CS4102 Recurrences Proofs

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1 Karatsuba Recurrence, Tree Method

Karatsuba Recurrence:

$$T(n) = 3T(\frac{n}{2}) + 8n$$

Using the tree method for solving the recurrence, we obtained the sum:

$$\begin{aligned}
T(n) &= 8n \sum_{i=0}^{\log_2 n} (\frac{3}{2})^i \\
&= 8n \frac{(\frac{3}{2})^{\log_2 n + 1} - 1}{\frac{3}{2} - 1} \\
&= 8n \frac{(\frac{3}{2})^{\log_2 n + 1} - 1}{\frac{1}{2}} \\
&= 16n((\frac{3}{2})^{\log_2 n + 1} - 1) \\
&= 16n(2^{\log_2 3 - 1})^{\log_2 n + 1} - 16n \\
&= 16n(2^{\log_2 3 \cdot \log_2 n - \log_2 n + \log_2 3 - 1}) - 16n \\
&= 16n((2^{\log_2 n})^{\log_2 3} \cdot 2^{-\log_2 n} \cdot 2^{\log_2 3} \cdot 2^{-1}) - 16n \\
&= 16n(n^{\log_2 3} \cdot \frac{1}{n} \cdot 3 \cdot \frac{1}{2}) - 16n \\
&= 24n^{\log_2 3} - 16n \\
&= \Theta(n^{\log_2 3}) \\
&\approx \Theta(n^{1.585})
\end{aligned}$$

2 Karatsuba, Guess and Check, Loose Bound

Karatsuba Recurrence:

$$T(n) = 3T(\frac{n}{2}) + 8n$$

Goal:

$$T(n) < 3000n^{1.6}$$

Base Case:

$$T(1) = 8 < 3000$$

Hypothesis:

$$\forall n < n_0, T(n) < 3000n^{1.6}$$

Inductive Step:

$$\begin{aligned}
T(n_0+1) &= 3T(\frac{n_0+1}{2}) + 8(n_0+1) \\
&< 3(3000(\frac{n_0+1}{2})^{1.6}) + 8(n_0+1) \\
&= \frac{3}{2^{1.6}} \cdot 3000(n_0+1)^{1.6} + 8(n_0+1) \\
&< 0.997 \cdot 3000(n_0+1)^{1.6} + 8(n_0+1) \\
&= (1-0.003) \cdot 3000(n_0+1)^{1.6} + 8(n_0+1) \\
&= 3000(n_0+1)^{1.6} + 8(n_0+1) - 0.003 \cdot 3000(n_0+1)^{1.6} \\
&= 3000(n_0+1)^{1.6} + 8(n_0+1) - 9(n_0+1)^{1.6} \\
&< 3000(n_0+1)^{1.6} + 8(n_0+1)
\end{aligned}$$

3 MergeSort, Guess and Check

MergeSort Recurrence:

$$T(n) = 2T(\frac{n}{2}) + n$$

Goal:

$$T(n) < n \log_2 n$$

Base Case: by inspection
Hypothesis:

$$\forall n < n_0, T(n) < n \log_2 n$$

Inductive Step:

$$\begin{aligned}
T(n_0+1) &= 2T(\frac{n_0+1}{2}) + (n_0+1) \\
&< 2(\frac{n_0+1}{2} \log_2 \frac{n_0+1}{2}) + n_0 + 1 \\
&= (n_0+1) \log_2 \frac{n_0+1}{2} + n_0 + 1 \\
&= (n_0+1)(\log_2(n_0+1) + \log_2 \frac{1}{2}) + n_0 + 1 \\
&= (n_0+1)(\log_2(n_0+1) - 1) + n_0 + 1 \\
&= (n_0+1) \log_2(n_0+1) - (n_0+1) + n_0 + 1 \\
&= (n_0+1) \log_2(n_0+1)
\end{aligned}$$

4 Alt. MergeSort, Guess and Check

MergeSort Recurrence:

$$T(n) = 2T(\frac{n}{2}) + 209n$$

Goal:

$$T(n) < 209n \log_2 n$$

Base Case: by inspection
Hypothesis:

$$\forall n < n_0, T(n) < 209n \log_2 n$$

Inductive Step:

$$\begin{aligned}
T(n_0 + 1) &= 2T(\frac{n_0 + 1}{2}) + 209(n_0 + 1) \\
&< 2(209\frac{n_0 + 1}{2} \log_2 \frac{n_0 + 1}{2}) + 209(n_0 + 1) \\
&= 209(n_0 + 1) \log_2 \frac{n_0 + 1}{2} + 209(n_0 + 1) \\
&= 209(n_0 + 1)(\log_2(n_0 + 1) + \log_2 \frac{1}{2}) + 209(n_0 + 1) \\
&= 209(n_0 + 1)(\log_2(n_0 + 1) - 1) + 209(n_0 + 1) \\
&= 209(n_0 + 1) \log_2(n_0 + 1) - 209(n_0 + 1) + 209(n_0 + 1) \\
&= 209(n_0 + 1) \log_2(n_0 + 1)
\end{aligned}$$

5 Karatsuba, Guess and Check, Tight Bound

Karatsuba Recurrence:

$$T(n) = 3T(\frac{n}{2}) + 8n$$

Goal:

$$T(n) < n^{\log_2 3} - 16n$$

Base Case: by inspection
Hypothesis:

$$\forall n < n_0, T(n) < n^{\log_2 3} - 16n$$

Inductive Step:

$$\begin{aligned}
T(n_0+1) &= 3T(\frac{n_0+1}{2}) + 8(n_0+1) \\
&< 3((\frac{n_0+1}{2})^{\log_2 3} - 16\frac{n_0+1}{2}) + 8(n_0+1) \\
&= 3(\frac{1}{3}(n_0+1)^{\log_2 3} - 8(n_0+1)) + 8(n_0+1) \\
&= (n_0+1)^{\log_2 3} - 24(n_0+1) + 8(n_0+1) \\
&= (n_0+1)^{\log_2 3} - 16(n_0+1)
\end{aligned}$$

6 Master Theorem Case 1

Recurrence:

$$T(n) = aT(\frac{n}{b}) + f(n)$$

Assumption:

$$f(n) = O(n^{\log_b a - \varepsilon}) \Rightarrow f(n) \leq c \cdot n^{\log_b a - \varepsilon}$$

To Show:

$$T(n) = O(n^{\log_b a})$$

Proof: (let $L = \log_b n$, i.e. the height of the recurrence tree)

$$
\begin{aligned}
T(n) &= f(n) + af(\frac{n}{b}) + a^2 f(\frac{n}{b^2}) + \ldots + a^L f(\frac{n}{b^L}) \\
&\le c((\frac{n}{b})^{\log_b a-\varepsilon} + a(\frac{n}{b})^{\log_b a-\varepsilon} + a^2(\frac{n}{b^2})^{\log_b a-\varepsilon} + \ldots + a^{L-1}(\frac{n}{b^{L-1}})^{\log_b a-\varepsilon}) + a^L f(1) \\
&= cn^{\log_b a-\varepsilon}(1 + \frac{a}{b^{\log_b a-\varepsilon}} + \frac{a^2}{b^{2\log_b a-\varepsilon}} + \ldots + \frac{a^{L-1}}{b^{(L-1)\log_b a-\varepsilon}}) + a^L f(1) \\
&= cn^{\log_b a-\varepsilon}(1 + b^{\varepsilon} + b^{2\varepsilon} + \ldots + b^{(L-1)\varepsilon}) + a^L f(1) \\
&= cn^{\log_b a-\varepsilon}(\frac{b^{\varepsilon L}-1}{b^{\varepsilon}-1}) + a^L f(1) \\
&= cn^{\log_b a-\varepsilon}(\frac{b^{\varepsilon\log_b n}-1}{b^{\varepsilon}-1}) + a^L f(1) \\
&= cn^{\log_b a-\varepsilon}((n^{\varepsilon}-1)\cdot\frac{1}{b^{\varepsilon}-1}) + a^{\log_b n} f(1) \\
&= cn^{\log_b a-\varepsilon}((n^{\varepsilon}-1)\cdot c_2) + n^{\log_b a}\cdot c_3 \\
&= c_4 n^{\log_b a} - c_4 n^{\log_b a-\varepsilon} + n^{\log_b a}\cdot c_3 \\
&= O(n^{\log_b a})
\end{aligned}
$$