

CSE 332 Winter 2026

Lecture 6: Recurrences

Nathan Brunelle

<http://www.cs.uw.edu/332>

Recursive Binary Search

```
public static boolean binarySearch(List<Integer> lst, int k){  
    return binarySearch(lst, k, 0, lst.size());  
}  
  
private static boolean binarySearch(List<Integer> lst, int k, int start, int end){  
    if(start == end)  
        return false;  
    int mid = start + (end-start)/2;  
    if(lst.get(mid) == k){  
        return true;  
    } else if(lst.get(mid) > k){  
        return binarySearch(lst, k, start, mid);  
    } else{  
        return binarySearch(lst, k, mid+1, end);  
    }  
}
```

5	8	13	42	75	79	88	90	95	99
0	1	2	3	4	5	6	7	8	9

$$T(n) = T\left(\frac{n}{2}\right)$$

$$T(1) = 1$$

Analysis of Recursive Algorithms

- Overall structure of recursion:
 - Do some non-recursive “work”
 - Do one or more recursive calls on some portion of your input
 - Do some more non-recursive “work”
 - Repeat until you reach a base case
- Running time: $T(n) = T(p_1) + T(p_2) + \dots + T(p_x) + f(n)$
 - The time it takes to run the algorithm on an input of size n is:
 - The sum of how long it takes to run the same algorithm on each smaller input
 - Plus the total amount of non-recursive work done in that stack frame
- Usually:
 - $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$
 - Called “divide and conquer”
 - $T(n) = T(n - c) + f(n)$
 - Called “chip and conquer”

How Efficient Is It?

- $T(\cancel{n}) = 1 + T\left(\left\lceil \frac{n}{2} \right\rceil\right)$
- Base case: $T(1) = 1$

$T(n)$ = “cost” of running the entire algorithm on an array of length n

$$T(\text{odd}-1) \leq T(\text{odd}) \leq T(2 \cdot \text{odd})$$

Let's Solve the Recurrence!

$$T(1) = 1$$

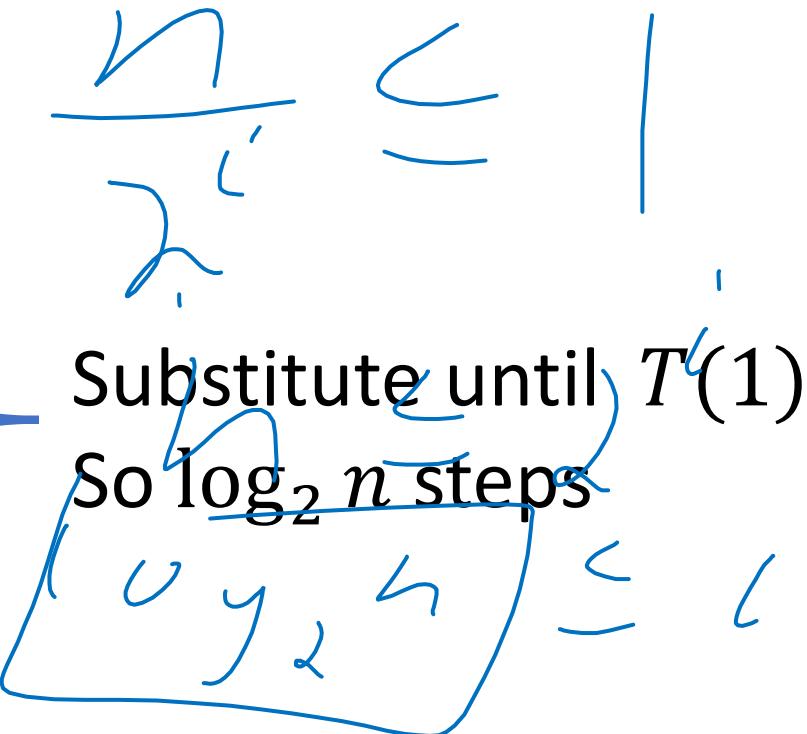
$$T(n) = 1 + T(\cancel{\frac{n}{2}})$$

$$1 + T(\cancel{\frac{n}{4}})$$

$$1 + T(\cancel{\frac{n}{8}})$$

⋮

1



$$T(n) = \sum_{i=1}^{\log_2 n} 1 = \log_2 n$$

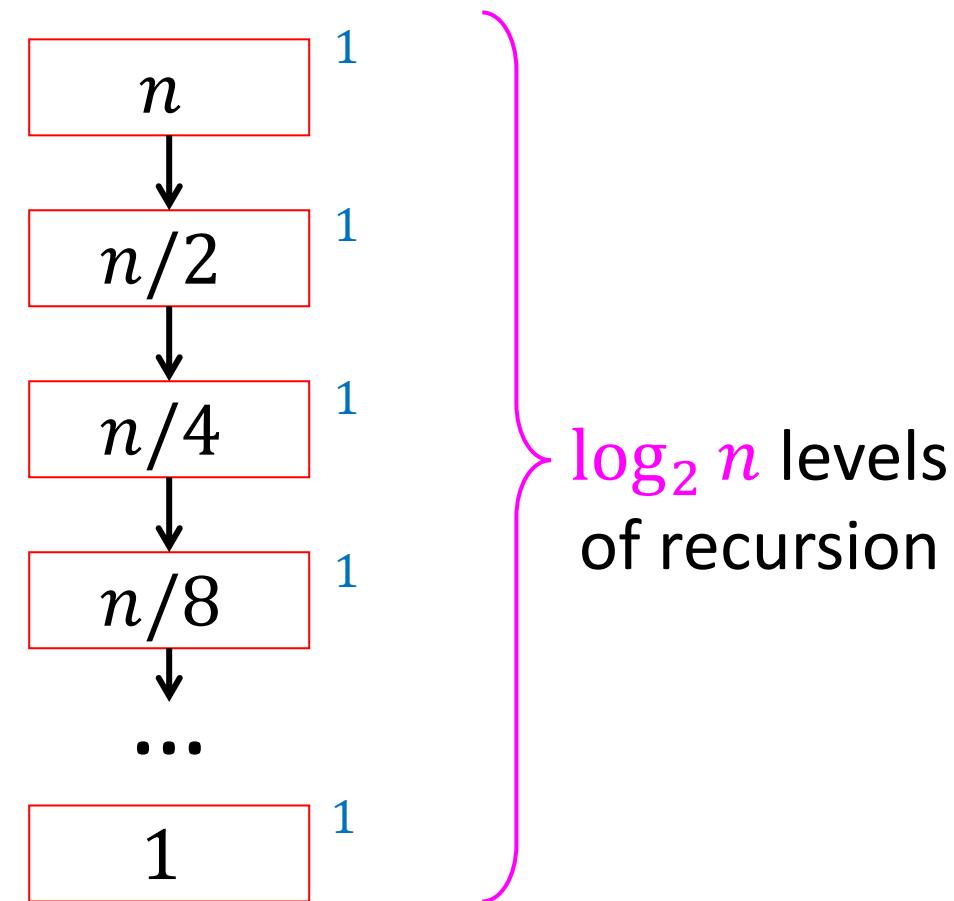
$$T(n) \in \Theta(\log n)$$

Make our process “prettier”

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

- Draw a picture of the recursion
- Identify the work done per stack frame
- Add up all the work!
 - Sum is the answer!
 - In this case $\Theta(\log_2 n)$

The “Tree Method”



Recursive Linear Search

5	8	13	42	75	79	88	90	95	99
0	1	2	3	4	5	6	7	8	9

```
public static boolean linearSearch(List<Integer> lst, int k){  
    return linearSearch(lst, k, 0, lst.size());  
}  
  
private static boolean linearSearch(List<Integer> lst, int k, int start, int end){  
    if(start == end){  
        return false;  
    } else if(lst.get(start) == k){  
        return true;  
    } else{  
        return linearSearch(lst, k, start+1, end);  
    }  
}
```

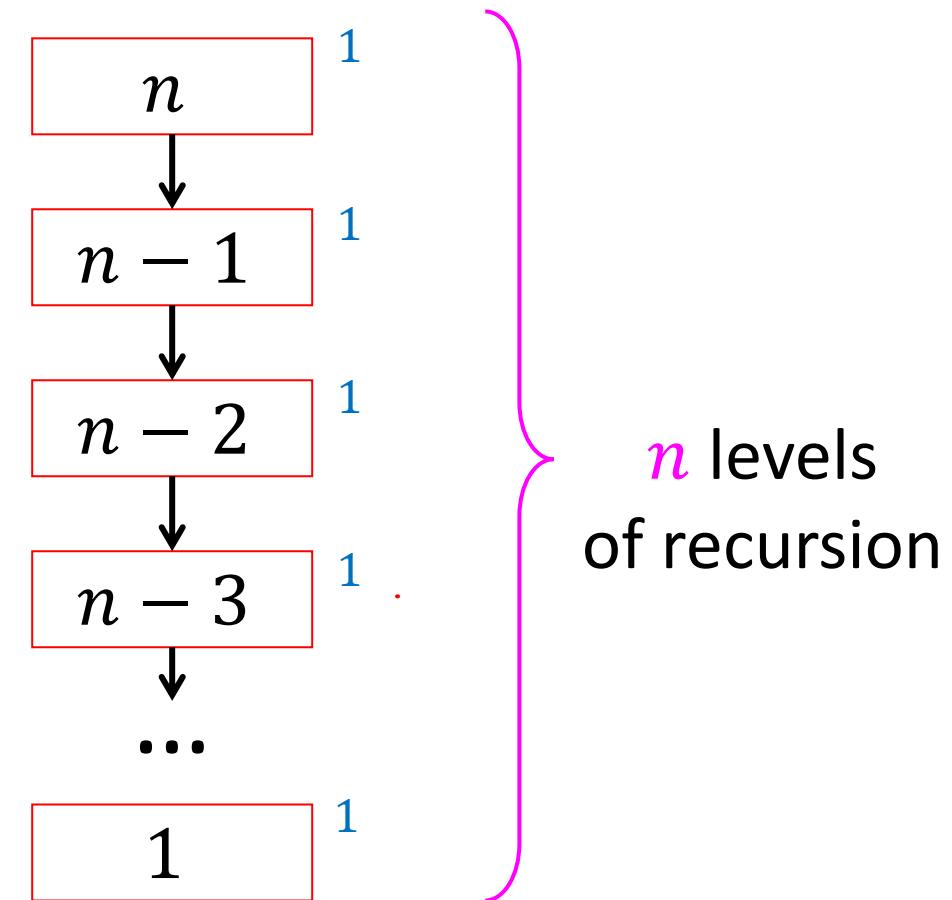
$$T(n) = T(n-1) + 1$$

Make our method “prettier”

$$T(n) = T(n - 1) + \underline{1}$$

- Draw a picture of the recursion
- Identify the work done per stack frame
- Add up all the work!

Running time: $\Theta(n)$



Recursive List Summation

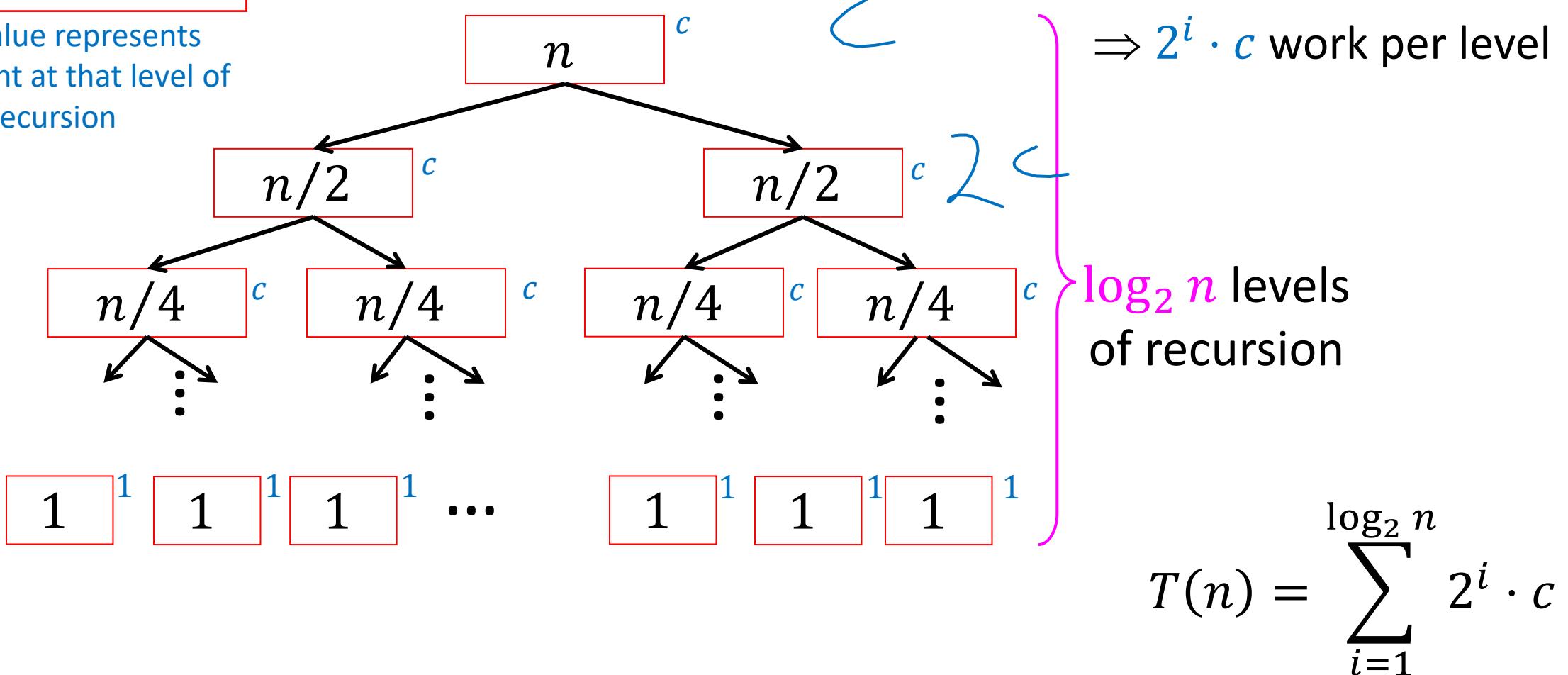
```
public int sum(int[] list){  
    return sum_helper(list, 0, list.size());  
}  
  
private int sum_helper(int[] list, int low, int high){  
    if (low == high){ return 0; }  
    if (low == high-1){ return list[low]; }  
    int middle = (high+low)/2;  
    return sum_helper(list, low, middle) + sum_helper(list, middle, high);  
}
```

$$T(n) = 2T\left(\frac{n}{2}\right) + 1$$

Tree Method

Red box represents a problem instance

Blue value represents time spent at that level of recursion



Recursive List Summation

$$T(n) = \sum_{i=1}^{\log_2 n} 2^i \cdot c$$

$$= c \cdot \sum_{i=1}^{\log_2 n} 2^i$$

$$= c \left(\frac{1 - 2^{\log_2 n}}{1 - 2} \right)$$

$$= c(n - 1) = \Theta(n)$$

Tree Method Summary: Chip and Conquer

- Recurrence looks like $T(n) = aT(n - b) + f(n)$
- Use the recurrence to draw a tree
 - a is the branching factor of the tree (e.g. if $a = 2$ then it's a binary tree)
 - Subtract b from the parent's input size to get children's input size
 - Work done per node is given by applying $f(n)$ to that node's input size
 - Height of the tree is $\frac{n}{b}$
 - Because that is the number of times we must subtract b until reaching a base case
 - Answer to the question "how many times must we subtract b until we reach 0?"
 - Any base case is a constant, so to reach a larger value would just be a constant change
- Use the tree to express running time as a series
 - Adding work done for each node level-by-level
 - Identify a pattern to express work done at level i as a function of i
 - Write a series using $i = 0$ up to $\frac{n}{b}$
- Solve the series

Tree Method Summary: Divide and Conquer

- Recurrence looks like $T(n) = aT\left(\frac{n}{b}\right) + f(n)$
- Use the recurrence to draw a tree
 - a is the branching factor of the tree (e.g. if $a = 2$ then it's a binary tree)
 - Divide the parent's input size by b to get children's input size
 - Work done per node is given by applying $f(n)$ to that node's input size
 - Height of the tree is $\log_b n$
 - Because that is the number of times we must divide by b until reaching a base case
 - Answer to the question "how many times must we divide by b until we reach 1?"
 - Any base case is a constant, so to reach a larger value would just be a constant change
- Use the tree to express running time as a series
 - Adding work done for each node level-by-level
 - Identify a pattern to express work done at level i as a function of i
 - Write a series using $i = 0$ up to $\log_b n$
- Solve the series

Let's do some more!

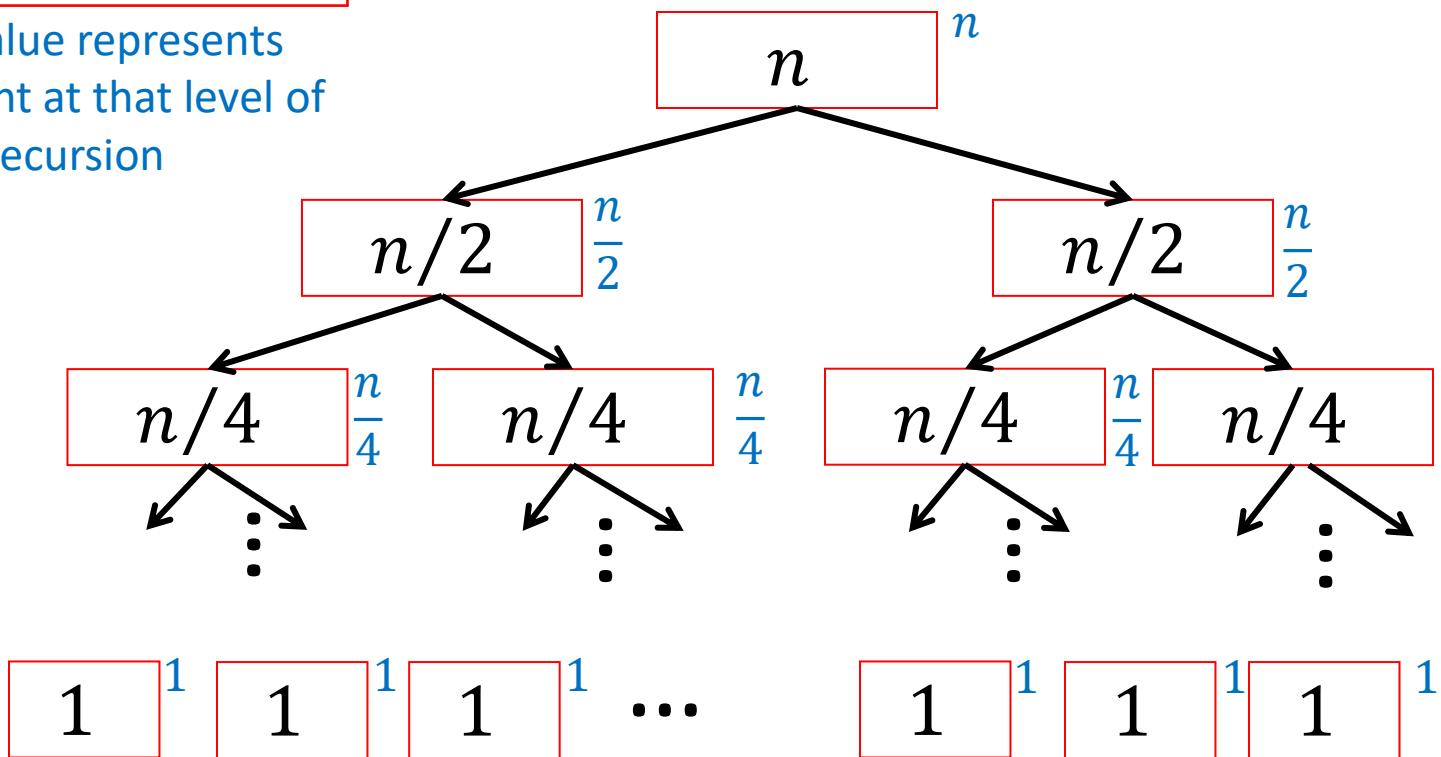
- For each, assume the base case is $n = 1$ and $T(1) = 1$
- $T(n) = 2T\left(\frac{n}{2}\right) + n$
- $T(n) = 2T\left(\frac{n}{2}\right) + n^2$
- $T(n) = 2T\left(\frac{n}{8}\right) + 1$

Tree Method

Red box represents a problem instance

Blue value represents time spent at that level of recursion

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



$\Rightarrow n$ work per level

$\log_2 n$ levels
of recursion

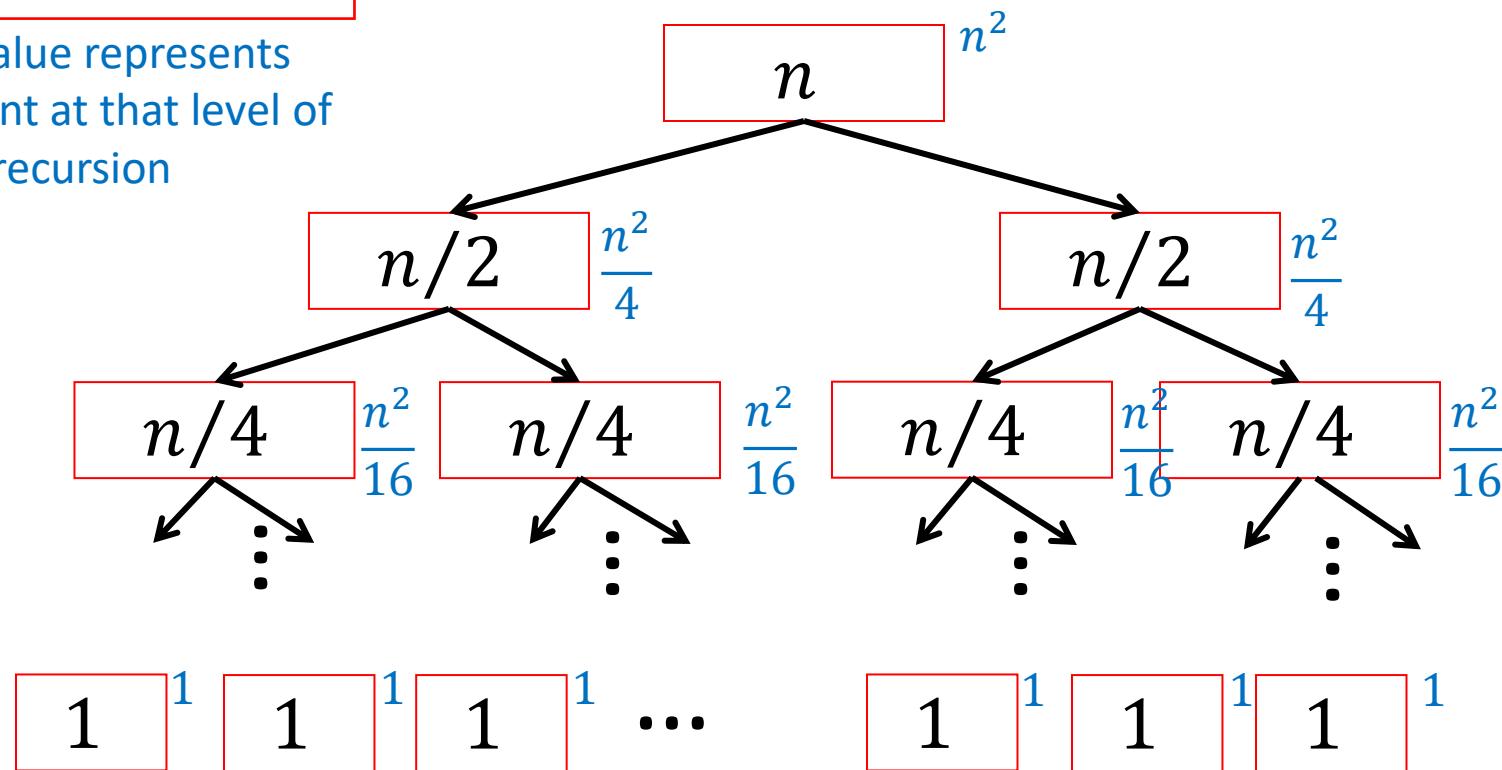
$$T(n) = \sum_{i=1}^{\log_2 n} n$$

Tree Method

Red box represents a problem instance

Blue value represents time spent at that level of recursion

$$T(n) = 2T\left(\frac{n}{2}\right) + n^2$$



⇒ ?? work per level

log₂ n levels
of recursion

$$T(n) = \sum_{i=1}^{\log_2 n} ??$$

Solving $T(n) = 2T\left(\frac{n}{2}\right) + n^2$

$$T(n) = \sum_{i=1}^{\log_2 n} \frac{n^2}{2^i}$$

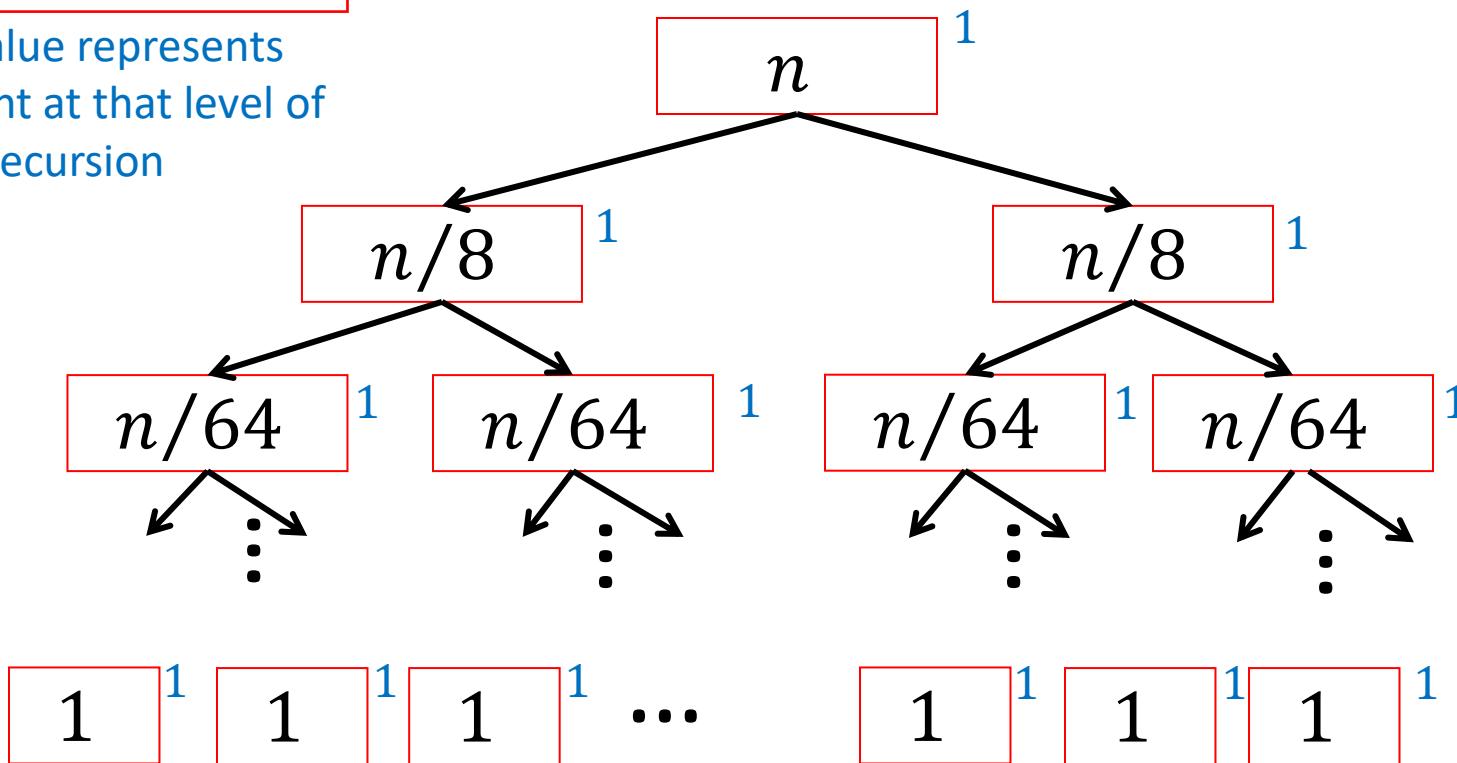
$$= n^2 \cdot \sum_{i=1}^{\log_2 n} \left(\frac{1}{2}\right)^i$$

Tree Method

Red box represents a problem instance

Blue value represents time spent at that level of recursion

$$T(n) = 2T\left(\frac{n}{8}\right) + 1$$



$\Rightarrow 2^i$ work per level

$\log_8 n$ levels
of recursion

$$T(n) = \sum_{i=1}^{\log_8 n} 2^i$$

Solving $T(n) = 2T\left(\frac{n}{8}\right) + 1$

$$T(n) = \sum_{i=1}^{\log_8 n} 2^i$$

$$= \left(\frac{1 - 2^{\log_8 n}}{1 - 2} \right)$$

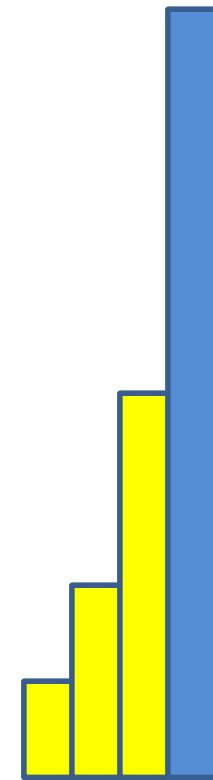
$$= 2^{\log_8 n} - 1$$

$$= n^{\log_8 2} = n^{\frac{1}{3}}$$

$$\sum_{i=0}^L a^i$$

Finite Geometric Series

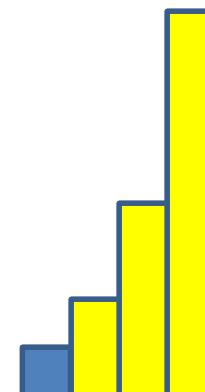
If $a > 1$



The series
multiplied by a

$$(1 + a + a^2 + \dots + a^L)a \quad (1 + a + a^2 + \dots + a^L)1$$

- =



The series



The next term
in the series
 a^{L+1}

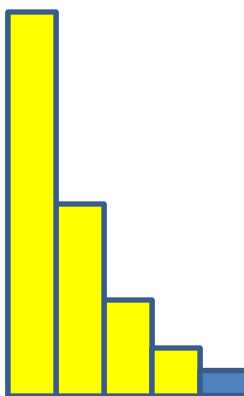


The first term

$$\sum_{i=0}^L a^i$$

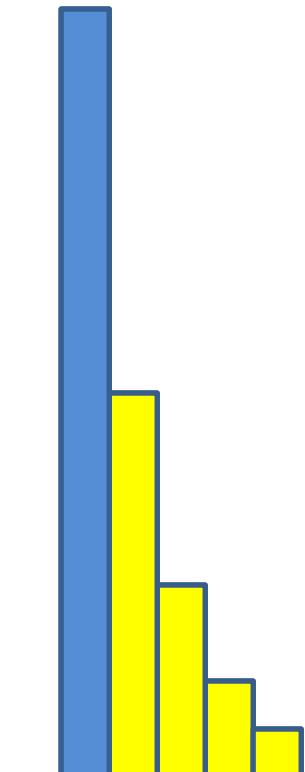
Finite Geometric Series

If $a < 1$



The series
multiplied by a

$$(1 + a + a^2 + \dots + a^L)a$$



The series

$$(1 + a + a^2 + \dots + a^L)1$$

The next term
in the series
 a^{L+1}

The first term

$$1$$

Solve for the series