Exercise 1: SOLUTIONS

Response by: **SOLUTIONS**

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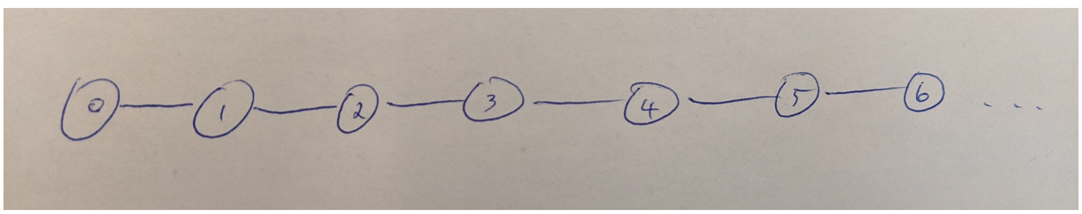
Exercise 1-4: Uncountable Sets

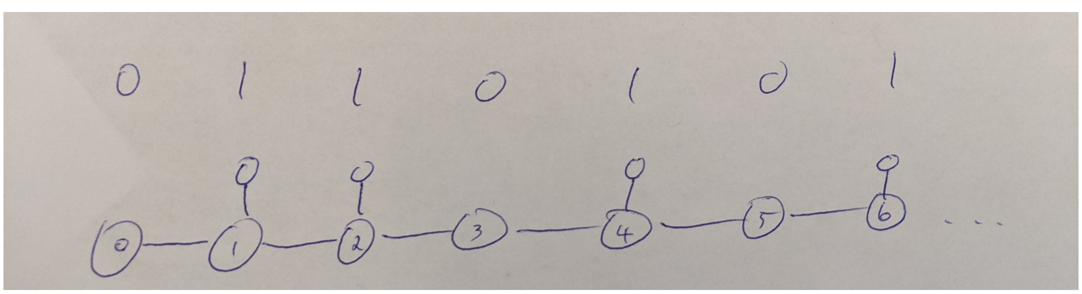
To show that the number of graphs (with a potentially infinite number of vertices) is uncountable, we will provide an injective (one-to-one) mapping from the inifinite binary strings (denote with $\{0,1\}^\infty$) to the infinite graphs (denote with G_∞).

To define this one-to-one mapping, the idea is to have a graph "gadget" to use as the basis to represent a string. We will consider a graph with an infinite number of vertices arranged in a "chain" like this:



In this case we add labels to each vertex for convenience. We will say that vertex v_0 is the (unique) node with degree 1 (i.e. only has one edge connected to it). The vertex adjacent to vertex v_0 we will call v_1, the vertex adjacent to v_1 that is not v_0 we ill call v_2, and in general will will label the vertices adjacent to v_i as v_{i+1} and v_{i-1} (with v_{i-1} as that vertex that is closer to v_0). In this way, we have defined an ordering of the vertices.

We can then map a binary string b a graph like the one above by adding additional vertices for each 1 that appears in b. In particular, if bit b_i in b is a 1, we will add an additional vertex to the graph that has an edge only to v_i. The following image illustrates an example of this mapping.



The mapping is one-to-one: To show that the mapping is one-to-one, we must show that two different strings map to different graphs. In this case, we know that if string $b' \neq b''$, then they must differ by at least one bit. Let's say that they differ at bit i, and without loss of generality we can say that $b' = 1$ and $b'' = 0$. In this case, if b' maps to graph g' and b'' maps to graph g'', we know that $g' \neq g''$ because vertex i in g' will have degree 3 whereas vertex i in graph g'' will have degree 2.