

CSE 332 Winter 2026

Lecture 7: Recurrences, Dictionaries, BSTs

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Analysis of Recursive Algorithms

- Overall structure of recursion:
 - Do some non-recursive “work”
 - Do one or more recursive calls on some portion of your input
 - Do some more non-recursive “work”
 - Repeat until you reach a base case
- Running time: $T(n) = T(p_1) + T(p_2) + \dots + T(p_x) + f(n)$
 - The time it takes to run the algorithm on an input of size n is:
 - The sum of how long it takes to run the same algorithm on each smaller input
 - Plus the total amount of non-recursive work done in that stack frame
- Usually:
 - $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$
 - Called “divide and conquer”
 - $T(n) = T(n - c) + f(n)$
 - Called “chip and conquer”

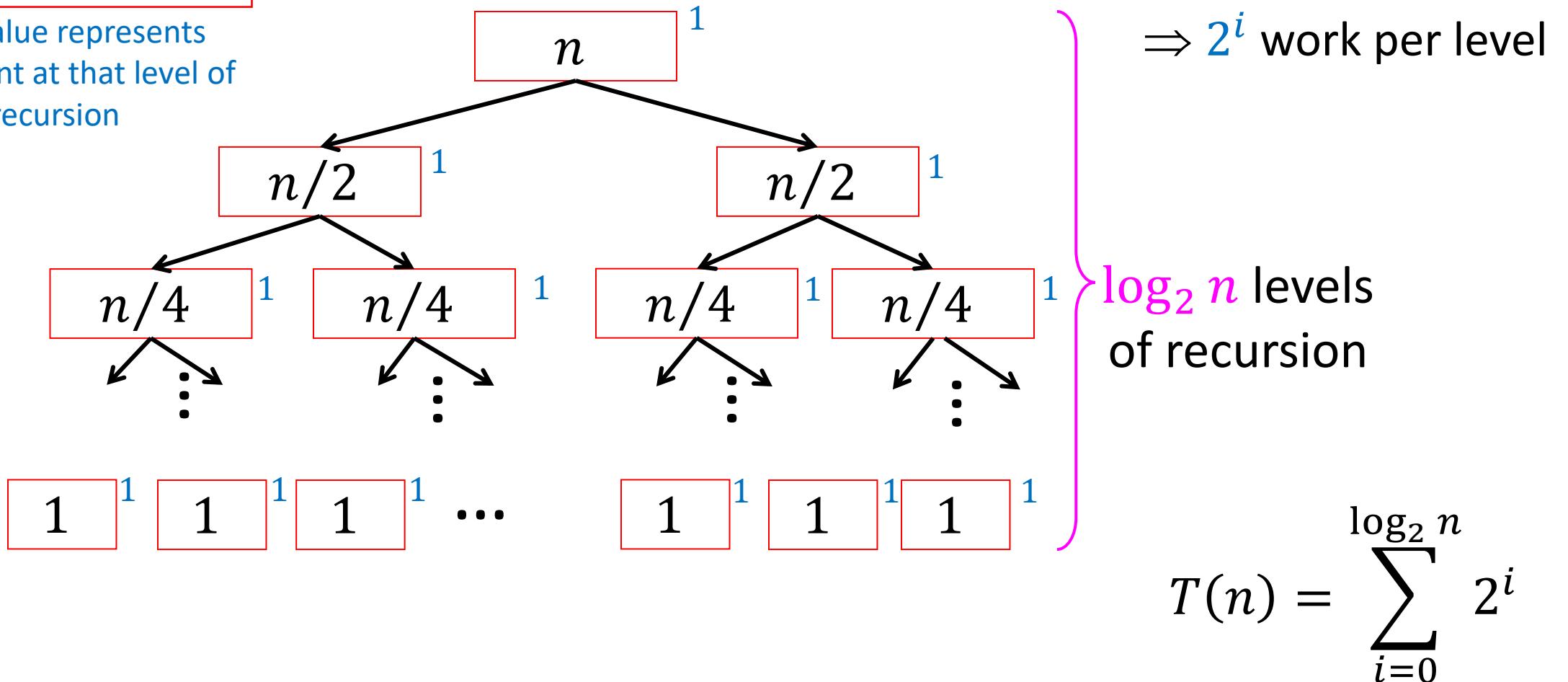
Recursive List Summation

```
public int sum(int[] list){  
    return sum_helper(list, 0, list.size);  
}  
  
private int sum_helper(int[] list, int low, int high){  
    if (low == high){ return 0; }  
    if (low == high-1){ return list[low]; }  
    int middle = (high+low)/2;  
    return sum_helper(list, low, middle) + sum_helper(list, middle, high);  
}
```

Tree Method

Red box represents a problem instance

Blue value represents time spent at that level of recursion



Tree Method Summary: Chip and Conquer

- Recurrence looks like $T(n) = aT(n - b) + f(n)$
- Use the recurrence to draw a tree
 - a is the branching factor of the tree (e.g. if $a = 2$ then it's a binary tree)
 - Subtract b from the parent's input size to get children's input size
 - Work done per node is given by applying $f(n)$ to that node's input size
 - Height of the tree is $\frac{n}{b}$
 - Because that is the number of times we must subtract b until reaching a base case
 - Answer to the question "how many times must we subtract b until we reach 0?"
 - Any base case is a constant, so to reach a larger value would just be a constant change
- Use the tree to express running time as a series
 - Adding work done for each node level-by-level
 - Identify a pattern to express work done at level i as a function of i
 - Write a series using $i = 0$ up to $\frac{n}{b}$
- Solve the series

Tree Method Summary: Divide and Conquer

- Recurrence looks like $T(n) = aT\left(\frac{n}{b}\right) + f(n)$
- Use the recurrence to draw a tree
 - a is the branching factor of the tree (e.g. if $a = 2$ then it's a binary tree)
 - Divide the parent's input size by b to get children's input size
 - Work done per node is given by applying $f(n)$ to that node's input size
 - Height of the tree is $\log_b n$
 - Because that is the number of times we must divide by b until reaching a base case
 - Answer to the question "how many times must we divide by b until we reach 1?"
 - Any base case is a constant, so to reach a larger value would just be a constant change
- Use the tree to express running time as a series
 - Adding work done for each node level-by-level
 - Identify a pattern to express work done at level i as a function of i
 - Write a series using $i = 0$ up to $\log_b n$
- Solve the series

Let's do some more!

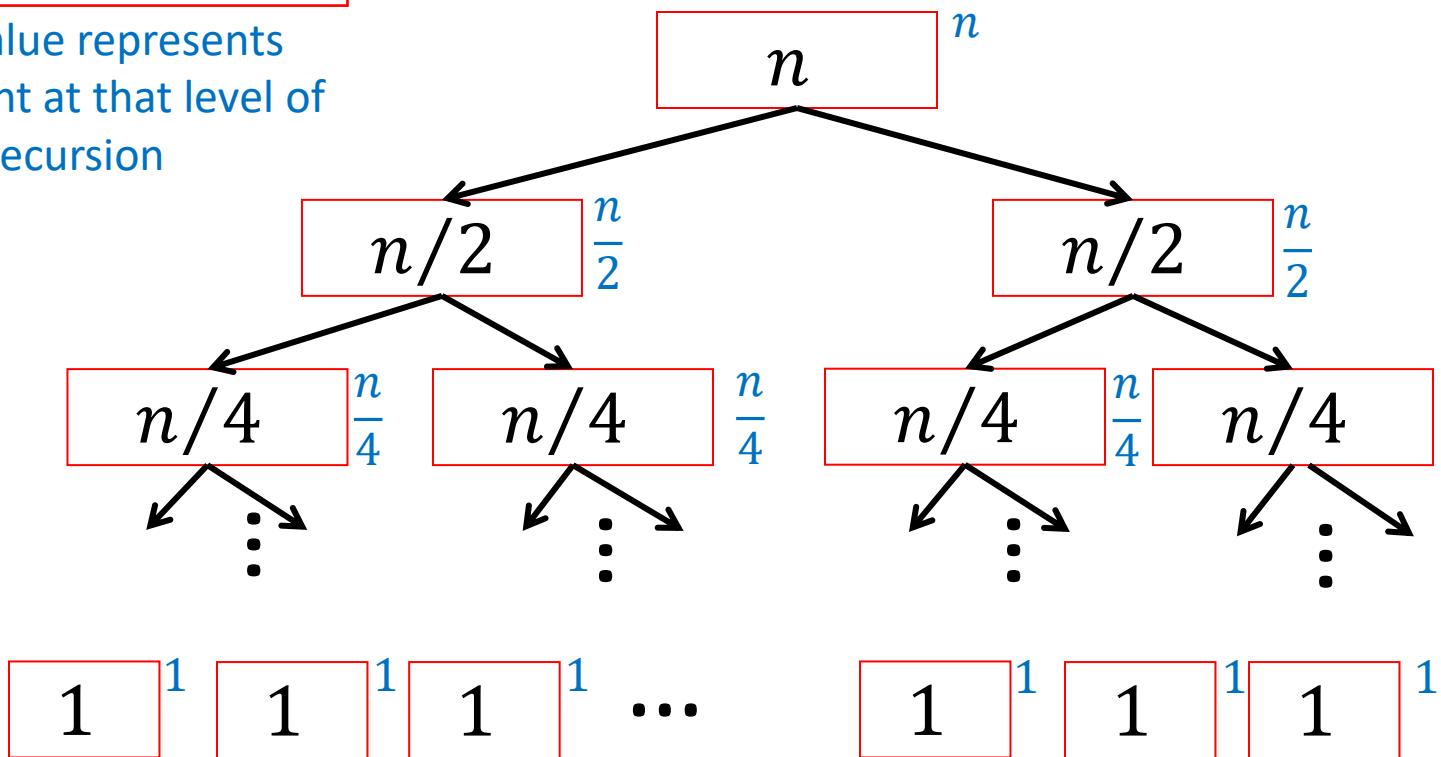
- For each, assume the base case is $n = 1$ and $T(1) = 1$
- $T(n) = 2T\left(\frac{n}{2}\right) + n$
- $T(n) = 2T\left(\frac{n}{2}\right) + n^2$
- $T(n) = 2T\left(\frac{n}{8}\right) + 1$

Tree Method

Red box represents a problem instance

Blue value represents time spent at that level of recursion

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



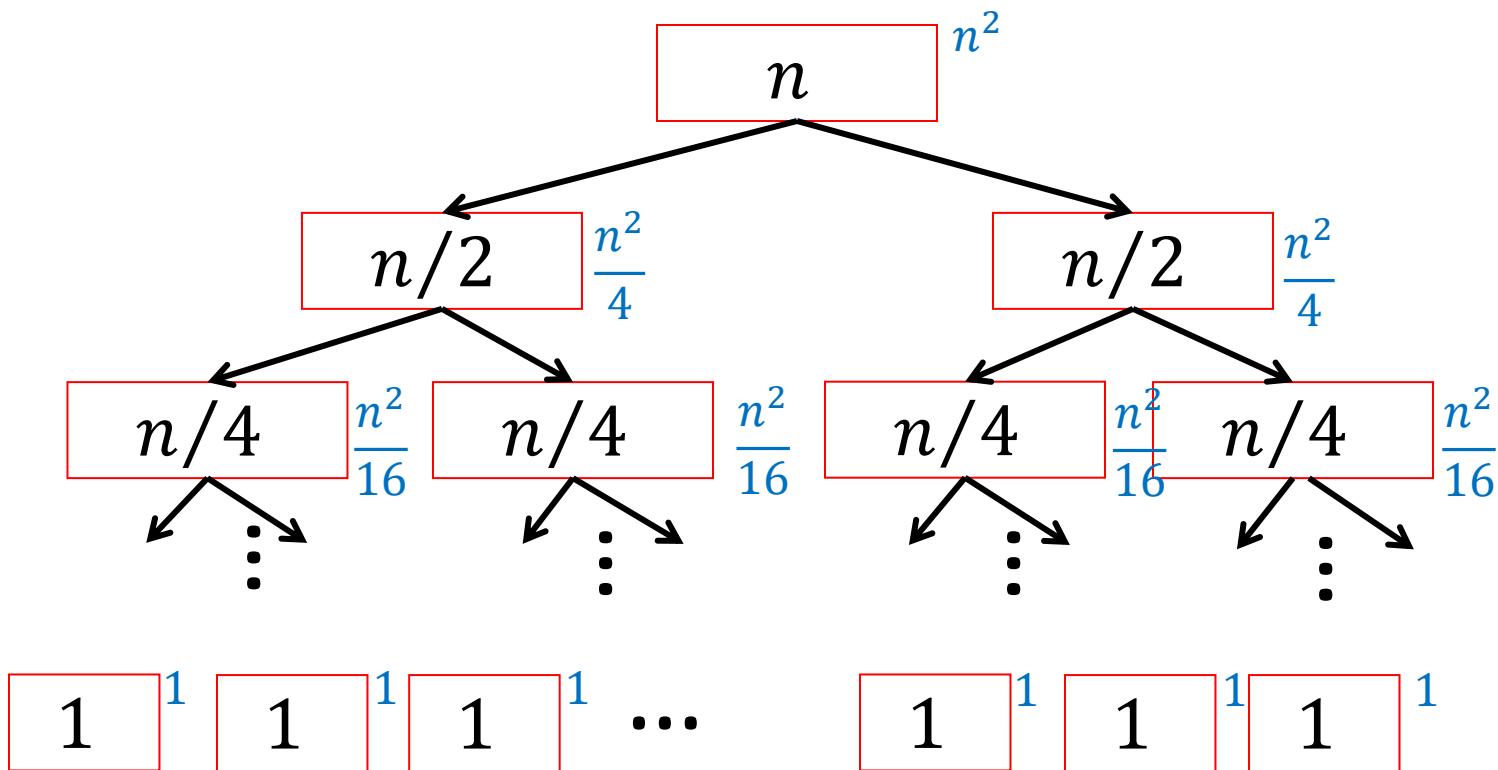
$\Rightarrow n$ work per level

$\log_2 n$ levels of recursion

$$T(n) = \sum_{i=0}^{\log_2 n} n$$

Tree Method

$$T(n) = 2T\left(\frac{n}{2}\right) + n^2$$



$$T(n) = \sum_{i=0}^{\log_2 n} ??$$

Solving $T(n) = 2T\left(\frac{n}{2}\right) + n^2$

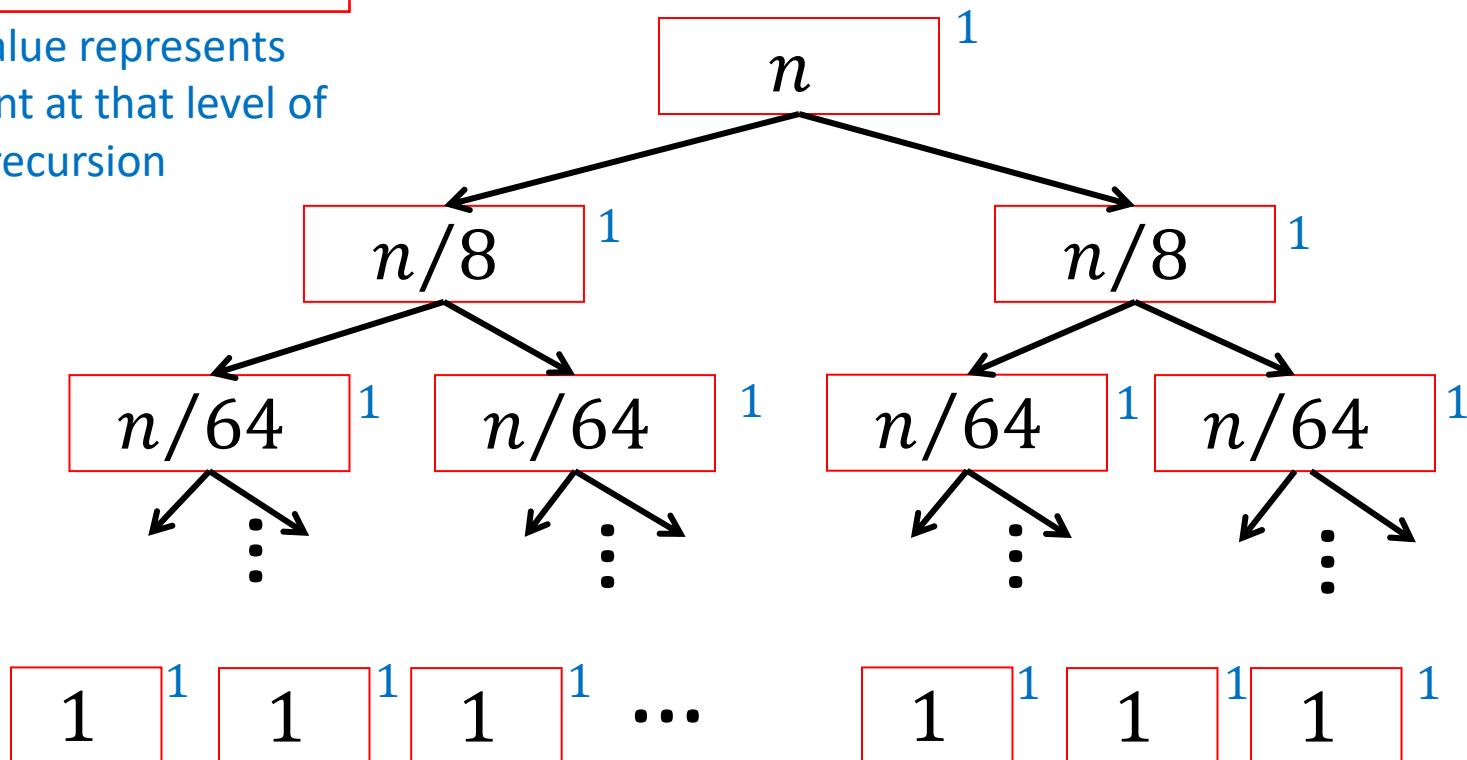
$$T(n) = \sum_{i=0}^{\log_2 n} \frac{n^2}{2^i}$$

$$= n^2 \cdot \sum_{i=0}^{\log_2 n} \left(\frac{1}{2}\right)^i$$

Tree Method

Red box represents a problem instance

Blue value represents time spent at that level of recursion



$$T(n) = 2T\left(\frac{n}{8}\right) + 1$$

$\Rightarrow 2^i$ work per level

$\log_8 n$ levels
of recursion

$$T(n) = \sum_{i=0}^{\log_8 n} 2^i$$

Solving $T(n) = 2T\left(\frac{n}{8}\right) + 1$

$$T(n) = \sum_{i=0}^{\log_8 n} 2^i$$

$$= \left(\frac{1 - 2^{\log_8 n}}{1 - 2} \right)$$

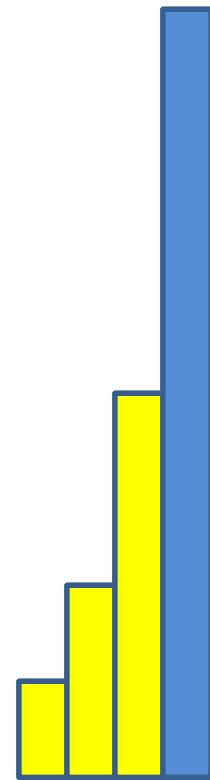
$$= 2^{\log_8 n} - 1$$

$$\begin{aligned} &= n^{\log_8 2} - 1 = n^{\frac{1}{3}} - 1 \\ &= \Theta(n^{\frac{1}{3}}) \end{aligned}$$

$$\sum_{i=0}^L a^i$$

Finite Geometric Series

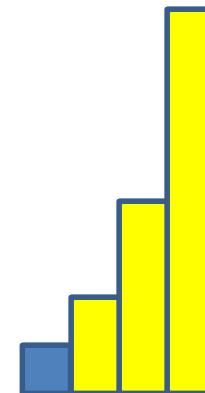
If $a > 1$



The series
multiplied by a

$$(1 + a + a^2 + \dots + a^L)a \quad (1 + a + a^2 + \dots + a^L)1$$

- =



The series



The next term
in the series
 a^{L+1}

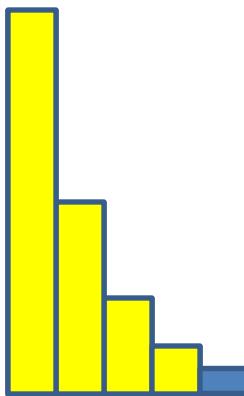


The first term

$$\sum_{i=0}^L a^i$$

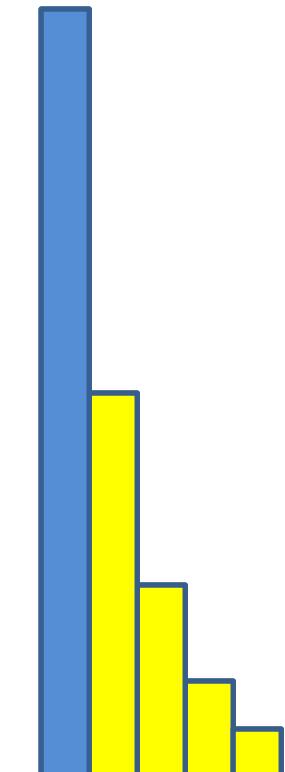
Finite Geometric Series

If $a < 1$



The series
multiplied by a

$$(1 + a + a^2 + \dots + a^L)a$$



The series

$$(1 + a + a^2 + \dots + a^L)1$$

The next term
in the series
 a^{L+1}

The first term

$$1$$

Solve for the series

Dictionary (Map) ADT

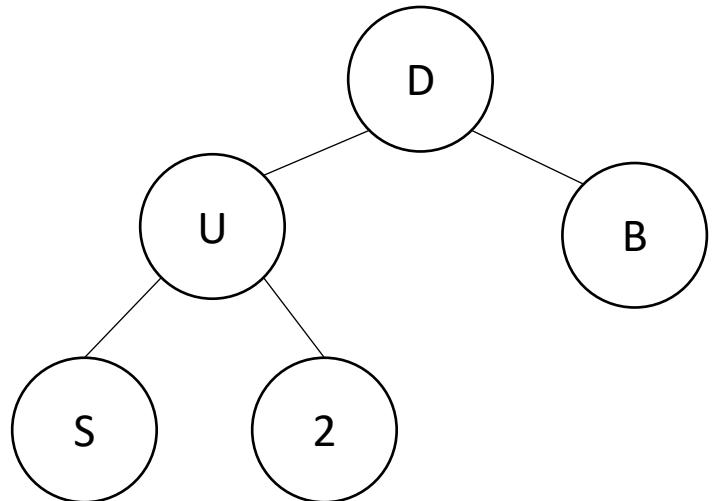
- **Contents:**
 - Sets of key+value pairs
 - Keys must be comparable
- **Operations:**
 - **insert(key, value)**
 - Adds the (key,value) pair into the dictionary
 - If the key already has a value, overwrite the old value
 - Consequence: Keys cannot be repeated
 - **find(key)**
 - Returns the value associated with the given key
 - **delete(key)**
 - Remove the key (and its associated value)

Naïve attempts

Data Structure	Time to insert	Time to find	Time to delete
Unsorted Array	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Unsorted Linked List	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Sorted Array	$\Theta(n)$	$\Theta(\log n)$	$\Theta(n)$
Sorted Linked List	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Heap	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Binary Search Tree (worst)	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Binary Search Tree (expected)	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$

More Tree “Vocab”

- Traversal:
 - An algorithm for “visiting/processing” every node in a tree
- Pre-Order Traversal:
 - Root, Left Subtree, Right Subtree
 - D U S 2 B
- In-Order Traversal:
 - Left Subtree, Root, Right Subtree
 - S U 2 D B
- Post-Order Traversal
 - Left Subtree, Right Subtree, Root
 - S 2 U B D



Name that Traversal!

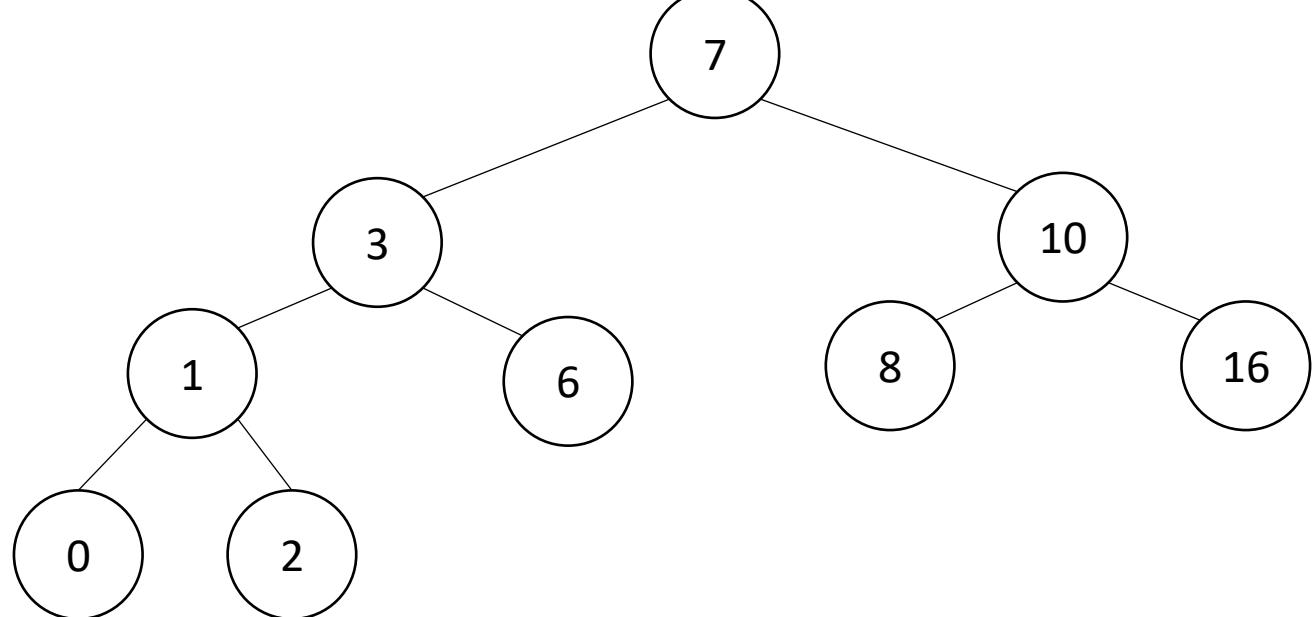
```
AorderTraversal(root){  
    if (root.left != Null){  
        process(root.left);  
    }  
    if (root.right != Null){  
        process(root.right);  
    }  
    process(root);  
}
```

```
BorderTraversal(root){  
    process(root);  
    if (root.left != Null){  
        process(root.left);  
    }  
    if (root.right != Null){  
        process(root.right);  
    }  
}
```

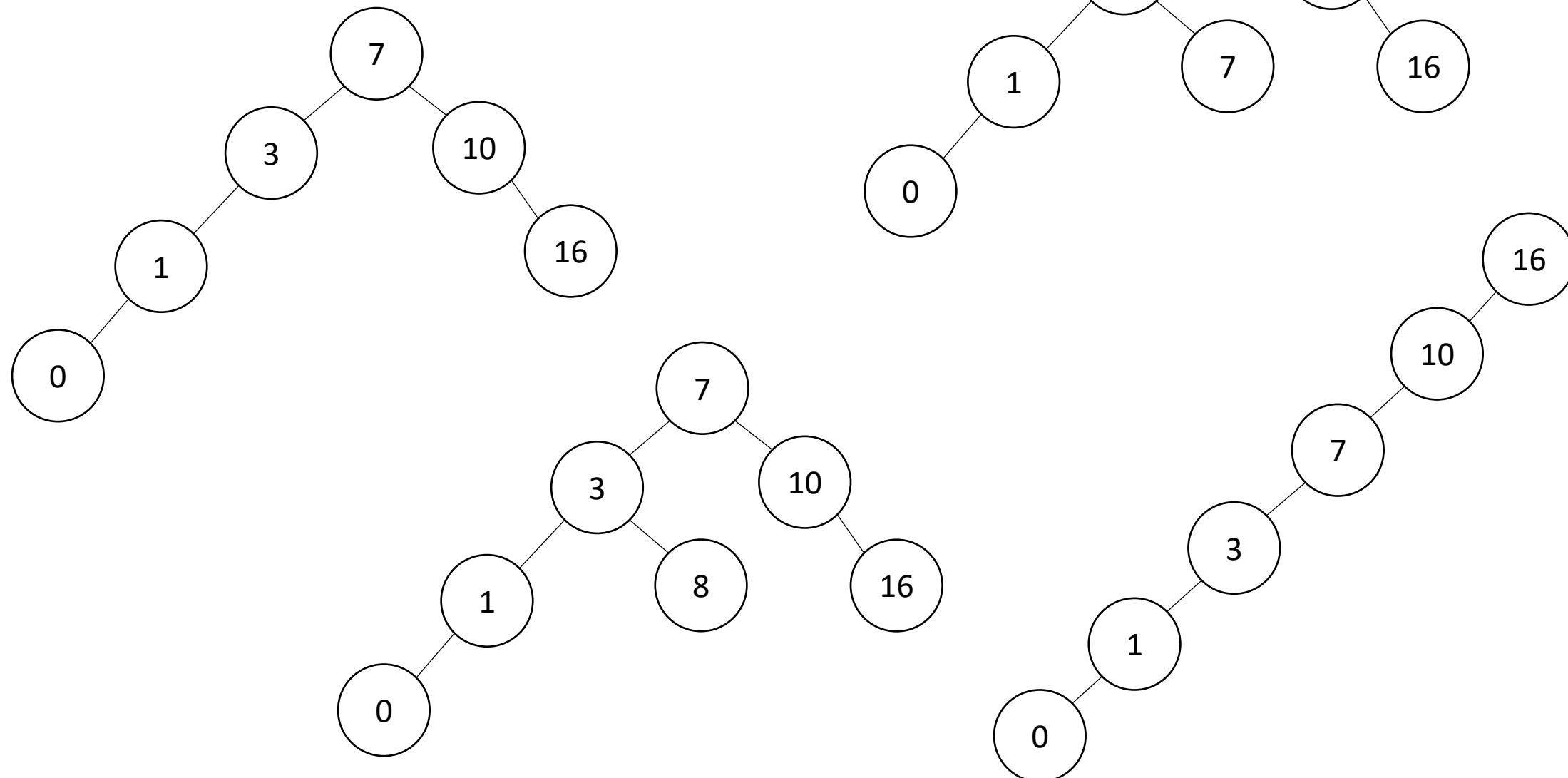
```
CorderTraversal(root){  
    if (root.left != Null){  
        process(root.left);  
    }  
    process(root)  
    if (root.right != Null){  
        process(root.right);  
    }  
}
```

Binary Search Tree

- Binary Tree
 - Definition:
 - Tree where each node has at most 2 children
- Order Property
 - All keys in the left subtree are smaller than the root
 - All keys in the right subtree are larger than the root
 - Consequence: cannot have repeated values



Are these BSTs?

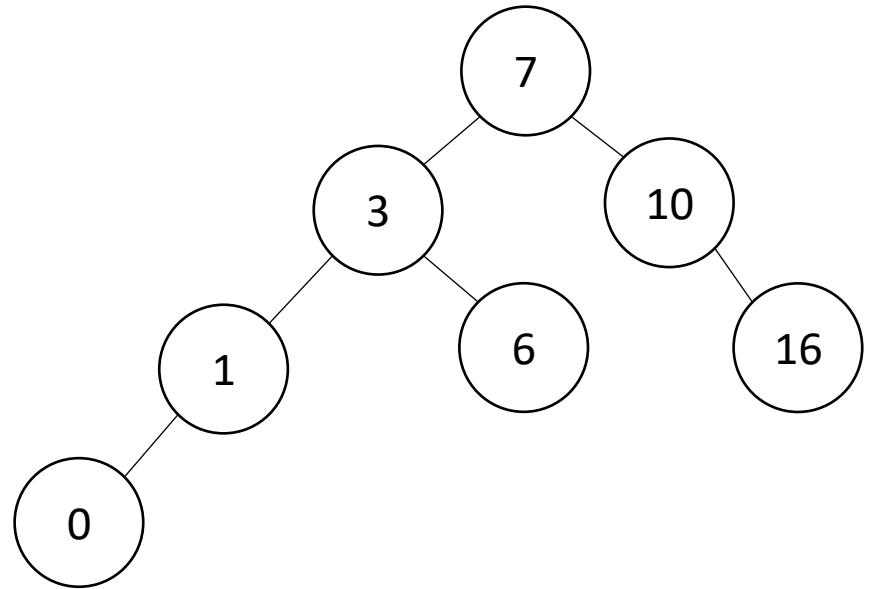


Aside: Why not use an array?

- We represented a heap using an array, finding children/parents by index
- We will represent BSTs with nodes and references. Why?
 - We might have “gaps” in our tree
 - Memory!
 - 2^n

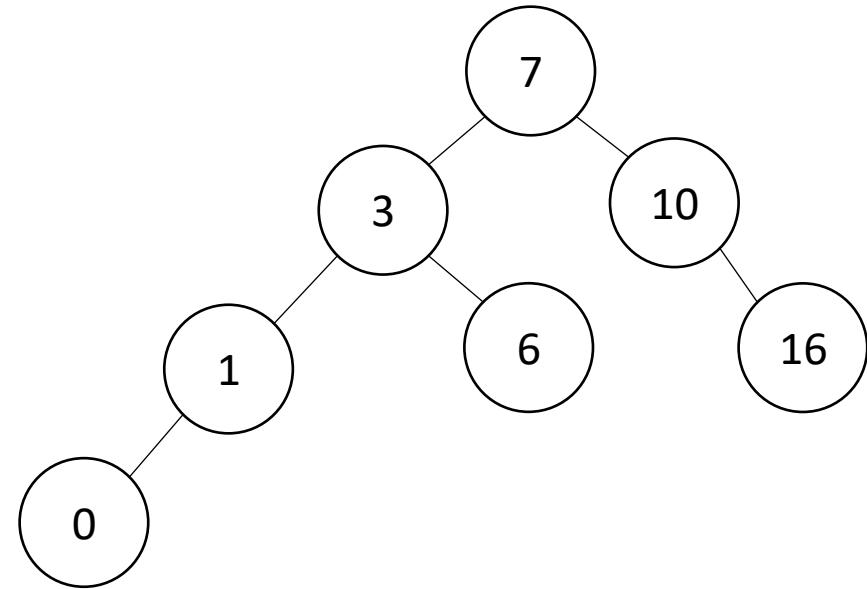
Find Operation (recursive)

```
find(key, root){  
    if (root == Null){  
        return Null;  
    }  
    if (key == root.key){  
        return root.value;  
    }  
    if (key < root.key){  
        return find(key, root.left);  
    }  
    if (key > root.key){  
        return find(key, root.right);  
    }  
    return Null;  
}
```



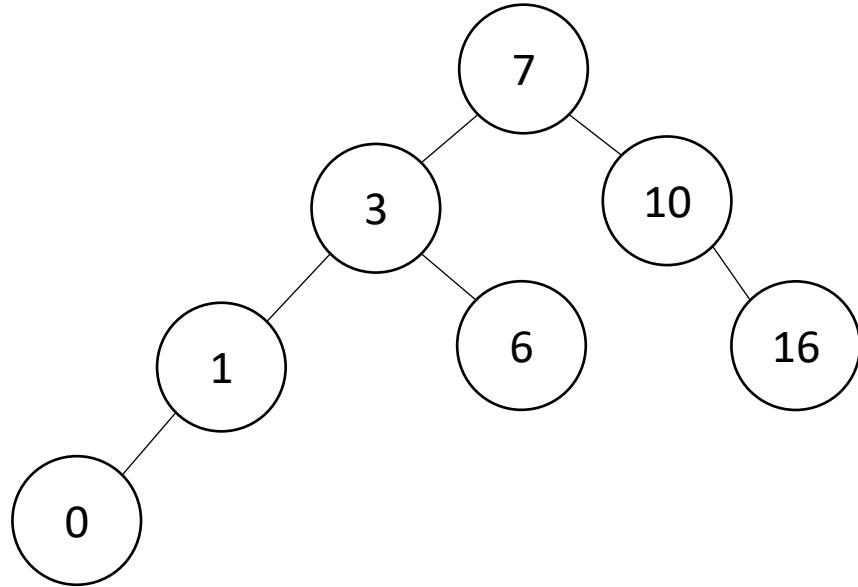
Find Operation (iterative)

```
find(key, root){  
    while (root != Null && key != root.key){  
        if (key < root.key){  
            root = root.left;  
        }  
        else if (key > root.key){  
            root = root.right;  
        }  
    }  
    if (root == Null){  
        return Null;  
    }  
    return root.value;  
}
```



Insert Operation (recursive)

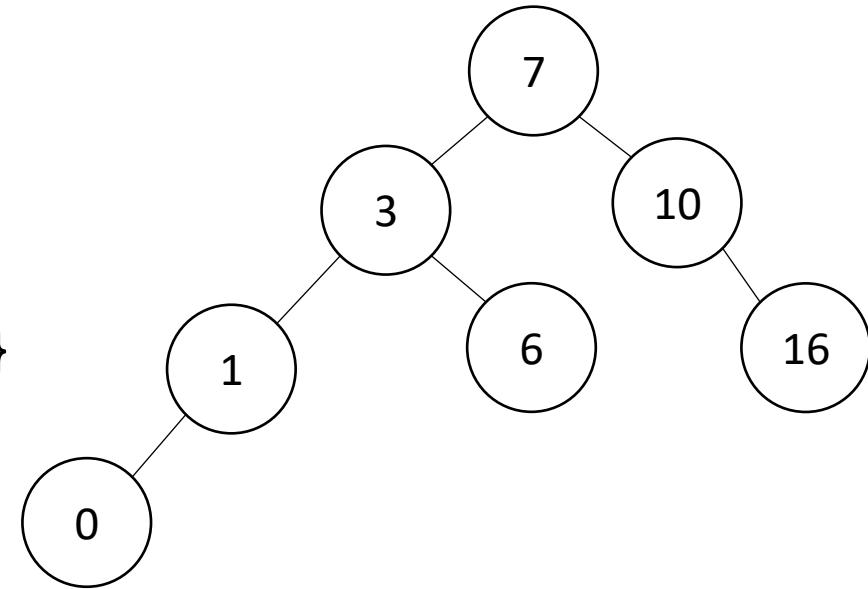
```
insert(key, value, root){  
    root = insertHelper(key, value, root);  
}  
insertHelper(key, value, root){  
    if(root == null)  
        return new Node(key, value);  
    if (root.key < key)  
        root.right = insertHelper(key, value, root.right);  
    else  
        root.left = insertHelper(key, value, root.left);  
    return root;  
}
```



Note: Insert happens only at the leaves!

Insert Operation (iterative)

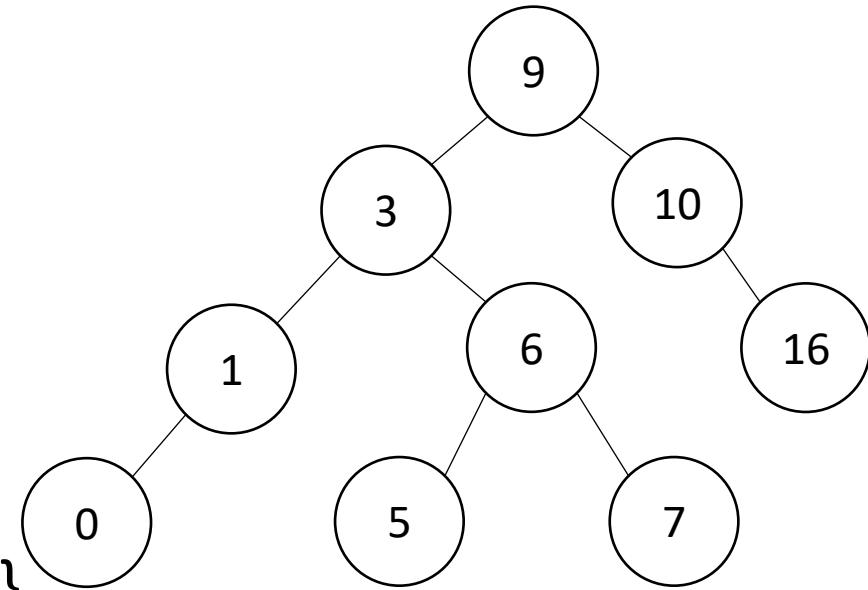
```
insert(key, value, root){  
    if (root == Null){ this.root = new Node(key, value); }  
    parent = Null;  
    while (root != Null && key != root.key){  
        parent = root;  
        if (key < root.key){ root = root.left; }  
        else if (key > root.key){ root = root.right; }  
    }  
    if (root != Null){ root.value = value; }  
    else if (key < parent.key){ parent.left = new Node(key, value); }  
    else{ parent.right = new Node (key, value); }  
}
```



Note: Insert happens only at the leaves!

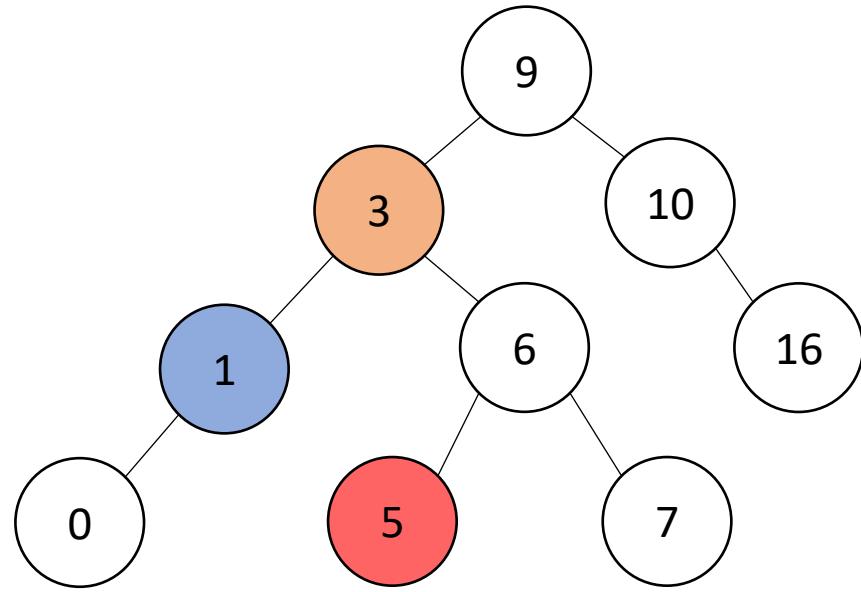
Delete Operation (iterative)

```
delete(key, root){  
    while (root != Null && key != root.key){  
        if (key < root.key){ root = root.left; }  
        else if (key > root.key){ root = root.right; }  
    }  
    if (root == Null){ return; }  
    // Now root is the node to delete, what happens next?  
}
```



Delete – 3 Cases

- 0 Children (i.e. it's a leaf)
- 1 Child
 - Replace the deleted node with its child
- 2 Children
 - Replace the deleted with the largest node to its left or else the smallest node to its right

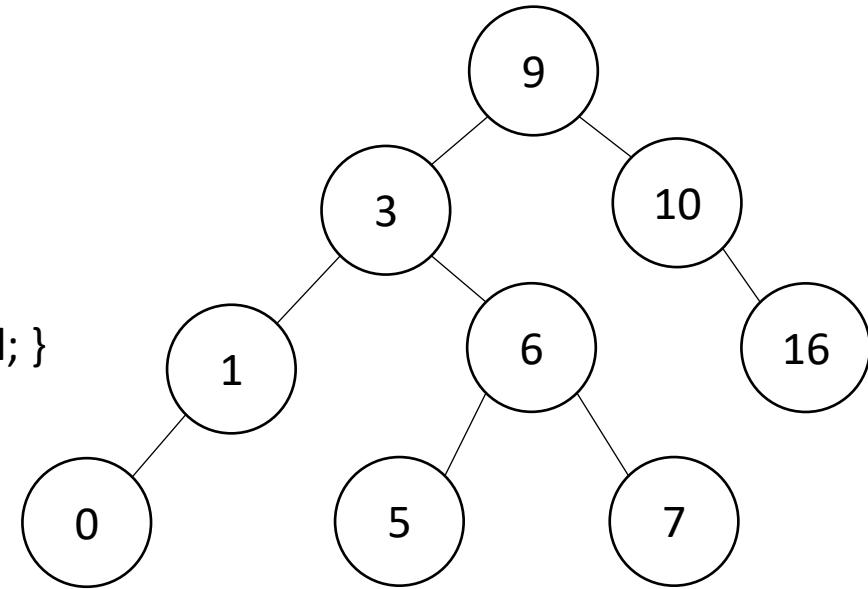


Finding the Max and Min

- Max of a BST:
 - Right-most Thing
- Min of a BST:
 - Left-most Thing

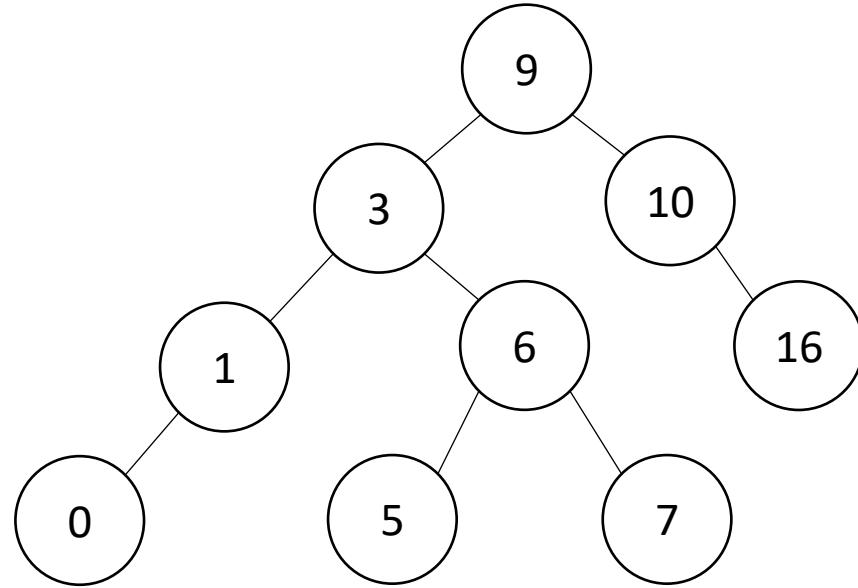
```
maxNode(root){  
    if (root == Null){ return Null; }  
    while (root.right != Null){  
        root = root.right;  
    }  
    return root;  
}
```

```
minNode(root){  
    if (root == Null){ return Null; }  
    while (root.left != Null){  
        root = root.left;  
    }  
    return root;  
}
```



Delete Operation (iterative)

```
delete(key, root){  
    while (root != Null && key != root.key){  
        if (key < root.key){ root = root.left; }  
        else if (key > root.key){ root = root.right; }  
    }  
    if (root == Null){ return; }  
    if (root has no children){  
        make parent point to Null Instead;  
    }  
    if (root has one child){  
        make parent point to that child instead;  
    }  
    if (root has two children){  
        make parent point to either the max from the left or min from the right  
    }  
}
```



Delete Operation (recursive)

```
delete(key, root){  
    if (root == Null){ return; } // key not present  
    if (root.key == key){  
        if (root has no children) { return Null; }  
        if (root has one child) { return that child; }  
        if (root has two children) {return removeMax(root.left);} }  
    if (root.key < key) { root.right = delete(key, root.right); }  
    else { root.left = delete(key, root.left); } }
```

