CS4102 Algorithms

Fall 2018

Warm up:

Grab cookies!

Start with 2, leftovers will be at regrade office hours

Vegan & Gluten-free are available

Courtesy of Nate and Robbie

Today’s Keywords

- Reductions
- Bipartite Matching
- Vertex Cover
- Independent Set
- NP-Completeness

CLRS Readings

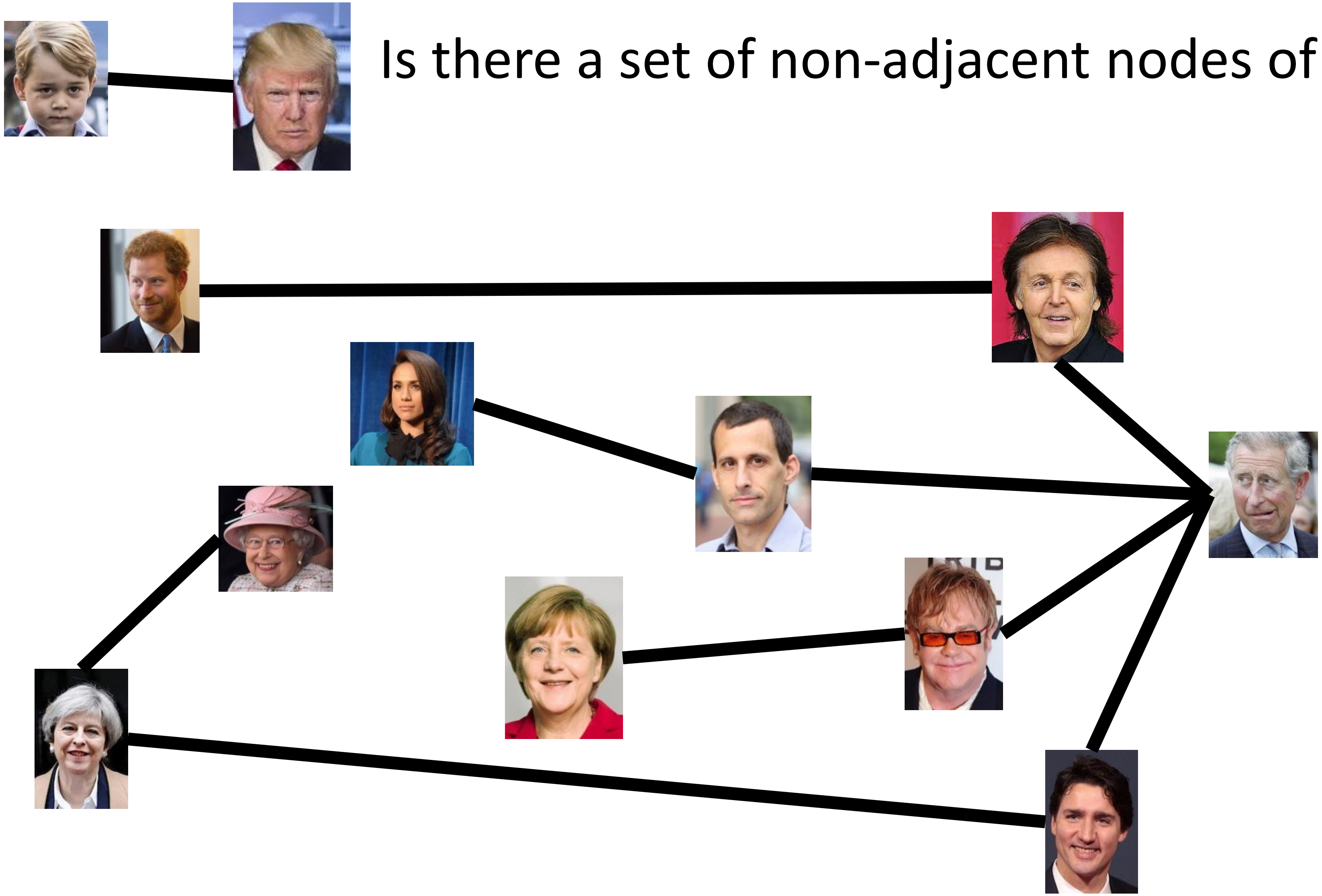
- Chapter 34

Homeworks

- HW9 due Friday 12/7 at 11pm
  - Written (use LaTeX)
  - Graphs

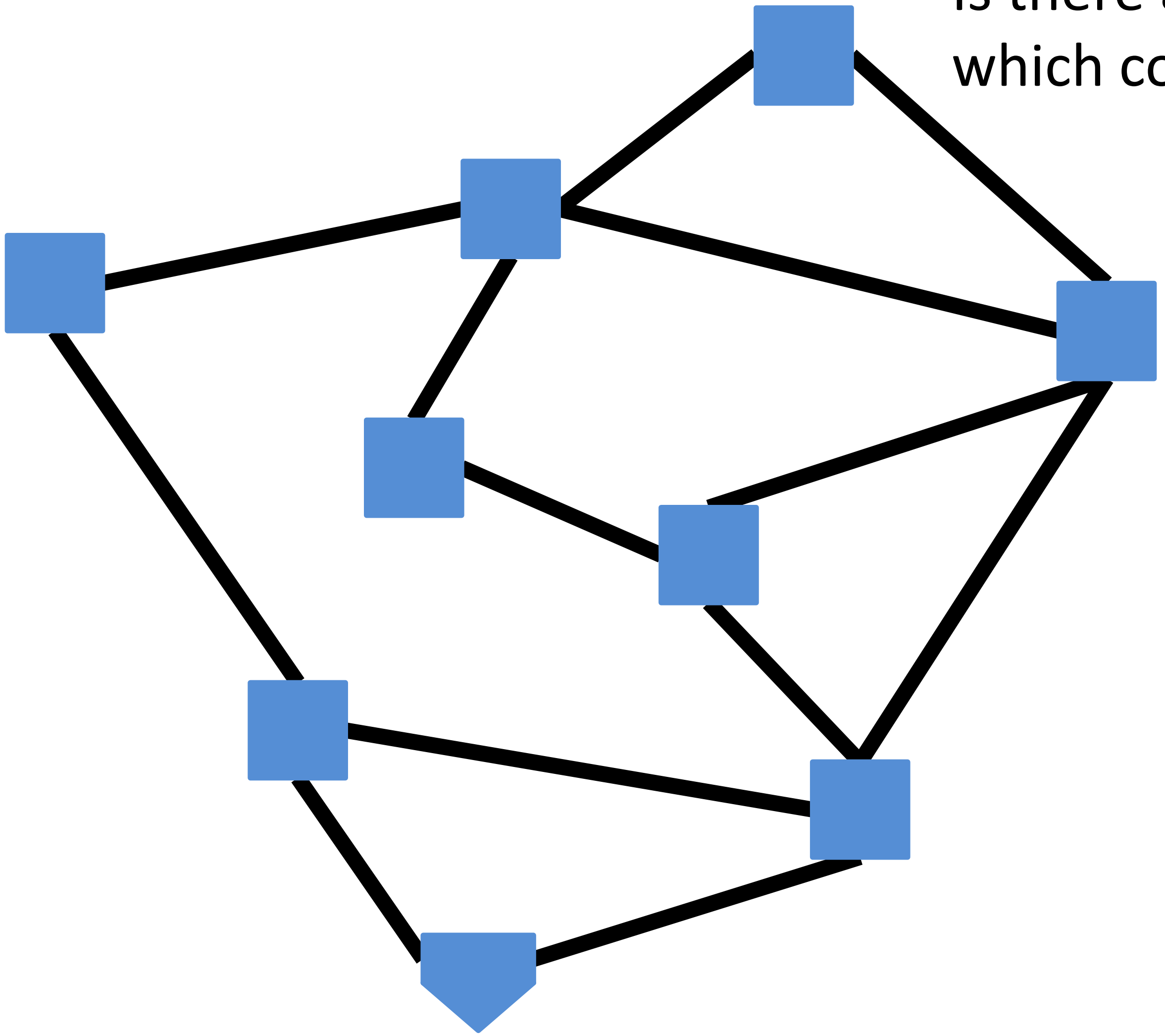
k Independent Set

Is there a set of non-adjacent nodes of size k?



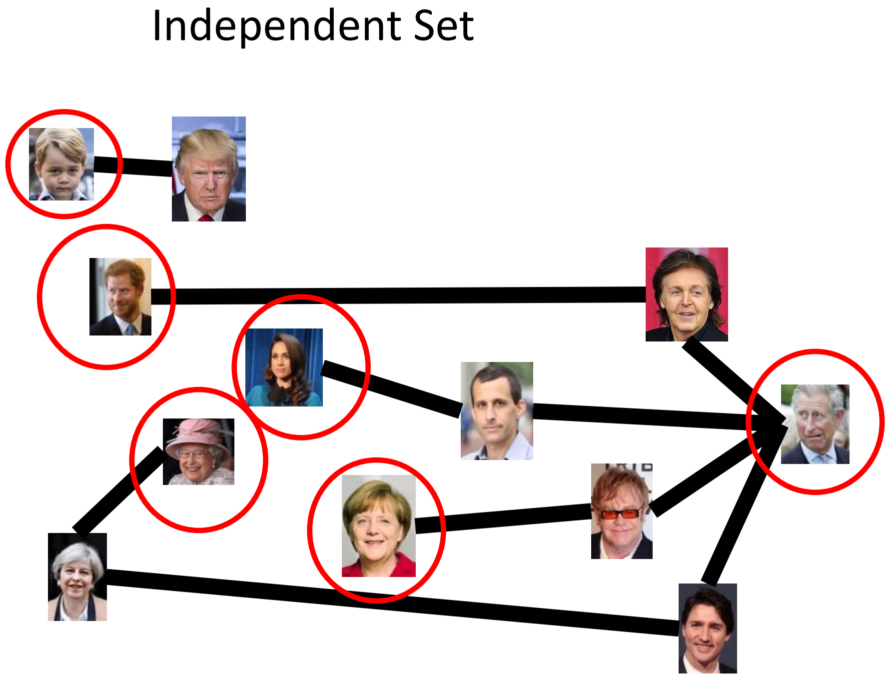
k Vertex Cover

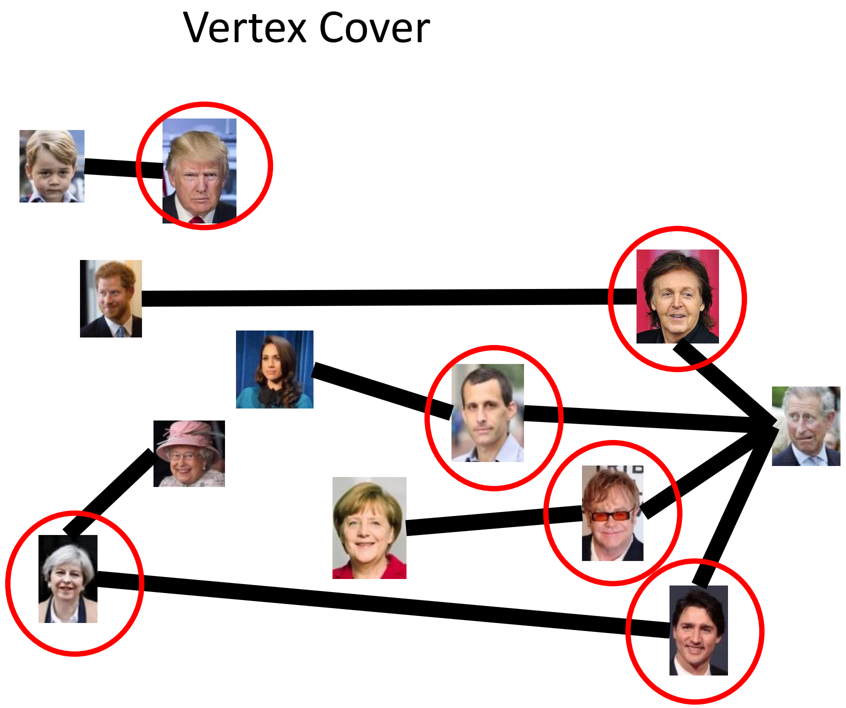
Is there a set of nodes of size k which covers every edge?



Reduction Idea

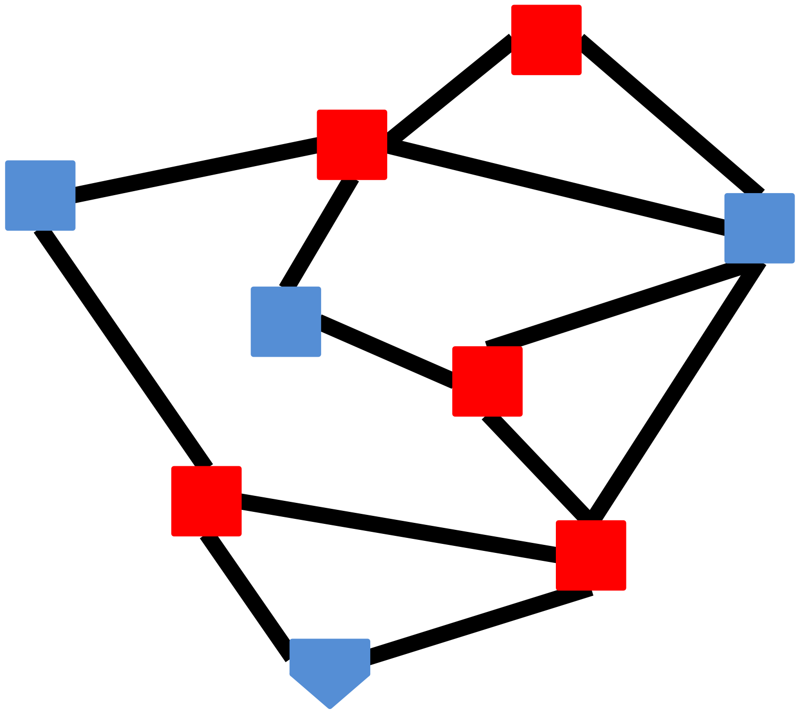
S is an independent set of G iff $V - S$ is a vertex cover of G

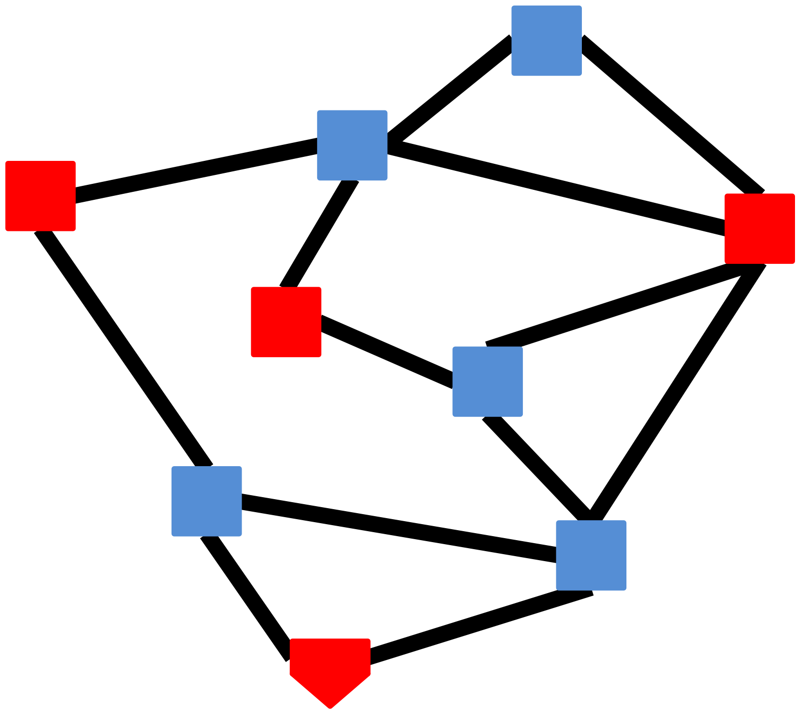




Reduction Idea

S is an independent set of G iff $V - S$ is a vertex cover of G





MacGyver’s Reduction

Problem we don’t know how to solve

A

Opening a door

Problem we do know how to solve

B

Lighting a fire

Aim duct at door, insert keg

How?

Solution for $\boldsymbol{A}$

Keg cannon battering ram

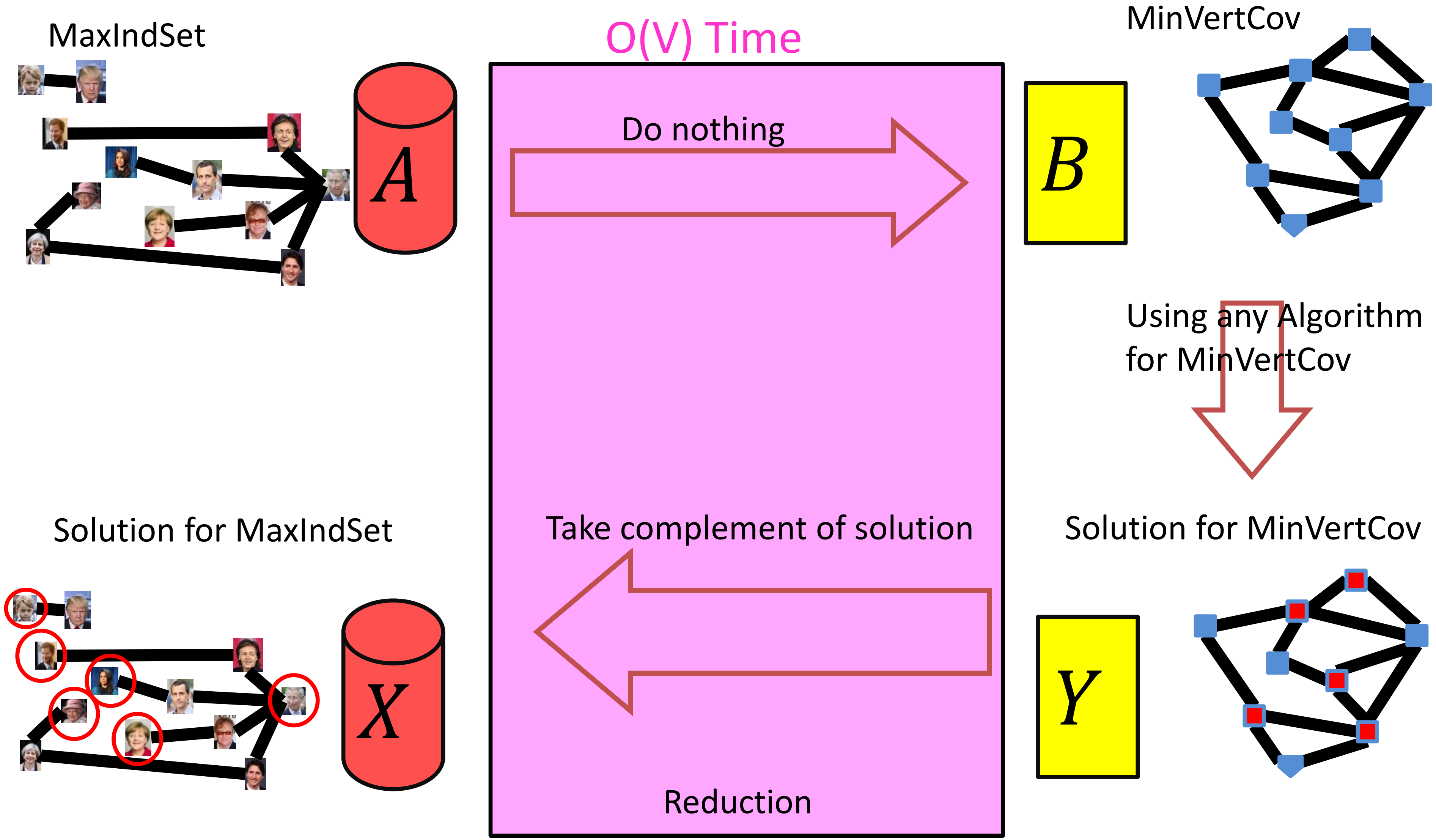
Put fire under the Keg

Reduction

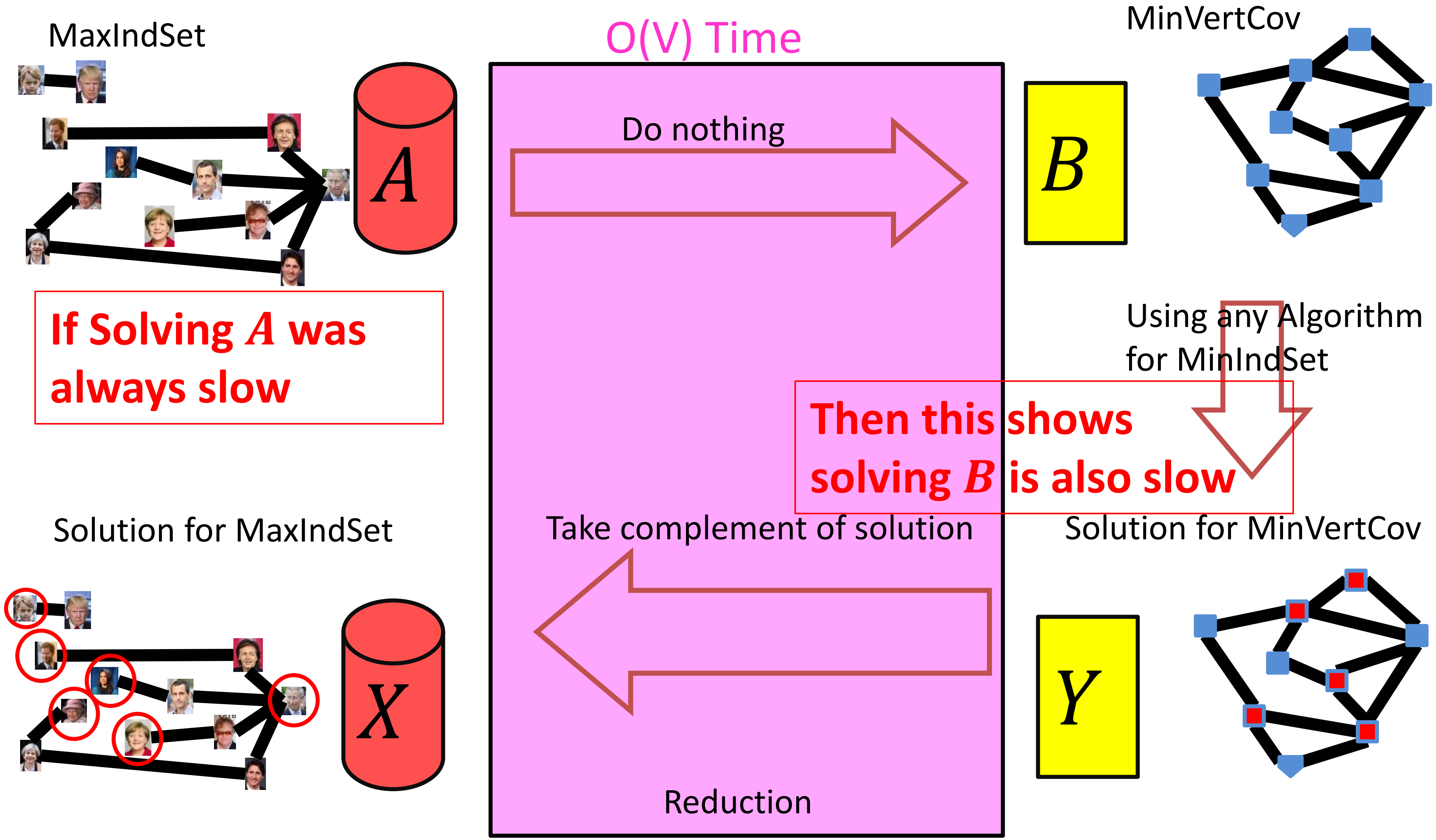
Solution for $\boldsymbol{B}$

Alcohol, wood, matches

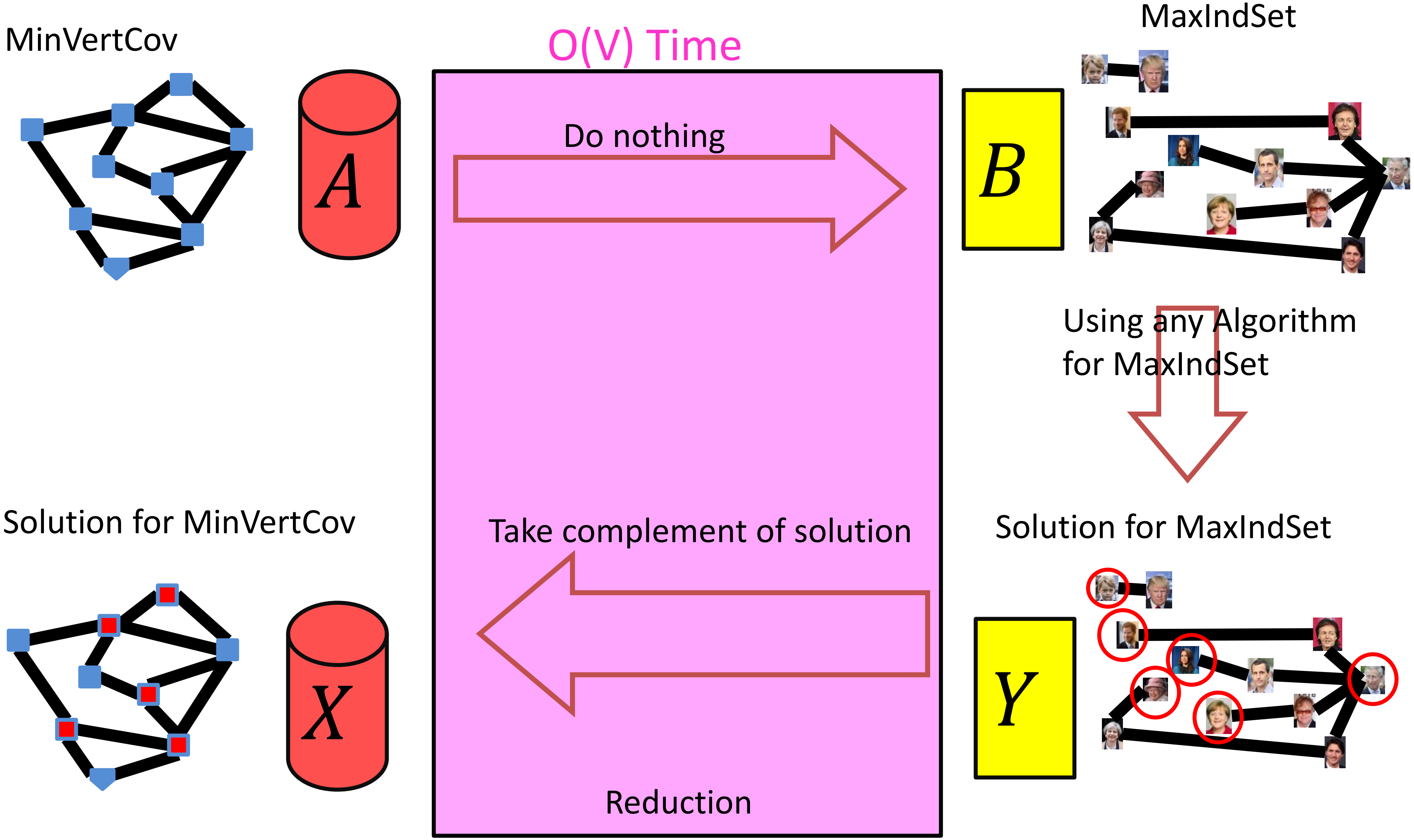
MaxVertCov V-Time Reducable to MinIndSet



Corollary

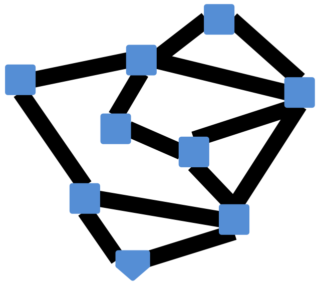

MaxIndSet
O(V) Time
MinVertCov
Do nothing
A
B
If Solving A was always slow
Using any Algorithm for MinIndSet
Then this shows solving B is also slow
Solution for MaxIndSet
Take complement of solution
Solution for MinVertCov
X
Y
Reduction

MaxIndSet V-Time Reducable to MinVertCov



Corollary

MinVertCov



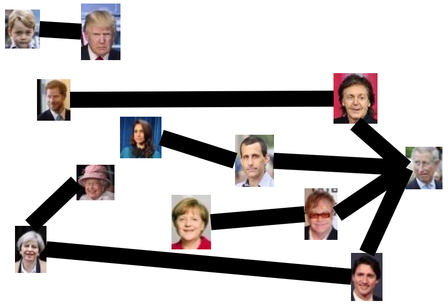
A

O(V) Time

Do nothing

B

MaxIndSet

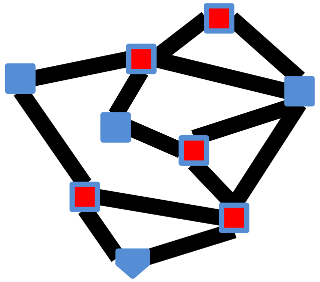


If Solving A was always slow

Using any Algorithm for MaxVertCov

Then this shows solving B is also slow

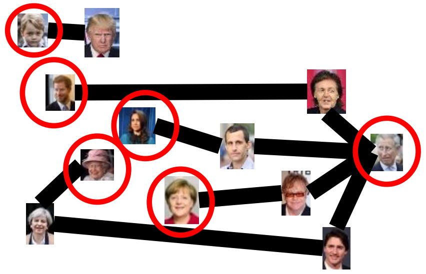
Solution for MinVertCov



X

Take complement of solution

Solution for MaxIndSet



Y

Reduction

Conclusion

- MaxIndSet and MinVertCov are either both fast, or both slow
  - Spoiler alert: We don’t know which!
    - (But we think they’re both slow)
  - Both problems are NP-Complete

Problem Types

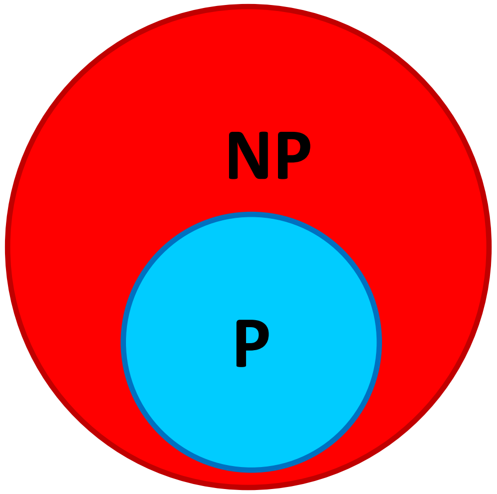
- Decision Problems:
  - Is there a solution?
    - Output is True/False
  - Is there a vertex cover of size k?
- Search Problems:
  - Find a solution
    - Output is complex
  - Give a vertex cover of size k
- Verification Problems:
  - Given a potential solution, is it valid?
    - Output is True/False
  - Is **this** a vertex cover of size k**?**

If we can solve this

Then we can solve this

P vs NP

- P
  - Deterministic Polynomial Time
  - Problems solvable in polynomial time
    - $O(n^p)$ for some number p
- NP
  - Non-Deterministic Polynomial Time
  - Problems verifiable in polynomial time
    - $O(n^p)$ for some number p
- Open Problem: Does P=NP?
  - Certainly $P \subseteq NP$



k-Independent Set is NP

- To show: Given a potential solution, can we verify it in $O(n^p)$? $[n = V + E]$

How can we verify it?

1. Check that it's of size k $O(V)$
2. Check that it's an independent set $O(V^2)$

k-Vertex Cover is NP

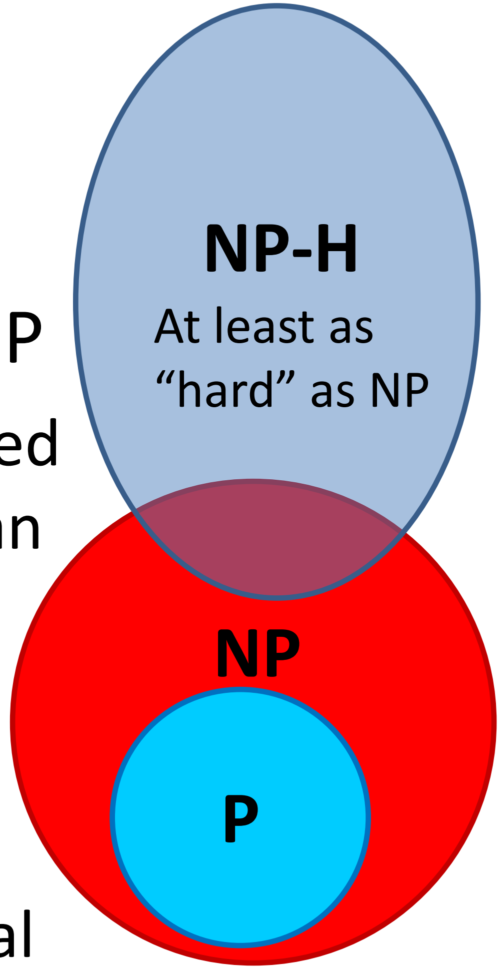
- To show: Given a potential solution, can we verify it in $O(n^p)$? $[n = V + E]$

How can we verify it?

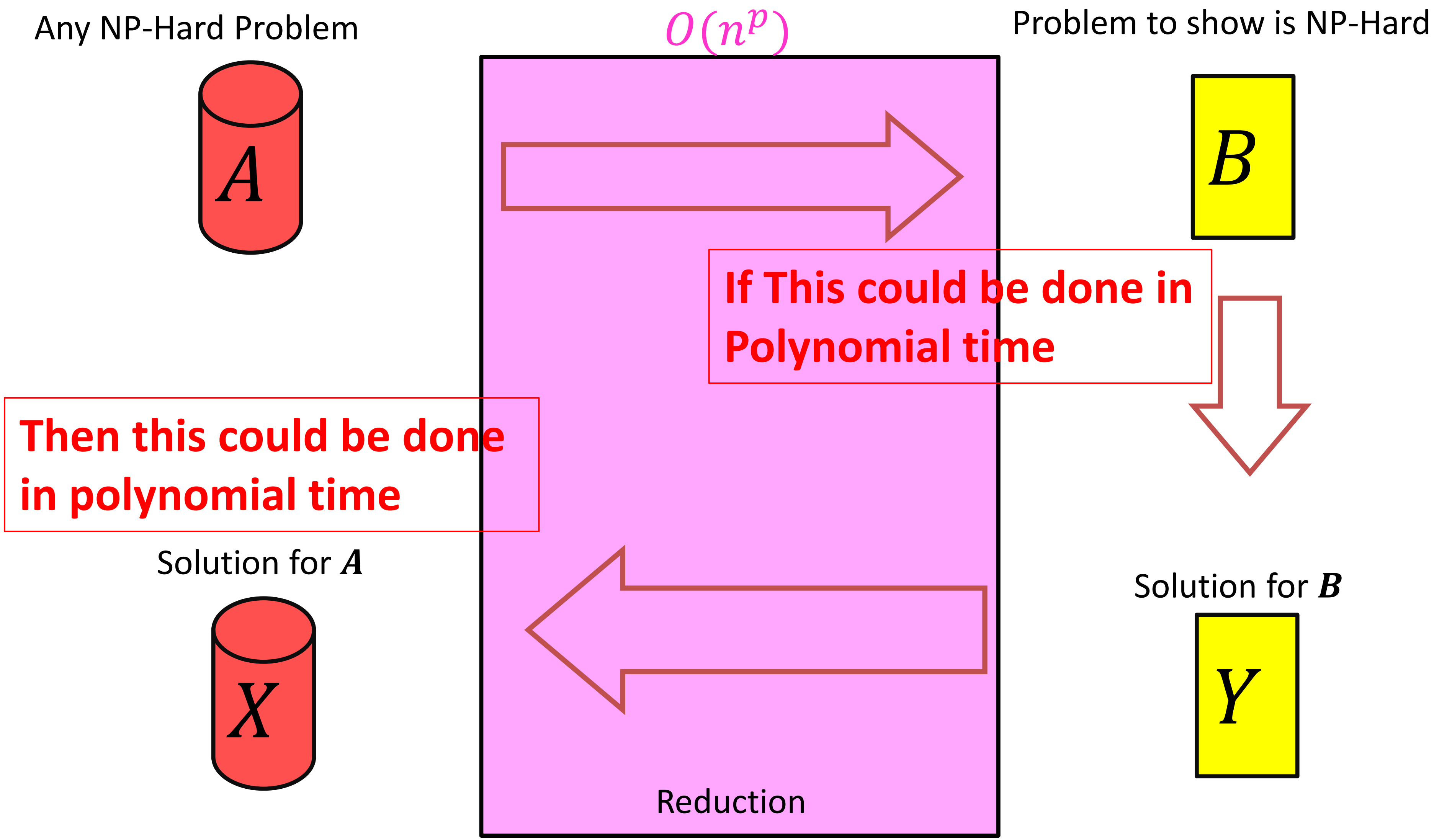
1. Check that it's of size k $O(V)$
2. Check that it's a Vertex Cover $O(E)$

NP-Hard

- How can we try to figure out if P=NP?
- Identify problems at least as “hard” as NP
  - If any of these “hard” problems can be solved in polynomial time, then all NP problems can be solved in polynomial time.
- Definition: NP-Hard:
  - B is NP-Hard if $\forall A \in NP, A \leq_p B$
  - $A \leq_p B$ means A reduces to B in polynomial time

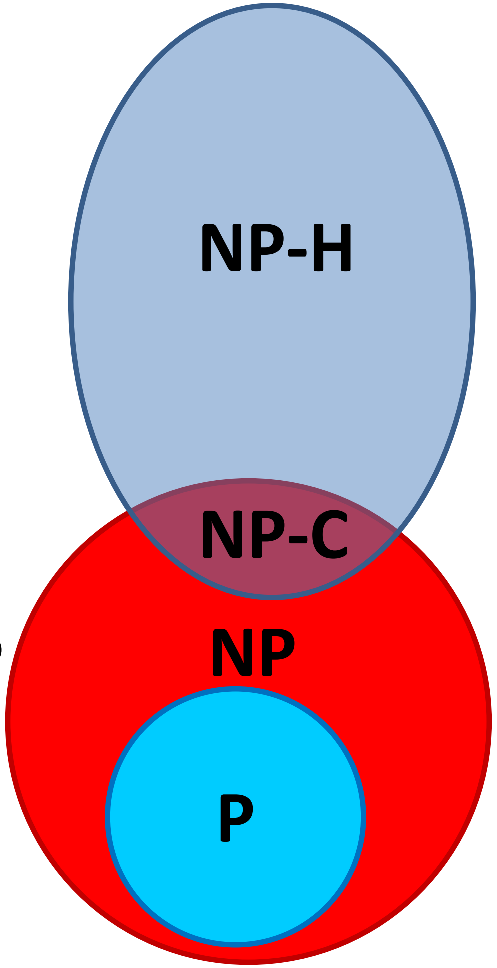


NP-Hardness Reduction



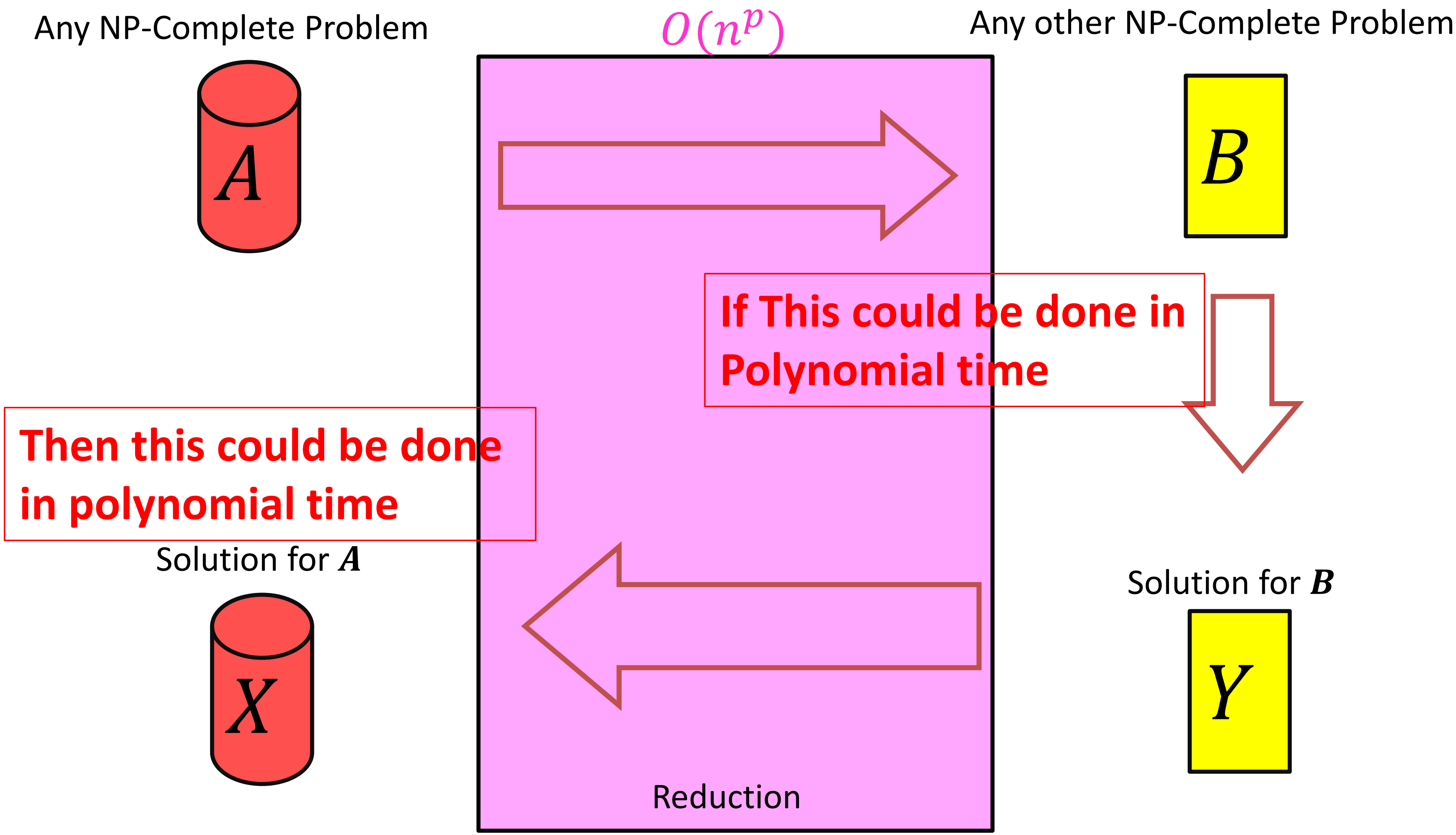
NP-Complete

- "Together they stand, together they fall"
- Problems solvable in polynomial time iff ALL NP problems are
- NP-Complete = NP ∩ NP-Hard
- How to show a problem is NP-Complete?
  - Show it belongs to NP
    - Give a polynomial time verifier
  - Show it is NP-Hard
    - Give a reduction from another NP-H problem

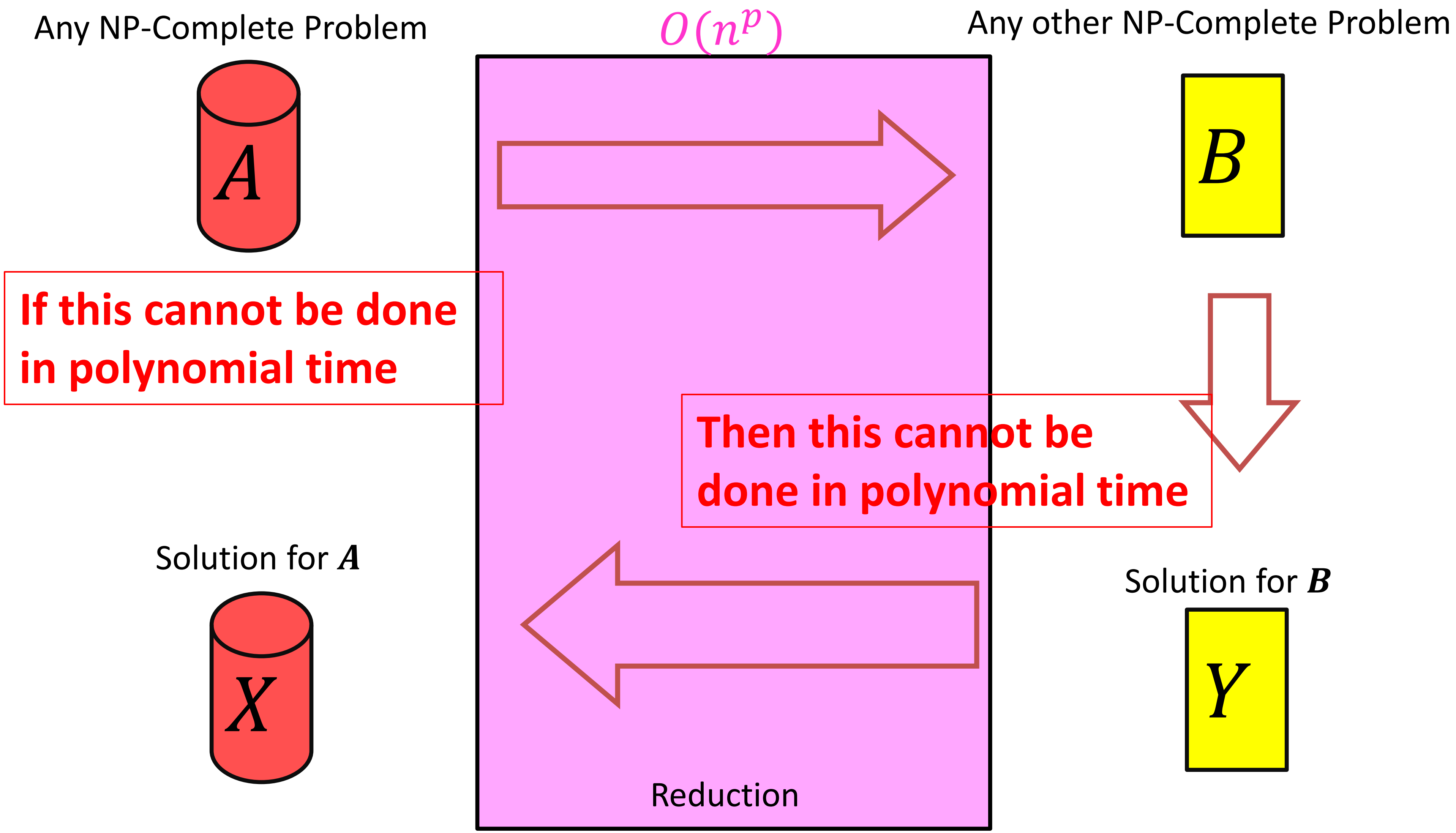


We now just need a FIRST NP-Hard problem

NP-Completeness



NP-Completeness



3-SAT

- Shown to be NP-Hard by Cook and Levin (independently)
- Given a 3-CNF formula (logical AND of clauses, each an OR of 3 variables), Is there an assignment of true/false to each variable to make the formula true?

$$\underbrace{(x \vee y \vee z)}_{\textbf{Clause}} \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$

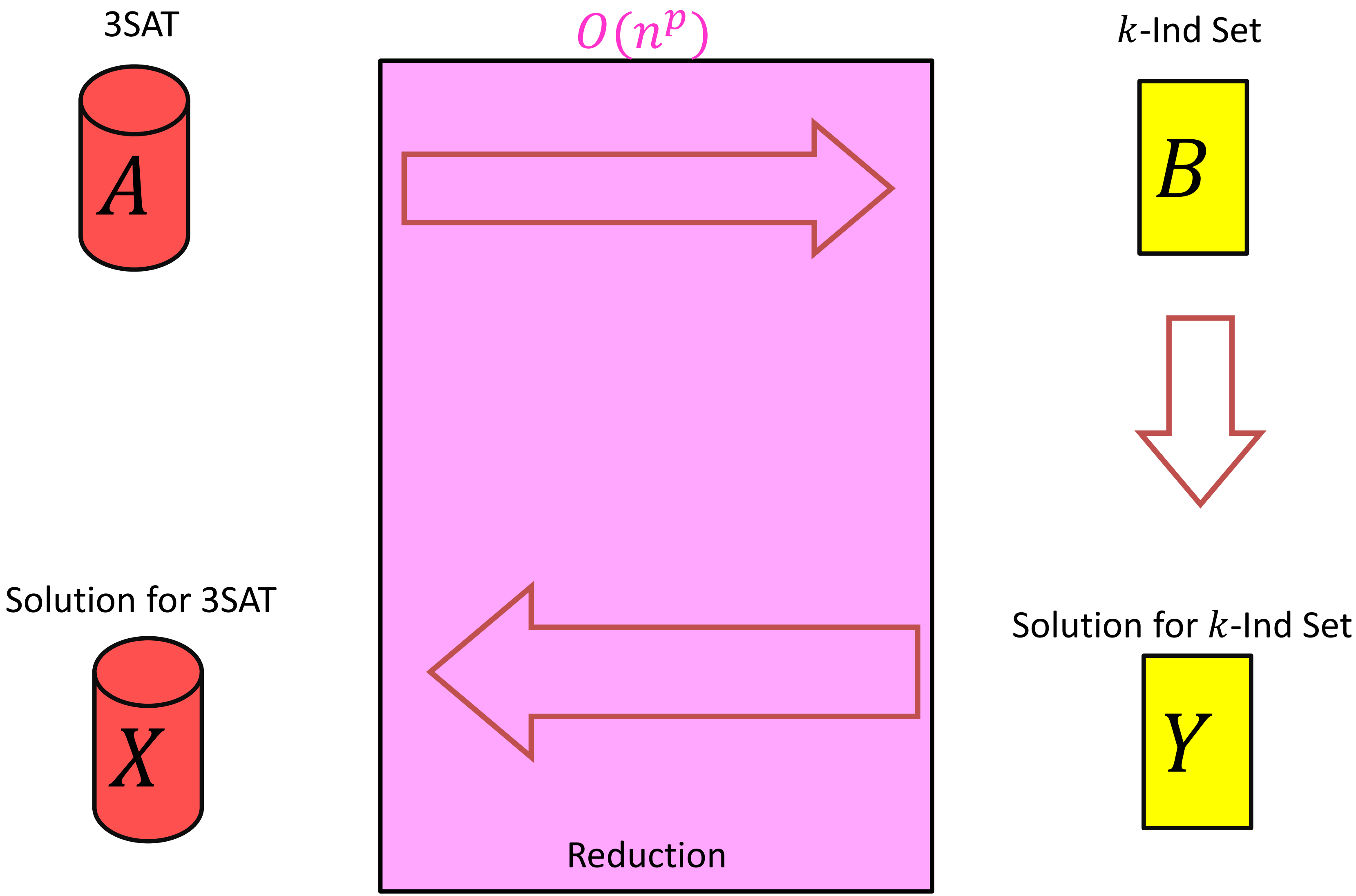
Variables

$$\boldsymbol{x = true}$$
$$\boldsymbol{y = false}$$
$$\boldsymbol{z = false}$$
$$\boldsymbol{u = true}$$

k-Independent Set is NP-Complete

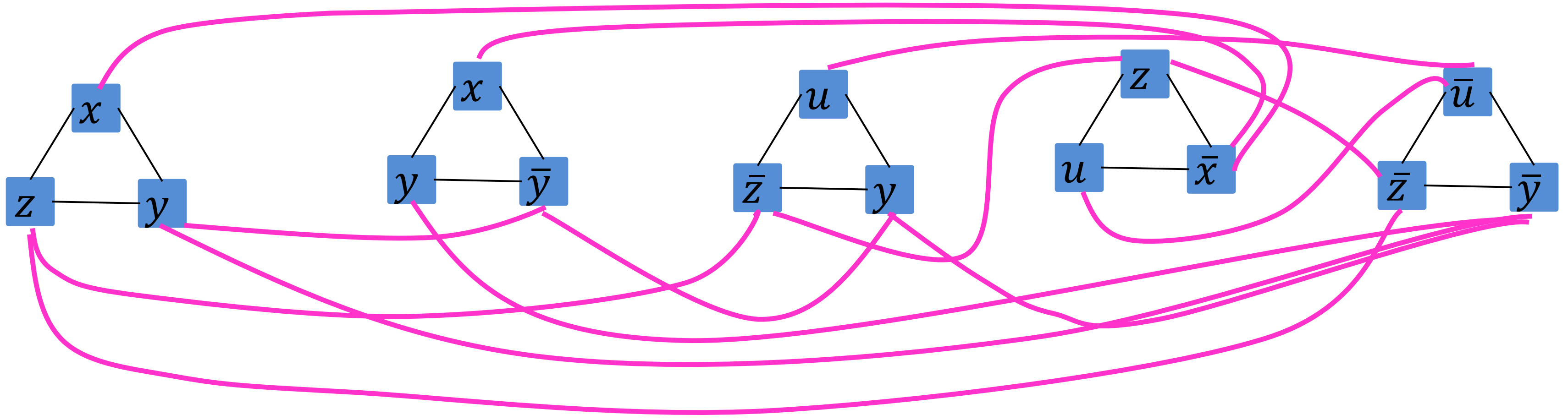
1. Show that it belongs to NP
   - Give a polynomial time verifier (slide 21)
2. Show it is NP-Hard
   - Give a reduction from a known NP-Hard problem
   - Show $3SAT \leq_p kIndSet$

$3SAT \leq_p kIndSet$



Instance of 3SAT to Instance of kIndSet

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



For each clause, produce a triangle graph with its three variables as nodes

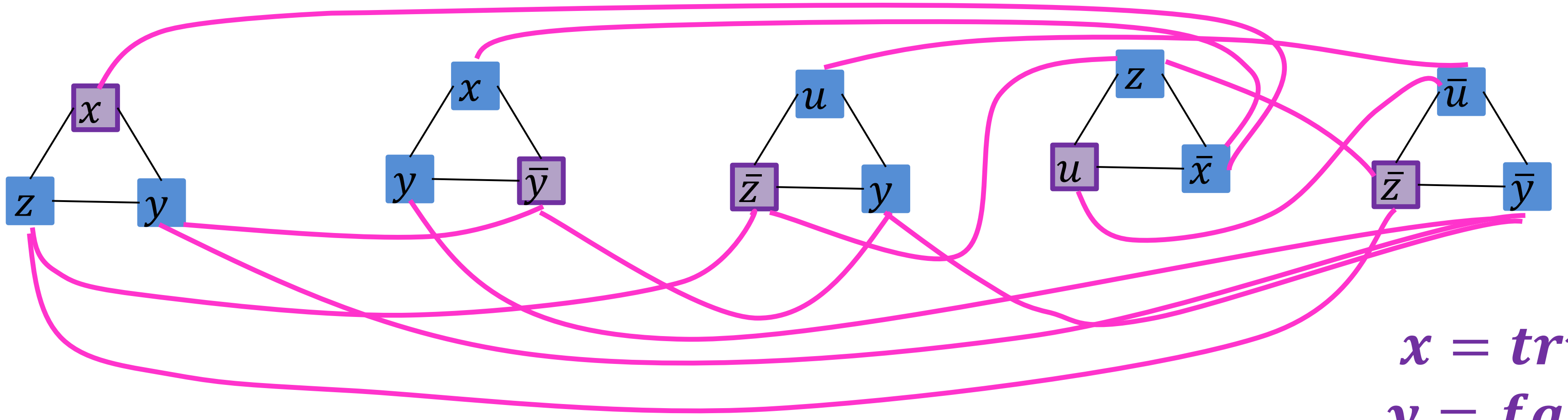
Connect each node to all of its opposites

Let k = number of clauses

There is a k-IndSet in this graph, iff there is a satisfying assignment

kIndSet ⇒ Satisfying Assignment

$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$



$\boldsymbol{x = true}$
$\boldsymbol{y = false}$
$\boldsymbol{z = false}$
$\boldsymbol{u = true}$

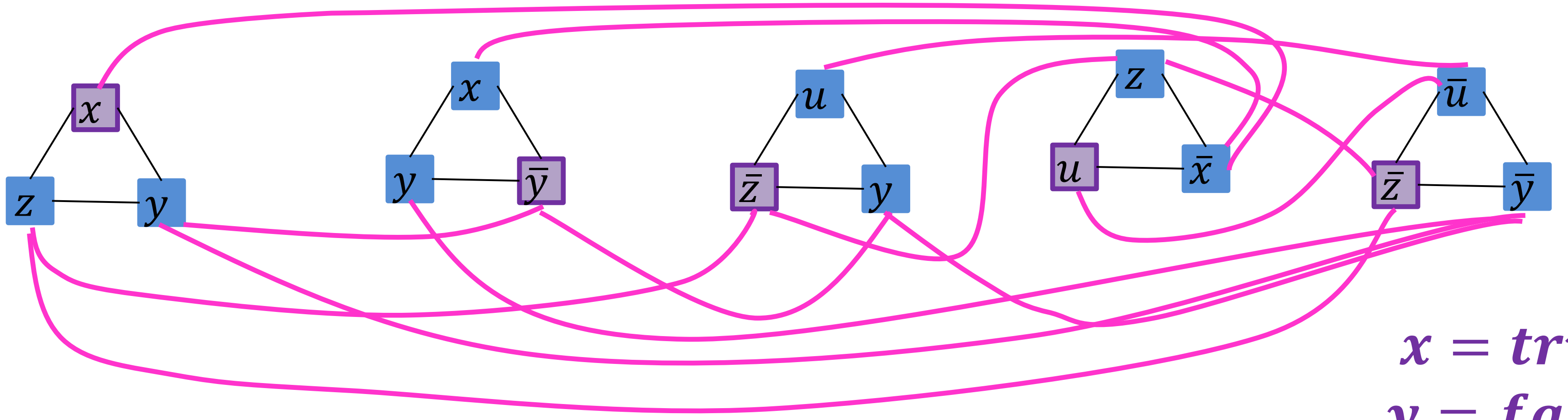
One node per triangle is in the Independent set: because we can have exactly k total in the set, and 2 in a triangle would be adjacent

If x is selected in some triangle, $\bar{x}$ is not selected in any triangle: Because every x is adjacent to every $\bar{x}$

Set the variable which each included node represents to "true"

Satisfying Assignment $\Rightarrow k$IndSet

$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$



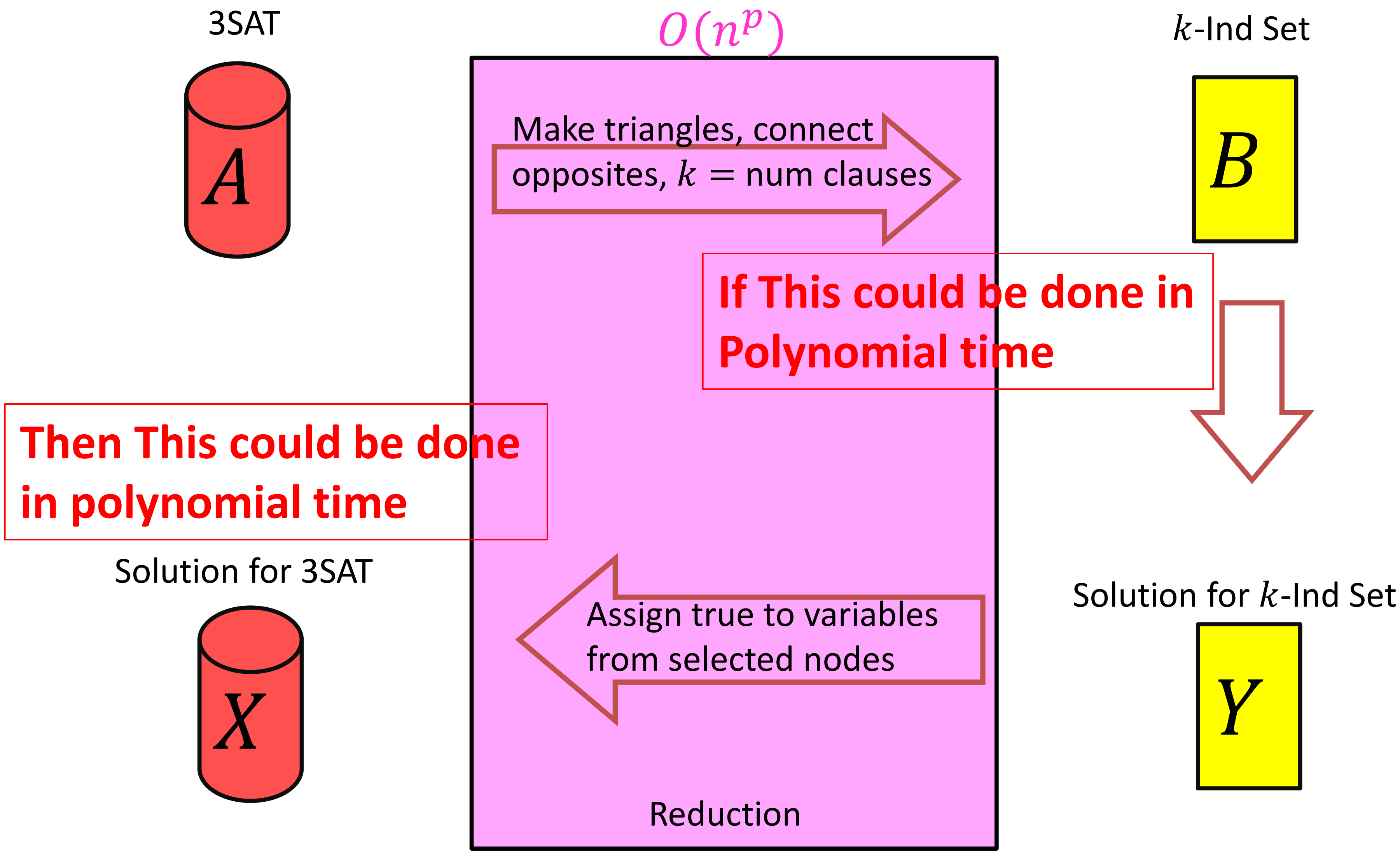
$\boldsymbol{x = true}$
$\boldsymbol{y = false}$
$\boldsymbol{z = false}$
$\boldsymbol{u = true}$

Use one true variable from the assignment for each triangle

The independent set has k nodes, because there are k clauses

If any variable x is true then $\bar{x}$ cannot be true

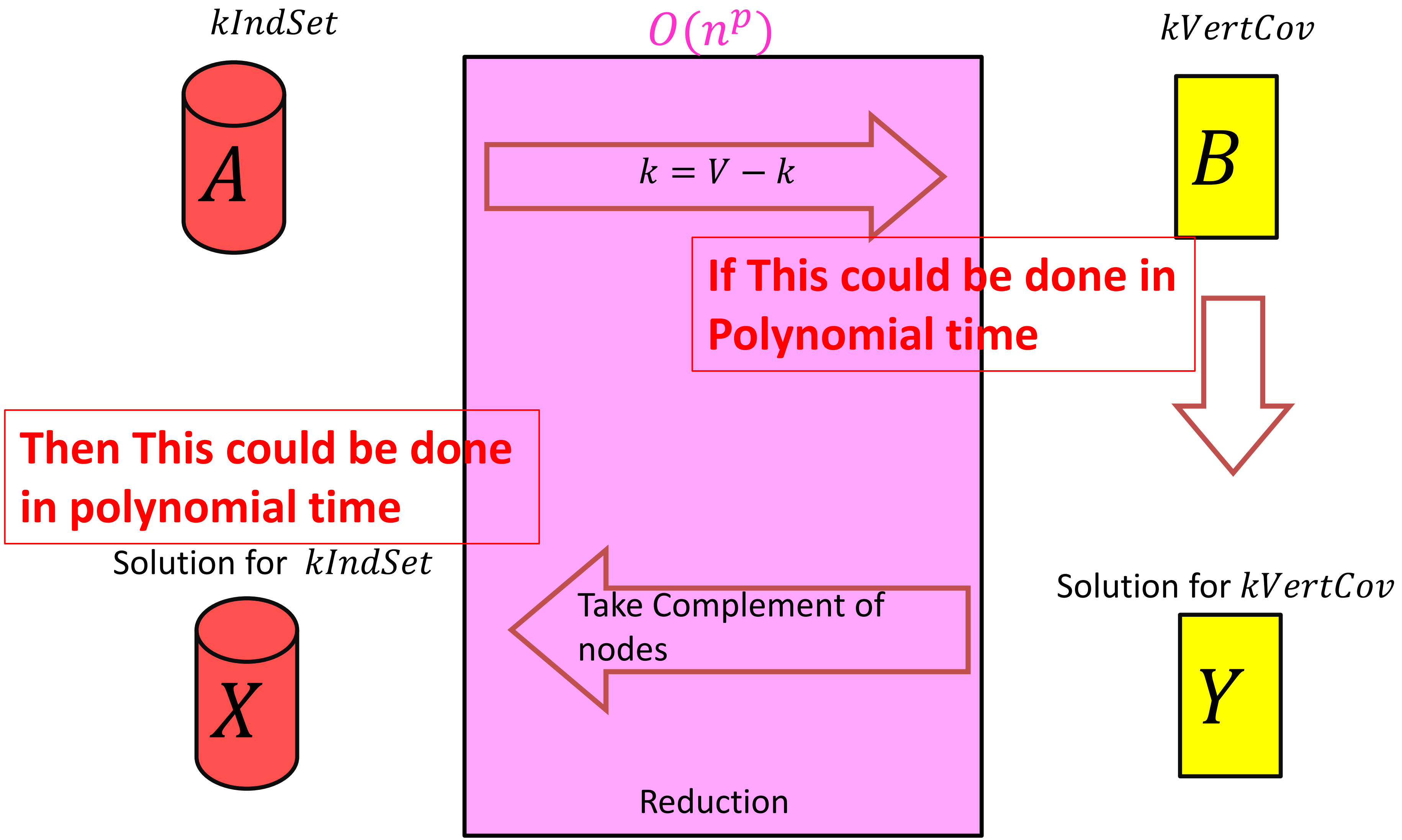
$3SAT \leq_p kIndSet$



k-Vertex Cover is NP-Complete

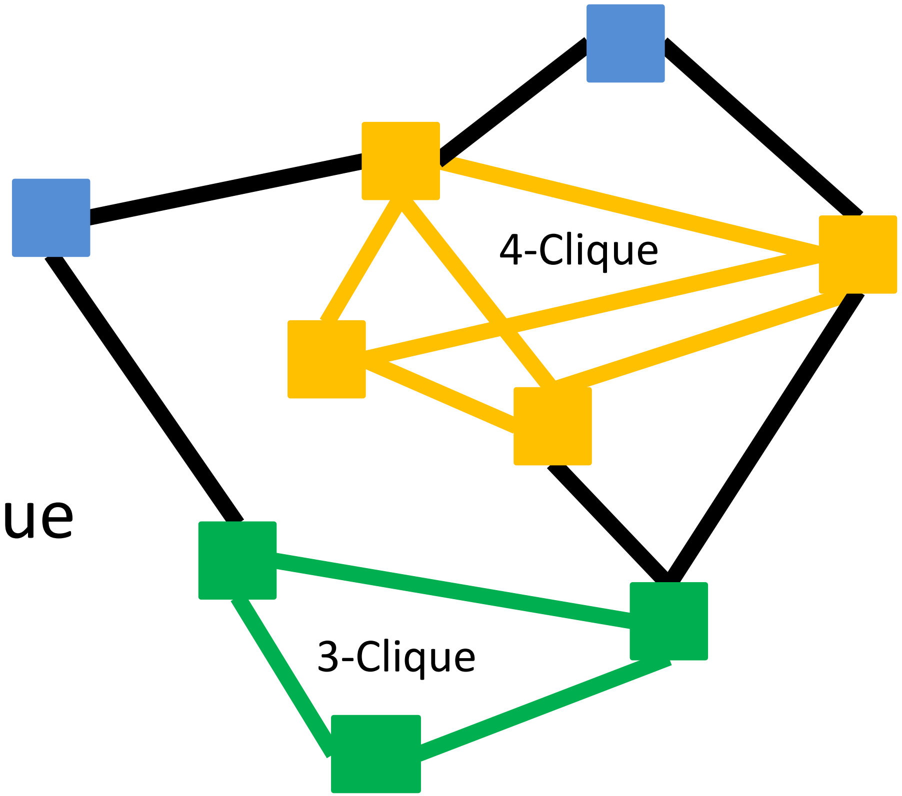
1. Show that it belongs to NP
   – Give a polynomial time verifier (slide 22)
2. Show it is NP-Hard
   – Give a reduction from a known NP-Hard problem
   – We showed $kIndSet \leq_p kVertCov$
      • (Last Class)

$kIndSet \leq_p kVertCov$



k-Clique Problem

- Clique: A complete subgraph
- k-Clique Problem:
  - Given a graph G and a number k, is there a clique of size k?

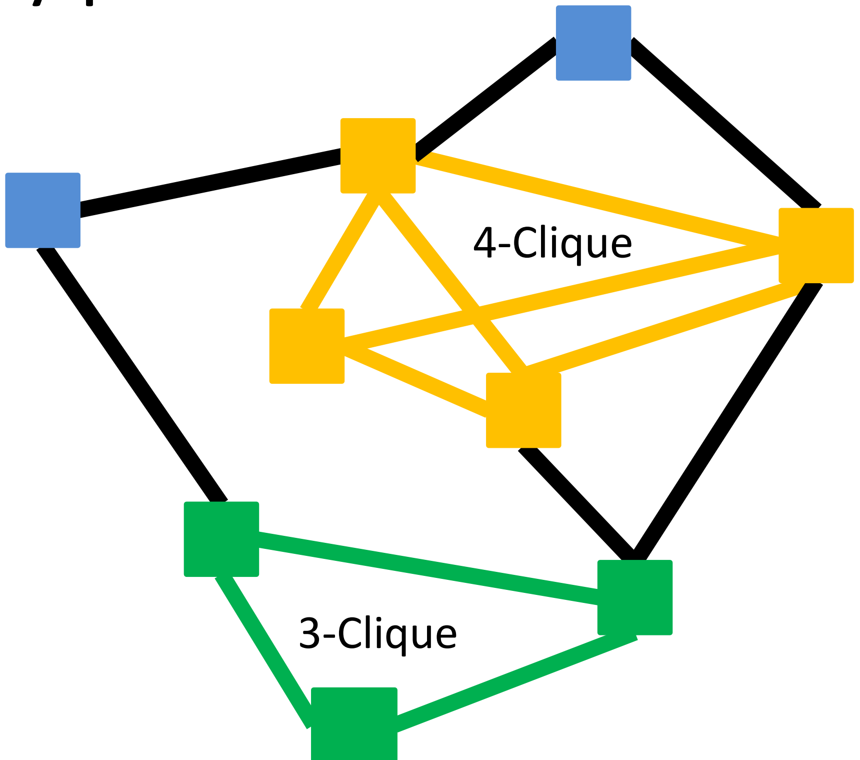


k-Clique is NP-Complete

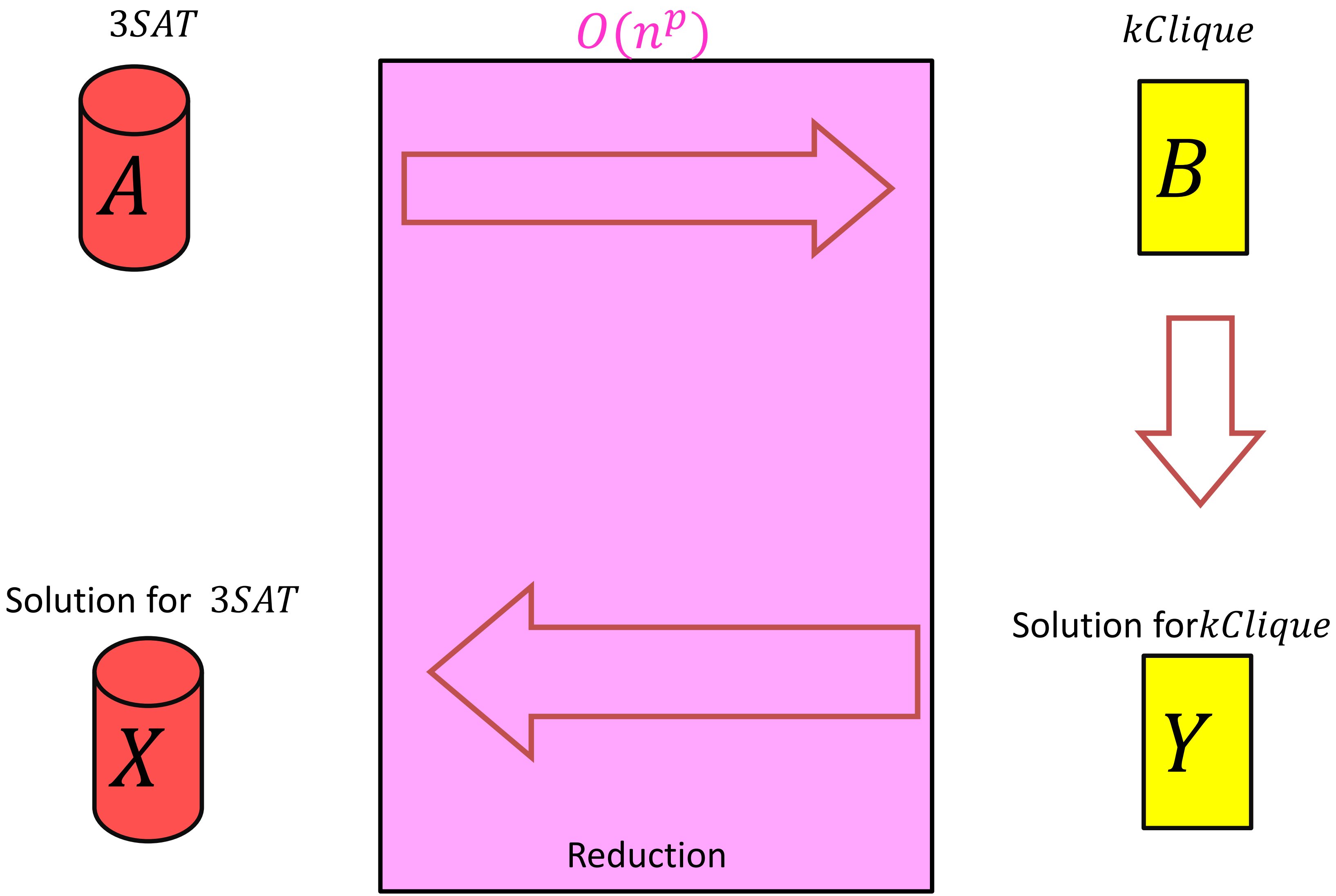
1. Show that it belongs to NP
   - Give a polynomial time verifier
2. Show it is NP-Hard
   - Give a reduction from a known NP-Hard problem
   - We will show $3SAT \leq_p kClique$

k-Clique is NP

1. Given a Graph and a potential solution
2. Check that the solution has k nodes
3. Check that every pair of nodes share an edge

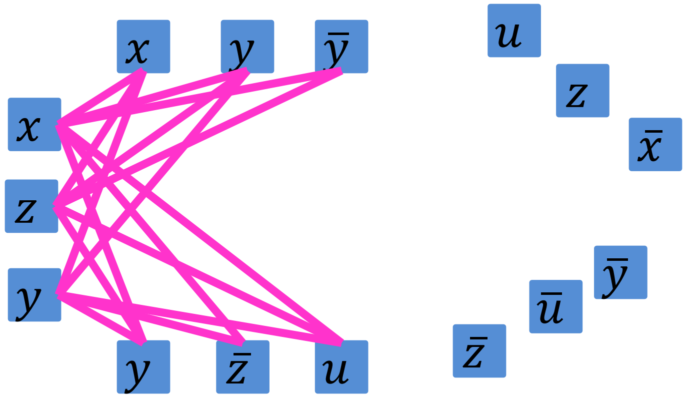


$3SAT \leq_p kClique$



Instance of 3SAT to Instance of kClique

$$(x \lor y \lor z) \land (x \lor \bar{y} \lor y) \land (u \lor y \lor \bar{z}) \land (z \lor \bar{x} \lor u) \land (\bar{x} \lor \bar{y} \lor \bar{z})$$



(also do this for the other clauses, omitted due to clutter)

For each clause, produce a node for each of its three variables

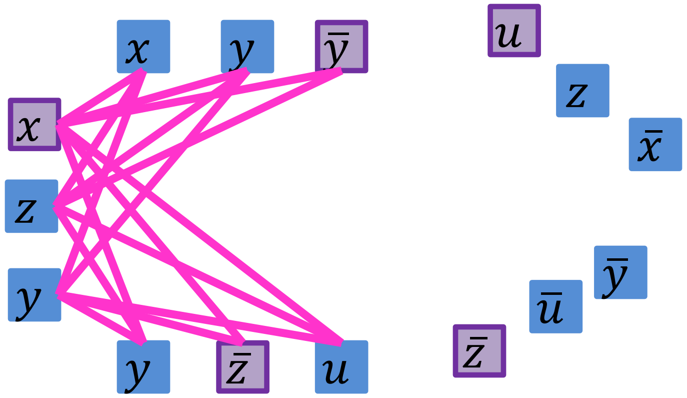
Connect each node to all non-contradictory nodes in the other clauses (i.e., anything that's not its negation)

Let k = number of clauses

There is a k-Clique in this graph, iff there is a satisfying assignment

kClique $\Rightarrow$ Satisfying Assignment

$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$



$\boldsymbol{x = true}$
$\boldsymbol{y = false}$
$\boldsymbol{z = false}$
$\boldsymbol{u = true}$

There are k triplets in the graph, and no two nodes in the same triplet are adjacent

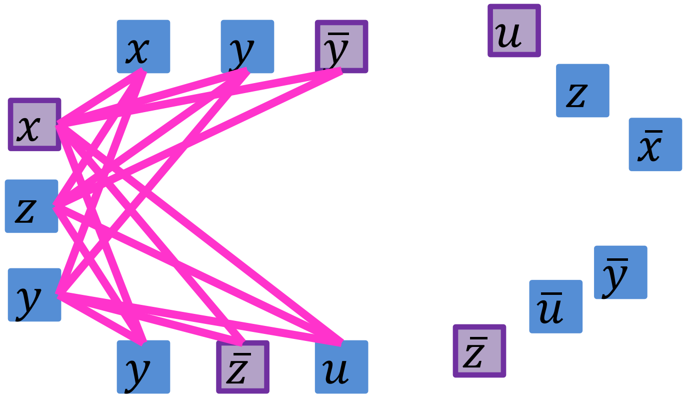
To have a k-Clique, must have one node from each triplet

Cannot select a node for both a variable and its negation

Therefore selection of nodes is a satisfying assignment

Satisfying Assignment$\Rightarrow$ kClique

$(x \lor y \lor z) \land (x \lor \bar{y} \lor y) \land (u \lor y \lor \bar{z}) \land (z \lor \bar{x} \lor u) \land (\bar{x} \lor \bar{y} \lor \bar{z})$

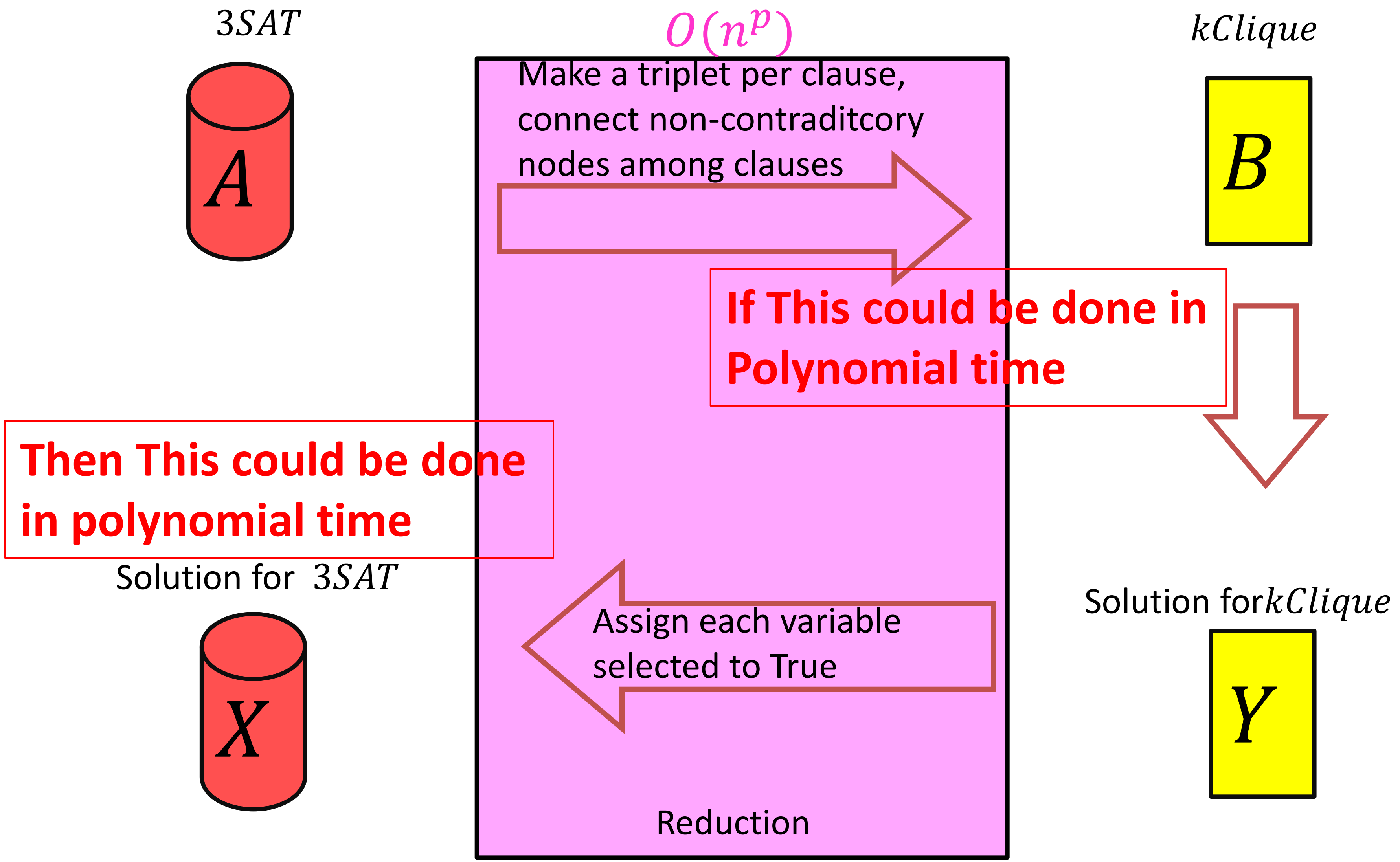


$\boldsymbol{x = true}$
$\boldsymbol{y = false}$
$\boldsymbol{z = false}$
$\boldsymbol{u = true}$

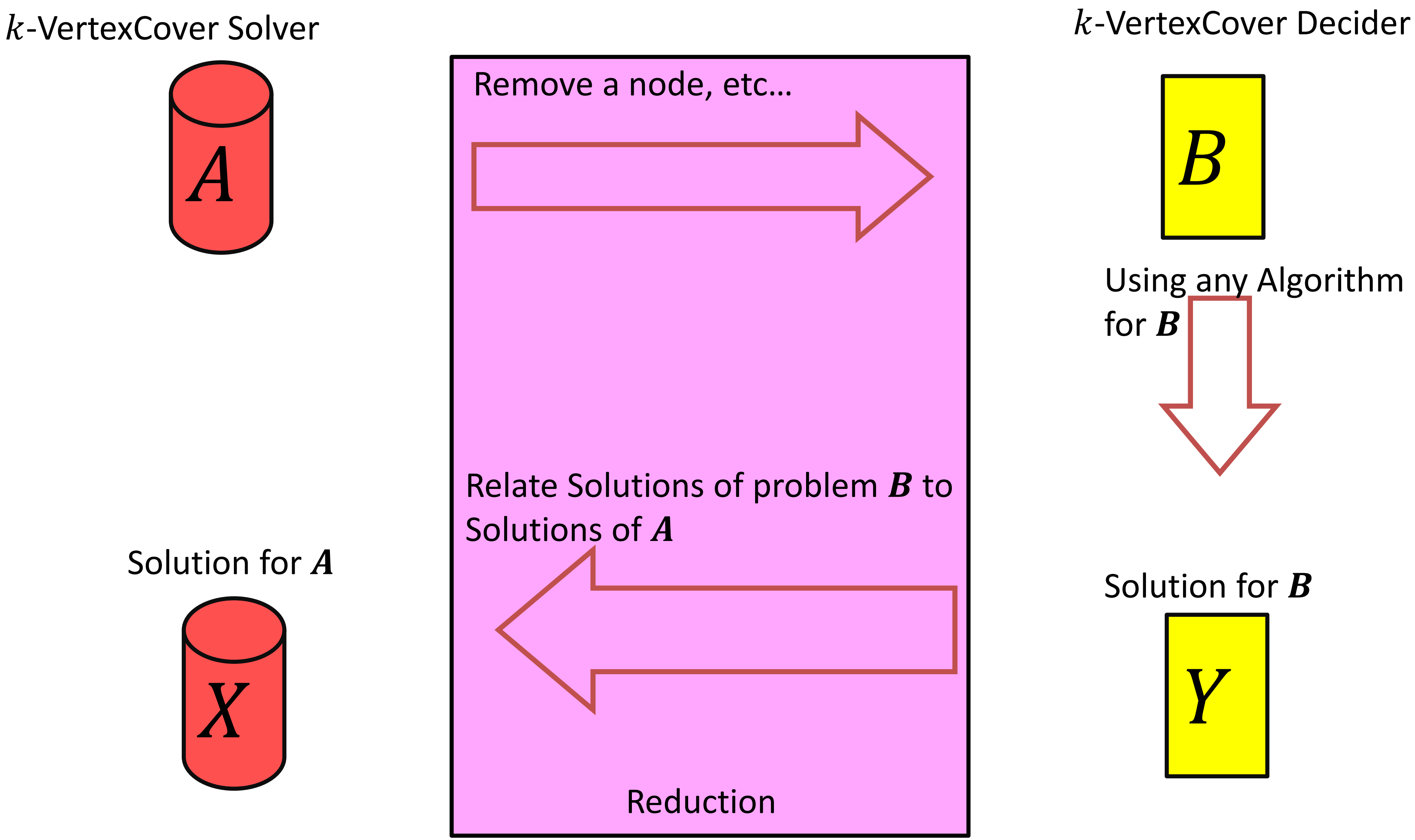
Select one node for a true variable from each clause

There will be k nodes selected
We can't select both a node and its negation
All nodes will be non-contradictory, so they will be pairwise adjacent

$3SAT \leq_p kClique$



Reduction


k-VertexCover Solver
A
Remove a node, etc…
k-VertexCover Decider
B
Using any Algorithm for $\boldsymbol{B}$
Relate Solutions of problem $\boldsymbol{B}$ to Solutions of $\boldsymbol{A}$
Solution for $\boldsymbol{A}$
X
Solution for $\boldsymbol{B}$
Y
Reduction

Problem Types

- Decision Problems:
  - Is there a solution?
    - Output is True/False
  - Is there a vertex cover of size k?
- Search Problems:
  - Find a solution
    - Output is complex
  - Give a vertex cover of size k
- Verification Problems:
  - Given a potential solution, is it valid?
    - Output is True/False
  - Is **this** a vertex cover of size k**?**

If we can solve this

Then we can solve this

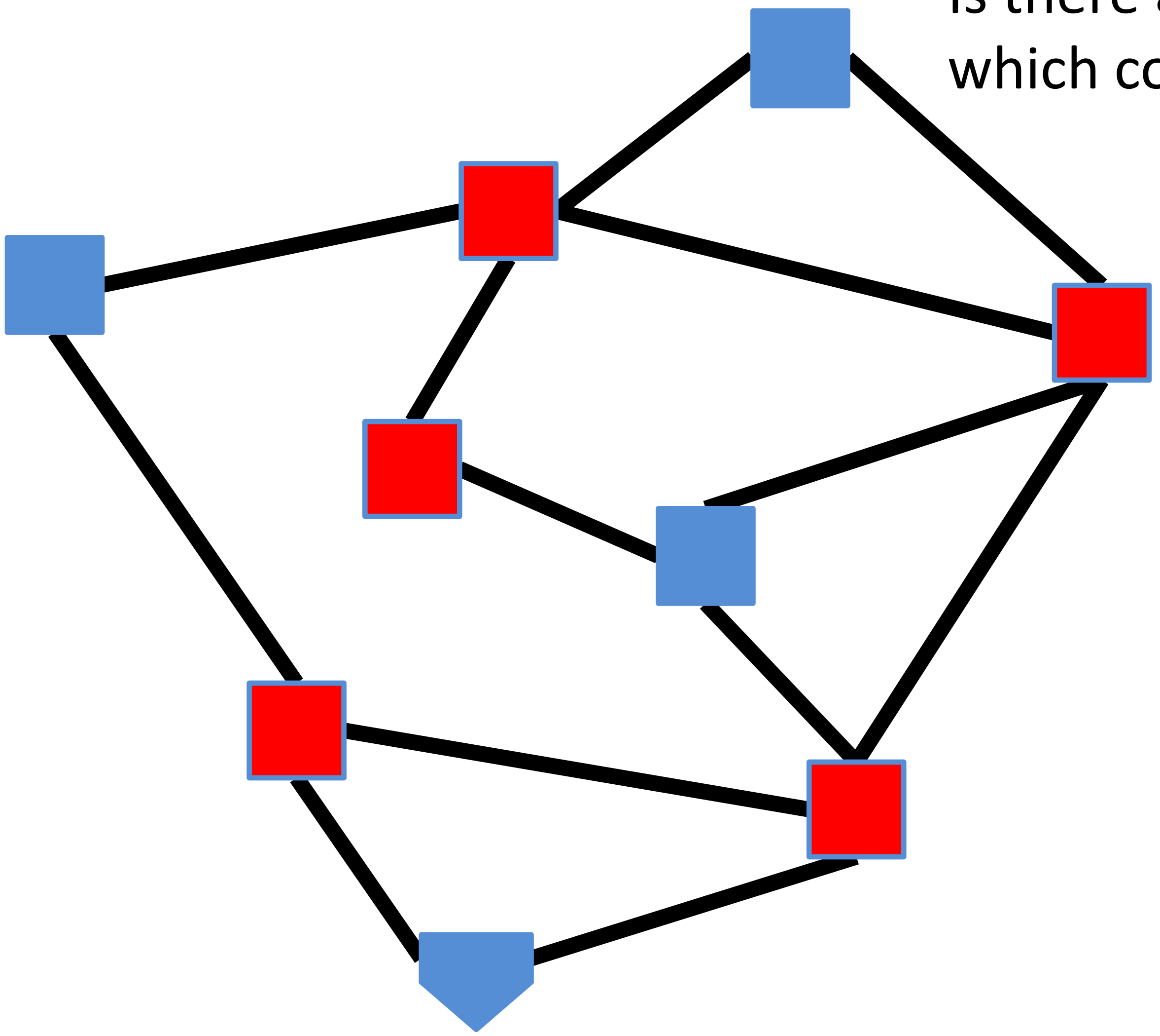
Using a k-VertexCover decider to build a searcher

- Set $i = k - 1$
- Remove nodes (and incident edges) one at a time
- Check if there is a vertex cover of size i
  - If so, then that removed node was part of the k vertex cover, set $i = i - 1$
  - Else, it wasn't

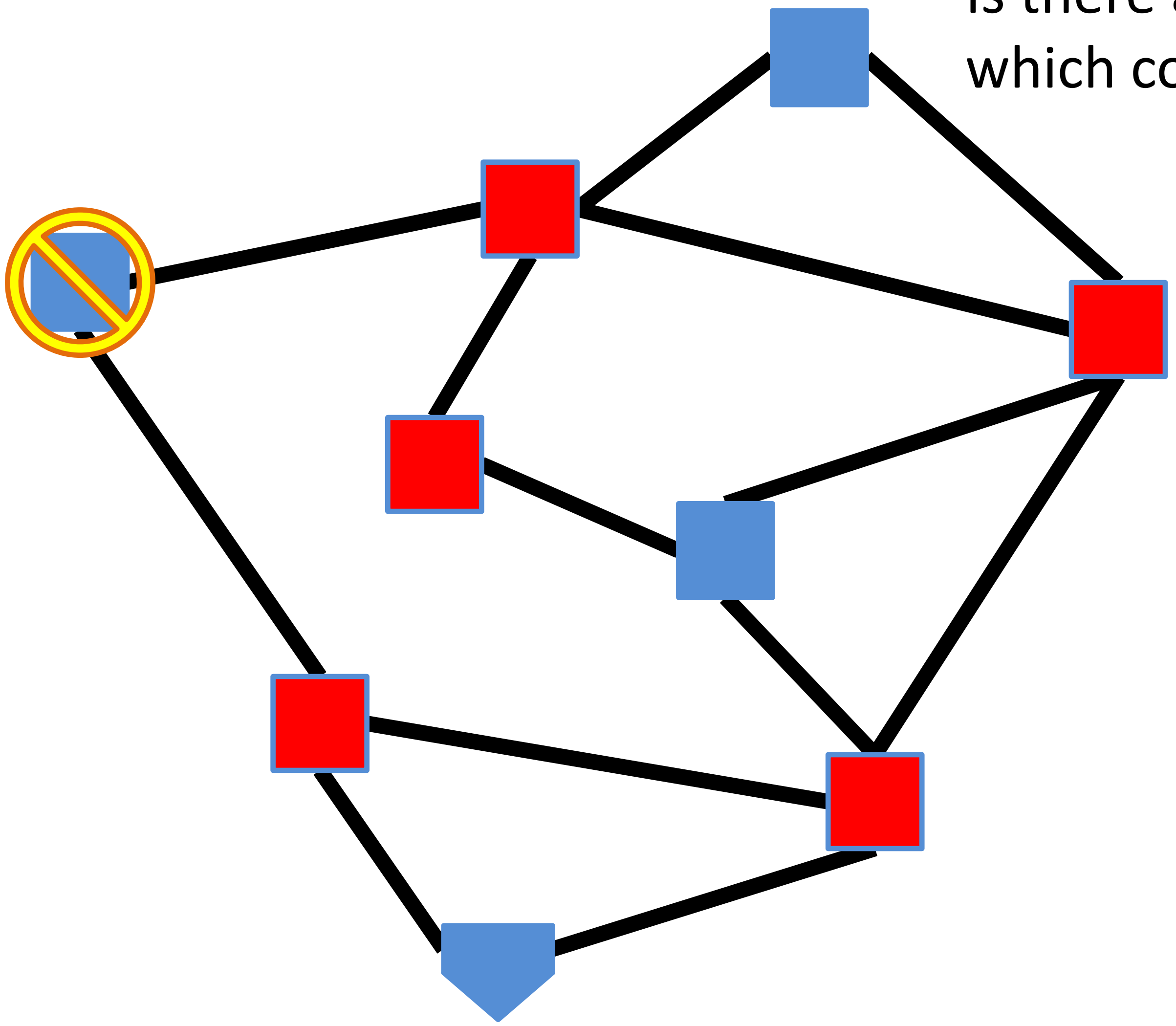
5 Vertex Cover (Decision)

Is there a set of nodes of size 5 which covers every edge?

Yes!



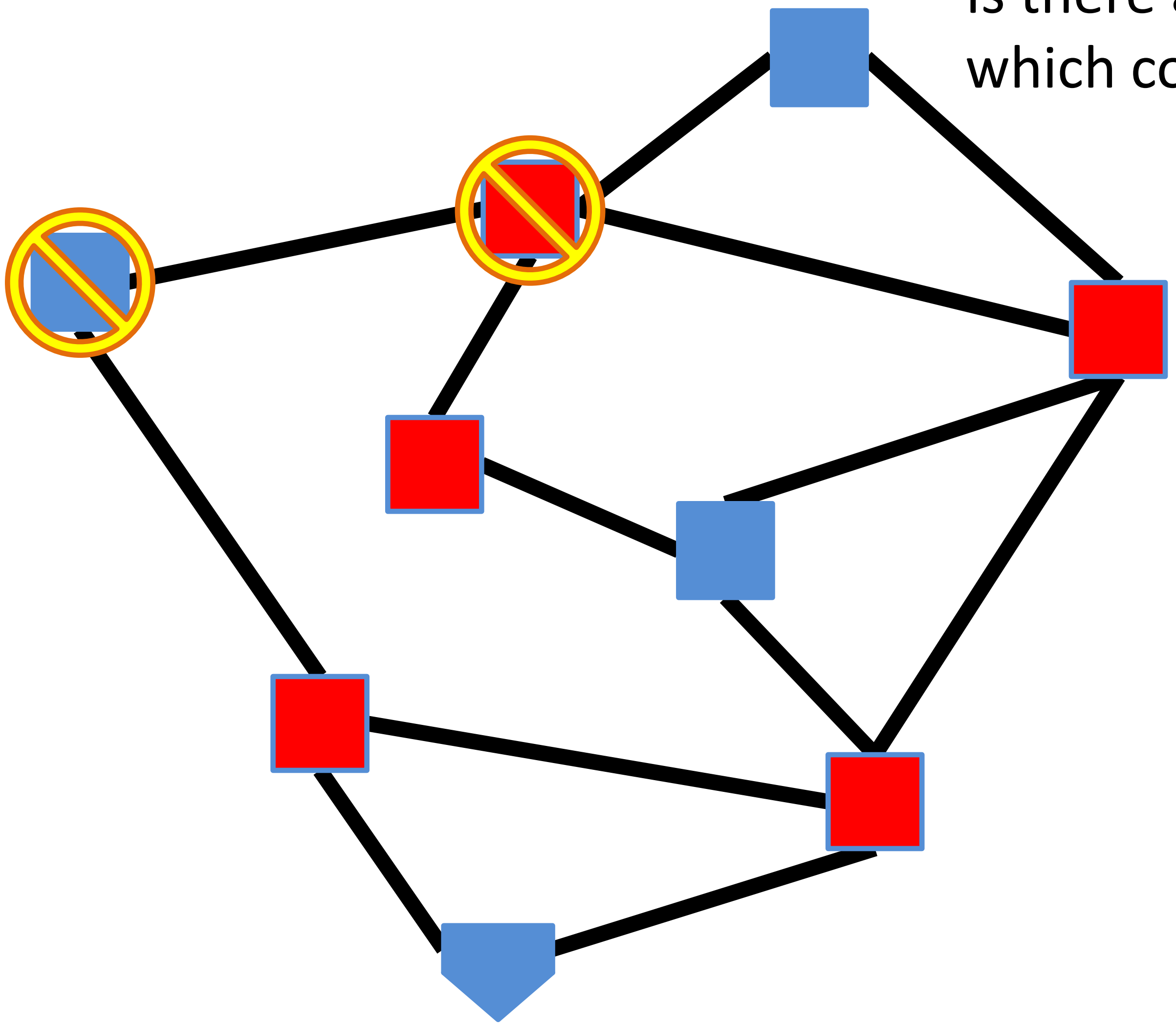
4 Vertex Cover (Decision)



Is there a set of nodes of size 4 which covers every edge?

No!

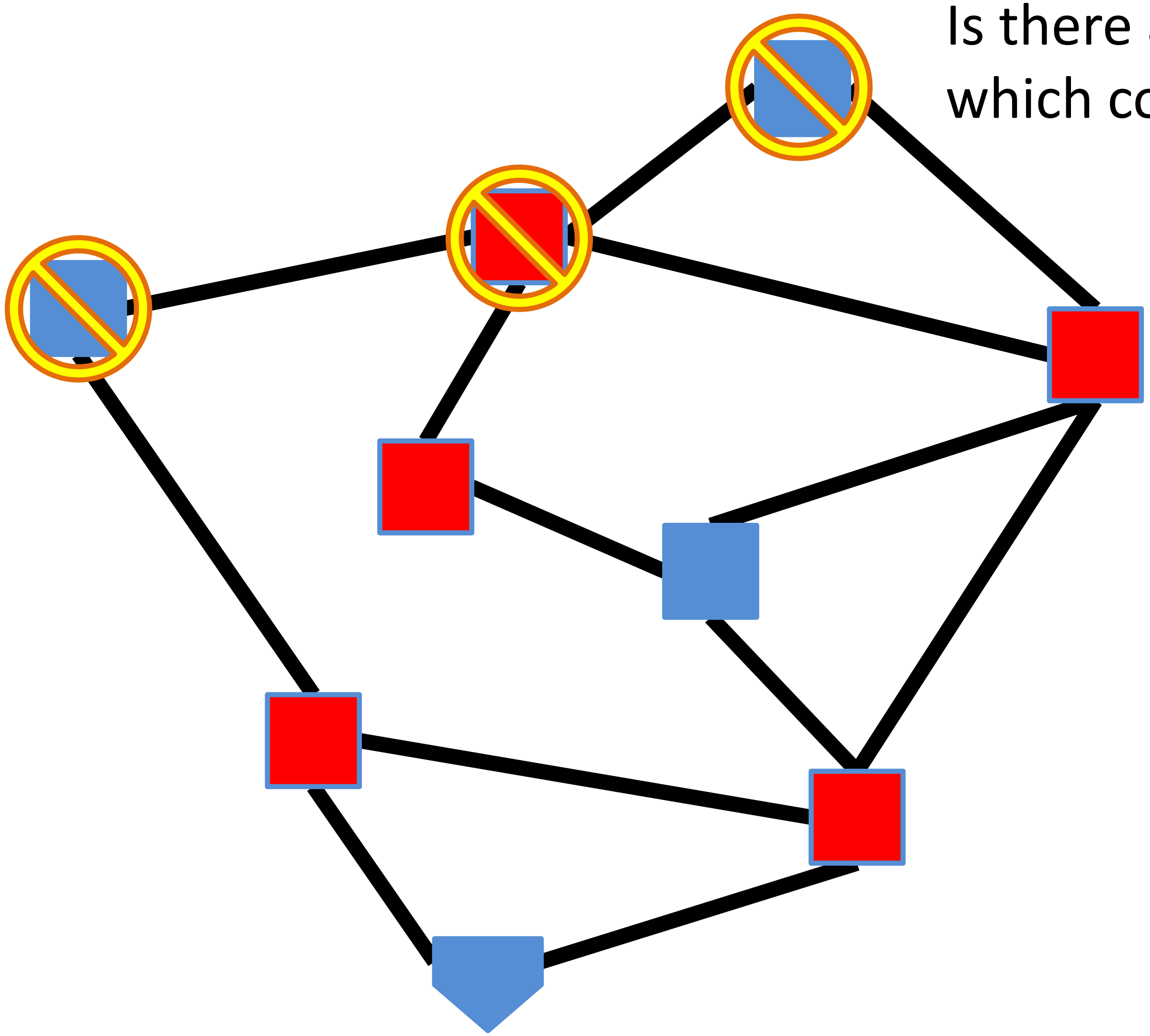
4 Vertex Cover (Decision)



Is there a set of nodes of size 4 which covers every edge?

Yes!

3 Vertex Cover (Decision)



Is there a set of nodes of size 3 which covers every edge?

No!