

CSE 332 Winter 2026

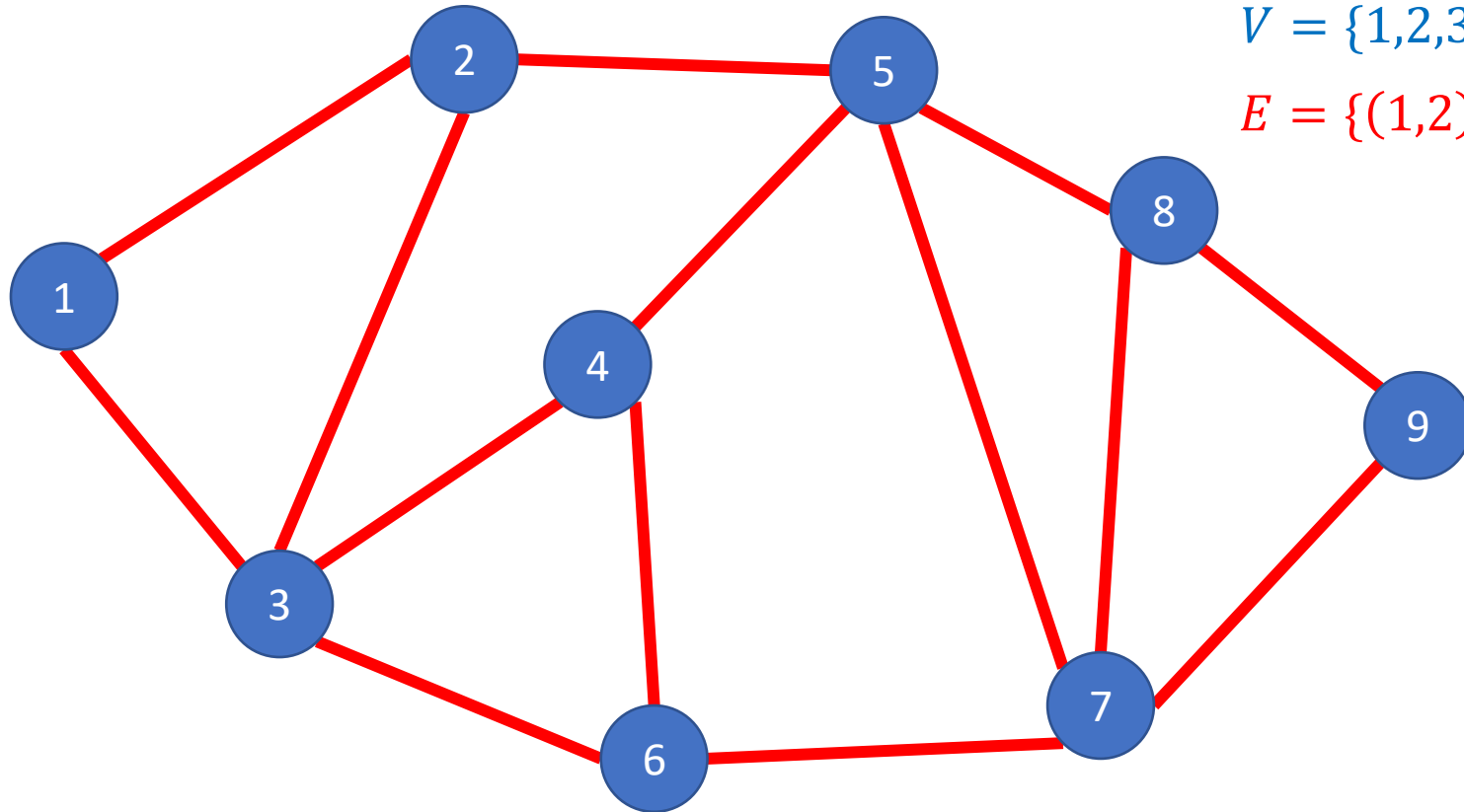
Lecture 15: Graphs 2

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<http://www.cs.uw.edu/332>

Undirected Graphs

Definition: $G = (\overset{\text{Vertices/Nodes}}{V}, \underset{\text{Edges}}{E})$

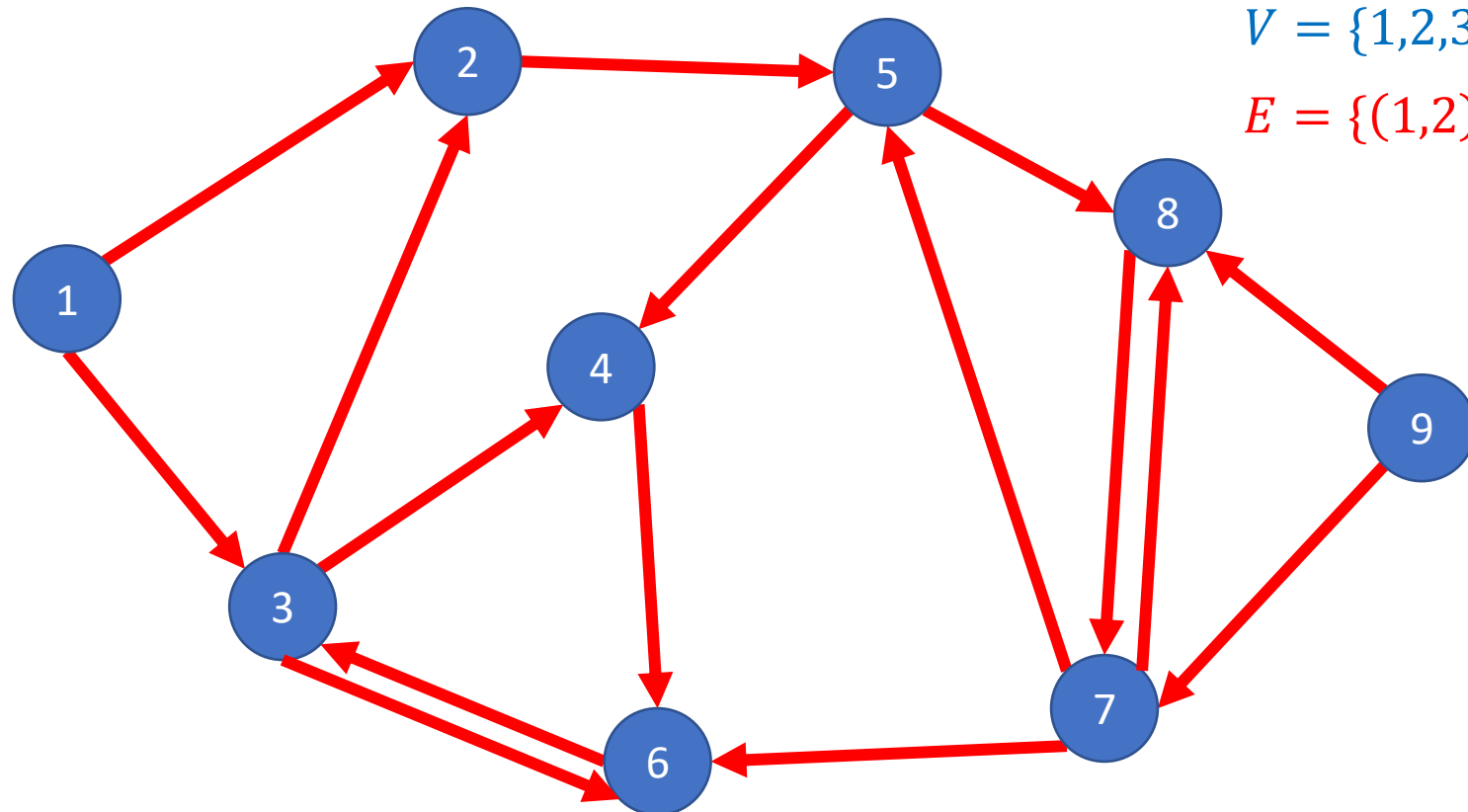


$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$E = \{(1, 2), (2, 3), (1, 3), \dots\}$

Directed Graphs

Definition: $G = (\overset{\text{Vertices/Nodes}}{V}, \underset{\text{Edges}}{E})$



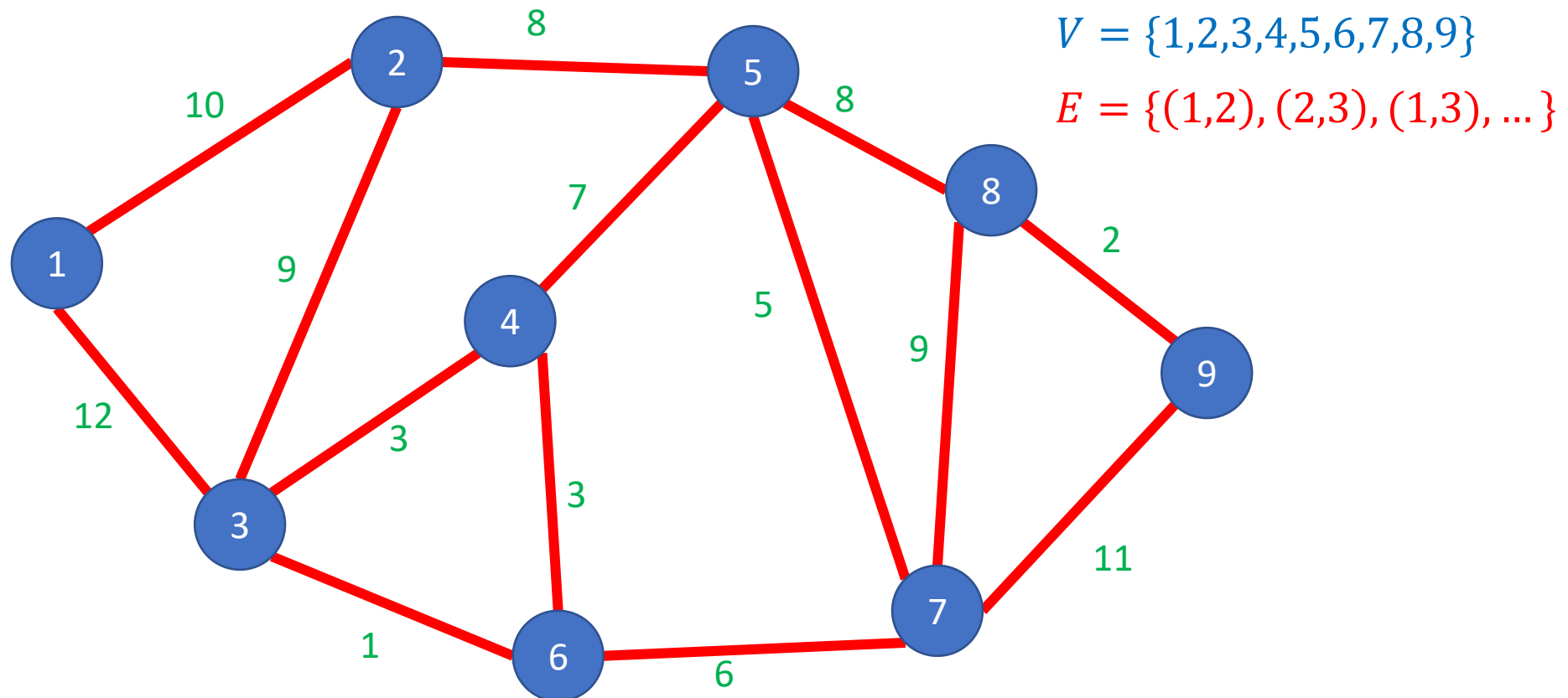
$V = \{1,2,3,4,5,6,7,8,9\}$

$E = \{(1,2), (2,3), (1,3), \dots\}$

Weighted Graphs

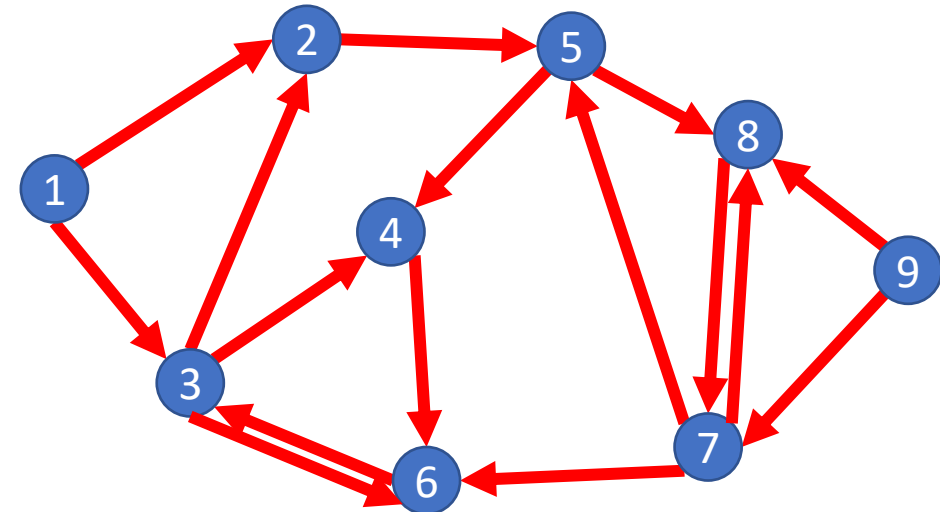
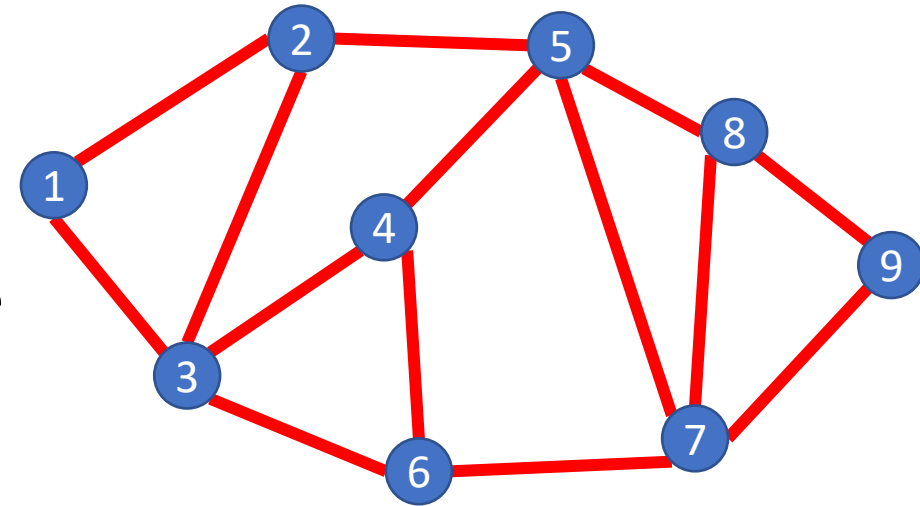
Definition: $G = (\overset{\text{Vertices/Nodes}}{V}, \overset{\text{Edges}}{E})$

$w(e)$ = weight of edge e



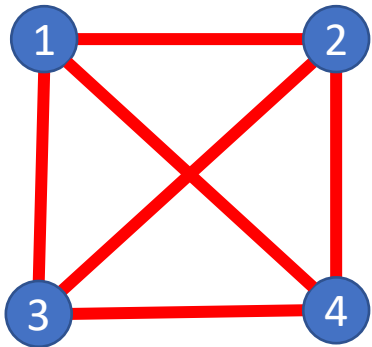
Some Graph Terms

- Adjacent/Neighbors
 - Nodes are adjacent/neighbors if they share an edge
- Degree
 - Number of edges “touching” a vertex
- Indegree
 - Number of incoming edges
- Outdegree
 - Number of outgoing edges

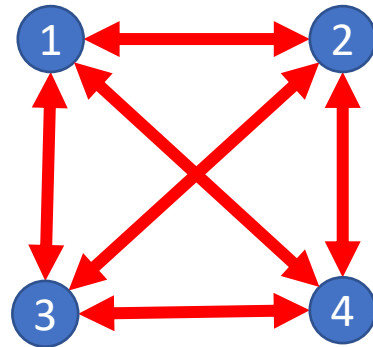


Definition: Complete Graph

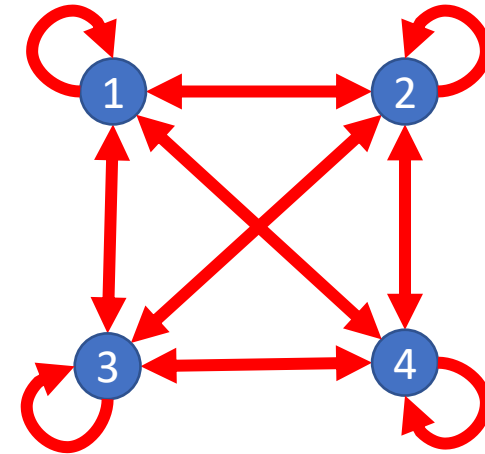
A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is an edge from v_1 to v_2



Complete
Undirected Graph



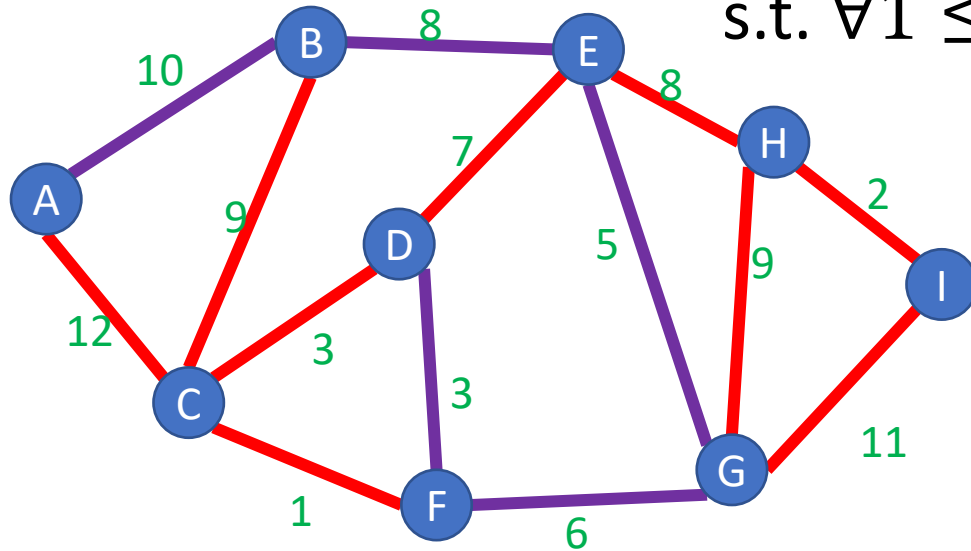
Complete
Directed Graph



Complete Directed
Non-simple Graph

Definition: Path

A sequence of nodes (v_1, v_2, \dots, v_k)
s.t. $\forall 1 \leq i \leq k - 1, (v_i, v_{i+1}) \in E$



Simple Path:

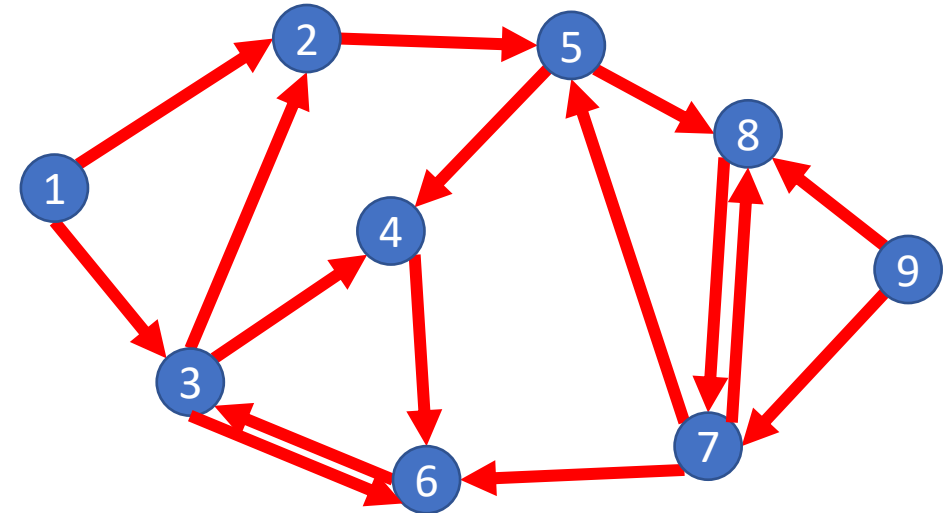
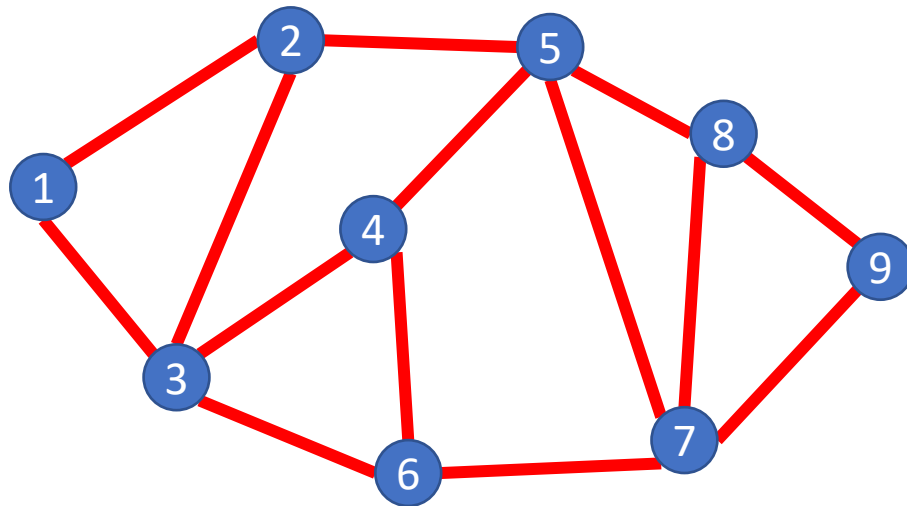
A path in which each node appears at most once

Cycle:

A path which starts and ends in the same place

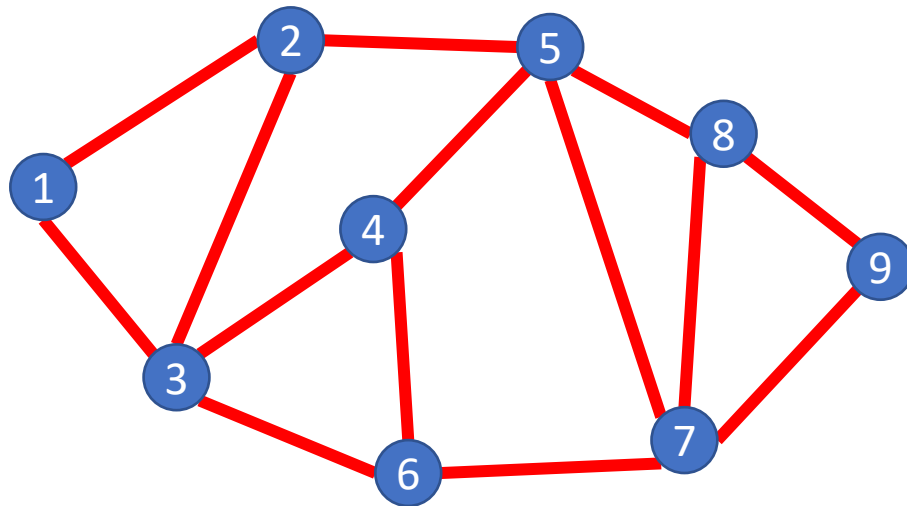
Definition: (Strongly) Connected Graph

A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is a path from v_1 to v_2

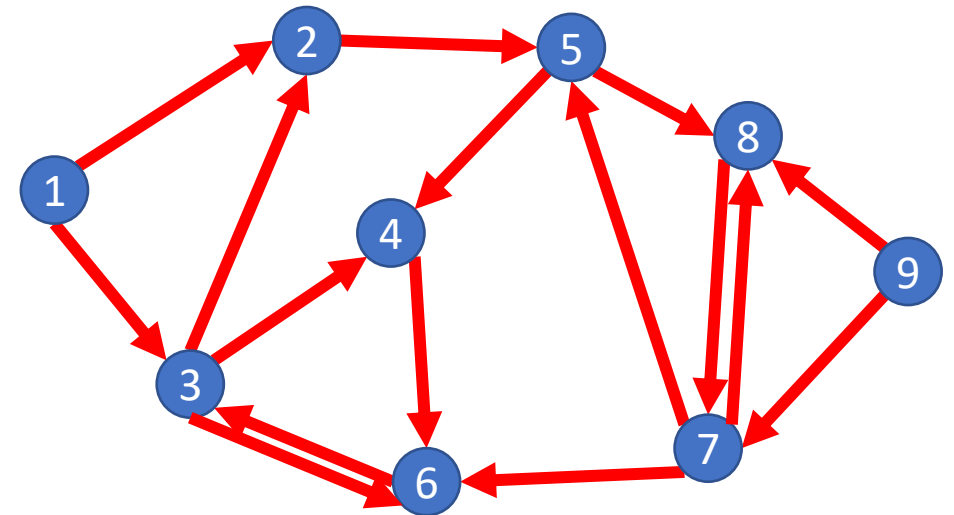


Definition: (Strongly) Connected Graph

A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is a path from v_1 to v_2



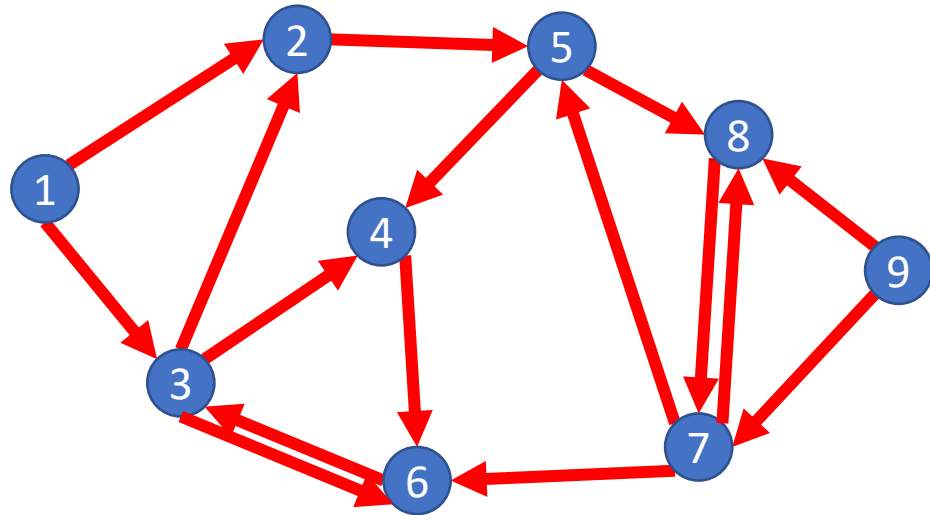
Connected



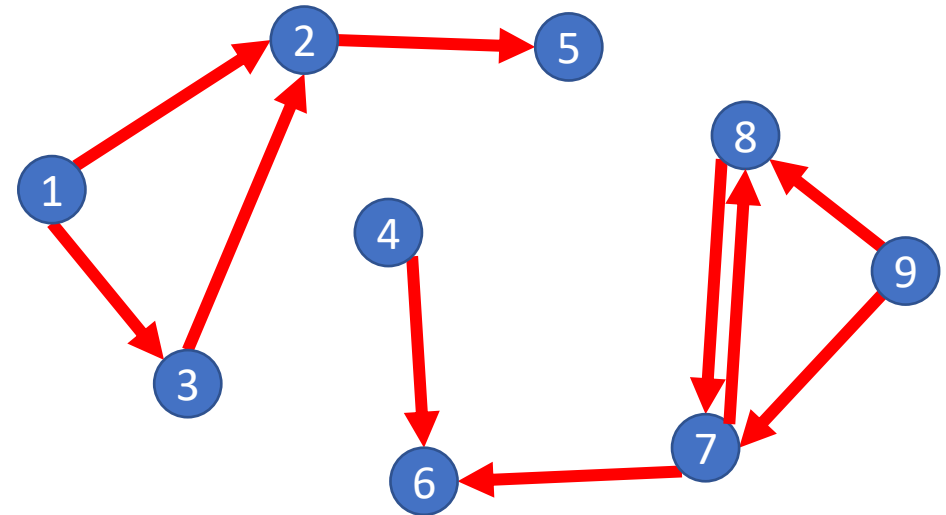
Not (strongly) Connected

Definition: Weakly Connected Graph

A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is a path from v_1 to v_2 ignoring direction of edges



Weakly Connected



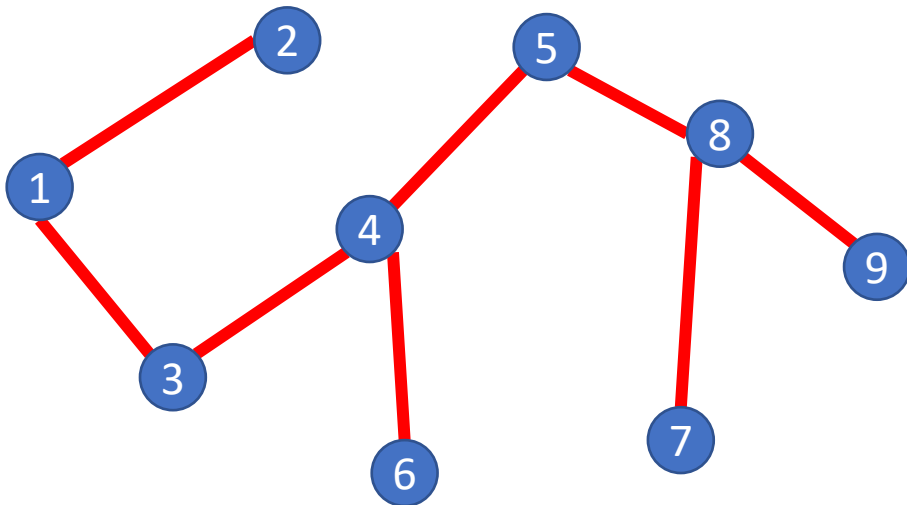
Not Weakly Connected

Graph Density, Data Structures, Efficiency

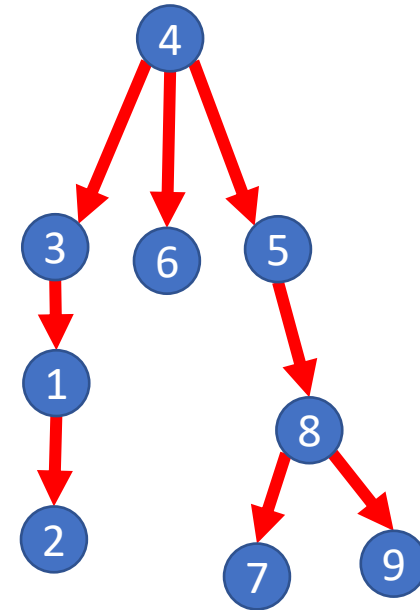
- The maximum number of edges in a graph is $\Theta(|V|^2)$:
 - Undirected and simple: $\frac{|V|(|V|-1)}{2}$
 - Directed and simple: $|V|(|V| - 1)$
 - Direct and non-simple (but no duplicates): $|V|^2$
- If the graph is connected, the minimum number of edges is $|V| - 1$
- If $|E| \in \Theta(|V|^2)$ we say the graph is **dense**
- If $|E| \in \Theta(|V|)$ we say the graph is **sparse**
- Because $|E|$ is not always near to $|V|^2$ we do not typically substitute $|V|^2$ for $|E|$ in running times, but leave it as a separate variable
 - However, $\log(|E|) \in \Theta(\log(|V|))$

Definition: Tree

A Graph $G = (V, E)$ is a tree if it is undirect, connected, and has no cycles (i.e. is acyclic). Often one node is identified as the “root”



A Tree

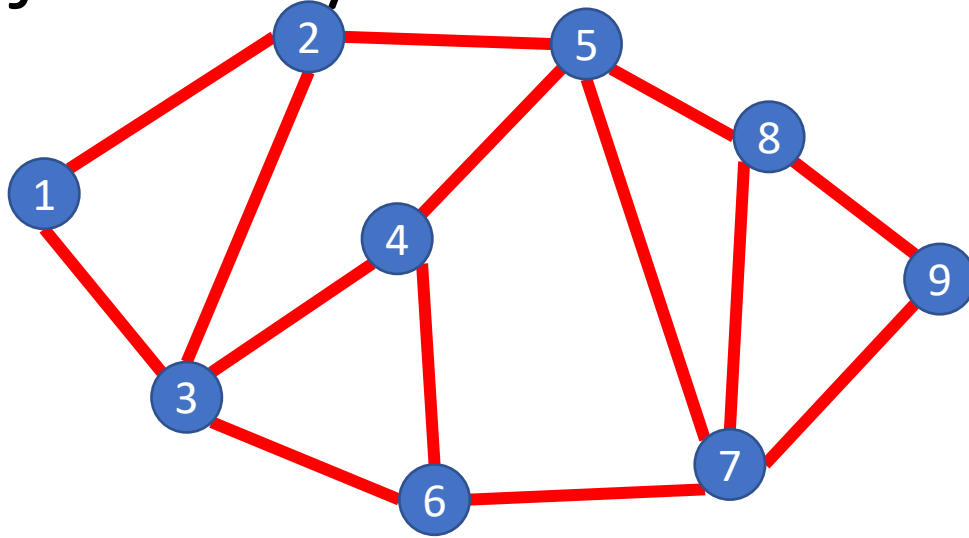


A Rooted Tree

Graph ADT

- Idea: Nodes with edges between them
 - Directed or undirected
 - Weighted or unweighted
- Operations we'll need:
 - addEdge: add a new edge between preexisting nodes
 - removeEdge: remove an edge
 - exists: Check if a particular edge exists
 - getNeighbors: give a list of all neighbors of a given node
 - For a directed graph, we also might want getNeighborsIncoming

Adjacency List Data Structure



Time/Space Tradeoffs

Space to represent: $\Theta(n + m)$

Add Edge (v, w) : $\Theta(\deg(v))$

Remove Edge (v, w) : $\Theta(\deg(v))$

Check if Edge (v, w) Exists: $\Theta(\deg(v))$

Get Neighbors (incoming): $\Theta(n + m)$

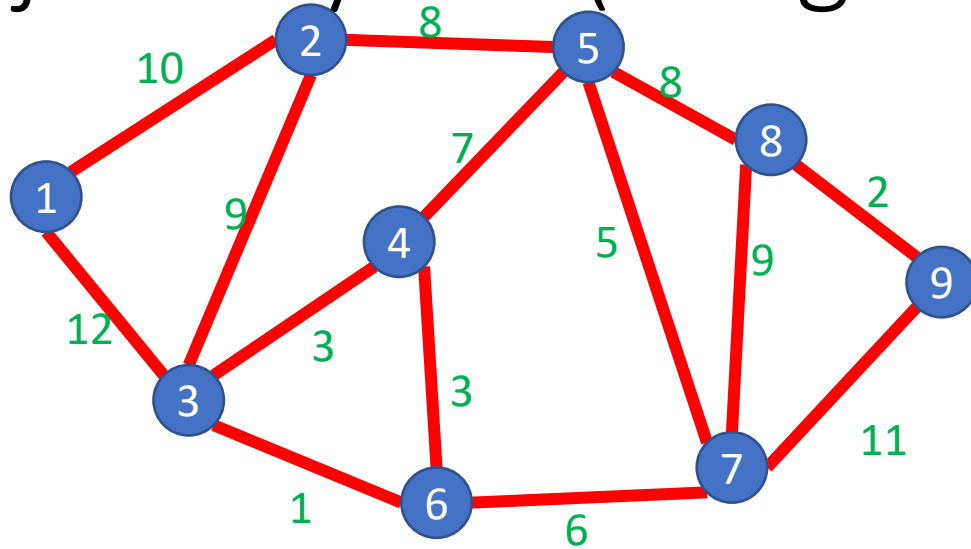
Get Neighbors (outgoing): $\Theta(\deg(v))$

$$|V| = n$$

$$|E| = m$$

1	2	3		
2	1	3	5	
3	1	2	4	6
4	3	5	6	
5	2	4	7	8
6	3	4	7	
7	5	6	8	9
8	5	7	9	
9	7	8		

Adjacency List (Weighted)



Time/Space Tradeoffs

Space to represent: $\Theta(n + m)$

Add Edge (v, w) : $\Theta(\deg(v))$

Remove Edge (v, w) : $\Theta(\deg(v))$

Check if Edge (v, w) Exists: $\Theta(\deg(v))$

Get Neighbors (incoming): $\Theta(n + m)$

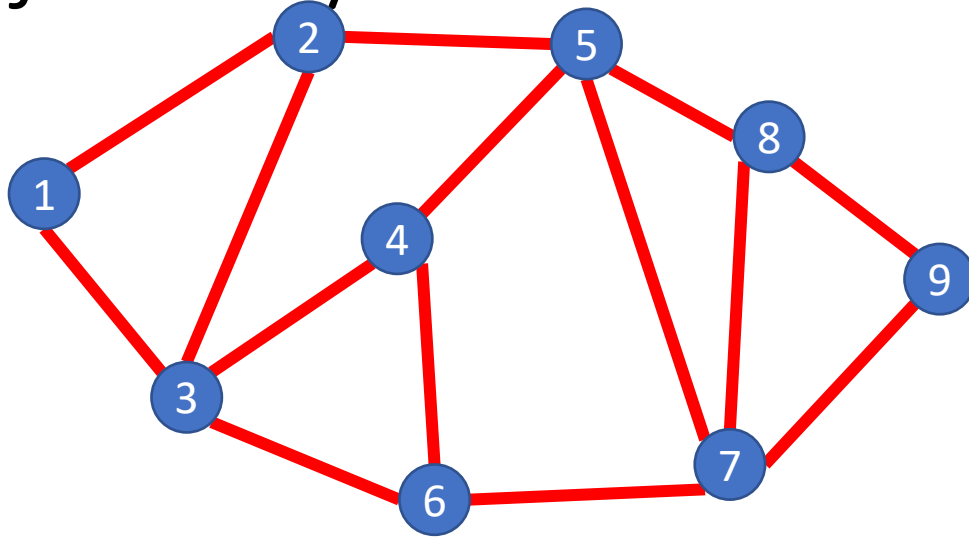
Get Neighbors (outgoing): $\Theta(\deg(v))$

$$|V| = n$$

$$|E| = m$$

1	2 (10)	3 (12)		
2	1 (10)	3 (9)	5 (8)	
3	1 (12)	2 (9)	4 (3)	6 (1)
4	3 (3)	5 (7)	6 (3)	
5	2 (8)	4 (7)	7 (5)	8 (8)
6	3 (1)	4 (3)	7 (6)	
7	5 (5)	6 (6)	8 (9)	9 (11)
8	5 (8)	7 (9)	9 (2)	
9	7 (11)	8 (2)		

Adjacency Matrix



Time/Space Tradeoffs

Space to represent: $\Theta(?)$

Add Edge (v, w) : $\Theta(?)$

Remove Edge (v, w) : $\Theta(?)$

Check if Edge (v, w) Exists: $\Theta(?)$

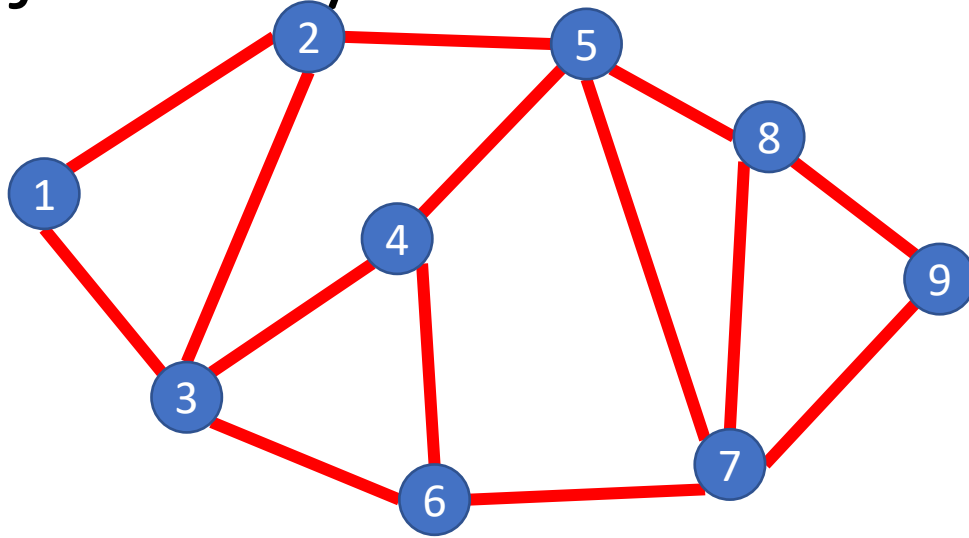
Get Neighbors (incoming): $\Theta(?)$

Get Neighbors (outgoing): $\Theta(?)$

$$\begin{aligned} |V| &= n \\ |E| &= m \end{aligned}$$

	1	2	3	4	5	6	7	8	9
1		1	1						
2	1		1		1				
3	1	1		1		1			
4			1		1	1			
5		1		1			1	1	
6			1	1			1		
7					1	1		1	1
8					1		1		1
9							1	1	

Adjacency Matrix



Time/Space Tradeoffs

Space to represent: $\Theta(n^2)$

Add Edge (v, w) : $\Theta(1)$

Remove Edge (v, w) : $\Theta(1)$

Check if Edge (v, w) Exists: $\Theta(1)$

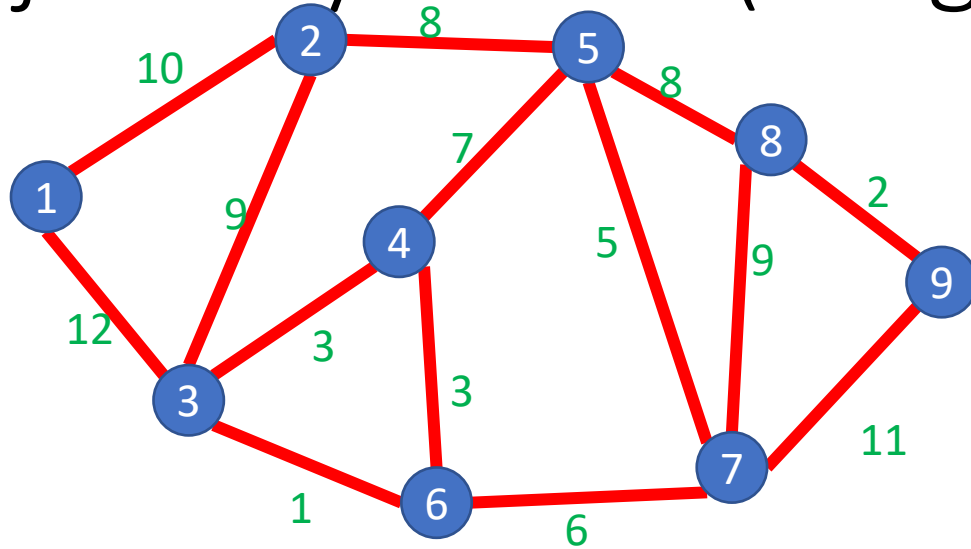
Get Neighbors (incoming): $\Theta(n)$

Get Neighbors (outgoing): $\Theta(n)$

$$\begin{aligned} |V| &= n \\ |E| &= m \end{aligned}$$

	1	2	3	4	5	6	7	8	9
1		1	1						
2	1		1		1				
3	1	1		1		1			
4			1		1	1			
5		1		1			1	1	
6			1	1			1		
7					1	1		1	1
8					1		1		1
9							1	1	

Adjacency Matrix (weighted)



	1	2	3	4	5	6	7	8	9
1		10	12						
2	10		9		8				
3	12	9		3		1			
4			3		7	3			
5		8		7			5	8	
6			1	3			1		
7					5	1		9	11
8					8		9		2
9							11	2	

Time/Space Tradeoffs

Space to represent: $\Theta(n^2)$

Add Edge (v, w) : $\Theta(1)$

Remove Edge (v, w) : $\Theta(1)$

Check if Edge (v, w) Exists: $\Theta(1)$

Get Neighbors (incoming): $\Theta(n)$

Get Neighbors (outgoing): $\Theta(n)$

$$|V| = n$$

$$|E| = m$$

Comparison

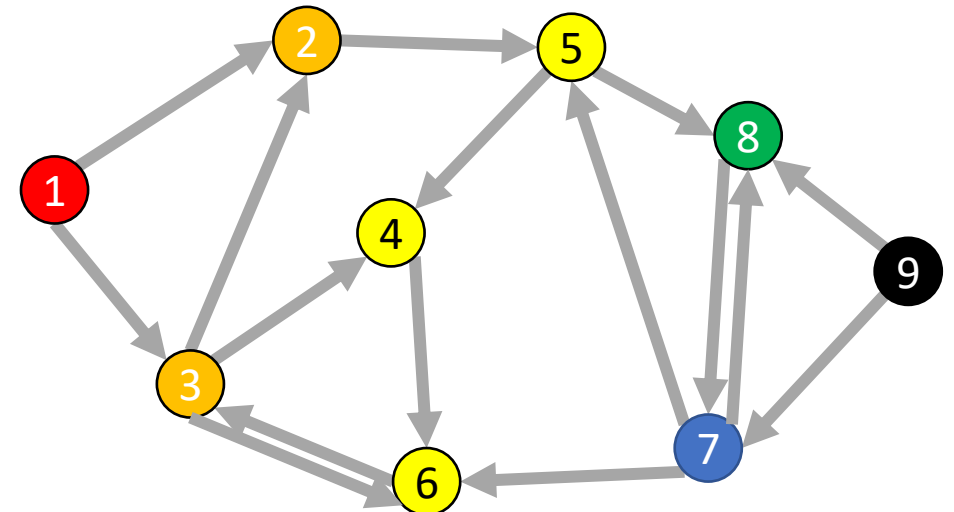
- Adjacency List:
 - Less memory when $|E| < |V|^2$
 - Operations with running time linear in degree of source node
 - Add an edge
 - Remove an edge
 - Check for edge
 - Get neighbors
- Adjacency Matrix:
 - Similar amount of memory when $|E| \approx |V|^2$
 - Constant time operations:
 - Add an edge
 - Remove an edge
 - Check for an edge
 - Operations running with linear time in $|V|$
 - Get neighbors

Adjacency List is more common in practice:

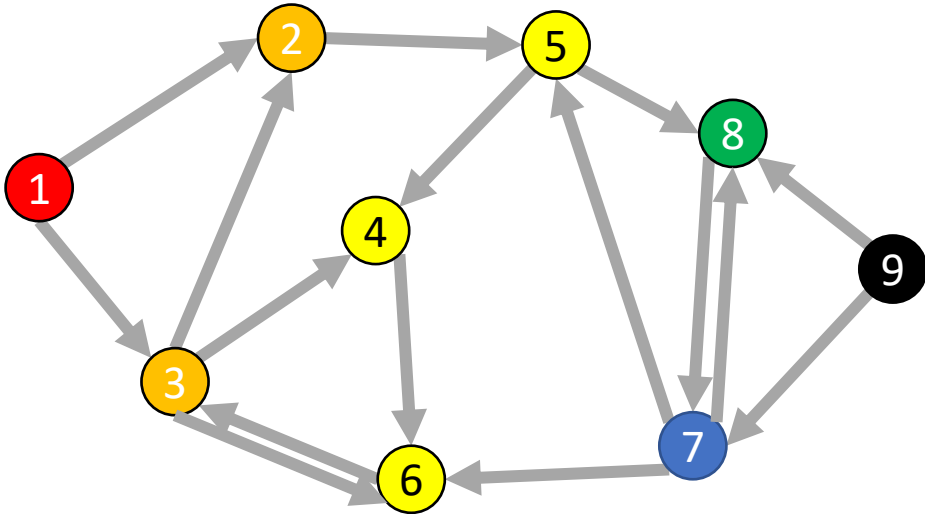
- Most graphs have $|E| \ll |V|^2$
 - Saves memory
 - Most nodes will have small degree
- Getting neighbors is a common operation
- Adjacency Matrix may be better if the graph is “dense” or if its edges change a lot

Breadth-First Search

- Input: a node s
- Behavior: Start with node s , visit all neighbors of s , then all neighbors of neighbors of s , ...
- Visits every node reachable from s in order of distance
- Output:
 - How long is the shortest path?
 - Is the graph connected?



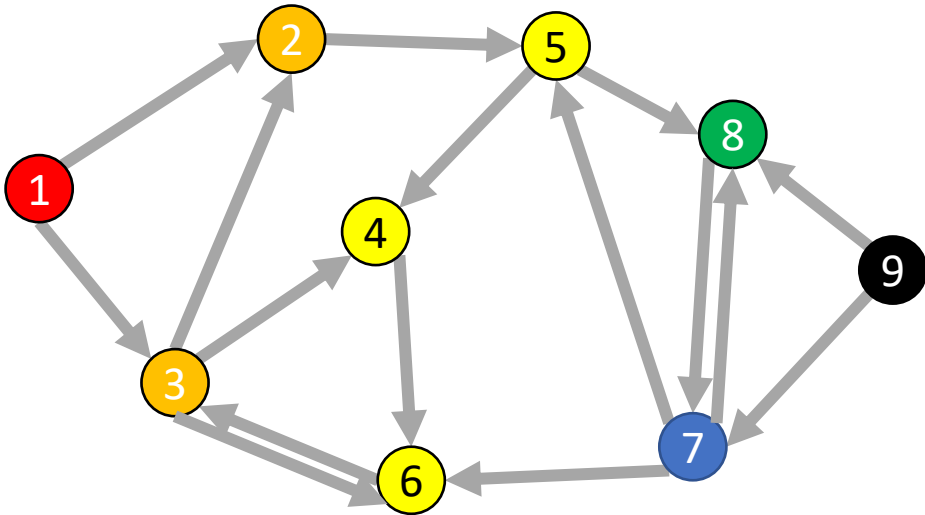
BFS



Running time: $\Theta(|V| + |E|)$

```
void bfs(graph, s){
    found = new Queue();
    found.enqueue(s);
    mark s as "visited";
    While (!found.isEmpty()){
        current = found.dequeue();
        for (v : neighbors(current)){
            if (! v marked "visited"){
                mark v as "visited";
                found.enqueue(v);
            }
        }
    }
}
```

BFS – Worked Example



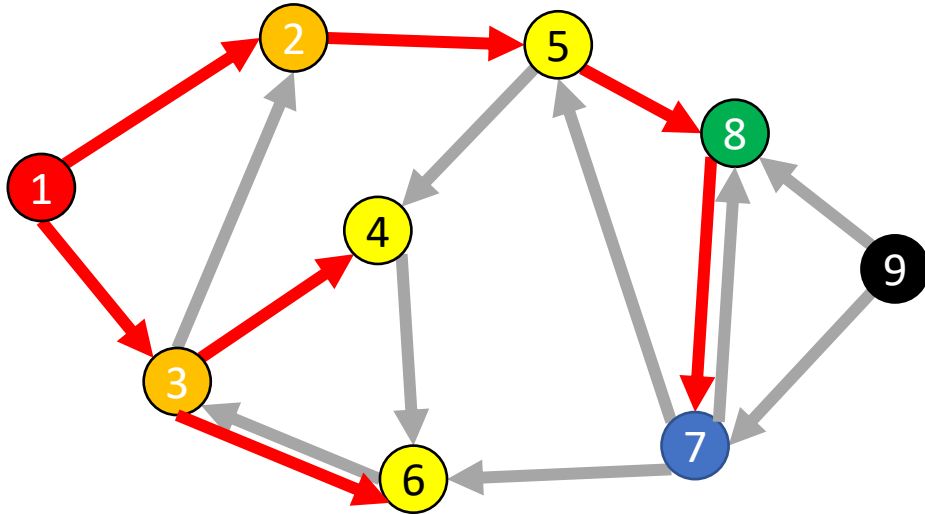
For each node:

For each unvisited neighbor:
add that neighbor to a queue
mark that neighbor as visited

Node	Visited?	Other Info
1	True	
2		
3		
4		
5		
6		
7		
8		
9		

Queue:

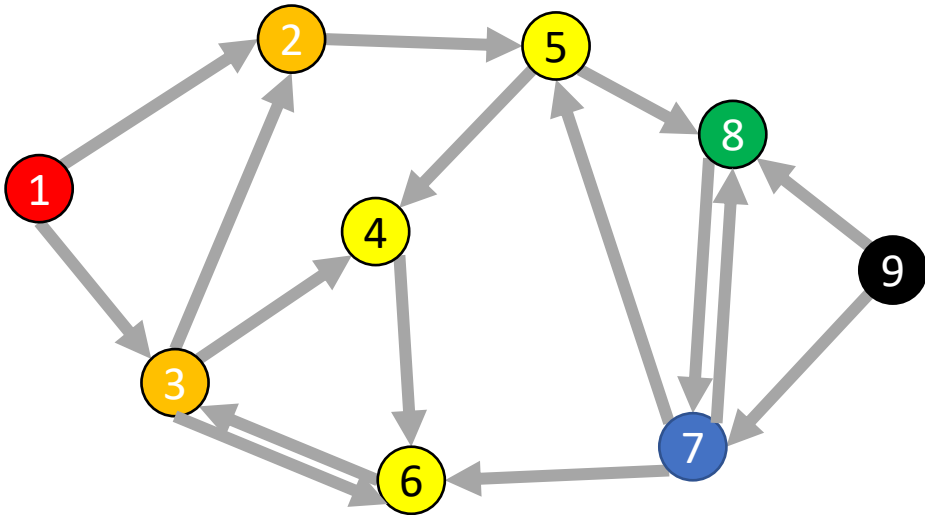
Find Distance (unweighted)



Idea: when it's seen, remember its "layer" depth!

```
int findDistance(graph, s, t){
    found = new Queue();
    layer = 0;
    found.enqueue(s);
    mark s as "visited";
    While (!found.isEmpty()){
        current = found.dequeue();
        layer = depth of current;
        for (v : neighbors(current)){
            if (! v marked "visited"){
                mark v as "visited";
                depth of v = layer + 1;
                found.enqueue(v);
            }
        }
    }
    return depth of t;
}
```

Find Distance – Worked Example



Node	Visited?	Layer
1		
2		
3		
4		
5		
6		
7		
8		
9		

For each node:

update current layer

For each unvisited neighbor:

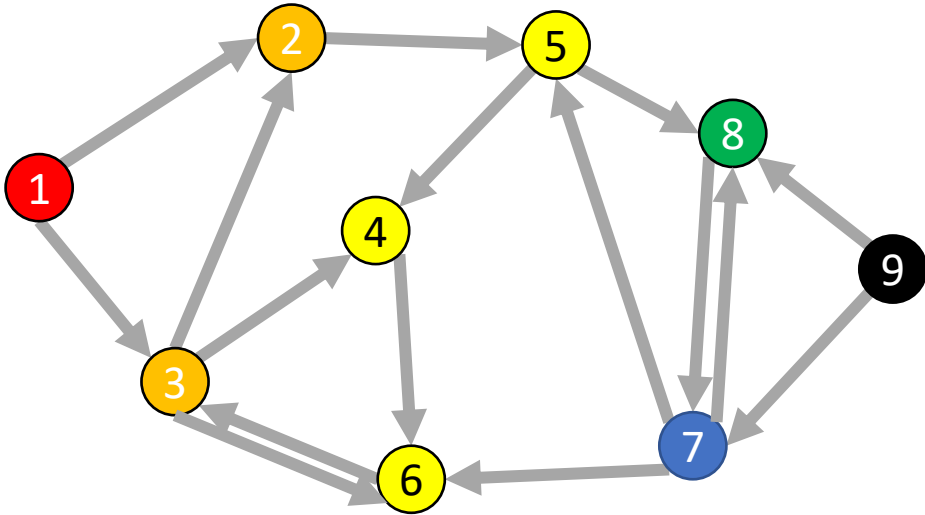
add that neighbor to a queue

mark that neighbor as visited

set neighbor's layer to be current layer + 1

Queue:

Shortest Path - Idea



Node	Visited?	Previous
1		
2		
3		
4		
5		
6		
7		
8		
9		

For each node:

For each unvisited neighbor:

add that neighbor to a queue

mark that neighbor as visited

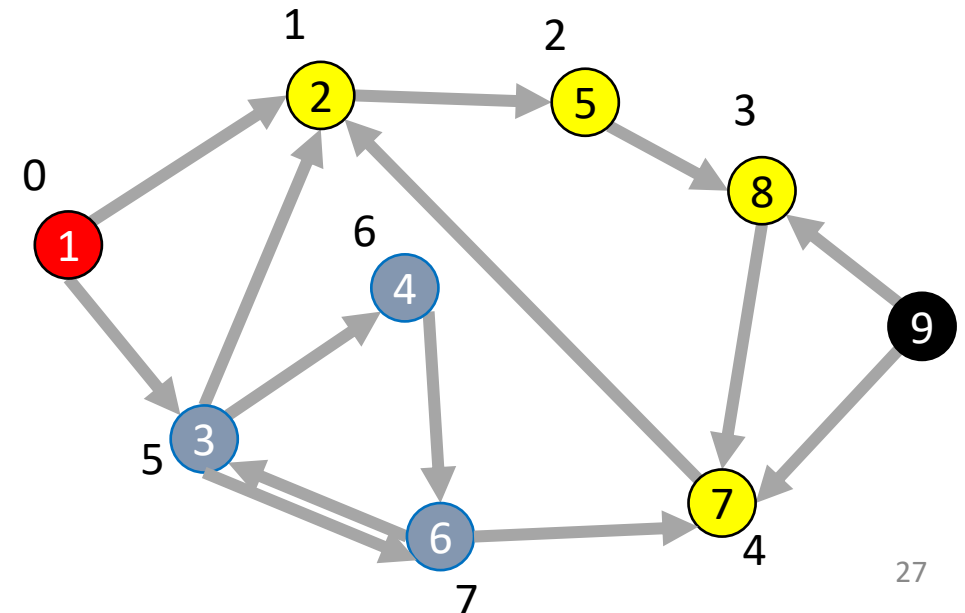
set neighbor's previous to be the current node

Queue:

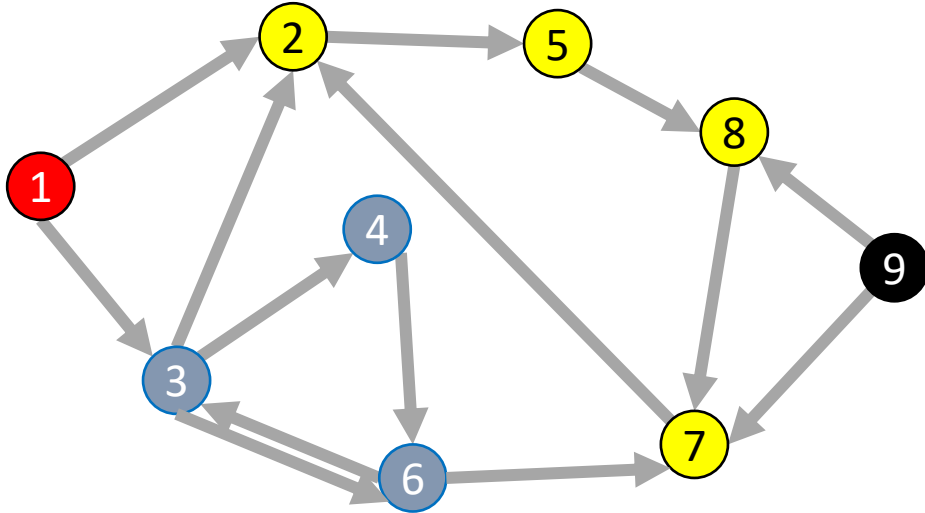
Depth-First Search

Depth-First Search

- Input: a node s
- Behavior: Start with node s , visit one neighbor of s , then all nodes reachable from that neighbor of s , then another neighbor of s ,...
 - Before moving on to the second neighbor of s , visit everything reachable from the first neighbor of s
- Output:
 - Does the graph have a cycle?
 - A **topological sort** of the graph.



DFS (non-recursive)

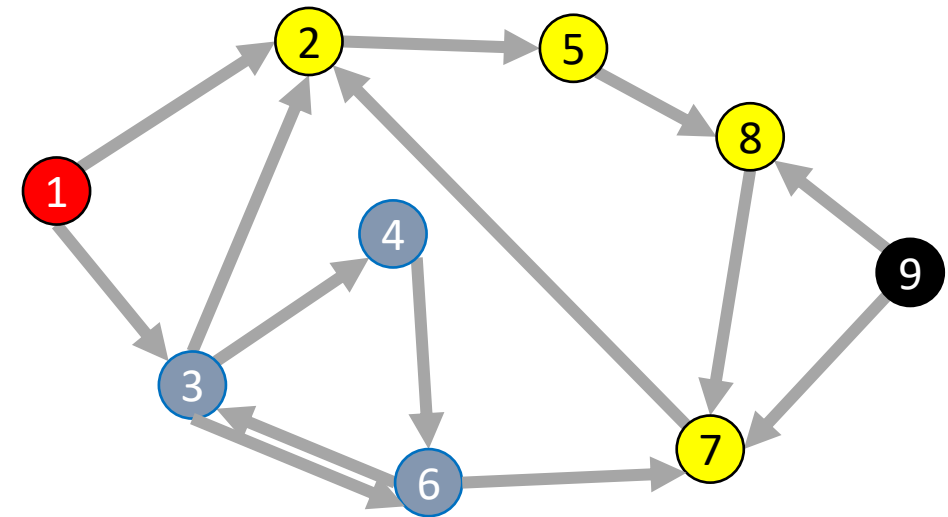


Running time: $\Theta(|V| + |E|)$

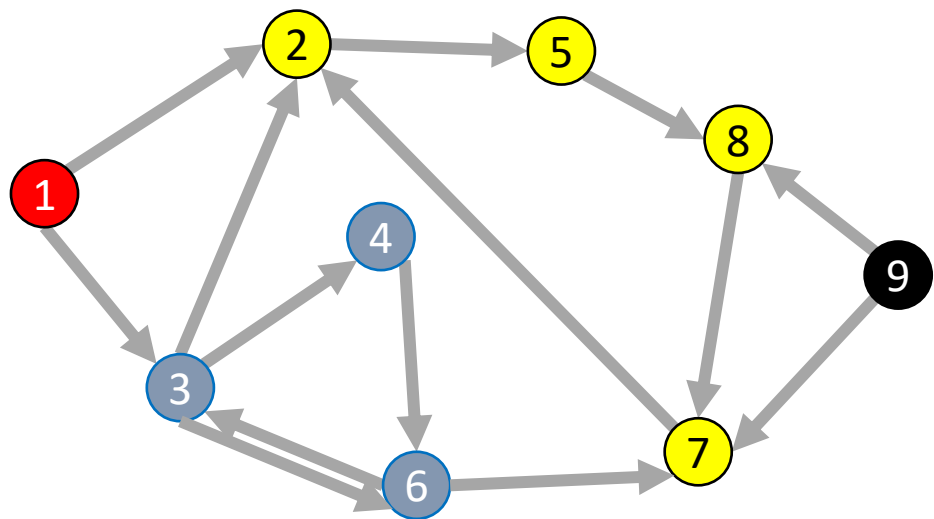
```
void dfs(graph, s){
    found = new Stack();
    found.pop(s);
    mark s as "visited";
    While (!found.isEmpty()){
        current = found.pop();
        for (v : neighbors(current)){
            if (! v marked "visited"){
                mark v as "visited";
                found.push(v);
            }
        }
    }
}
```

DFS Recursively (more common)

```
void dfs(graph, curr){  
    mark curr as "visited";  
    for (v : neighbors(current)){  
        if (! v marked "visited"){  
            dfs(graph, v);  
        }  
    }  
    mark curr as "done";  
}
```



DFS – Worked Example



Starting from the current node:
for each unvisited neighbor:
mark the neighbor as visited
do a DFS from the neighbor
mark the current node as done

Node	Visited?	Done?	Other Info
1			
2			
3			
4			
5			
6			
7			
8			
9			

(Call)
Stack:

Using DFS

- Consider the “visited times” and “done times”

- Edges can be categorized:

- Tree Edge

- (a, b) was followed when pushing
- (a, b) when b was unvisited when we were at a

- Back Edge

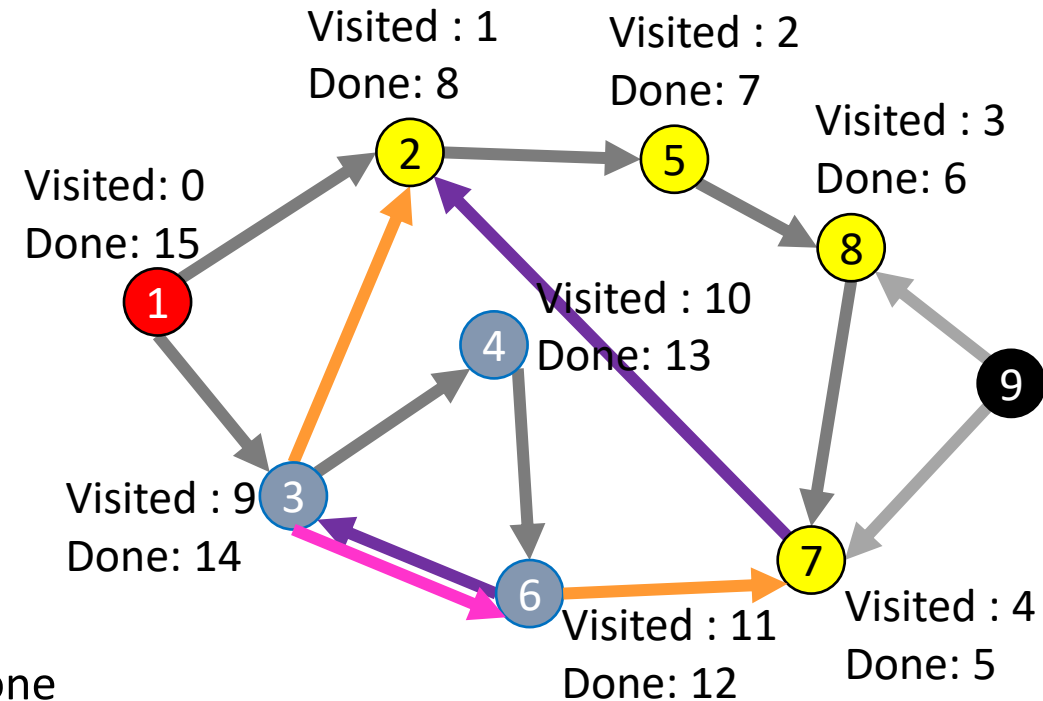
- (a, b) goes to an “ancestor”
- a and b visited but not done when we saw (a, b)
- $t_{visited}(b) < t_{visited}(a) < t_{done}(a) < t_{done}(b)$

- Forward Edge

- (a, b) goes to a “descendent”
- b was visited and done between when a was visited and done
- $t_{visited}(a) < t_{visited}(b) < t_{done}(b) < t_{done}(a)$

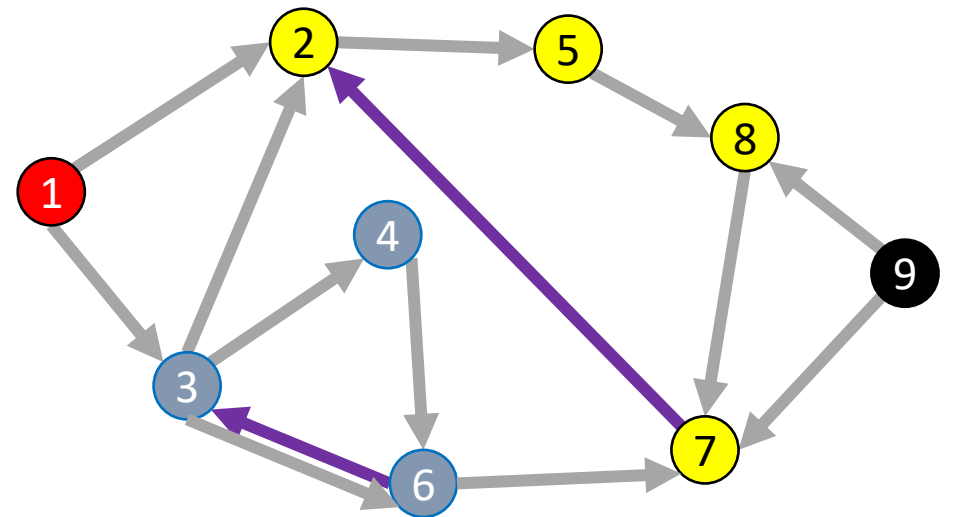
- Cross Edge

- (a, b) goes to a node that doesn't connect to a
- b was seen and done before a was ever visited
- $t_{done}(b) < t_{visited}(a)$



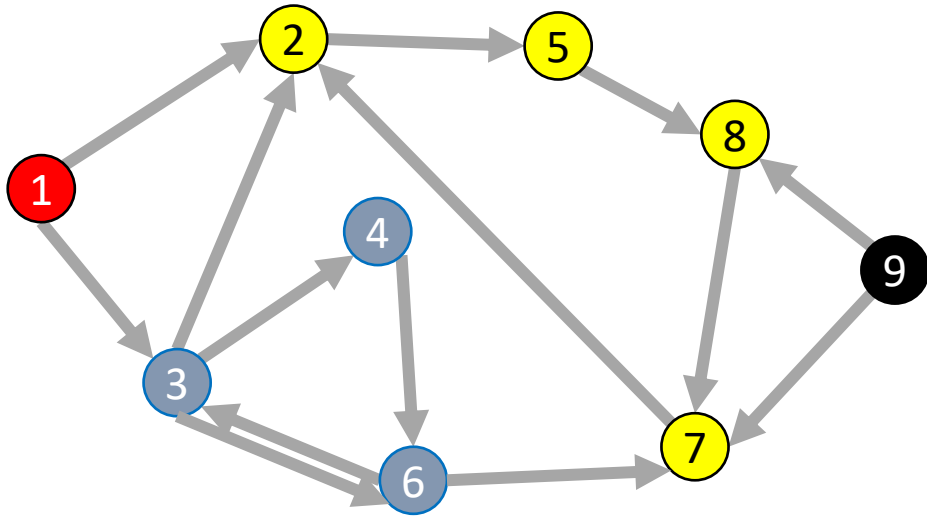
Back Edges

- Behavior of DFS:
 - “Visit everything reachable from the current node before going back”
- Back Edge:
 - The current node’s neighbor is an “in progress” node
 - Since that other node is “in progress”, the current node is reachable from it
 - The back edge is a path to that other node
 - **Cycle!**



Cycle Detection

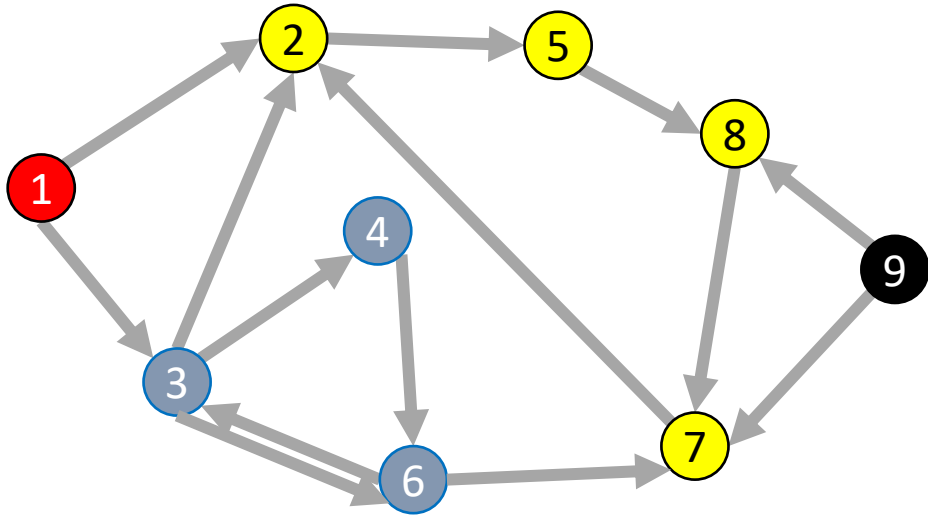
Idea: Look for a back edge!



```
Boolean hasCycle(graph){
    for(v : graph.vertices){
        if( ! v marked "done"){
            if(hasCycle(graph, v)){ return true; }
        }
    }
    return false;
}

boolean hasCycle(graph, curr){
    mark curr as "visited";
    cycleFound = false;
    for (v : neighbors(current)){
        if (v marked "visited" && ! v marked "done"){
            cycleFound=true;
        }
        if (! v marked "visited" && !cycleFound){
            cycleFound = hasCycle(graph, v);
        }
    }
    mark curr as "done";
    return cycleFound;
}
```

Cycle Detection – Worked Example



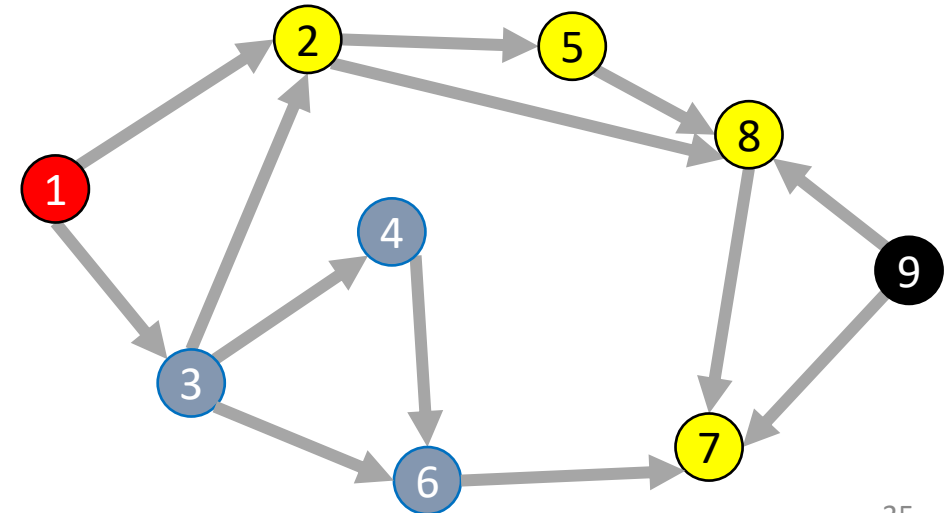
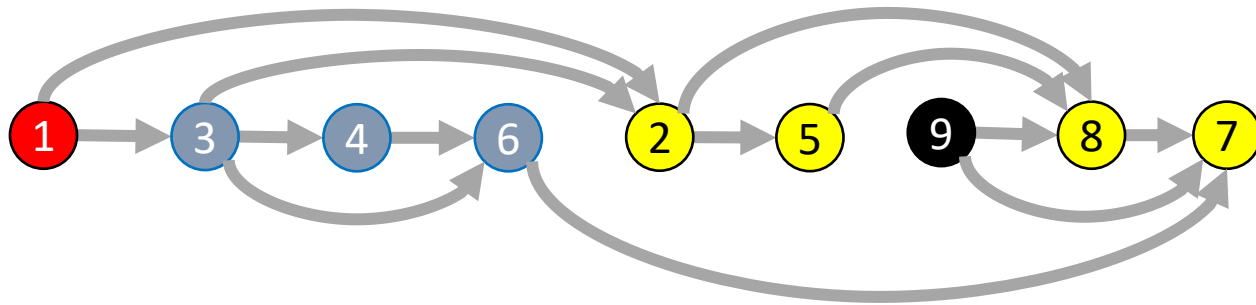
Starting from the current node:
for each non-done neighbor:
if the neighbor is visited:
we found a cycle!
else:
mark the neighbor as visited
do a DFS from the neighbor
mark the current node as done

Node	Visited?	Done?	Other Info
1			
2			
3			
4			
5			
6			
7			
8			
9			

(Call)
Stack:

Topological Sort

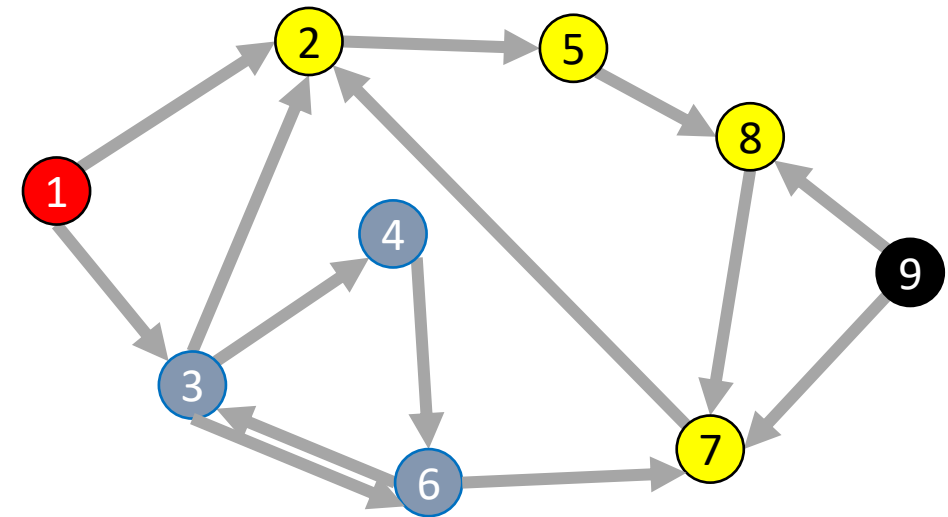
- A Topological Sort of a **directed acyclic graph** $G = (V, E)$ is a permutation of V such that if $(u, v) \in E$ then u is before v in the permutation



DFS Recursively

```
void dfs(graph, curr){  
    mark curr as "visited";  
    for (v : neighbors(current)){  
        if (! v marked "visited"){  
            dfs(graph, v);  
        }  
    }  
    mark curr as "done";  
}
```

Idea: List in reverse
order by "done" time



DFS: Topological sort

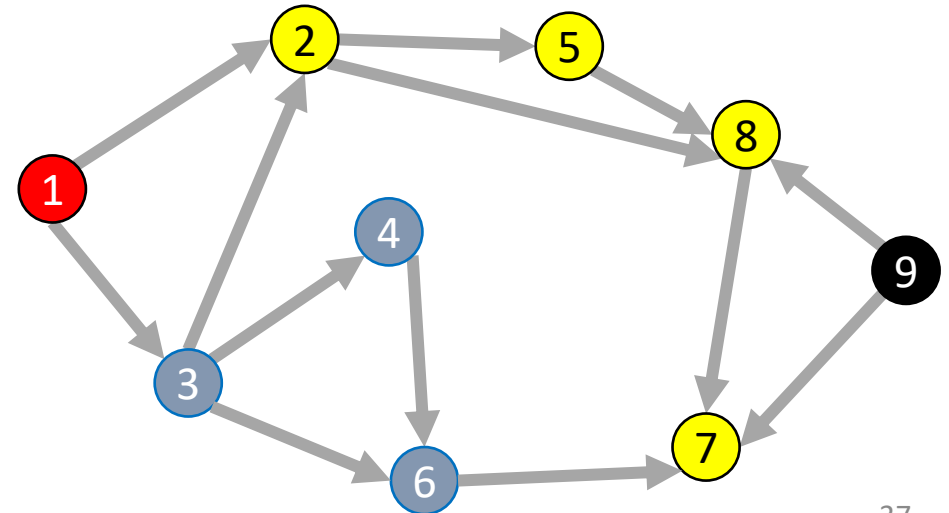
```
List topSort(graph){  
    List<Nodes> done = new List<>();  
    for (Node v : graph.vertices){  
        if (!v.visited){  
            finishTime(graph, v, finished);  
        }  
    }  
    done.reverse();  
    return done;  
}
```

Idea: List in reverse
order by “done” time

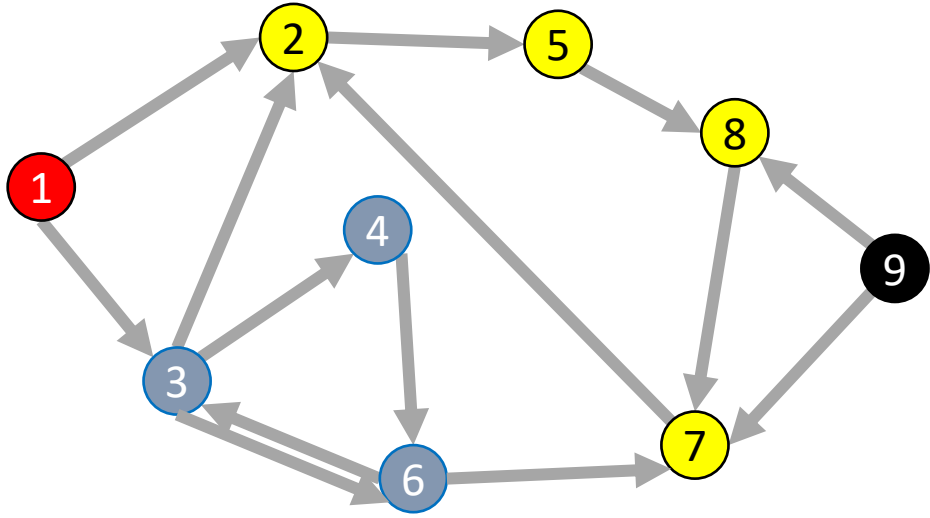
finished:



```
void finishTime(graph, curr, finished){  
    curr.visited = true;  
    for (Node v : curr.neighbors){  
        if (!v.visited){  
            finishTime(graph, v, finished);  
        }  
    }  
    done.add(curr)  
}
```



Topological Sort– Worked Example



Starting from the current node:
for each non-done neighbor:
if the neighbor is visited:
we found a cycle!
else:
mark the neighbor as visited
do a DFS from the neighbor
mark the current node as done
add current node to finished

Node	Visited?	Done?	Other Info
1			
2			
3			
4			
5			
6			
7			
8			
9			

(Call)

Stack:

finished:

--	--	--	--	--	--	--	--	--