CSE 332 Winter 2026
Lecture 12: Sorting

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Linear Probing: Delete

- **Option 1 (harder)**: Plug the hole with other items in a way that makes probes behave correctly
- **Option 2 (easier)**: "Tombstone" deletion. Leave a special object that indicates an something was deleted from there
  - The tombstone does not act as an open space when finding (so keep looking after its reached)
  - When inserting you can replace a tombstone with a new item

k, v		k, v	k, v		k, v				
0	1	2	3	4	5	6	7	8	9

Linear Probing + Tombstone: Find

- To find key k
  - Calculate `i = h(k) % table.length`
  - While `table[i]` has a key other than k, set `i = (i+1) % table.length`
  - If you come across k return `table[i]`
  - If you come across an empty index, the find was unsuccessful

0	1	2	3	4	5	6	7	8	9

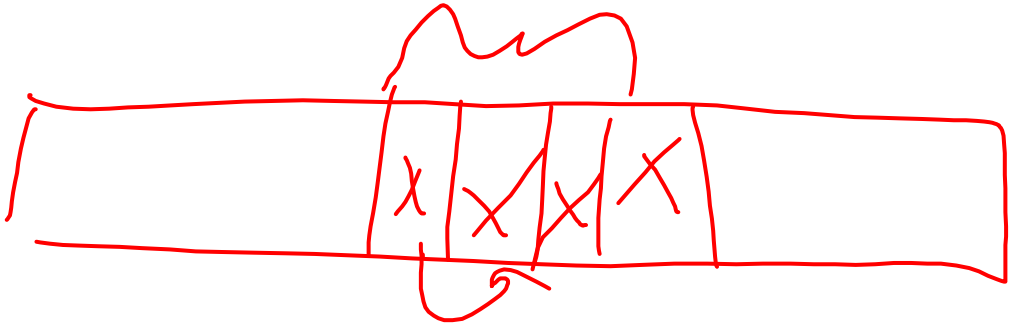
Linear Probing + Tombstone: Insert

- To insert `k,v`
  - Calculate `i = h(k) % table.length`
  - While `table[i]` has a key other than k, set `i = (i+1) % table.length`
    - If `table[i]` has a tombstone, set `x = i`
      - That is where we will insert if the find is unsuccessful
  - If you come across k, set `table[i] = k,v`
  - If you come across an empty index, the find was unsuccessful
    - Set `table[x] = k,v` if we saw a tombstone
    - Set `table[x] = k,v` otherwise

0	1	2	3	4	5	6	7	8	9

Downsides of Linear Probing

- What happens when λ approaches 1?
  - Get longer and longer contiguous blocks
  - A collision is guaranteed to grow a block
    - Larger blocks experience more collisions
    - Feedback loop!
- What happens when λ exceeds 1?
  - Impossible!
  - You can't insert more stuff



Quadratic Probing: Insert Procedure

- To insert k, v
  - Calculate `i = h(k) % table.length`
  - If `table[i]` is occupied then try `(i+1²) % table.length`
  - If that is occupied try `(i+2²) % table.length`
  - If that is occupied try `(i+3²) % table.length`
  - If that is occupied try `(i+4²) % table.length`
  - …

4
9
16
25

0	1	2	3	4	5	6	7	8	9

Quadratic Probing: Example

- Insert:
  - ~~76~~
  - ~~40~~
  - ~~48~~
  - 5 → 5 → $5+1^2$ → $5+2^2$
  - 55 → 6 → $6+1^2$ → $6+2^2$
  - 47 → 5 → $5+1^2$ → $5+2^2$ → $5+3^2$ → $5+4^2$

6 2

0	1	2	3	4	5	6
48		5	55		40	76

Using Quadratic Probing

- If you probe `table.length` times, you start repeating indices
- If `table.length` is prime and $\lambda < \frac{1}{2}$ then you're guaranteed to find an open spot in at most `table.length/2` probes

- Helps with the clustering problem of linear probing, but does not help if many things hash to the same value

Double Hashing: Insert Procedure

$h(h(x), y)$

- Given h and g are both good hash functions
- To insert k, v
  - Calculate `i = h(k) % table.length`
  - If `table[i]` is occupied then try `(i+g(k)) % table.length`
  - If that is occupied try `(i+2*g(k)) % table.length`
  - If that is occupied try `(i+3*g(k)) % table.length`
  - If that is occupied try `(i+4*g(k)) % table.length`
  - …

$i + c \cdot g(x) = h(x)$

$g(x) = 0$

$g(x)$ can't be $= 0$ or length

0	1	2	3	4	5	6	7	8	9

Sorting

- Rearrangement of items into some defined sequence
  - Usually: reordering a list from smallest to largest according to some metric
- Why sort things?
  - Enable things like binary search
    - It makes some algorithms faster
  - Nicer for human algorithms too
  - Data organization

More Formal Definition

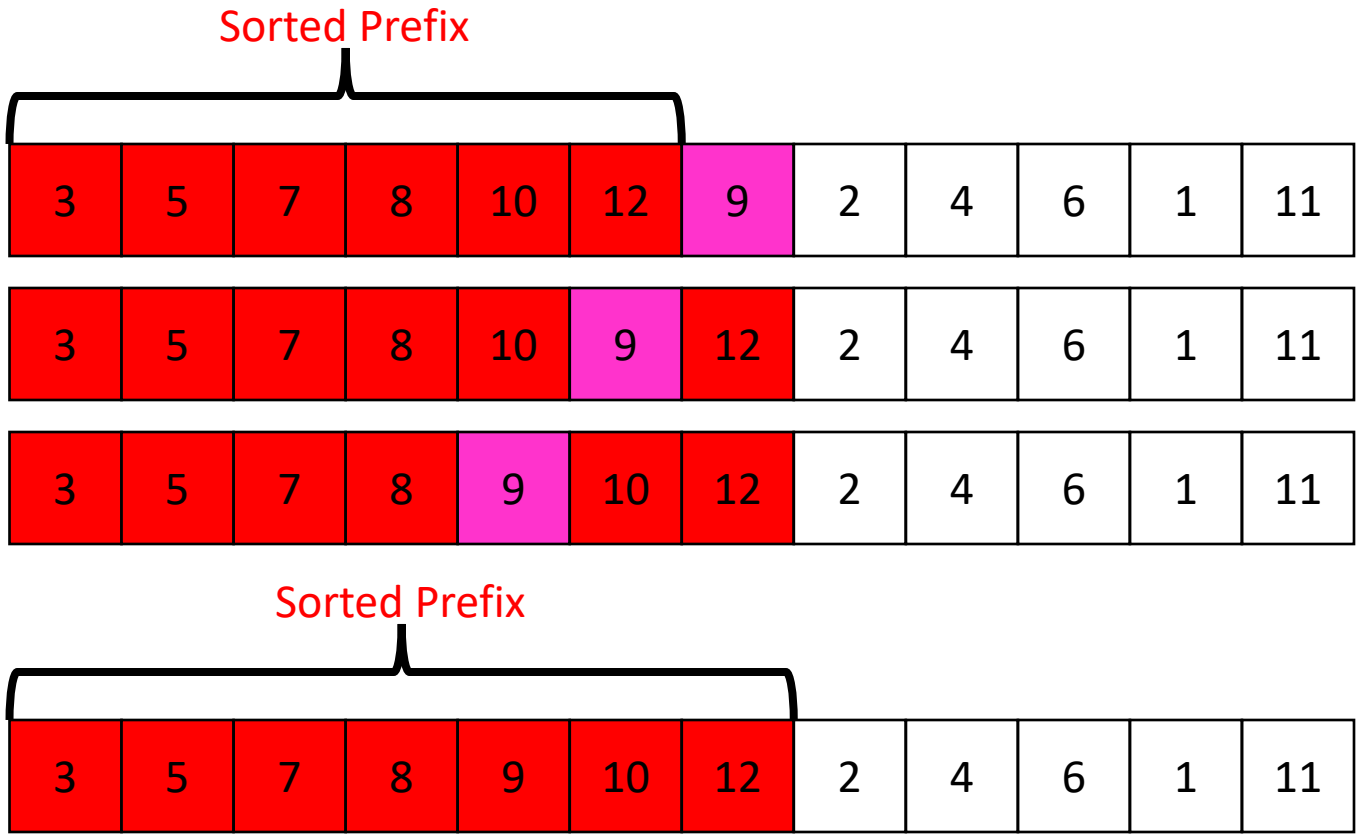
- Input:
  - An array A of items
  - A comparison function for these items
    - Given two items x and y, we can determine whether $x < y$, $x > y$, or $x = y$
- Output:
  - A permutation of A such that if $i \leq j$ then $A[i] \leq A[j]$
  - Permutation: a sequence of the same items but perhaps in a different order

Sorting “Landscape”

- There is no singular best algorithm for sorting
- Some are faster, some are slower
- Some use more memory, some use less
- Some are super extra fast if your data matches particular assumptions
- Some have other special properties that make them valuable
- No sorting algorithm can have only all the “best” attributes

Insertion Sort

- Idea: Maintain a sorted list prefix, extend that prefix by "inserting" the next element



Insertion Sort

- If the items at index 0 and 1 are out of order, swap them
- Keep swapping the item at index 2 with the thing to its left as long as the left thing is larger
- …
- Keep swapping the item at index i with the thing to its left as long as the left thing is larger

```
for (i=1; i<a.length; i++){
    prev = i-1;
    while(a[i] < a[prev] && prev > -1){
        temp = a[i];
        a[i] = a[prev];
        a[prev] = a[i];
        i--;
        prev--;
    }
}
```

Running Time:

Worst Case: $\Theta(\cdot)$ n^2

Best Case: $\Theta(\cdot)$ n

10	77	5	15	2	22	64	41	18	19	30	21	3	24	23	33
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Insertion Sort

- If the items at index 0 and 1 are out of order, swap them
- Keep swapping the item at index 2 with the thing to its left as long as the left thing is larger
- …
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    prev = i-1;
    while(a[i] < a[prev] && prev > -1){
        temp = a[i];
        a[i] = a[prev];
        a[prev] = a[i];
        i--;
        prev--;
    }
}
```

Running Time:

Worst Case: $\Theta(n^2)$

Best Case: $\Theta(n)$

10	77	5	15	2	22	64	41	18	19	30	21	3	24	23	33
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Aside: Bubble Sort – we won’t cover it

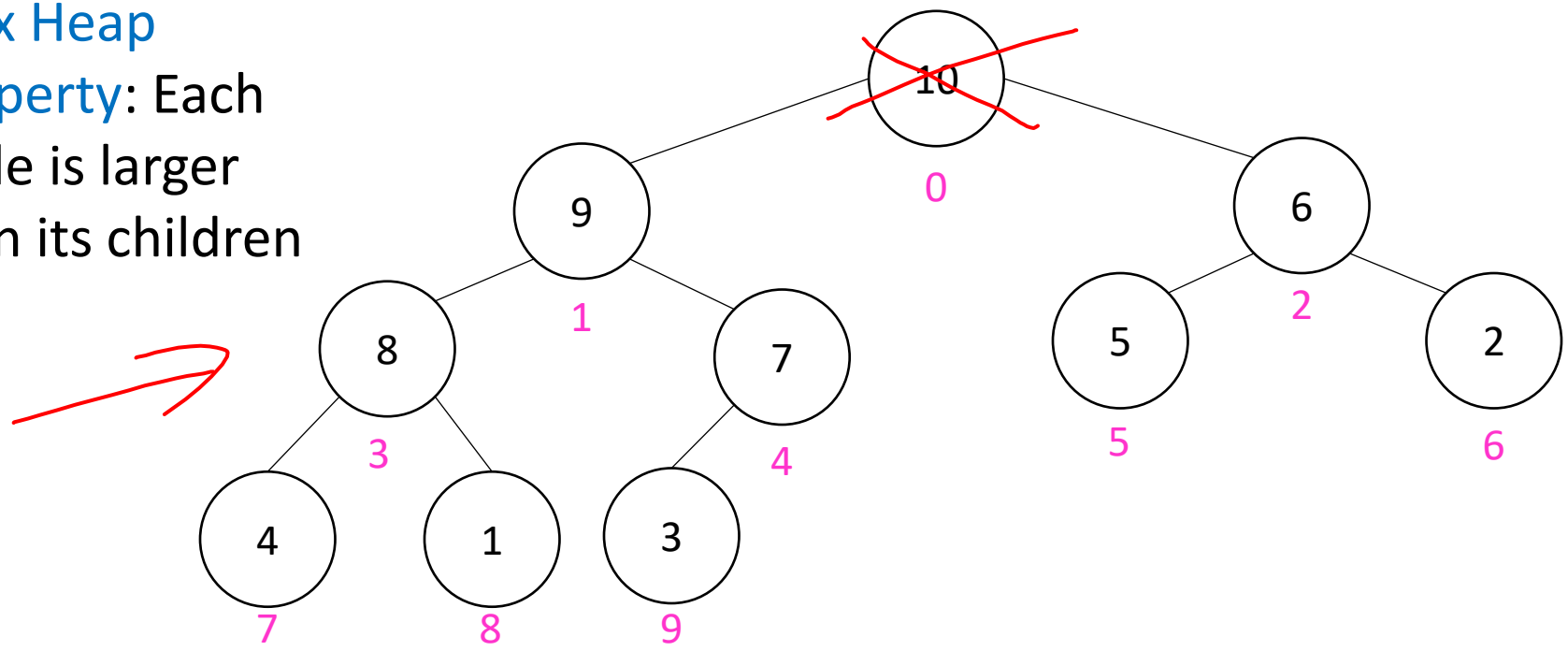
"the bubble sort seems to have nothing to recommend it, except a catchy name and the fact that it leads to some interesting theoretical problems” –Donald Knuth, The Art of Computer Programming

Heap Sort

- Idea: Build a maxHeap, repeatedly extract the max element from the heap to build sorted list Right-to-Left

10	9	6	8	7	5	2	4	1	3
0	1	2	3	4	5	6	7	8	9

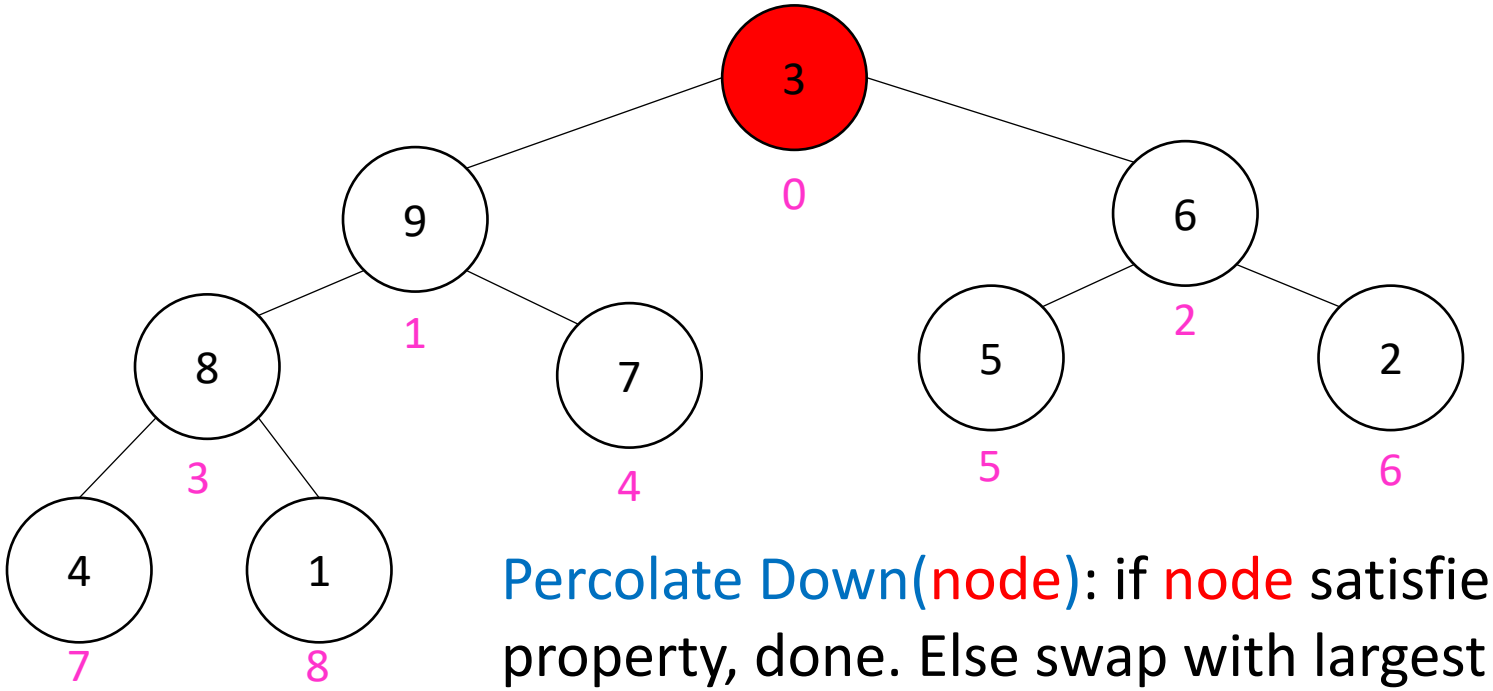
Max Heap Property: Each node is larger than its children



Heap Sort

- Remove the Max element (i.e. the root) from the Heap: replace with last element, call percolateDown(root)

3	9	6	8	7	5	2	4	1	
0	1	2	3	4	5	6	7	8	9

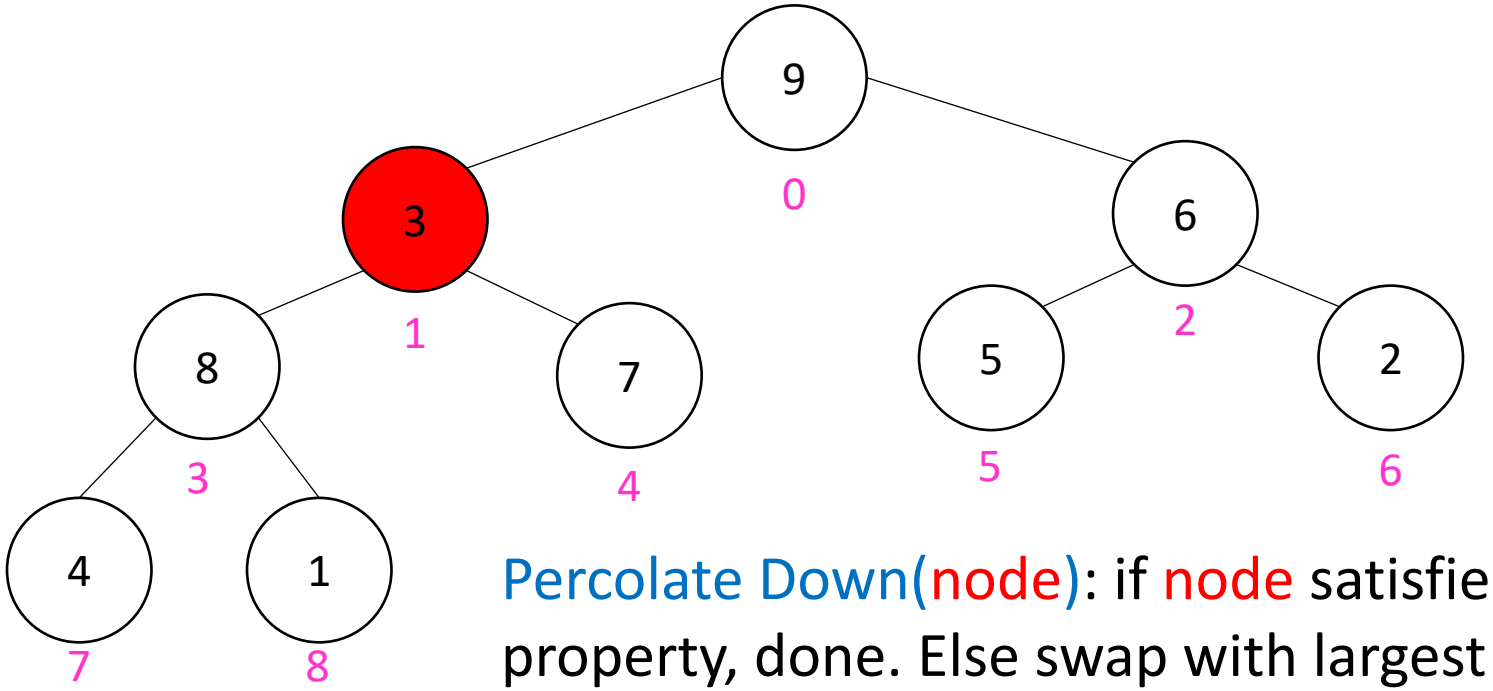


Percolate Down(node): if node satisfies heap property, done. Else swap with largest child and repeat on that subtree

Heap Sort

- Remove the Max element (i.e. the root) from the Heap: replace with last element, call percolateDown(root)

9	3	6	8	7	5	2	4	1	
0	1	2	3	4	5	6	7	8	9



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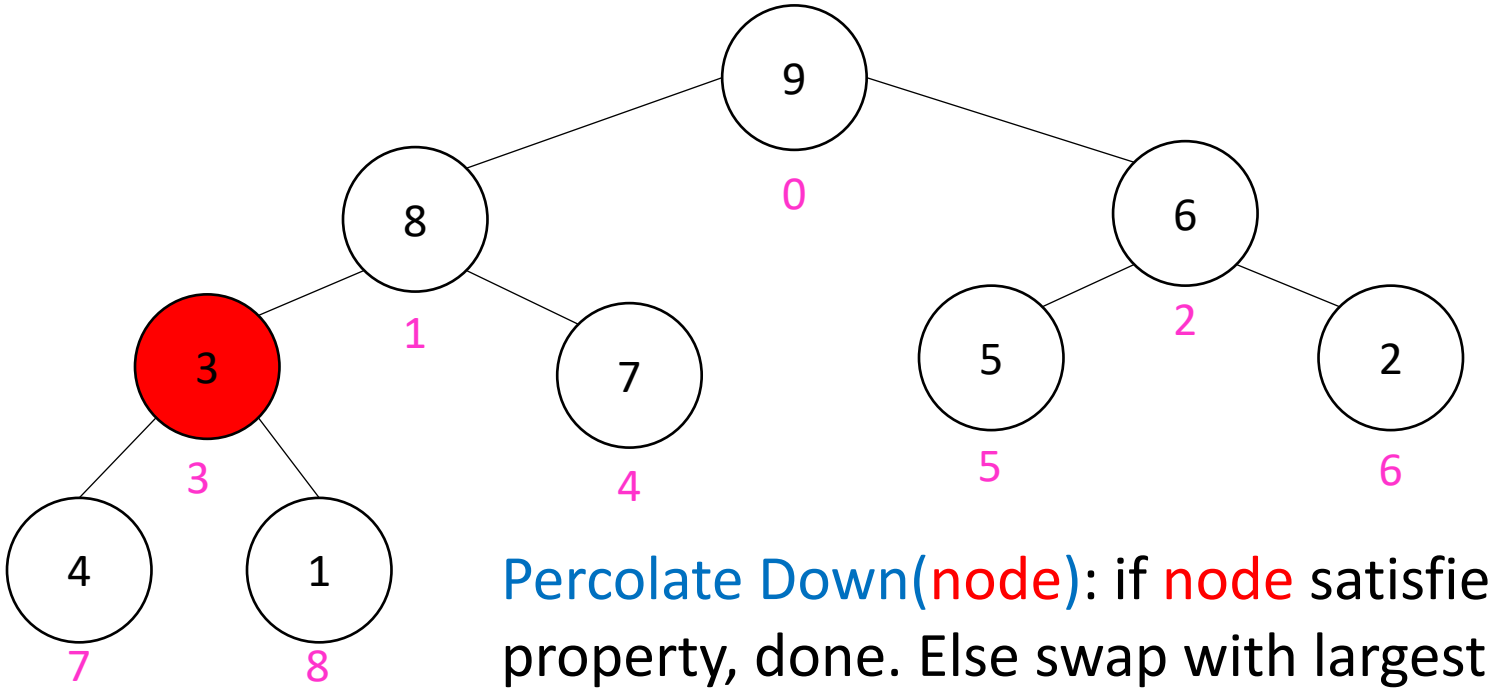

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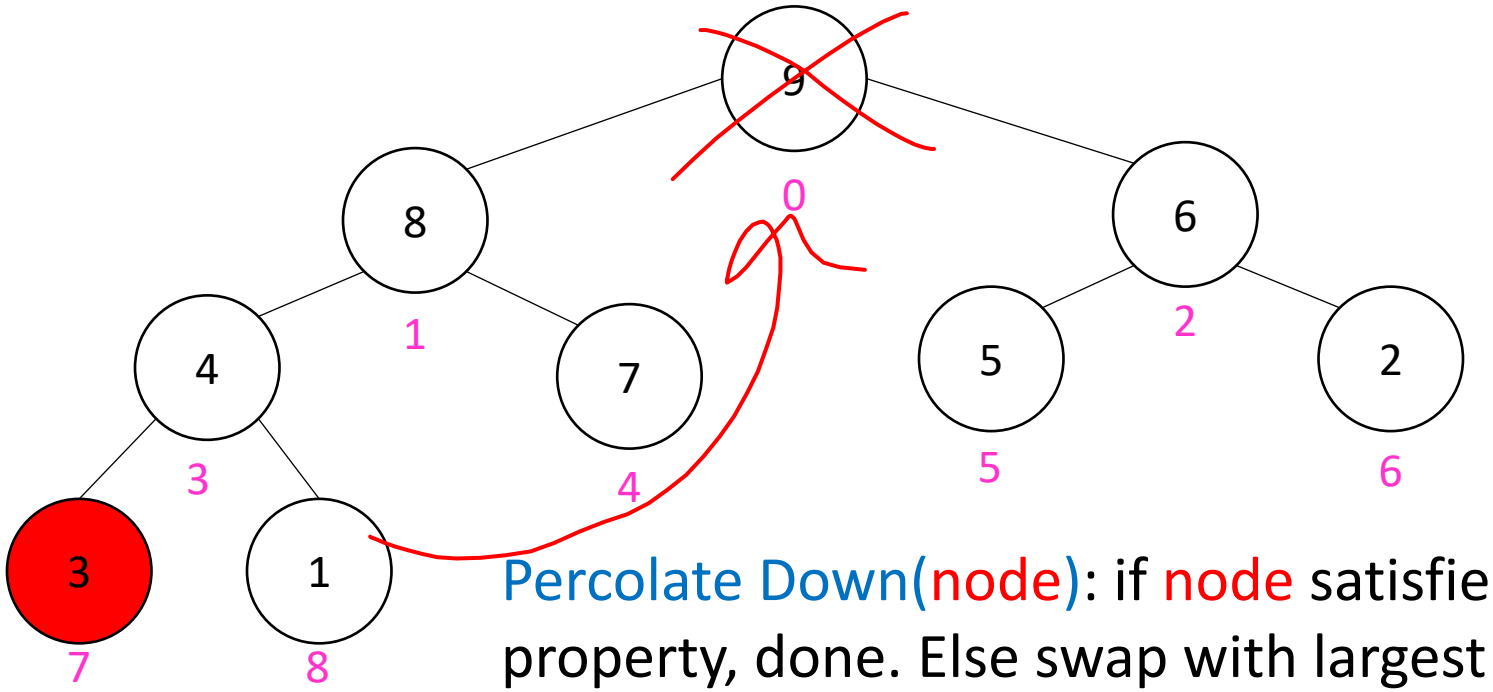


Percolate Down(node): if node satisfies heap property, done. Else swap with largest child and repeat on that subtree

Heap Sort

- Build a max heap
- Call extract
- Put that at the end of the array

```
myHeap = buildMaxHeap(a);
for (int i = a.length-1; i>=0; i--){
    item = myHeap.extract();
    a[i] = item;
}
```

Running Time:

Worst Case: $\Theta(\cdot)$ $n \log n$

Best Case: $\Theta(\cdot)$ n

“In Place” Sorting Algorithm

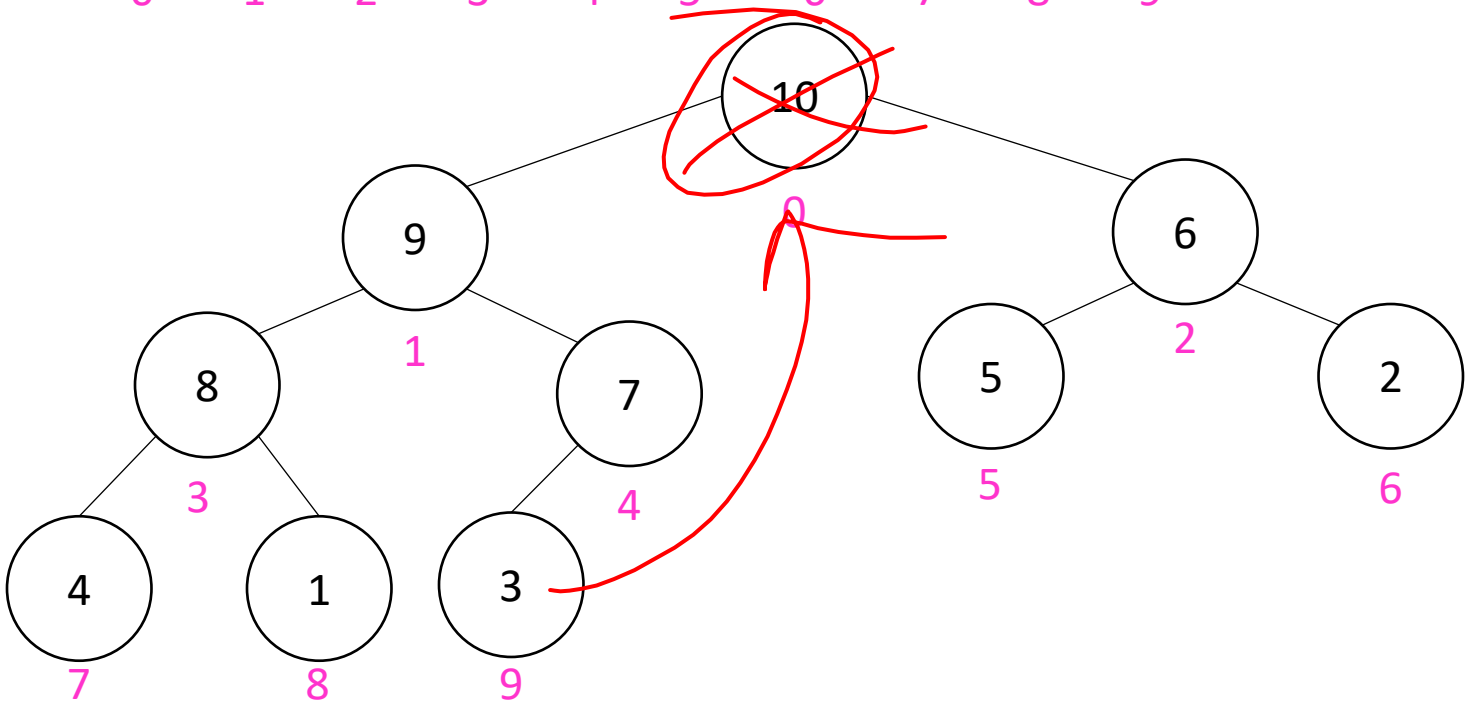
- A sorting algorithm which requires no extra data structures
- Idea: It sorts items just by swapping things in the same array given
- Definition: it only uses $\Theta(1)$ extra space

- Selection sort: In Place!
- Insertion sort: In Place!
- Heap sort: Not In Place!
  - But we can fix that!

In Place Heap Sort

- Idea: When “removing” an element from the heap, swap it with the last item of the heap then “pretend” the heap is one item shorter

10	9	6	8	7	5	2	4	1	3
0	1	2	3	4	5	6	7	8	9



Heap Sort

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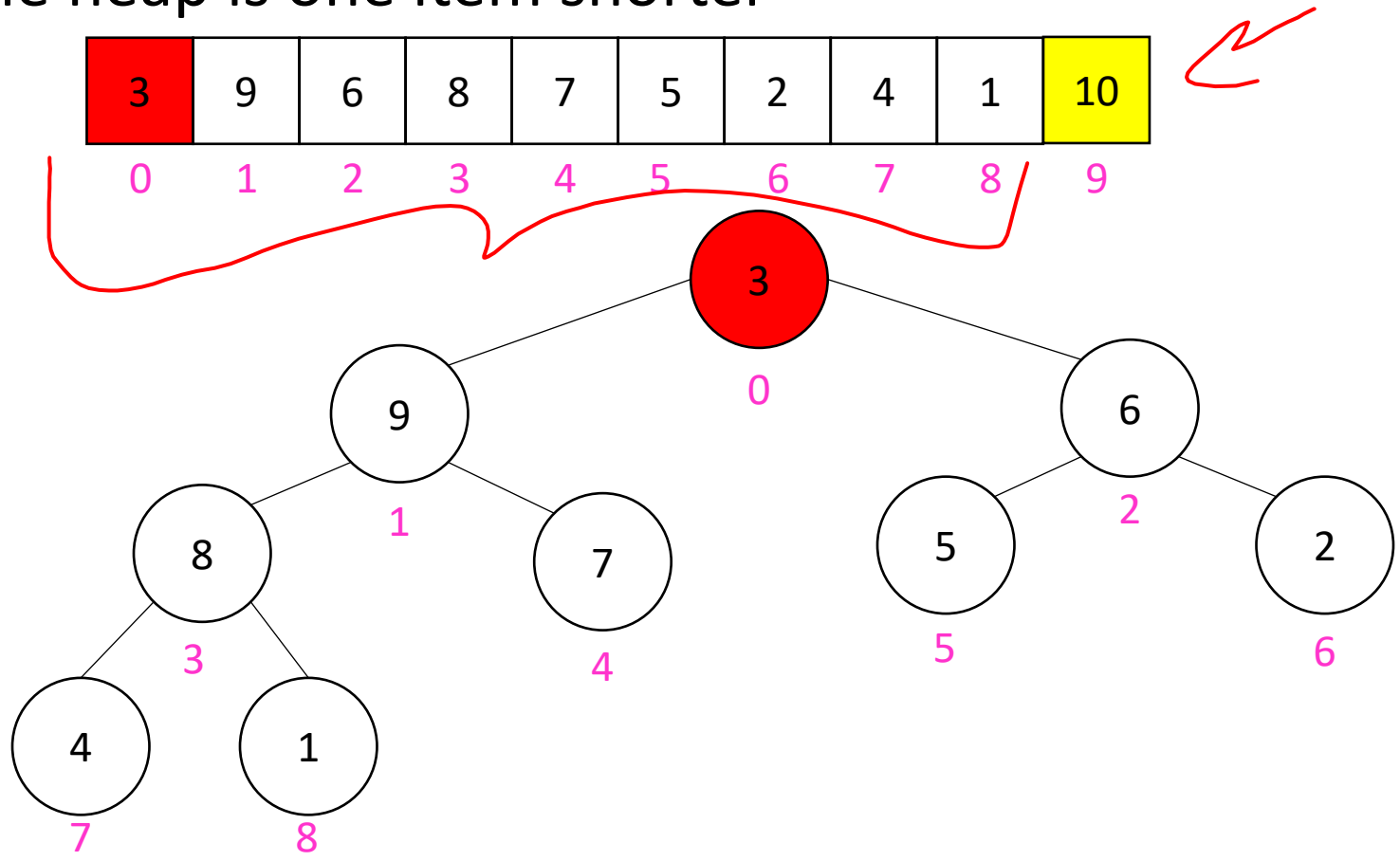
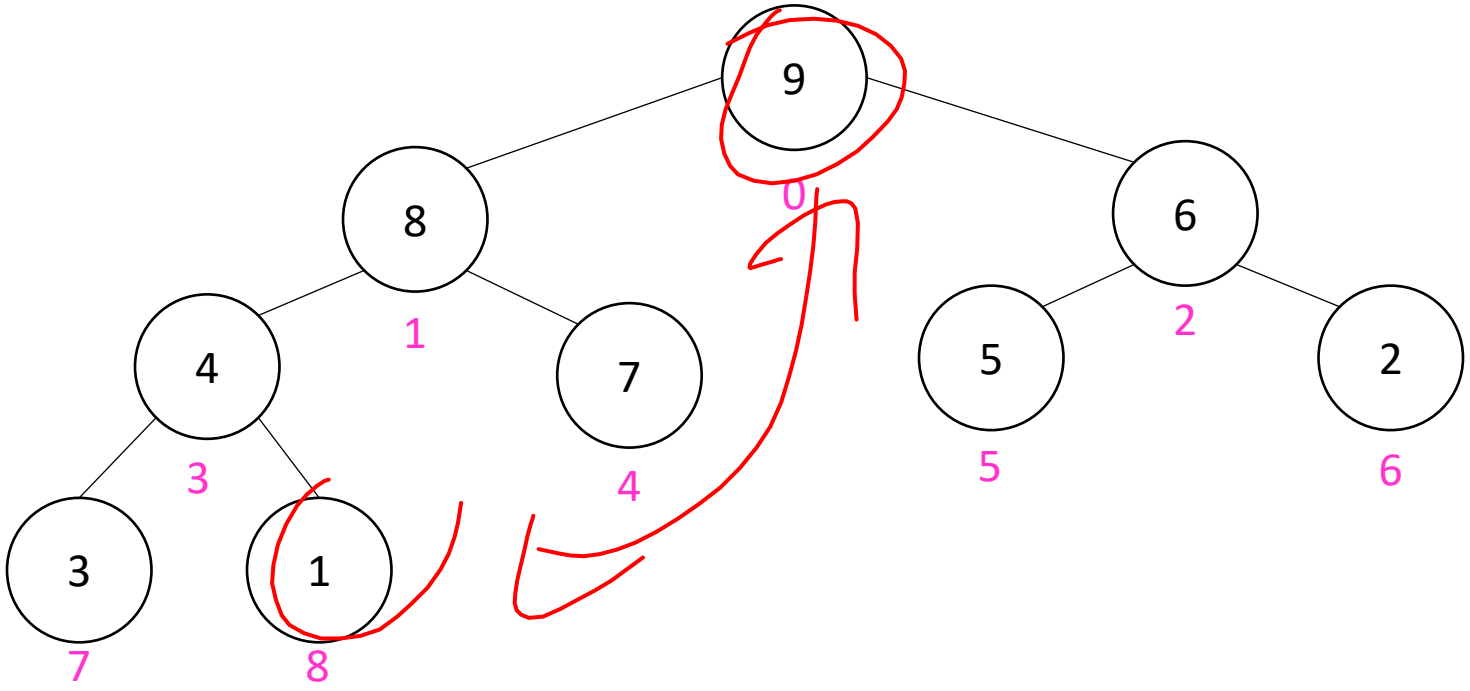


Heap Sort

- Idea: When “removing” an element from the heap, swap it with the last item of the heap then “pretend” the heap is one item shorter

9	8	6	4	7	5	2	3	1	10
0	1	2	3	4	5	6	7	8	9



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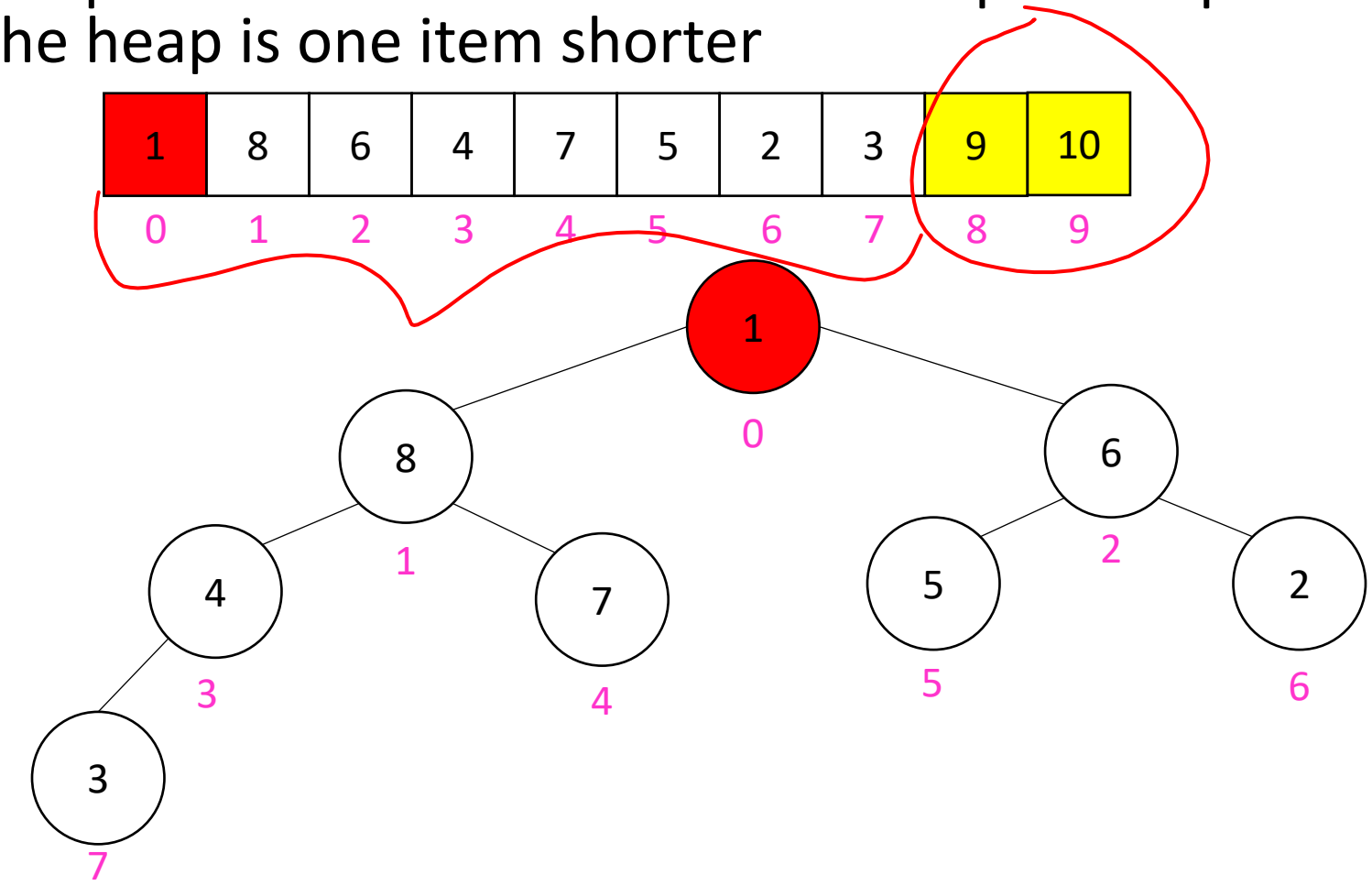
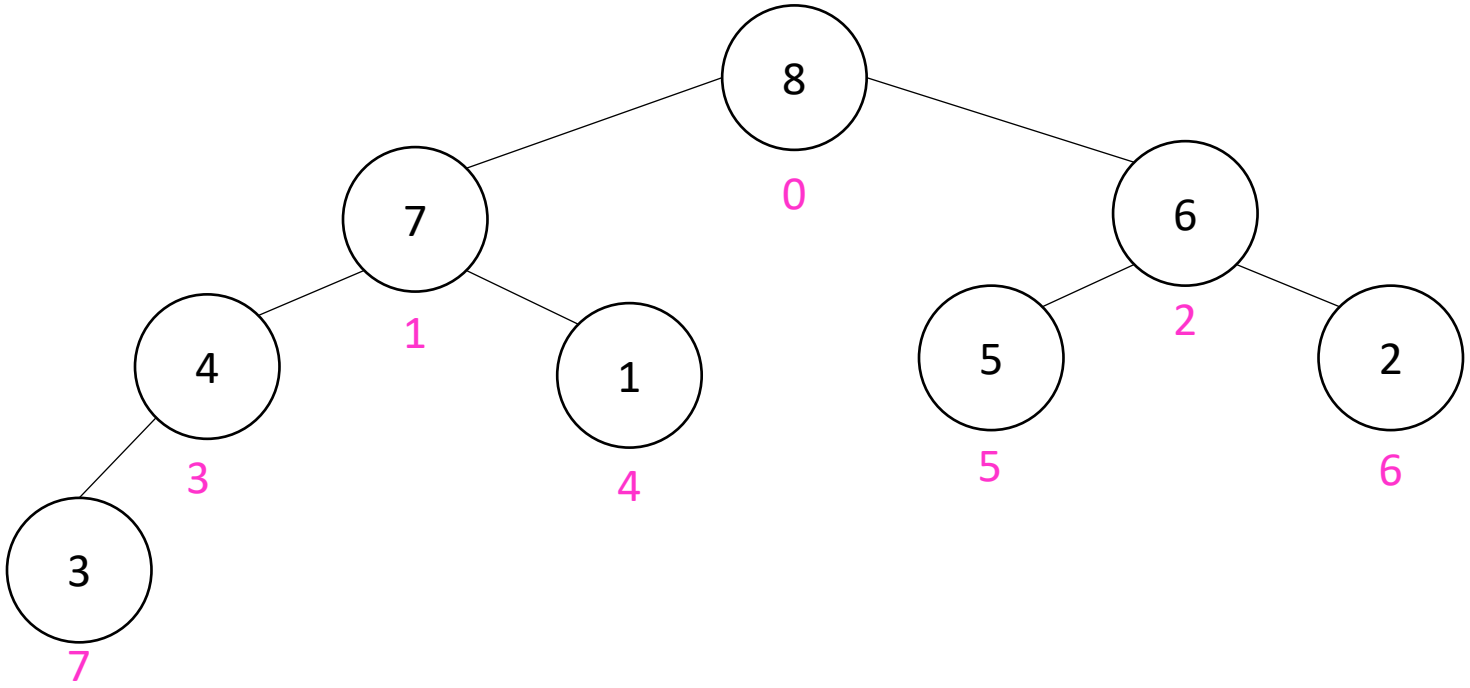


Heap Sort

- Idea: When “removing” an element from the heap, swap it with the last item of the heap then “pretend” the heap is one item shorter

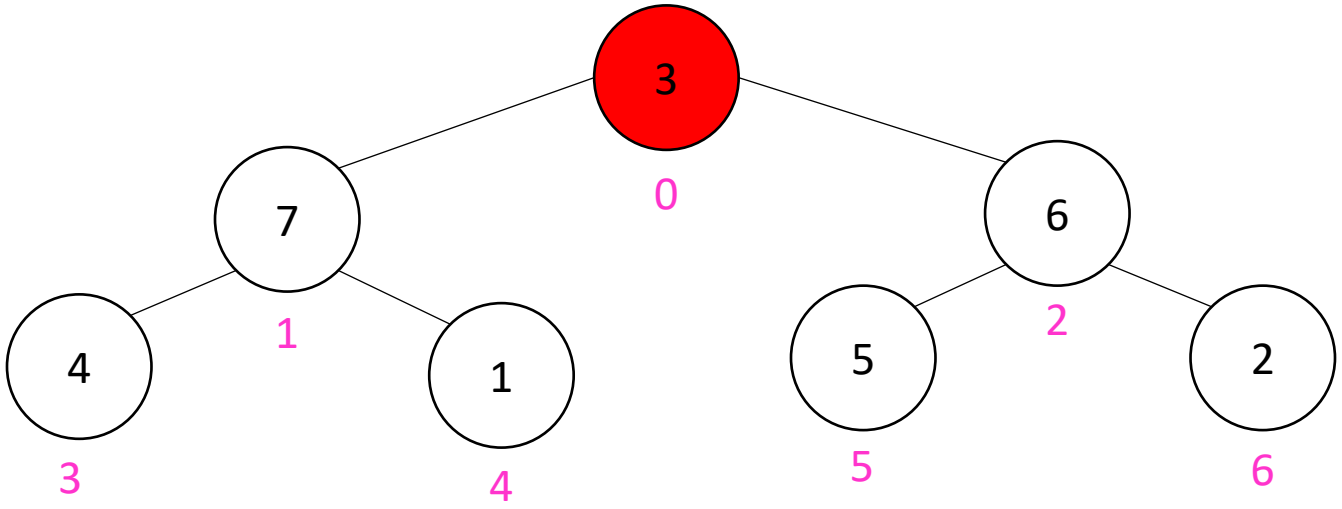
1	8	6	4	7	5	2	3	9	10
0	1	2	3	4	5	6	7	8	9



Heap Sort

- Idea: When “removing” an element from the heap, swap it with the last item of the heap then “pretend” the heap is one item shorter

3	8	6	4	7	5	2	8	9	10
0	1	2	3	4	5	6	7	8	9



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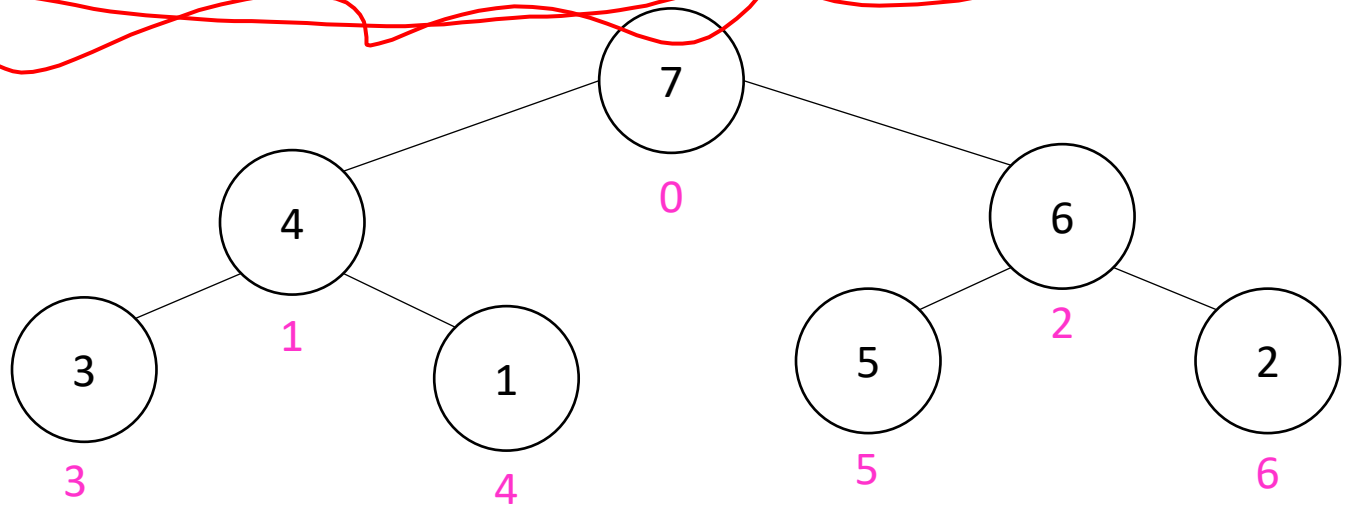


In Place Heap Sort

- Build a heap using the same array (Floyd's build heap algorithm works)
- Call deleteMax
- Put that at the end of the array

```
buildHeap(a);
for (int i = a.length-1; i>=0; i--){
    temp=a[i]
    a[i] = a[0];
    a[0] = temp;
    percolateDown(0);
}
```

Running Time:

Worst Case: $\Theta(\cdot)$

Best Case: $\Theta(\cdot)$

Floyd’s buildHeap method

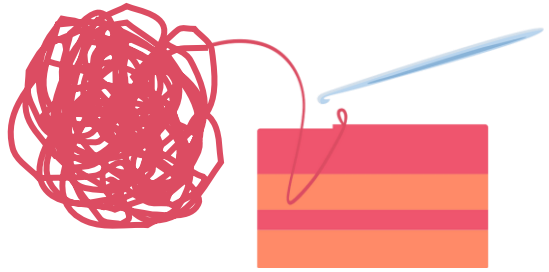
- Working towards the root, one row at a time, percolate down

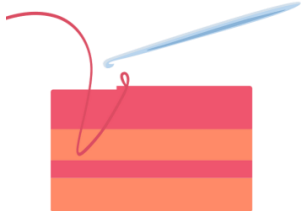
```
buildHeap(){
    for(int i = size; i>0; i--){
        percolateDown(i);
    }
}
```

Divide And Conquer Sorting

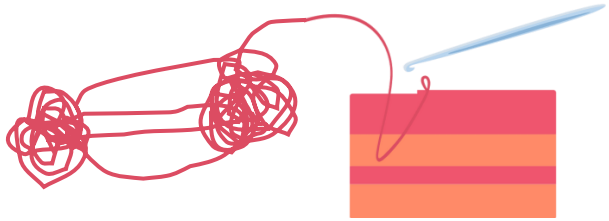
- Divide and Conquer:
  - Recursive algorithm design technique
  - Solve a large problem by breaking it up into smaller versions of the same problem

Divide and Conquer

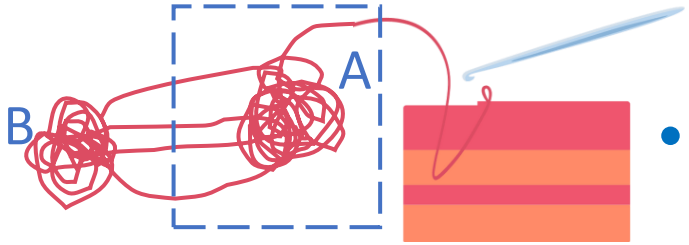




- **Base Case**:
  - If the problem is “small” then solve directly and return



- **Divide**:
  - Break the problem into subproblem(s), each smaller instances



- **Conquer**:
  - Solve subproblem(s) recursively

- **Combine**:
  - Use solutions to subproblems to solve original problem

Divide and Conquer Template Pseudocode

```
def my_DandC(problem){
   // Base Case
   if (problem.size() <= small_value){
      return solve(problem);  // directly solve (e.g., brute force)
   }
   // Divide
   List subproblems = divide(problem);

   // Conquer
   solutions = new List();
   for (sub : subproblems){
      subsolution = my_DandC(sub);
      solutions.add(subsolution);
   }
   // Combine
   return combine(solutions);
}
```

Merge Sort

5	8	2	9	4	1

5

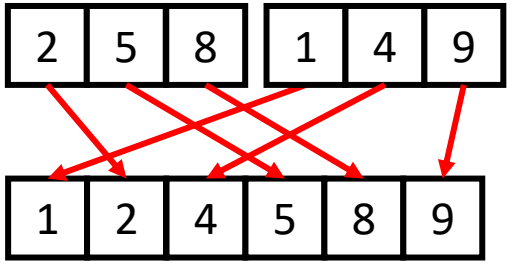
- **Base Case:**
  - If the list is of length 1 or 0, it's already sorted, so just return it

5	8	2	9	4	1

- **Divide:**
  - Split the list into two "sublists" of (roughly) equal length

2	5	8	1	4	9

- **Conquer:**
  - Sort both lists recursively



- **Combine:**
  - **Merge** sorted sublists into one sorted list

Merge Sort In Action!

Sort between indices low and $high$

5	8	2	9	4	1	3	7

low $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $high$

Base Case: if low == $high$ then that range is already sorted!

Divide and Conquer: Otherwise call mergesort on ranges $\left(low, \frac{low+high}{2}\right)$ and $\left(\frac{low+high}{2} + 1, high\right)$

5	8	2	9	4	1	3	7

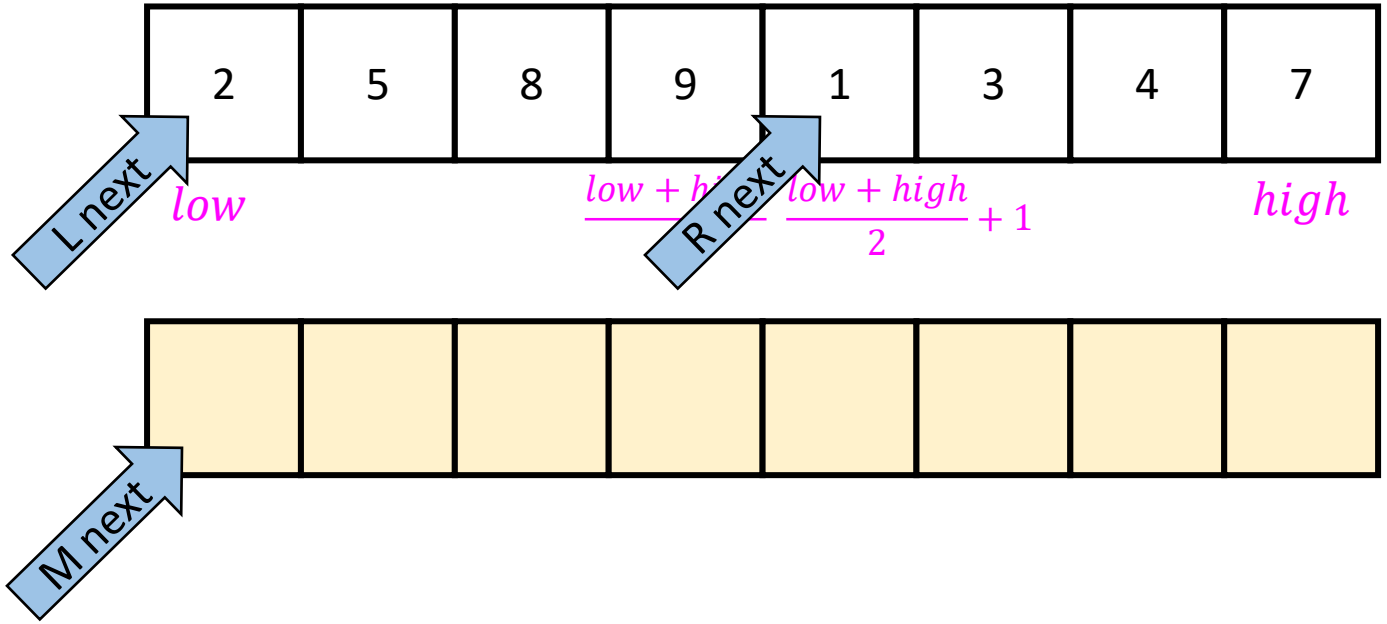
low $\qquad$ $\frac{low + high}{2}$ $\qquad$ $\frac{low + high}{2} + 1$ $\qquad$ $high$

After Recursion:

2	5	8	9	1	3	4	7

low $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $high$

Merge (the combine part)



Create a new array to merge into, and 3 pointers/indices:

- L_next: the smallest "unmerged" thing on the left
- R_next: the smallest "unmerged" thing on the right
- M_next: where the next smallest thing goes in the merged array

One-by-one: put the smallest of L_next and R_next into M_next,
then advance both M_next and whichever of L/R was used.

Merge Sort Pseudocode

```
void mergesort(myArray){
        ms_helper(myArray, 0, myArray.length());
}
void mshelper(myArray, low, high){
        if (low == high){return;}  // Base Case
        mid = (low+high)/2;
        ms_helper(low, mid);
        ms_helper(mid+1, high);
        merge(myArray, low, mid, high);
}
```

Merge Pseudocode

```
void merge(myArray, low, mid, high){
        merged = new int[high-low+1]; // or whatever type is in myArray
        l_next = low;
        r_next = high;
        m_next = 0;
        while (l_next <= mid && r_next <= high){
                if (myArray[l_next] <= myArray[r_next]){
                        merged[m_next++] = myArray[l_next++];
                }
                else{
                        merged[m_next++] = myArray[r_next++];
                }
        }
        while (l_next <= mid){ merged[m_next++] = myArray[l_next++]; }
        while (r_next <= high){  merged[m_next++] = myArray[r_next++]; }
        for(i=0; i<=merged.length; i++){ myArray[i+low] = merged[i];}
}
```

Analyzing Merge Sort

1. Identify time required to Divide and Combine
2. Identify all subproblems and their sizes
3. Use recurrence relation to express recursive running time
4. Solve and express running time asymptotically

- **Divide**: 0 comparisons
- **Conquer**: recursively sort two lists of size $\frac{n}{2}$
- **Combine**: n comparisons
- **Recurrence:**

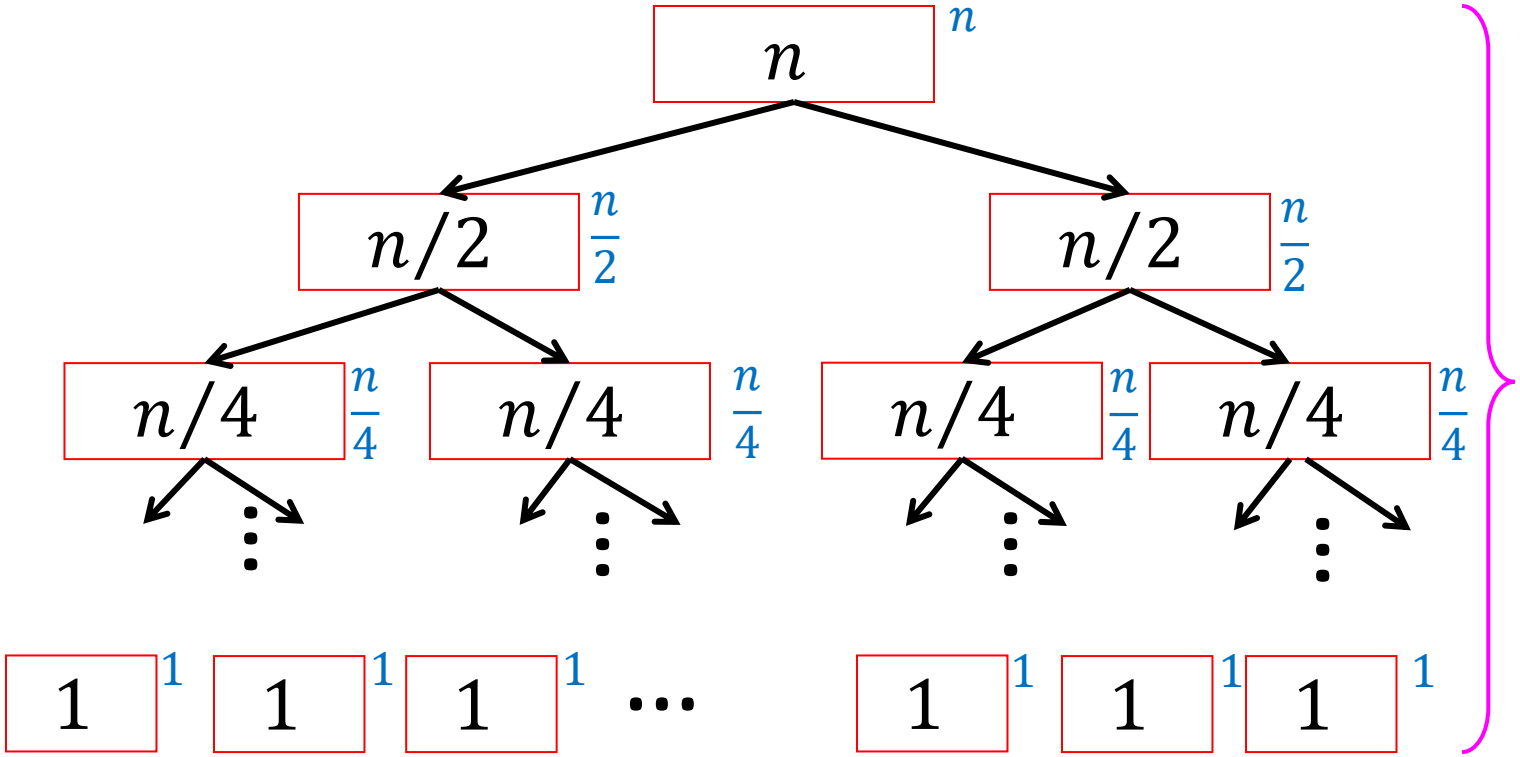
$$T(n) = 0 + T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + n$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Red box represents a problem instance

Blue value represents time spent at that level of recursion

$$T(n) = 2T(\frac{n}{2}) + n$$



$\Rightarrow n$ comparisons / level

$\log_2 n$ levels of recursion

$$T(n) = \sum_{i=1}^{\log_2 n} n = n \log_2 n$$

Properties of Merge Sort

- Worst Case Running time:
  - $\Theta(n \log n)$
- In-Place?
  - No!
- Adaptive?
  - No!
- Stable?
  - Yes!
  - As long as in a tie you always pick l_next

Quicksort

- Like Mergesort:
  - Divide and conquer
  - $O(n \log n)$ run time (kind of…)
- Unlike Mergesort:
  - Divide step is the “hard” part
  - *Typically* faster than Mergesort

Quicksort

Idea: pick a pivot element, recursively sort two sublists around that element

- Divide: select pivot element p, Partition(p)
- Conquer: recursively sort left and right sublists
- Combine: Nothing!

Partition (Divide step)

Given: a list, a pivot p

Start: unordered list

8	5	7	3	12	10	1	2	4	9	6	11

Goal: All elements $< p$ on left, all $> p$ on right

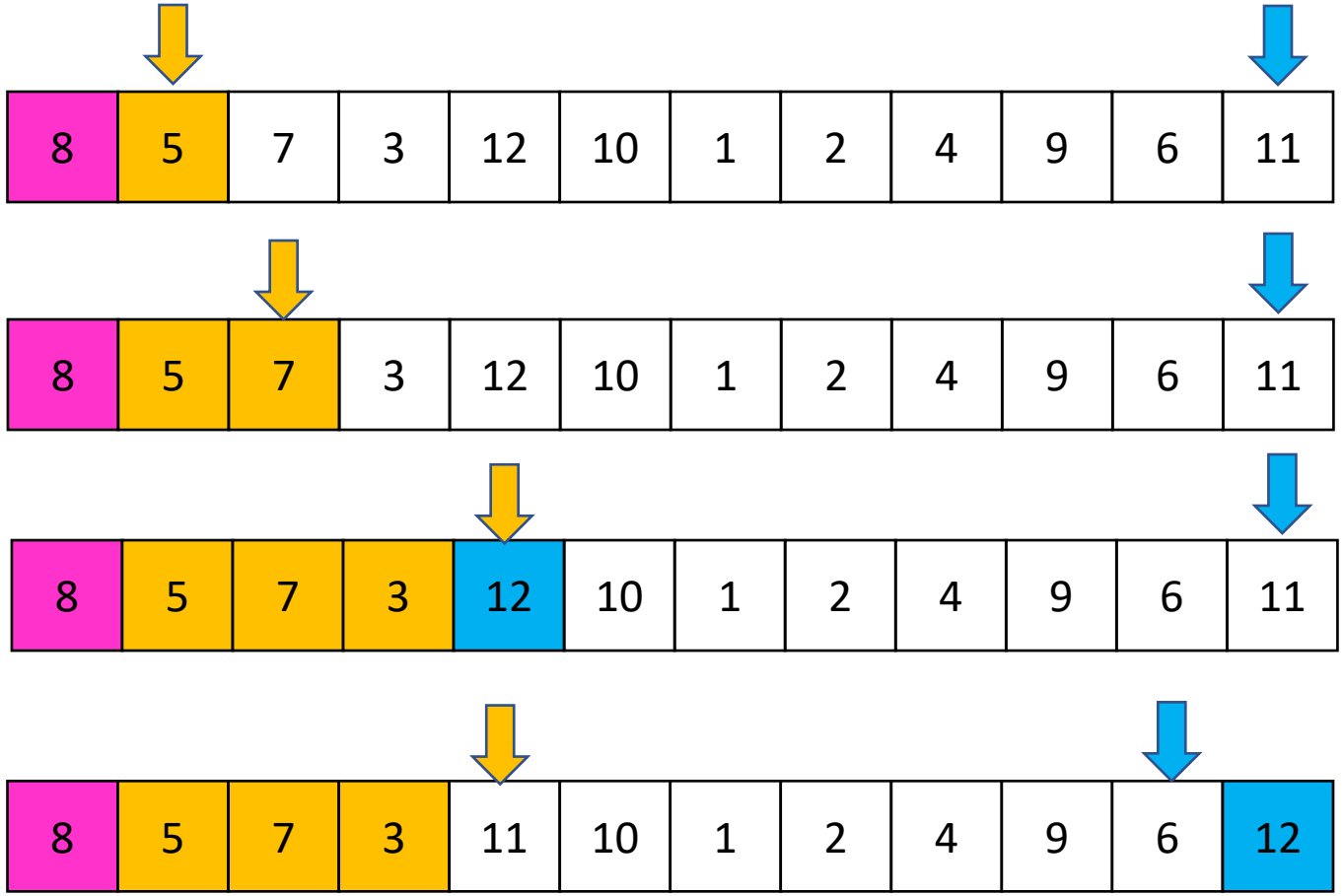
5	7	3	1	2	4	6	8	12	10	9	11

Partition, Procedure

If Begin value $< p$, move Begin right

Else swap Begin value with End value, move End Left

Done when Begin = End

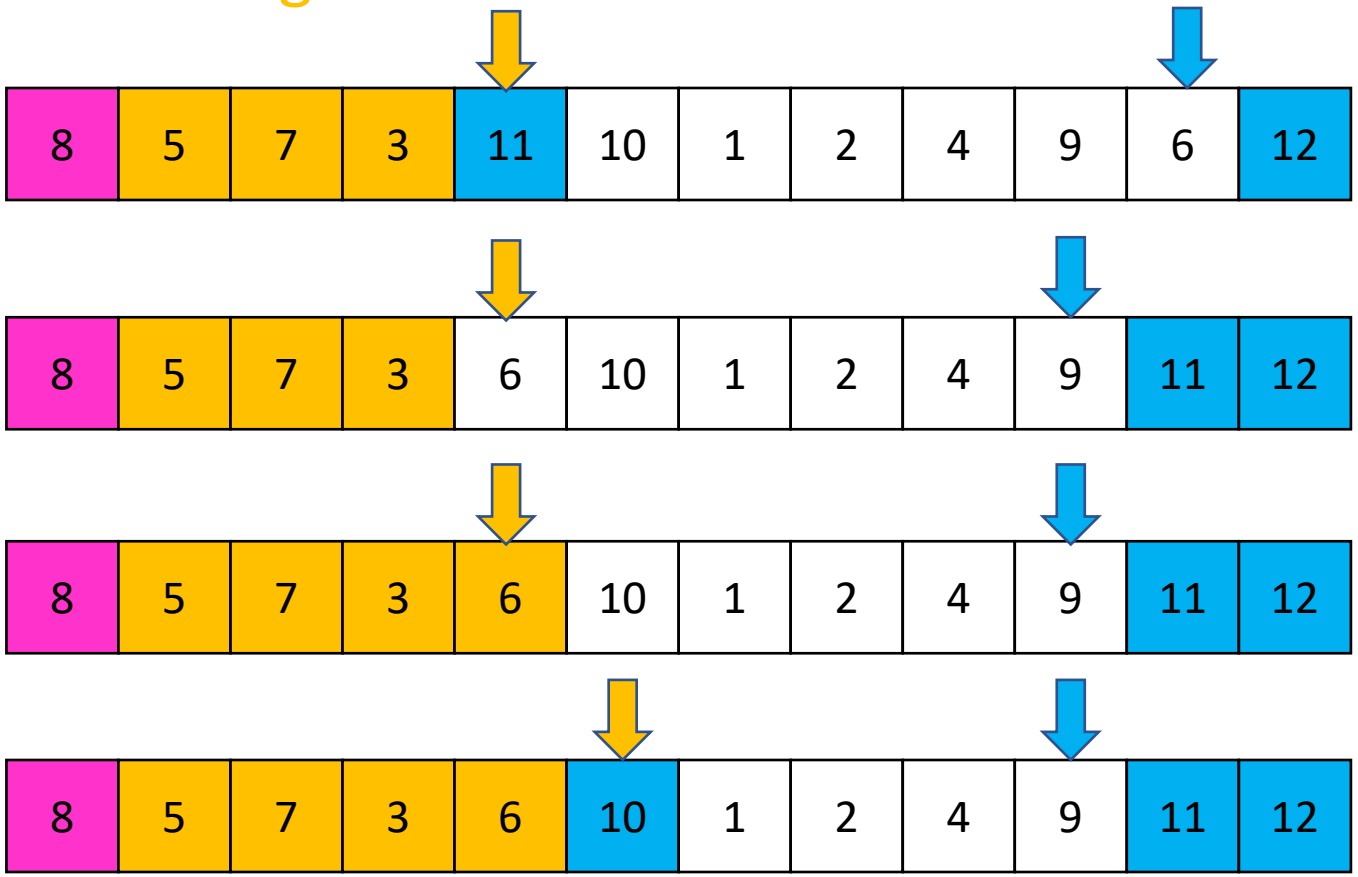


Partition, Procedure

If Begin value $< p$, move Begin right

Else swap Begin value with End value, move End Left

Done when Begin = End

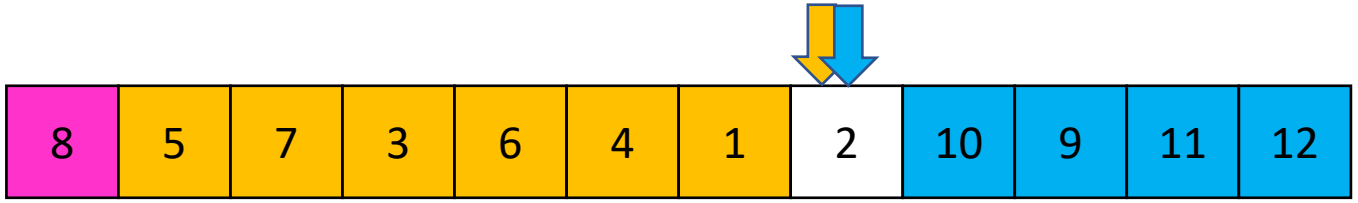


Partition, Procedure

If Begin value $< p$, move Begin right

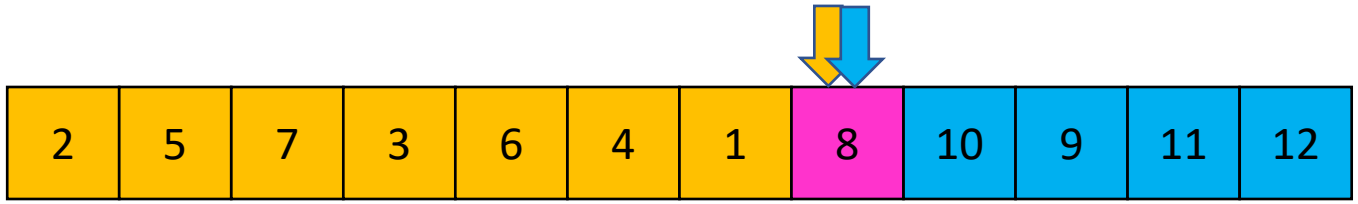
Else swap Begin value with End value, move End Left

Done when Begin = End



Case 1: meet at element $< p$

Swap p with pointer position (2 in this case)

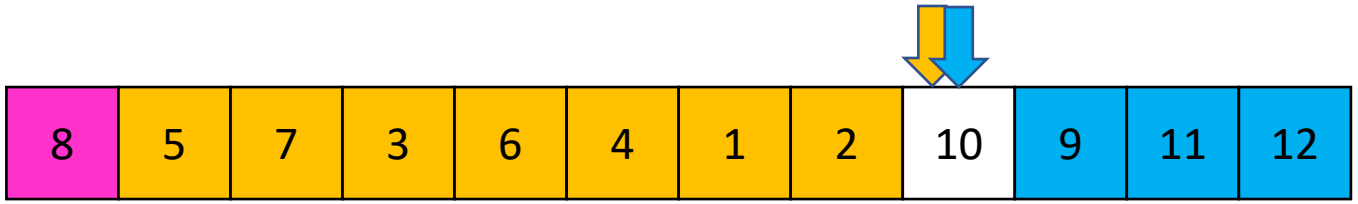


Partition, Procedure

If Begin value $<$ p, move Begin right

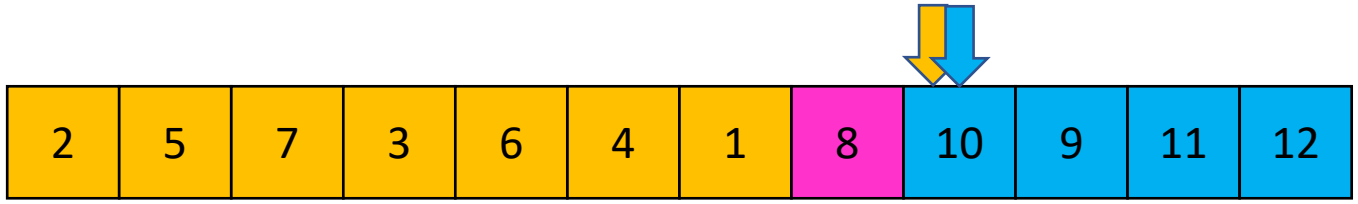
Else swap Begin value with End value, move End Left

Done when Begin = End



Case 2: meet at element $> p$

Swap p with value to the left (2 in this case)

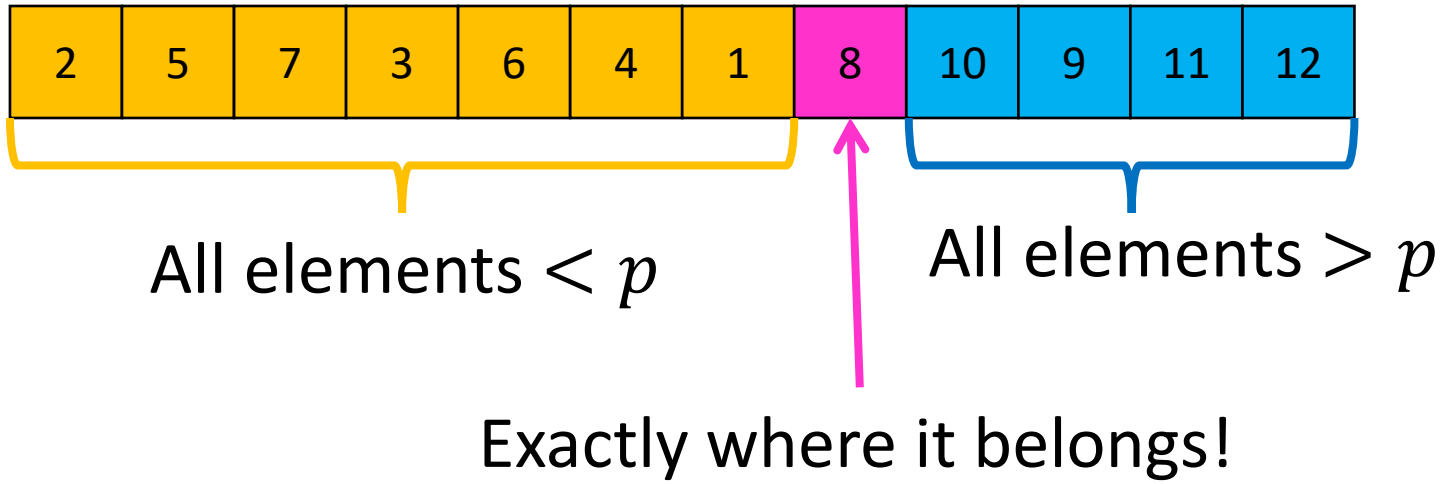


Partition Summary

1. Put p at beginning of list
2. Put a pointer (Begin) just after p, and a pointer (End) at the end of the list
3. While Begin < End:
   1. If Begin value $< p$, move Begin right
   2. Else swap Begin value with End value, move End Left
4. If pointers meet at element $< p$: Swap p with pointer position
5. Else If pointers meet at element $> p$: Swap p with value to the left

Run time? $O(n)$

Conquer



Recursively sort Left and Right sublists

Quicksort Run Time (Best)

If the pivot is always the median:

2	5	1	3	6	4	7	8	10	9	11	12

2	1	3	5	6	4	7	8	9	10	11	12

Then we divide in half each time

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$T(n) = O(n \log n)$$

Quicksort Run Time (Worst)

If the pivot is always at the extreme:

1	5	2	3	6	4	7	8	10	9	11	12

1	2	3	5	6	4	7	8	10	9	11	12

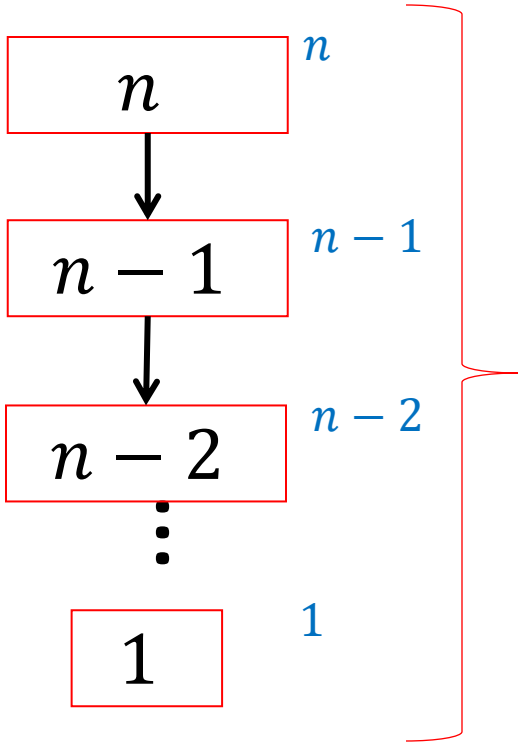
Then we shorten by 1 each time

$$T(n) = T(n-1) + n$$

$$T(n) = O(n^2)$$

Quicksort Run Time (Worst)

$$T(n) = T(n-1) + n$$



$$T(n) = 1 + 2 + 3 + \cdots + n$$

$$T(n) = \frac{n(n+1)}{2}$$

$$T(n) = O(n^2)$$

Quicksort on a (nearly) Sorted List

First element always yields unbalanced pivot

1	2	3	4	5	6	7	8	9	10	11	12

1	2	3	4	5	6	7	8	9	10	11	12

So we shorten by 1 each time

$$T(n) = T(n-1) + n$$

$$T(n) = O(n^2)$$

Good Pivot

- What makes a good Pivot?
  - Roughly even split between left and right
  - Ideally: median
- There are ways to find the median in linear time, but it’s complicated and slow and you’re better off using mergesort
- In Practice:
  - Pick a random value as a pivot
  - Pick the middle of 3 random values as the pivot

Properties of Quick Sort

- Worst Case Running time:
  - $\Theta(n^2)$
  - But $\Theta(n \log n)$ average! And typically faster than mergesort!
- In-Place?
  - ….Debatable
- Adaptive?
  - No!
- Stable?
  - No!

Improving Running time

- Recall our definition of the sorting problem:
  - Input:
    - An array A of items
    - A comparison function for these items
      - Given two items x and y, we can determine whether $x < y$, $x > y$, or $x = y$
  - Output:
    - A permutation of A such that if $i \leq j$ then $A[i] \leq A[j]$
- Under this definition, it is impossible to write an algorithm faster than $n \log n$ asymptotically.
- Observation:
  - Sometimes there might be ways to determine the position of values without comparisons!