Problem Set 2: Cirque Du Ligne Droite

The first thing you should do in exercise2.tex is set up your name as the author of the submission by replacing the line, `\submitter{TODO: your name}`, with your name and UVA email id, e.g., `\submitter{Grace Hopper (gmh1a)}`.

Before submitting , also remember to:

- List your collaborators and resources, replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. (Remember to update this before submitting if you work with more people.)
- Replace the second line in exercise2.tex, `\usepackage{uvatoc}` with `\usepackage[response2]{uvatoc}`, `\usepackage[response3]{uvatoc}`, `\usepackage[response4]{uvatoc}`, `\usepackage[response5]{uvatoc}` for the appropriate problem submission.

Collaborators and Resources: TODO: replace this with your collaborators and resources (if you did not have any, replace this with *None*)

Exercise 2-2: Maximum number of Inputs

The *depth* of a circuit is the length of the longest path (in the number of gates) from the an input to an output in the circuit. Prove using induction that the maximum number of inputs for a Boolean circuit (as defined by Definition 3.5 in the book) that produces one output that depends on all of its inputs with depth d is 2^d for all $d \geq 0$. (Note: there are ways to prove this without using induction, but the purpose of this problem is to provide induction practice, so only solutions that are well constructed proofs using the induction principle will be worth full credit.

Exercise 2-3: Compare n bit numbers (Exercise 3.2 in TCS book)

Prove that there exists a constant c such that for every n there is a Boolean circuit (using only *AND*, *OR*, and *NOT* gates) C of at most $c \cdot n$ gates that computes the function $CMP_{2n} : \{0,1\}^{2n} \rightarrow \{0,1\}$ such that $CMP_{2n}(a_0 \cdots a_{n-1} b_0 \cdots b_{n-1}) = 1$ if and only if the number represented by $a_0 \cdots a_{n-1}$ is larger than the number represented by $b_0 \cdots b_{n-1}$.

Exercise 2-4: *NOR* is universal (Exercise 3.7 in TCS book)

Let *NOR* $: \{0,1\}^2 \rightarrow \{0,1\}$ defined as *NOR*$(a,b) =$ *NOT*(*OR*(a,b)). Prove that $\{$*NOR*$\}$ is a universal set of gates (i.e., anything that can be computed using *AND,OR,NOT* can also be computed using just *NOR*).

Exercise 2-5: XOR is not universal (based on Exercise 3.5 in TCS book)

Prove that the set $\{$*XOR*$, 0, 1\}$ is not universal. (You can use any strategy you want to prove this; see the book for one hint of a possible strategy, but we think you may be able to find easier ways to prove this, and it is not necessary to follow the strategy given in the book.