PS2: Counting

Collaboration: You should work on the problems yourself, before discussing with others, including your cohorts at your cohort meeting. In addition to discussing with your cohortmates, you may discuss the problems with any other current CS3102 students you want, and use any resources you want except for any materials from previous offerings of this course or complete solutions that might be available on the web, which are not permitted. Sharing and subsequently submitting any text, code, images, figures, etc. constitutes plagiarism, so make sure all submitted materials are created exclusively by members of your cohort.

Problem 1: Function Properties Practice

Recall the following definitions:

Definition 1 (Injective) *A function $f : A \to B$ is injective provided that $\forall a_1 \in A. \forall a_2 \in A.(a_1 \neq a_2) \to (f(a_1) \neq f(a_2))$. Informally, a function is injective if different inputs always give different outputs.*

Definition 2 (Surjective) *A function $f : A \to B$ is surjective provided that $\forall b \in B. \exists a \in A. f(a) = b$. Informally, a function is surjective every member of the co-domain is the output for some member of the domain.*

Which of the following statements are True? For each statement that is not True, give a counterexample. For each statement that is, justify it with a proof (all of the true things below can be proven in one or 2 sentences, but slightly longer proofs are also acceptable).

1. If a function's domain is $\emptyset$ then that function is injective.
2. If a function's domain is $\emptyset$ then that function is **not** surjective.
3. For every set A, there exists an injective function $f : A \to \mathcal{P}(A)$
4. For every set A, there exists a surjective function $f : A \to \mathcal{P}(A)$
5. If A is countable and B is countable then there exists a bijection $f : A \leftrightarrow B$
6. If A is countably infinite and B is countably infinite then there exists a bijection $f : A \leftrightarrow B$

Problem 2: Countable or not?

For each of the following sets, indicate whether it is finite, countably infinite, or uncountable. Justify your choice with a proof (except as specified below). As a reminder, unless otherwise specified, strings are defined to have finite length (all parts can be proven in one or 2 sentences, but slightly longer proofs are also acceptable).

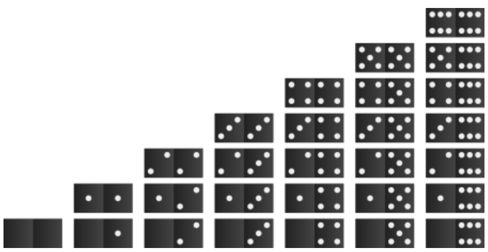
1. The set of all binary strings that represent the value 3102 (using the method described in class).
2. The set of all infinitely long trinary strings (i.e. the set $\{0, 1, 2\}^\infty$).
3. The set of all **finite** subsets of $\mathbb{N}$ (i.e. the set $\{s | s \in \mathcal{P}(\mathbb{N}) \wedge |s| \in \mathbb{N}\}$
4. The set of all python programs that you can run on your computer (as in the one that you're actually using right now as you're writing up the solution to this problem). A convincing but informal argument is acceptable here.

Problem 3: Fingers

The Cantorvanian creatures from the planet Cantorvania have only one hand, but it has a countably infinite number of fingers. A human glove has only 5 holes for fingers, so when a Cantorvanian wears one it will put many fingers into the same finger hole. To wear a glove Cantorvanians do not need to put their fingers into the glove's holes contiguously. For example, fingers 4 and 7 may go into hole 2, with finger 5 going into hole 5. It is ok if some glove holes have no fingers, but all fingers must be in a hole.[1]

Present a diagonalization argument to prove that there is an uncountable number of ways for a Cantorvanian to wear a human glove.

Problem 4: Infinite Dominoes



A domino is a tile with an **unordered** pair of numbers on it (e.g. $[0,5]$ or $[3,3]$). Dominoes come in sets containing all pairs of natural numbers less than or equal to some upper bound.

A pack of "double 3" dominoes will contain all unordered pairs of values from the set $\{0,1,2,3\}$. In particular, the set of all double 3 dominoes is $\{[0,0],[0,1],[0,2],[0,3],[1,1],[1,2],[1,3],[2,2],[2,3],[3,3]\}$. Note that $[1,2]$ is the same domino as $[2,1]$. A pack of "double 6" dominoes will contain all unordered pairs of values from the set $\{0,1,2,3,4,5,6\}$ (see image for the complete set).

A *domino chain* is a sequence of dominoes, with no duplicates, ordered so that the second value of each domino matches the first value of the next. The following domino sequences are valid chains:

– $[1,2][2,5][5,5][5,0]$

– $[10,9][9,8][8,7]$

The following domino sequences are not valid chains:

– $[1,2][2,5][5,4][0,1]$ - not valid because the second value of $[5,4]$ does not match the first value of $[0,1]$

– $[1,2][2,3][3,2][2,0]$ - not valid because $[3,2]$ is the same domino as $[2,3]$ and chains may not have duplicates.

Consider a pack of "double $\mathbb{N}$" dominoes, which contains all of the infinitely-many unordered pairs of natural numbers. Show that there is an uncountable number of infinite-length domino chains that can be constructed from a pack of "double $\mathbb{N}$" dominoes.

[1] I don't know about you, but after reading this question "finger" no longer sounds like a word, and I'm strangely aware of the way mine move.

Problem 5: Getting Help

If you're in a cohort, discuss each of the following with your cohort-mates.

Please include at least one response to each question (if you're in a cohort, try your best to give responses which best summarize the sentiment of your entire cohort). You're welcome to include multiple answers if you have more to share or else cannot reach a singular consensus.

- If you're in need of help on a problem set, where can you seek help (aside from TA and Professor office hours, though you're of course welcomed and encouraged to go there as well!). Please additionally share your thoughts on this question on #ps2 in discord.
- If you have gone to office hours already this semester, how did it go? If you haven't, why?
- Are there times where we currently do not offer office hours but you wish we did? If we were to adjust the office hours schedule, are there times we currently have office hours that you'd like us to continue offering them?
- Problem sets are released on Mondays. When do you first read them? When do you first start working on them? How well does this schedule work for you?

