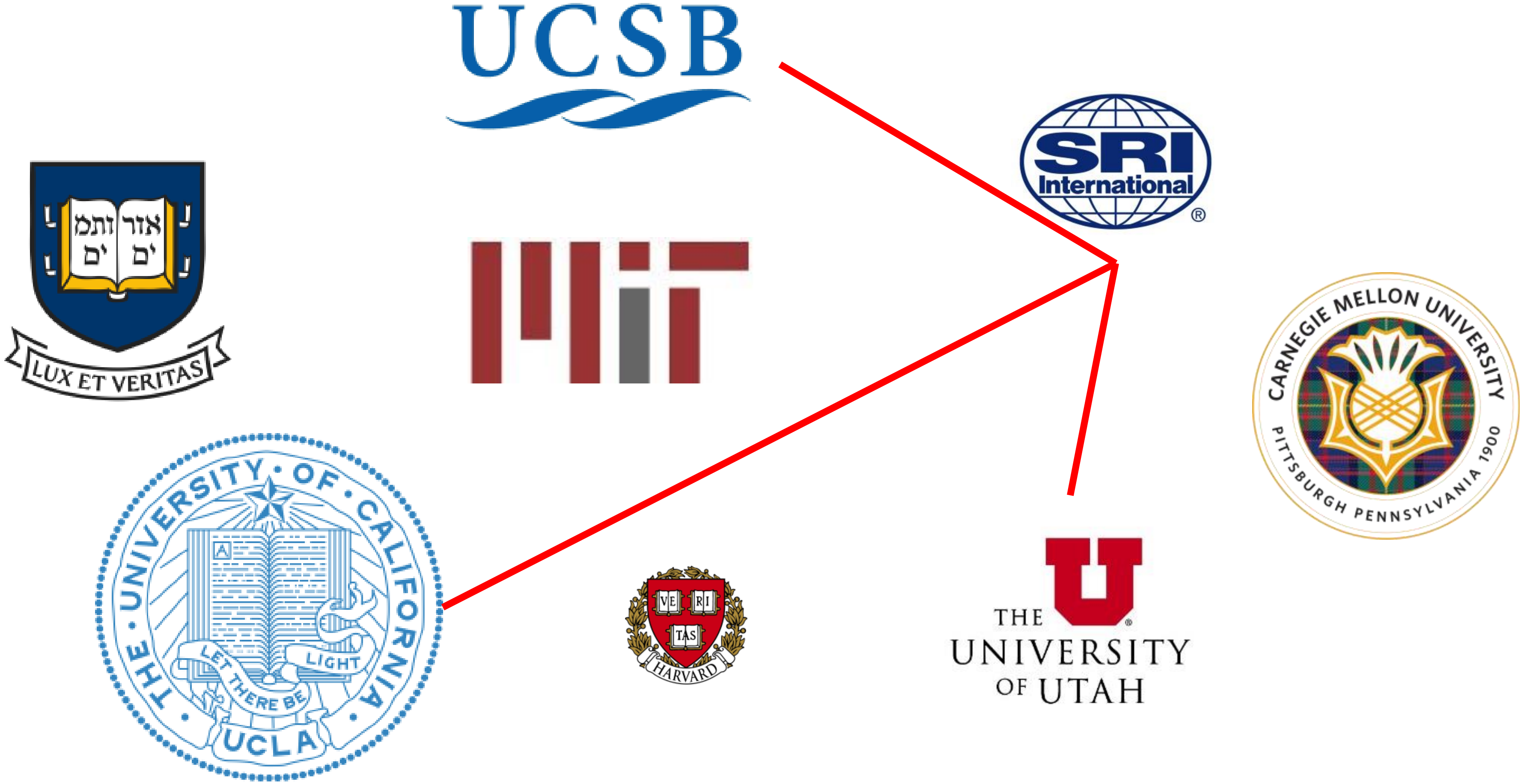
Minimum Spanning Trees (MST)

ETHAN BLASER (GRAD TA)

MODIFIED SLIDES FROM PROF. BRUNELLE

ARPANET


UCSB
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CARNEGIE MELLON UNIVERSITY
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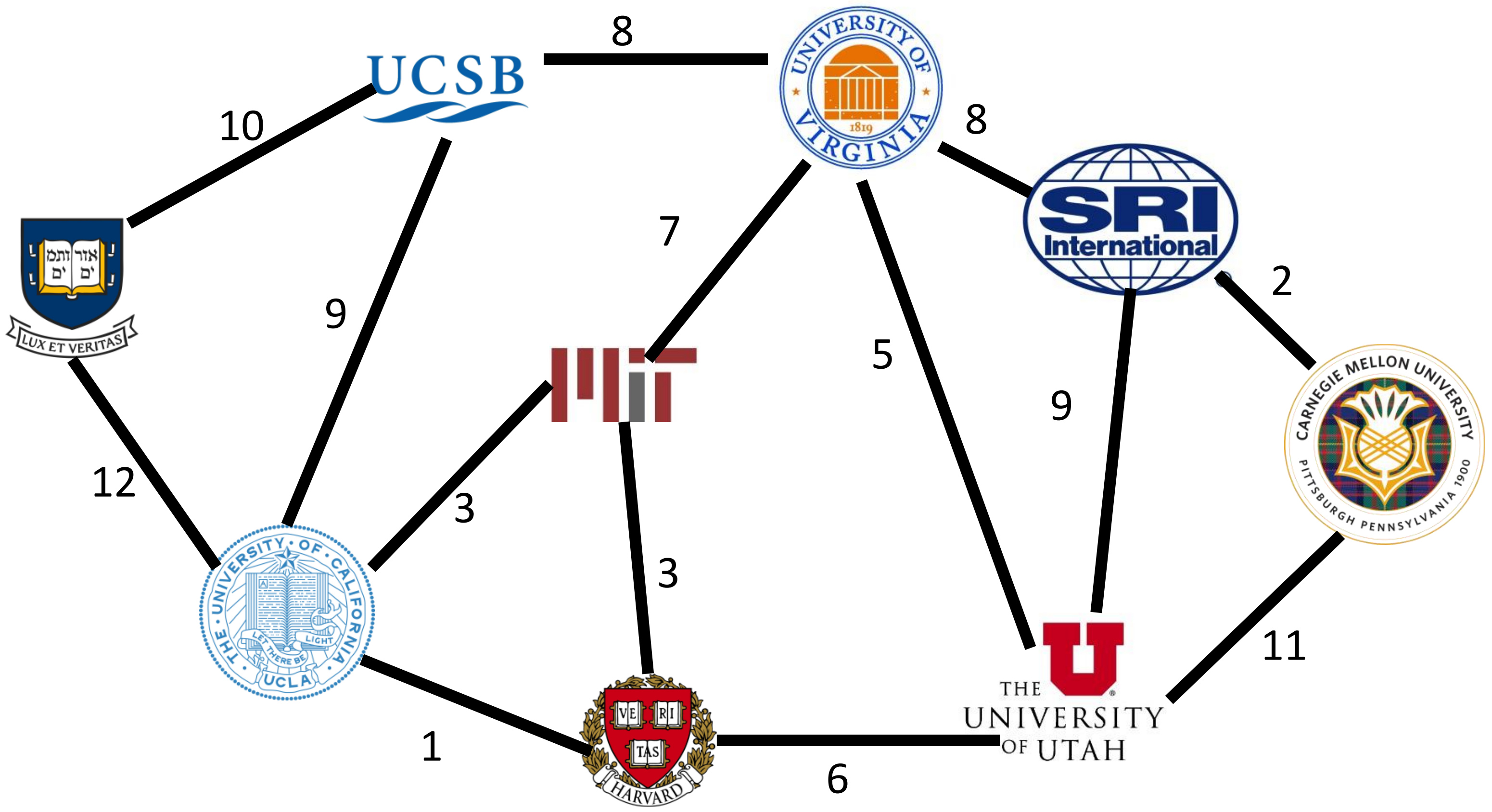
THE UNIVERSITY OF UTAH

Problem



Find a
Minimum
Spanning Tree

We need to connect together all these places into a network
We have feasible wires to run, plus the cost of each wire
Find the cheapest set of wires to run to connect all places

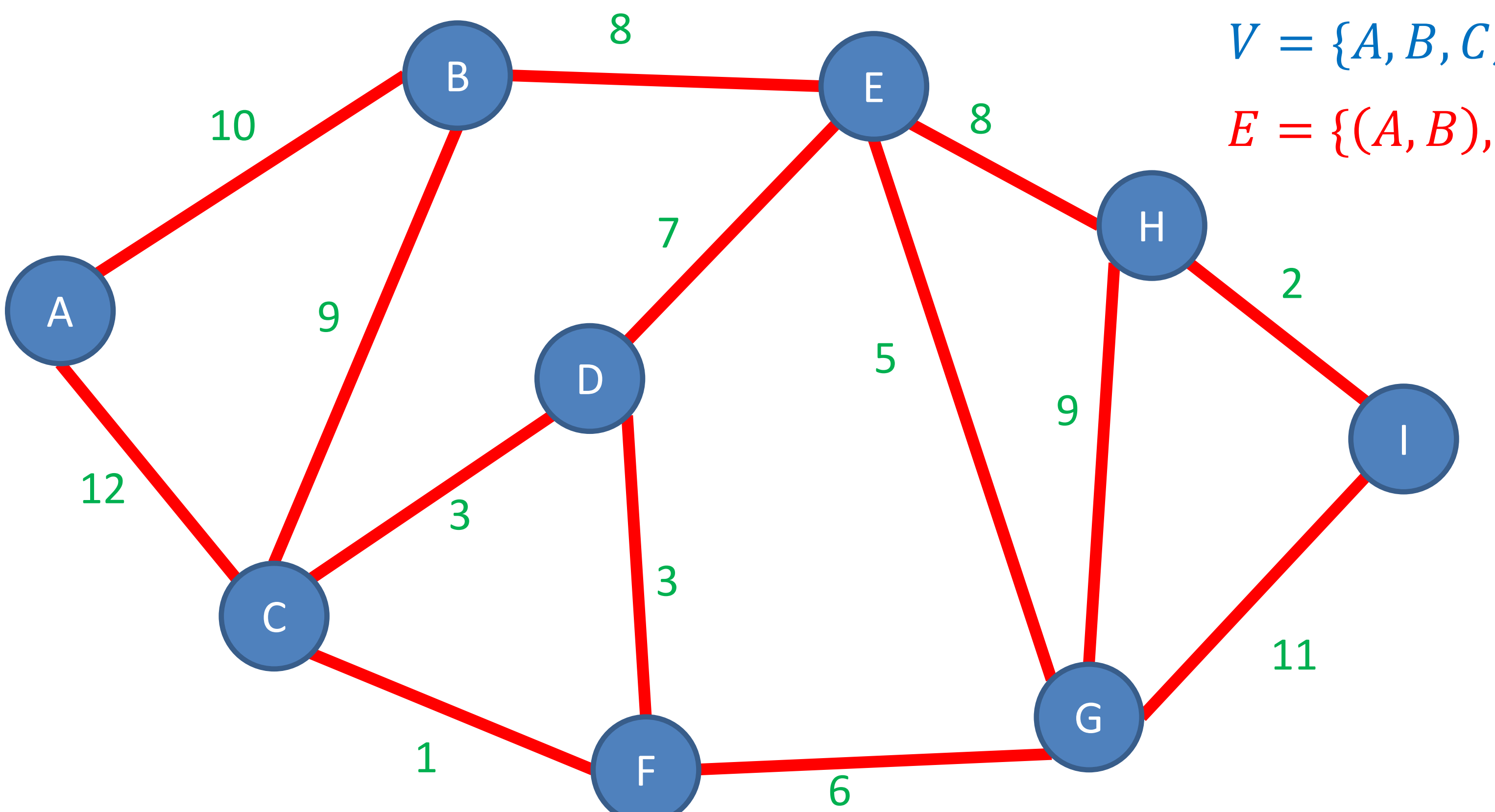
Undirected Graphs

Vertices/Nodes

Definition: $G = (V, E)$

Edges

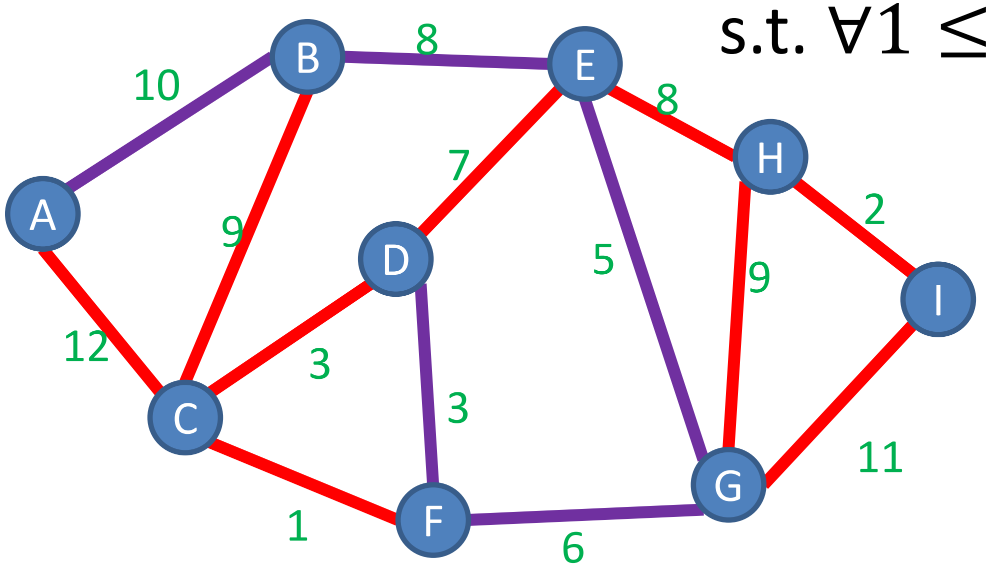
$w(e)$ = weight of edge e



$V = \{A, B, C, D, E, F, G, H, I\}$

$E = \{(A, B), (A, C), (B, C), \dots\}$

Definition: Path



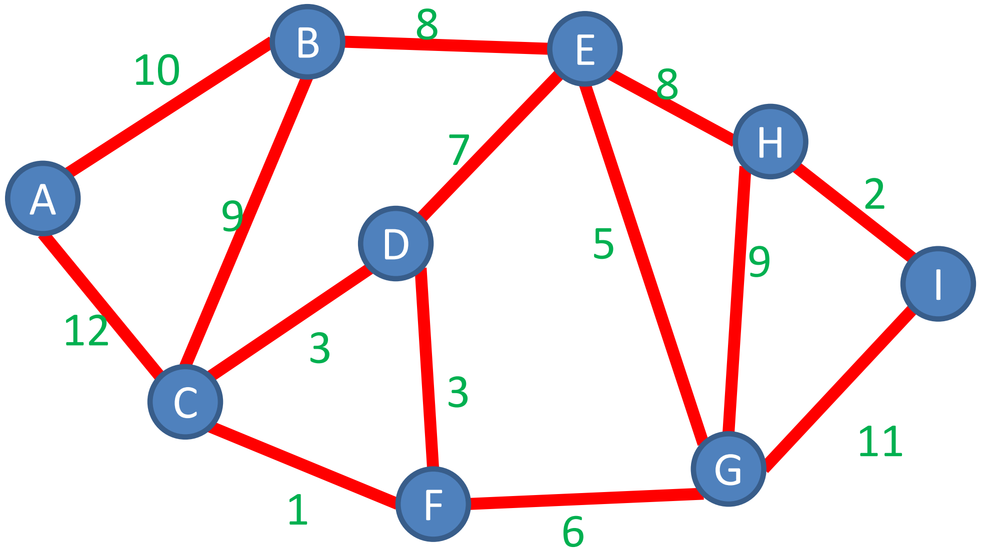
A sequence of nodes $(v_1, v_2, \dots, v_k)$
s.t. $\forall 1 \le i \le k-1, (v_i, v_{i+1}) \in E$

<u>Simple Path:</u>
A path in which each node appears at most once

<u>Cycle:</u>
A path of > 2 nodes in which $v_1 = v_k$

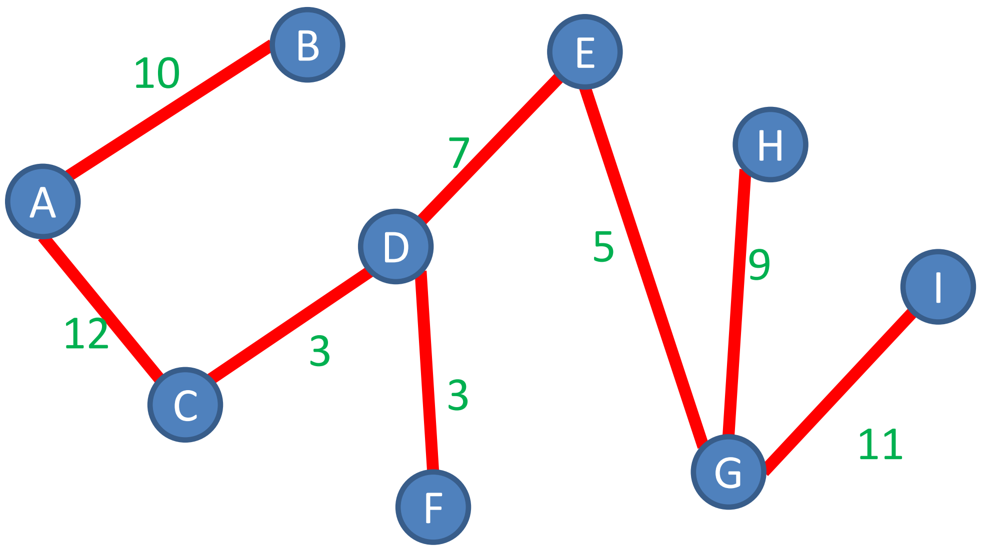
Definition: Connected Graph

A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is a path from v_1 to v_2



Definition: Tree

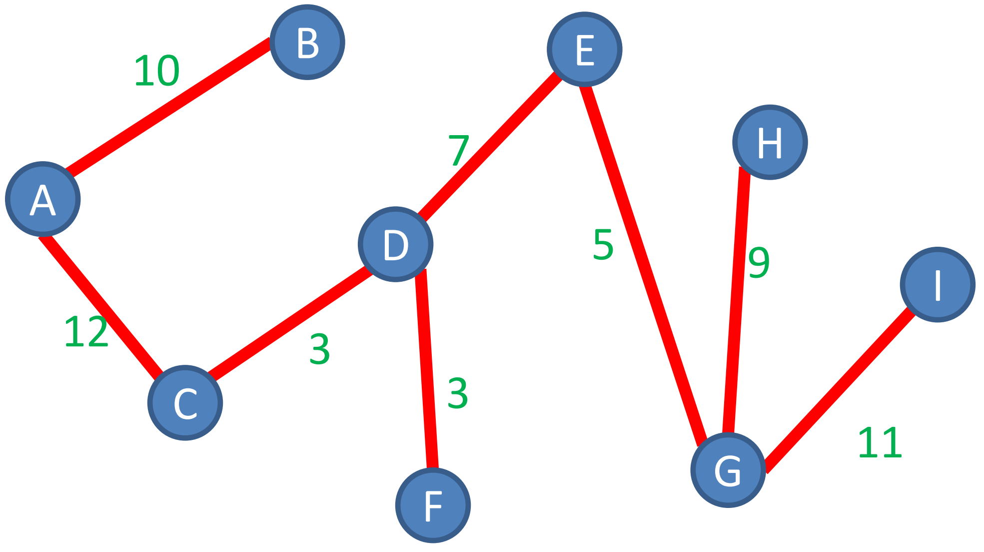
A connected graph with no cycles



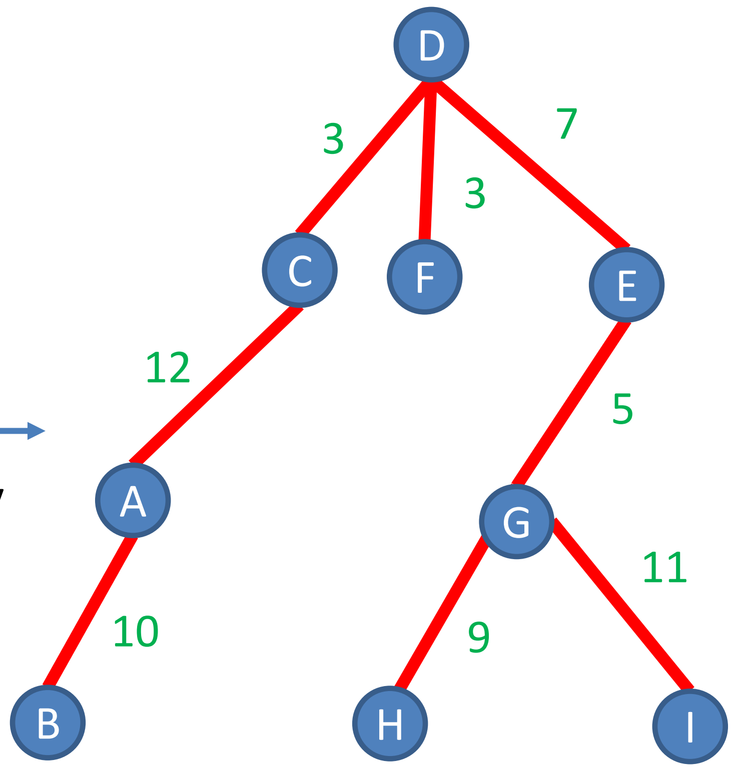
Note: A tree does not need a root, but they often do!

Definition: Tree

A connected graph with no cycles

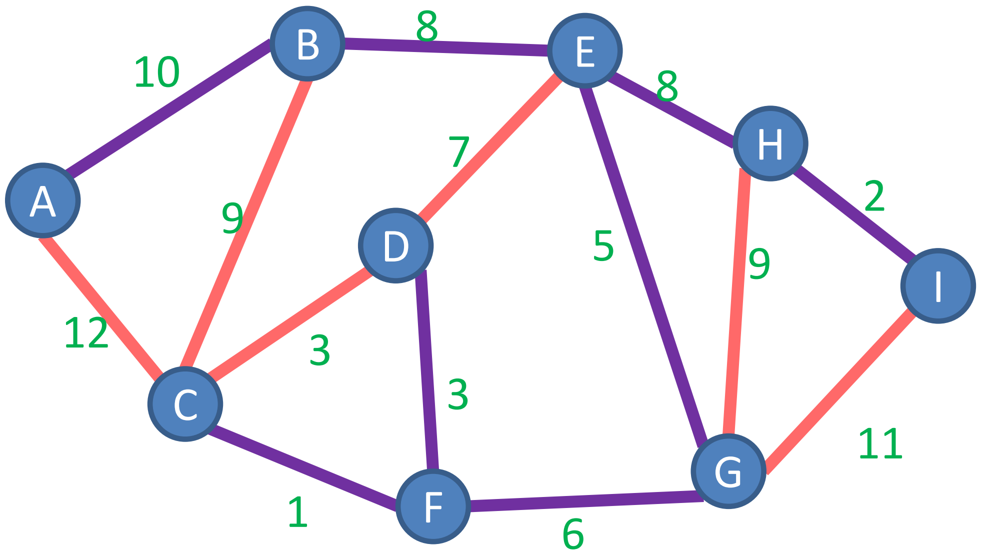


Pick some arbitrary root node and rearrange tree



Definition: Spanning Tree

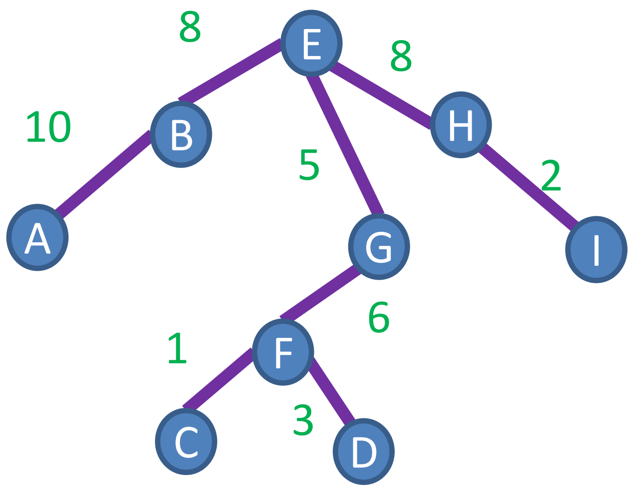
A Tree $\boldsymbol{T} = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$



How many edges does T have?

$V - 1$

Pick some arbitrary root node and rearrange tree

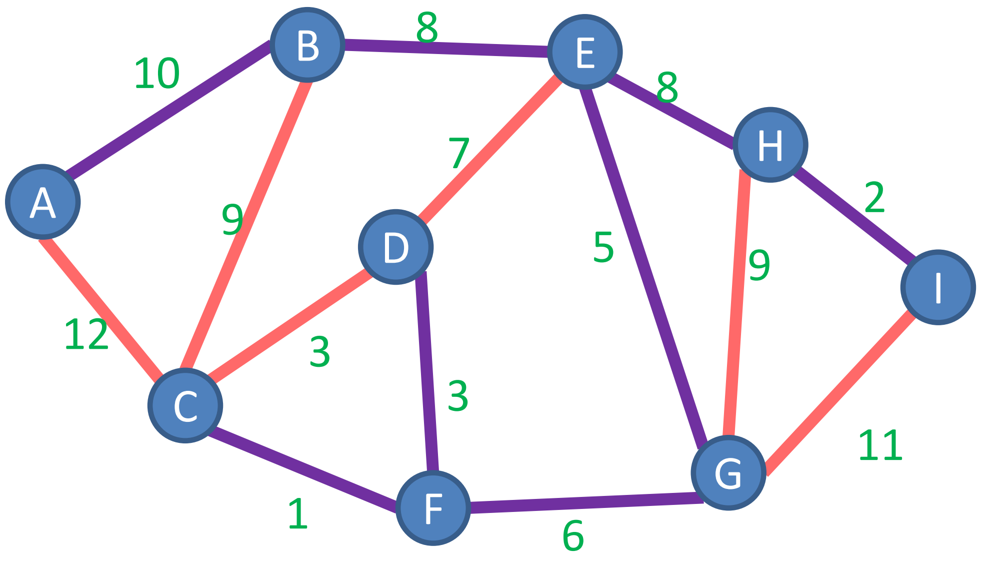


Any set of V-1 edges in the graph that doesn't have any cycles is guaranteed to be a spanning tree!

Any set of V-1 edges that connects all the nodes in the graph is guaranteed to be a spanning tree!

Definition: Minimum Spanning Tree

A Tree $\boldsymbol{T} = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$, that has minimal cost



$$Cost(T) = \sum_{e \in E_T} w(e)$$

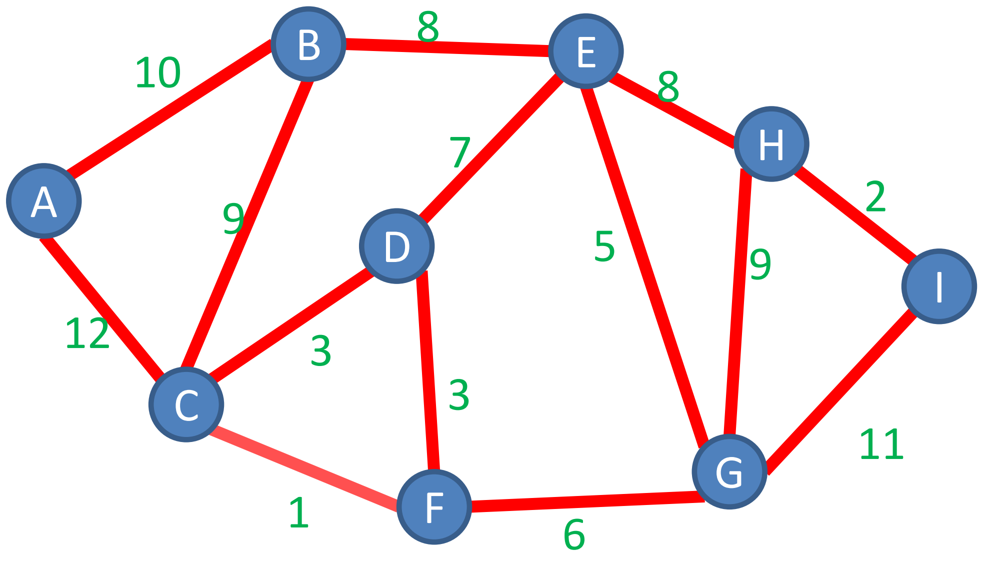
Greedy Algorithms

- Require Optimal Substructure
  - Solution to larger problem contains the solution to a smaller one
  - Only one subproblem to consider!
- Idea:
  1. Identify a greedy choice property
     - How to make a choice guaranteed to be included in some optimal solution
  2. Repeatedly apply the choice property until no subproblems remain

Kruskal’s Algorithm

Start with an empty tree A

Add to A the lowest-weight edge that does not create a cycle



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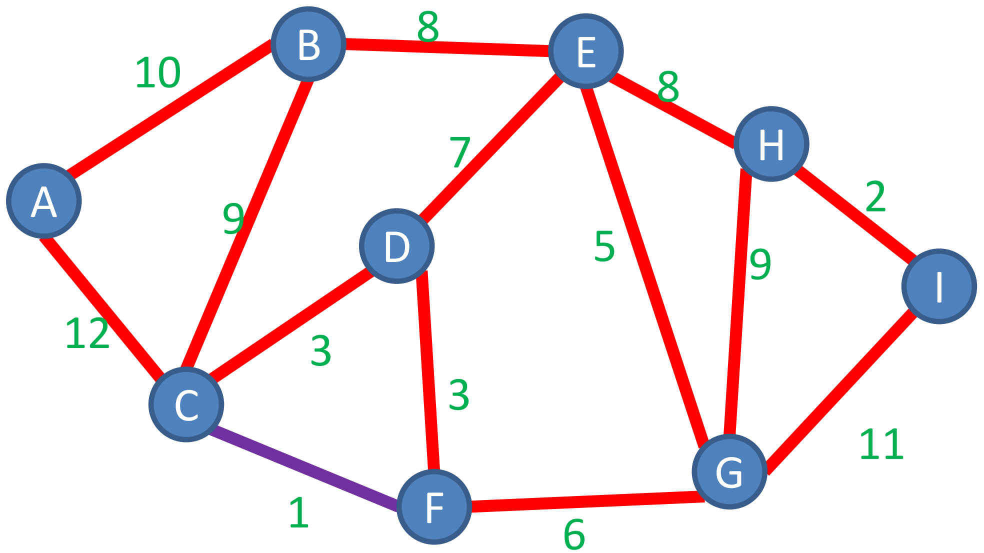

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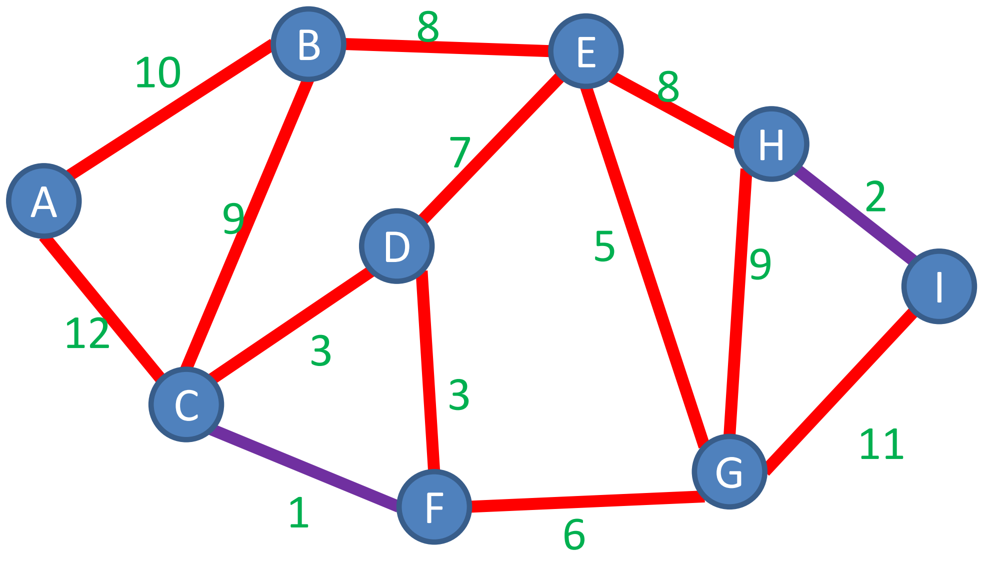

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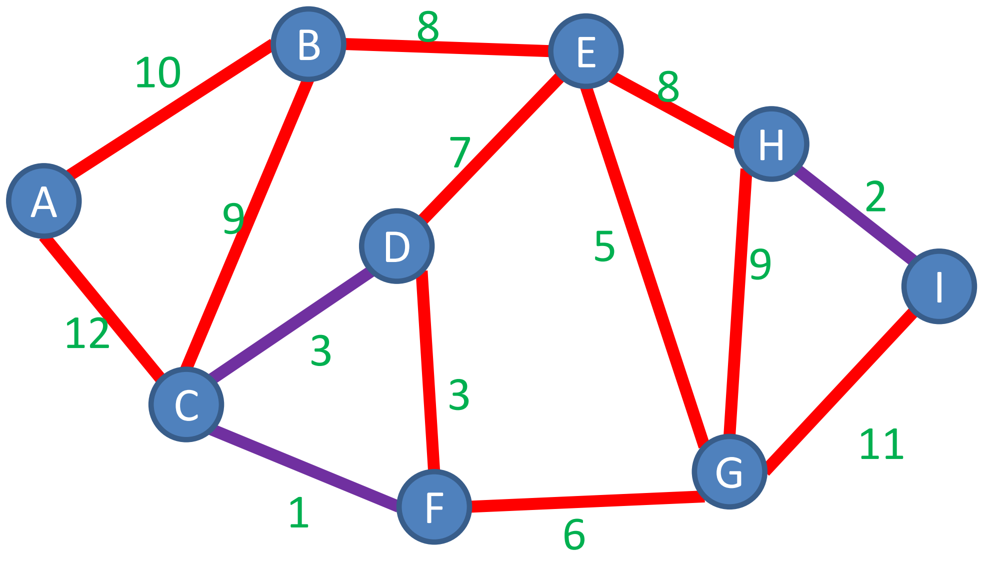

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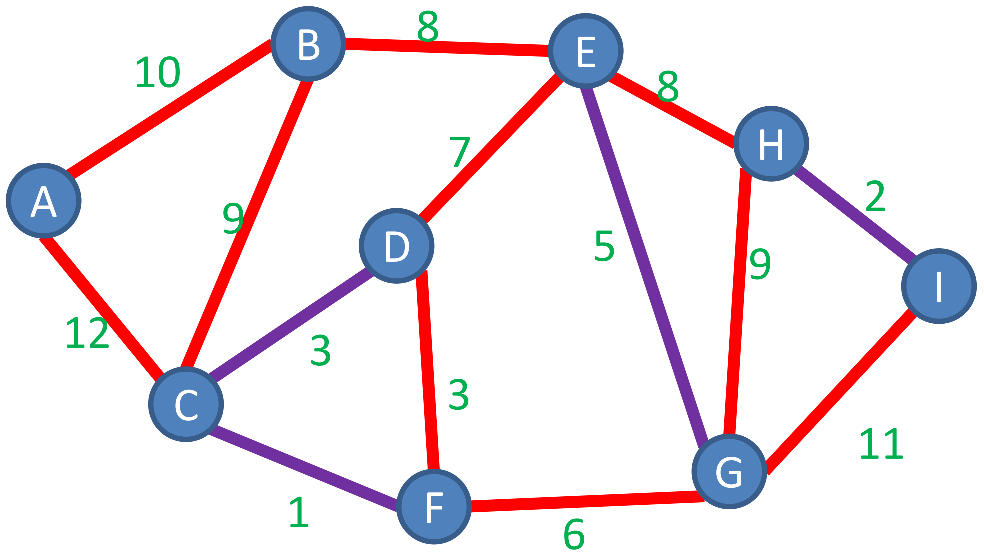

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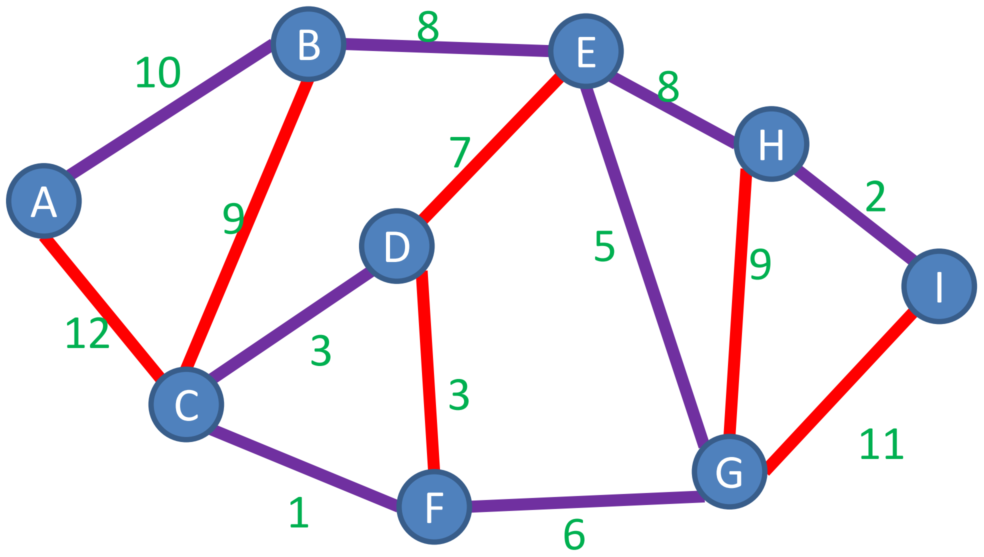
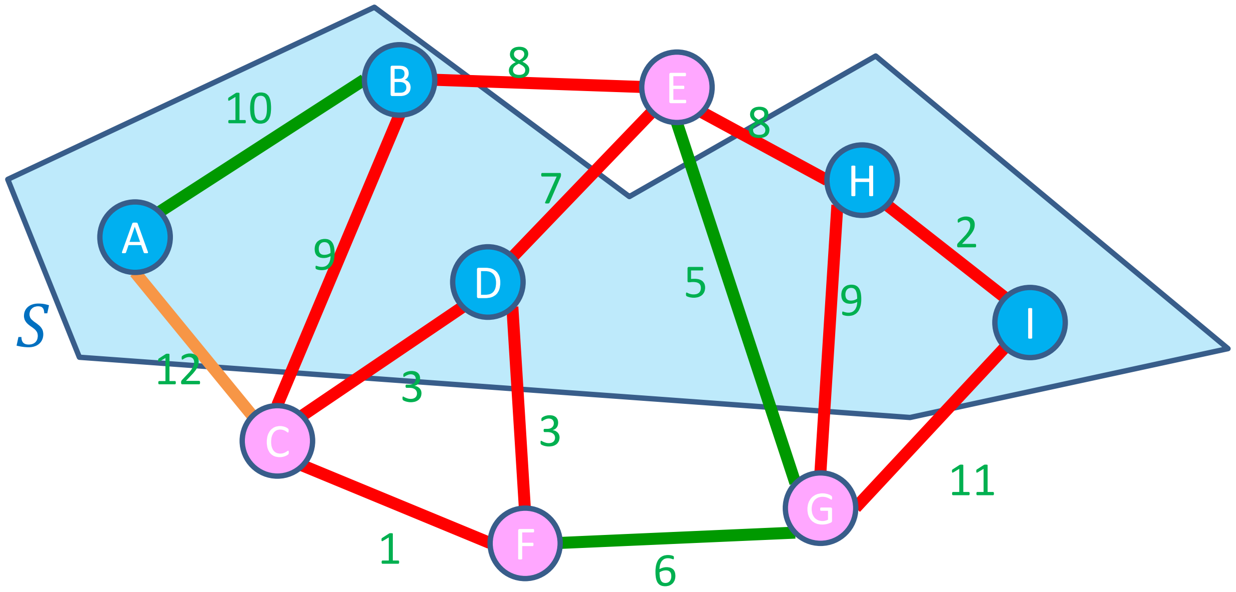


Definition: Cut

A Cut of graph $G = (V, E)$ is a partition of the nodes into two sets, S and $V - S$



Edge $(v_1, v_2) \in E$ crosses a cut if $v_1 \in S$ and $v_2 \in V - S$ (or opposite), e.g. (A, C)

A set of edges R Respects a cut if no edges cross the cut e.g. $R = \{(A, B), (E, G), (F, G)\}$

Exchange argument

- Shows correctness of a greedy algorithm
- Idea:
  – Show exchanging an item from an arbitrary optimal solution with your greedy choice makes the new solution no worse
  – How to show my sandwich is at least as good as yours:
    - Show: “I can remove any item from your sandwich, and it would be no worse by replacing it with the same item from my sandwich”

Cut Theorem

If a set of edges A is a subset of a minimum spanning tree T, let $(S, V - S)$ be any cut which A respects. Let e be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.

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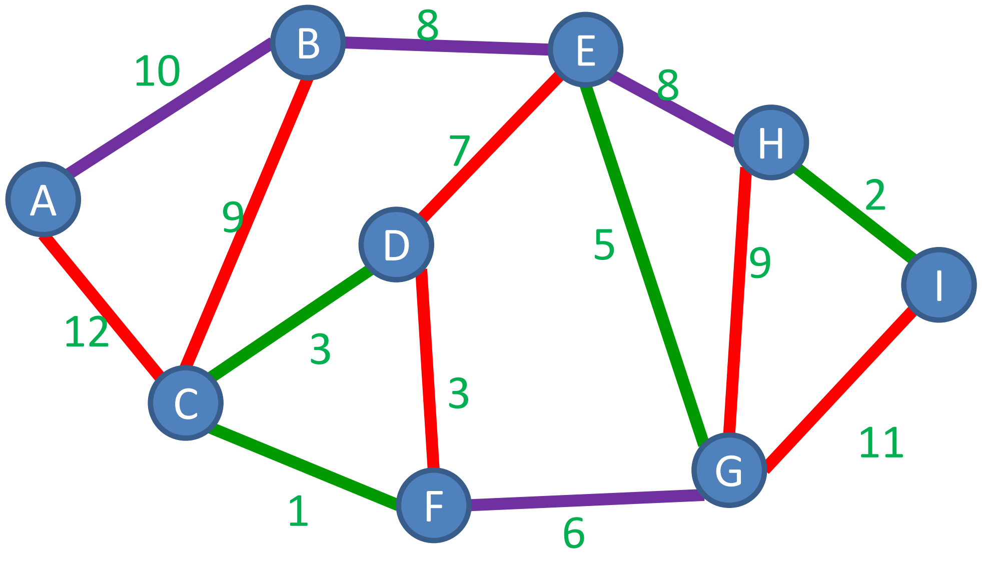

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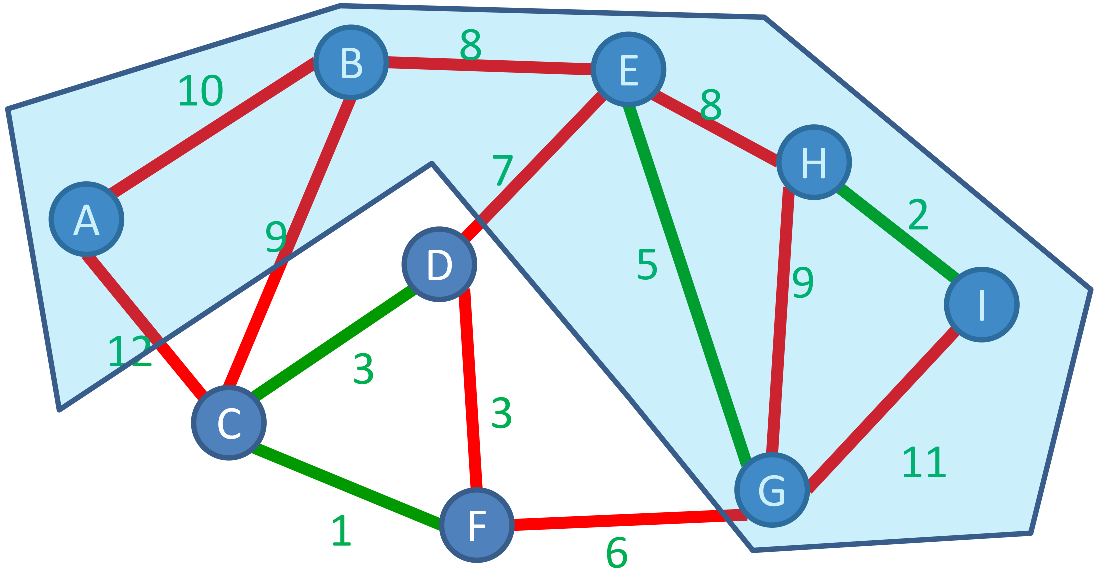

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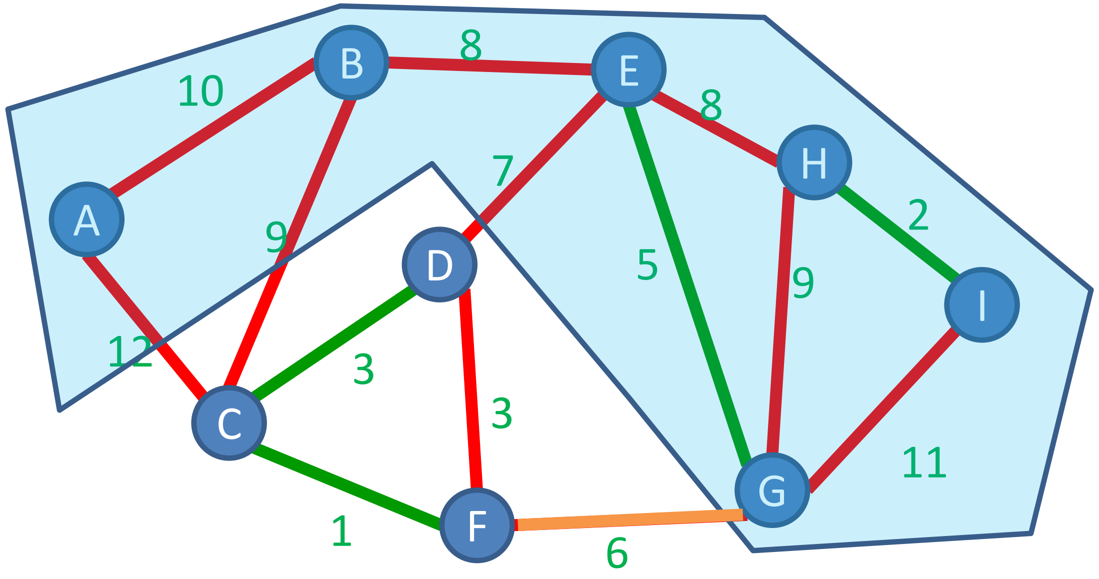
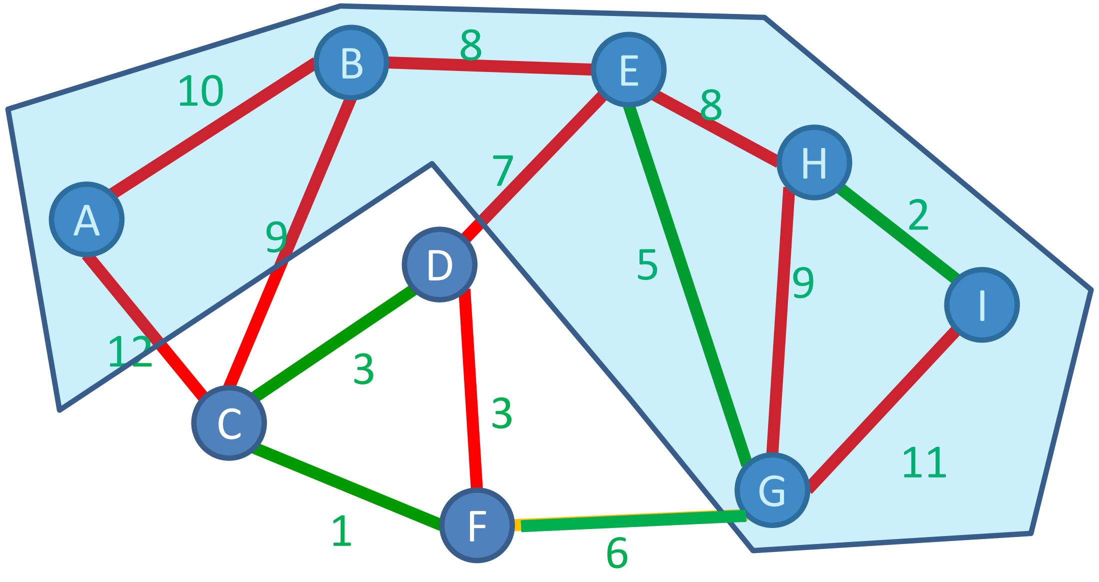


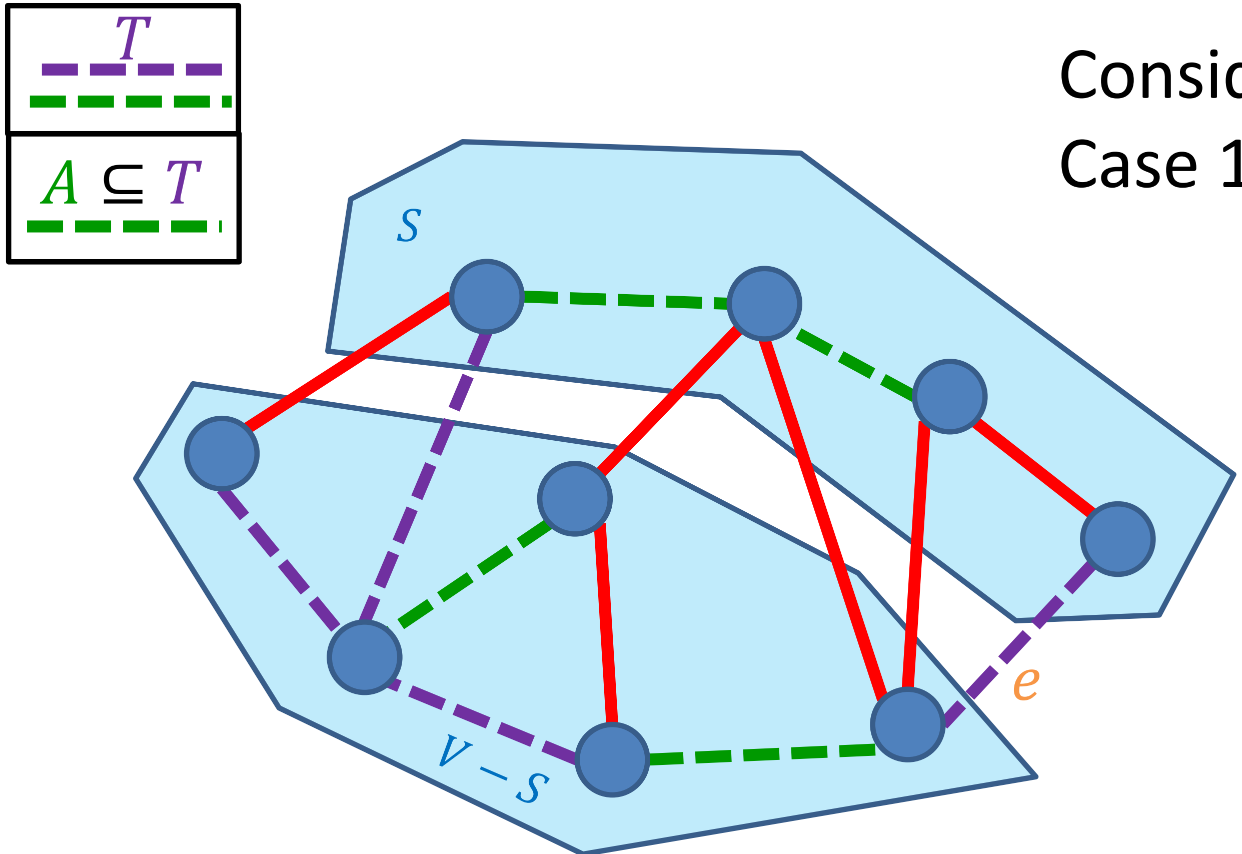
Cut Theorem

If a set of edges A is a subset of a minimum spanning tree T, let $(S, V - S)$ be any cut which A respects. Let e be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.



Proof of Cut Theorem

Claim: If A is a subset of a MST T, and e is the least-weight edge which crosses cut $(S, V - S)$ (which A respects) then $A \cup \{e\}$ is also a subset of a MST.



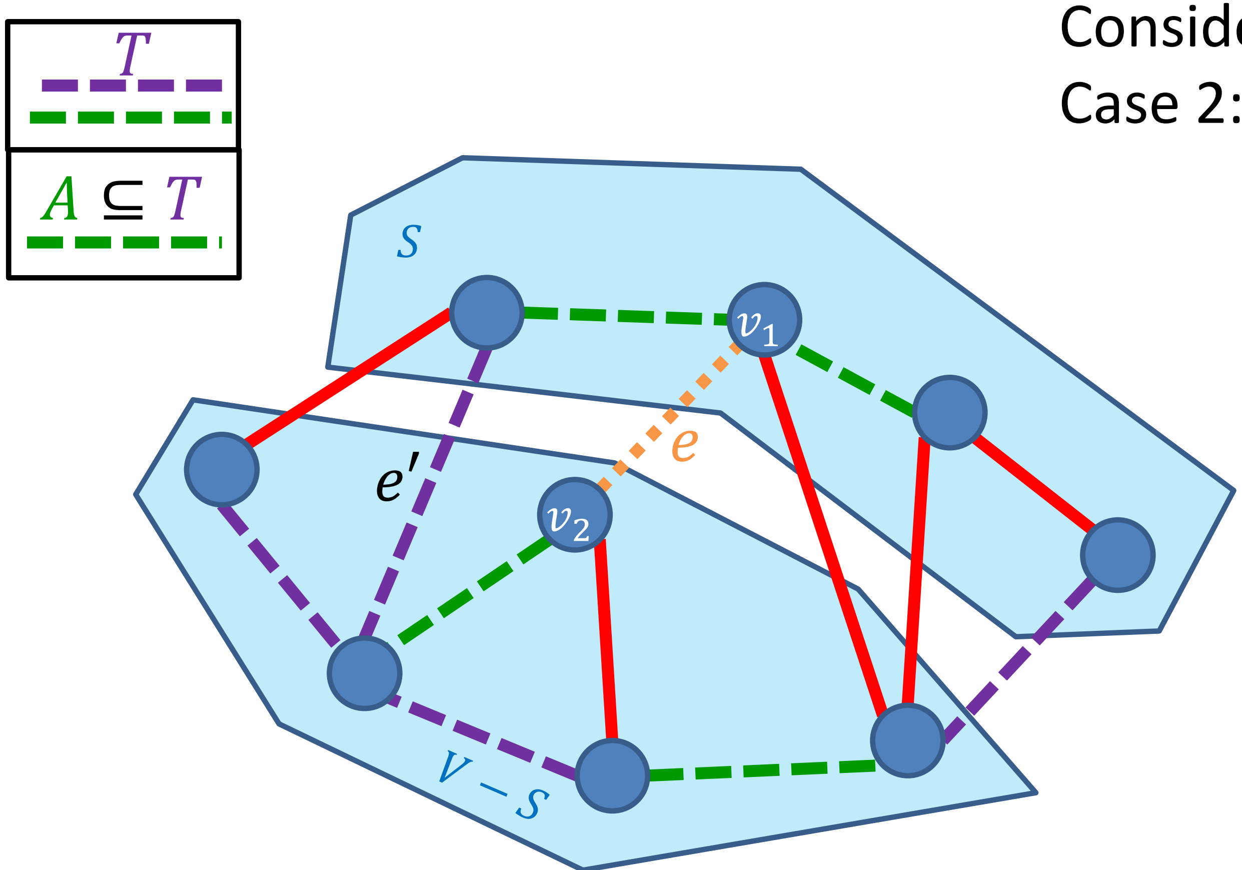
Consider some MST T,

Case 1: (the easy case)

If $e \in T$ Then claim holds

Proof of Cut Theorem

Claim: If A is a subset of a MST T, and e is the least-weight edge which crosses cut $(S, V - S)$ (which A respects) then $A \cup \{e\}$ is also a subset of a MST.



Consider some MST T,

Case 2:

Consider if $e = (v_1, v_2) \notin T$

Since T is a MST, there is some path from v_1 to v_2.

Let e' be the first edge on this path which crosses the cut

Build tree T' by exchanging e' for e

Proof of Cut Theorem

- We must show the following 2 things after building a new tree T':

1. The sum of the weights did not go up
   - $w(T') \leq w(T)$
2. T' is still a spanning tree

Proof of Cut Theorem

Claim: If A is a subset of a MST T, and e is the least-weight edge which crosses cut $(S, V - S)$ (which A respects) then $A \cup \{e\}$ is also a subset of a MST.

Consider some MST T,

Case 2:

Consider if $e = (v_1, v_2) \notin T$

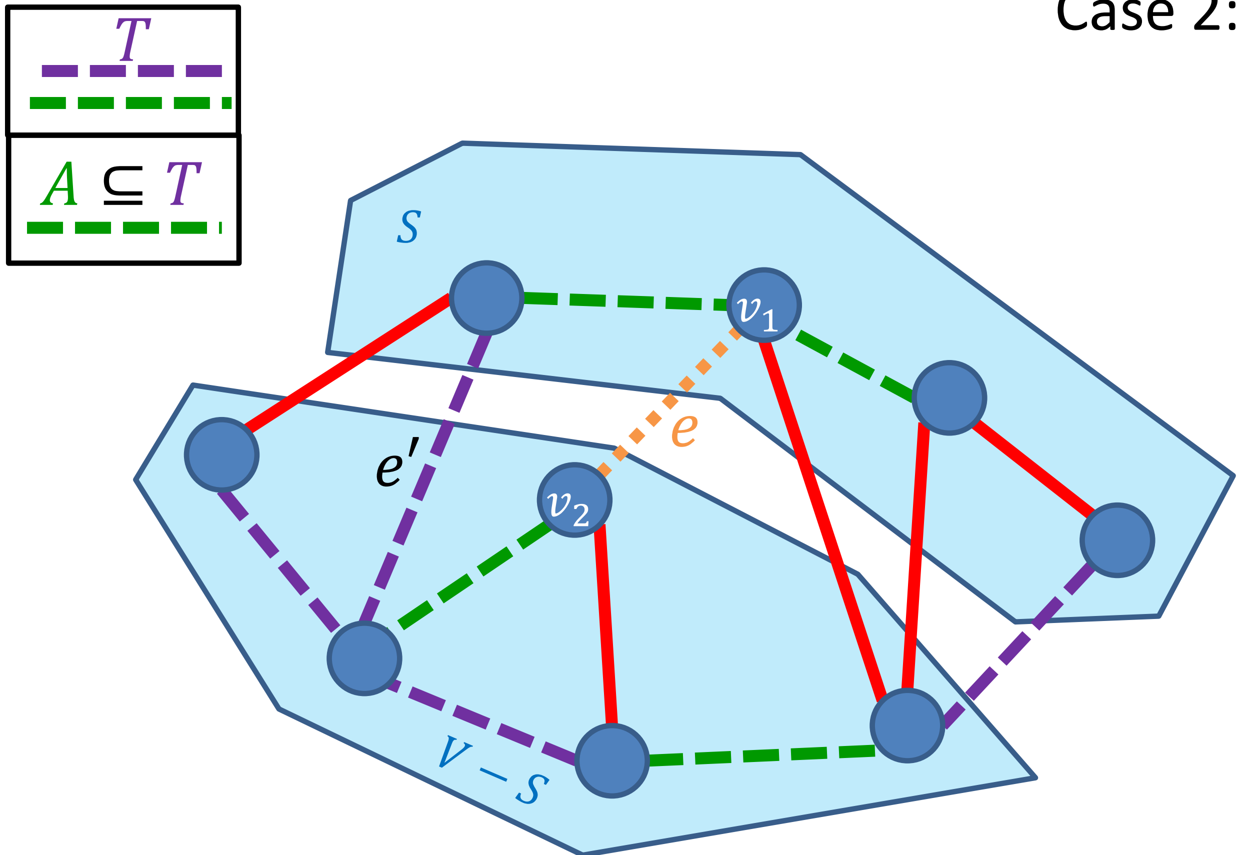
$T' = T$ with edge e instead of e'

We assumed $w(e) \leq w(e')$

$w(T') = w(T) - w(e') + w(e)$

$w(T') \leq w(T)$

We have proven the cost of T' is less than or equal to T



Proof of Cut Theorem

- We must show the following 2 things after building a new tree T':

1. The sum of the weights did not go up ✓
   - We proved that $w(T') \leq w(T)$
2. T' is still a spanning tree. How do we prove this?

Any set of V-1 edges that connects all the nodes in the graph is guaranteed to be a spanning tree!

Any set of V-1 edges in the graph that doesn't have any cycles is guaranteed to be a spanning tree!

Proof of Cut Theorem

- We must show the following 2 things after building a new tree T':

1. The sum of the weights did not go up
   – We proved that $w(T') \leq w(T)$
2. T' is still a spanning tree
   – T' has V-1 edges
   – T' connects all the nodes in the graph

Proof of Cut Theorem

Claim: If A is a subset of a MST T, and e is the least-weight edge which crosses cut $(S, V - S)$ (which A respects) then $A \cup \{e\}$ is also a subset of a MST.

Consider some MST T,

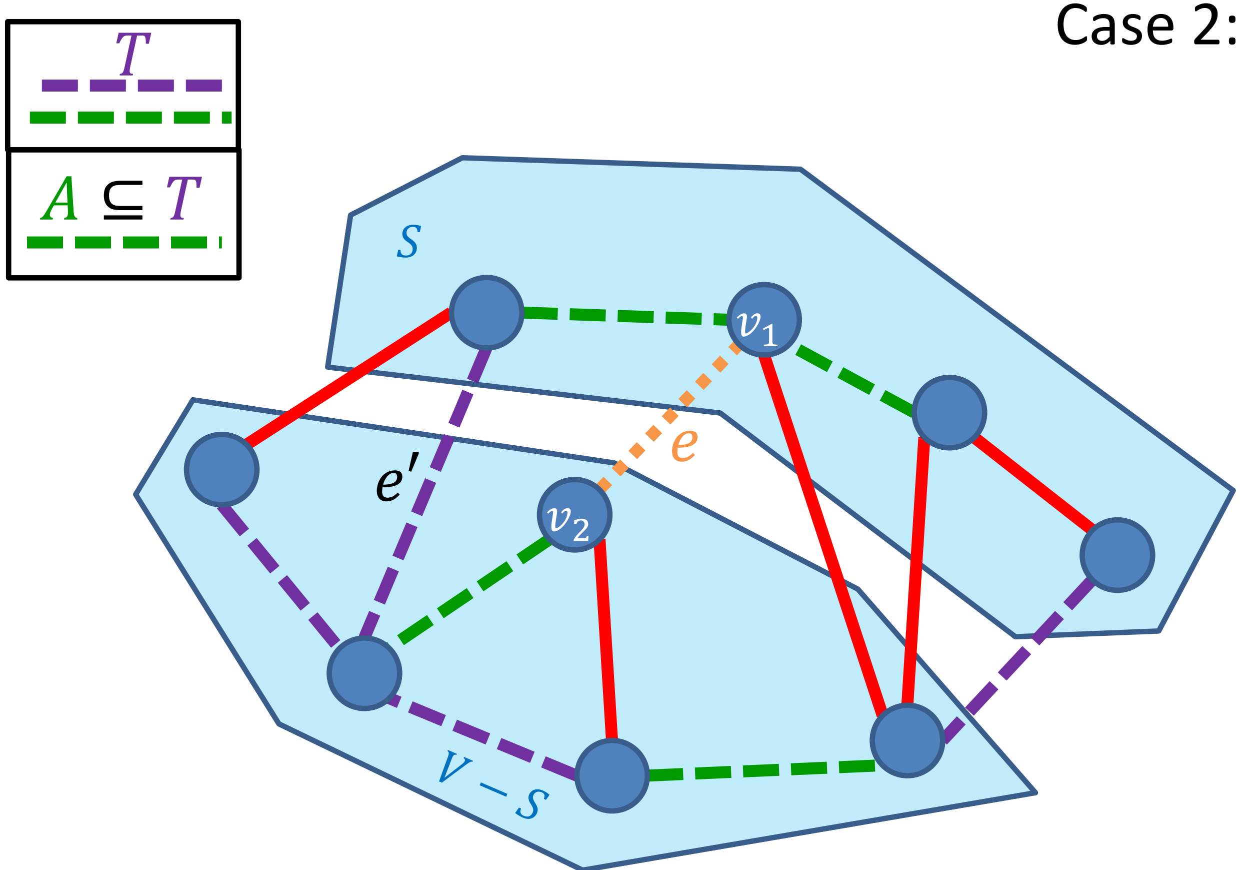
Case 2:

Consider if $e = (v_1, v_2) \notin T$

$T' = T$ with edge e instead of e'

We know T has $V - 1$ edges

We did a 1 for 1 edge swap so it must be the case that T' also has $V - 1$ edges



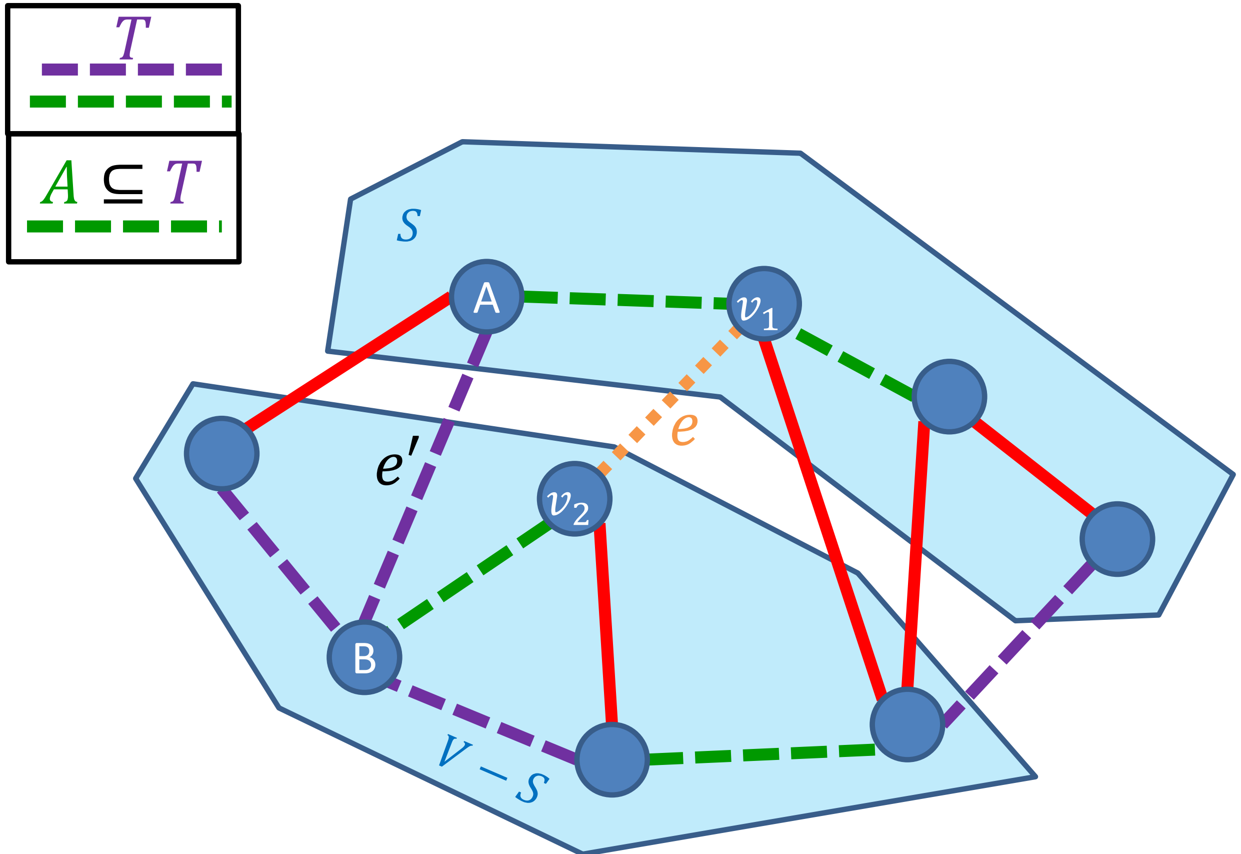
Proof of Cut Theorem

- We must show the following 2 things after building a new tree T':

1. The sum of the weights did not go up ✓
   - We proved that $w(T') \leq w(T)$
2. T' is still a spanning tree
   - T' has V-1 edges ✓
   - T' connects all the nodes in the graph

Proof of Cut Theorem

Claim: If A is a subset of a MST T, and e is the least-weight edge which crosses cut $(S, V - S)$ (which A respects) then $A \cup \{e\}$ is also a subset of a MST.



Consider some MST T,

Consider if $e = (v_1, v_2) \notin T$

$T' = T$ with edge e instead of e'

Show that we didn't disconnect the end points of e' which we will call (A, B):

There was some path in T from $v_1$to v_2 that uses e'. So, there must be a path from v_1 to A and B to v_2.

We have connected v_1 and v_2 with e, so we're good!

Proof of Cut Theorem

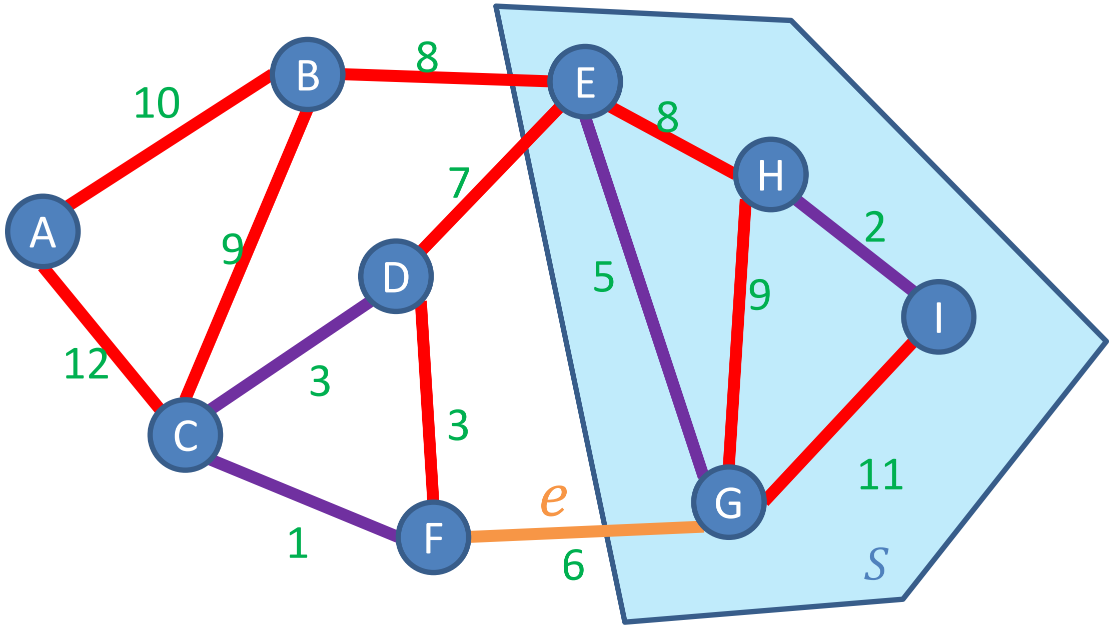
- We must show the following 2 things after building a new tree T’:

1. The sum of the weights did not go up ✓
   - We proved that $w(T') \leq w(T)$
2. T' is still a spanning tree ✓
   - T’ has V-1 edges
   - T’ connects all the nodes in the graph

Therefore T’ is also a MST on G

Proof of Kruskal’s Algorithm

Start with an empty tree A
Repeat $V - 1$ times:
Add the min-weight edge that doesn’t cause a cycle



Proof: Suppose we have some arbitrary set of edges A that Kruskal’s has already selected to include in the MST. $e = (F, G)$ is the edge Kruskal’s selects to add next

We know that there cannot exist a path from F to G using only edges in A because e does not cause a cycle

We can cut the graph therefore into 2 disjoint sets:

- nodes reachable from G using edges in A
- nodes reachable from F using edges in A

e is the minimum cost edge that crosses this cut, so by the Cut Theorem, Kruskal’s is optimal!

Kruskal's Algorithm Runtime

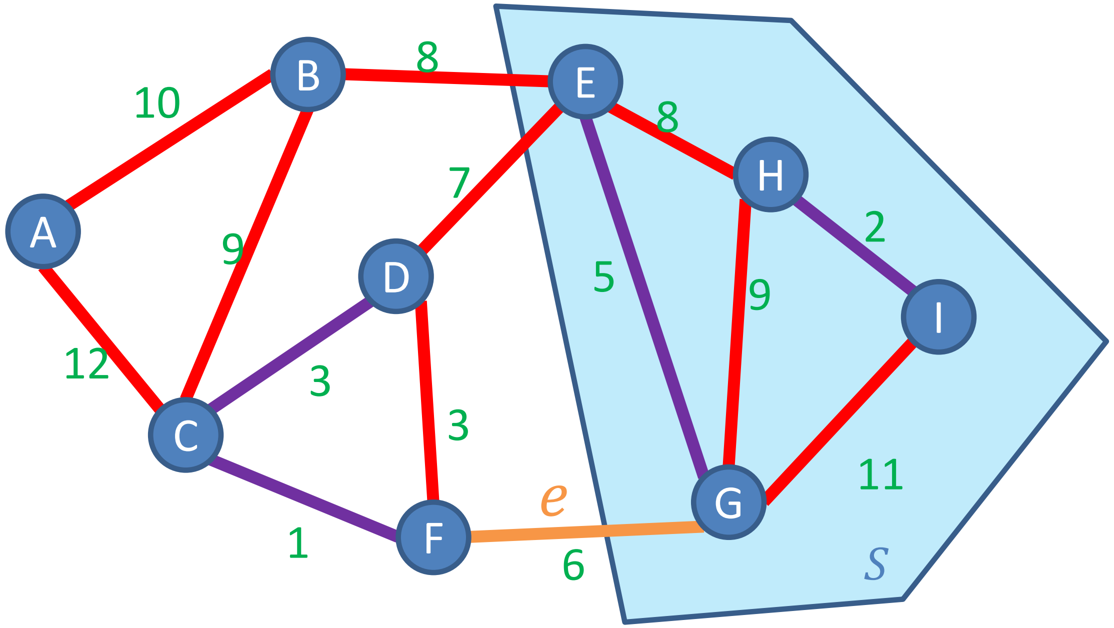
Start with an empty tree A

Repeat $V - 1$ times:

Add the min-weight edge that doesn't cause a cycle

Keep edges in a Disjoint-set data structure (very fancy)

$O(E \log V)$



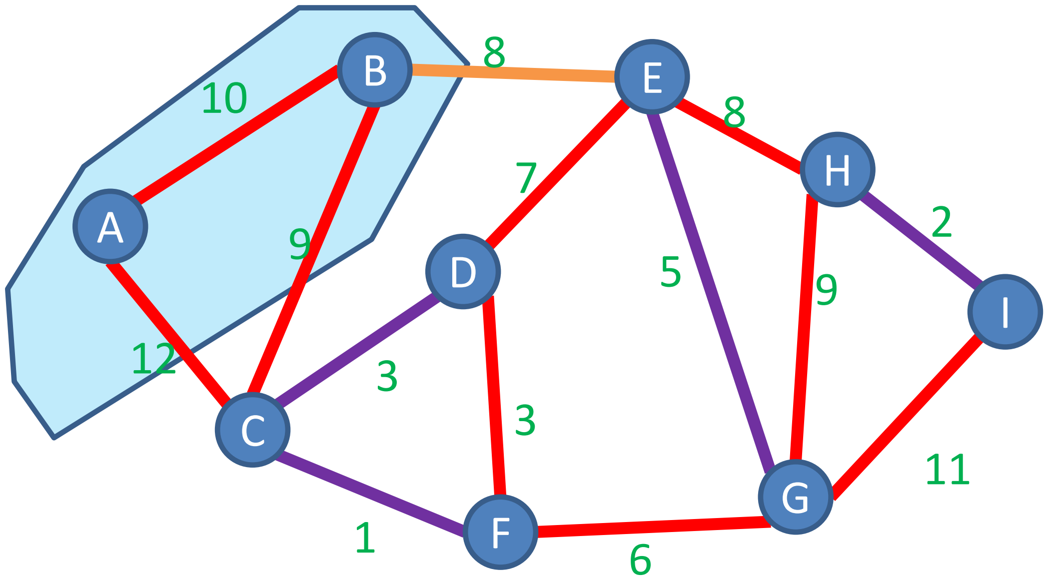
General MST Algorithm

Start with an empty tree A

Repeat $V-1$ times:

Pick a cut $(S, V-S)$ which A respects

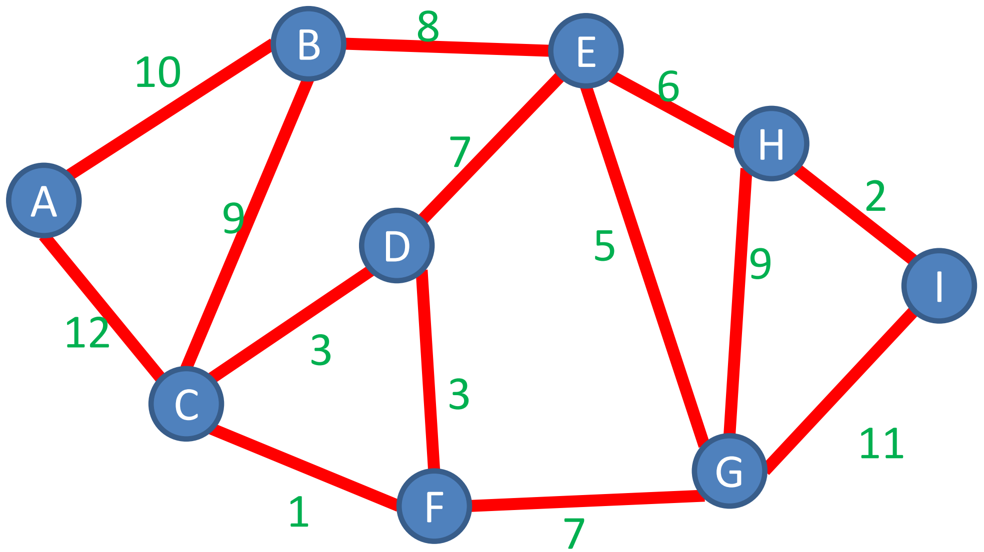
Add the min-weight edge which crosses $(S, V-S)$



Prim's Algorithm

Start with an empty tree A
Repeat $V - 1$ times:
	Pick a cut $(S, V - S)$ which A respects
	Add the min-weight edge which crosses $(S, V - S)$

S is all endpoint of edges in A
e is the min-weight edge that grows the tree



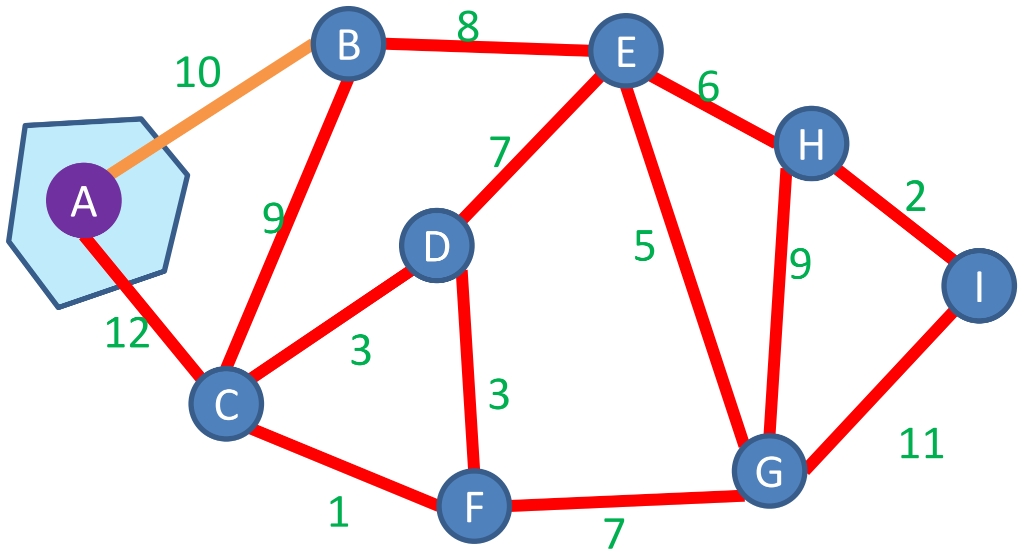
Prim's Algorithm

Start with an empty tree A

Pick a start node

Repeat $V - 1$ times:

Add the min-weight edge which connects to node in A with a node not in A



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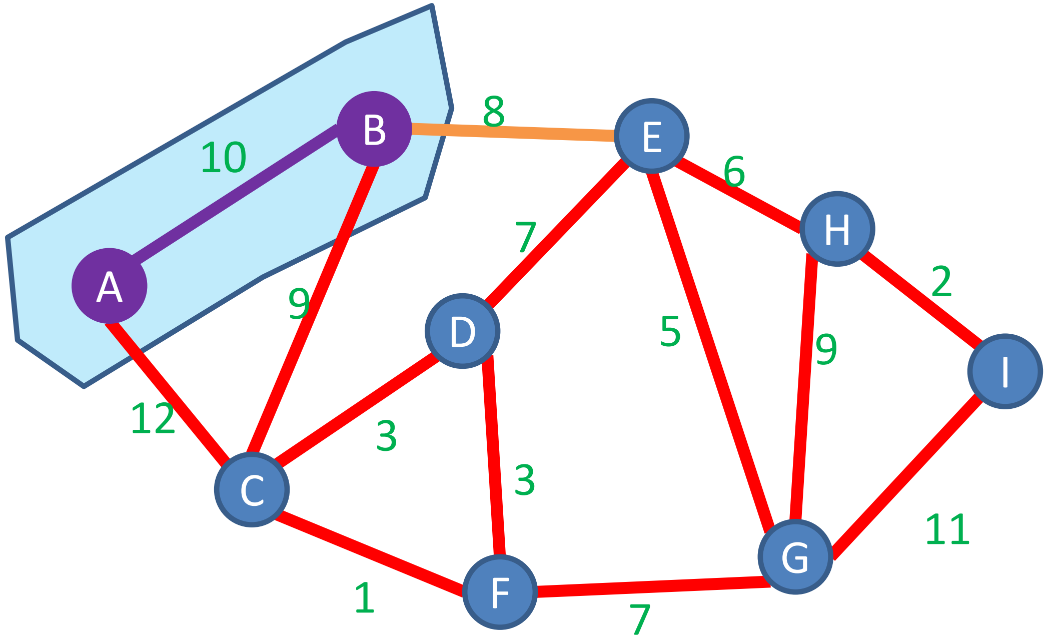

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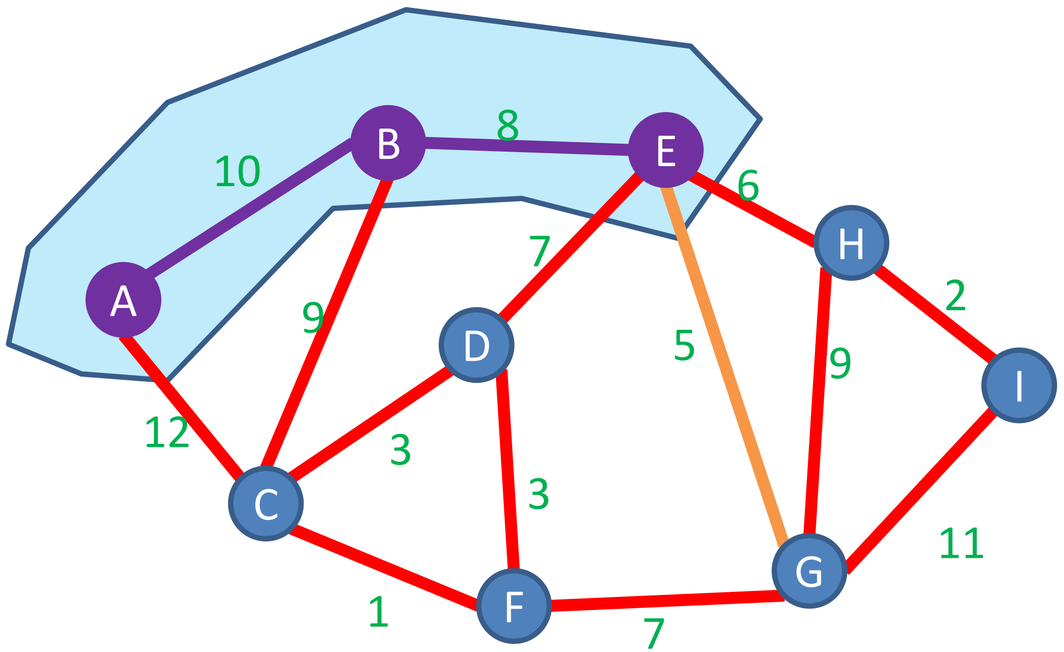

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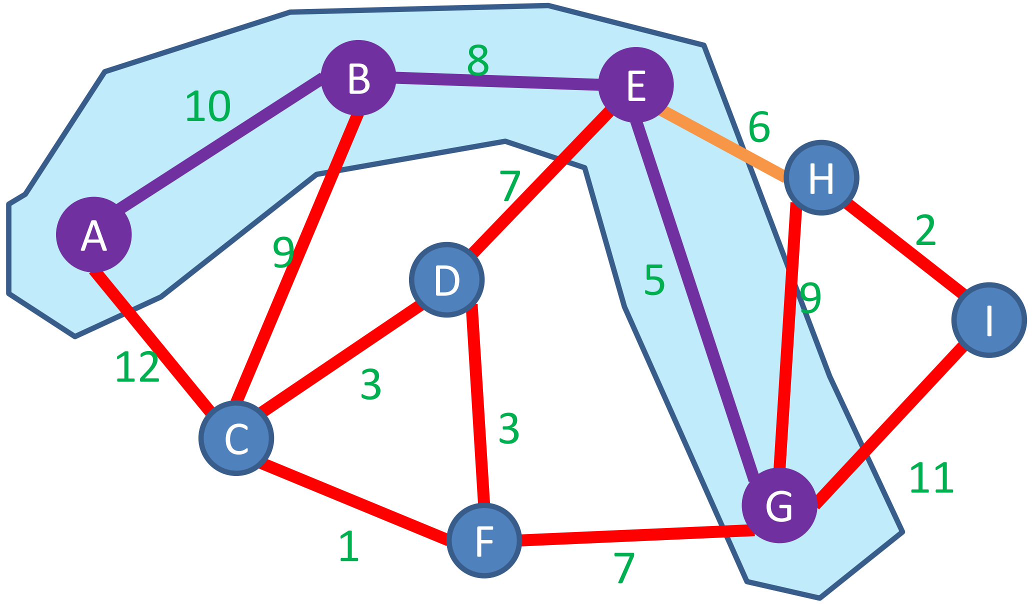


Prim's Algorithm

Start with an empty tree A

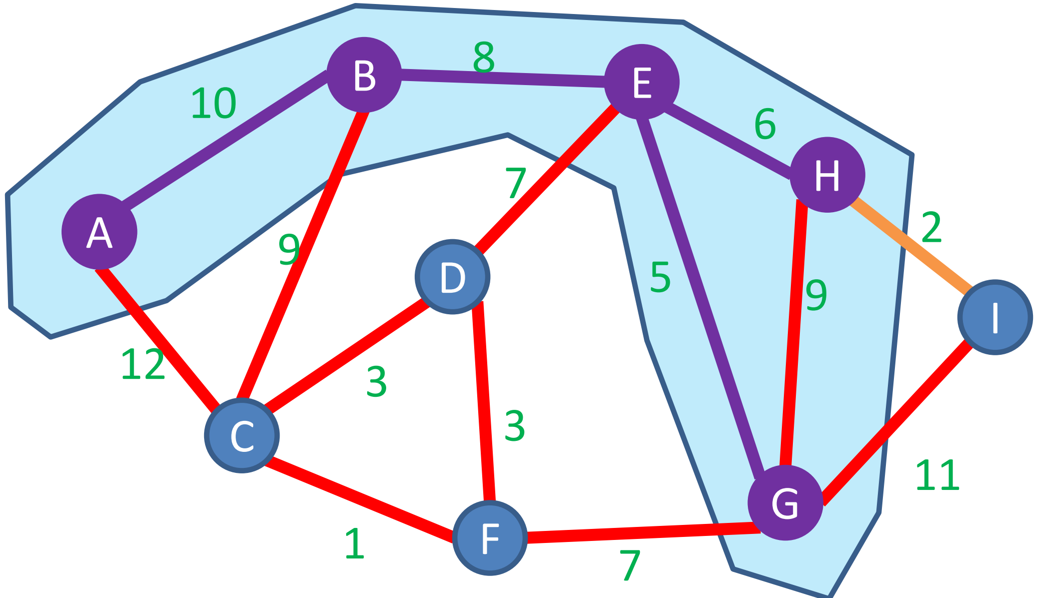
Pick a start node

Repeat $V - 1$ times:

Add the min-weight edge which connects to node in A with a node not in A

Keep edges in a Heap

$O(E \log V)$



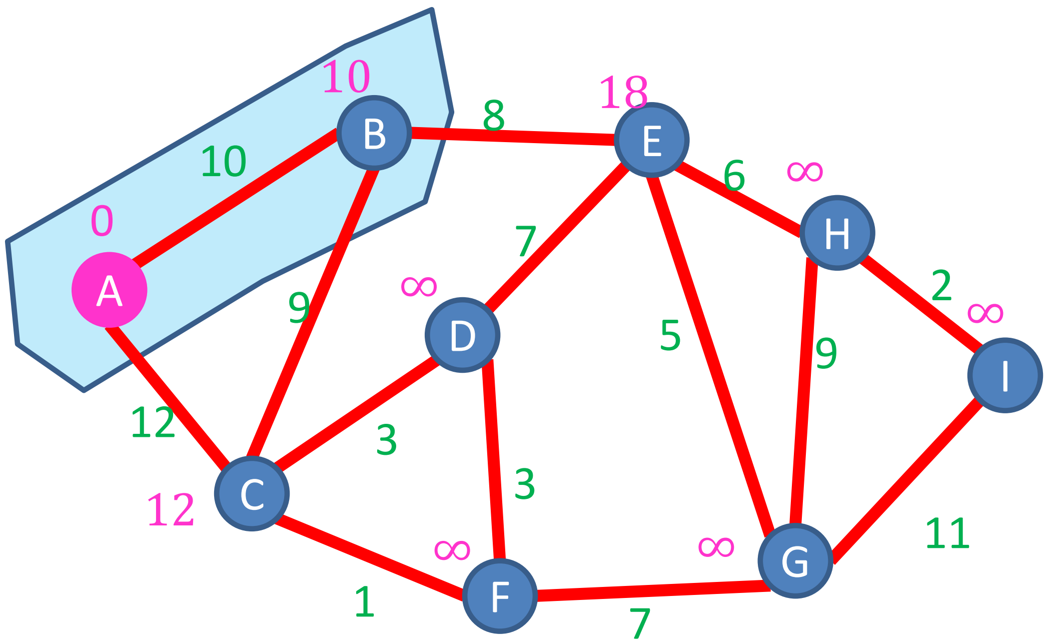
Dijkstra's Algorithm

Given some start node s

Start with an empty tree A

Repeat $V - 1$ times:

  Add the "nearest" node to s not yet in A



Dijkstra's Algorithm

Given some start node s

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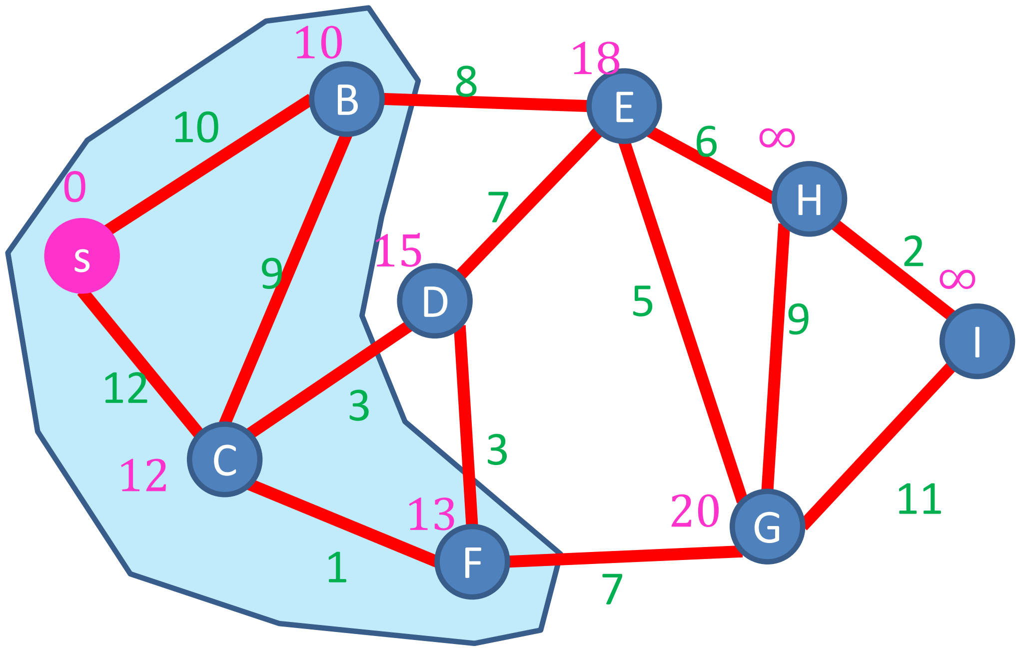
Repeat $V - 1$ times:

Add the "nearest" node to s not yet in A

VERY similar to Prim's!



Dijkstra's Algorithm

Initialize $d_v = \infty$ for each node v

Keep a priority queue PQ of nodes, using d_v as key

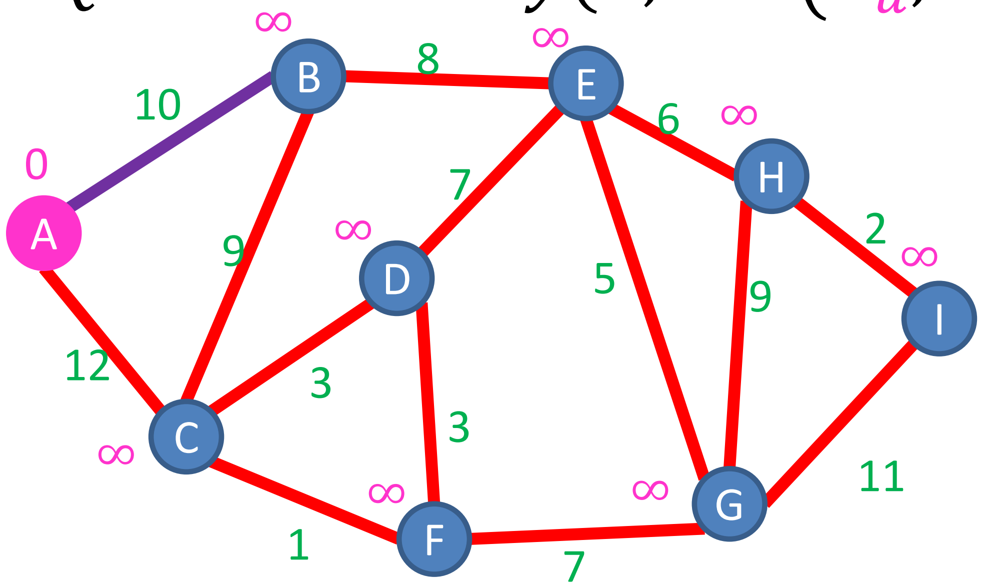
Pick a start node s, set $d_s = 0$

While PQ is not empty:

  $v = PQ.extractmin()$

  for each $u \in V$ s.t. $(v, u) \in E$:

    $PQ.decreaseKey(u, \min(d_u, d_v + w(v, u)))$



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Keep a priority queue PQ of nodes, using d_v as key

Pick a start node s, set $d_s = 0$

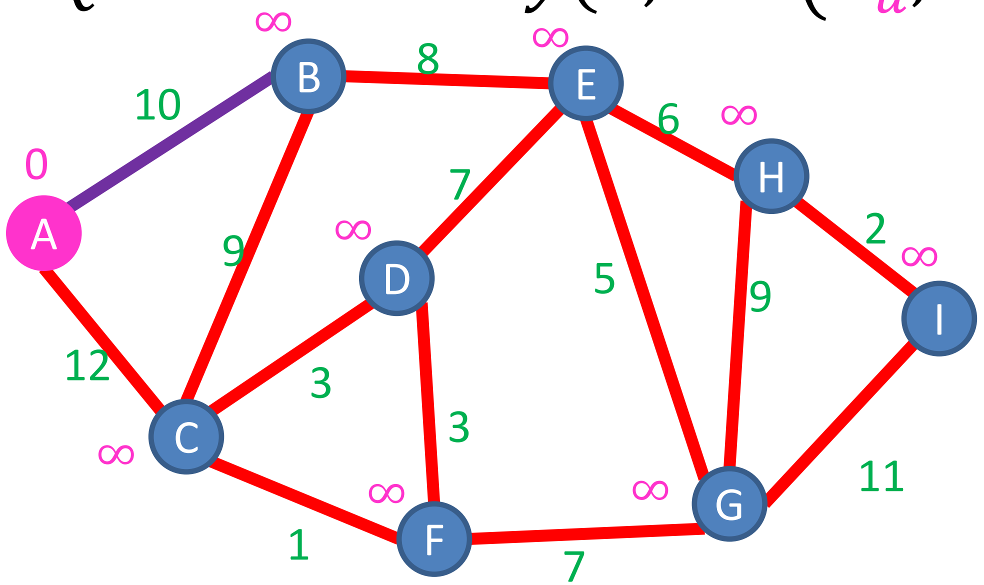
While PQ is not empty:

    $v = PQ.extractmin()$

    for each $u \in V$ s.t. $(v, u) \in E$:

        $PQ.decreaseKey(u, \min(d_u, w(v, u)))$

Can use min heap dataset to make each decreaseKey() operation faster!



Summary of MST results

- Fredman-Tarjan ‘84: $\Theta(E + V \log V)$
- Gabow et al ‘86: $\Theta(E \log \log^* V)$
- Chazelle ‘00: $\Theta(E\alpha(V))$
- Pettie-Ramachandran ’02: $\Theta(?)$ (optimal)
- Karger-Klein-Tarjan ‘95: $\Theta(E)$ (randomized)