CS4102 Algorithms

Fall 2018

Warm up

Simplify:

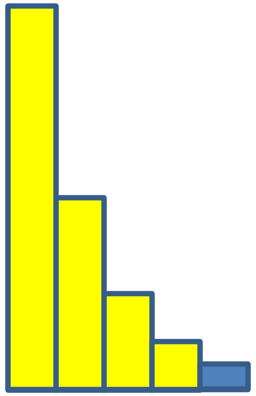
$$(1 + a + a^2 + a^3 + a^4 + \cdots + a^L)(a - 1) = ?$$

$$(\cancel{a} + \cancel{a^2} + \cancel{a^3} + \cancel{a^4} + \cancel{a^5} + \cdots + \cancel{a^L} + a^{L+1}) +$$
$$(-\cancel{a} - \cancel{a^2} - \cancel{a^3} - \cancel{a^4} - \cancel{a^5} - \cdots - \cancel{a^L} - 1) =$$
$$a^{L+1} - 1$$

$$\sum_{i=0}^{L} a^i = \frac{a^{L+1} - 1}{a - 1}$$

Finite Geometric Series

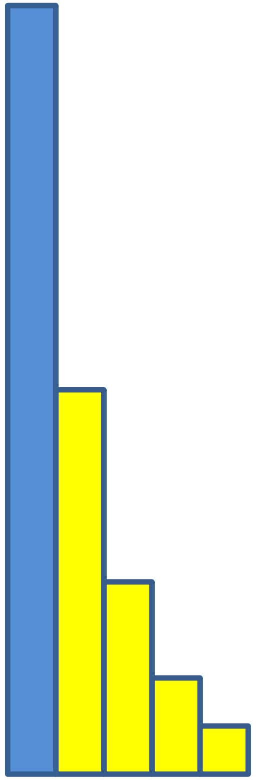
$a < 1$



The series multiplied by a

$(1 + a + a^2 + \cdots + a^L)a$

$-$



The series

$(1 + a + a^2 + \cdots + a^L)1$

$=$

The next term in the series

a^{L+1}

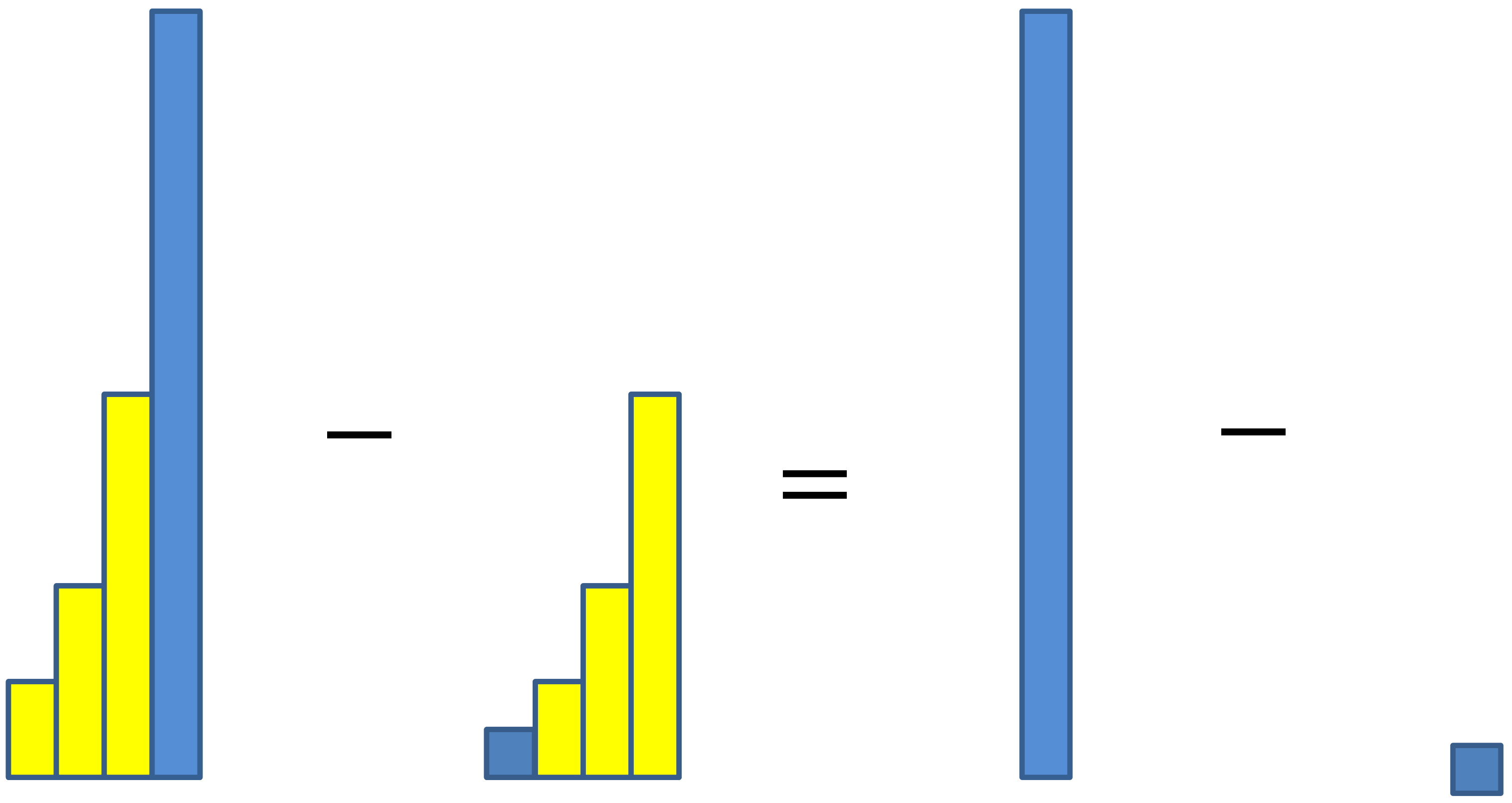
$-$

The first term

1

Finite Geometric Series

$a > 1$



The series multiplied by a

$(1 + a + a^2 + \cdots + a^L)a$

The series

$(1 + a + a^2 + \cdots + a^L)1$

The next term in the series

a^{L+1}

The first term

1

Today’s Keywords

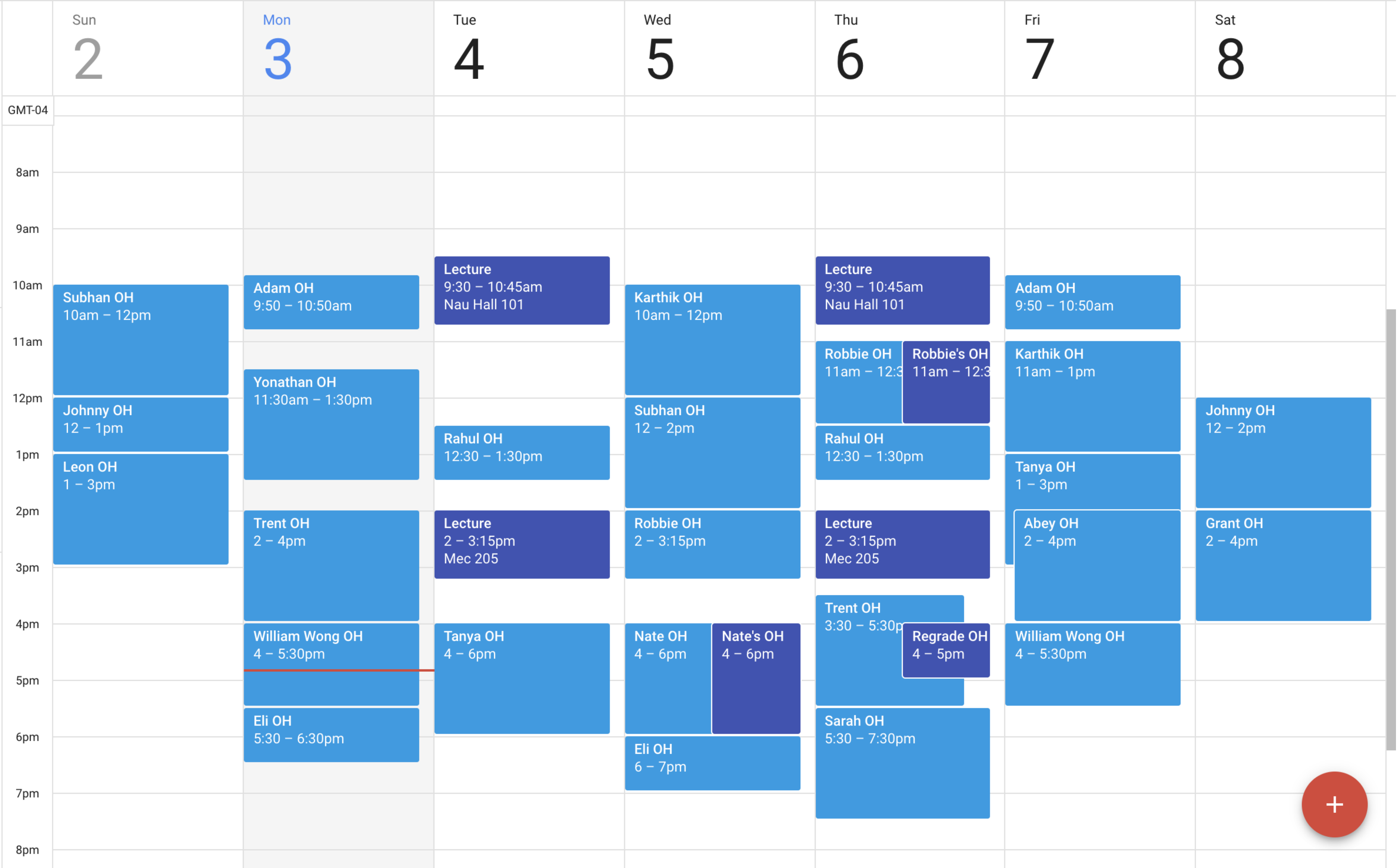
- Divide and Conquer
- Recurrences
- Merge Sort
- Karatsuba
- Tree Method

CLRS Readings

- Chapter 4

Homeworks

- Hw0 due 11pm Wednesday, Sept 5
  - Submit BOTH pdf and zip!
- Hw1 Due Wednesday, Sept 12
  - Start early!
  - Written (use Latex!)
  - Asymptotic notation
  - Recurrences
  - Divide and Conquer



Sun 2
Mon 3
Tue 4
Wed 5
Thu 6
Fri 7
Sat 8
GMT-04
8am
9am
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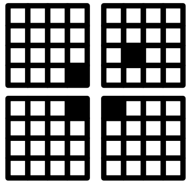
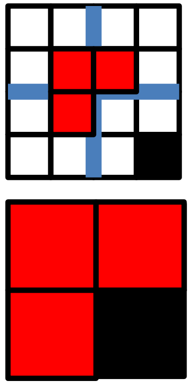
3pm
4pm
5pm
6pm
7pm
8pm
Subhan OH 10am – 12pm
Johnny OH 12 – 1pm
Leon OH 1 – 3pm
Adam OH 9:50 – 10:50am
Yonathan OH 11:30am – 1:30pm
Trent OH 2 – 4pm
William Wong OH 4 – 5:30pm
Eli OH 5:30 – 6:30pm
Lecture 9:30 – 10:45am Nau Hall 101
Rahul OH 12:30 – 1:30pm
Lecture 2 – 3:15pm Mec 205
Tanya OH 4 – 6pm
Karthik OH 10am – 12pm
Subhan OH 12 – 2pm
Robbie OH 2 – 3:15pm
Nate OH 4 – 6pm
Nate's OH 4 – 6pm
Eli OH 6 – 7pm
Lecture 9:30 – 10:45am Nau Hall 101
Robbie OH 11am – 12:3
Robbie's OH 11am – 12:3
Rahul OH 12:30 – 1:30pm
Lecture 2 – 3:15pm Mec 205
Trent OH 3:30 – 5:30p
Regrade OH 4 – 5pm
Sarah OH 5:30 – 7:30pm
Adam OH 9:50 – 10:50am
Karthik OH 11am – 1pm
Tanya OH 1 – 3pm
Abey OH 2 – 4pm
William Wong OH 4 – 5:30pm
Johnny OH 12 – 2pm
Grant OH 2 – 4pm

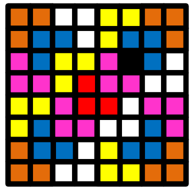
Divide and Conquer*



- **Divide**:
  - Break the problem into multiple subproblems, each smaller instances of the original
- **Conquer**:
  - If the subproblems are “large”:
    - Solve each subproblem recursively
  - If the subproblems are “small”:
    - Solve them directly (base case)



- **Combine**:
  - Merge together solutions to subproblems



*CLRS Chapter 4

Analyzing Divide and Conquer

1. Break into smaller subproblems
2. Use recurrence relation to express recursive running time
3. Use asymptotic notation to simplify

- **Divide**: $D(n)$ time,
- **Conquer**: recurse on small problems, size s
- **Combine**: $\mathrm{C}(n)$ time
- **Recurrence:**
  - $T(n) = D(n) + \sum T(s) + C(n)$

Recurrence Solving Techniques

Tree

Guess/Check

"Cookbook"

Substitution

Merge Sort

- **Divide**:
  - Break n-element list into two lists of ${}^{n}/_{2}$ elements
- **Conquer**:
  - If $n > 1$:
    - Sort each sublist recursively
  - If $n = 1$:
    - List is already sorted (base case)
- **Combine**:
  - Merge together sorted sublists into one sorted list

Merge

- **Combine:** Merge sorted sublists into one sorted list
- We have:
  - 2 sorted lists (L_1, L_2)
  - 1 output list (L_{out})

While (L_1 and L_2 not empty):

    If $L_1[0] \leq L_2[0]$:

        L_{out}.append(L_1.pop())

    Else:

        L_{out}.append(L_2.pop())

L_{out}.append(L_1)

L_{out}.append(L_2)

Analyzing Merge Sort

1. Break into smaller subproblems
2. Use recurrence relation to express recursive running time
3. Use asymptotic notation to simplify

- **Divide**: 0 comparisons
- **Conquer**: recurse on 2 small problems, size $\frac{n}{2}$
- **Combine**: n comparisons
- **Recurrence:**
  - $T(n) = 2T(\frac{n}{2}) + n$

Recurrence Solving Techniques

Tree

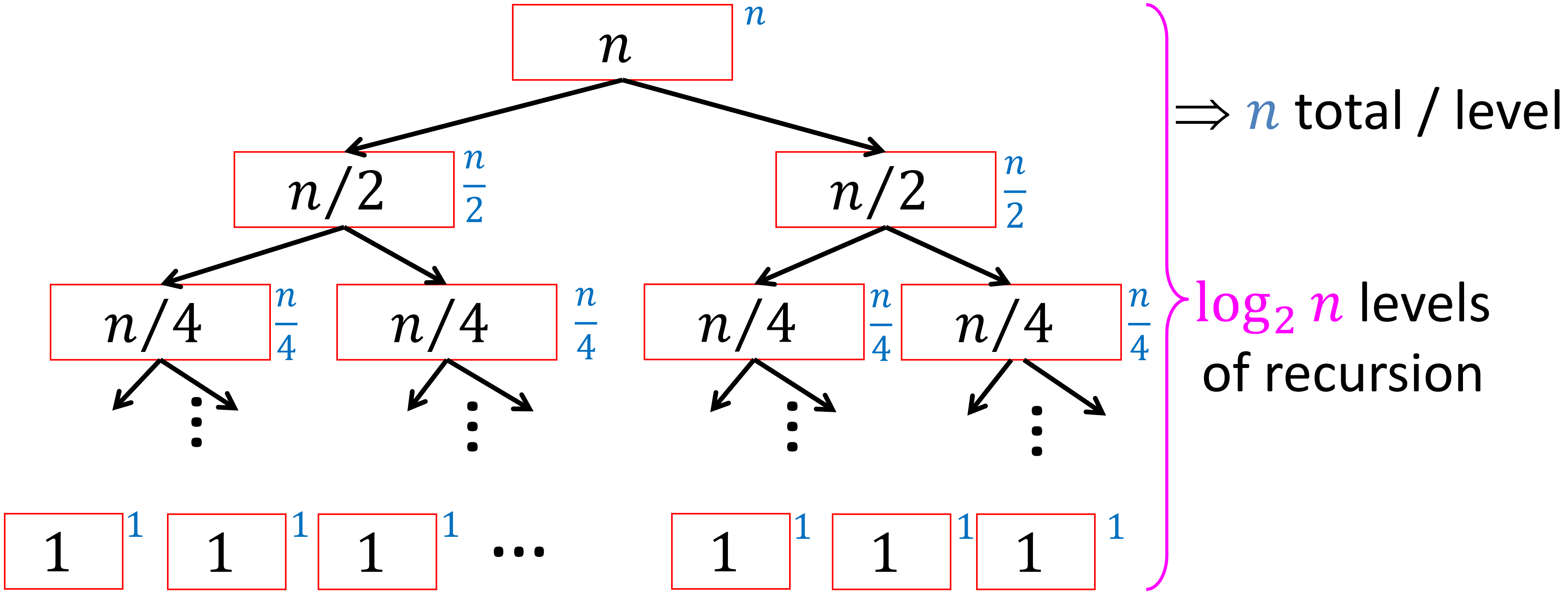
Guess/Check

"Cookbook"

Substitution

Tree method

$$T(n) = 2T(\frac{n}{2}) + n$$



$$T(n) = \sum_{i=1}^{\log_2 n} n = n \log_2 n$$

Multiplication

- Want to multiply large numbers together

$$\begin{array}{r} 4\,1\,0\,2 \\ \times\ 1\,8\,1\,9 \\ \hline \end{array}$$

n-digit numbers

- What makes a "good" algorithm?
- How do we measure input size?
- What do we "count" for run time?

"Schoolbook" Method

How many total multiplications?

n-digit numbers

$$
\begin{array}{r}
4\,1\,0\,2 \\
\times\ \boxed{1}\boxed{8}\boxed{1}\boxed{9} \\
\hline
3\,6\,9\,1\,8 \\
4\,1\,0\,2 \\
3\,2\,8\,1\,6 \\
+\ 4\,1\,0\,2 \\
\hline
7\,4\,6\,1\,5\,3\,8
\end{array}
\quad
\left.
\begin{array}{l}
n \text{ mults} \\
n \text{ mults} \\
n \text{ mults} \\
n \text{ mults}
\end{array}
\right\} n \text{ levels} \Rightarrow \theta(n^2)
$$

Divide and Conquer method

1. Break into smaller subproblems

$$\begin{array}{r} \boxed{a}\,\boxed{b} \\ \times\ \boxed{c}\,\boxed{d} \\ \hline \end{array} \quad \begin{array}{l} = 10^{\frac{n}{2}}\,\boxed{a} + \boxed{b} \\ = 10^{\frac{n}{2}}\,\boxed{c} + \boxed{d} \end{array}$$

$$10^{n}\left(\boxed{a} \times \boxed{c}\right) +$$

$$10^{\frac{n}{2}}\left(\boxed{a} \times \boxed{d} + \boxed{b} \times \boxed{c}\right) +$$

$$\left(\boxed{b} \times \boxed{d}\right)$$

Divide and Conquer Multiplication

- **Divide**:
  - Break n-digit numbers into four numbers of ${}^{n}/_{2}$ digits each (call them a, b, c, d)
- **Conquer**:
  - If $n > 1$:
    - Recursively compute ac, ad, bc, bd
  - If $n = 1$: (i.e. one digit each)
    - Compute ac, ad, bc, bd directly (base case)
- **Combine**:
  - $10^{n}(ac) + 10^{\frac{n}{2}}(ad + bc) + bd$

Divide and Conquer method

2. Use recurrence relation to express recursive running time

$$10^{n}(\boxed{ac}) + 10^{\frac{n}{2}}(\boxed{ad} + \boxed{bc}) + \boxed{bd}$$

Recursively solve

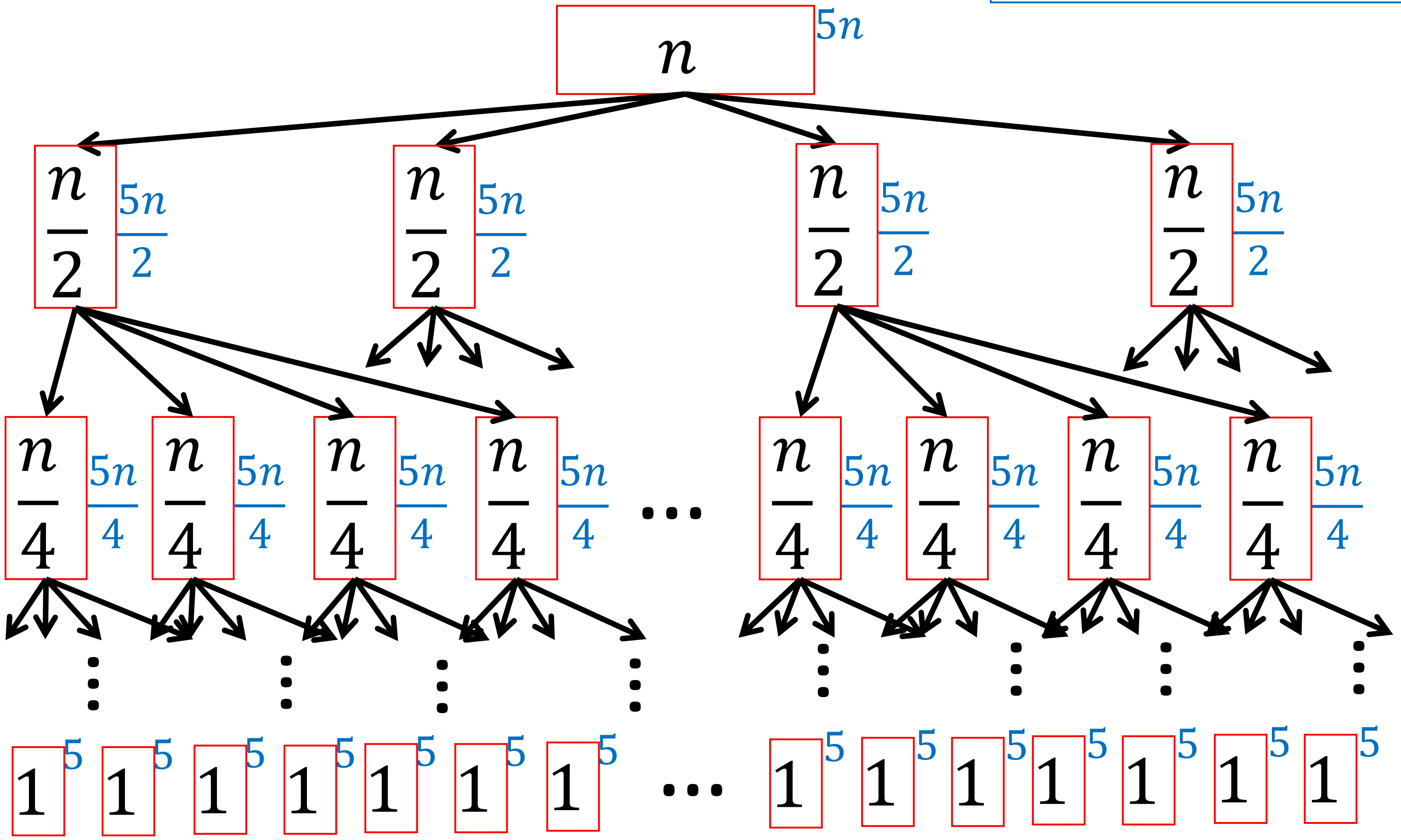
$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

Divide and Conquer method

3. Use asymptotic notation to simplify

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

$$T(n) = 5n \sum_{i=0}^{\log_2 n} 2^i$$



$5n$

$\frac{4}{2} \cdot 5n$

$\frac{16}{4} \cdot 5n$

$\vdots$

$2^{\log_2 n} \cdot 5n$

Divide and Conquer method

3. Use asymptotic notation to simplify

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

$$T(n) = 5n \sum_{i=0}^{\log_2 n} 2^i$$

$$T(n) = 5n \frac{2^{\log_2 n+1} - 1}{2 - 1}$$

$$T(n) = 5n(2n - 1) \; = \Theta(n^2)$$

Karatsuba

1. Break into smaller subproblems

$$\boxed{a}\boxed{b} = 10^{\frac{n}{2}}\boxed{a} + \boxed{b}$$

$$\times\ \boxed{c}\boxed{d} = 10^{\frac{n}{2}}\boxed{c} + \boxed{d}$$

$$10^{n}\left(\boxed{a} \times \boxed{c}\right) +$$

$$10^{\frac{n}{2}}\left(\boxed{a} \times \boxed{d} + \boxed{b} \times \boxed{c}\right) +$$

$$\left(\boxed{b} \times \boxed{d}\right)$$

$$\begin{array}{r} a \quad b \\ \times \; c \quad d \\ \hline \end{array}$$

Karatsuba

$$10^{n}\left(\boxed{ac}\right) + 10^{\frac{n}{2}}\left(\boxed{ad + bc}\right) + \boxed{bd}$$

Can’t avoid these

This can be simplified

$$(a+b)(c+d) =$$

$$\boxed{ac} + \boxed{ad + bc} + \boxed{bd}$$

$$\boxed{ad + bc} = \boxed{(a+b)(c+d) - \boxed{ac} - \boxed{bd}}$$

Two multiplications

One multiplication

Karatsuba

2. Use recurrence relation to express recursive running time

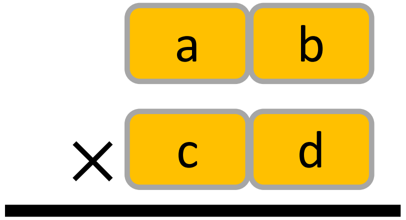
$$10^n\boxed{(ac)} + 10^{\frac{n}{2}}\left(\boxed{(a+b)(c+d)} - ac - bd\right) + \boxed{bd}$$

Recursively solve

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Karatsuba

- **Divide**:
  - Break n-digit numbers into four numbers of ${}^{n}/_{2}$ digits each (call them a, b, c, d)
- **Conquer**:
  - If $n > 1$:
    - Recursively compute ac, bd, $(a+b)(c+d)$
  - If $n = 1$:
    - Compute ac, bd, $(a+b)(c+d)$ directly (base case)
- **Combine**:
  - $10^n(ac) + 10^{\frac{n}{2}}\big((a+b)(c+d) - ac - bd\big) + bd$



Karatsuba Algorithm

1. Recursively compute: $ac, bd, (a+b)(c+d)$
2. $(ad+bc) = (a+b)(c+d) - ac - bd$
3. Return $10^{n}(ac) + 10^{\frac{n}{2}}(ad+bc) + bd$

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

<u>Pseudo-code</u>

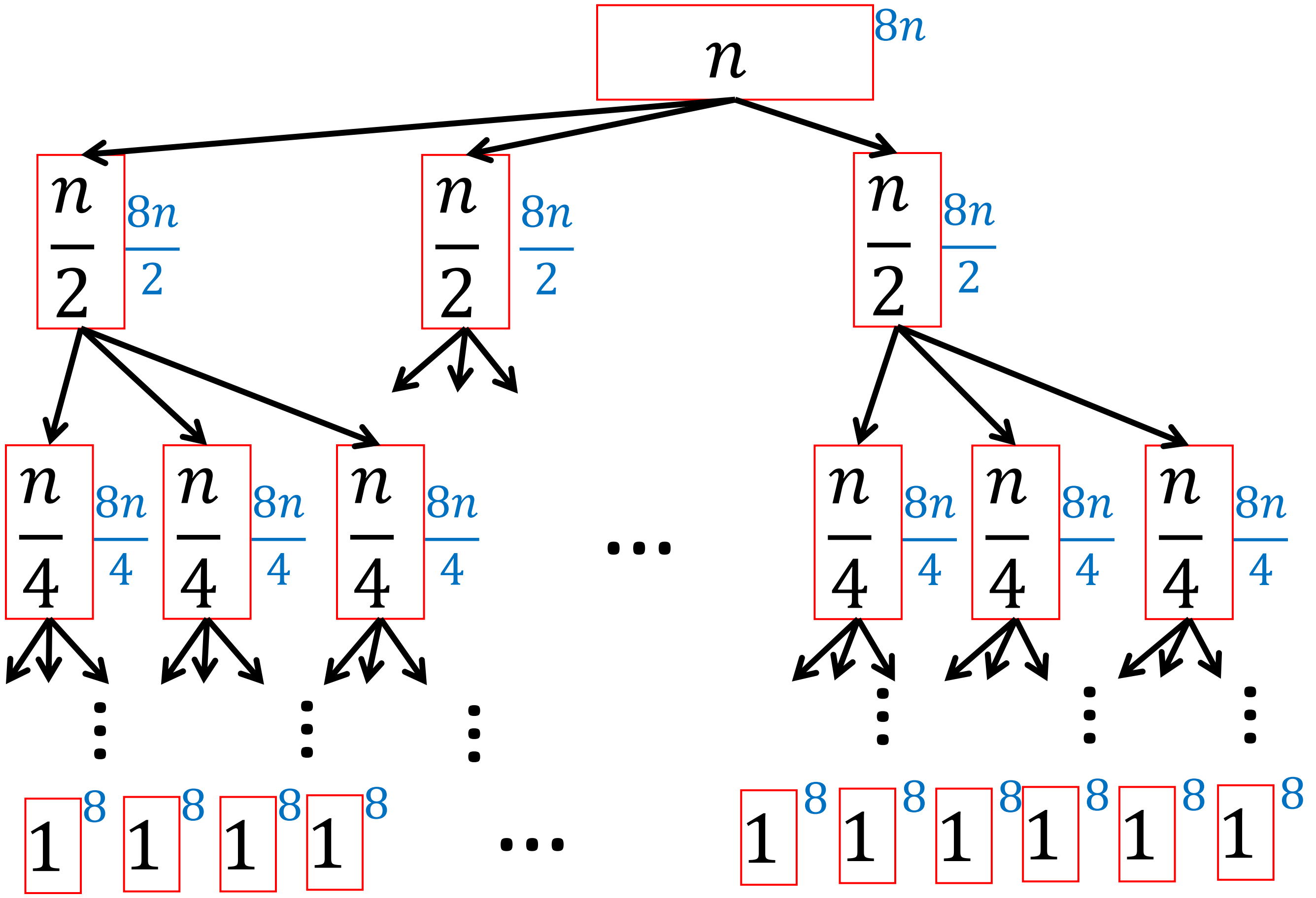
1. x = Karatsuba(a,c)
2. y = Karatsuba(a,d)
3. z = Karatsuba(a+b,c+d)-x-y
4. Return 10^{n}x +$10^{n/2}$z+y

Karatsuba

3. Use asymptotic notation to simplify

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

$$\boxed{T(n) = 8n \sum_{i=0}^{\log_2 n} \left({}^{3}/_{2}\right)^{i}}$$



$$8 \cdot 1n$$

$$\frac{8}{2} \cdot 3n$$

$$\frac{8}{4} \cdot 9n$$

$$\vdots$$

$$\frac{8}{2^{\log_2 n}} \cdot 3^{\log_2 n} n$$

Karatsuba

3. Use asymptotic notation to simplify

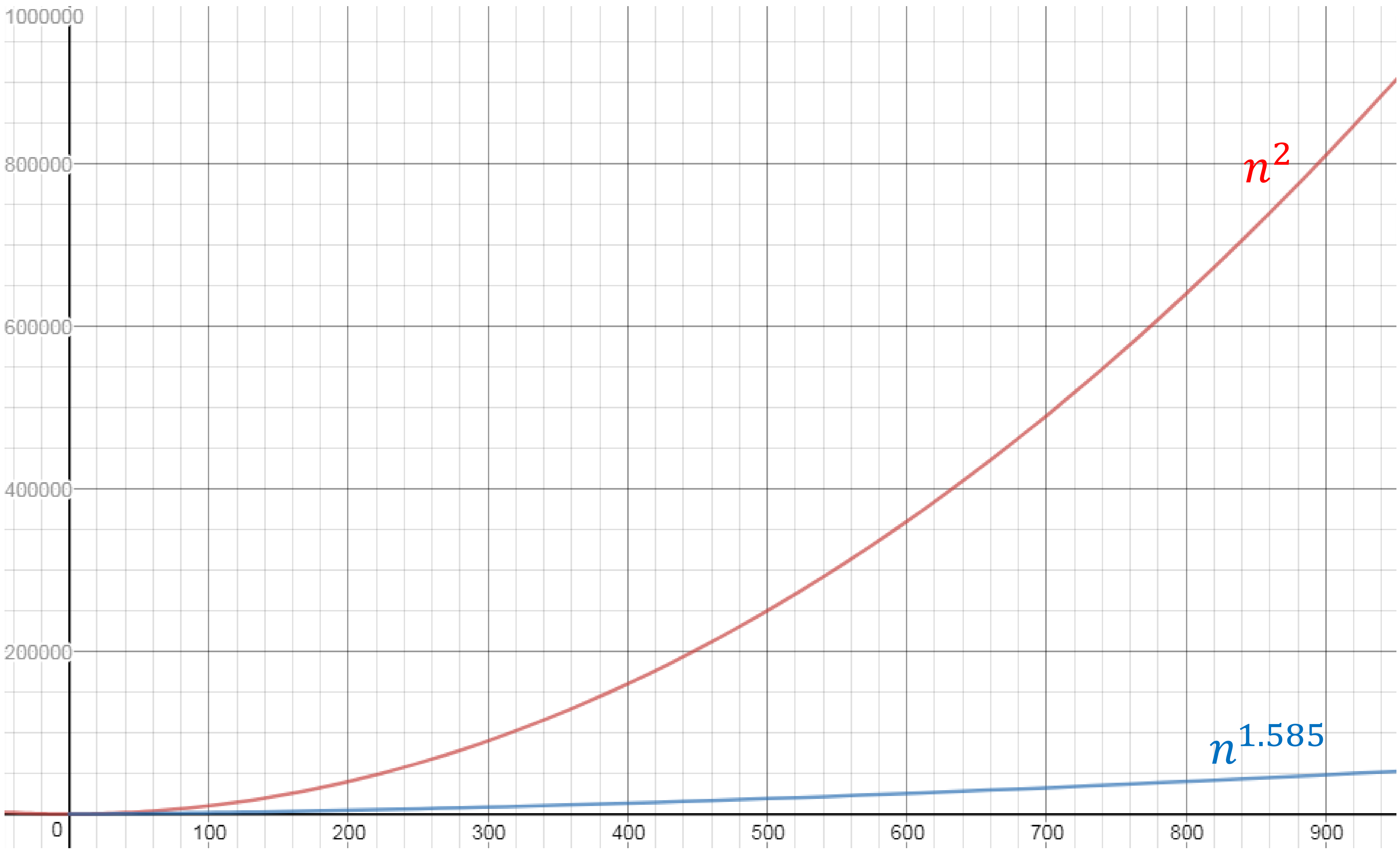
$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

$$T(n) = 8n \sum_{i=0}^{\log_2 n} \left({}^{3}/_{2}\right)^{i}$$

$$T(n) = 8n \frac{\left({}^{3}/_{2}\right)^{\log_2 n+1} - 1}{{}^{3}/_{2} - 1}$$

Math, math, and more math…(on board, see lecture supplement)

$$T(n) = 24\left(n^{\log_2 3}\right) - 16n \;= \Theta\left(n^{\log_2 3}\right)$$
$$\approx \Theta\left(n^{1.585}\right)$$


1000000
800000
600000
400000
200000
0
100
200
300
400
500
600
700
800
900
n^2
$n^{1.585}$