CSE 417 Autumn 2025

Lecture 4: Gale–Shapley analysis

Glenn Sun

Announcements

HW1 due Friday @ 11:59pm!

HW1 LaTeX template fix: Check Ed!

Continue to get those Concept Checks in on time, please 🙂

Review of Gale–Shapley

Gale–Shapley algorithm

```
while there is a free proposer p ∈ P do
    Let r be the top remaining person on p's preference list.
    if r is also free then
        Have r accept p.
    else if r is paired but prefers the new proposer p then
        Have r accept p and reject their current match p′.
    else (if r is paired and prefers their current match)
        Have r reject p.
return all matches
```

Proposer optimality

Proposer optimality theorem: The Gale–Shapley algorithm always finds the unique stable matching that is both **best for proposers** and **worst for receivers**.

"best": Every proposer is happier in this matching than any other stable matching (or equally as happy).

"worst": Similar.

Proofs with while loops

Proofs with while loops

Input: A positive integer n

Goal: An integer k such that $(k-1)^2 < n \leq k^2$

1. Let $k = 1$.
2. **while** $k^2 < n$ **do**
3. Update $k = k + 1$.
4. **return** k

Proofs with while loops

“Loops terminate”: For a while loop, we give an **upper bound on the number of iterations.**

Scratch work (working backwards):

- We need $\boldsymbol{k^2 \geq n}$ for termination.
- This happens when $\boldsymbol{k = \lceil \sqrt{n} \rceil}$.
- At the end of the $\boldsymbol{i}$th iteration, $\boldsymbol{k = i + 1}$. (Loop invariant!)
- Thus, we should be good after $\boldsymbol{\lceil \sqrt{n} \rceil - 1}$ iterations.

Proofs with while loops

"**Loops terminate**": For a while loop, we give an **upper bound on the number of iterations.**

Claim. The loop terminates within $\lceil \sqrt{n} \rceil - 1$ iterations.

Proof. It is a loop invariant that after i iterations, we have $k = i + 1$.

After $\lceil \sqrt{n} \rceil - 1$ iterations, $k = \lceil \sqrt{n} \rceil$. Thus $k^2 = \lceil \sqrt{n} \rceil^2 \geq n$, so the while loop exits.

Proofs with while loops

To prove "meets specification" with a while loop, it's usually not enough to use loop invariants. Must use

loop invariant + while exit condition

Example:

- loop invariant: $(\boldsymbol{k}-\mathbf{1})^2 < \boldsymbol{n}$
- while exit condition: $\boldsymbol{k}^2 \geq \boldsymbol{n}$

Recall final goal: an integer $\boldsymbol{k}$ such that $(\boldsymbol{k}-\mathbf{1})^2 < \boldsymbol{n} \leq \boldsymbol{k}^2$

Proving Gale–Shapley correct

Three requirements for correctness

"No exceptions": In line 2 when p picks the next top person on their preference list, p has not yet exhausted the entire list.

"Loops terminate": Every proposer gets eventually matched.

"Meets specification": The final set of matches is a perfect matching with no unstable pairs.

Proving no exceptions

Claim. In line 2 when p picks the next top person on their preference list, p has not yet exhausted the entire list.

Proof. Suppose for contradiction p has exhausted the entire list.

In other words, they have proposed to everyone and also got rejected by everyone.

Q: What must be true if p was rejected by everyone?

Proving no exceptions

while there is a free proposer $p \in P$ **do**
    Let r be the top remaining person on p's preference list.
    if r is also free **then**
        Have r accept p.
    else if r is paired but prefers the new proposer p **then**
        Have r accept p and reject their current match p'.
    else (if r is paired and prefers their current match)
        Have r reject p.
return all matches

A: Every r was already paired when they rejected p!

Proving no exceptions

We know: Every r was already paired when they rejected p.

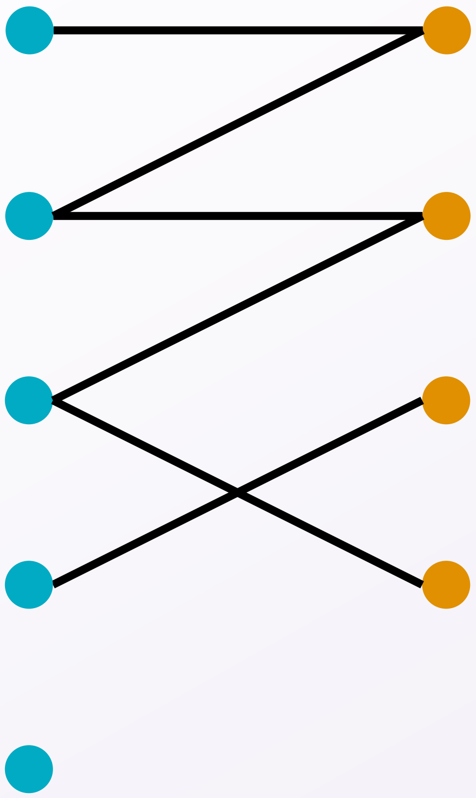
Loop invariant 1: At the end of every iteration, if r was ever paired to someone in any previous iteration, it is still paired to someone.

- Essentially because we only unpaired r in this iteration if we immediately paired them with someone else.

Conclusion: Every r is still paired.

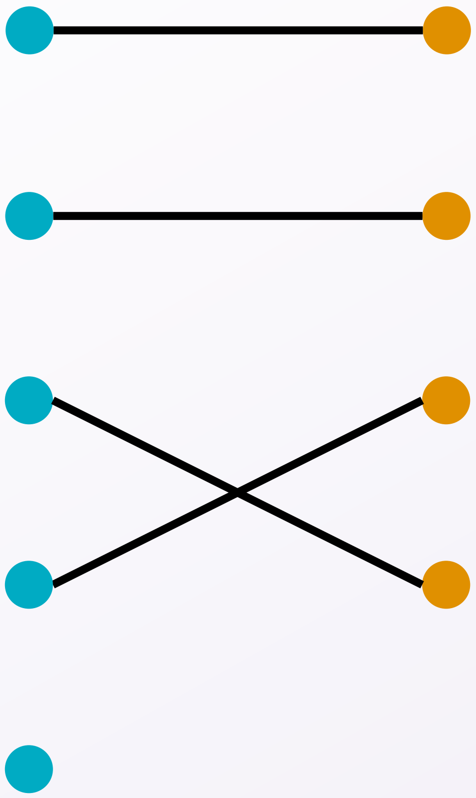
Proving no exceptions

We know: Every r is still paired.



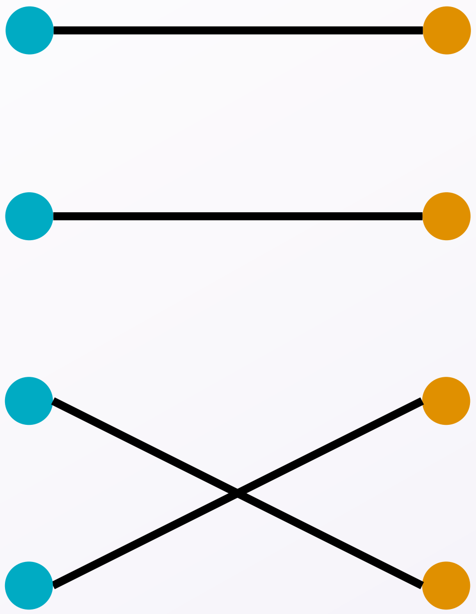
Proving no exceptions

Loop invariant 2: After every iteration, each person is matched to at most one other person.



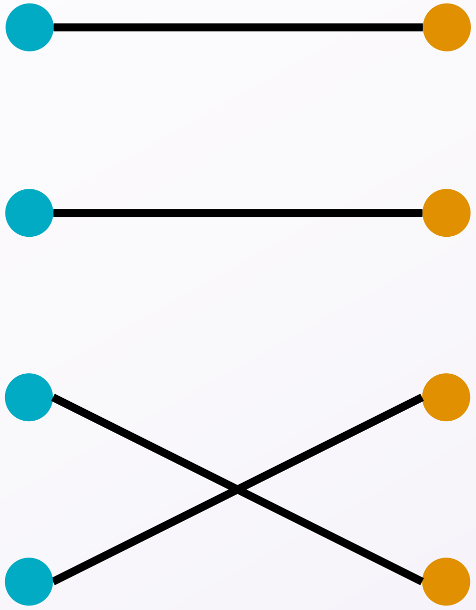
Proving no exceptions

But $|P| = |R| = n$ (equal number of proposers and receivers).



Proving no exceptions

Every proposer must be paired, contradiction with p being free.



Proving loops terminate

Claim. Every proposer gets matched within n^2 iterations.

Proof. There are n^2 possible proposals (each $p \in P$ to each $r \in R$).

Because line 2 always picks a new proposal and never throws an error, the while loop must end within n^2 iterations.

Proving perfect matching

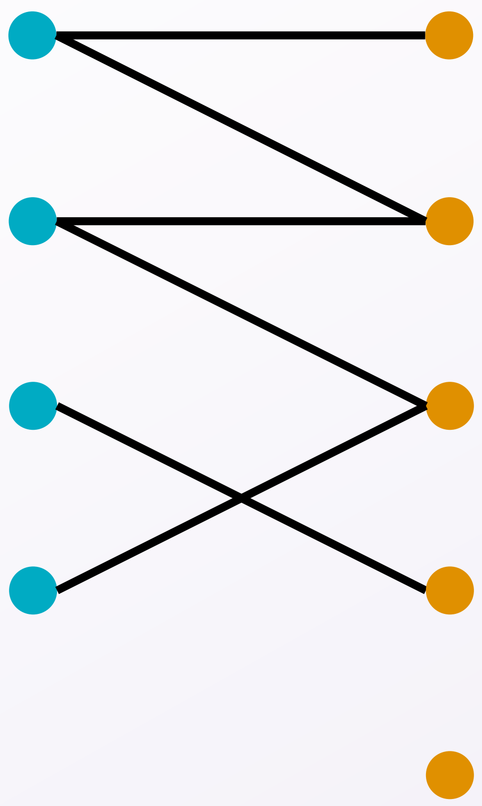
Remember, when there is a while loop, correctness should use both a **loop invariant** and the **while exit condition**.

Claim. The output is a perfect matching.

Proof. Very similar to before — in following slides.

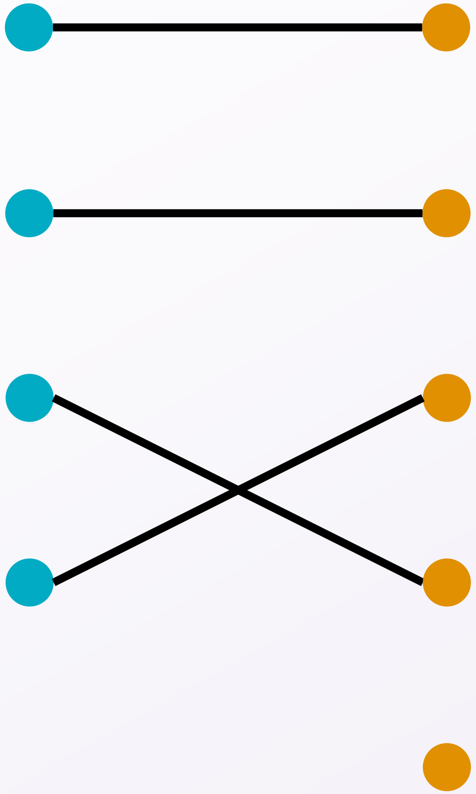
Proving perfect matching

While exit condition: Every $\boldsymbol{p}$ is paired.



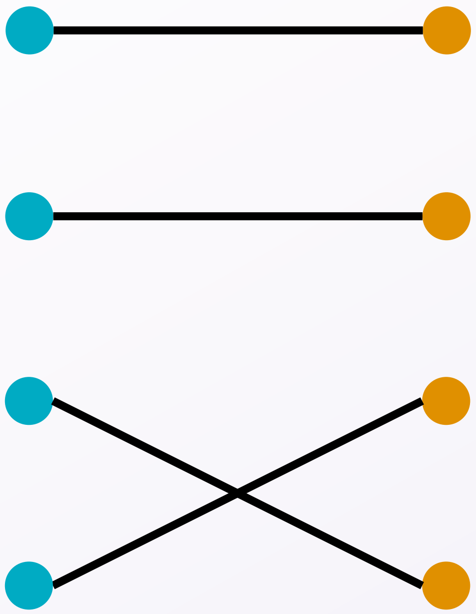
Proving perfect matching

Loop invariant 2: After every iteration, each person is matched to at most one other person.



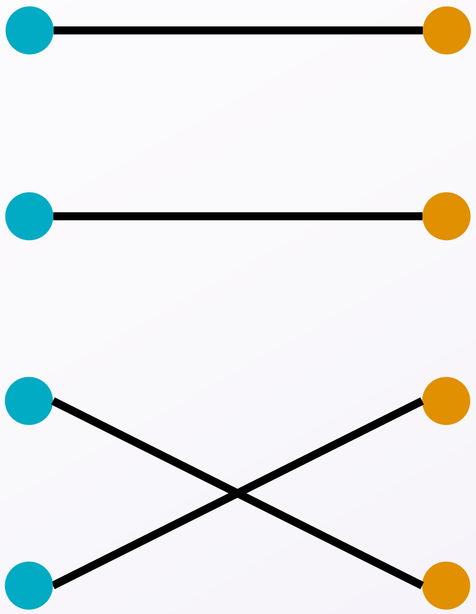
Proving perfect matching

But $|P| = |R| = n$ (equal number of proposers and receivers).



Proving perfect matching

Thus, everyone is paired with exactly one other, so this is a perfect matching.



Proving no unstable pairs

Claim. The output has no unstable pairs.

Proof. Let (p, r) be any proposer-receiver pair.

Case 1: (p, r) is in the output matching.

Then (p, r) is not unstable.

Proving no unstable pairs

Claim. The output has no unstable pairs.

Proof. Let (p, r) be any proposer-receiver pair.

Case 2: (p, r) is not in the output matching.

Why would a pair not be matched? Two possibilities:

- **Case 2a:** p stopped proposing before getting to r.
- **Case 2b:** p proposed to r, but got rejected or later unpaired.

Proving no unstable pairs

Case 2a: p stopped proposing before getting to r.

Because we output a perfect matching, p is matched to someone.

Because p proposes in order of preference, p must be matched to someone they prefer over r.

Then (p, r) is not unstable.

Proving no unstable pairs

Case 2b: p proposed to r, but got rejected or later unpaired.

```
    else if r is paired but prefers the new proposer p then
        Have r accept p and reject their current match p'.
    else (if r is paired and prefers their current match)
        Have r reject p.
```

Conclusion: At the time of rejection, r must have been matched to someone that they rank higher than p.

Proving no unstable pairs

Case 2b: p proposed to r, but got rejected or later unpaired.

Loop invariant 3 ("trading up"): At the end of every iteration, if r is paired, it prefers its current match over all previous matches.

Because we output a perfect matching, r is matched to someone.

By the loop invariant, r is still matched to someone that they rank higher than p.

So (p, r) is not unstable.

What’s next?

We proved correctness. What other questions can we ask?

- How fast is the algorithm? —wait for Friday!
- What about many-to-one matchings? —on your HW2!
- *Which* stable matching does it produce? —our next topic
- Can people gain advantage by lying on their preferences lists?
  What about “stable roommates” (matching within one group)?
  etc. —if we have time

Proposer optimality/receiver pessimality

Proposer optimality theorem

Proposer optimality theorem: The Gale–Shapley algorithm always finds the unique stable matching that is both **best for proposers** and **worst for receivers**.

Say that a pair (p, r) are called **valid partners** if there is *some* stable matching where they are matched together.

Proof of proposer optimality

By contradiction. Suppose the output is not proposer-optimal.

This means that for some p, the output matches (p, r), but r' is also a valid partner and p prefers $r' > r$.

Let p' be the reason that r' rejected p.

$$p\colon \; \ldots > r' > \ldots > r > \ldots$$
$$p'\colon \; \ldots > \ldots > r' > \ldots > \ldots$$

$$r'\colon \; \ldots > p' > \ldots > p > \ldots$$

Proof of proposer optimality

Since (p, r') are valid partners, consider that matching.

Q: Where is p''s partner?

p: $\ldots > r' > \ldots > r > \ldots$

p': $\ldots > \ldots > r' > \ldots > \ldots$

r': $\ldots > p' > \ldots > p > \ldots$

Proof of proposer optimality

Since (p, r') are valid partners, consider that matching.

Q: Where is p''s partner?

If p''s partner is worse than r', then (p', r') is unstable, contradiction.

p: $\ldots > r' > \ldots > r > \ldots$

p': $\ldots > \ldots > r' > \ldots > \ldots$

r': $\ldots > p' > \ldots > p > \ldots$

Proof of proposer optimality

What if p''s partner is better than r'?

p: … > r' > … > r > …

p': … > … > r' > … > …

r': … > p' > … > p > …

Proof of proposer optimality

What if p''s partner is better than r'?

Common technique: Upgrade the original assumption so that the situation was the *first* time a valid partner was rejected.

Then p''s match in the new matching cannot be better than r'!

$$\begin{array}{llllllllll}
p: & \dots & > & r' & > & \dots & > & r & > & \dots \\
p': & \dots & > & \dots & > & r' & > & \dots & > & \dots \\
 & & & & & & & & & \\
r': & \dots & > & p' & > & \dots & > & p & > & \dots
\end{array}$$

Proof of receiver pessimality

By contradiction. Suppose the output is not receiver-pessimal.

This means that for some r, the output matches (p, r), but p' is also a valid partner and r prefers $p > p'$.

r: … > p > … > p' > …

p: … > … > r > … > …

Proof of receiver pessimality

Since (p', r) are valid partners, consider that matching.

r: … > p > … > p' > …

p: … > … > r > … > …

Proof of receiver pessimality

Since (p', r) are valid partners, consider that matching.

By proposer optimality, r is the best possible match for p.

$$r: \quad \ldots > p > \ldots > p' > \ldots$$
$$p: \quad \ldots > \ldots > r > \ldots > \ldots$$

Proof of receiver pessimality

Since (p', r) are valid partners, consider that matching.

By proposer optimality, r is the best possible match for p.

So (p, r) is an unstable pair, contradiction.

$$r\colon \quad \ldots > p > \ldots > p' > \ldots$$
$$p\colon \quad \ldots > \ldots > r > \ldots > \ldots$$

Miscellaneous notes (just for fun)

Lying can help receivers

	Proposers						Receivers' true prefs				
1:	A	>	B	>	C	A:	2	>	1	>	3
2:	B	>	A	>	C	B:	1	>	2	>	3
3:	A	>	B	>	C	C:	1	>	2	>	3

With true preferences, A gets their second choice.

Lying can help receivers

Proposers					
1:	A	>	B	>	C
2:	B	>	A	>	C
3:	A	>	B	>	C

Receivers' true prefs					
A:	2	>	1	>	3
B:	1	>	2	>	3
C:	1	>	2	>	3

With A lying					
A:	2	>	3	>	1
B:	1	>	2	>	3
C:	1	>	2	>	3

With true preferences, A gets their second choice.

When A lies about second/third choices, A gets their top choice!

(Lying does not help proposers because of proposer optimality.)

Stable roommates problem

One group of $2n$ people and preferences over all $2n-1$ others.

1:	2	>	3	>	4
2:	3	>	1	>	4
3:	1	>	2	>	4
4:	1	>	2	>	3

(1, 2) and (3, 4)

(2, 3) unstable

(1, 3) and (2, 4)

(1, 2) unstable

(1, 4) and (2, 3)

(1, 3) unstable

Stable roommates problem

Stable solution may not always exist.

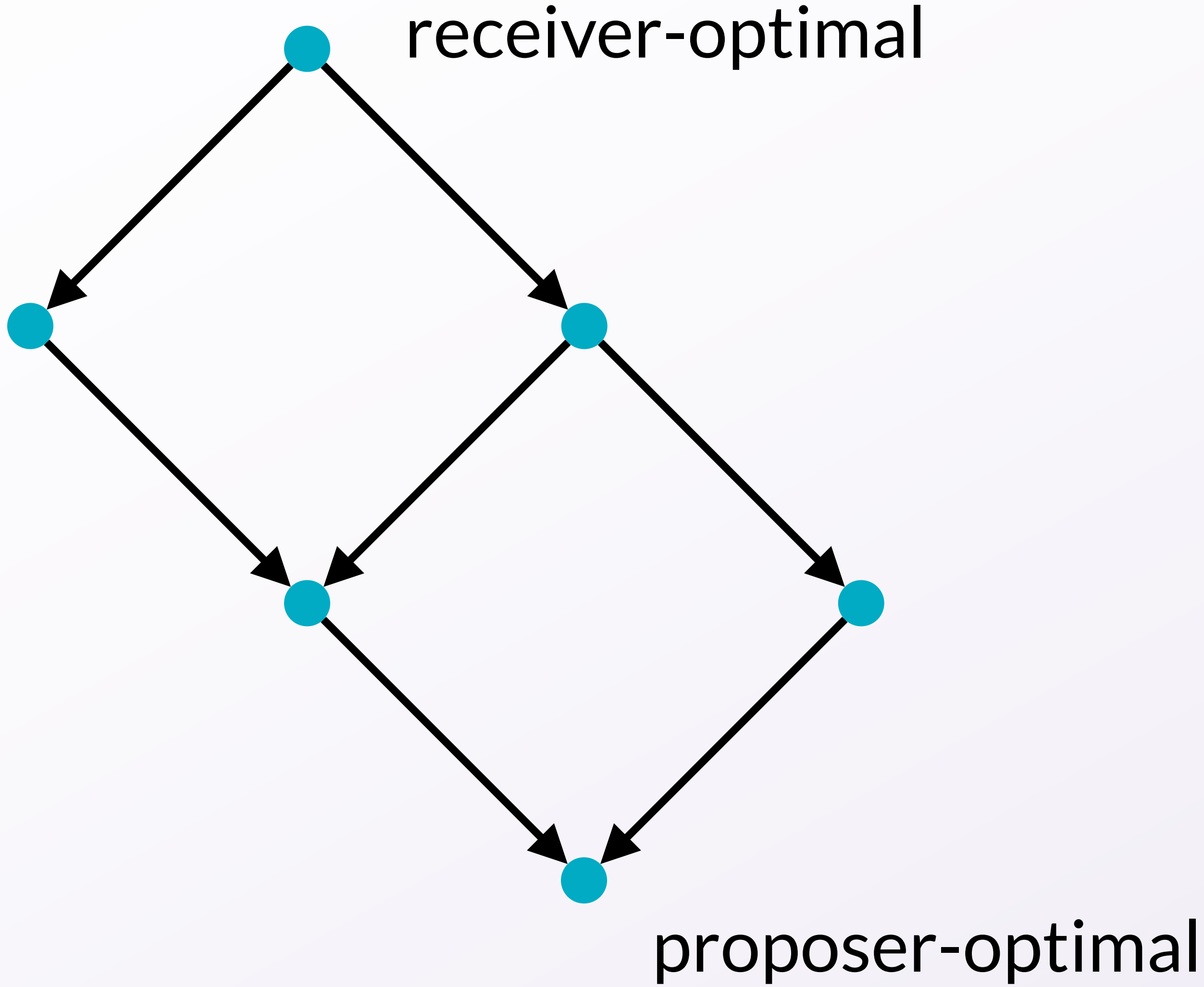
Irving's algorithm (1985) extends Gale–Shapley to determine if a stable solution exists, and if so, find it!

Characterizing all stable matchings

The set of all stable matchings forms an interesting structure called a **lattice**:

- Draw an arrow between from matching M_1 to M_2 if every proposer is happier in M_2 over M_1 (or equally happy).
- Then, every pair of matchings has a latest common ancestor and earliest common descendant.
- Gives alternate proof that a proposer-optimal matching exists!

Characterizing all stable matchings



Final reminders

HW1 due Friday @ 11:59pm.

I have OH now-12:30pm:

- Meet at front of classroom, we’ll walk over together
- CSE (Allen) 214 if you’re coming later

Nathan has online OH 12–1pm:

- Link on Canvas/course website
- https://washington.zoom.us/my/nathanbrunelle