Exercise 1: SOLUTIONS

Response by: **SOLUTIONS**

Exercise 1-2: LED Bus Displays

Part 1) We will show that $97 \cdot 17 = 1649$ bits are required. To do this, we will find a bijection between the images displayable on the bus display, and binary strings of length 1649.

Mapping from images to $\{0,1\}^{1649}$: We begin by describing how to map images to binary strings.

Consider that we have a 97×17 image p. I will first number the pixels of p row-by-row (top-to-bottom, left-to-right) so that the top-left most pixel is labelled with 0, and the bottom-right most pixel is labelled with 1648. I will use p_i to refer to the pixel labelled with natural number i.

The image p will map to a bitstring b by assigning $b_i = 1$ if p_i is on (orange), and $b_i = 0$ otherwise (i.e. if p_i is black). Next we will show that this mapping is a bijection.

The mapping is one-to-one:

Consider that we have two different images p' and p''. We will show that the binary strings they map to (call them b' and b'' respectively) must be different as well.

For images p' and p'' to be different, they must differ by at least one pixel. Let i represent the index of this pixel. Since $p'_i \neq p''_i$ it must be the case according to our mapping that $b'_i \neq b''_i$, and so $b' \neq b''$.

The mapping is onto:

Consider an arbitrary binary string $b \in \{0,1\}^{1649}$, we will show that some image p maps to it. Essentially, this will be done with the “inverse” mapping of the above. The pixel p_i in p be on (orange) if $b_i = 1$ and black if $b_i = 0$. The produced image will map to b.

Conclusion:

Since there is a bijection between images and binary strings of length 1649, it must be that those two sets have the same cardinality. If we were use binary strings shorter than 1649, the set of binary strings must be smaller than the set of images, and so no onto mapping could exist. Thus we conclude that 1649 is the smallest number of bits which allow us to encode all binary strings.

Part 2) For this section we will show that we need 1648 (i.e. 1 fewer) bits to represent all images if we require the majority be off. We will do this by showing that the number of majority-on images is equal to the number of majority-off images. Since there are an odd number of pixels, every image must be either majority-on or majority-off, and so each of those sets must have half the cardinality of the set of all images. Since $2^{1648} = 2^{1649} \cdot \frac{1}{2}$, we can conclude that 1648 bits suffice.

The Mapping: To be able to conclude the above, we will show that there is a bijection between the set of majority-on images and the set of majority-off images. To do this, we will take a majority-on image and invert all of its pixels. Since that image was majority-on, and all orange pixels become black (and vice-versa), the new image is majority black.

Demonstrating it's a bijection: This mapping is one-to-one because if we have majority-orange images that differ by some pixel, those pixels will still be different in the majority-black images they map two (since both were inverted, they remain different). This mapping is onto because any majority-black image can be converted to a majority-orange image which maps to it by inverting the bits (the inverse of the inverse will get you back to the original image).