Retake of Quiz 1: Induction and Uncountability

Collaboration: This quiz is open note but individual. You may use any resources that were provided to you by the course, or that you had recorded as part of your own notes prior to when you first viewed this quiz. You may not discuss this quiz with your cohort-mates, class-mates, tutor, friends, family, or anyone else except the course staff (who will answer clarification questions only).

Problem 1: Induction

Prove using induction that the number of binary strings of length $3n$ that match the regular expression $(001|110|111)^*$ is 3^n.

Problem 2: Diagonalization

Using a proof by diagonalization, prove that the following set is uncountable:

$$F = \left\{ f : \mathbb{N} \to \mathbb{N} \middle| (a > b) \to \Big(f(a) > f(b) \Big) \right\}$$

In other words, this is the set of all strictly increasing functions that map natural numbers to natural numbers. A function is strictly increasing if larger inputs are guaranteed to produce larger outputs.

For example, $f(x) = x^2$ is strictly increasing since if $a > b$ then $f(a) > f(b)$. However, $f(x) = (x - 5)^2$ is not strictly increasing since $1 < 2$ but $f(1) > f(2)$. Similarly, $f(x) = \lceil \frac{x}{2} \rceil$ is not strictly increasing since $0 < 1$ but $f(0) = f(1)$.