Exercise 1: SOLUTIONS

Response by: **SOLUTIONS**

HONOR CODE NOTICE: SOLUTION SET, for use by CS3102 at UVA students in Spring 2020 only. Anyone reading this outside of that course for that semester is in violation of the honor code.

Exercise 1-3: Countable Graphs

To demonstrate that the number of finite graphs is countable, we will show a surjection from the Natural Numbers onto the set of finite graphs. Since a surjection from $\mathbb{N}$ to finite graphs (call the set of finite graphs G_f) means $|\mathbb{N}| \geq |G_f|$, and $\mathbb{N}$ is countable, we can conclude that G_f is countable as well. To then show that G_f is countably infinite, we will show that G_f is infinite.

G_f **is countable:** Consider that we have a graph of n vertices. For an undirected graph, any (unordered) pair of vertices defines a unique edge. We next derive an upper bound on the number of distinct graphs of n vertices. The number of unordered pairs of n vertices is given by $\binom{n}{2} = \frac{n(n-1)}{2}$. Any graph of n nodes can be defined by a subset of the edges (identifying which edges belong to that graph), and therefore the number of unique graphs is upper-bounded by the cardinality of the powerset of the set of all unordered pairs of vertices, i.e. $2^{\frac{n(n-1)}{2}}$. Most importantly, this implies there are finitely many graphs of fixed size n.

Since for any n, there are only finitely many graphs, we can define a surjection from $\mathbb{N}$ to G_f by mapping 0 to the graph of 0 nodes, then 1 to the graph of 1 node, then $2, 3$ to each of the graphs with 2 nodes (one where the edge is present, the other where the edge is not), and for each n we will map the next $2^{\frac{n(n-1)}{2}}$ natural numbers to all the graphs of n nodes. Since every graph will appear on this list, by construction, this defines a surjection from the natural numbers to the finite graphs, and so the number of finite graphs is countable.

G_f **is infinite:** Since for any natural number n we can find a graph with n nodes (e.g. the graph with n nodes and 0 edges), and there are an infinite number of natural numbers, the number of graphs is also infinite.