CS4102 Algorithms

Spring 2023

Seinfeld

Movie Time!

In Season 9 Episode 7 “The Slicer” of the hit 90s TV show *Seinfeld*, George discovers that, years prior, he had a heated argument with his new boss, Mr. Kruger. This argument ended in George throwing Mr. Kruger’s boombox into the ocean. How did George make this discovery?

https://www.youtube.com/watch?v=pSB3HdmLcY4

Seam Carving

- Method for image resizing that doesn't scale/crop the image

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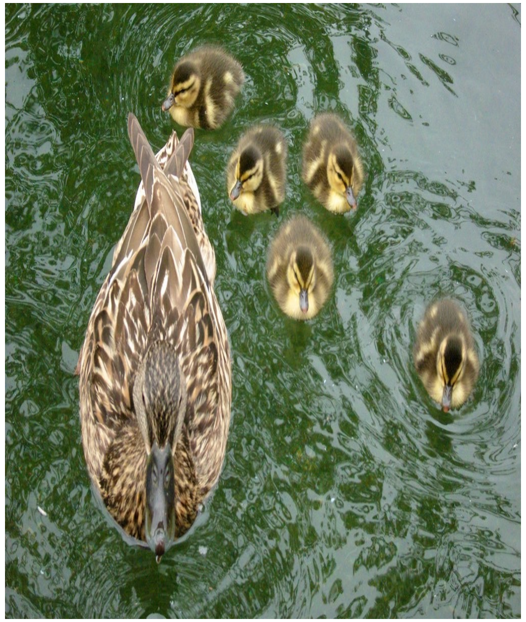
Cropped

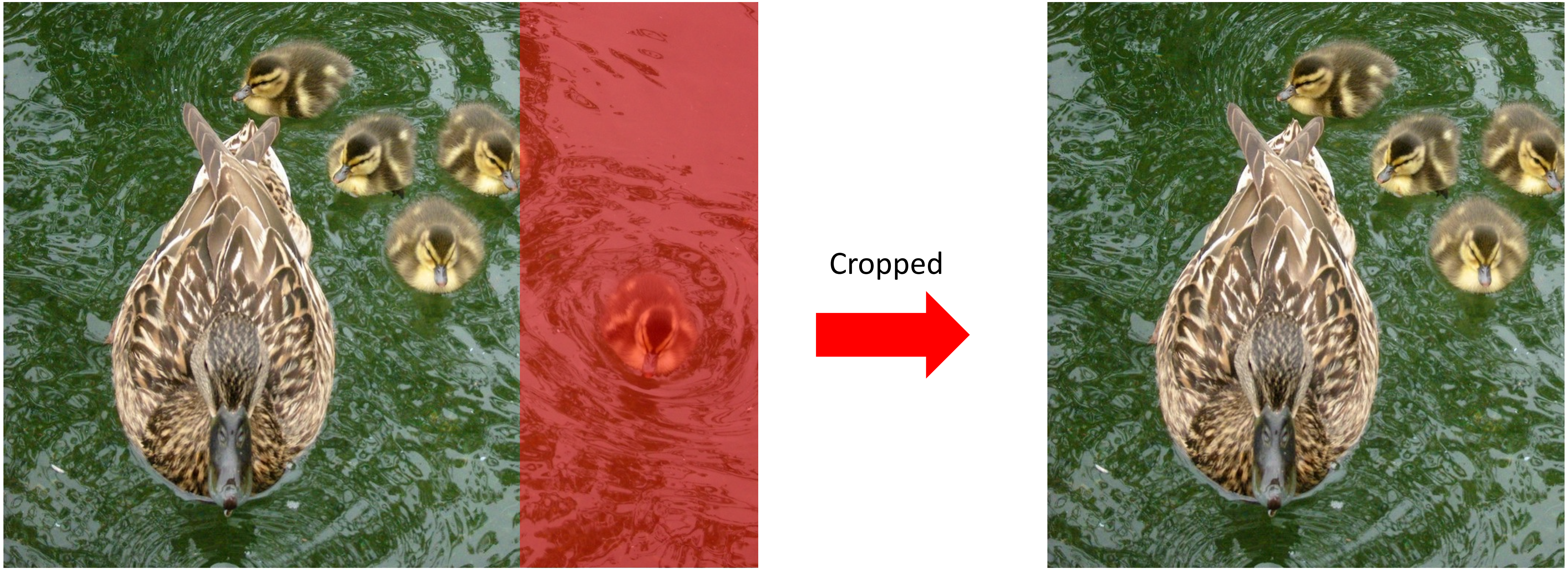
Scaled



Carved

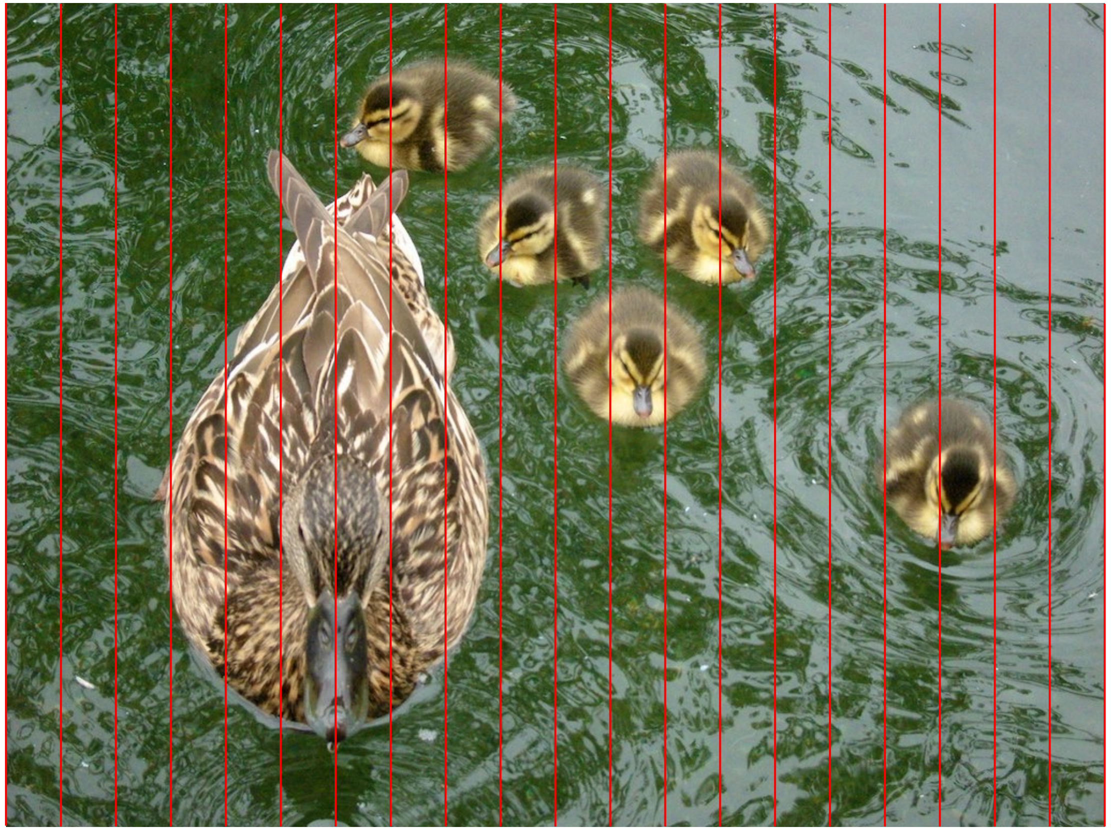
Cropping

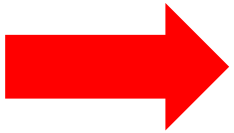
- Removes a “block” of pixels

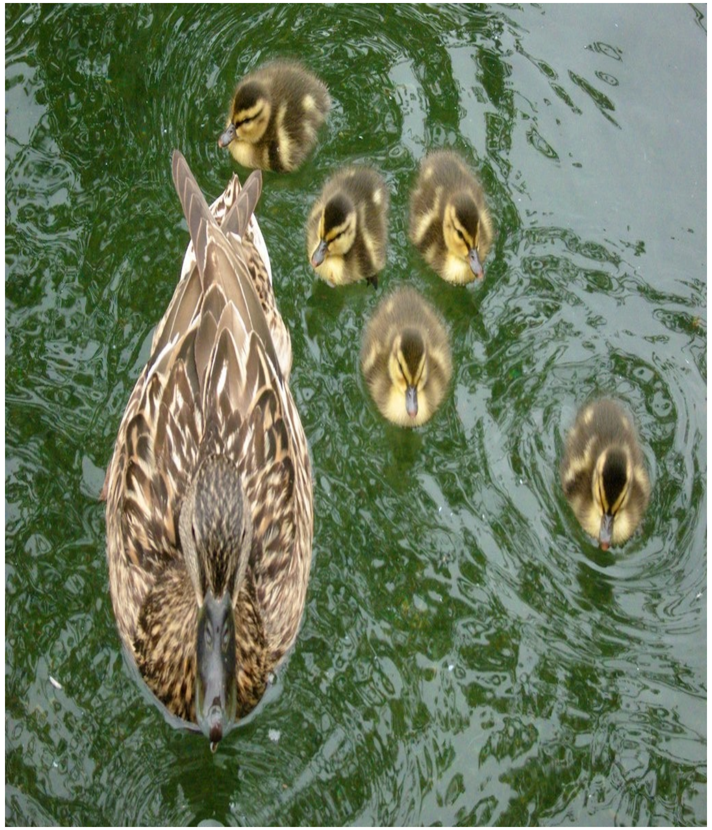


Scaling

- Removes “stripes” of pixels

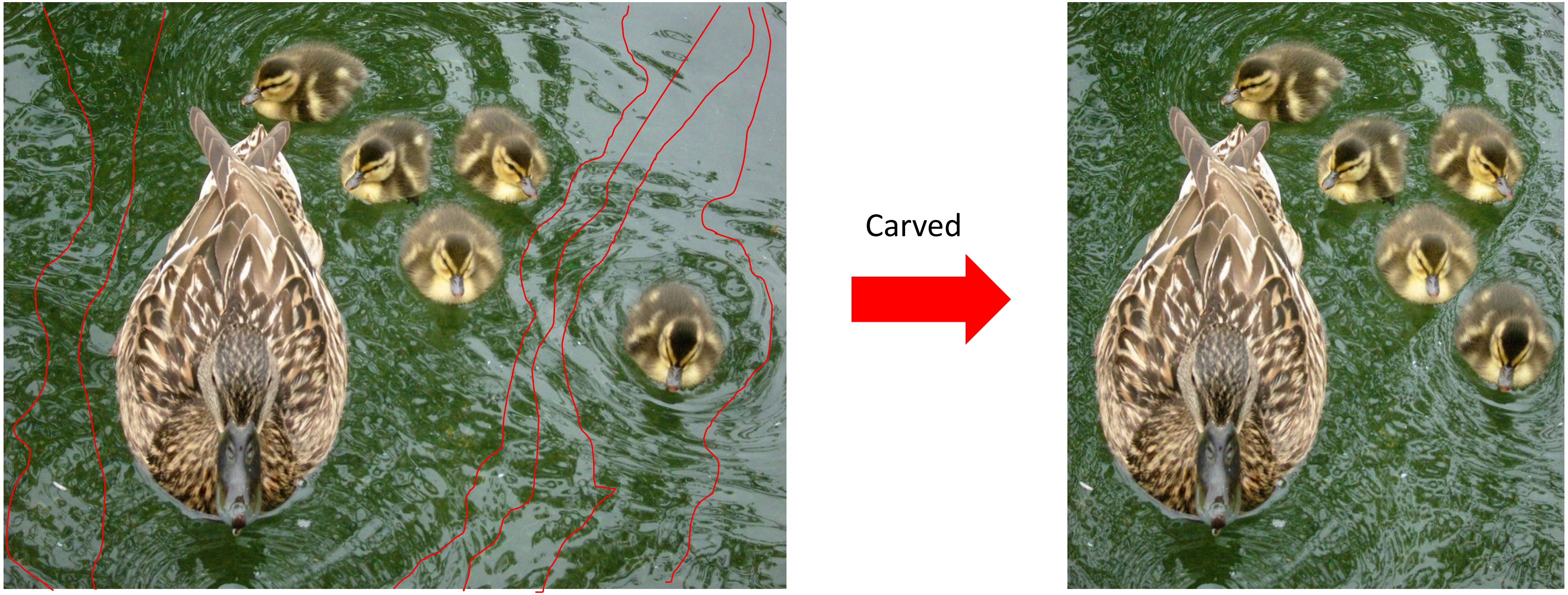






Seam Carving

- Removes “least energy seam” of pixels
- https://trekhleb.dev/js-image-carver/



Seattle Skyline

Energy of a Seam

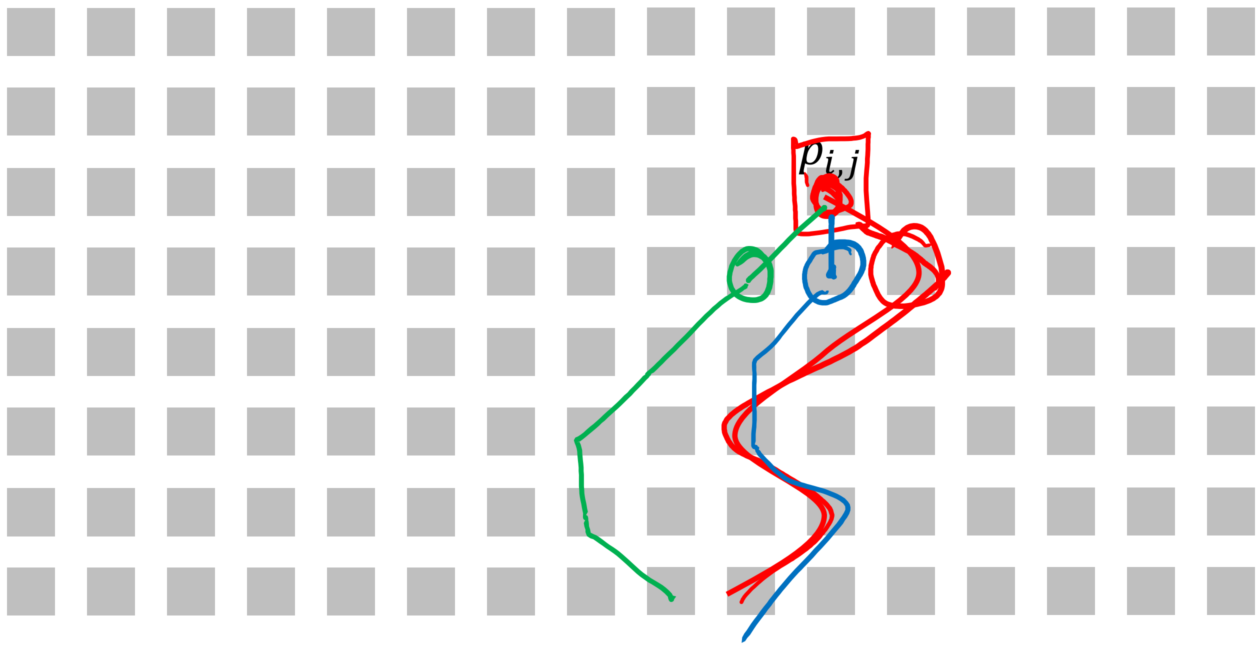
- $e(p) =$ energy of pixel p
  - Many choices
  - E.g.: change of gradient
    - how much the color of this pixel differs from its neighbors
  - Particular choice doesn't matter, we use it as a "black box"
- Energy of seam: sum of the energies of each pixel on the seam

Dynamic Programming

- Requires Optimal Substructure
  - Solution to larger problem is the (optimal) solutions to a smaller one plus one “decision”
- Idea:
  1. Identify the substructure of the problem
     - What are the options for the “last thing” done? What subproblem comes from each?
  2. Save the solution to each subproblem in memory
  3. Select an order for solving subproblems
     - “Top Down”: Solve each recursively
     - “Bottom Up”: Iteratively solve smallest to largest

Identify Recursive Structure

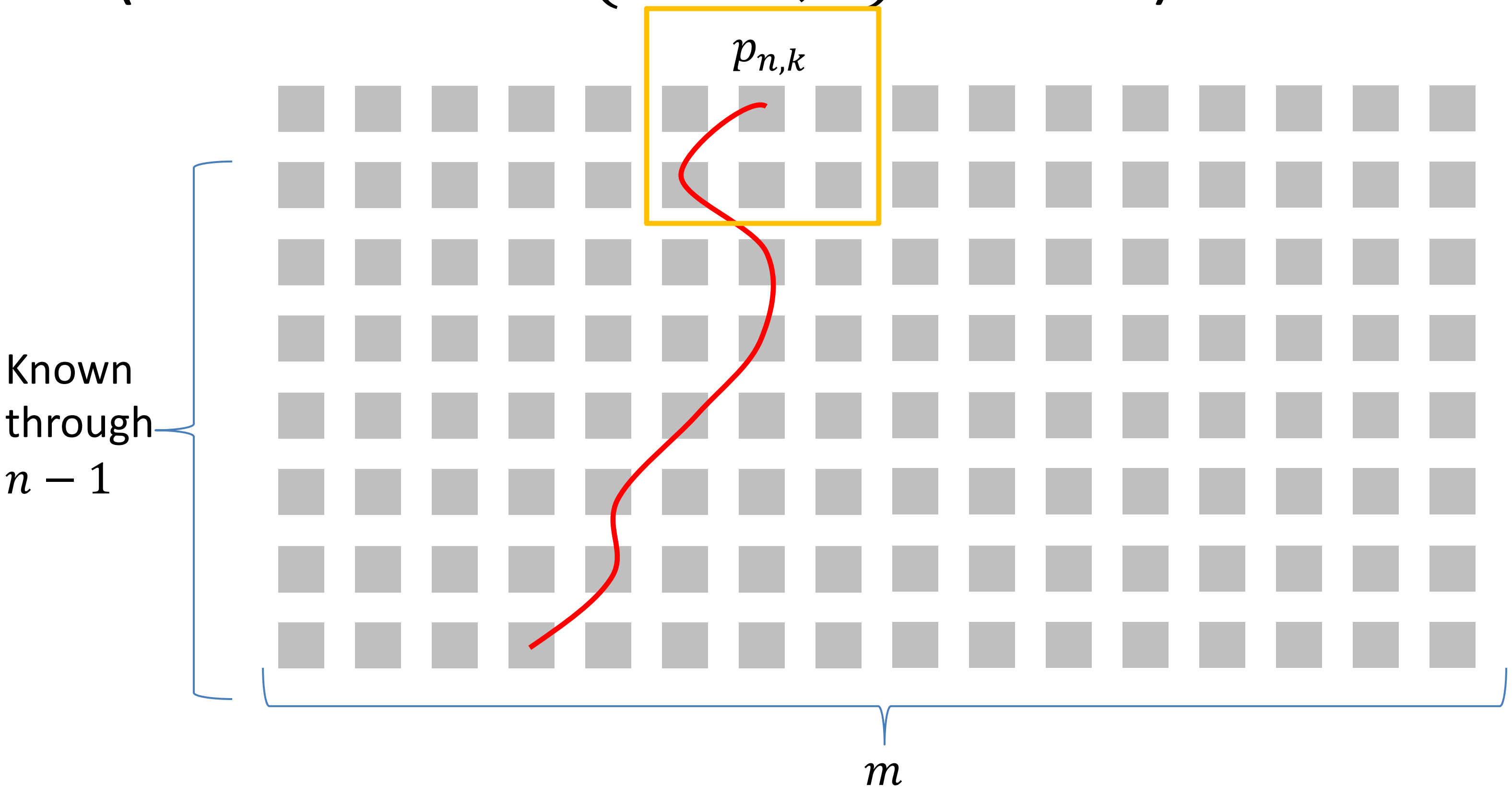
Let $S(i,j)$ = weight least energy seam from the bottom of the image up to pixel $p_{i,j}$



Computing $S(n, k)$

Assume we know the least energy seams for all of row $n - 1$

(i.e. we know $S(n-1, \ell)$ for all ℓ)



Computing $S(n, k)$

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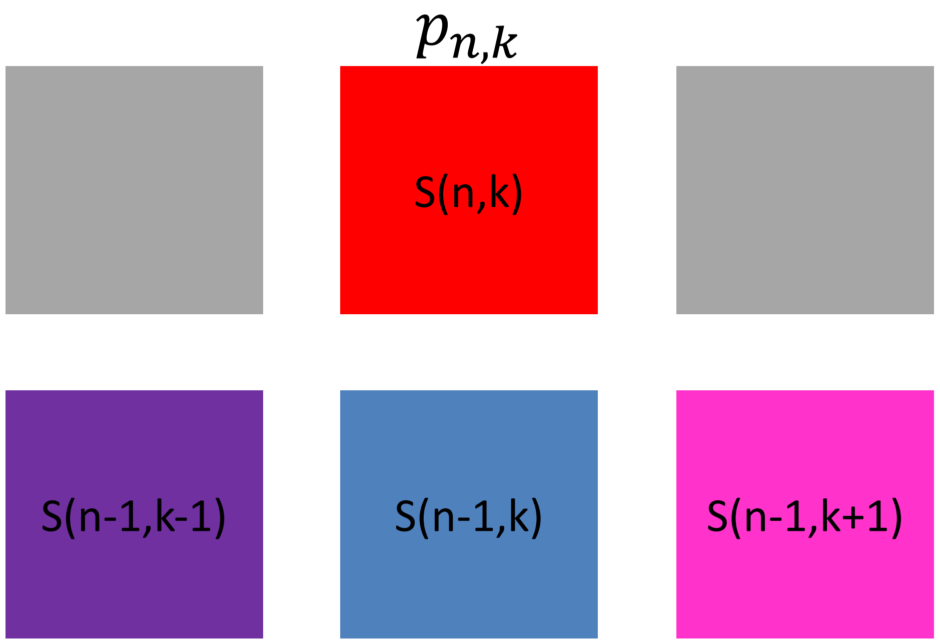
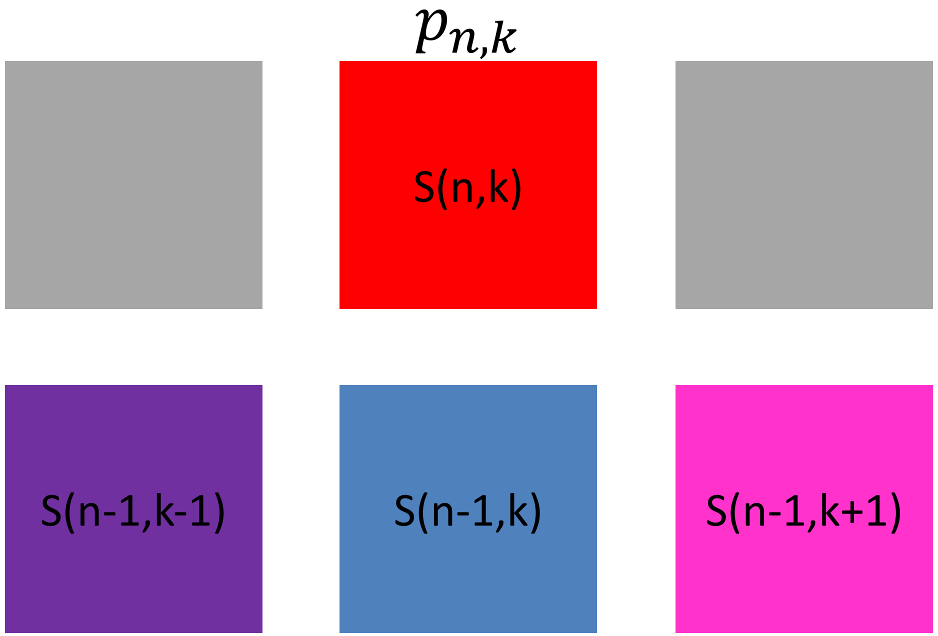


Computing $S(n,k)$

Assume we know the least energy seams for all of row $n-1$ (i.e. we know $S(n-1,\ell)$ for all ℓ)

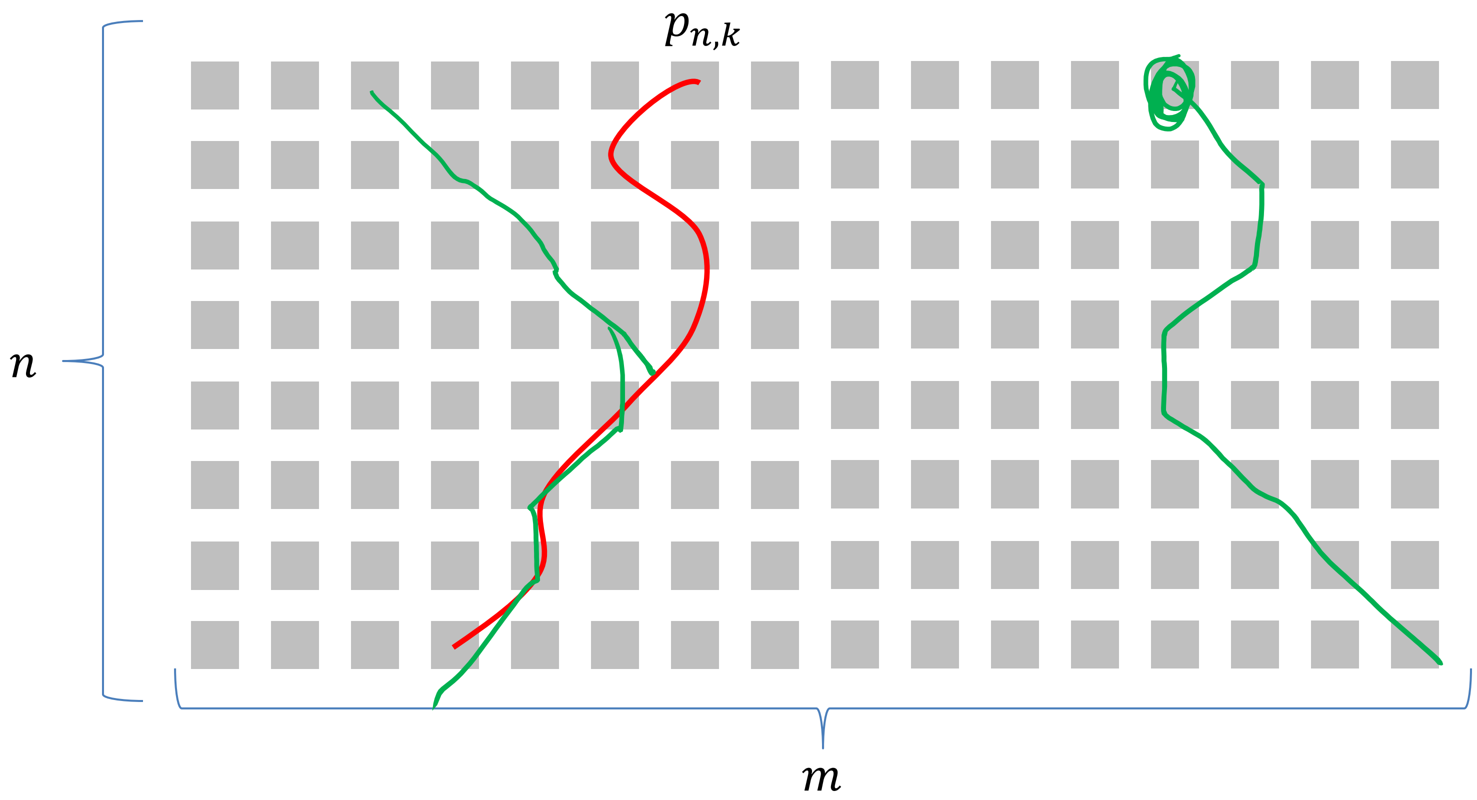
$$S(n,k) = min \begin{cases} S(n-1,k-1) + e(p_{n,k}) \\ S(n-1,k) + e(p_{n,k}) \\ S(n-1,k+1) + e(p_{n,k}) \end{cases}$$



Finding the Least Energy Seam

Want the least energy seam going from bottom to top, so delete:

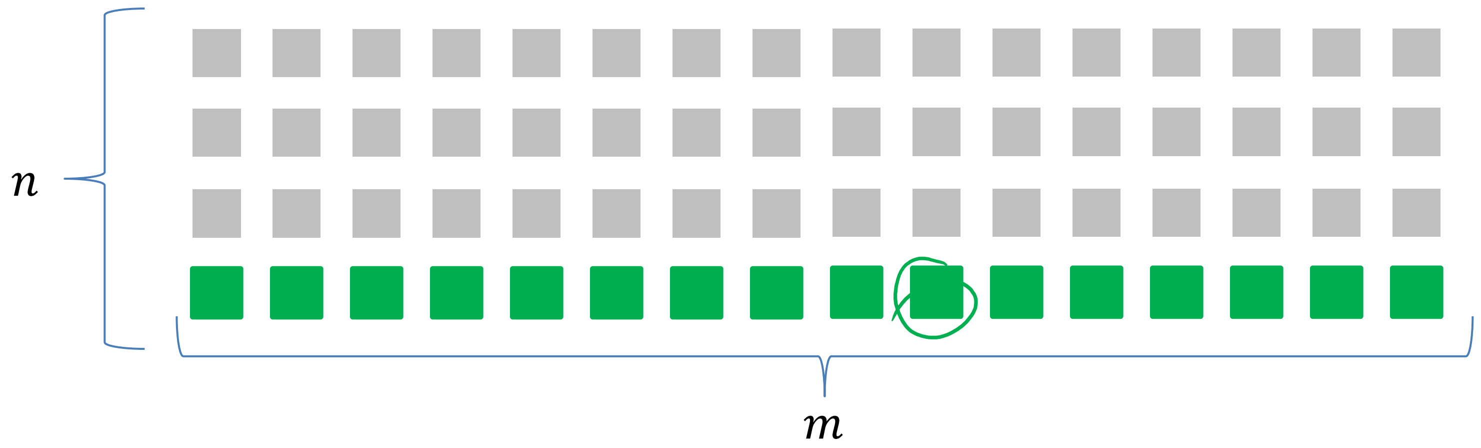
$$\min_{k=1}^{m}(S(n,k))$$



Bring It All Together

Start from bottom of image (row 1), solve up to top

Initialize $S(1,k) = e(p_{1,k})$ for each pixel in row 1



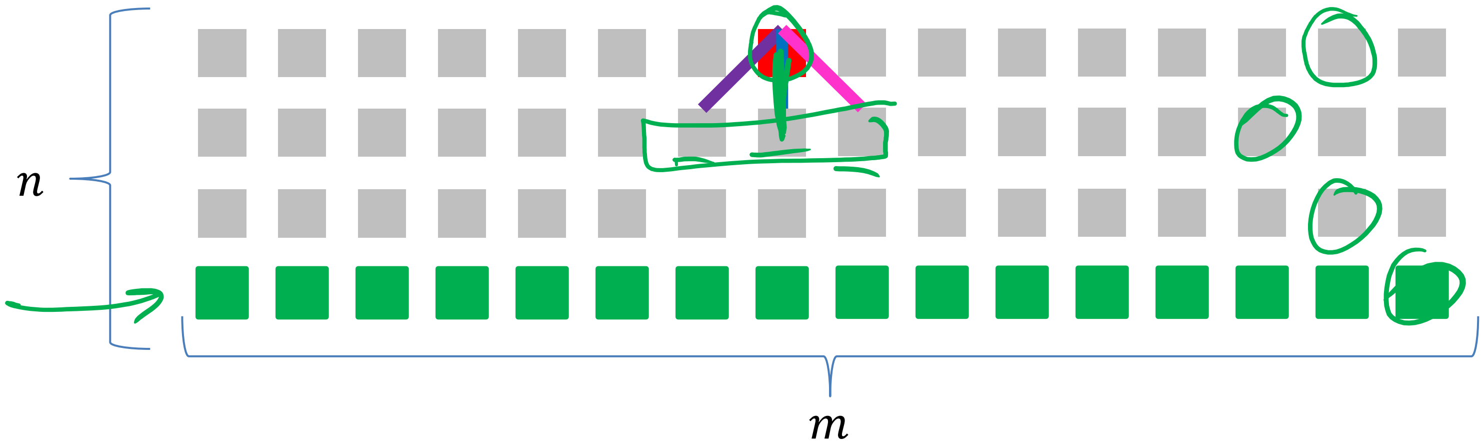
Energy of the seam initialized to the energy of that pixel

Bring It All Together

Start from bottom of image (row 1), solve up to top

Initialize $S(1,k) = e(p_{1,k})$ for each pixel $p_{1,k}$

For $i > 2$ find $S(i,k) = \min_{k=1}^{m} \begin{cases} S(i-1,k-1) + e(p_{i,k}) \\ S(i-1,k) + e(p_{i,k}) \\ S(i-1,k+1) + e(p_{i,k}) \end{cases}$



Energy of the seam initialized to the energy of that pixel

Top Down Least Energy Seam Pseudocode

```
init M[n][n] to -1
def S(i,j):
    if M[i][j] > -1
        return M[i][j]
    if i == 1:
        M[i][j] = e(1,j)
        return e(1,j)
    best = infty
    best = min(best, S(i-1,j-1) + e(i,j))
    best = min(best, S(i-1,j) + e(i,j))
    best = min(best, S(i-1,j+1) + e(i,j))
    M[i][j] = best
    return best
```

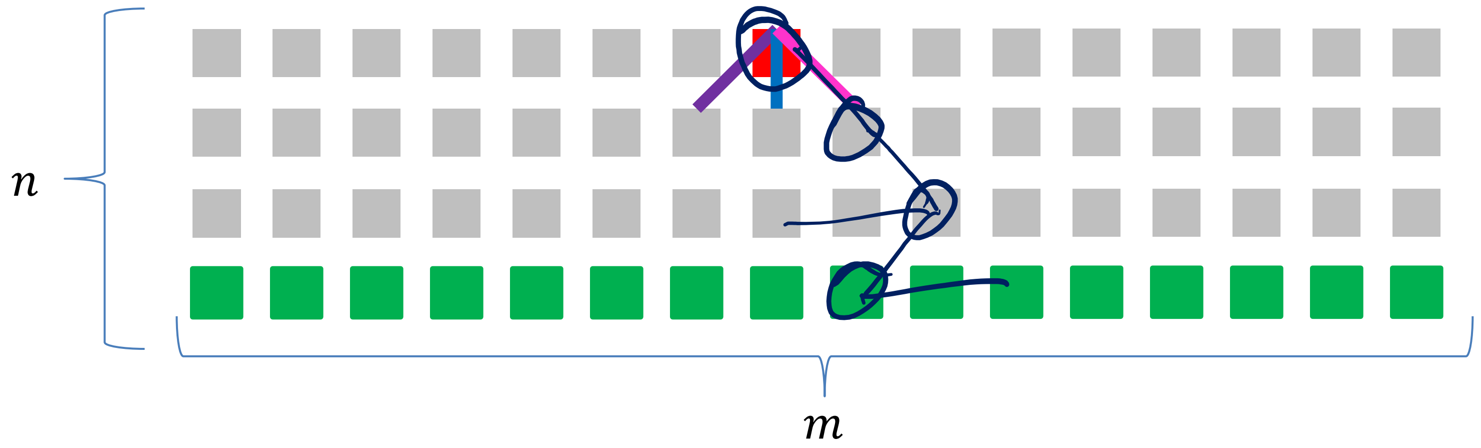
Bring It All Together

Start from bottom of image (row 1), solve up to top

Initialize $S(1,k) = e(p_{1,k})$ for each pixel $p_{1,k}$

For $i \geq 2$ find $S(i,k) = \min\limits_{k=1}^{m} \begin{cases} S(i-1,k-1) + e(p_{i,k}) \\ S(i-1,k) + e(p_{i,k}) \\ S(i-1,k+1) + e(p_{i,k}) \end{cases}$

Pick smallest from top row, backtrack, removing those pixels



Energy of the seam initialized to the energy of that pixel

Seam Carving Pseudocode

```
def  remove_seam(pic):
    seam_ws = []
    for(i in range(pic.width)):
        seam_ws.append(S(pic.height, i))
    min_seam_end = min_index(seam_ws)
    min_seam = backtrack(min_seam_end)
    picture.remove(min_seam)
```

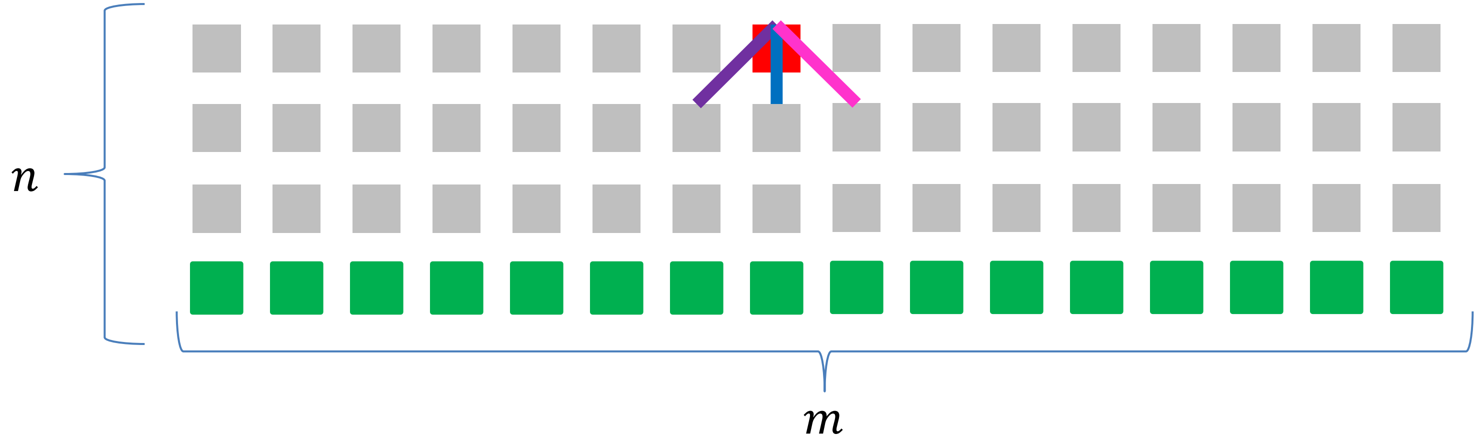
Run Time?

Start from bottom of image (row 1), solve up to top

Initialize $S(1,k) = e(p_{1,k})$ for each pixel $p_{1,k}$ $\quad\Theta(m)$

For $i \geq 2$ find $S(i,k) = \min_{k=1}^{m} \begin{cases} S(i-1,k-1) + e(p_{i,k}) \\ S(i-1,k) + e(p_{i,k}) \\ S(i-1,k+1) + e(p_{i,k}) \end{cases}$ $\quad\Theta(n \cdot m)$

Pick smallest from top row, backtrack, removing those pixels $\quad\Theta(n + m)$

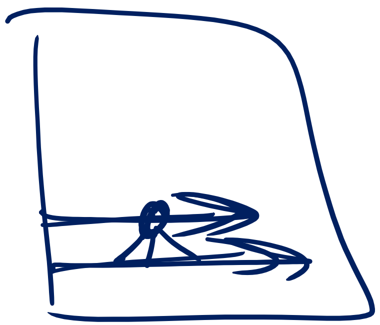


Energy of the seam initialized to the energy of that pixel

Seam Carving Bottom Up

$\Theta(m)$

for i in range(pic.width):
    S[1][i] = e(1,i)



$\Theta(n \cdot m)$

for i in range 1 : pic.height
    for j in range 1 : pic.width
        S[i][j] = min(S[i-1][j-1], S[i-1][j], S[i-1][j+1]) + e(i,j)

do remove-seam

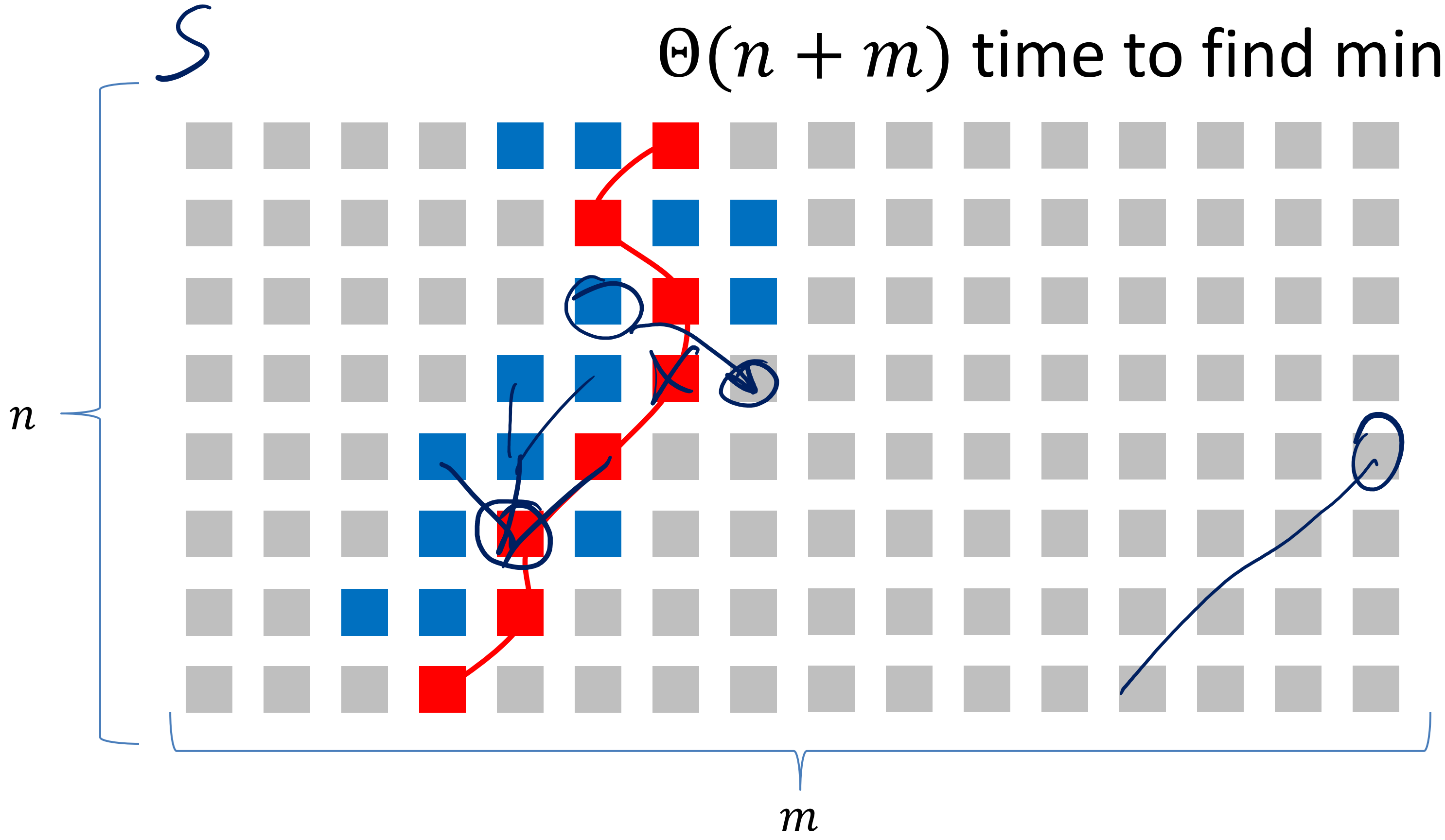
Repeated Seam Removal

Only need to update pixels dependent on the removed seam

$2n$ pixels change

$\Theta(2n)$ time to update pixels

$\Theta(n+m)$ time to find min+backtrack



Longest Common Subsequence

Given two sequences X and Y, find the length of their longest common subsequence

Example:
$X = ATCTGAT$
$Y = TGCATA$
$LCS = TCTA$

Brute force: Compare every subsequence of X with Y
$\Omega(2^n)$

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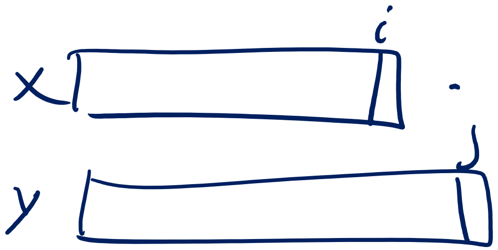
1. Identify Recursive Structure



Let $LCS(i,j)$ = length of the LCS for the first i characters of X, first j character of Y

Find $LCS(i,j)$:

Case 1: $X[i] = Y[j]$

$X = ATCTGCGT$
$Y = TGCATAT$
$LCS(i,j) = LCS(i-1, j-1) + 1$

Case 2: $X[i] \neq Y[j]$

$X{=}ATCTGCGA$
$Y{=}TGCATAT$
$LCS(i,j) = LCS(i, j-1)$

$X{=}ATCTGCGT$
$Y{=}TGCATAC$
$LCS(i,j) = LCS(i-1, j)$

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

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Save to M[i,j]

Read from M[i,j] if present

```
X = "alkjdflaksjdf"
Y = "lakjsdflkasjdlfs"
M = 2d array of len(X) rows and len(Y) columns, initialized to -1
def LCS(int i, int j):
        # returns the length of the LCS shared between the length-i prefix of  X and length-j prefix of Y
        # memoization
        if M[i,j] > -1:
                return M[i,j]
        #base case:
        if i == 0 or j == 0:
                ans = 0
        elif X[i] == Y[j]:
                ans = LCS(i-1, j-1) + 1
        else:
                ans = max( LCS(i, j-1), LCS(i-1, j) )
        M[i,j] = ans
        return ans
print(LCS(len(X)+1, len(Y)+1)) # the answer for the entirety of X and Y
```

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

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3. Solve in a Good Order

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

$Y =$ \ $X =$		0	A 1	T 2	C 3	T 4	G 5	A 6	T 7
	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
T	1	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**
G	2	**0**	**0**	**1**	**1**	**1**	**2**	**2**	**2**
C	3	**0**	**0**	**1**	**2**	**2**	**2**	**2**	**2**
A	4	**0**	**1**	**1**	**2**	**2**	**2**	**3**	**3**
T	5	**0**	**1**	**2**	**2**	**3**	**3**	**3**	**4**
A	6	**0**	**1**	**2**	**2**	**3**	**3**	**4**	**4**

To fill in cell (i, j) we need cells $(i-1, j-1), (i-1, j), (i, j-1)$
Fill from Top->Bottom, Left->Right (with any preference)

Run Time?

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

$X =$, $Y =$

		0	A 1	T 2	C 3	T 4	G 5	A 6	T 7
	0	0	0	0	0	0	0	0	0
T	1	0	0	1	1	1	1	1	1
G	2	0	0	1	1	1	2	2	2
C	3	0	0	1	2	2	2	2	2
A	4	0	1	1	2	2	2	3	3
T	5	0	1	2	2	3	3	3	4
A	6	0	1	2	2	3	3	4	4

Run Time: $\Theta(n \cdot m)$ (for $|X| = n$, $|Y| = m$)

Reconstructing the LCS

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

$Y =$	$X =$	0	A 1	T 2	C 3	$\boxed{T}$ 4	$\boxed{G}$ 5	$\boxed{A}$ 6	$\boxed{T}$ 7
	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
$\boxed{T}$	1	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**
$\boxed{G}$	2	**0**	**0**	**1**	**1**	**1**	**2**	**2**	**2**
C	3	**0**	**0**	**1**	**2**	**2**	**2**	**2**	**2**
$\boxed{A}$	4	**0**	**1**	**1**	**2**	**2**	**2**	**3**	**3**
$\boxed{T}$	5	**0**	**1**	**2**	**2**	**3**	**3**	**3**	**4**
A	6	**0**	**1**	**2**	**2**	**3**	**3**	**4**	**4**

Start from bottom right,

if symbols matched, print that symbol then go diagonally

else go to largest adjacent

Reconstructing the LCS

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$Y =$	$X =$	0	A 1	$\boxed{T}$ 2	$\boxed{C}$ 3	T 4	G 5	$\boxed{A}$ 6	$\boxed{T}$ 7
	0	0	0	0	0	0	0	0	0
$\boxed{T}$	1	0	0	1	1	1	1	1	1
G	2	0	0	1	1	1	2	2	2
$\boxed{C}$	3	0	0	1	2	2	2	2	2
$\boxed{A}$	4	0	1	1	2	2	2	3	3
$\boxed{T}$	5	0	1	2	2	3	3	3	4
A	6	0	1	2	2	3	3	4	4

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Y \ X	0	A 1	$\boxed{T}$ 2	$\boxed{C}$ 3	$\boxed{T}$ 4	G 5	$\boxed{A}$ 6	T 7
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
$\boxed{T}$ 1	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**
G 2	**0**	**0**	**1**	**1**	**1**	**2**	**2**	**2**
$\boxed{C}$ 3	**0**	**0**	**1**	**2**	**2**	**2**	**2**	**2**
A 4	**0**	**1**	**1**	**2**	**2**	**2**	**3**	**3**
$\boxed{T}$ 5	**0**	**1**	**2**	**2**	**3**	**3**	**3**	**4**
$\boxed{A}$ 6	**0**	**1**	**2**	**2**	**3**	**3**	**4**	**4**

Start from bottom right,

if symbols matched, print that symbol then go diagonally

else go to largest adjacent