Week 2: Cantor's Proof

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This is a pdf version of the version of Cantor's proof presented in lecture.

Cantor's Proof

To Show: For any set S, $|S|< |2^S|$

Proof:

Strategy: By contradiction

Idea: Show that there cannot be a surjective (onto) mapping from S to 2^S

Assume toward a contradiction that $|S|\geq |2^S|$. This must mean that there exists a function $g : S \to 2^S$ that is surjective (onto).

We will use the function g to construct the set $P = \{a \in S | a \notin g(a)\}$, containing all elements of S that are not members of the subset of S which g maps them to. Note that $P \subseteq S$ and therefore $P \in 2^S$.

Since g is onto, and P is a member of the co-domain of g, it must be that there is some element $x \in S$ where $g(x) = P$. Since $x \in S$ it must be that either a) $x \in P$ or else b) $x \notin P$. We will show that both choices lead to a contradiction.

a) $x \in P$: If $x \in P$, then by definition of P (as the set of all things not in the set g maps them to) it must be that $x \notin g(x)$. By definition of x (as the element which maps to P), we can therefore conclude that $x \notin P$, which is a contradiction

b) $x \notin P$: If $x \notin P$, the by definition of P it must be that $x \in g(x)$. By definition of x, we can therefore conlcude that $x \in P$, which is a contradiction

We must therefore conclude that there cannot be a surjective (onto) mappin from S to 2^S, so therefore our original assumption that $|S|\geq |2^S|$ was incorrect. We thus conclude that $|S|< |2^S|$