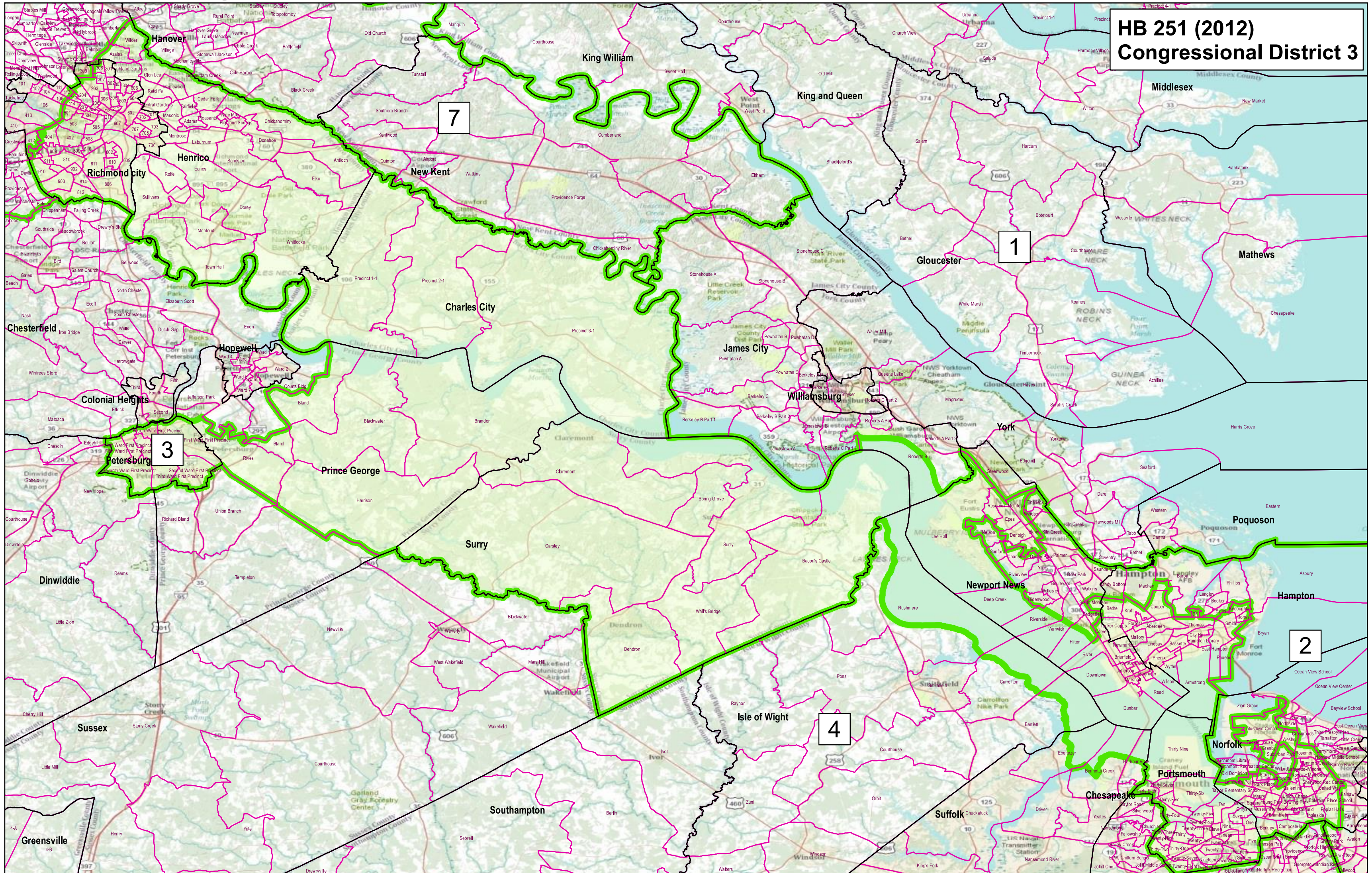
- Gerrymandering cannot be used to:
  - Disadvantage racial/ethnic/religious groups
- It can be used to:
  - Disadvantage political parties

VA 5th District



Gerrymandering Today

- Computers make it really effective



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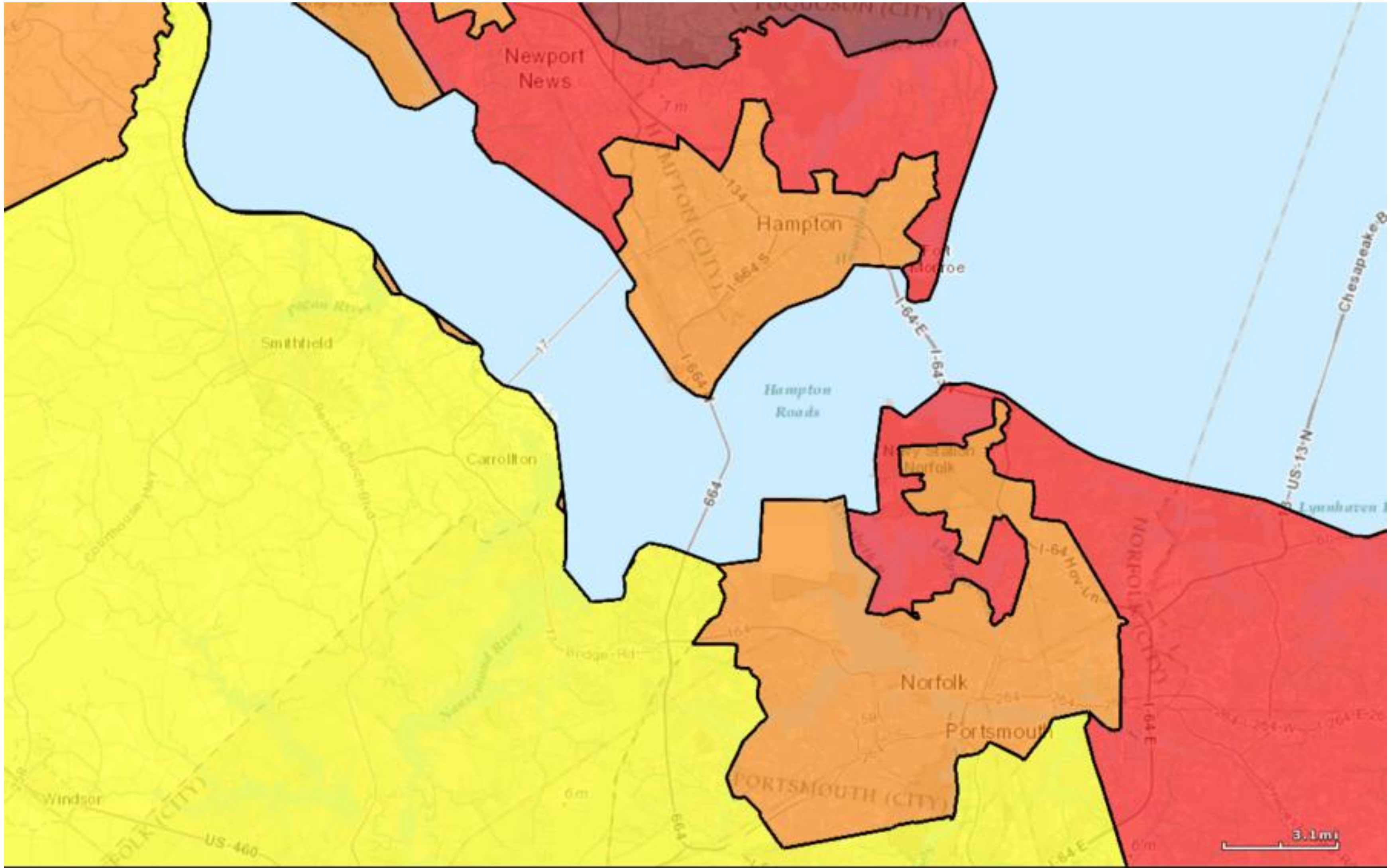

Gerrymandering Today

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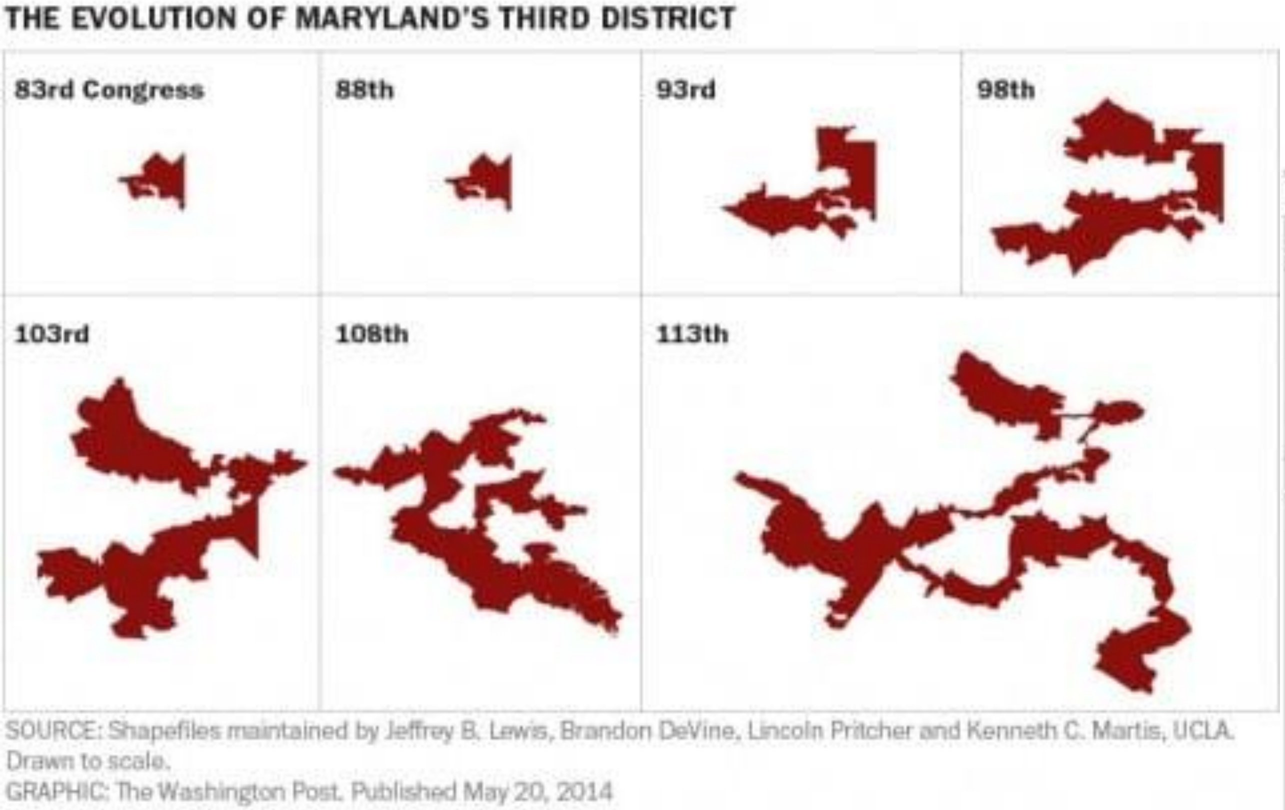
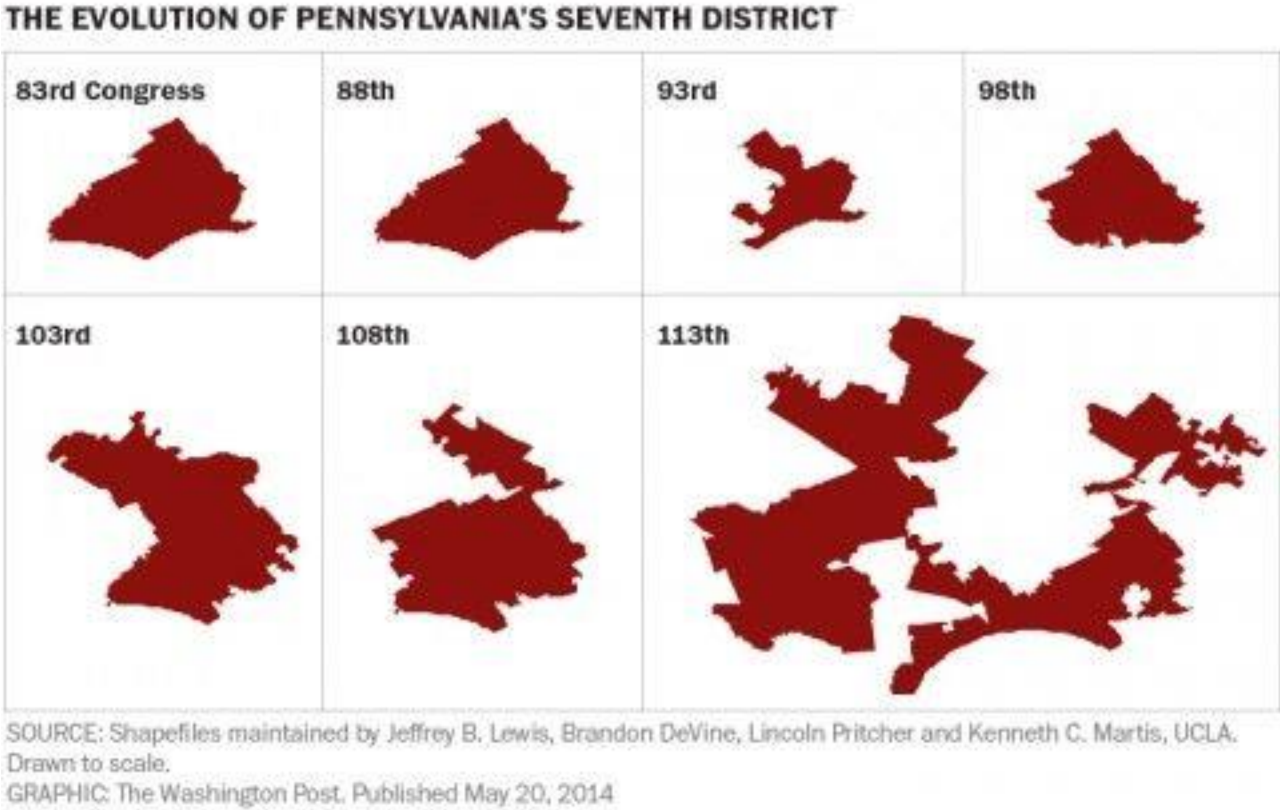
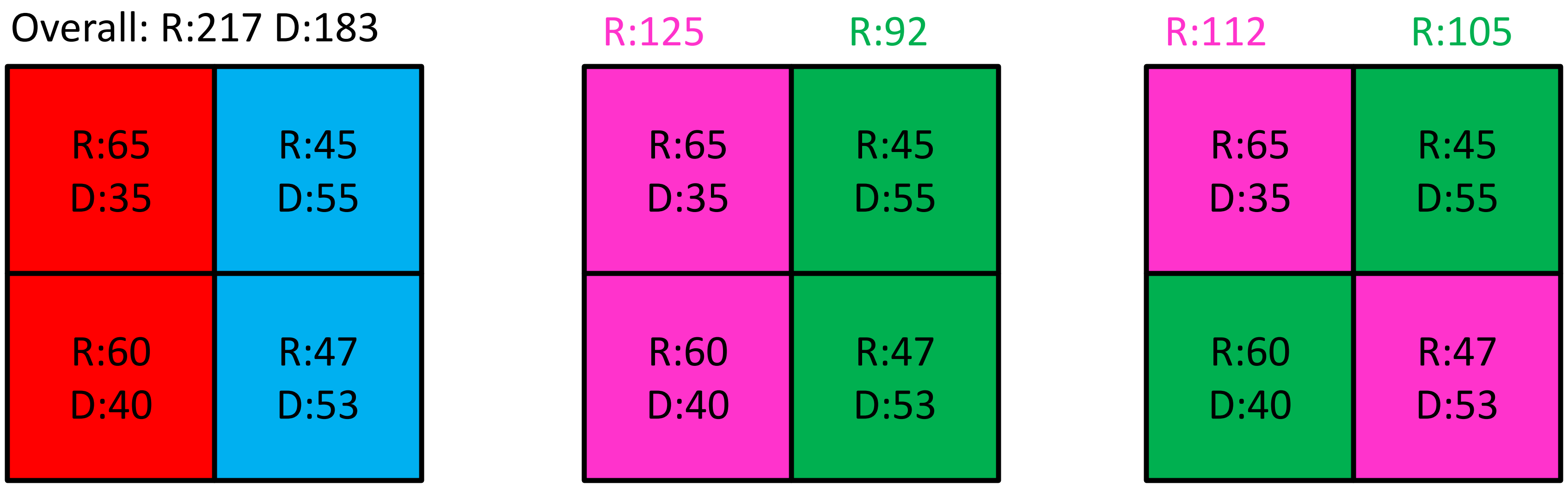




How does it work?

- States are broken into precincts
- All precincts have the same size
- We know voting preferences of each precinct
- Group precincts into districts to maximize the number of districts won by my party



Gerrymandering Problem Statement

- Given:
  - A list of precincts: $p_1, p_2, \ldots, p_n$
  - Each containing m voters
- Output:
  - Districts $D_1, D_2 \subset \{p_1, p_2, \ldots, p_n\}$
  - Where $|D_1| = |D_2|$
  - $R(D_1), R(D_2) > \frac{mn}{4}$
    - $R(D_i)$ gives number of "Regular Party" voters in D_1
    - $R(D_i) > \frac{\mathrm{mn}}{4}$ means D_i is majority "Regular Party"
  - "failure" if no such solution is possible

Dynamic Programming

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- Idea:
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Consider the last precinct

D_1
k precincts
x voters for R

D_2
$n-k-1$ precincts
y voters for R

After assigning the first $n-1$ precincts

p_n

If we assign p_n to D_1

D_1
$k+1$ precincts
$x+R(p_n)$ voters for R

Valid gerrymandering if:
$k+1=\frac{n}{2}$,
$x+R(p_n), y>\frac{mn}{4}$

If we assign p_n to D_2

D_2
$n-k$ precincts
$y+R(p_n)$ voters for R

Valid gerrymandering if:
$\mathrm{n}-k=\frac{n}{2}$,
$x, y+R(p_n)>\frac{mn}{4}$

Define Recursive Structure

$S(j,k,x,y) =$ True if from among the first $\boldsymbol{j}$ precincts:

$\boldsymbol{k}$ are assigned to D_1

exactly $\boldsymbol{x}$ vote for R in D_1

exactly $\boldsymbol{y}$ vote for R in D_2

$n \times n \times mn \times mn$

4D Dynamic Programming!!!

Two ways to satisfy $S(j,k,x,y)$:

$S(j,k,x,y)$ = True if:
from among the first j precincts
k are assigned to D_1
exactly x vote for R in D_1
exactly y vote for R in D_2

D_1
$k-1$ precincts
$x-R(p_j)$ voters for R

D_2
$j-k$ precincts
y voters for R

p_j

Then assign p_j to D_1

D_1
k precincts
x voters for R

D_2
$j-k$ precincts
y voters for R

OR

D_1
k precincts
x voters for R

D_2
$j-1-k$ precincts
$y-R(p_j)$ voters for R

p_j

Then assign p_j to D_2

$$S(j,k,x,y) = S(j-1,k-1,x-R(p_j),y) \vee S\left(j-1,k,x,y-R(p_j)\right)$$

Final Algorithm

$$S(j,k,x,y) = S(j-1,k-1,x-R(p_j),y) \vee S\left(j-1,k,x,y-R(p_j)\right)$$

Initialize $S(0,0,0,0) = $ True
for $j = 1, \ldots, n$:
  for $k = 1, \ldots, \min(j, \frac{n}{2})$:
    for $x = 0, \ldots, jm$:
      for $y = 0, \ldots, jm$:
        $S(j,k,x,y) =$
          $S(j-1,k-1,x-R(p_j),y)$
          $\vee\, S\left(j-1,k,x,y-R(p_j)\right)$

Search for True entry at $S(n, \frac{n}{2}, > \frac{mn}{4}, > \frac{mn}{4})$

$S(j,k,x,y) = $ True if: from among the first j precincts k are assigned to D_1 exactly x vote for R in D_1 exactly y vote for R in D_2

Run Time

$$S(j,k,x,y) = S(j-1,k-1,x-R(p_j),y) \vee S\left(j-1,k,x,y-R(p_j)\right)$$

Initialize $S(0,0,0,0) = $ True

n for $j = 1, \dots, n$:

$\frac{n}{2}$ for $k = 1, \dots, \min(j, \frac{n}{2})$:

$nm$for $x = 0, \dots, jm$:

nm for $y = 0, \dots, jm$:

$$S(j,k,x,y) =$$

$$S(j-1,k-1,x-R(p_j),y)$$

$$\vee\, S\left(j-1,k,x,y-R(p_j)\right)$$

Search for True entry at $S(n, \frac{n}{2}, > \frac{mn}{4}, > \frac{mn}{4})$

$\Theta(n^4m^2)$

$\Theta(n^4m^2)$

- Runtime depends on the *value* of m, not *size* of m
- Run time is exponential in *size* of input
- Note: Gerrymandering is NP-Complete