CS4102 Recurrences Proofs

Nathan Brunelle

Fall 2018

1 Karatsuba Recurrence, Tree Method

Karatsuba Recurrence:

$$T(n) = 3T(\frac{n}{2}) + 8n$$

Using the tree method for solving the recurrence, we obtained the sum:

$$\begin{aligned}
T(n) &= 8n\sum_{i=0}^{\log_2 n}(\frac{3}{2})^i \\
&= 8n\frac{(\frac{3}{2})^{\log_2 n+1} - 1}{\frac{3}{2} - 1} \\
&= 8n\frac{(\frac{3}{2})^{\log_2 n+1} - 1}{\frac{1}{2}} \\
&= 16n((\frac{3}{2})^{\log_2 n+1} - 1) \\
&= 16n(2^{\log_2 3-1})^{\log_2 n+1} - 16n \\
&= 16n(2^{\log_2 3\cdot\log_2 n-\log_2 n+\log_2 3-1}) - 16n \\
&= 16n((2^{\log_2 n})^{\log_2 3}\cdot 2^{-\log_2 n}\cdot 2^{\log_2 3}\cdot 2^{-1}) - 16n \\
&= 16n(n^{\log_2 3}\cdot\frac{1}{n}\cdot 3\cdot\frac{1}{2}) - 16n \\
&= 24n^{\log_2 3} - 16n \\
&= \Theta(n^{\log_2 3}) \\
&\approx \Theta(n^{1.585})
\end{aligned}$$

2 Karatsuba, Guess and Check, Loose Bound

Karatsuba Recurrence:

$$T(n) = 3T(\frac{n}{2}) + 8n$$

Goal:

$$T(n) \leq 3000n^{1.6}$$

Base Case:

$$T(1) = 8 \leq 3000$$

Hypothesis:

$$\forall n < x_0, T(n) \leq 3000n^{1.6}$$

Inductive Step:

$$\begin{aligned}
T(x_0 + 1) &= 3T(\frac{x_0 + 1}{2}) + 8(x_0 + 1) \\
&\leq 3\left(3000\left(\frac{x_0 + 1}{2}\right)^{1.6}\right) + 8(x_0 + 1) \\
&= \frac{3}{2^{1.6}} \cdot 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) \\
&\leq 0.997 \cdot 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) \\
&= (1 - 0.003) \cdot 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) \\
&= 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) - 0.003 \cdot 3000(x_0 + 1)^{1.6} \\
&= 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) - 9(x_0 + 1)^{1.6} \\
&\leq 3000(x_0 + 1)^{1.6}
\end{aligned}$$

3 MergeSort, Guess and Check

MergeSort Recurrence:

$$T(n) = 2T(\frac{n}{2}) + n$$

Goal:

$$T(n) \leq n \log_2 n$$

Base Case: by inspection
Hypothesis:

$$\forall n < x_0, T(n) \leq n \log_2 n$$

Inductive Step:

$$\begin{aligned}
T(x_0 + 1) &= 2T(\frac{x_0 + 1}{2}) + (x_0 + 1) \\
&\leq 2(\frac{x_0 + 1}{2} \log_2 \frac{x_0 + 1}{2}) + x_0 + 1 \\
&= (x_0 + 1) \log_2 \frac{x_0 + 1}{2} + x_0 + 1 \\
&= (x_0 + 1)(\log_2(x_0 + 1) + \log_2 \frac{1}{2}) + x_0 + 1 \\
&= (x_0 + 1)(\log_2(x_0 + 1) - 1) + x_0 + 1 \\
&= (x_0 + 1) \log_2(x_0 + 1) - (x_0 + 1) + x_0 + 1 \\
&= (x_0 + 1) \log_2(x_0 + 1)
\end{aligned}$$

4 Alt. MergeSort, Guess and Check

MergeSort Recurrence:

$$T(n) = 2T(\frac{n}{2}) + 209n$$

Goal:

$$T(n) \leq 209n \log_2 n$$

Base Case: by inspection
Hypothesis:

$$\forall n < x_0, T(n) \leq 209n \log_2 n$$

Inductive Step:

$$\begin{aligned}
T(x_0 + 1) &= 2T(\frac{x_0 + 1}{2}) + 209(x_0 + 1) \\
&\leq 2(209\frac{x_0 + 1}{2} \log_2 \frac{x_0 + 1}{2}) + 209(x_0 + 1) \\
&= 209(x_0 + 1) \log_2 \frac{x_0 + 1}{2} + 209(x_0 + 1) \\
&= 209(x_0 + 1)(\log_2(x_0 + 1) + \log_2 \frac{1}{2}) + 209(x_0 + 1) \\
&= 209(x_0 + 1)(\log_2(x_0 + 1) - 1) + 209(x_0 + 1) \\
&= 209(x_0 + 1) \log_2(x_0 + 1) - 209(x_0 + 1) + 209(x_0 + 1) \\
&= 209(x_0 + 1) \log_2(x_0 + 1)
\end{aligned}$$

5 Karatsuba, Guess and Check, Tight Bound

Karatsuba Recurrence:

$$T(n) = 3T(\frac{n}{2}) + 8n$$

Goal:

$$T(n) \leq 24n^{\log_2 3} - 16n$$

Base Case: by inspection
Hypothesis:

$$\forall n < x_0, T(n) \leq 24n^{\log_2 3} - 16n$$

Inductive Step:

$$\begin{aligned}
T(x_0+1) &= 3T(\frac{x_0+1}{2}) + 8(x_0+1) \\
&\leq 3(24(\frac{x_0+1}{2})^{\log_2 3} - 16\frac{x_0+1}{2}) + 8(x_0+1) \\
&= 3(\frac{24}{3}(x_0+1)^{\log_2 3} - 8(x_0+1)) + 8(x_0+1) \\
&= 24(x_0+1)^{\log_2 3} - 24(x_0+1) + 8(x_0+1) \\
&= 24(x_0+1)^{\log_2 3} - 16(x_0+1)
\end{aligned}$$

6 Master Theorem Case 1

Recurrence:

$$T(n) = aT(\frac{n}{b}) + f(n)$$

Assumption:

$$f(n) = O(n^{\log_b a - \varepsilon}) \Rightarrow f(n) \leq c \cdot n^{\log_b a - \varepsilon}$$

To Show:

$$T(n) = O(n^{\log_b a})$$

Proof: (let $L = \log_b n$, i.e. the height of the recurrence tree)

$$
\begin{aligned}
T(n) &= f(n) + af(\frac{n}{b}) + a^2 f(\frac{n}{b^2}) + \ldots + a^L f(\frac{n}{b^L}) \\
&\leq c((n)^{\log_b a-\varepsilon} + a(\frac{n}{b})^{\log_b a-\varepsilon} + a^2(\frac{n}{b^2})^{\log_b a-\varepsilon} + \ldots + a^{L-1}(\frac{n}{b^{L-1}})^{\log_b a-\varepsilon}) + a^L f(1) \\
&= cn^{\log_b a-\varepsilon}(1 + \frac{a}{b^{\log_b a-\varepsilon}} + \frac{a^2}{b^{2\log_b a-\varepsilon}} + \ldots + \frac{a^{L-1}}{b^{(L-1)\log_b a-\varepsilon}}) + a^L f(1) \\
&= cn^{\log_b a-\varepsilon}(1 + b^{\varepsilon} + b^{2\varepsilon} + \ldots + b^{(L-1)\varepsilon}) + a^L f(1) \\
&= cn^{\log_b a-\varepsilon}(\frac{b^{\varepsilon L}-1}{b^{\varepsilon}-1}) + a^L f(1) \\
&= cn^{\log_b a-\varepsilon}(\frac{b^{\varepsilon \log_b n}-1}{b^{\varepsilon}-1}) + a^L f(1) \\
&= cn^{\log_b a-\varepsilon}((n^{\varepsilon}-1) \cdot \frac{1}{b^{\varepsilon}-1}) + a^{\log_b n} f(1) \\
&= cn^{\log_b a-\varepsilon}((n^{\varepsilon}-1) \cdot c_2) + n^{\log_b a} \cdot c_3 \\
&= c_4 n^{\log_b a} - c_4 n^{\log_b a-\varepsilon} + n^{\log_b a} \cdot c_3 \\
&= O(n^{\log_b a})
\end{aligned}
$$